Embezzlement of entanglement
Approx violation of conservation laws
& Entanglement in nonlocal games

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w/ Jesse Wang[3] (1311.6842 + ongoing work)

Built on initial results by van Dam & Hayden (0201041)

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Plan:

- Quantum mechanics notations
- Locality and correlations
- Schmidt decomposition and entanglement

- Embezzling of entanglement by reordering Schmidt coeffs
- Embezzling of entanglement by superposing different # of entangled states

- Violating conservation law by superposing different # of conserved quantities

- Limitations to embezzlement
- Nonlocal games that cannot be won with finite amount of entanglement

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e.g. $|\psi\rangle = \sum_i \alpha_i |i\rangle$, $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ : \\ \\ \alpha_d \end{pmatrix}$

$\sum_i |\alpha_i|^2 = 1$

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6. A measurement along comp basis	f: $C^d \to \Delta^d$ $\sum_i \alpha_i \|i\rangle \mapsto (\|\alpha_1\|^2,\|\alpha_2\|^2,\ldots\|\alpha_d\|^2)$

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- 2 measurements applied separately to the two sys result in independent outcomes (no mutual information)

- holds with any local operation applied before the meas

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6. Entangled states	e.g. $\sum_k \alpha_k \vert k\rangle\vert k\rangle$

- completely correlated measurement outcomes

Schmidt decomposition

Theorem. Let $|\psi\rangle \in C^{dA} \otimes C^{dB}$, $N = d_A \leq d_B$

Then, $\exists$ U, V s.t. $|\psi\rangle = \sum_{k=1}^{N} \alpha_k \, (U|k\rangle) \otimes (V|k\rangle)$.

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Obs 1: Local operations leave the Schmidt coeffs invariant

Obs 2: Conversely, if $|\psi_1\rangle$, $|\psi_2\rangle$ have the same set of Schmidt coeffs, then, $|\psi_1\rangle = U \otimes V |\psi_2\rangle$ for some isometries U,V.

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Relation to entanglement:

1. $|\psi\rangle$ entangled iff Schmidt rank ≥ 2.
2. "Amount" of entanglement $E(|\psi\rangle)$

   $= \text{entropy of } \{|\alpha_\kappa|^2\} = -\sum_k |\alpha_k|^2 \log |\alpha_k|^2$ (conserved)

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Schmidt coeffs	α_1	$a\alpha_1$	$b\alpha_1$
	α_2	$a\alpha_2$	$b\alpha_2$
	:	:	:
	α_N	$a\alpha_N$	$b\alpha_N$

Schmidt coeffs when Alice and Bob hold multiple systems:

If $|\psi\rangle_{AB} = \sum_k \alpha_k |k\rangle_A |k\rangle_B$

then $|\psi\rangle_{AB} |00\rangle_{A'B'} = \sum_k \alpha_k |k0\rangle_{AA'} |k0\rangle_{BB'}$

Schmidt coeffs: $\alpha_1, \alpha_2, \ldots \alpha_N$

If $|\phi\rangle = a|00\rangle + b|11\rangle$

then $|\psi\rangle_{AB} |\phi\rangle_{A'B'} = \sum_k a\alpha_k |k0\rangle_{AA'} |k0\rangle_{BB'} + b\alpha_k |k1\rangle_{AA'} |k1\rangle_{BB'}$

Schmidt coeffs: $a\alpha_1, a\alpha_2, \ldots a\alpha_N, b\alpha_1, \ldots b\alpha_N$

Octave demonstration with $\alpha_k \propto 1/\sqrt{k}$.

N=8;

α_1 through α_8 :

0.607	0.429	0.350	0.303	0.271	0.248	0.229	0.214

a = 0.8; b = 0.6;

a α_1 through a α_8, b α_1 through b α_8 :

0.485	0.343	0.280	0.243	0.217	0.198	0.183	0.172
0.364	0.257	0.210	0.182	0.163	0.149	0.138	0.129

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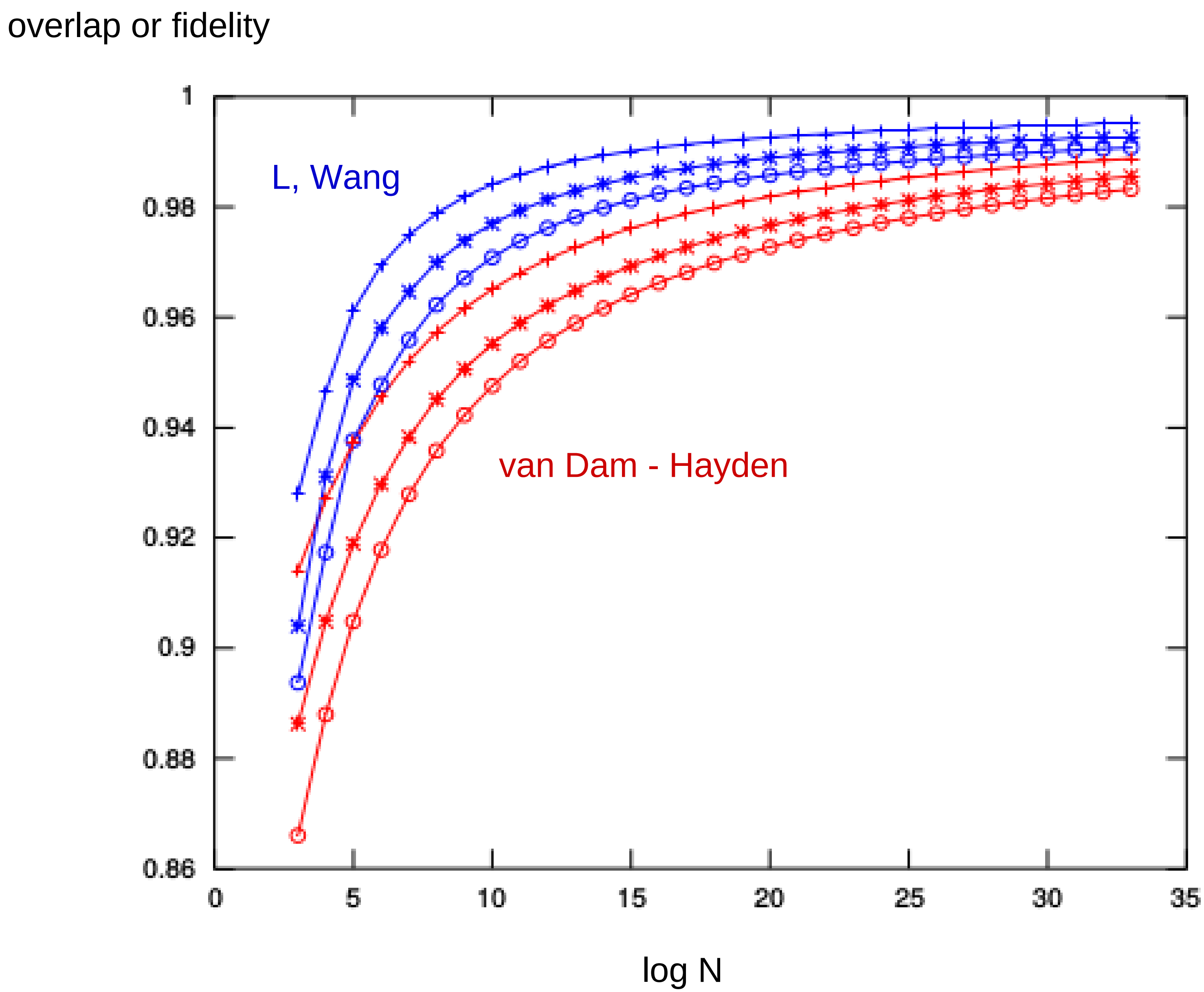
N=20; overlap = 0.90500
N=45; overlap = 0.92070
N=300; overlap = 0.94378


overlap or fidelity
L, Wang
van Dam - Hayden
1
0.98
0.96
0.94
0.92
0.9
0.88
0.86
0
5
10
15
20
25
30
35
log N

Embezzlement of entanglement (I)

Theorem. $\forall\ \varepsilon > 0$, $\forall\ d$, $\forall\ |\phi\rangle_{A'B'} \in C^d \otimes C^d$

$\exists\ N$, $\exists\ |\psi\rangle_{AB} \in C^N \otimes C^N$, $\exists\ U, V$

s.t. $(U_{AA'} \otimes V_{BB'})\ |\psi\rangle_{AB}\ |00\rangle_{A'B'} \approx^{\varepsilon} |\psi\rangle_{AB}\ |\phi\rangle_{A'B'}$!

(while $|\psi\rangle_{AB}\ |00\rangle_{A'B'} \not\leftrightarrow |\psi\rangle_{AB}\ |\phi\rangle_{A'B'}$)

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van Dam & Hayden (0201041)

- conceived such possibility !!
- proved the stronger, universal case where the same $|\psi\rangle$ works for all $|\phi\rangle$ (exchanging the red & blue quantifiers)
- relies heavily on the Schmidt decompositions for the bipartite setting

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Goal: $(U_{AA'} \otimes V_{BB'})\, |\psi\rangle_{AB}\, |00\rangle_{A'B'} \approx^{\varepsilon} |\psi\rangle_{AB}\, |\phi\rangle_{A'B'}$!

2nd method / interpretation:

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1. Choose $A = A_1 \ldots A_n$, $B = B_1 \ldots B_n$, $\dim(A_i) = \dim(B_i) = d$

$$|\psi\rangle_{AB} = C \sum_{r=1}^{n-1} |00\rangle_{A1B1}\, |00\rangle_{A2B2} \ldots |00\rangle_{ArBr}\, |\phi\rangle_{Ar+1Br+1} \ldots |\phi\rangle_{AnBn}$$

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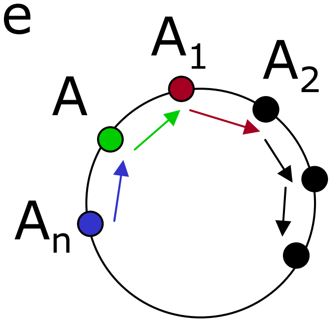
2. Choose $U_{AA'}$ as:

$$U |i_1\rangle_{A1} |i_2\rangle_{A2} \dots |i_n\rangle_{An} |i\rangle_{A'}$$
$$= |i\rangle_{A1} |i_1\rangle_{A2} \dots |i_{n-1}\rangle_{An} |i_n\rangle_{A'}$$

i.e. U permutes the systems cyclicly.



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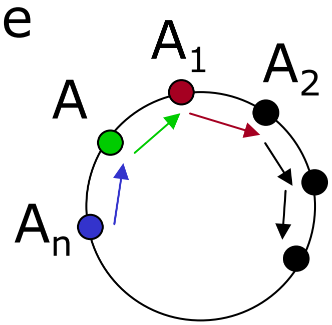
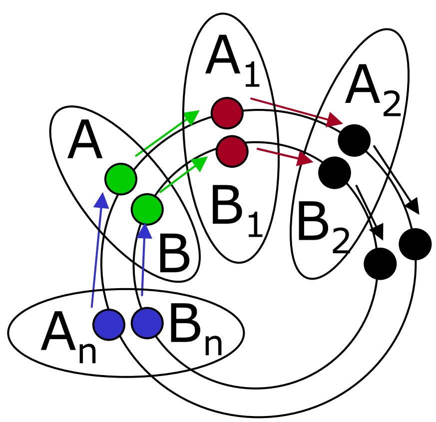


3. $V_{BB'}$ acts similarly.



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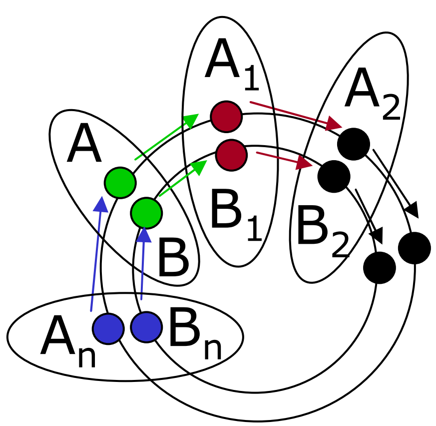

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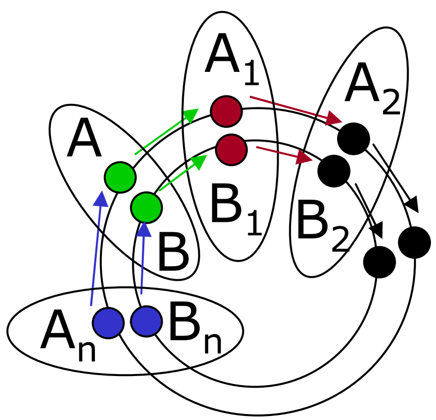

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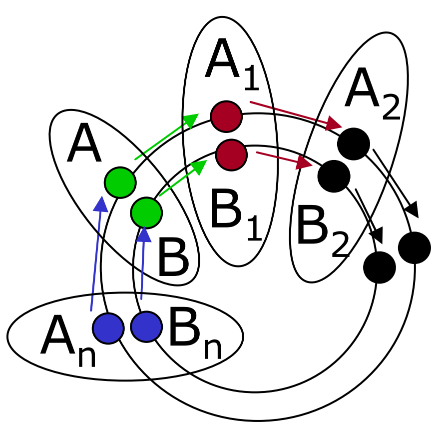
   $= \left(C \sum_{r=1}^{n-1} |00\rangle^{\otimes r+1}\, |\phi\rangle^{\otimes n-r-1}\right)_{AB} |\phi\rangle_{A'B'}$



Embezzlement of entanglement (II)

Goal: $(U_{AA'} \otimes V_{BB'})\, |\psi\rangle_{AB}\, |00\rangle_{A'B'} \approx^{\varepsilon} |\psi\rangle_{AB}\, |\phi\rangle_{A'B'}$!

2nd method / interpretation:

1. $|\psi\rangle_{AB} = C \sum_{r=1}^{n-1} |00\rangle_{A1B1}\, |00\rangle_{A2B2} \ldots |00\rangle_{ArBr}\, |\phi\rangle_{Ar+1Br+1} \cdots |\phi\rangle_{AnBn}$

   $= C \sum_{r=1}^{n-1} |00\rangle^{\otimes r}\, |\phi\rangle^{\otimes n-r}$

2. $(U_{AA'} \otimes V_{BB'})\, |\psi\rangle_{AB}\, |00\rangle_{A'B'}$

   $= (U_{AA'} \otimes V_{BB'})\, C \sum_{r=1}^{n-1} |00\rangle_{A1B1} |00\rangle_{A2B2} .. |00\rangle_{ArBr}\, |\phi\rangle_{Ar+1Br+1} ..\, |\phi\rangle_{AnBn}\, |00\rangle_{A'B'}$

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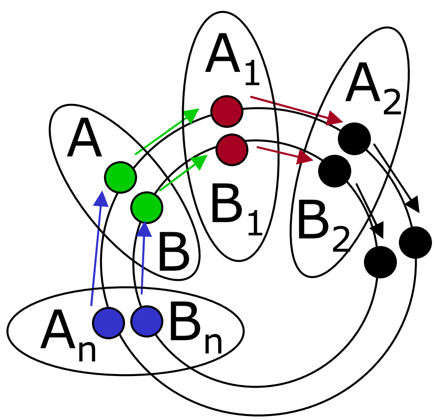

inner product with $|\psi\rangle_{AB}$ is $\geq 1-1/n$

$\therefore n = 1/\varepsilon$ suffices.

Embezzlement of entanglement (II)

Achieves $(U_{AA'} \otimes V_{BB'})\ |\psi\rangle_{AB}\ |00\rangle_{A'B'} \approx^{\varepsilon} |\psi\rangle_{AB}\ |\phi\rangle_{A'B'}$!

Summary of 2nd method:

1. $|\psi\rangle_{AB}\ |00\rangle_{A'B'} = C \sum_{r=1}^{n-1} |00\rangle^{\otimes r}\ |\phi\rangle^{\otimes n-r}\ |00\rangle_{A'B'}$
2. $(U_{AA'} \otimes V_{BB'})\ |\psi\rangle_{AB}\ |00\rangle_{A'B} = (C \sum_{r=1}^{n-1} |00\rangle^{\otimes r+1}\ |\phi\rangle^{\otimes n-r-1})_{AB}\ |\phi\rangle_{A'B'}$

Note: U, V do not depend on $|\phi\rangle_{A'B'}$.

Embezzlement of entanglement (II)

Achieves $(U_{AA'} \otimes V_{BB'}) \, |\psi\rangle_{AB} \, |00\rangle_{A'B'} \approx^{\varepsilon} |\psi\rangle_{AB} \, |\phi\rangle_{A'B'}$!

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Extension 1: works for any m-party states $|\phi\rangle_{A'B'}$

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Note: U, V do not depend on $|\phi\rangle_{A'B'}$.

Extension 1: works for any m-party states $|\phi\rangle_{A'B'}$

Extension 2: works for any initial state not just $|00\rangle_{A'B'}$

So, $|\psi\rangle = C \sum_{r=1}^{n-1} |\eta\rangle^{\otimes r}\, |\phi\rangle^{\otimes n-r}$ enables $|\psi\rangle_{AB}\, |\eta\rangle_{A'B'} \leftrightarrow |\psi\rangle_{AB}\, |\phi\rangle_{A'B'}$

Embezzlement of entanglement (II)

Achieves $(U_{AA'} \otimes V_{BB'})\, |\psi\rangle_{AB}\, |00\rangle_{A'B'} \approx^{\varepsilon} |\psi\rangle_{AB}\, |\phi\rangle_{A'B'}$!

Extension 3: Coherent state exchange,

i.e., embezzle / not in superposition

$(a\, |00\rangle_{AcBc}\, |\gamma\rangle_{A'B'} + b\, |11\rangle_{AcBc}\, |\eta\rangle_{A'B'})\, |\psi\rangle_{AB}$

$\rightarrow (a\, |00\rangle_{AcBc}\, |\gamma\rangle_{A'B'} + b\, |11\rangle_{AcBc}\, |\phi\rangle_{A'B'})\, |\psi\rangle_{AB}$

systems that controls whether to embezzle or not

<u>Extension 4</u> (approx violation of conservation laws)

Suppose operations are restricted and $|\eta\rangle \not\leftrightarrow |\phi\rangle$.

e.g., restricted to local operation, $|\eta\rangle=|00\rangle$, $|\phi\rangle=(|00\rangle+|11\rangle)/\sqrt{2}$

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e.g., $|\eta\rangle$ and $|\phi\rangle$ contain different amount of a *conserved* quantity

then $|\psi\rangle = C \sum_{r=1}^{n-1} |\eta\rangle^{\otimes r} |\phi\rangle^{\otimes n-r}$ enables the approx transformation

$$(a|0\rangle|\gamma\rangle + b|1\rangle|\eta\rangle)\, |\psi\rangle \leftrightarrow^{\varepsilon} (a|0\rangle|\gamma\rangle + b|1\rangle|\phi\rangle)\, |\psi\rangle$$

(Applying method 2 conditioned on the control register being 1,

note that conditioned permutation respect global conservation.)

Extension 5 (macroscopically-controlled q gates)

e.g., $|0\rangle_S$, $|1\rangle_S$ correspond to spin down and up respectively.

$|1\rangle_S$ is at a higher energy level than $|0\rangle_S$.

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Allowed: $|r\rangle_L\,|0\rangle_S \leftrightarrow |r\text{-}1\rangle_L\,|1\rangle_S$

where $|k\rangle_L$ represents k photos in a laser beam.

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Idea: use $|\psi\rangle_L = \sum_{r=1}^{n-1} |r\rangle_L$ so

$|\psi\rangle_L\,(a|0\rangle_S + b|1\rangle_S) \leftrightarrow \sum_{r=1}^{n-1} |r\text{-}1\rangle_L\;a|1\rangle_S + \sum_{r=1}^{n-1} |r+1\rangle_L\,b|0\rangle_S$

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both nearly indistinguishable from $|\psi\rangle_L$ so X gate nearly coherent.

Limits to embezzlement of entanglement

Qualitative no-go theorem: $|\psi\rangle_{AB}\,|00\rangle_{A'B'} \not\leftrightarrow |\psi\rangle_{AB}\,|\phi\rangle_{A'B'}$

Embezzlement: $\forall\,\varepsilon > 0$, $\forall$ d, $\forall\ |\phi\rangle_{A'B'} \in C^d \otimes C^d$

$\exists$ N, $\exists\ |\psi\rangle_{AB} \in C^N \otimes C^N$, $\exists$ U, V

s.t. $\langle\psi|_{AB}\,\langle\phi|_{A'B'}\,(U_{AA'} \otimes V_{BB'})\,|\psi\rangle_{AB}\,|00\rangle_{A'B'} \geq 1-\varepsilon$

So No-go theorem is not robust or continuous enough.

Idea: obtain lower bound on ε as a function of N by continuity of von Neumann entropy.

Limits to embezzlement of entanglement

Theorem:

If $\varepsilon > 0$, $|\phi\rangle_{A'B'} \in C^d \otimes C^d$, $|\psi\rangle_{AB} \in C^N \otimes C^N$,

and $\exists$ U, V s.t. $\langle\psi|_{AB}\ \langle\phi|_{A'B'}\ (U_{AA'} \otimes V_{BB'})\ |\psi\rangle_{AB}\ |00\rangle_{A'B'} \geq 1 - \varepsilon$

then $\varepsilon \geq 8\ [\ E(|\phi\rangle) / (\log N + \log d)\ ]^2$

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then $\varepsilon \geq 8 [E(|\phi\rangle) / (\log N + \log d)]^2$

Proof: Let $|\omega\rangle = (U_{AA'} \otimes V_{BB'}) |\psi\rangle_{AB} |00\rangle_{A'B'}$

Then $1- \varepsilon \leq \langle\psi|_{AB} \langle\phi|_{A'B'} |\omega\rangle_{AA'BB'}$

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$\Rightarrow 2\sqrt{2} \sqrt{\varepsilon}$

$\geq || \, |\psi\rangle\langle\psi|_{AB} \otimes |\phi\rangle\langle\phi|_{A'B'} - |\omega\rangle\langle\omega|_{AA'BB'} \, ||_1$

by relating fidelity and trace
distance between pure states

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monotonicity of trace distance
under quantum operations

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Fannes inequality for von Neumann entropy

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$\geq E(|\phi\rangle) / (\log N + \log d)$

So, embezzlement (and coherent state exchange) can be approximated better and better with larger and larger local dimensions, but never possible exactly.

Applications to nonlocal games.

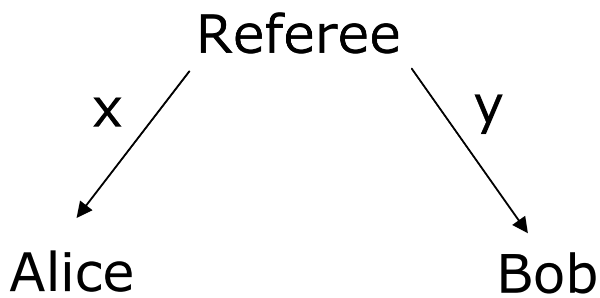
Nonlocal game:

Referee

Alice

Bob

<u>Nonlocal game:</u>



$x,y \sim p_{xy}$

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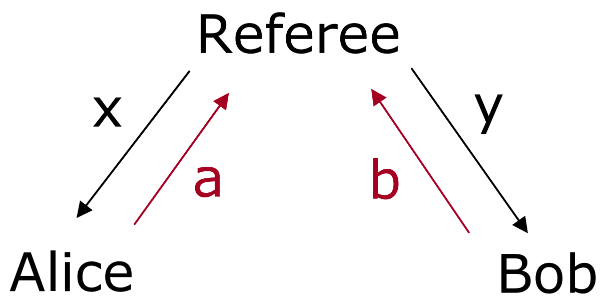

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R_{xy} set of (a,b) that wins if x,y are inputs

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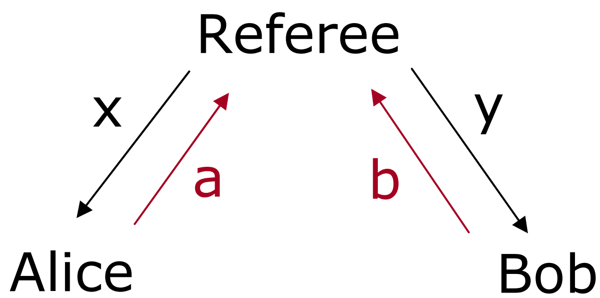

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Alice and Bob win if $(a,b) \in R_{xy}$

Value of the game G = $\omega(G)$ = max win probability

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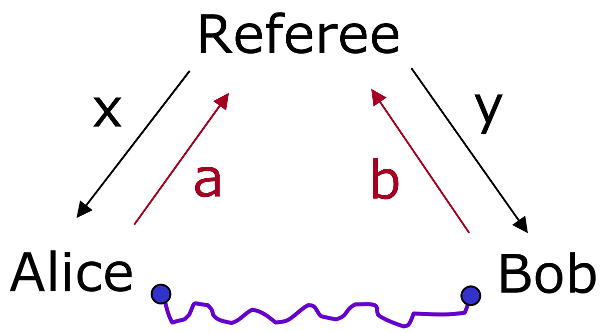

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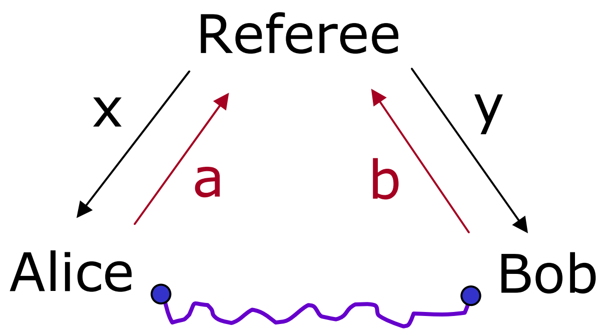

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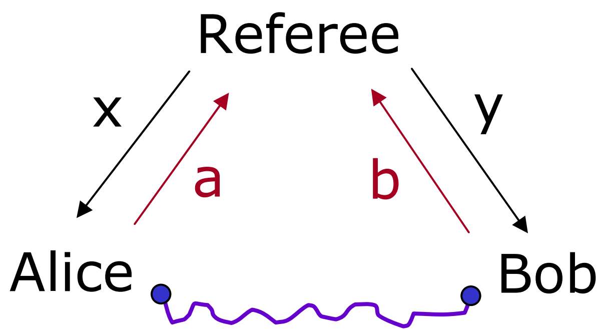
If $\omega^*(G) > \omega(G)$, the game corresponds to a Bell's inequality where x,y are measurement settings and a,b are outcomes.

e.g., $x,y,a,b \in \{0,1\}$, $(a,b) \in R_{xy}$ iff $ab = x \oplus y$ corr to CHSH ineq

Nonlocal game:



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Qn: how much and what type of entanglement is needed to attain the supremum?

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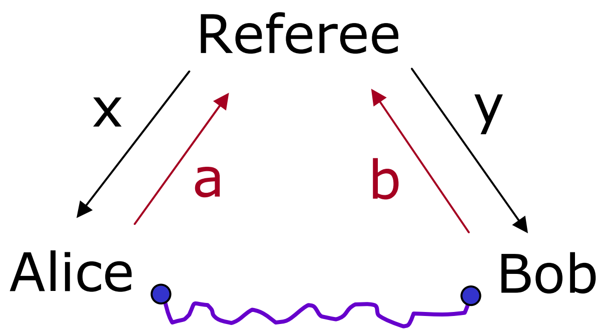

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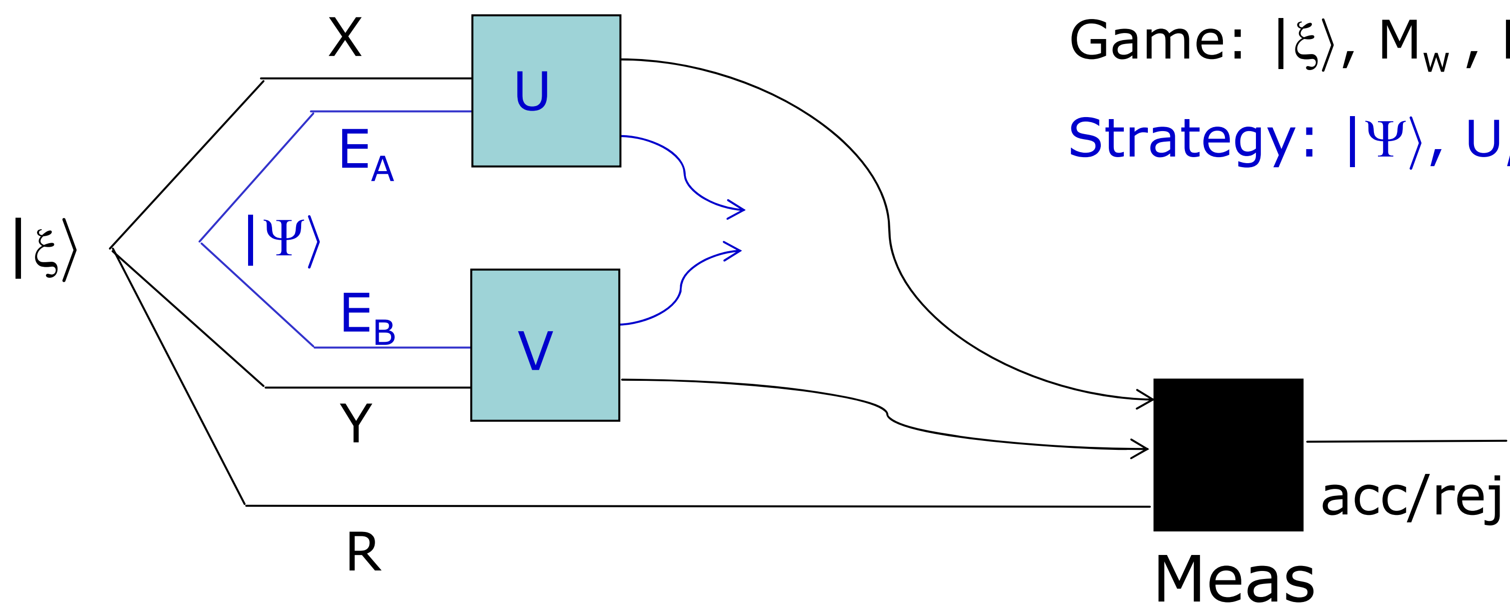
Open if sup attained with finite dim if $x,y \in \{0,1,2\}$, $a,b \in \{0,1\}$

Quantum cooperative game:

Referee prepares a quantum state $|\xi\rangle_{XYR}$,
sends X to Alice and Y to Bob
receives A from Alice and B from Bob
measures ABR according to POVM $\{M_w , M_l\}$

Alice and Bob win if outcome is w.

Qn: does sharing entangled state $|\Psi\rangle$ increases the winning prob?
how much and what entangled state are needed?



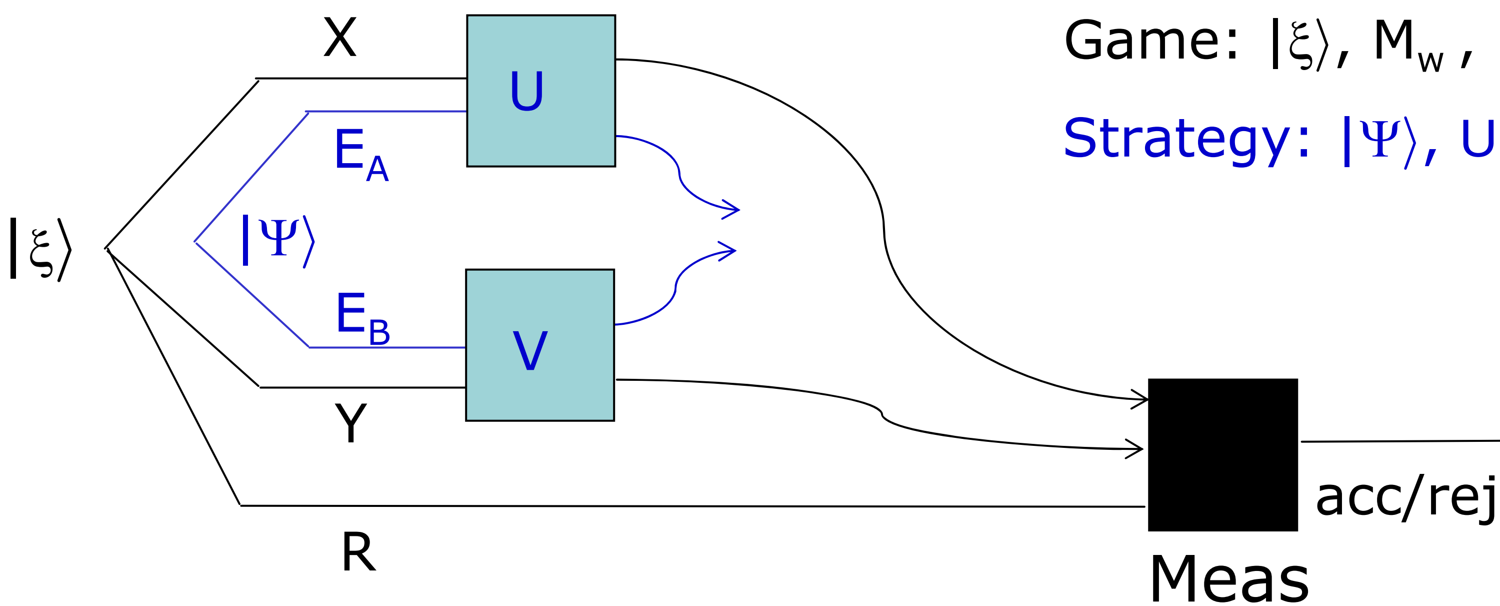
Game: $|\xi\rangle$, M_w , M_l

Strategy: $|\Psi\rangle$, U, V

Game that cannot be won with finite entanglement:

$$|\xi\rangle_{XYR} = \frac{1}{\sqrt{2}}\,(|0\rangle|00\rangle + |1\rangle|\Phi\rangle)_{RXY} \quad \text{where } |\Phi\rangle := (|11\rangle + |22\rangle)/\sqrt{2}$$

Let $|\gamma\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{RAB}$, POVM: $M_w = |\gamma\rangle\langle\gamma|$, $M_l = I - M_{acc}$



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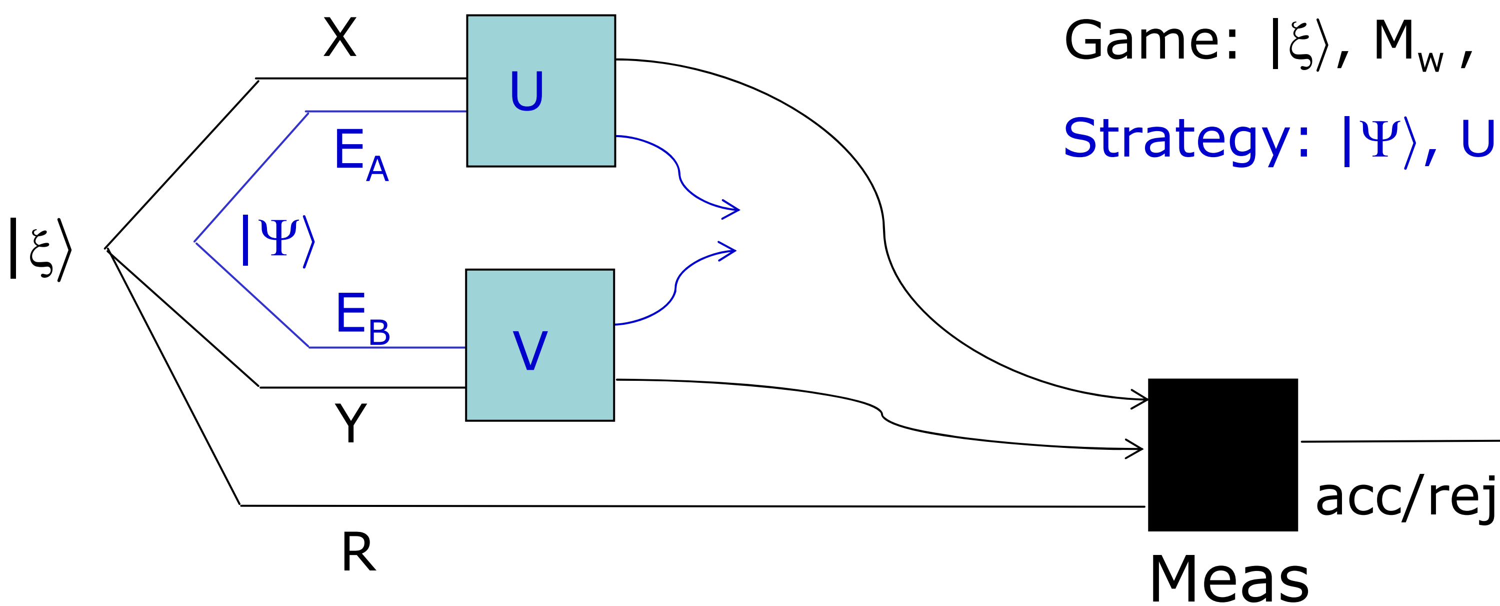
Let $|\gamma\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)_{RAB}$, POVM: $M_w = |\gamma\rangle\langle\gamma|$, $M_l = I - M_{acc}$

Then, with coherent state exchange, prob(win) increases with dim($E_{A,B}$) but never reaches 1.



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Strategy: $|\Psi\rangle$, U, V

Open problem 1

Now that we know there is no bound on the entanglement needed in the optimal prover strategy in general for quantum multi-prover interactive proof system ....

if we allow a small deviation from optimal, is there a bound on the amount of entanglement?

Simpler question: for cooperative games with fixed small (constant) system dimensions and ϵ, is there a universal (indep of game) upper bound on amt of entanglement that is sufficient to achieve accepting probability ϵ-close to optimal?

Open problem 2

The coherent state exchange protocol for 3 or more parties can be made universal (just like embezzlement of entanglement) but it is very inefficient. Is there a more efficient universal protocol?