COMBINATORIAL MODELS FOR SURFACE AND FREE GROUP SYMMETRIES

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COMBINATORIAL MODELS FOR SURFACE AND FREE GROUP SYMMETRIES

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SUMMARY

The curve complex of Harvey allows combinatorial representation of a surface mapping class group by describing its action on simple closed curves. Similar complexes of spheres, free factors, and free splittings allow combinatorial representation of the automorphisms of a free group. We consider a Birman exact sequence for combinatorial models of mapping class groups and free group automorphisms. We apply this and other extension techniques to compute the automorphism groups of several simplicial complexes associated with mapping class groups and automorphisms of free groups.

CHAPTER 1

INTRODUCTION

Ivanov's Metaconjecture and Free Group Automorphisms

This thesis considers two parallel simplicial complexes. The first is the complex of curves in a surface. The second is the complex of spheres in 3-space with wormholes. Both are *combinatorial models* for their respective spaces: any symmetry of the graph comes from a symmetry of the space itself.

Studies of the mapping class group of a surface make critical use of the curve complex. The curve complex $\mathcal{C}S$ has homotopy classes of simple closed curves as vertices and with simplices for disjoint collections of curves. Harvey first defined the curve complex $\mathcal{C}S$ in [1] to describe a compactification of Teichmüller space and study the mapping class group action. In the follwing years, the curve complex itself has become *the* space for mapping class group actions. Harer demonstrated the curve complex is simply connected [2], and Masur and Minsky showed that it is δ-hyperbolic [3], to the delight of Gromov-enthusiasts. Ivanov showed that the curve $\mathcal{C}S$ is an exact combinatorial model for the mapping class group; the automorphism group of $\mathcal{C}S$ is the mapping class group [4]. The resulting literary explosion of curve-complex rigidity results led Ivanov to propose his now infamous metaconjecture:

Metaconjecture 1. *Every sufficiently rich object associated to a surface S has as its group of automorphisms the mapping class group $\mathrm{MCG}^{\pm} S$. Moreover, this can be proved by a reduction to the theorem about the automorphisms of the curve complex $\mathcal{C}S$.*

The first statement of the metaconjecture is reasonable, natural, and just vague enough to leave no possibility of gainsay. The second part proved prophetic, as the literature has flourished with rigidity results that rely on the underlying automorphism group of the curve complex. We refer to Brendle and Margalit for a survey of such results [5]. Brendle and Margalit have made a heroic attempt to unify these results in [5]. There they show that the class of "sufficiently rich objects" includes any normal subgroup of a surface mapping class group containing an element with small support, and any connected simplicial complex of regions in S that does not have pairs of "exchangable" vertices.

Less is known about automorphisms of free groups than is known about mapping class groups, and some promising analogies between the two has translated geometric strategies of Thurston and others into strategies tackling the algebraic monster $\mathrm{Out}\,F$ of outer automorphisms of a free group F. Since finite graphs have a free group F as their fundamental group, one analog suggests considering $\mathrm{Out}\,F$ as the mapping class of a graph, though homotopy equivalences must substitute for diffeomorphisms. Just as the mapping class group acts on the Teichmüller space of hyperbolic metrics, Culler and Vogtmann defined an *outer space* of metrics of graphs [6]. There are several contenders for a curve complex analog. Culler-Vogtmann outer space itself retracts to a spine whose simplicial automorphisms are given by $\mathrm{Out}\,F$ [7]. Hatcher and Vogtmann suggested the poset of free factors [8], and Hatcher also considered the complex of free splittings [9].

A more direct geometric analog of the curve complex is given by the complex of spheres in 3-space with wormholes. By removing open balls of S^3 and identifying boundary spheres to form wormholes, we can obtain a manifold M with free fundamental group F. According to Laudenbach the diffeomorphism of M (up to homotopy) contains $\mathrm{Out}\,F$ as a finite index subgroup [10]. In fact the complex of free splittings of F is naturally isomorphic to the sphere complex $\mathcal{S}$ of M, with the spheres of M specifying conjugacy classes of splittings of $\pi_1(M) = F$ via Van Kampen's Theorem. Aramayana and Souto proved that the automorphism group of the sphere complex $\mathcal{S}$ is in fact $\mathrm{Out}\,F$ [11], by showing that automorphisms of $\mathcal{S}$ biject equivariantly to automorphisms of Culler-Vogtmann outer space. The analog with curves of a surface can be seen more concretely by considering a subsurface of M that tunnels through the wormholes. In minimal position, spheres are specified by curves of the surface. This suggests a route to proof of $\operatorname{Out} F$ theorems and an $\operatorname{Out} F$ analog to Ivanov's metaconjecture: when a proof calls for curves of a surface S, consider instead corresponding spheres of M.

This will be our major strategy. Here we advance the goal of an $\operatorname{Out} F$ analog to the Brendle-Margalit theorem, considering the following question:

Question 1. What combinatorial objects associated to a free group F have as their group of automorphisms the outer automorphism group $\operatorname{Out} F_n$, and when can this be proved by a reduction to the theorem about the automorphisms of Culler-Vogtmann outer space?

The results herein all adhere to this reduction by passing through the complex $\mathcal{S}$ of spheres in M_n. We consider this a particular incarnation of Margalit and Brendle's generalized metaconjecutre.

Metaconjecture 2. *Suppose that X is a nice space. Every object naturally associated to X and having sufficiently rich structure has $\operatorname{Out} \pi_1(X)$ as its group of automorphisms.*

Outline of Results

The novel technical results presented here are largely $\operatorname{Out} F$ analogs to theorems regarding the curve complex. We divide these results into three chapters. Chapter 3 considers the role of point pushing and the Birman exact sequence in the complex of curves of a surface and in the complex of spheres in M. Chapter 4 considers some subcomplexes of the sphere complex whose automorphism group is $\operatorname{Out} F$. Chapter 5 considers low complexity cases for the complex of strongly separating spheres.

Birman Point Pushing

In Chapter 3 we consider how adding or removing punctures affects the curve complex or the complex of spheres. In Section 3.1 we reprove the known result

Theorem 1.1. *Let $S_{g,p}$ be the orientable genus g surface with p punctures. If the natural map*

$$\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\, \mathcal{C}S_{g,p}$$

is an isomorphism, then so is

$$\mathrm{MCG}^{\pm} S_{g,p+1} \to \mathrm{Aut}\, \mathcal{C}S_{g,p+1}.$$

We do so by a new method considering the role of the Birman exact sequence for point pushing in the complex of curves. The proof follows the following outline.

1. For each puncture q there is a puncture-forgetting projection map $\rho_q : \mathcal{C}S_{g,p+1} \to \mathcal{C}S_{g,p}$ that parallels the Birman exact sequence for the mapping class group $\mathrm{MCG}\, S_{g,p}$, so that automorphisms which preserve the fibration of ρ_q must arise from mapping classes.

2. The fibers of the projection ρ_q are subtrees of $\mathcal{C}S_{g,p}$, with the projection ρ_q collapsing edges between curves that cobound punctured annuli.

3. The punctured annuli biject to an arc complex of $S_{g,p}$, which we show to be uniquely colored (in the graph-theoretic sense) by the punctures of the surface $S_{g,p}$ so that the fibers of the projection ρ_q for various punctures q biject to the coloring partition of the arc complex

4. Automorphisms of $\mathcal{C}S_{g,p}$ act by automorphism on this arc complex, so that the arc complex coloring, and thus the fibers of ρ_q, are maintained.

The main result of Section 3.2 uses an analogous proof. We show that

Theorem 1.2. *The natural map $\mathrm{Out}_{n,p} \to \mathrm{Aut}\, \mathcal{S}_{n,p}$ is an isomorphism for $n \geq 3$ and $p \geq 0$.*

where $\mathrm{Out}_{n,p}$ is a relative outer automorphism group and $\mathcal{S}_{n,p}$ is the complex of spheres in the punctured manifold $M_{n,p}$ with n "wormholes" and p punctures. The proof is fully analogous to the surface case. Automorphisms of $\mathcal{S}_{n,p}$ are shown to respect the fibration of a point-forgetting projection, so that the proof reduces to considering automorphisms of the sphere complex $\mathcal{S}$ of M.

Further Out

In Chapter 4 we consider subcomplexes and associated complexes of the sphere complex. The main technique of these proofs is to extend automorphisms from a subcomplex $\mathcal{S}'$ of $\mathcal{S}$ to the full sphere complex by finding a combinatorial characterization of spheres absent from $\mathcal{S}'$. Typically this is a *sharing pair* of spheres in $\mathcal{S}'$ that intersect in minimal position to bound spheres of $\mathcal{S}$. In Section 4.1 we prove

Theorem 1.3. *The natural map* $\mathrm{Out}(F_n) \to \mathrm{Aut}\,\mathcal{S}_n^{sep}$ *is an isomorphism for* $n \geq 3$.

The proof works by extending automorphism of the separating spheres complex to the nonseparating spheres by observing that small separating spheres contain a unique nonseparating sphere. In Section 4.2 we prove

Theorem 1.4. *For* $n \geq 3k$*, the natural map* $\mathrm{Out}\,F_n \to \mathrm{Aut}\,\mathcal{S}_n^{sep,k}$ *is an isomorphism.*

where $\mathcal{S}_n^{sep,k}$ is the complex of spheres freely splitting the rank n free group $\pi(M_n) = F_n$ into factors of rank at least k. The proof is by a sharing pair extension proceeding inductively on the rank k.

In Section 4.3 we prove the free factor complex $\mathcal{FF}_n$ is also a combinatorial model.

Theorem 1.5. *For* $n \geq 3$*, the natural map* $\mathrm{Out}\,F_n \to \mathrm{Aut}\,\mathcal{FF}_n$ *is an isomorphism.*

Strongly Separating Curves

In Chapter 5 we consider the complex of strongly separating curves. A curve is strongly separating if it is separating but does not bound a twice punctured disk of $S_{g,p}$. In [12]

Bowditch shows that the automorphisms of $\mathcal{C}^{ss} S_{g,p}$ are induced by the mapping class group $\mathrm{MCG}^{\pm} S_{g,p}$ in all but finitely many cases, and asks whether this is true of the remaining few. In Section 5.1 we use the point-forgetting projection techniques of Chapter 3 to obtain a few remaining cases. In Section 5.2 we give computational evidence for the undecided cases.

CHAPTER 2

PRELIMINARIES

Combinatorial Models

Our basic objects of study will be graphs and simplicial complexes. In general, by a combinatorial model for a group G, we mean any graph or simplicial complex whose automorphism group is naturally isomorphic to G.

Graphs and Simplicial Complexes

In the discussion below a *graph* will be a collection of vertices equipped with a collection of edges that are unordered pairs of vertices. We will allow graphs to have multiple, parallel copies of edges between a given pair of vertices, which we call multi-edges, as well as edges vv for a single vertex v, which we call self-loops. We call any graph *simple* if it has no multi-edges or self-loops. The *simplification* of any graph is the simple graph obtained by removing any self-loops and identifying the parallel copies of each multi-edge into a single edge. We say two vertices v and w are *adjacent* if vw is an edge, and we say two edges are *incident* if they share a vertex, or else say an edge is incident to its two vertices.

We will typically consider a *simplicial complex* $\mathcal{C}$ purely combinatorially as a set of vertices $\mathcal{C}^{(0)}$ equipped with a set of simplices, so an n-simplex is a set of $n+1$ vertices, and the faces of the simplex are the nonempty subsets. The n-skeleton $\mathcal{C}^{(n)}$ of the complex $\mathcal{C}$ is the subcomplex of all k-simplices of $\mathcal{C}$ for $k \leq n$. For any subset V of vertices, the *induced* subcomplex on V contains a simplex of $\mathcal{C}$ if and only if it is a subset of V. By the *link* of a vertex v, we mean the induced subcomplex of the vertices adjacent to v.

Occasionally we will abuse notation by failing to differentiate between a simplicial complex and its geometric realization. In particular, dimensions, interior points of simplices, and homotopy properties all refer to the geometric realization of a graph or a complex, rather than the combinatorial object itself.

Most of the simplicial complexes here considered are formed roughly like this:

1. Choose a space X.

2. Take as vertices of the complex all the homotopy classes of particular subspaces of X of a particular topological type

3. Declare a collection of homotopy classes to span a simplex if they have representatives that can be homotoped so that they are all mutually disjoint in S

Such a simplicial complex is typically *flag*, meaning that a set of vertices form a simplex if and only if they span a clique in the 1-skeleton $\mathcal{C}^{(1)}$. Since the one skeleton is a graph, we will frequently refer to the 1-simplices of a flag complex as edges.

Just as it is convenient to reduce higher simplices to their edges, so too do actions on the edges frequently determine actions on vertices. We recall Whitney's Graph Isomorphism Theorem [13], which states that the edge-incidence relation determines a simple graph, with a single exceptional pair.

Theorem 2.1. *An edge-incidence preserving bijection between two simple, connected graphs is a isomorphism, provided neither is the complete graph K_3.*

There is an edge-incidence preserving edge bijection between the complete graph K_3 and the complete bipartite graph $K_{1,3}$.

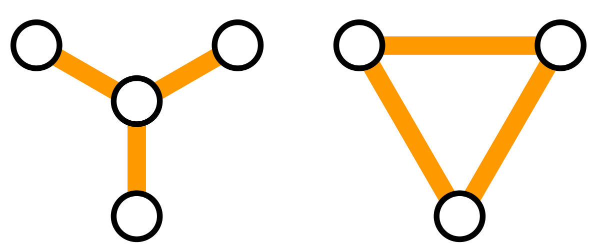


Figure 2.1: The complete bipartite graph $K_{1,3}$ and the triangle K_3 are the only simple graphs are not isomorphic yet have an incidence preserve bijection of edges.

One classical consideration of graph theory is the problem of chromatic numbers: How many colors are required to tag the vertices of a graph so that adjacent vertices have distinct colors? We will require a slight generalization where each vertex instead requires a given number of colors and adjacent vertices must share no color. One could replace vertices by cliques and consider classical colorings, but this introduces additional automorphisms to a graph, so we will instead consider colorings by disjoint sets of specified size.

Definition 2.2. A k, η-coloring of a graph $G = (V, E)$ is an assignment to each vertex v of a number of colors $\eta(v)$ and a choice $f(v) \subset \{1, \ldots, k\}$ of $\eta(v)$ colors such that two adjacent vertices have no common colors. I.e. a function $\eta : V \to \mathbb{Z}_+$ and $f : V \to 2^{\{1,\ldots,k\}}$ so that $|f(v)| = \eta(v)$ and

$$f(v) \cap f(u) = \varnothing$$

if v is adjacent to u in G. Call G k, η-colorable if it admits a k, η-coloring, and uniquely k, η-colorable if there is only one k, η-coloring up to bijection of the color set k. We will abusively refer to a k, η-coloring as a coloring if k and η are clear in context.

We will see in Chapter 3 that complexes of based loops and spheres are uniquely colorable by the base points.

Putman's Lemma

In [14] Putman outlines a clever and versatile argument for establishing the connectivity of a simplicial complex that admits a group action. Putman points out that it suffices to show that there is a special basepoint v_0 whose orbit intersects every connected component, and a special generating set H such that $H \cdot v_0$ has paths to v_0. This trick reduces many connectivity arguments to a few trivial checks—if the generating set can be chosen so that most elements leave the base point v_0 fixed or move it only a short distance. We will make abundant use of the following Lemma.

Lemma 2.3. *Let group G have generating set H and act on simplicial set X. Fix a basepoint $v \in X^{(0)}$. If*

1. *for all $v' \in X^{(0)}$ the orbit $G \cdot v$ intersects the connected component of v' in X and*

2. *for all $h \in H^{\pm}$ there is some path from v to $h \cdot v$*

then X is connected.

Proof. By *(1.)* it suffices to show that there is a path from v to $g \cdot v$ for every $g \in G$. Writing g as a word of H we may factor $g = h_w \cdots h_1 \cdot v$ as a word of $\in H^{\pm}$. By hypothesis there is a path s_j from v to $h_j \cdot v$ Then

$$s_1\ (h_1 h_2 h_1^{-1} \cdot s_2) \ldots (h_1 \cdots h_{k-1}) h_k (h_1 \cdots h_{k-1})^{-1} \cdot s_k$$

is a path from v to $g \cdot v$. □

We suggest a modification of Putman's Lemma that we will use to establish the uniqueness of the coloring of a complex. Let Γ be a graph with sets V and V' of vertices. We say that a set V *forces* a coloring on V' if for every k, η-coloring f of the induced subgraph of γ with vertices V, the extension of f to γ restricts to the same coloring on V'.

Lemma 2.4. *Let group G with generating set H act on a graph X. Fix a collection $V \subset X^{(0)}$ of k vertices. If*

1. *for every vertex $v \in X$ the orbit $G \cdot V$ forces a coloring on v, and*

2. *for all $h \in H^{\pm}$ we have V forces a coloring on $h \cdot V$*

then V forces a coloring on X, and in particular X is uniquely k, η-colorable.

Proof. Observe that forcing a coloring is a transitive relation on subsets of vertices. It suffices to see that V forces a coloring on its orbit $G \cdot V$, since the orbit forces a coloring on X. Suppose that W and W' are vertex sets such that W forces a coloring on W'. Let

$g \in G$. Then $g \cdot W$ must force a coloring on $g \cdot W$, or else we would have colorings f of W and two distinct colorings f' and f'' of W such that f extends to colorings restricting to f' and f'', and these pullpack to colorings $f \circ g$ of W, and $f' \circ g$, $f'' \circ g$ of W' contradicting that W forces a coloring on W'. Then if $g = h_w \cdots h_1$ as a word of $\in H^{\pm}$ we have $h_j \cdots h_1 \cdot V$ forces a coloring on $h_{j+1} \cdots h_1 \cdot V$, so that V forces a coloring on $g \cdot V$ by transitivity. $\square$

Bass-Serre Theory

We refer the reader to Serre [15] and for a fuller treatment of the theory of groups acting on trees. These trees provide an algebraic abstraction of covering spaces. In essence, if X is a space with subspace Y, by considering all lifts of Y to the universal cover $\tilde{X}$ we can form a tree whose vertices are the components of $\tilde{X}$ cut along all lifts of Y and equipped with an action by $\pi_1(X)$ as the deck transformations of the cover $\tilde{X}$.

A *graph of groups* Γ is a connected graph (V, E) together with a collection of vertex groups $\{G_v\}_{v \in V}$ and a collection of edge groups $\{G_e\}_{e \in E}$ together with inclusions of the edge groups into their incident vertex groups, that is for each edge $e = uv$ there are injections

$$G_u \xleftarrow[i_{uv}]{} G_e \xrightarrow{i_{vu}} G_v.$$

Then for any spanning tree T of Γ the fundamental group $\pi_1(\Gamma, T)$ is the group generated by $\{x_e\}_{e \in E}$ and the vertex groups G_v for $v \in V$, together with the relations $i_{uv}(g) = x_e i_{vu}(g) x_e^{-1}$ for all $e = uv$ and $g \in G_e$, and $x_e = 1$ for all $e \in T$.

The universal cover $\tilde{\Gamma}$ of the graph of groups Γ (with respect to $\pi_1(\Gamma, T)$) is the tree with vertices given by the left cosets of vertex groups in $\pi_1(\Gamma, T)$. The edges are given by the left cosets of edge groups in $\pi_1(\Gamma, T)$. So if gG_e is the left coset of edge group G_e with $g \in \pi_1(\Gamma, T)$ and edge $e = uv$, then gG_e is an edge between the $\tilde{\Gamma}$ vertices gG_u and gx_eG_v. The quotient $p : \tilde{\Gamma} \to \Gamma$ is given by $gG_x \mapsto x$ for any vertex or edge. In fact $\tilde{\Gamma}$ is a tree, and is equipped with action of $\pi_1(\Gamma, T)$ by left multiplication

$$h \cdot (gG_x) = (hg)G_x$$

for any $g, h \in \pi_1(\Gamma, T)$. The action of $\pi_1(\Gamma, T)$ on the tree $\tilde{\Gamma}$ thus has

$$\text{stab}_{\pi_1(\Gamma,T)}(gG_x) = gG_xg^{-1}$$

and respects the projection p and acts without inverting any edges of the tree. This tree action is unique in the sense described by the Fundamental Theorem of Bass-Serre Theory:

Theorem 2.5. *Let T be a tree with group G acting without inversions. If Γ is the quotient graph of groups with T any spanning tree, then G is isomorphic to $\pi(\Gamma, T)$, and there is an G-equivariant isomorphism betweeen T and the universal cover $\tilde{\Gamma}$ of Γ.*

Surface Models

We refer the reader to Farb and Margalit [16] as the definitive treatment of surface topology and surface group algebra, and to Hatcher [17] for theory and notation of homotopy, but we here establish notation and recall some relevant results.

Let $S_{g,p}$ be the orientable genus g surface with a set finite set P of p punctures. The mapping class group is

$$\mathrm{MCG}(S_{g,p}) = \pi_0\text{Diff}^+(S_{g,p})$$

the group of isotopy classes of orientation preserving homeomorpisms of $S_{g,p}$. We will also consider the extended mapping class group of orientation reversing homeomorphisms

$$\mathrm{MCG}^{\pm}(S_{g,p}) = \pi_0\text{Diff}(S_{g,p}).$$

By a *curve* of $S_{g,p}$ we mean the homotopy class of a simple closed curve, an embedded copy of the circle S^1. We will frequently abuse notation by refering to a curve both as the embedding $S^1 \hookrightarrow S_{g,p}$ and its homotopy class, as dictated by context. The same will be true of *loops* which we consider to be pointed embeddings $(S^1, s_0) \hookrightarrow (S_{g,p}, q)$ considered up to homotopy of $S_{g,p}$ fixing the basepoint q, and *arcs* which we consider to be embedded intervals $([0,1], 0, 1) \hookrightarrow (S_{g,p}, q_0, q_1)$ considered up to homotopy of $S_{g,p}$ fixing the endpoints q_0 and q_1 which we often allow to be punctures. We will often refer to what Farb and Margalit call the *change of coordinates principle*: curves x and y lie in the same MCG orbit if and only if their complements in $S_{g,p}$ are homeomorphic. Thus the topological types of curves are *nonsepararing*, and *separating* curves whose regular neighborhood complement is $S_{g'p'} \sqcup S_{g'',p''}$ where $g = g' + g''$ and $p + 2 = p' + p''$. For a separating curve x we refer to the connected components of its complement as the *sides* of x, and the *small side* as whichever has a less negative Euler characteristic.

The Curve Complex

Harvey defined the complex of curves $\mathcal{C}S_{g,p}$ as follows [1]. Take as vertices all homotopy classes of simple closed curves. A collection of curves forms a simplex if and only if they are mutually disjoint. Farb and Margalit give an extensive treatment in [16].

The works of Ivanov [4], Korkmaz [18], and Luo [19], describe the automorphisms of complexes of curves. Their theorem states that (except in a few low complexity cases) the curve complex is a combinatorial model for the mapping class group.

Theorem 2.6. *The natural map*

$$\mathrm{MCG}^{\pm}\, S_{g,p} \to \mathrm{Aut}\, \mathcal{C}S_{g,p}$$

is surjective whenever the curve complex $\mathcal{C}S_{g,p}$ has positive dimension $3g + p - 4$ and $(g,p) \neq (1,2)$, and an isomorphism if $(g,p) \notin \{(1,2),(2,0)\}$.

The Birman Exact Sequence

The Birman exact sequence describes the mapping class group of a punctured surface as a fibration over the mapping class group of the unpunctured surface, where the fundamental group is the fiber. Given a specified point $q \in S_{g,p}$ and loop $\alpha : ([0,1], 0, 1) \to (S_{g,p}, q, q)$ based at q we can construct a *point pushing map* that pushes q along the loop α. Construct the push map by taking an isotopy $H : [0,1] \times S_{g,p} \to S_{g,p}$ that is the identity outside of a neighborhood of α and so that $H(t, q) = \alpha(t)$ for all $t \in [0,1]$. Then $H(0, \cdot)$ and $H(1, \cdot)$ are isotopic homeomophisms of $S_{g,p}$ relative to P, but are not isotopic relative $P \cup \{q\}$. This embeds the fundamental group in the mapping class group as a subgroup of point-pushing maps in the group $\mathrm{MCG}^{\pm}(S_{g,p+1}, q)$ of mapping classes fixing the point q. The relationship is fully described by the following exact sequence due to Birman [20].

Theorem 2.7. *Let $q \in S_{g,p}$ be a puncture for negative Euler-characteristic $S_{g,p}$. The surface inclusion $S_{g,p+1} = S_{g,p} - \{q\} \hookrightarrow S_{g,p}$ induces the following short exact sequence*

$$1 \to \pi_1(S_{g,p}, q) \to \mathrm{MCG}^{\pm}(S_{g,p+1}, q) \to \mathrm{MCG}^{\pm} S_{g,p} \to 1.$$

Surprisingly, while the fundamental group of the surface is normal in the mapping class, we will see in the next section that the extended mapping class group itself is never normal in any supergroup, except as a direct product.

Mapping class group are only trivial normal subgroups.

In general, a fibration of groups is an exact sequence

$$1 \longrightarrow A \longrightarrow E \longrightarrow B \longrightarrow 1$$

and these are classified up to isomorphism by the outer automorphisms $\mathrm{Out}\, A$ of the fiber A and the cohomology $H^*(B; Z(A))$ of the base B. We refer to Brown for a general theory

[21]. But mapping class groups are centerless and out-less. We prove here centerless and out-less fibers always make for trivial fibrations, so fibrations with MCG as the fiber are always trivial for sufficiently complex surfaces. See Farb and Margalit [16] for centers of mapping class groups.

Theorem 2.8. *The center of* $\mathrm{MCG}^{\pm} S_{g,p}$ *is trivial, unless*

$$(g, p) \in \{(0, 2), (1, 0), (1, 1), (1, 2), (2, 0)\}$$

and in all these cases the center is isomorphic to $\mathbb{Z}/2$*.*

The curve complex $\mathcal{C}S_{g,p}$ plays a large role in the proof that $\mathrm{Aut}\,\mathrm{MCG}^{\pm} S_{g,p} \cong \mathrm{MCG}^{\pm} S_{g,p}$. The work of McCarthy [22], Ivanov [4], Korkmaz [18] demonstrates the following theorem showing that $\mathrm{Out}\,\mathrm{MCG}^{\pm} S_{g,p} = 1$.

Theorem 2.9. *Let* (g, p) *have* $g \geq 2$ *and* $g + p \geq 3$*, or* $g = 1$ *and* $p \geq 3$*, or* $g = 0$ *and* $p \geq 5$*. Let* G *and* G' *be finite index subgroups of* $\mathrm{MCG}^{\pm} S_{g,p}$*. Then any isomorphism* $G \to G'$ *is induced by an inner automorphism of* $\mathrm{MCG}^{\pm} S_{g,p}$*.*

Theorem 2.10. *A centerless, out-less group always fibers trivially.*

Suppose that

$$1 \longrightarrow A \hookrightarrow E \xrightarrow{\rho} B \longrightarrow 1$$

is a short exact sequence of groups. If A *has trivial center and outer automorphism group, then* $E \cong A \times B$*.*

Proof. Let $B = \langle S | R \rangle$ be a presentation for B. For each $s \in S$ choose $e_s \in E$ so that $\pi(e_s) = s$. Note that $A = \ker \rho$ is normal in E. So the conjugation $x \mapsto e_s x e_s^{-1}$ restricts to an automorphism of A. By hypothesis $\mathrm{Aut}\, A = \mathrm{Inn}\ A$, so there must be $a_s \in A$ such that $e_s x e_s^{-1} = a_s x a_s^{-1}$ for all $x \in A$. Since

$$\rho(e_s) = \rho(e_s a_s^{-1}) = s$$

we may replace e_s with $e_s a_s^{-1}$ so that $e_s x e_s^{-1} = x$ for all $x \in A$. So $\langle e_s \rangle_{s\in S}$ commutes with A in E. But then if $\prod_i s_i \in R$ is a relation of B, we have $\prod_i e_{s_i} \in A$ by the exact sequence, so $\prod_i e_{s_i} \in Z(A)$ is in the center of A. But by hypothesis the center is trivial $Z(A) \cong 1$, so $\prod_i e_{s_i} = 1$ in E. So $s \mapsto e_s$ extends to a homomorphism $B \to E$ that splits the exact sequence, and whose image commutes with A. It must be that $E \cong A \times B$. $\square$

Corollary 11. *The extended mapping class group has only trivial extensions. Let (g, p) have $g \geq 2$ and $g + p \geq 3$, or $g = 1$ and $p \geq 3$, or $g = 0$ and $p \geq 5$. Suppose that*

$$1 \longrightarrow \mathrm{MCG}^{\pm} S_{g,p} \hookrightarrow E \twoheadrightarrow B \longrightarrow 1$$

is an exact sequence of groups. Then $E \cong B \times \mathrm{MCG}^{\pm} S_{g,p}$.

In particular the normalizer N of $\mathrm{MCG}^{\pm} S_{g,p}$ in any group is a direct product $\mathrm{MCG}^{\pm} S_{g,p} \times N/\mathrm{MCG}^{\pm} S_{g,p}$.

Free Group Automorphism Models

We write F_n for the free group generated by n elements. The inner automorphisms given by conjugation are denoted Inn F_n so that

$$\mathrm{Out}\, F_n = \frac{\mathrm{Aut}\, F_n}{\mathrm{Inn}\, F_n}$$

We refer to Vogtmann for an excellent survery on what is currently known about $\mathrm{Out}\, F_n$ [23], but here recall some of the most relevant facts.

Let $a_1, \ldots, a_n$ be a generating set for F_n. Elements of $\mathrm{Aut}\, F_n$ include

1. a *permutation* is an automorphism that permutes the generating set $\{a_1, \ldots, a_n\}$
2. an *inversion* at a_i is an automorphism μ_i extended from $\mu_i(a_i) = a_i^{-1}$ and $\mu_i(a_j) = a_j$ for $j \neq i$.

3. a *transvection* is an automorphism τ_{ij} extended from $\tau_{ij}(a_i) = a_i a_j$ and $\tau_{ij}(a_k) = a_k$ for $k \neq i$

Nielson [24] showed that $\mathrm{Aut}\, F_n$ can be generated by taking any basis $a_1, \ldots, a_n$ of F_n and taking an inversion, a transposition, and the permutations of $\{a_1, \ldots, a_n\}$.

Many results of $\mathrm{Out}\, F_n$ are defined and proved analogously to results on surface mapping class groups. There are in fact several productive such analogies. The first considers elements of $\mathrm{Out}\, F_n$ as the mapping class group of graphs of rank n. Since the graphs are one dimensional, to obtain $\mathrm{Out}\, F_n$ the homotopy equivalences but be considered, rather than homeomorphism. This is perhaps best explored by Culler and Vogtmann [6], who define the *outer space* of metrics on a graph that functions for $\mathrm{Out}\, F_n$ just as Teichmüller space does for surface mapping class groups. Bridson-Vogtmann use techniques similar to Ivanov to show that $\mathrm{Out}\,\mathrm{Out}\, F_n = \mathrm{Out}\,\mathrm{Aut}\, F_n = 1$ [25].

A second analog instead considers $\mathrm{Out}\, F_n$ as the mapping class group of a doubled handlebody. Let M_n be the compact 3-manifold that is the connect sum $\#^n\,(S^1 \times S^2)$. Since $\pi_1(M_n, x_0) = F_n$, the diffeomorphisms of M_n act on F_n and provide a model for $\mathrm{Out}\, F_n$. The 3-manifold M_n makes an even closer analog to the surface S_g since one way to construct M_n is to take two copies of a genus n handlebody and glue their boundary surfaces by the identity map. To include boundary spheres, we let $M_{n,p}$ is the compact 3-manifold obtained from n copies of $S^1 \times S^2$ with the interiors of p disjoint balls removed. We take the convention that $M_{0,p}$ is S^3 with the interiors of p disjoint balls removed. Diff$(M_{n,p})$ is the group of orientation-preserving diffeomorphisms of $M_{n,s}$. Then the mapping class group of M_n π_0Diff$(M_{n,p}, \partial M_{n,p})$ can provide a model for $\mathrm{Out}\, F_n$. The group π_0Diff$(M_{n,p}, \partial M_{n,p})$ contains a finite normal subgroup N generated by order 2 Dehn-twists about nonseparating spheres. Laudenbach showed in [10] that

$$\frac{\pi_0 \mathrm{Diff}(M_n)}{N} \cong \mathrm{Out}\, F_n$$

and

$$\frac{\pi_0\textbf{Diff}(M_{n,1}, \partial M_{n,1})}{N} \cong \operatorname{Aut} F_n$$

Since neither these groups nor the sphere complexes distinguish between removing a point or a ball from M_n, we will abusively also refer M_n with a set P of p distinct points removed as $M_{n,p}$ where convenient, and refer to the set P as the punctures of $M_{n,p}$, since this sometimes unifies notation with the surface case.

For $p \geq 1$, discussion of *relative* free groups or free groups with boundary can be found in Meucci [26] and Hatcher and Wahl [27].

Consider F_{n+p} with basis $\{a_1, \dots, a_n, b_1, \dots, b_p\}$. Let $\mathrm{Aut}_{n,p}$ be the subgroup $\mathrm{Aut}_{n,p} < F_{n+p}$ with $\phi \in \mathrm{Aut}_{n,p}$ if ϕ preserves the conjugacy class of $\langle a_1, \dots, a_n\rangle$ and $\phi(b_i)$ is conjugate to b_j for some j. Let $\mathrm{Out}_{n,p} = \mathrm{Aut}_{n,p} / \mathrm{Inn}\, F_{n+p}$. An immediate consequence of the work of Hatcher and Vogtmann is that $\pi_0\textbf{Diff}(M_{n,p})/N \cong \mathrm{Out}_{n,p}$ [28]. In analogy with the pure mapping class group we will write

$$\mathrm{POut}_{n,p} \cong \pi_0\textbf{Diff}(M_{n,p}, \partial M_{n,p})/N$$

to be the subgroup $\mathrm{POut}_{n,p} < \mathrm{Out}_{n,p}$ that is the quotient from the subgroup of $\mathrm{Aut}_{n,p}$ with $\phi(b_i)$ conjugate to b_i for all i. So we have an exact sequence

$$1 \longrightarrow \mathrm{POut}_{n,p} \longrightarrow \mathrm{Out}_{n,p} \longrightarrow \mathrm{Sym}(p) \longrightarrow 1$$

where $\mathrm{Sym}(p)$ is the symmetric group on p symbols. Similarly let $\mathrm{Out}_{n,p}^{(q)}$ be the quotient of the $\mathrm{Aut}_{n,p}$ subgroup with $\phi(b_q)$ conjugate to b_q, so

$$1 \longrightarrow \mathrm{POut}_{n,p} \longrightarrow \mathrm{Out}_{n,p}^{(q)} \longrightarrow \mathrm{Sym}(p-1) \longrightarrow 1$$

Hatcher and Vogtmann provide a Birman type exact sequence for M_n. Let P be p marked points in M_n and $q \in M_n - P$. There is a fibration

$$\begin{array}{ccc} \text{Diff}(M_n, P \cup \{q\}) & \longrightarrow & \text{Diff}(M_n, P) \\ & & \downarrow \\ & & M_n - P \end{array}$$

where the projection is given by evaluation at q. The long exact sequence of homotopy groups associated to the fibration yields a Birman-like short exact sequence

$$1 \longrightarrow F_n \longrightarrow \pi_0\text{Diff}(M_n, P \cup \{q\})) \longrightarrow \pi_0\text{Diff}(M_n, P)) \longrightarrow 1$$

which after a quotient by the finite normal Dehn-twist subgroup yields an exact sequence

$$1 \longrightarrow F_n \longrightarrow \text{Out}^{(q)}_{n,p+1} \longrightarrow \text{Out}_{n,p} \longrightarrow 1.$$

So $\text{Out}_{n,p}$ is generated by

1. permutations of $\{a_1, \ldots, a_n\}$
2. permutations of $\{b_1, \ldots, b_p\}$
3. an a-inversion at a_1
4. an a-transvection τ_{ij} with $\tau_{ij}(a_i) = a_i a_j$ and τ_{ij} the identity on the other elements of the generating set
5. a conjugation of b_i by a_j with $\gamma_{ij}(b_i) = a_j b_i a_j^{-1}$ and γ_{ij} the identity on the other elements of the generating set

With this model of $\text{Out}\, F_n$ the analog of the curve complex is the complex of embedded spheres in M_n. Let $\mathcal{S}_{n,p}$ be the complex of spheres in $M_{n,p}$. The vertices of $\mathcal{S}_{n,p}$ are homotopy classes of essential, embedded 2-spheres $S^2 \hookrightarrow M_{n,s}$ in $\mathcal{S}_{n,p}$. We will abuse notation and say *sphere* to mean both a particular embedding $S^2 \hookrightarrow M_{n,p}$ and its homotopy

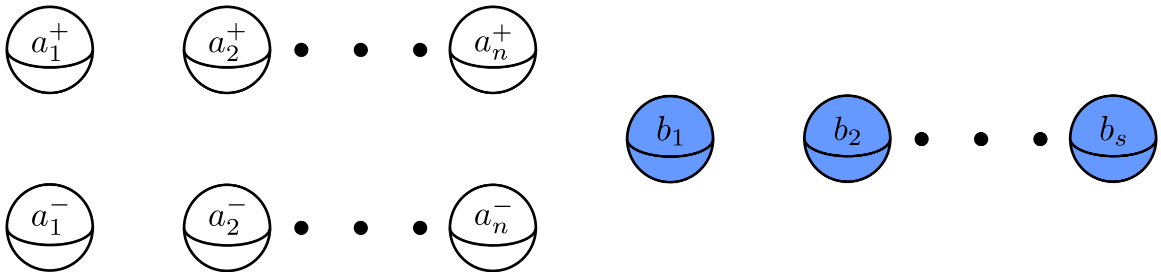


Figure 2.2: The manifold $M_{n,p}$ can be contained from S^3 by deleting the interior of $2n + p$ balls then identifying n pairs of spheres via the antipodal map.

class, as dictated by context. A set of spheres forms a simplex if there are mutually disjoint embeddings of the homotopy classes. Hatcher showed that the realization of $\mathcal{S}_n$ contains a dense subspace homeomorphic to Culler-Vogtmann outer space [9]. Aramayona and Souto showed that the sphere complex itself is a combinatorial model for $\mathrm{Out}\, F_n$ [11].

Theorem 2.12. *The natural map* $\mathrm{Out}\, F_n \to \mathrm{Aut}\, \mathcal{S}_n$ *is an isomorphism for* $n \geq 3$.

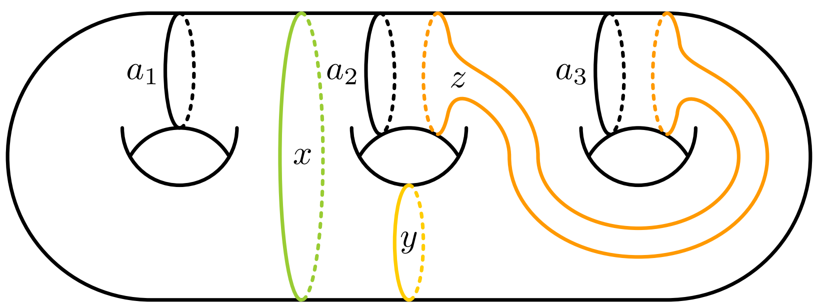


Figure 2.3: A maximal collection of disjoint curves bounding disks in the handlebody. The prescribed doubling gives spheres of M_3 specifying a maximal simplex of the sphere complex $\mathcal{S}_3$.

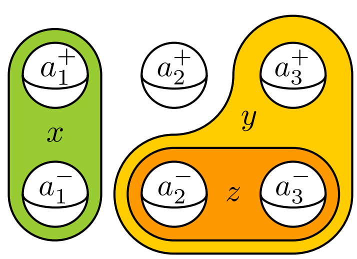


Figure 2.4: A maximal collection of disjoint spheres of S^3 with spheres removed. The prescribed gluing gives spheres of M_3 specifying a maximal simplex of the sphere complex $\mathcal{S}_n$.

There are two helpful diagramatic approaches to considering spheres in M_n, shown in Figures 2.3 and 2.4. The first represents a sphere by considering a disk-bounding curve in a genus n handlebody H_n,

$$x : (D^2, S^1) \hookrightarrow (H_n, S_n).$$

Then x induces an embedding $x : S^2 \hookrightarrow M_n$ by taking two copies of H_n and identifying the boundary $\partial H_n = S_n$ of the two copies

$$\begin{array}{ccccc} (D^2, S^1) & \longrightarrow & D^2 \sqcup_{S^1} D^2 & \longrightarrow & S^2 \\ \downarrow & & \downarrow & & \downarrow \\ (H_n, S_n) & \longrightarrow & H_n \sqcup_{S_n} H_n & \longrightarrow & M_n. \end{array}$$

This gives a surjection from homotopy classes of (D^2, S^1) in (H_n, S_n) to homotopy classes of S^2 in M_n, but this representation by disks is not unique. As in Figure 2.5, Dehn twists about disk bounding curves intersecting x give distinct disks that glue up to give homotopic spheres in M_n. The punctured manifold $M_{n,p}$ is similarly obtained by gluing a handlebody with p half balls removed along the surface S_n^p.

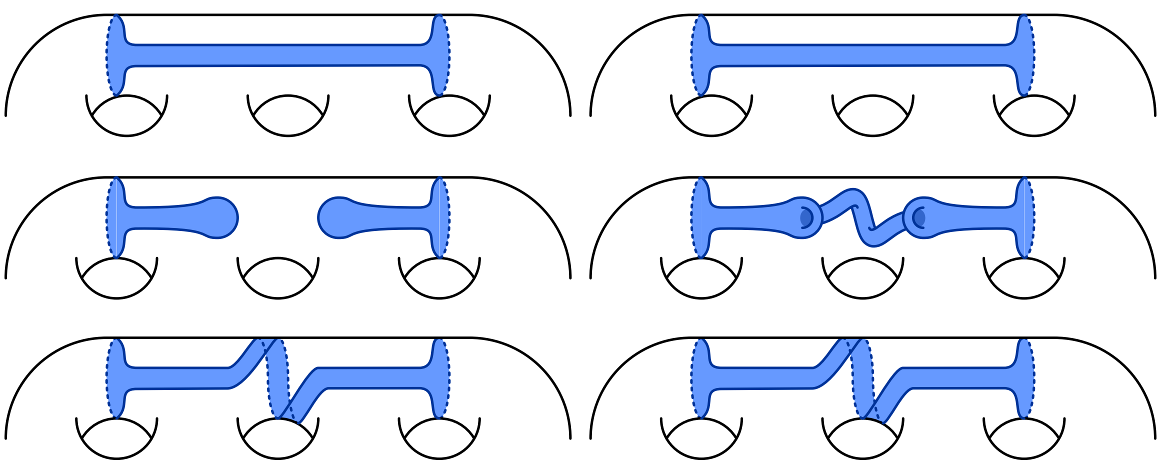


Figure 2.5: Identifying the two copies of a handlebody along their boundary obtains M_n and glues disks into spheres. Disks differing by Dehn twists about curves bounding disks in the handlebody glue to the same sphere.

A second diagramatic representation is to consider cutting $M_{n,p}$ along a collection of n disjoint nonsepating spheres to obtain S^3 with $2n + p$ open balls removed. Label the resulting boundary S^2 spheres $x^+_{a_1}, \dots, x^+_{a_n}$ and $x^-_{a_1}, \dots, x^-_{a_n}$ and $x_{b_1}, \dots, x_{b_p}$. Then identifying $x^+_{a_i}$ and $x^-_{a_i}$ via the S^2 antipodal map obtains $M_{n,p}$ again. Then for any basepoint $q \in M_{n,p}$ we have a basis for $F_n \cong \pi_1(M_{n,p}, q)$ as $a_1, \dots, a_n$ with a_i the loop disjoint from x_{a_j} for $j \neq i$ and intersecting x_{a_i} once by traveling into $x^+_{a_i}$ and out of $x^-_{a_i}$, which we defined to be positive intersection. Diffeomorphisms of M_n, p realizing $\mathrm{Out}_{n,p}$ can be described using this model. By capping every boundary component of $M_{n,p}$ with a copy of $M_{1,1}$ we can include $\mathrm{Out}_{n,p} \hookrightarrow \mathrm{Out}\, F_{n+p}$ and consider diffeomorphisms of $M_{n+p,0}$ that preserve (setwise and with orientation) the set of separating spheres where we glued the capping $M_{1,1}$ and the nonseparating spheres contained inside.

1. Permutations σ of $\{a_1, \dots, a_n\}$ can be realized by any diffeomorphism sending sphere x_{a_i} to $x_{\sigma(a_i)}$

2. Inversion $\iota_1 : a_1 \mapsto a_1^{-1}$ can be realized by cutting M_n along x_{a_1} exchanging the spheres $x^+_{a_1}$ and $x^-_{a_1}$ and then regluing $x^+_{a_1}$ and $x^-_{a_1}$

3. Transposition $\tau_{12} : a_1 \mapsto a_1 a_2$ can be realized by cutting M_n along x_{a_1}, pushing $x^-_{a_1}$ along a loop that intersects x_{a_2} once negatively and disjoint from x_{a_i} with $i \neq 1, 2$, and finally regluing $x^+_{a_1}$ and $x^-_{a_1}$ We will also refer to this as the push of $x^-_{a_1}$ through $x^-_{a_2}$. See Figure 2.6.

4. Conjugation $\eta : a_1 \mapsto a_1 b_1 a_1^{-1}$ can be realized by pushing x_{b_1} along a loop that intersects x_{a_1} once negatively and is disjoint from x_{a_i} for $i \neq 1$

By homotoping a sphere so that it is based at q, we obtain a splitting of $\pi_1(M_n, q) \cong F_n$, and in fact conjugancy classes of splittings of F_n are in bijection with spheres of M_n. By considering these splittings, Handel and Mosher show that $\mathcal{S}_n$ is δ-hyperbolic [29].

In [9] Hatcher shows $\mathcal{S}_n$ is contractible and describes a normal form for spheres embedded in M_n. A maximal collection Σ of disjoint spheres has $3n - 3$ spheres. (One could, for example take the disks of a pants decomposition in the handlebody.) Cutting M_n along Σ

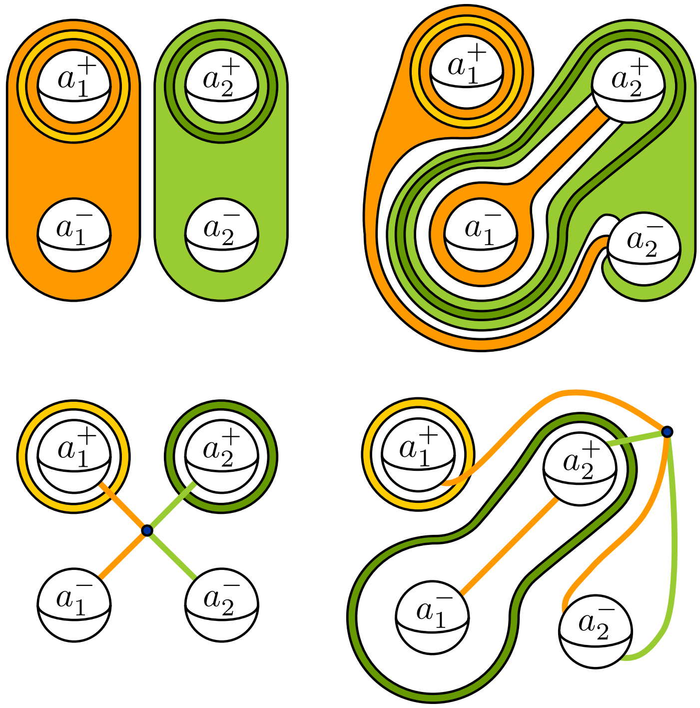


Figure 2.6: Pushing $x_{a_1}^-$ through $x_{a_2}^-$ induces the transvection $a_1 \mapsto a_1 a_2$ on the fundamental group π_1.

such a collection of spheres produces $2n-2$ copies of $M_{0,3}$. Hatcher shows that any sphere x of M_n can be homotoped so that x is parallel to a sphere of Σ or meets them tranversely in a nonempty colleciton of circles splitting x into components x_i such that

1. Each component x_i meets any sphere of Σ in at most one circle.

2. No component x_i is a disk isotopic by an isotopy fixing its boundary to a disk in a sphere of Σ

Further the homotopy class of x is uniquely determined by the data of the components x as a sphere parallel to a sphere of Σ, or else the components x_i as a disk, annulus, or pair of pants in each component of M_n cut along Σ, and the spheres of Σ that x_i boundary components intersect.

Just as in the case of curves of the surface, the spheres of a surface have a change of coordinates principle: spheres x and y lie in the same $\mathrm{Out}_{n,p}$ orbit if and only if their complements in $M_{n,p}$ are homeomorphic. Thus the topological types of spheres are *non-*

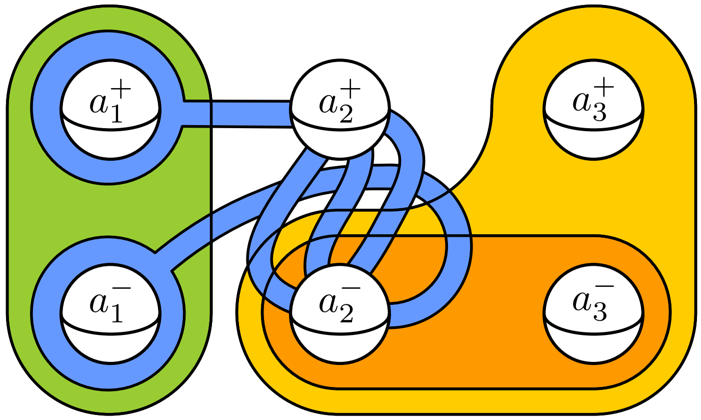


Figure 2.7: The sphere for the splitting $\langle a_1^{-1}a_2^4\rangle * \langle a_2, a_3\rangle$ in Hatcher normal form with respect to the maximal collection $\{a_1, a_2, a_3, x, y, z\}$ in M_3.

separaring, and *separating* spheres whose complement is $M_{n'p'} \sqcup M_{n'',p''}$ where $n = n'+n''$ and $p + 2 = p' + p''$. For a separating sphere x we refer to the connected components of its complement as the *sides* of x, and the *small side* as whichever has a less negative Euler characteristic. We refer to a sphere y on the small side of x as *engulfed* by x, or *engulfed* if it lies on the large side of x.

Pandit has shown that the nonseparating spheres also constitute a combinatorial model for $\mathrm{Out}\, F_n$ [30]. That is let $\mathcal{S}_n^{nonsep} \subset \mathcal{S}_n$ be the induced subcomplex spanned by nonseparating spheres. Pandit gives

Theorem 2.13. *For $n \geq 3$ the natural map*

$$\mathrm{Out}(F_n) \to \mathrm{Aut}\,(\mathcal{S}_n^{nonsep})$$

is an isomorphism.

CHAPTER 3

BIRMAN POINT PUSHING

In this chapter we consider the relationship of combinatorial models after adding or deleting punctures. We will see in Section 3.1 how the Birman exact sequence appears in the automorphisms of the curve complex. Section 3.2 follows a parallel outline to show that adding punctures to $M_{n,p}$ creates an analogous fibration of the sphere complex.

We recall the Theorem 2.6 of Ivanov [4], Korkmaz [18], and Luo [19].

Theorem 2.6. *The natural map*

$$\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\,\mathcal{C}S_{g,p}$$

is surjective whenever the curve complex $\mathcal{C}S_{g,p}$ has positive dimension $3g + p - 4$ and $(g, p) \neq (1, 2)$, and an isomorphism if $(g, p) \notin \{(1, 2), (2, 0)\}$.

Although their methods of proof are general and do not require separate consideration of the closed and punctured cases, we will demonstrate that additional punctures of the surface leave the isomorphism $\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\,\mathcal{C}S_{g,p}$ intact. We do so by attempting to substitute this ismorphism into the Birman exact sequence. Recall Birman's Theorem 2.7

Theorem 2.7. *Let $q \in S_{g,p}$ be a puncture for negative Euler-characteristic $S_{g,p}$. The surface inclusion $S_{g,p+1} = S_{g,p} - \{q\} \hookrightarrow S_{g,p}$ induces the following short exact sequence*

$$1 \to \pi_1(S_{g,p}, q) \to \mathrm{MCG}^{\pm}(S_{g,p+1}, q) \to \mathrm{MCG}^{\pm} S_{g,p} \to 1.$$

Curves and Punctures

Our goal in this section will be an independent proof of the following weaker version of Theorem 2.6, in preparation for the free group analog in Section 3.2.

Theorem 1.1. *Let $S_{g,p}$ be the orientable genus g surface with p punctures. If the natural map*

$$\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\, \mathcal{C}S_{g,p}$$

is an isomorphism, then so is

$$\mathrm{MCG}^{\pm} S_{g,p+1} \to \mathrm{Aut}\, \mathcal{C}S_{g,p+1}.$$

Remark 1. Every simplex Δ of $\mathcal{C}S_{g,p}$ is a collection of disjoint curves that cuts up the surface $S_{g,p}$ into a number of connected components. This gives a Bass-Serre graph of groups for $\pi_1 S_{g,p}$ induced by the Δ specified splitting. The underlying simple graph is the adjacency graph studied in Margalit, Behrstock [31] and Shackleton [32]. These also appear as graphs associated to pants decompositions in [33].

Definition 3.1. Let $\Delta \subset \mathcal{C}S_{g,p}$ be a simplex. The *region adjacency graph* $\mathcal{G}_\Delta$ of Δ is the graph whose vertices are the connected components of the cut surface

$$S_{g,p} - \bigcup_{c \in \Delta} c$$

with an edge for every curve c incident to the regions it bounds.

We will also consider the graph simplification $\mathcal{G}_\Delta^{simp}$ obtained from the (possibly looped, multi-edged) graph $\mathcal{G}_\Delta$ by removing any self-loops and identifying multi-edges.

Automorphisms of the curve complex act naturally on the set of adjacency graphs by isomorphism. Similar Lemmas are due to Margalit and Behrstock [31], though the graphs considered are simple graphs without multiedges or loops.

Lemma 3.2. *Curve complex automorphisms preserve the edge incidence of region adjacency graphs.*

Let $\phi \in \mathrm{Aut}\, \mathcal{C}S_{g,p}$ and let Δ be a simplex of $\mathcal{C}S_{g,p}$ with adjacency graph $\mathcal{G}_\Delta$. Then $e_c, e_{c'}$ are incident edges of $\mathcal{G}_\Delta$ if and only if $e_{\phi(c)}, e_{\phi(c')}$ are incident edges of $\mathcal{G}_{\phi(\Delta)}$.

Proof. We will argue that ϕ induces a bijection $\phi_* : E_{\mathcal{G}_\Delta} \to E_{\mathcal{G}_{\phi(\Delta)}}$ on the set of edges that preserves the incidence and non-incidence of edges.

Let e_c be an edge of $\mathcal{G}_\Delta$ given by curve c. Then $\phi_*(e_c) = e_{\phi(c)}$ defines a bijection between the edges of $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$. We will show e_c is incident to $e_{c'}$ if and only if there is a curve of $\mathcal{C}S_{g,n}$ intersecting c and c', but no other curve of Δ. Then $e_{\phi(c)}$ is incident to $e_{\phi(c')}$ if and only if there is a curve of $\mathcal{C}S_{g,n}$ intersecting $\phi(c)$ and $\phi(c')$, but no other curve of $\phi(\Delta)$.

Suppose e_c is incident to $e_{c'}$. Observe every region of $S_{g,p} - \bigcup_{c\in\Delta} c$ contains an embedded pair of pants $S_{0,3}$. So if we consider regluing regions along c and c', we obtain the component R of $S_{g,p} - \bigcup_{b\neq c,c'} b$ with $c, c' \subset R$. Then R must contain an embedded $S_{0,5}$ or an $S_{1,1}$. So R contains a curve c'' intersecting c and c', and since $c'' \subset R$, it does not intersect any other curve of Δ.

Suppose e_c is not incident to $e_{c'}$ in $\mathcal{G}_\Delta$. Then there is a multicurve $\Delta' \subset \Delta$ that separates c from c' in $S_{g,p}$. But then every curve that intersects c and c' must intersect a curve of Δ'. $\square$

Example 3. Edge incidence preservation is not always enough to guarantee a graph isomorphism.

Recall the Whitney Graph Isomorphism Theorem 2.1 states that for simple graphs an edge bijection preserving incidence is an isomorphism, except in the case of K_3. However, for non-simple graphs, edge incidence can be preserved by swapping a loop with a multiedge. This is the case for some automorphisms of $\mathcal{C}S_{1,2}$ acting on region adjacency graphs.

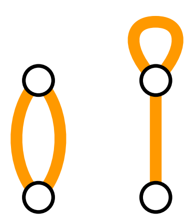


Figure 3.1: An edge bijection preserving incidence may not be an isomorphism for multigraphs. A self-loop might swap with a multiedge if the multiedge is not incident to additional edges.

Luo describes how a quotienting of $S_{1,2}$ by a hyperelliptic involution gives an isomorphism $\operatorname{Aut}\mathcal{C}S_{1,2} \to \operatorname{Aut}\mathcal{C}S_{0,5}$ [19]. An automorphism of $\mathcal{C}S_{1,2}$ bijects edges and preserves incidence of the region adjacency graph, but may not induce an isomorphism. The corresponding automorphism of $\mathcal{C}S_{0,5}$ induces an isomorphism of the region adjacency graph, as in Figure 3.2.

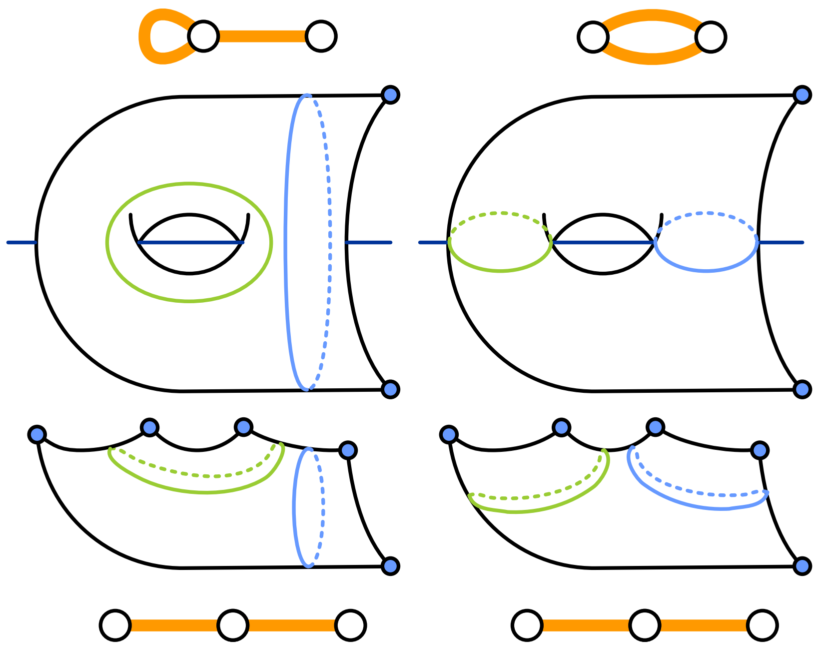


Figure 3.2: (Top) Two nonisomorphic region adjacency graphs can be exchanged by an automorphism of $\mathcal{C}_{1,2}$, though the edge incidence relation is preserved. (Bottom) Such an automorphism of $\mathcal{C}S_{1,2}$ corresponds via hyperelliptic involution to a homeomorphism of $S_{0,5}$.

Corollary 4. *Curve complex automorphisms induce isomorphisms of region adjacency graphs of maximal simplices.*

Suppose that $3g+p \geq 5$ and $(g,p) \neq (1,2)$. Let $\phi \in \operatorname{Aut}\mathcal{C}S_{g,p}$ and let Δ be a maximal simplex of $\mathcal{C}S_{g,p}$. Then $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$ are isomorphic graphs.

Proof. Any maximum simplex Δ gives a pants decomposition of the surface $S_{g,p}$ with $3g+p-3$ curves and $2g+p-2$ pairs of pants. So $\mathcal{G}_\Delta^{simp}$ and $\mathcal{G}_{\phi(\Delta)}^{simp}$ are simple, connected graphs with the same number of vertices and the same edge-incidence relations. Then by Whitney's Theorem 2.1, $\mathcal{G}_\Delta^{simp}$ is isomorphic to $\mathcal{G}_{\phi(\Delta)}^{simp}$.

To see that self-loops are preserved, observe that as Δ cuts $S_{g,p}$ into pairs of pants, every vertex of $\mathcal{G}_\Delta$ has degree at most 3. Then if e_c is a self-loop at vertex v_R, it is incident to exactly one other edge e_x that cannot be a self-loop, so e_x is uniquely represented in $\mathcal{G}_\Delta^{simp}$. If $(g,p) \neq (1,2)$, then e_x is incident to another edge e_y. Then $e_\phi(x)$ is also uniquely represented in the isomorphic graph $\mathcal{G}_\Delta^{simp}$ and has a degree 1 vertex. Then in $\mathcal{G}_\Delta^{simp}$, $e_\phi(x)$ is incident to $e_\phi(y)$, and $e_\phi(c)$ is incident to $e_\phi(x)$, but not $e_\phi(y)$ or any other edge, so $e_\phi(c)$ must be a loop at the vertex that is degree 1 in $\mathcal{G}_{\phi(\Delta)}^{simp}$. □

Lemma 3.5. *Automorphisms of the curve complex preserve the sides and topological type of curves.*

Suppose that $3g + p \geq 5$ and $(g,p) \neq (1,2)$. Let $\phi \in \operatorname{Aut}\mathcal{C}S_{g,p}$ and let x be a curve. Then there is a homeomorphism of $S_{g,p}$ exchanging x and $\phi(x)$. Furthermore, if x is separating and y, y' lie on the same side of x, then $\phi(y), \phi(y')$ lie on the same side of $\phi(x)$.

Proof. We will characterize each topological type of curve by a combinatorial property of a corresponding region adjacency graph, and apply Lemmas 3.2 and 4.

- Nonseparating curves: Observe that a curve c is nonseparating if and only if there is a maximal simplex Δ such that e_c is a self-loop in the region adjacency graph $\mathcal{G}_\Delta$.

- Separating curves: Observe that if curve x separates $S_{g,p}$

$$S_{g,p} = S_{g',p'} \sqcup_c S_{g-g',p-p'+2}$$

then the corresponding edge e_x of the region adjacency graph $\mathcal{G}_\Delta$ is a cut edge. More specifically, if

$$\Delta = \Delta_+ \cup \{x\} \cup \Delta_-$$

with Δ_+ and Δ_- the curves on each side of the separating curve c, then e_x separates $\mathcal{G}_\Delta$

$$\mathcal{G}_\Delta - \{e_x\} = \mathcal{G}_{\Delta_+} \sqcup \mathcal{G}_{\Delta_-}$$

into the components $\mathcal{G}_{\Delta_+}$, with $3g' + p' - 3$ edges and genus g', and $\mathcal{G}_{\Delta_-}$ with $3(g-g') + p - p' - 1$ edges and genus $g - g'$.

$\square$

Remark 2. For the closed surface S_g the inclusion $S_g - \{q\} \hookrightarrow S_g$ induces a well defined puncture-forgetting projection map

$$\rho_q : \mathcal{C}S_{g,1} \to \mathcal{C}S_g.$$

by sending curves to their image. Since we do not allow peripheral curves in $\mathcal{C}S_{g,1}$, no curve becomes nullhomotopic. However, in the case of multiple punctures P, the surface $S_{g,p}$ has curves bounding twice-punctured disks, that may become peripheral if a puncture is forgotten. Excluding these curves gives a subcomplex $\mathcal{C}(S_{g,p}, q) \subset \mathcal{C}S_{g,p}$ where the puncture-forgetting map is well-defined for puncture q.

$$\rho_q : \mathcal{C}(S_{g,p}, q) \to \mathcal{C}S_{g,p-1}$$

Kent, Leininger, and Schleimer [34] show that this forgetful projection has fibers described by Bass-Serre trees of the surface fundamental group so that there is a fibration of the form

$$\mathcal{T} \to \mathcal{C}(S_{g,p}, q) \to \mathcal{C}S_{g,p-1}.$$

More rigorously,

Theorem 3.6. *Let $\Delta \subset \mathcal{C}S_{g,p}$ be a simplex with interior point $x \in \Delta$. Then the fiber $\rho_q^{-1}(x)$ is $\pi_1(S_{g,p}, q)$-equivariantly homeomorphic to the tree $\mathcal{T}_\Delta$, the Bass-Serre tree for* *the splitting of* $\pi_1(S_{g,p}, q)$ *determined by the multicurve* Δ.

Remark 3. Observe that $\mathcal{C}(S_{g,p}, q)$ is not characteristic in $CS_{g,p}$, since in general automorphisms of $CS_{g,p}$ will permute the punctures. Let $\mathrm{Aut}(\mathcal{C}S_{g,p}, q) < \mathrm{Aut}\,\mathcal{C}S_{g,p}$ be the subgroup of $\mathrm{Aut}\,\mathcal{C}S_{g,p}$ that preserves the fibration $\mathcal{C}(S_{g,p}, q) \to \mathcal{C}S_{g,p-1}$, that is

$$\phi\left(\rho_q^{-1}\rho_q(x)\right) = \rho_q^{-1}\rho_q(\phi(x))$$

for every $x \in C(S_{g,p}, q)$.

If $\phi \in \mathrm{Aut}\,\mathcal{C}(S_{g,p}, q)$ then there is a well defined push-forward automorphism of the less-punctured quotient $\mathcal{C}S_{g,p-1}$. Define $\rho_q^*\phi \in \mathrm{Aut}\,\mathcal{C}S_{g,p-1}$ by

$$\left(\rho_q^*\phi\right)(x) = \rho_q\left(\phi(y)\right)$$

for any choice of $y \in \rho_q^{-1}(x)$. This is well defined since if $y' \in \rho_q^{-1}(x) = \rho_q^{-1}\rho_q(y)$ then

$$\phi(y') \in \phi(\rho_q^{-1}\rho_q y) = \rho_q^{-1}\rho_q\phi(y)$$

by definition of $\mathrm{Aut}\,\mathcal{C}(S_{g,p}, q)$. Then this gives a pushforward map

$$\rho_q^* : \mathrm{Aut}\left(CS_{g,p}, q\right) \to \mathrm{Aut}\,\mathcal{C}S_{g,p-1}$$

given by $\phi \mapsto \rho_q^*\phi$ as above.

These automorphisms display the structure of the Birman exact sequence.

Lemma 3.7. *This diagram commutes*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S_{g,p-1}, q) & \longrightarrow & \mathrm{MCG}^{\pm}(S_{g,p}, q) & \xrightarrow{f_q} & \mathrm{MCG}^{\pm}\, S_{g,p-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(S_{g,p-1}, q) & \xrightarrow{\alpha} & \mathrm{Aut}\,\mathcal{C}(S_{g,p}, q) & \xrightarrow{\rho_q^*} & \mathrm{Aut}\,\mathcal{C}S_{g,p-1} & \longrightarrow & 1
\end{array}$$

and has exact rows when ρ_q is surjective.

Proof. The pushforward $\rho_q^* : \operatorname{Aut}\mathcal{C}(S_{g,p}, q) \to \operatorname{Aut}\mathcal{C}S_{g,p-1}$ is defined by

$$(\rho_q^*\phi)(x) = \rho_q(\phi(y))$$

where $y \in \rho_q^{-1}(x)$. This is well defined since if $z \in \rho_q^{-1}(x)$ then $\phi(z)) \in \rho_q^{-1}\rho_q(\phi(y))$ by definition of $\operatorname{Aut}\mathcal{C}(S_{g,p}, q)$.

The map α is defined by the first square, so it certainly commutes. And α gives a well defined injection, since for any nontrivial loop γ there is a nonseparating curve c that intersects γ so that the point pushing map $\alpha(\gamma)$ moves c, and $c \in \mathcal{C}(S_{g,p}, q)$.

The second square must commute, since if $[\psi] \in \mathrm{MCG}^{\pm}(S_{g,p}, q)$ is a mapping class and c a curve of $S_{g,p}$, the homotopy class of $\psi(c)$ is the same if we first allow homotopies of the homeomorphism ψ which do not fix q, or if we first consider the homeomorphism ψ up to homotopy fixing q, then homotope the curve $\psi(c)$ forgetting q.

As in Theorem 3.6, the fiber $\rho_q^{-1}(x)$ of the projection $\rho_q : \mathcal{C}(S_{g,p}, q) \to \mathcal{C}S_{g,p}$ for a curve x is homeomorphic to the Bass-Serre tree $\mathcal{T}_x$ given by the splitting x specifies on $\pi_1(S_{g,p}, q)$. Then the kernel $\ker\rho_q$ is a group acting faithfully on the tree $\mathcal{T}_\Delta$, so by the Fundamental Theorem of BassSerre Theory 2.5, $\ker\rho_q^*$ is isomorphic to the fundamental group π_1 of the quotient graph of groups, but the corresponding graph of groups is exactly the Van Kampen splitting of π_1 induced by x. Thus

$$\ker\rho_q^* = \text{image}\ \alpha \cong \pi_1(S_{g,p}, q)$$

and the second row is exact. □

We will show that, though curve complex automorphisms might not preserve the fibers of ρ_q for any particular puncture q, they do permute the fibers of the puncture-forgetting projections $(\rho_q)_{q\in P}$. We do so by proving the unique colorability of an arc complex. Korkmaz's proof of Theorem 2.6 utilizes a slightly more general arc complex allowing peripheral arcs [18] and simplices of arcs that share endpoints.

Definition 3.8. Define the pointed arc complex $\mathcal{A}S_{g,p}$ to be the complex of homotopy classes of embedded non-peripheral arcs in $S_{g,p}$ with endpoints in P, where an arc is peripheral if it is a separating loop based at a single puncture and one of its sides is a punctured monogon of $S_{g,p}$. Two arcs or disks are adjacent in $\mathcal{A}S_{g,p}$ if their homotopy classes have disjoint representatives and share no punctures as endpoints.

Remark 4. The pointed arc complex $\mathcal{A}S_{g,p}$ has as vertices both arcs with two distinct endpoints and loops based at a single puncture. Since loops that are disjoint are always based at distinct punctures, there is an obvious way to color (in the graph-theoretic sense) the vertices of $\mathcal{A}S_{g,p}$ that are loops: assign a color to each puncture and all the loops based at that puncture. The arcs with distinct endpoints require two colors, however. We make a slight generalization of k-colorings to allow a privileged set of vertices that that require two colors.

Recall from Definition 2.2 that a k, η-coloring assigns to each vertex x a set of $\eta(x)$ colors from k options so that adjacent vertices have disjoint color sets.

Remark 5. Margalit in [35] discusses the trinion pants complex. Pants decompositions of $S_{g,p}$ are maximal simplices of $\mathcal{C}S_{g,p}$, with two such pants decompositions giving sharing an edge in the pants complex if they differ by a single pair of minimally intersecting curves. Hatcher and Thurston demonstrated the pants complex (which they call *markings*) of a surface is connected and simply connected in [33]. We recall their result as the following lemma.

Lemma 3.9. *Let Δ, Δ' be maximal k-simplicies of $\mathcal{C}S_{g,p}$. Then there is a sequence $\Delta = \Delta_0, \Delta_1, \ldots, \Delta_n = \Delta'$ of maximal simplices such that $\Delta_i \cap \Delta_{i+1}$ is a $k-1$ simplex and the curves $c_i \in \Delta_i - \Delta_{i+1}$ and $c_i' \in \Delta_{i+1} - \Delta_i$ are contained in a single component R of $S_{g,p} - \bigcup_{x \in \Delta_i \cap \Delta_{i+1}} x$. Further, c_i, c_i' can be chosen to intersect once if $R \cong S_{1,1}$ and twice if $R \cong S_{0,4}$.*

Remark 6. Lemma 3.9 in particular implies that for any two maximal k-simplicies Δ, Δ' contained in a k-colorable subgraph of $\mathcal{C}S_{g,p}$ the simplex Δ forces a coloring on Δ'. Bestvina, Bromberg, and Fujiwara note bounds on the chromatic number of the curve graph $\mathcal{C}S_{g,p}$ in [36]. Gaster, Greene, and Vlamis further develop the theory of in [37], where they additionally consider colorings of a related arc complex that contains $\mathcal{A}S_{g,p}$ as a proper subcomplex, albeit with many more edge relations

We will show that $\mathcal{A}S_{g,p}$ is uniquely colorable.

Definition 3.10. Fix an ordering $\sigma : P \to \{1, \ldots, p\}$ of the punctures and let x be a curve of S_g. A *σ-nest* of curves parallel to x is the homotopy class in $S_{g,p}$ of an embedding

$$N : S^1 \times I \hookrightarrow S_{g,p}$$

such that N is a homotopic to x in S_g by homotopy forgetting the punctures, and image of $S^1 \times \{\sigma(i) - 1/(p-1)\}$ is a loop based at puncture $\sigma^{-1}(i)$ that we refer to as the rib N_i. The nest also gives a collection of $p-1$ arcs. The i^{th} *vertebra* of N is the arc from $\sigma^{-1}(i)$ to $\sigma^{-1}(i+1)$ and given by

$$N\left(\{s_0\} \times \left[\frac{\sigma(i)-1}{p-1}, \frac{\sigma(i+1)-1}{p-1}\right]\right)$$

for the basepoint $s_0 \in S^1$. We refer to $N(s_0 \times I)$ as the *spine* of N.

Example 11. Observe that any nest N in $S_{g,p}$ specifies a maximal simplex Δ_N in $\mathcal{A}S_{g,p}$. Further two σ-nests N, N' respectively parallel to disjoint curves x, x' of $S_{g,p}$ specify a length p sequence from Δ_N to $\Delta_{N'}$ of maximal simplices intersecting in codimension one faces as by replacing N_i with N_i', as in Figure 3.3. So Δ_N forces a p-coloring on $\Delta_{N'}$.

In fact even if x and y intersect Δ_N forces a p-coloring on $\Delta_{N'}$ if there are enough punctures. Consider two paths of maximal simplices intersecting in codimension one faces shown in Figure 3.4. First replace N_1, N_2 with the first vertebra $\alpha_{1,2} = N(s_0 \times [\sigma(1)/(p -$

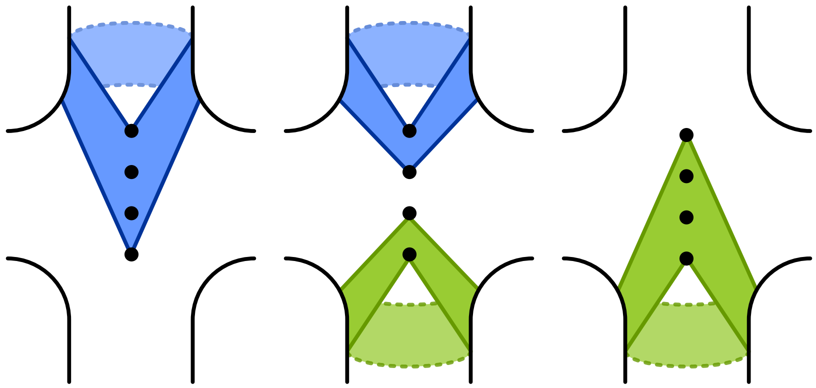


Figure 3.3: A color forcing path between nests parallel to disjoint curves.

$1), \sigma(2)/(p-1)])$ between the punctures $\sigma(1)$ and $\sigma(2)$. Then iteratively we replace N_i with the loop α_i based at $\sigma(i)$ and parallel to the boundary of a regular neighborhood of α_{i-1} for $i = 3$ to $i = p$. Then replace α_i with N'_i for $i = p$ to $i = 3$ to obtain a simplex $\{\alpha_2, N'_3, \ldots, N'_p\}$. Starting from the other end of the spine we construct a similar path with the puncture order reversed to obtain a simplex $\{N'_1, \ldots, N'_p, \alpha_{p-1,p}\}$. Since $\{\alpha_2, N'_3, \ldots, N'_p\}$ and $\{N'_1, \ldots, N'_p, \alpha_{p-1,p}\}$ together contain $\Delta_{N'}$, it must be that Δ_N forces a p-coloring on $\Delta_{N'}$.

We will use this technique to demonstrate that Δ_N forces a coloring on all of $\mathcal{A}S_{g,p}$.

Lemma 3.12. *The pointed arc complex is uniquely colored by the punctures.*

Let $3g + p \geq 6$. The pointed arc complex $\mathcal{A}S_{g,p}$ admits a unique p, η-coloring, up to permutation of the colors, where $\eta(x)$ is the number of endpoints of the arc x.

Proof. We will argue with a modification of the Putman trick 2.4. Observe that (since we exclude peripheral arcs) every maximal simplex of $\mathcal{A}S_{g,p}$ has an arc with an end at every puncture. Observe that if $p \leq 2$ the result is trivial, so we assume $p \geq 3$. Let f be any p-coloring of the arc complex $\mathcal{A}S_{g,p}$.

We first fix a collection X of curves. If $g \geq 1$, then $S_{g,p}$ has a nonseparating curve, and we let X contain p parallel nonseparating loops based at the punctures as in Figure 3.5.

If $g = 0$ we take as $X = \{x_i\}_{i=1}^{p}$ to be as in Figure 3.6 with $p-4$ parallel loops based at $p_2, \ldots, p_{p-2}$ and 4 additional curves x_1, x_2, x_{p-1}, x_p so that x_1, x_2 and x_3 pairwise intersect

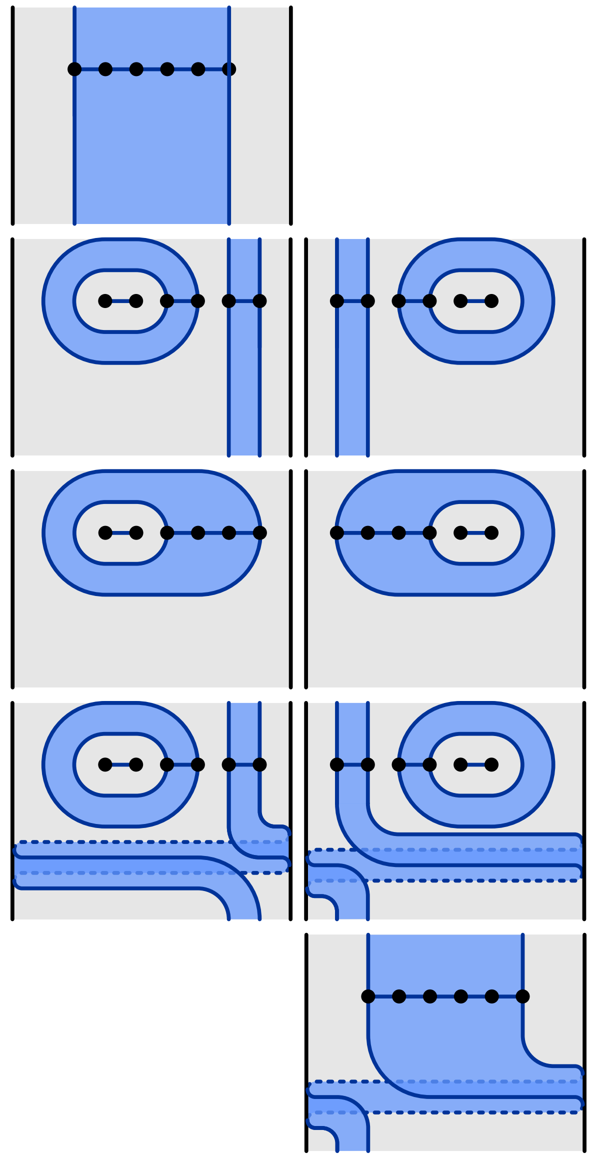


Figure 3.4: Two paths of simplices forcing a coloring between nests parallel to intersecting curves. “Pack” curves on one boundary component of the nest into curves surrounding punctured disks, then unpack them parallel to a different curve of the unpunctured surface. Repeat this with the other boundary component of the nest.

twice and are disjoint from all other curves of X, and similarly x_p, x_{p-1}, x_{p-2} pairwise intersect twice and are disjoint from all other curves of X. We will argue that each loop of X must be colored differently. Observe that there is an arc α_{12} from p_1 to p_2 that is disjoint from all loops of X except x_1 and x_2. Similarly there is $\alpha_{p-1,p}$ disjoint from all loops of X except x_p and x_{p-1}. So the collection $\alpha_{12}, x_3, \ldots, x_{p-2}, \alpha_{p-1,p}$ require all p-colors to

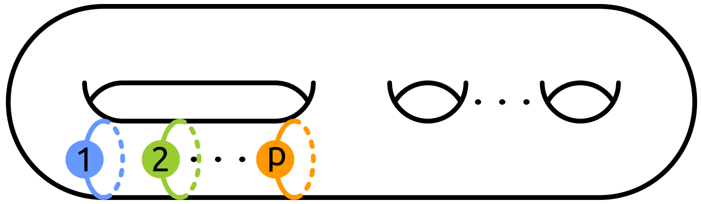


Figure 3.5: A base collection of parallel nonseparating loops, as in a nest.

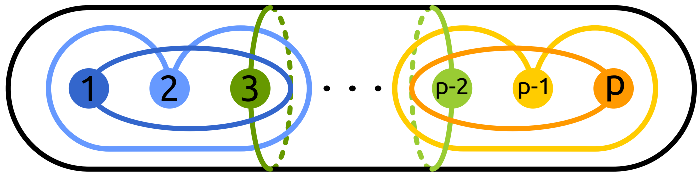


Figure 3.6: A base collection of mostly parallel loops in the punctured sphere.

paint, and it must be that $f(x_1) \cup f(x_2) = f(\alpha_{12})$ and $f(x_{p-1}) \cup f(x_p) = f(\alpha_{p,p-1})$. So X requires p-colors to paint.

We may then assume, possibly after relabeling the colors, that f colors the arcs of X by their punctures. Applying Lemma 2.4, we will show that the coloring on X forces the coloring on all of $\mathcal{A}S_{g,p}$. Our technique will be to contruct paths with the group action of $\mathrm{MCG}\, S_{g,p}$ on $\mathcal{A}S_{g,p}$ and show that f is determined along these paths. Let $\alpha_{i,i+1}$ be an arc that is disjoint from all loops of X except x_i and x_{i+1} and contained in the annulus they bound if they are disjoint. Take as a generating set of $\mathrm{MCG}\, S_{g,p}$ the Dehn half-twists along the arcs $\alpha_{i,i+1}$ and the usual Humphrey's generators of Dehn twists about nonseparating curves that are disjoint from X, except for one curve z that intersects each loop of X exactly once.

We now claim that for $h \in H$ the coloring f is determined on the loops $h \cdot X$ by the coloring on X. We must consider several cases depending on whether h is a twist or half-twist, and how $\alpha_{i,i+1}$ intersects X. These cases are considered in Figures 3.7-3.10.

Observe that if $g \in \mathrm{MCG}\, S_{g,p}$ we can write $g = h_n \cdots h_1$ with $h_i \in H$. And if the loops of $h_k \cdots h_1 \cdot X$ are painted by their punctures, then so are the loops of $h_{k+1} \cdots h_1 \cdot X$ by the above argument.

Observe in the case of $g = 0$ every curve is in $\mathrm{MCG}\, S_{g,p} \cdot X$. If $g \geq 1$ then $\mathrm{MCG}\, S_{g,p} \cdot X$

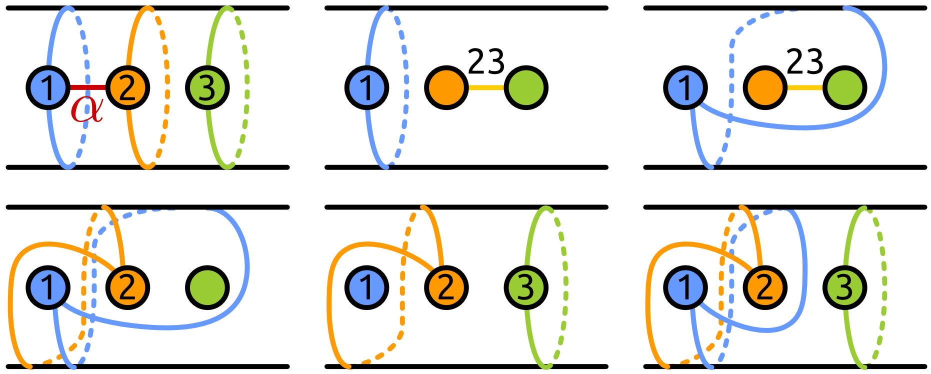


Figure 3.7: Case 1: The half-twist h is about an arc α in an annulus between x_i and x_{i+1} and disjoint from the other loops of X, as in the top left. The lower right shows $h(x_1)$, $h(x_2)$, and $h(x_3) = x_3$. Then if $f(x_i) = p_i$ the sequence of curve replacements shows that $f(h(x_i)) = p_i$.

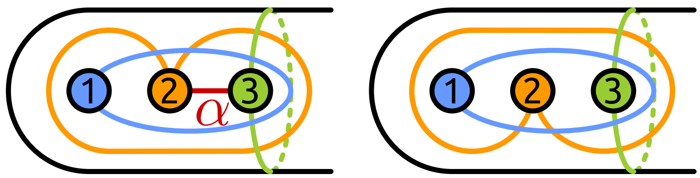


Figure 3.8: Case 2: The half-twist h is about an arc α x_i and x_{i+1} and disjoint from the other loops of X, where x_i and x_{i+1} intersect twice. We may assume the configuration on the left and note that $h(x_2) = x_3$, so that $f(h(x_3))$ must be p_2.

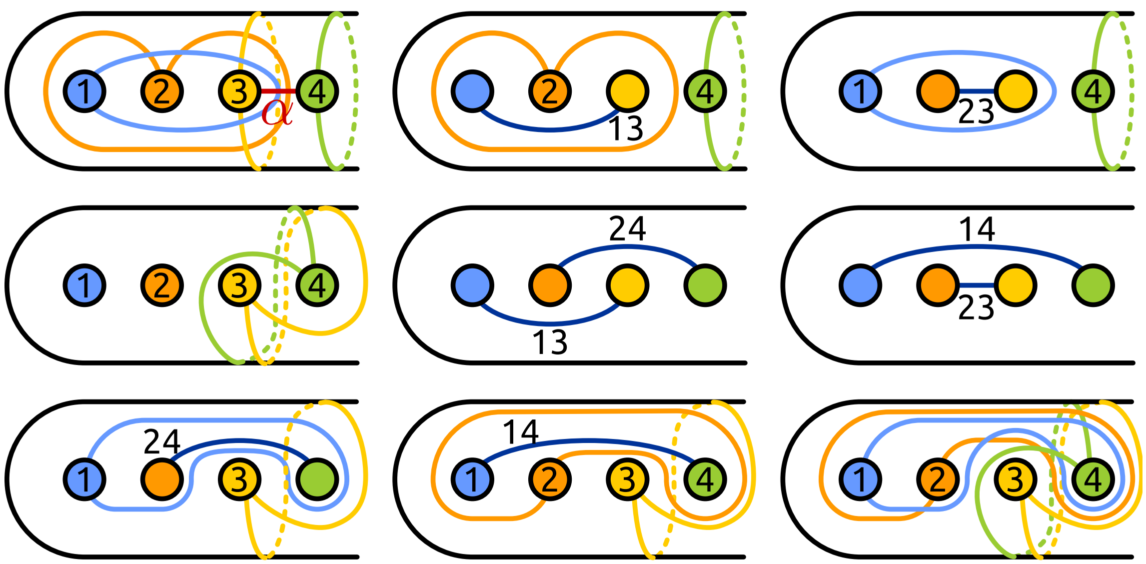


Figure 3.9: Case 3: The half-twist h is about arc α between disjoint curves x_i and x_{i+1}, and α intersects other curves of X. We may assume the configuration on the top left. Then the image of h is shown in the bottom right. Note that by Case 1, $f(h(x_3)) = p_3$ and $f(h(x_4)) = p_4$ so that we need only determine $f(h(x_1))$ and $f(h(x_2))$.

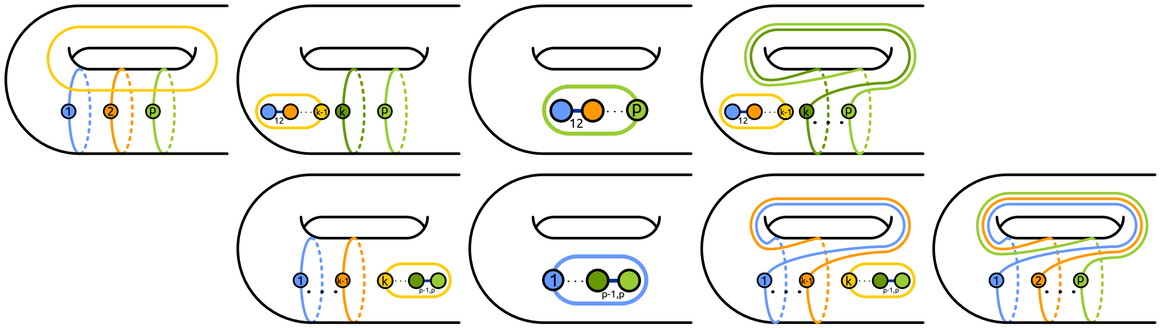


Figure 3.10: Case 4: The Dehn twist $h = T_z$ about a nonseparating curve that links with the loops of X. We may assume the configuration of X and z as in the top left. The image $T_z(X)$ is shown in the bottom right. In the top row: Replace x_1 and x_2 with $\alpha_{1,2}$ so that $f(\alpha_{12}) = \{p_1, p_2\}$. Then iteratively replace x_k with the loop y_k separating $\alpha_{1,2}, y_3, \ldots, y_{k-1}$ from the other punctures, so that $f(y_k) = p_k$ for $k = 3, \ldots, p$. Then replace y_k with $h(x_k)$ so that $f(T_z(x_k)) = p_k$ for $k = p, \ldots, 3$. A similar process reversing the order of the punctures as in the bottom row shows $f(T_z(x_k)) = p_k$ for $k = 1, \ldots, p-2$.

includes all nonseparating loops, and any separating loop x is disjoint from $p-1$ mutually disjoint loops $x_1, \ldots, x_{p-1}$ so that the color $f(x)$ is determined by the colors $f(x_i)$ and so $f(x)$ must be colored by its puncture. □

Lemma 3.13. *Curve complex automorphisms induce arc complex automorphisms.*

There is a natural $\mathrm{MCG}^{\pm}\ S_{g,p}$ *equivariant map*

$$\mathrm{Aut}\,\mathcal{C}S_{g,p} \to \mathrm{Aut}\,\mathcal{A}S_{g,p}.$$

Proof. By Lemma 3.5 automorphisms of $\mathcal{S}_{g,p}$ preserve the class of two curves. Observe that curves c, c' of $S_{g,p}$ bound a punctured annulus if and only if there is a maximal simplex $\Delta \subset \mathcal{C}S_{g,p}$ containing c, c' so that in the region adjacency graph $\mathcal{G}_\Delta$, the edges e_c and $e_{c'}$ are incident at a degree 2 vertex v_a. Then applying Lemma 4 $\phi(c)$ and $\phi(c')$ must cobound a punctured annulus.

Let $\phi \in \mathrm{Aut}\,\mathcal{C}S_{g,p}$ and let a be an punctured annulus $\mathcal{A}S_{g,p}$ with bounding curves c, c'. If $(g,p) \neq (1,2)$ then c, c' uniquely specify the annulus. Then $\phi(c), \phi(c')$ specify cobound a punctured annulus $\phi_*(a)$.

Finally a, a' are disjoint arcs if and only if there is a maximal simplex $\Delta \subset \mathcal{C}S_{g,p}$ such that the bounding curves of regular neighborhoods of a and a' are in Δ. Then a and a' are represented by distinct vertices v_a and $v_{a'}$ of $\mathcal{G}_\Delta$. So a and a' are disjoint if and only if $\phi_*(a)$ and $\phi_*(a')$ are disjoint.

Thus $\phi_* : \mathcal{A}S_{g,p} \to \mathcal{A}S_{g,p}$ is an isomorphism. □

Lemma 3.14. *Curve complex automorphism permute puncture-projection fibers.*

Let $\phi \in \operatorname{Aut} \mathcal{C}S_{g,p}$ for $3g + p \geq 6$. Then if x, y are curves of $S_{g,p}$ with $\rho_q(x) = \rho_q(y)$, then there is a puncture $q' \in P$ such that $\rho_{q'}(\phi(x)) = \rho_{q'}(\phi(y))$.

Proof. Consider the structure of $\rho_q^{-1}(\rho_q(x))$. We have it is a subtree of $\mathcal{C}S_{g,p}$ with $x, x' \in \rho_q^{-1}(\rho_q(x))$ adjacent if and only if they bound an annulus punctured by q. Then we have a path $x = x_0, \ldots, x_n = y$ such that x_i, x_{i+1} bound an annulus punctured by q. Using 3.13 we have $\phi(x_i), \phi(x_{i+1})$ bound an annulus punctured by some q_i. Then since x_i, x_{i+1}, x_{i+2} bound annuli punctured by q that are the regular neighborhoods of loops $a_i, a_{i+1} \in \mathcal{A}S_{g,p}$ based at q. Then $\phi(x_i), \phi(x_{i+1}), \phi(x_{i+2})$ bound annuli that are the regular neighborhoods of loops $\phi_*(a_i), \phi_*(a_{i+1})$. By Lemma 3.12 $\phi_*(a_i)$ and $\phi_*(a_{i+1})$ are based at a common point q'. □

Proof of Theorem 1.1. Assume that the natural map

$$\operatorname{MCG}^{\pm} S_{g,p-1} \xrightarrow{\gamma} \operatorname{Aut} \mathcal{C}S_{g,p-1}$$

is an isomorphism. Then by Lemma 3.7 the following diagram commutes

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S_{g,p-1}, q) & \longrightarrow & \operatorname{MCG}^{\pm}(S_{g,p}, q) & \xrightarrow{f_q} & \operatorname{MCG}^{\pm} S_{g,p-1} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
1 & \longrightarrow & \pi_1(S_{g,p-1}, q) & \longrightarrow & \operatorname{Aut} \mathcal{C}(S_{g,p}, q) & \xrightarrow{\rho_q} & \operatorname{Aut} \mathcal{C}S_{g,p-1} & \longrightarrow & 1.
\end{array}$$

and since $\gamma f_q = \rho_q \beta$ is a surjection we have that ρ_q is a surjection and the rows are exact.

By the Five Lemma β is an isomorphism.

Let $\phi \in \operatorname{Aut}\mathcal{C}S_{g,p}$. We have that by Lemma 3.14 that ϕ permutes the fibers $\{\rho_q^{-1}\}$, so there is $\psi \in \mathrm{MCG}^{\pm} S_{g,p}$ so that $\psi \in \operatorname{Aut}\mathcal{C}S_{g,p}$ is such that $\psi\phi$ maintains the fibers ρ_q^{-1}. So $\psi\phi \in \operatorname{Aut}\mathcal{C}(S_{g,p}, q)$. But then there is $\psi' \in \mathrm{MCG}^{\pm} S_{g,p}$ so that $\psi\phi = \psi'$. But then $\phi = (\psi^{-1}\psi')$ is also induced by a mapping class we have the natural map

$$\mathrm{MCG}^{\pm} S_{g,p} \longrightarrow \operatorname{Aut}\mathcal{C}S_{g,p}$$

is an isomorphism. □

Spheres and Punctures

The main theorem of this section is the following

Theorem 1.2. *The natural map* $\mathrm{Out}_{n,p} \to \operatorname{Aut}\mathcal{S}_{n,p}$ *is an isomorphism for* $n \geq 3$ *and* $p \geq 0$.

The proof is directly analogous to our proof of Theorem 1.1. We will use a the structure of the puncture forgetful map and induct on the number of punctures. A base, unpunctured case is considered by the Theorem of Aramayona and Souto [11].

Theorem 2.12. *The natural map* $\operatorname{Out} F_n \to \operatorname{Aut}\mathcal{S}_n$ *is an isomorphism for* $n \geq 3$.

We will show that automorphisms of the punctured sphere complex respect the fibration induced by forgetting punctures, so that adding additional punctures expands the automorphism group of the complex of spheres according to a Birman exact sequence.

Theorem 3.15. *If the natural map*

$$\mathrm{Out}_{n,p} \to \operatorname{Aut}\mathcal{S}_{n,p}$$

is an isomorphism, then so is

$$\mathrm{Out}_{n,p+1} \to \mathrm{Aut}\,\mathcal{S}_{n,p+1}$$

Definition 3.16. As in Definition 3.1 for surfaces, if $\Delta \subset \mathcal{S}_{g,p}$ is a simplex the *region adjacency graph* $\mathcal{G}_\Delta$ of Δ is the graph whose vertices are the connected components of the cut manifold

$$M_{n,p} - \bigcup_{x \in \Delta} x$$

with an edge e_x for every sphere x and with the edge e_x incident to the connected components it bounds. We will also consider the graph simplification $\mathcal{G}_\Delta^{simp}$.

Lemma 3.17. *Sphere complex automorphisms preserve edge incidence of region adjacency graphs.*

Let $\phi \in \mathrm{Aut}\,\mathcal{S}_{n,p}$ and let Δ be a simplex of $\mathcal{S}_{n,p}$ with adjacency graph $\mathcal{G}_\Delta$. Then $e_x, e_{x'}$ are incident edges of $\mathcal{G}_\Delta$ if and only if $e_{\phi(x)}, e_{\phi(x')}$ are incident edges of $\mathcal{G}_\phi(\Delta)$.

Proof. We will argue that the induced bijection ϕ_* between the edges of $\mathcal{G}_\Delta$ and the edges of $\mathcal{G}_{\phi(\Delta)}$ preserves incidence.

Let $x, x' \in \Delta$ be distinct spheres of $M_{n,p}$. Suppose that the associated edges e_x and e_x' of $\mathcal{G}_\Delta$. It suffices to show that e_x and $e_{x'}$ are incident if and only if there is a third $y \in \Delta$ with y intersecting x and x' but no other sphere of Δ. In that case $e_{\phi(x)}$ and $e_{\phi(x')}$ are incident if and only if there is no third $\phi(y) \in \phi(\Delta)$ with $\phi(y)$ intersecting $\phi(x)$ and $\phi(x')$ but no other sphere of $\phi(\Delta)$. So ϕ induces an incidence-preserving edge bijection between $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$.

Suppose that e_x and $e_{x'}$ are incident. Then there is a region R of $M_{n,p} - \bigcup_{z \neq x, x'} z$ containing x and x', and since every region of $M_{n,p} - \bigcup_{z \in \Delta} z$ contains at least an $M_{0,3}$, it must be that R contains an $M_{1,2}$ or $M_{0,5}$.

If R contains an $M_{1,2}$ the subcomplex of spheres in R contains a copy of $\mathcal{S}_{1,2}$ and so must have infinite diameter. So there must be a sphere y in R that intersects both x and x'. Since y is in R, it intersects no other sphere of Δ.

If R is simply connected then it must have a copy of $M_{0,5}$ and x and x' are essential and separating in R. Then there are two boundary spheres y', y'' of R with a path α between them that passes through both x and x'. Let y be the boundary of a regular neighborhood of $\alpha \cup y' \cup y''$ in R. Then y intersects both x and x', but since y is in R, y does not intersect any other sphere of Δ.

Suppose that e_x and $e_{x'}$ are not incident. Then there is a collection of spheres $\Delta' \subset \Delta$ that separate x from x' in $M_{n,p}$. So any sphere intersecting x and x' must also intersect a sphere of Δ. $\square$

Corollary 18. *Sphere complex automorphisms preserve the adjacency graphs of maximal simplices.*

Let $3n + p \geq 6$. Let $\phi \in \operatorname{Aut} \mathcal{S}_{n,p}$ and let Δ be a maximal simplex. Then $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$ are isomorphic.

Proof. Any maximum simplex Δ contains $3n + p - 3$ spheres and cuts $M_{n,p}$ $2n + p - 2$ copies of $M_{0,3}$. So $\mathcal{G}_\Delta^{simp}$ and $\mathcal{G}_{\phi(\Delta)}^{simp}$ are simple, connected graphs with the same number of vertices and the same edge incidence relations. So by Whitney's Theorem 2.1, $\mathcal{G}_\Delta^{simp}$ and $\mathcal{G}_{\phi(\Delta)}^{simp}$ are isomorphic.

To see that self-loops are preserved, observe that as Δ cuts $M_{n,p}$ into copies of $M_{0,3}$, every vertex of $\mathcal{G}_\Delta$ has degree at most 3. Then if e_x is a self-loop at vertex v_R it is incident to exactly one other edge $e_{x'}$ that cannot be a self-loop or have a parallel edge since v_R is degree 3. So $e_{\phi(x')}$ has a degree one vertex in $\mathcal{G}_{\phi(\Delta)}^{simp}$. Since $3n + p - 3 \geq 3$ we have $\mathcal{G}_{\phi(\Delta)}$ has at least 3 edges. So if both vertices of $e_{\phi(x')}$ are degree one in $\mathcal{G}_{\phi(Delta)}^{simp}$, then $\mathcal{G}_{\phi(Delta)}$ has two vertices with a self loop at each. So $e_{\phi(x)}$ is a self loop. If $e_{\phi(x')}$ has only one degree one vertex in $\mathcal{G}_{\phi(Delta)}^{simp}$, then $e_{\phi(x)}$ is only incident to $e_{\phi(x')}$. So $e_{\phi(x)}$ must be a self loop in $\mathcal{G}_{\phi(\Delta)}$.

If ϕ preserves both the graph simplification and the self loops of $\mathcal{G}_\Delta$, it must be that ϕ also preserves the multi-edges, and so ϕ induces a graph isomorphism. $\square$

Lemma 3.19. *Sphere complex automorphisms preserve the topological type of spheres, and the sides of the spheres.*

Let $3n + p \geq 6$. Let $\phi \in \operatorname{Aut} \mathcal{S}_{n,p}$. Let x be a sphere of $M_{n,p}$. Then x and $\phi(x)$ have the same topological type. Further if x is separating and y, y' are spheres in the same connected component of $M_{n,p} - x$, then $\phi(x)$ is separating and $\phi(y), \phi(y')$ are in the same connected component of $M_{n,p} - \phi(x)$.

Proof. By Lemma 18 it suffices to characterise the topological type and sides of a sphere in terms of the region adjacency graph of a maximal simplex.

· Nonseparating spheres: Observe that x is a nonseparating sphere if and only if there is a maximal simplex Δ in that the corresponding edge e_x is a self-loop in the region adjacency graph $\mathcal{G}_\Delta$.

· Separating spheres: Observe that if x separates $M_{n,p}$

$$M_{n,p} = M_{n',p'} \sqcup_x M_{n-n',p-p'+2}$$

if and only if the corresponding edge e_x of the region adjacency graph $\mathcal{G}_\Delta$ is a cut edge. More specifically, if

$$\Delta = \Delta_+ \cup \{x\} \cup \Delta_-$$

with Δ_+ and Δ_- the spheres on each side of x, then e_x separates $\mathcal{G}_\Delta$

$$\mathcal{G}_\Delta - e_x = \mathcal{G}_{\Delta_+} \sqcup \mathcal{G}_{\Delta_-}$$

into the components $\mathcal{G}_{\Delta_+}$ with $3n' + p' - 3$ edges and rank n', and $\mathcal{G}_{\Delta_-}$ with with $3(n - n') + p - p' - 1$ edges and rank $n - n'$.

$\square$

Remark 7. Consider the inclusion that ignores the puncture q

$$i_q : M_{n,p} \hookrightarrow M_{n,p-1}.$$

Then for the homotopy class $[x]$ of a sphere in $M_{n,p}$ we have the class $[i_q(x)]$ in $M_{n,p-1}$ by forgetting the puncture q. Separating spheres of $M_{n,p}$ bounding a copy of S^3 containing only q and one other puncture will become non-essential in this inclusion, but other homotopy classes of spheres have well defined essential representatives up to homotopy forgetting q.

Let $\mathcal{S}^{(q)}_{n,p} \subset \mathcal{S}_{n,p}$ be the subcomplex for which the puncture forgetful map $\rho_q : [x] \mapsto [i_q(x)]$ is well defined. So we have a surjective projection map

$$\rho_q : \mathcal{S}^{(q)}_{n,p} \to \mathcal{S}_{n,p-1}.$$

As in the case for surfaces, the fibers of this map are Bass-Serre trees. Let x be a sphere of $\mathcal{S}_{n,p-1}$. Homotope x in $M_{n,p-1}$ so that it is pointed at q. Then the two boundary spheres of a regular neighborhood of x in $M_{n,p-1}$ gives a well edge of $\rho_q^{-1}(x) \subset \mathcal{S}_{n,p}$. Let Γ be the corresponding be the graph of groups given by the splitting of $F_n = \pi_1(M_{n,p-1}, q)$. So the vertices of Γ are the components of $M_{n,p-1} - x$ with vertex groups given by the π_1 of the component, and an edge with trivial edge group between two components if x is separating, or a self-loop if x is nonseparating. Then there is a an isomorphism between $\rho_q^{-1}(x)$ and the Bass-Serre tree $\tilde{\Gamma}$ given by associating every edge xx' of $\rho_q^{-1}(x)$ with the edge of $\tilde{\Gamma}$ given by uv if

$$\text{stab}_{\pi_1(M_{n,p-1},q)}(u) * \text{stab}_{\pi_1(M_{n,p-1},q)}(v)$$

is the splitting specified by the pointed sphere at q whose regular neighborhood in $M_{n,p}$ has boundary spheres x and x'.

Definition 3.20. Let $\mathrm{Aut}(\mathcal{S}_{n,p}, q) < \mathrm{Aut}\,\mathcal{S}_{n,p}$ be the subgroup preserving the fibration of the forgetful map ρ_p. That is $\phi \in \mathrm{Aut}(\mathcal{S}_{n,p}, q)$ if

$$\phi\left(\rho_p^{-1}\rho_p(x)\right) = \rho_p^{-1}\rho_p(\phi(x))$$

for all spheres x.

Lemma 3.21. *This diagram commutes*

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(M_{n,p-1}, q) \arrow[r] \arrow[d] & \mathrm{Out}_{n,p}^{(q)} \arrow[r, "f_q"] \arrow[d] & \mathrm{Out}_{n,p-1} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(M_{n,p-1}, q) \arrow[r, "\alpha"] & \mathrm{Aut}(\mathcal{S}_{n,p}, q) \arrow[r, "\rho_q^*"] & \mathrm{Aut}\,\mathcal{S}_{n,p-1} \arrow[r] & 1
\end{tikzcd}$$

and has exact rows when ρ_q is surjective.

Proof. The map α is defined by the first square, so it commutes. The map α is injective, since for any loop γ based at q, there is a nonseparating sphere x intersecting γ so that the push map $\alpha(\gamma)$ acts non-trivially on x and so cannot be the identity on $\mathcal{S}_{n,p}^{(q)}$.

The second square must commute, since if $[\psi] \in \mathrm{Out}_{n,p}^{(q)}$ is a mapping class of $M_{n,p}$ and x a sphere of $M_{n,p}$, the homotopy class of $\psi(x)$ is the same if we first forget that the homeomorphism ψ fixes q, or if we first allow ψ with q fixed then homotope the sphere $\psi(x)$ forgetting q.

A fiber $\rho_q^{-1}(x)$ of the forgetful map $\rho_q^* : \mathcal{S}_{n,p}^{(q)} \to \mathcal{S}_{n,p-1}$ is isomorphic to the Bass-Serre tree associated to the splitting. Then the kernel $\ker \rho_q^*$ is a group acting on the tree $\mathcal{T}_\Delta$, so by the Fundamental Theorem of BassSerre Theory 2.5, $\ker \rho_q^*$ is isomorphic to the fundamental group π_1 of the quotient graph of groups, but the corresponding graph of groups is exactly the Van Kampen splitting of π_1 induced by x. Thus

$$\ker \rho_q^* = \text{image}\ \alpha \cong \pi_1(S_{g,p}, q)$$

and the second row is exact. $\square$

Remark 8. The edges of the fibers of the forgetful map are between two spheres that cobound a $S^2 \times I$ punctured by q. Such spheres are specified by the boundaries of regular neighborhoods of pointed spheres of $M_{n,p}$, so we consider the associated complex.

Definition 3.22. Let $\mathcal{PS}_{n,p}$ be the *pointed sphere complex* defined as follows. Let the vertices of $\mathcal{PS}_{n,p}$ be either

(1.) pointed spheres in $M_{n,p}$, i.e. homotopy classes of maps $(S^2, s_0) \to (M_{n,p}, P)$ where s_0 is a basepoint of the 2-sphere S^2 or else

(2.) unpointed spheres that bound a twice punctured ball

A collection of pointed spheres in $M_{n,p}$ span a simplex of $\mathcal{PS}_{n,p}$ if they have disjoint representatives, including the basepoints.

Definition 3.23. As in Definintion 3.10, fix an order $\sigma : P \to \{1, \dots, p\}$ let a *σ-nest* of curves parallel to a sphere x as the homotopy class in $M_{n,p}$ of an embedding

$$N : S^2 \times I \hookrightarrow S_{g,p}$$

such that N is a homotopic to x in S_g by homotopy forgetting the punctures. The image of $S^1 \times \{\sigma(i)/(p-1)\}$ is a pointed sphere based at puncture $\sigma(i)$ that we refer to as the pointed sphere N_i.

Lemma 3.24. *The pointed sphere complex is uniquely colorable.*

Let $\eta(x)$ be 1 if x is a pointed sphere and 2 if x is an unpointed sphere that bounds a twice punctured ball of $M_{n,p}$. There is a unique k, η-coloring of $\mathcal{PS}_{n,p}$ given by the puncture labels. Further it is the only k, η-coloring up to relabeling of the k colors.

Proof. The argument is by the color modified Putman Lemma 2.4. Observe that if $p \leq 2$ or $n = 0$ the result is trivial, so we assume $p \geq 3$ and $n \geq 1$. Let f be any p, η-coloring of the pointed sphere complex $\mathcal{PS}_{n,p}$.

Choose a maximal collection of nonseparating spheres $x_{a_1}, \dots, x_{a_n}$ of $M_{n,p}$ and let $x_{b_1}, \dots, x_{b_p}$. Let H be the generating set of $\mathrm{Out}_{n,p}$ consisting of the transpositions $\sigma_i^a = (a_i a_{i+1})$ and $\sigma_i^b = (b_i b_{i+1})$, the inversion ι_1 at a_1 if $n = 1$ or else the inversion ι_2, the transvection τ_{12} with $a_1 \mapsto a_1 a_2$ if $n \geq 2$, and the conjugation γ_{11} with $b_1 \mapsto a_1 b_1 a_1^{-1}$.

Let V be the nest of p nonseparating spheres all parallel to x_{a_1} so that v_i is based at x_{b_i} and separates $x_{a_1}^+, x_{b_1}, \dots, x_{b_{i-1}}$ from the other spheres of $M_{n,p}$ cut along $x_{a_1}, \dots, x_{a_p}$, as in Figure 3.11.

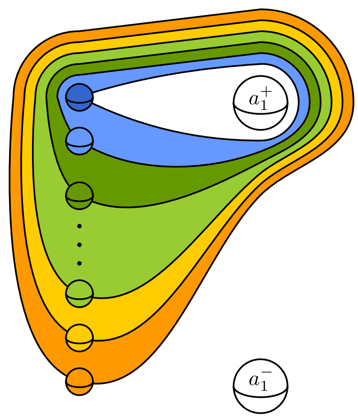


Figure 3.11: A nest of p pointed spheres requires p distinct colors.

Since they are all disjoint they form a p-clique that requires they must be distinctly colored. We may assume, possibly after relabeling, that f colors each pointed sphere of V by the label of its puncture; so $V = \{v_i\}_{i \in P}$ and $f(v_i) = \{i\}$.

We first show that V forces a coloring on $h \cdot V$ for all $h \in H^{\pm}$. We consider the cases of the different types of generators.

1. Transvection. Observe that the choice of transvection is realized by the push of $x_{a_1}^-$ through $x_{a_2}^-$ and along a path disjoint from $v_1, \dots, v_p$. So $V = \tau_{12} \cdot V$.

2. a Transposition σ_i^a. Only σ_1^a does not fix V. Let $v_{i,j}$ be the sphere separating x_{b_i} and x_{b_j} from the other spheres x_{a_k} and x_{b_ℓ} for $\ell \neq j, k$. Figure 3.12 shows two sequences of forced colorings between k-colored simplices intersecting in $k-1$-colored simplices. The first sequence forces a coloring on $\{v_{1,2}, \sigma_1^a v_3, \dots, \sigma_1^a v_p\}$.

The first sequence forces a coloring on $\{\sigma_1^a v_1, \ldots \sigma_1^a v_{p-2}, v_{p-1,p}\}$. So V forces a coloring on $\sigma_1^a \cdot V$.

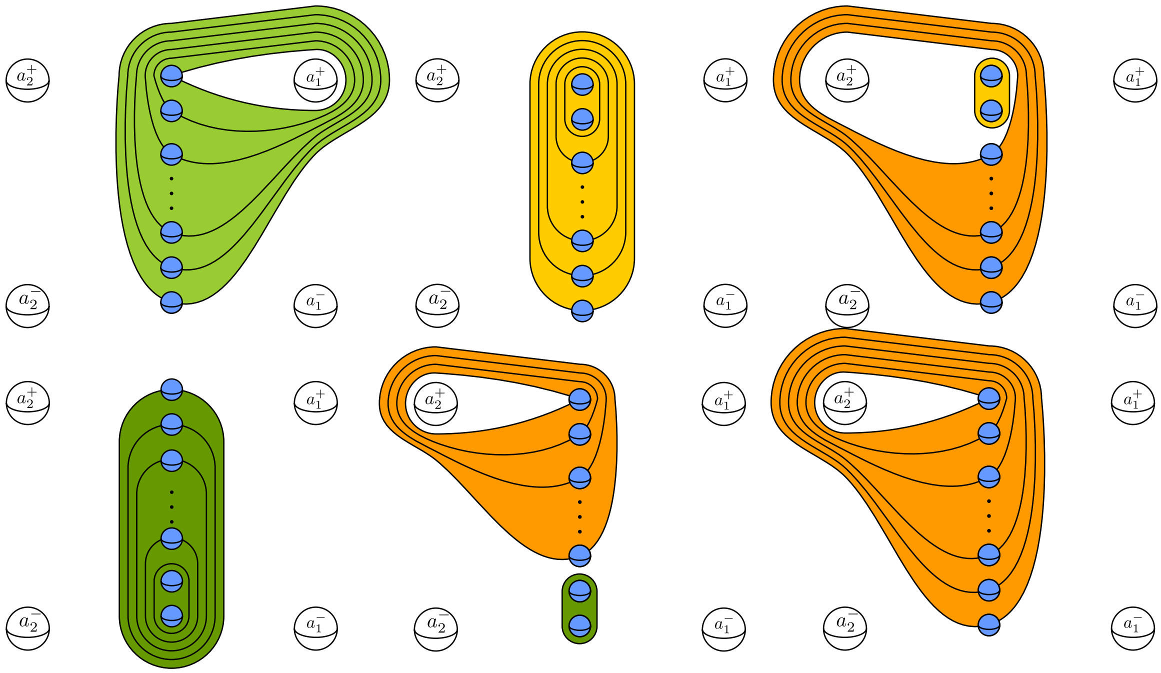


Figure 3.12: A σ-nest forces a coloring on $\sigma_1^b\sigma$ nest for transposition σ_1^a.

3. b-transposition. Consider first the transposition σ_1^b. Figure 3.13 shows a sequence of forcing colorings σ_1^b The argument for σ_i^b is similar.

4. Inversion. If $n > 1$ then the inversion ι_2 leaves V fixed.

   If $n = 1$ then Figure 3.14 shows a sequence of coloring forcing. The first sequence forces a coloring on $\{v_{1,2}, \iota_1 v_3, \ldots, \iota_1 v_p\}$. The first sequence forces a coloring on $\{\iota_1 v_1, \ldots \iota_1 v_{p-2}, v_{p-1,p}\}$. So V forces a coloring $\iota_1 \cdot V$.

5. Conjugation. Figure 3.15 shows a sequences of forced colorings between k-colored simplices intersecting in $k-1$-colored simplices. from V to $\tau_{12} \cdot V$. The case for τ_{12}^{-1} is similar.

From this we have that V forces a coloring on its orbit $\mathrm{Out}_{n,p} \cdot V$.

Finally we argue that $\mathrm{Out}_{n,p} \cdot V$ forces a coloring on $\mathcal{S}_{n,p}$. Let $V_k \subset \mathcal{PS}_{n,p}^{(0)}$ be the set of all unpointed separating spheres bounding $M_0, 3$ and pointed separating spheres bounding

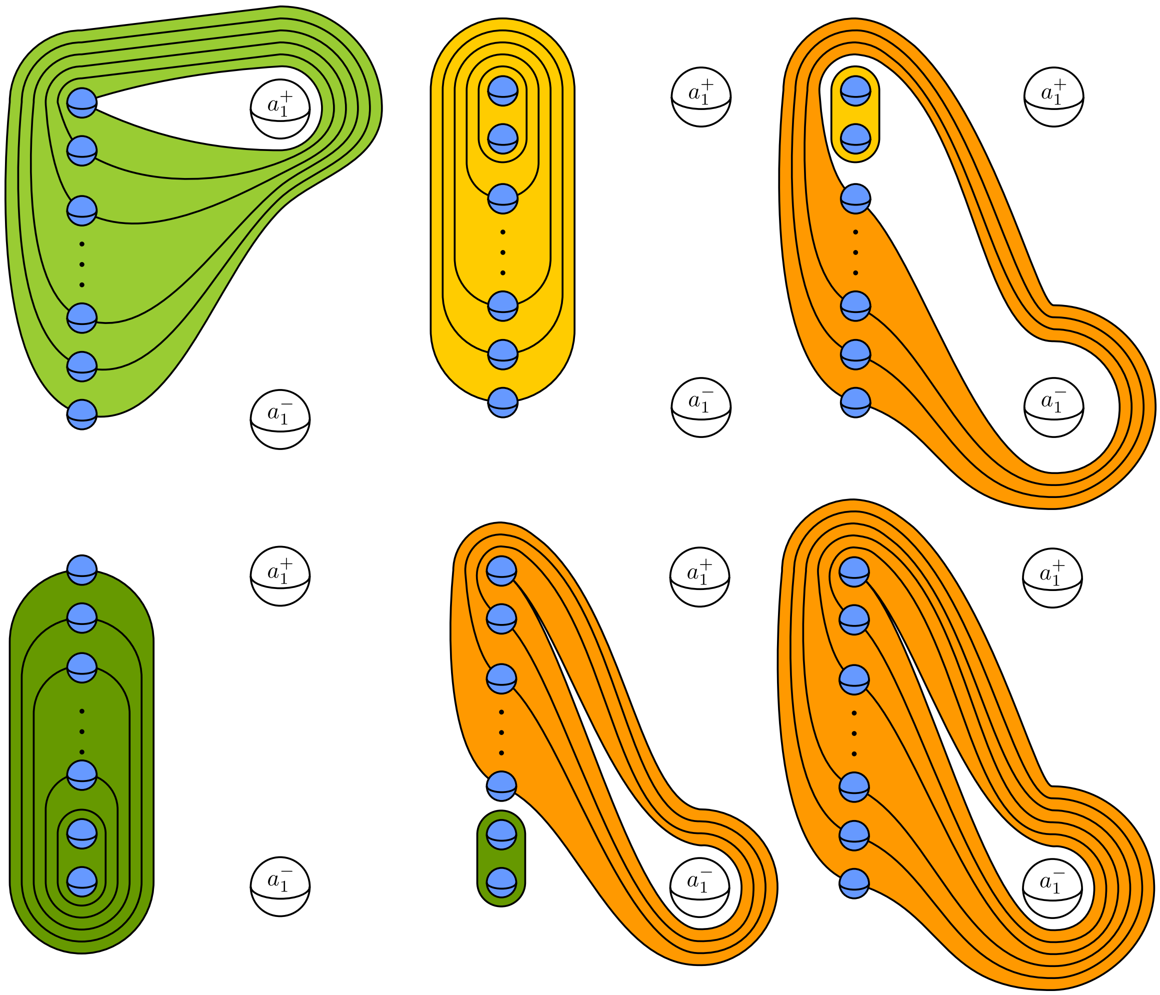


Figure 3.13: A σ-nest forces a coloring on $\sigma_1^b\sigma$ nest for transposition σ_1^b.

$M_{0,j}$ for $j \leq k$. If $x \in V_3$, then we may find a collection of $k-2$ pointed nonseparating spheres that are disjoint from each other and from x. Then since $\mathrm{Out}_{n,p} \cdot V$ contains all pointed nonseparating spheres the coloring is determined on x. So $\mathrm{Out}_{n,p} \cdot V$ forces a coloring on V_3. Inductively we have that $V_k \cup \mathrm{Out}_{n,p} \cdot V$ forces a coloring on $V_{k+1} \cup \mathrm{Out}_{n,p} \cdot V$. So V forces a coloring on $V_{p-1} \cup \mathrm{Out}_{n,p} \cdot V$. Then if x is any pointed separating sphere, there are $p-1$ pointed curves or separating spheres bounding an $M_{0,3}$ that are mutually disjoint and disjoint from x. So $V_{p-1} \cup \mathrm{Out}_{n,p} \cdot V$ forces a coloring on x. Since x was an arbitrary separating sphere, $V_{p-1} \cup \mathrm{Out}_{n,p} \cdot V$. The result then follow from Lemma 2.4. $\square$

Lemma 3.25. *Sphere complex automorphisms induce automorphisms of the based sphere*

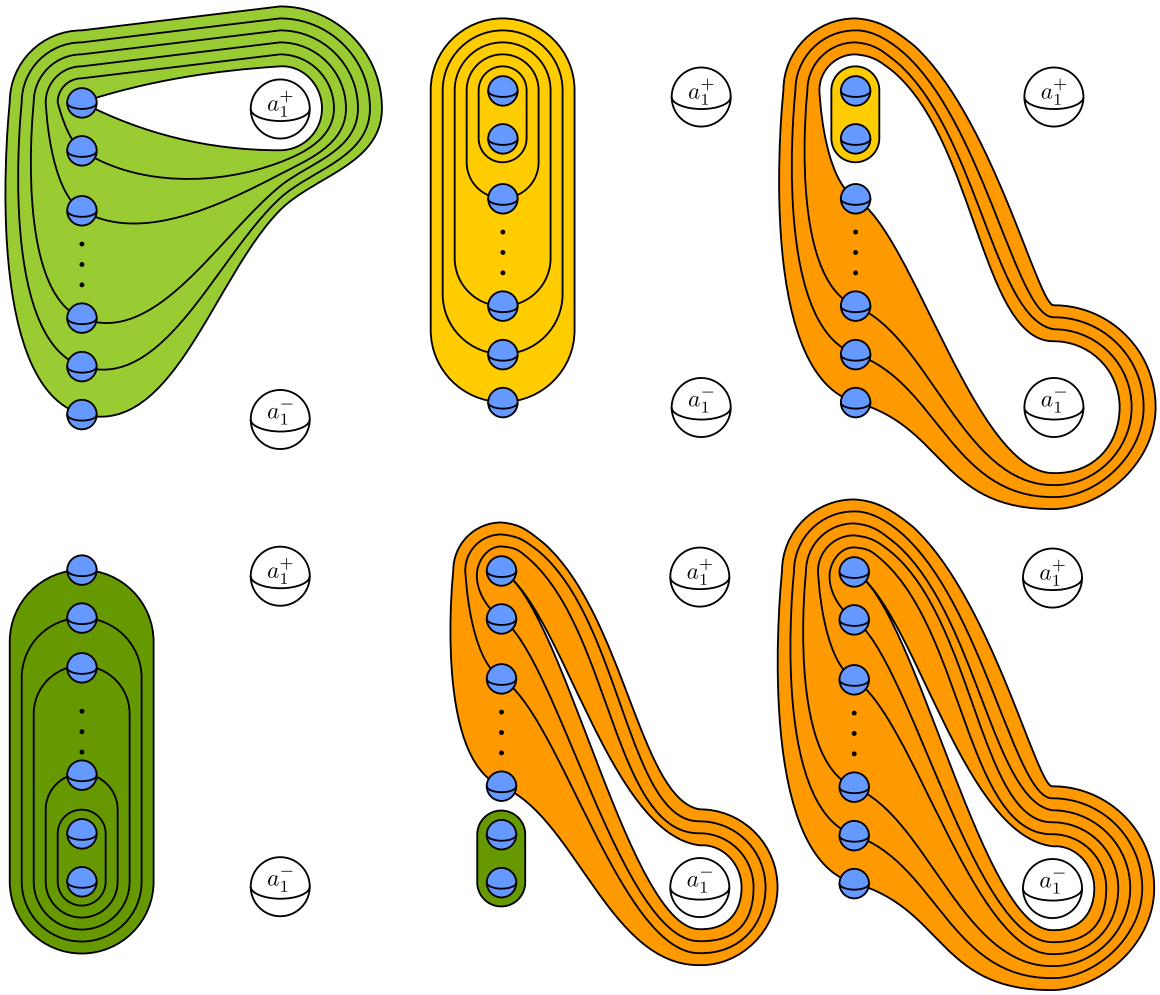


Figure 3.14: Two sequences together show the nest V forces a coloring on $\iota_1 V$.

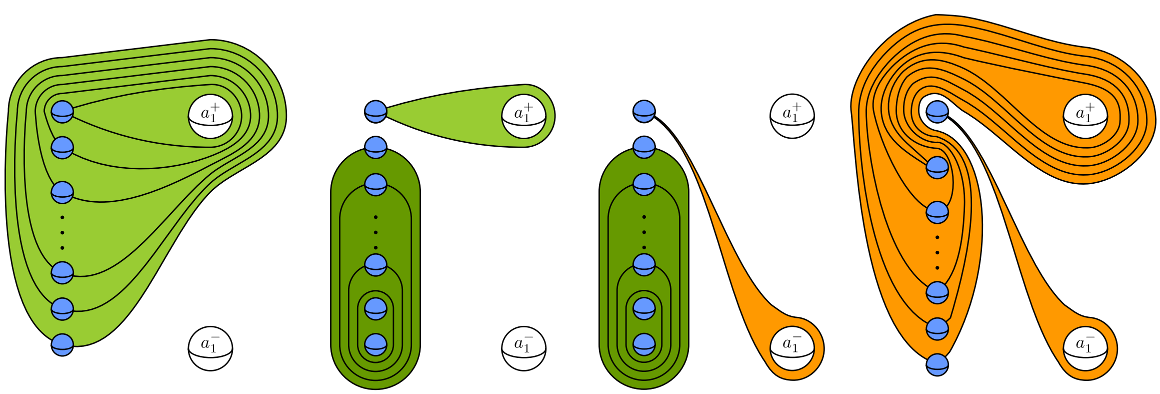


Figure 3.15: The nest V forces a coloring on $\tau_{12} V$.

complex.

There is a natural $\mathrm{Out}_{n,p}$*-equivariant map* $\mathrm{Aut}\,\mathcal{S}_{n,p} \to \mathrm{Aut}\,\mathcal{PS}_{n,p}$*.*

Proof. Let $\phi \in \operatorname{Aut} \mathcal{S}_{n,p}$ by an automorphism of the sphere complex.

According to Lemma 3.19 the automorphism ϕ preserves the class of separating spheres bounding $M_{0,3}$. Suppose that x is a pointed sphere of $M_{n,p}$. Then a regular R neighborhood of x is an $M_{0,3}$ with a puncture and bounded by two spheres x', x'' of $M_{n,p}$. Observe that two spheres of $\mathcal{S}_{n,p}$ cobound an $M_{0,3}$ with a puncture if and only if they are in a maximal simplex Δ such that the corresponding vertex in the region adjacency graph $\mathcal{G}_\Delta$ is degree 2. By Lemma 18 the adjacency graph $\mathcal{G}_{\phi(\Delta)}$ is isomorphic to $\mathcal{G}_\Delta$, and $\phi(x')$ and $\phi(x'')$ bound a regular neighborhood of a punctured sphere $\hat{\phi}(x)$, and the map $x \mapsto \hat{\phi}(x)$ gives an isomorphism of $\mathcal{PS}_{n,p}$. Further $\phi \mapsto \hat{\phi}$ is an injection since every sphere of $M_{n,p}$ is a boundary component for a regular neighborhood of some pointed sphere of $M_{n,p}$. Hence if $\hat{\phi}$ is the identity, so must ϕ be. □

Lemma 3.26. *Sphere complex automorphisms permute the fibers of the puncture forgetful map. Let $\phi \in \operatorname{Aut} \mathcal{S}_{n,p}$ and $x \in \mathcal{S}_{n,p}$. Then*

$$\phi\left(\rho_q^{-1}\rho_q(x)\right) = \rho_{q'}^{-1}\rho_{q'}(\phi(x))$$

for some $q' \in P$.

Proof. Observe that an edge of $\mathcal{S}_{n,p}$ in $\rho_q^{-1}\rho_q(x)$ specifies a punctured sphere of $\mathcal{PS}_{n,p}$ colored by q. If $x, x' \in \rho_q^{-1}\rho_q(x)$ then we have a path $x = x_0, \ldots, x_n = y$ with x_{i-1}, x_i cobounding an $M_{0,3}$ that is the regular neighbodhood of punctured sphere y_i. Then by Lemma 3.25 ϕ induces an automorphism $\hat{\phi} \in \mathcal{PS}_{n,p}$ such that $\phi(x_{i-1}), \phi(x_i)$ cobound a neighborhood of $\hat{\phi}(y_i)$. By Lemma 3.24 $\mathcal{PS}_{n,p}$ is uniquely colored by the punctures, so since y_i are all colored by q and $\hat{\phi}$ must permute the colors we have that $\hat{\phi}(y_i)$ are all punctured spheres based at the same point q' and give the edges for the path $\phi(x_0), \ldots, \phi(x_n)$ in $\rho_{q'}^{-1}\rho_{q'}(x)$. □

Proof of Theorem 3.15. Suppose that the natural map

$$\mathrm{Out}_{n,p-1} \xrightarrow{\gamma} \mathrm{Aut}\,\mathcal{S}_{n,p-1}$$

is an isomorphism. Then according to Lemma 3.21 the following diagram commutes

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(M_{n,p-1},q) & \longrightarrow & \mathrm{Out}^{(q)}_{n,p} & \xrightarrow{f_q} & \mathrm{Out}_{n,p-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow & & \\
1 & \longrightarrow & \pi_1(M_{n,p-1},q) & \xrightarrow{\alpha} & \mathrm{Aut}(\mathcal{S}_{n,p},q) & \xrightarrow{\rho_q} & \mathrm{Aut}\,\mathcal{S}_{n,p-1} & \longrightarrow & 1
\end{array}$$

and has exact rows since ρ_q is a surjection as $\gamma f_q = \rho_q\beta$ is. By the Five Lemma we have β is an isomorphism. Let $\phi \in \mathrm{Aut}\,\mathcal{S}_{n,p}$. By 3.26 ϕ permutes the fibers of the maps ρ_q, so there is $\psi \in \mathrm{Out}_{n,p}$ such that $\psi\phi$ preserves $\rho_q^{-1}\rho_q$. But then $\psi\phi \in \mathrm{Aut}\,\mathcal{S}^{(q)}_{n,p}$, and by the exact sequence there is $\psi' \in \mathrm{Out}_{n,p}$ such that $\psi\phi = \psi'$. But then $\phi = \psi^{-1}\psi'$ is also in $\mathrm{Out}_{n,p}$. It follows that the natural map

$$\mathrm{Out}_{n,p} \xrightarrow{\gamma} \mathrm{Aut}\,\mathcal{S}_{n,p}$$

is an isomorphism. $\square$

CHAPTER 4

FURTHER OUT

In this chapter we advance the goal of an $\operatorname{Out} F_n$ analog to the Brendle-Margalit Theorem [5] by considering subcomplexes of the sphere complex $\mathcal{S}_n$ that are themselves combinatorial models. We will largely argue by extending automorphisms from subcomplexes of $\mathcal{S}_n$ to automorphisms of $\mathcal{S}_n$ by finding a combinatorial characterization of any absent type of sphere.

In Section 4.1 we consider the complex of separating spheres. In Section 4.2 we consider the complex of separating sphere with sufficiently complex sides. In Section 4.3 we consider the complex of coconnected spheres and its relation to the free factor complex.

Complex of Separating Spheres

Let $\mathcal{S}_{n,p}^{sep} \subset \mathcal{S}_{n,p}$ be the complex of embedded homotopy classes of separating spheres in $M_{n,p}$. In this section we will aim to show that the complex of separating spheres is a combinatorial model for $\mathrm{Out}_{n,p}$.

Theorem 1.3. *The natural map $\operatorname{Out}(F_n) \to \operatorname{Aut} \mathcal{S}_n^{sep}$ is an isomorphism for $n \geq 3$.*

We begin by computing the dimension.

Lemma 4.1. *$\mathcal{S}_{n,p}^{sep}$ is a flag complex of dimension $2n + p - 4$.*

Proof. $\mathcal{S}_{n,p}^{sep}$ is the induced subcomplex of $\mathcal{S}_{n,p}$, which is known to be flag [11]. We show by induction that any collection Σ of disjoint spheres in $M_{n,p}$ is a subset of a maximal collection of $\max(2n + p - 3, 0)$ disjoint spheres.

Suppose for a base case that $\Sigma = \varnothing$. Any pants decomposition of the surface $S_{n,p} = \partial H_n - P$ by curves surrounding disks in the handlebody H_n with punctures P has $3n-3+p$

curves with n of them nonseparating. Taking an identical copy of $\partial H_n - P$ and gluing along $S_{n,p}$ promotes the pants decomposition of $S_{n,p}$ into a decomposition of $M_{n,p}$ into $M_{0,3}$s. Of these $3n-3+p$ spheres, n are nonseparating and $2n+p-3$ are separating.

Assume that any collection of k or fewer disjoint spheres in $M_{n,p}$ is a subset of a maximal collection of $2n+p-3$ disjoint spheres. Let Σ be a collection of k disjoint spheres and let x be a sphere disjoint from all spheres of Σ. Then cutting $M_{n,p}$ along x yields two components homeomorphic to $M_{n',p'}$ and $M_{n'',p''}$ where $n'+n''=n$ and $p'+p''=p+2$. By inductive hypothesis, the set spheres of Σ in each component can extended to maximal sets Σ_1 and Σ_2 of size $2n'+p'-3$ and $2n''+p''-3$ respectively. Then $\Sigma \cup \{x\}$ is contained in the maximal set $\Sigma_1 \cup \Sigma_2 \cup \{x\}$ of size

$$(2n'+p'-3)+(2n''+p''-3)+1 = 2n+p-3.$$

□

Lemma 4.2. *$\mathcal{S}^{sep}_{n,p}$ is connected whenever it has positive dimension, except if $(n,p)=(2,1)$.*

Proof. Consider first the case where $n = 0$. There is a deformation retraction of $M_{0,s}$ away from the puncture p_s to a wedge product $\bigvee_{i\in s-1} S^2_i$ of $p-1$ copies of p^2. We thus have $\pi_2(M_{0,s}) \cong \mathbb{Z}^{s-1}$. If x is an embedded sphere of $M_{0,s}$ separating the set of punctures $\{p_{i_1},\ldots,p_{i_k}\}$ from the other punctures, then the map

$$x \hookrightarrow M_{0,s} \longrightarrow \bigvee\nolimits_{i\in s-1} S^2_i \longrightarrow S^2_{p_k}$$

is degree 1 if x separates p_k from p_s and 0 otherwise. (See degree theory of [17].) So there are $2^{s-1}-1$ homotopy classes of spheres in $M_{0,s}$, with each sphere totally determined by its bipartition of the punctures. Let P be the punctures. So $\mathcal{S}^{sep}_{0,s}$ is the isomorphic to the complex of bipartitions of P with size with two bipartitions adjacent if their sides give nested subsets of P. Then if $p \geq 5$ there is a path in $\mathcal{S}^{sep}_{0,s}$ between any two spheres made by moving elements between the partitions one at a time.

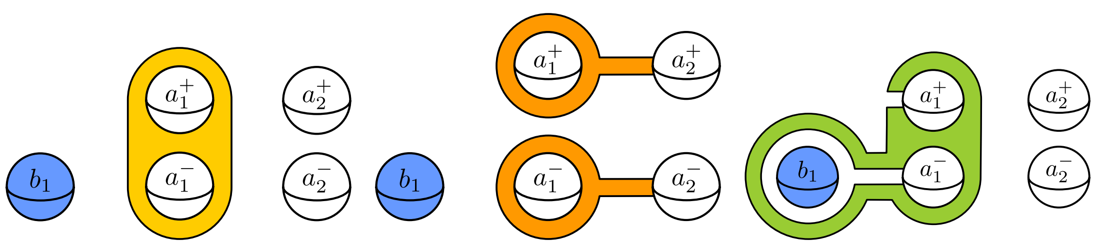
Consider the case where $n \geq 1$. We make use of Putman's Lemma 2.3 where $G = \mathrm{Out}_{n,p}$ and $X = \mathcal{S}^{sep}_{n,p}$. Fix a sphere v that bounds an embedded copy of $M_{1,1}$ in $M_{n,p}$. Note that $\mathrm{Out}_{n,p}$ acts transitively on such spheres, and every separating sphere that does not bound an embedded copy of $M_{1,1}$ is disjoint from such a sphere. So the orbit $\mathrm{Out}_{n,p} \cdot v$ intersects the connected component of every separating, considered as a vertex of $\mathcal{S}^{sep}_{n,p}$.

Consider a free basis $a_1, \ldots, a_n, b_1, \ldots, b_s$ of the free group F_{n+p} with v disjoint from the spheres representing the basis and v separating a_1 from $a_2, \ldots, a_n$. Take as a generating set of $\mathrm{Out}_{n,p}$ the transpositions of $\{b_1, \ldots, b_s\}$, the transpositions and inversions of $\{a_1, \ldots, a_s\}$, the transvection $a_1 \mapsto a_1 a_2^{-1}$, and the conjugation $b_1 \mapsto a_1 b_1 a_1^{-1}$.

Observe that transpositions of $\{b_1, \ldots, b_s\}$ and inversions of $\{a_1, \ldots, a_n\}$ leave v fixed. Further, transpositions of $\{a_1, \ldots, a_n\}$ either fix a_1 and thus v, or move v to a disjoint separating sphere at distance 1 from v in $\mathcal{S}^{sep}_{n,p}$.

Consider the image v' of v under the conjugation $b_1 \mapsto a_1 b_1 a_1^{-1}$, as shown in green in Figure 4.1.

Then v' and v are contained in a copy of $M_{1,2}$ bounded by b_1 and the sphere u separating a_1 and b_1 from $a_2, \ldots, a_n$ and $b_2, \ldots, b_s$. If $n \geq 2$ or $p \geq 3$ we have that u is essential and this gives a length 2 path v to u to v' in $\mathcal{S}^{sep}_{n,p}$. The inverse conjugation $b_1 \mapsto a_1^{-1} b_1 a_1$ similarly moves v distance 2 in $\mathcal{S}^{sep}_{n,p}$. Appealing to Putman's Lemma 2.3, we conclude that



(a) The basepoint sphere v separates a_1 from $a_2, \ldots, a_n$ and $b_1, \ldots, b_s$.

(b) The transvection $a_1 \mapsto a_1 a_2^{-1}$ corresponds to pushing a_1^+ through a_2^-.

(c) The conjugation $b_1 \mapsto a_1 b_1 a_1^{-1}$ corresponds to pushing b_1 through a_1^+.

Figure 4.1: Nontrivial $\mathrm{Out}_{n,p}$ generator actions on the base sphere v move v at most distance 2 in $\mathcal{S}^{sep}_{n,p}$.

$\mathcal{S}_{n,p}^{sep}$ is connected for $n = 1$ and $p \geq 3$.

Suppose $n \geq 2$. Then generation of $\mathrm{Out}_{n,p}$ also requires transvection. Consider the image v' of v under the diffeomorphism corresponding to the transvection $a_1 \mapsto a_1 a_2^{-1}$, as shown in orange in Figure 4.1. Then v' and v are contained in a copy of $M_{2,1}$ bounded by the sphere u separating a_1 and a_2 from $a_3, \ldots, a_n$ and $b_1, \ldots, b_s$. If $n \geq 3$ or $p \geq 2$ we have that u is essential and this gives a length 2 path v to u to v' in $\mathcal{S}_{n,p}^{sep}$. The inverse transvection $b_1 \mapsto a_1 a_2$ similarly moves v distance 2 in $\mathcal{S}_{n,p}^{sep}$. Appealing to Putman's Lemma 2.3, we conclude that $\mathcal{S}_{n,p}^{sep}$ is connected for $n = 2$ and $p \geq 2$ or $n \geq 3$.

Finally, we show that $\mathcal{S}_{2,1}^{sep}$ is disconnected. By capping the boundary component with a sphere we obtain a map

$$\phi : \left(\mathcal{S}_{2,1}^{sep}\right)^{(0)} \to \left(\mathcal{S}_{2,0}^{sep}\right)^{(0)}$$

Observe that if u and v are disjoint spheres of $\mathcal{S}_{2,1}^{sep}$ then $\phi(u) = \phi(v)$. So ϕ gives a surjective simplicial map

$$\mathcal{S}_{2,1}^{sep} \to \mathcal{S}_{2,0}^{sep}.$$

But as $\mathcal{S}_{2,0}^{sep}$ is totally disconnected it must be that $S_{2,1}^{sep}$ is disconnected. $\square$

We say that a sphere is $M_{n',p'}$-bounding if it bounds an embedded copy of $M_{n',p'} \subset M_{n,p}$.

Lemma 4.3. *For $k \leq n/2$, $M_{k,1}$-bounding spheres are characteristic in $\mathcal{S}_n^{sep}$ for $n \geq 3$.*

Proof. Suppose that $x \in \mathcal{S}_n^{sep}$ bounds an $M_{k,1}$. Observe the link of x is isomorphic to a join $\mathcal{S}_{k,1}^{sep} * \mathcal{S}_{n-k,1}^{sep}$. By Lemma 4.1 the dimensions of the sides of the join are $2k - 3$ and $2n - 2k - 3$, so any automorphism of $\mathcal{S}_n^{sep}$ must send x to a genus k-bounding sphere. $\square$

Observe that $M_{1,0} = S_1 \times S_2$ so that $\pi_2(M_{1,0}, p) \cong \mathbb{Z}$. Using the long exact sequence of the pair $(M_{1,1}, \partial M_{1,1})$ we compute $\pi_2(M_{1,0}) \cong \pi_2(M_{1,1}, S^2)$, so $M_{1,1}$ contains a unique homotopy class of nonseparating sphere generating the second homotopy group.

Then for any automorphism $\phi \in \mathrm{Aut}\left(\mathcal{S}_{n,p}^{sep}\right)$ we can extend ϕ to a map $\hat{\phi} : \mathcal{S}_{n,p} \to \mathcal{S}_{n,p}$. If x is a separating sphere we assign $\hat{\phi}(x) = \phi(x)$. If a is a nonseparating sphere, then

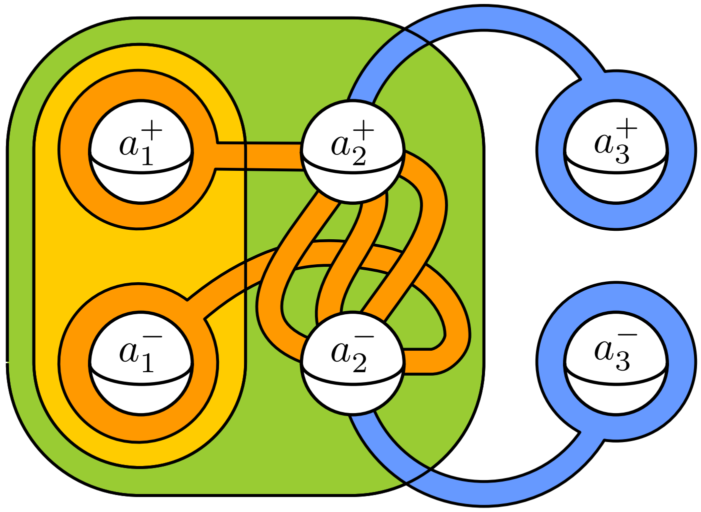


Figure 4.2: The sharing pair x and x' bound $M_{1,1}$ shown in yellow and orange. They are contained in the green $M_{2,1}$ bounded by y. The blue $M_{1,1}$ is bounded by z. Observe an $M_{1,1}$-bounding sphere containing a_1 can be represented by drawing two parallel copies a_1^+ and a_1^- and then connecting them by attaching a handle given by the regular neighborhood of an arc from a_1^- to a_1^+ disjoint from a_1. Fixing a_1, the spheres x and x' are determined by their respective intersection numbers with a_2.

there is there is an $M_{1,1}$-bounding sphere x bounding an $M_{1,1}$ that contains a. Then $\phi(x)$ bounds an $M_{1,1}$ by Lemma 4.3. We define $\hat{\phi}(a)$ to be the nonseparating sphere in the $M_{1,1}$ bounded by $\phi(x)$. We must first demonstrate that $\hat{\phi}$ is well defined.

Fix a nonseparating sphere a. Define a *sharing pair* $\{x, x'\}$ (sharing a) to be $M_{1,1}$-bounding spheres x and x' such that x and x' each bound an $M_{1,1}$ containing a and are contained in a common $M_{2,1}$ bounded by separating sphere y.

Lemma 4.4. *If* $\{x, x'\}$ *is a sharing pair, then* $\{\phi(x), \phi(x')\}$ *is a sharing pair for any automorphism* $\phi \in \mathrm{Aut}\,(\mathcal{S}_n^{sep})$.

Proof. Let a_1 be a nonseparating sphere of $M_{n,p}$ and let $\{x, x'\}$ be a sharing pair for a_1. Then x and x' are adjacent to an $M_{2,1}$-bounding sphere y, but not each other in $\mathcal{S}_n^{sep}$. Let a_2 be a nonseparating sphere disjoint from a_1 in the $M_{2,1}$ bounded by y. Observe further we may find an $M_{1,1}$-bounding sphere z that intersects y, but not x or x'. Then, appealing to Lemma 4.4, $\phi(x)$ and $\phi(x')$ must be intersecting $M_{1,1}$-bounding spheres. Let A be the $M_{2,1}$ bounded by $\phi(y)$. $\phi(z)$ is disjoint from $\phi(x)$ and $\phi(x')$, but not $\phi(y)$. Consider the image of $\phi(z)$ in the A. If $\phi(z)$ bounded a region containing a nonseparating sphere in A, there would only be one class of separating sphere in A disjoint from $\phi(z)$. Then $\phi(z)$ must bound in A a handle given by the boundary of a regular neighborhood of an arc of $\pi_1(A, \partial A)$ that must pass through a nonseparating sphere a of A. But then $\phi(x)$ and $\phi(x')$ must both bound the nonseparating sphere of A disjoint from a. So $\{\phi(x), \phi(x')\}$ is a sharing pair. $\square$

Let a be a nonseparating sphere of M_n. We will show that any two $M_{1,1}$-bounding spheres that contain a on their $M_{1,1}$-side are connected by a sequence of sharing pairs. Let $\mathcal{P}_a$ be the *sharing pair* graph defined as follows. The vertices of $\mathcal{P}_a$ are genus 1-bounding separating spheres of M_n that bound an $M_{1,1}$ containing a. Two vertices of $\mathcal{P}_a$ are adjacent if they form a sharing pair for a.

Lemma 4.5. *The sharing pair graph $\mathcal{P}_a$ is connected.*

Proof. We appeal to Putman's Lemma 2.3 using the graph $X = \mathcal{P}_a$ and the group $G \leq \mathrm{Out}_n$ fixing a setwise. Let $a_1, \ldots, a_n$ be a basis for F_n. Then G is generated by diffeomorphisms corresponding to permutations of $\{a_2, \ldots, a_n\}$, inversions, and the transvections $a_1 \mapsto a_1 a_2^{-1}$ and $a_2 \mapsto a_2 a_3^{-1}$.

Observe that G acts transitively on $M_{1,1}$-bounding spheres that contain a on their genus 1-side. Let v be the sphere separating a_1 from $a_2, \ldots, a_n$. Observe that of the chosen generators only the transvection $\phi : a_1 \mapsto a_1 a_2^{-1}$ has nontrivial action on v. But, as can be seen in figure 4.1, v and $\phi(v)$ are contained in an $M_{2,1}$ so that $\{v, \phi(v)\}$ is a sharing pair.

It follow by Putman's Lemma 2.3 that $\mathcal{P}_a$ is connected. $\square$

The previous Lemma shows that $\hat{\phi}$ is well defined. If a is a nonseparating sphere of M_n and x and x' $M_{1,1}$-bounding sphere bounding an $M_{1,1}$ containing a, then as P_a is connected there is a sequence of sharing pairs from x to x'. By Lemma 4.4 this gives a sequence of sharing pairs from $\phi(x)$ to $\phi(x')$. But then $\phi(x)$ and $\phi(x')$ share the same nonseparating sphere so that $\hat{\phi}(a)$ is well defined.

Certainly $\hat{\phi}$ is simplicial. If a and a' are disjoint nonseparating spheres then there are disjoint $M_{1,1}$-bounding spheres x and x' bounding disjoint copies of $M_{1,1}$ separating a and a', respectively. Since $\phi(x)$ and $\phi(x')$ are disjoint $M_{1,1}$-bounding spheres, $\hat{\phi}(a)$ and $\hat{\phi}(a')$ are also disjoint. If y is a separating sphere disjoint from a, then either there is an $M_{1,1}$-bounding sphere separating a from y or y is an $M_{1,1}$-bounding sphere, so that $\hat{\phi}(a)$ is disjoint from $\hat{\phi}(y) = \phi(y)$.

Proof of Theorem 1.3. The map constructed above

$$\Phi : \mathrm{Aut}\,(\mathcal{S}_n^{sep}) \to \mathrm{Aut}\,(\mathcal{S}_n)$$

with $\phi \mapsto \hat{\phi}$ is an isomorphism with the map simply restricting automorphisms

$$\mathrm{Aut}\,(\mathcal{S}_n) \to \mathrm{Aut}\,(\mathcal{S}_n^{sep})$$

giving an inverse to Φ. Then the result follows from Theorem 2.12. $\square$

Complexes of High Genus Separating Spheres

For $k \leq n/2$, we call a sphere $x : S^2 \hookrightarrow M_{n,p}$ k-separating if both components of $M_{n,p} - x$ contain either a boundary component or at least k disjoint separating spheres. If x bounds a copy of $M_{j,1}$ with $j < n/2$, we refer to it as to as x^{in}, or the *inside* of x. We will also describe objects disjoint from and inside of x as *engulfed* by x, and disjoint objects on the outside as *exgulfed* by x.

Let $\mathcal{S}_{n,p}^{sep,k} \subset \mathcal{S}_{n,p}^{sep}$ be the subcomplex spanned by homotopy classes of essential k-separating spheres. In this section we show that $\mathcal{S}_{n,p}^{sep,k}$ is a combinatorial model for $\mathrm{Out}_{n,p}$.

Theorem 1.4. *For $n \geq 3k$, the natural map $\mathrm{Out}\,F_n \to \mathrm{Aut}\,\mathcal{S}_n^{sep,k}$ is an isomorphism.*

Observe that for $k > 1$, $\mathcal{S}_{n,p}^{sep,k}$ does not have a uniform dimension. For example, in the case with no boundary components, $p = 0$, we can construct a maximal (with respect to inclusion) simplex of maximal dimension $n-2k$ as in Figure 4.3. If we write $n = qk + r$ with $q = \lfloor n/k \rfloor$ and $0 < r < k$, then we can construct maximal (with respect to inclusion) simplices of smaller dimension $2q + r - 4$ as in Figure 4.4.

If there are boundary spheres we can construct a maximal dimension simplex similar to Figure 4.3 by replacing the $M_{k,1}$-bounding spheres with boundary spheres. Similar linear nesting shows any k-separating sphere can be contained in a maximum dimension simplex of dimension for $k > 1$

$$\max_n \left\{ \Delta^n \hookrightarrow \mathcal{S}^{sep,k}_{n,p} \right\} = \begin{cases} n - 2k & \text{if } p = 0 \\ n - k & \text{if } p = 1 \\ n + p - 3 & \text{if } s \geq 2 \end{cases} \cdot$$

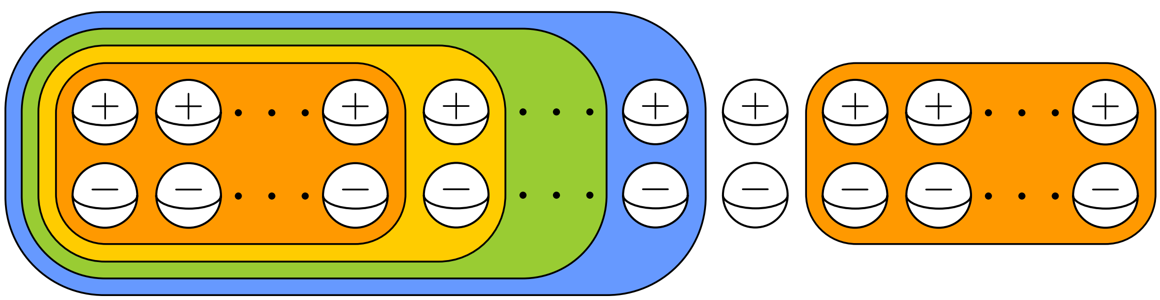


Figure 4.3: A maximal dimension maximal simplex of $\mathcal{S}^{sep,k}_n$ for $k > 1$ is spanned by $n - 2k + 1$ spheres and cuts M_n into 2 copies of $M_{k,1}$ and $n - 2k$ copies of $M_{1,2}$. The corresponding graph of M_n components is an unbranched tree with 2 leaves of weight k and $n - 2k$ internal vertices of weight 1.

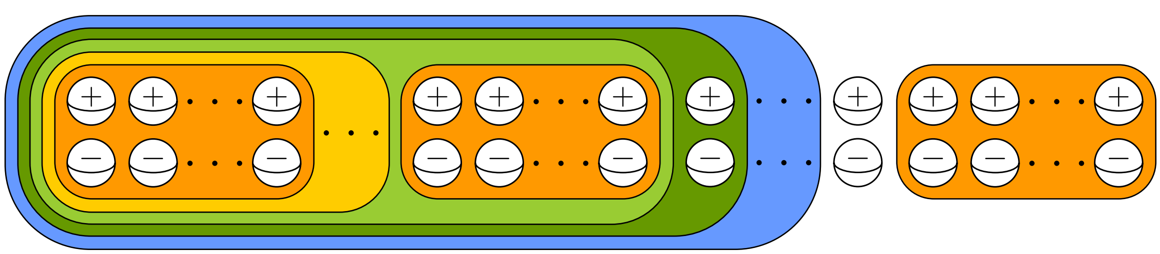


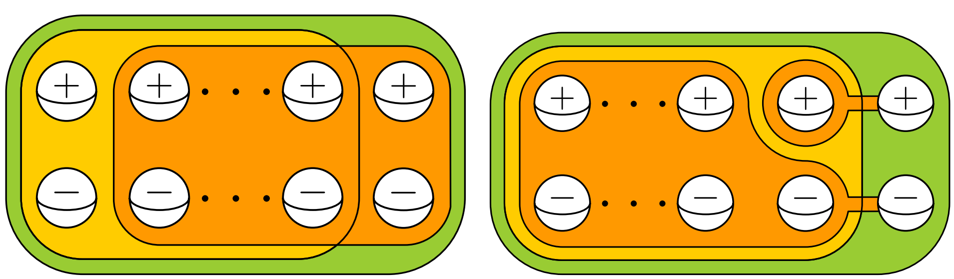
Figure 4.4: A minimal dimension maximal simplex of $\mathcal{S}^{sep,k}_n$ for $k > 1$ is spanned by $2q + r - 3$ spheres and cuts M_n into q copies of $M_{k,1}$ and r copies of $M_{1,2}$ and $q - 2$ copies of $M_{0,3}$. The corresponding graph of M_n components is a tree with q leaves of weight k and $q - 2$ internal vertices weight 0 and r internal vertices of weight 1.

Lemma 4.6. *For $1 < k < n/2$, the complex of k-separating spheres $\mathcal{S}_{n,p}^{sep,k}$ is connected whenever it has positive dimensional simplices if $p = 0$, and whenever it has 2 dimensional simplices if $p > 0$.*

Proof. The proof is by Putman's Lemma 2.3 with the group $\mathrm{Out}_{n,p}$.

Consider first the case with $p = 0$ and suppose that $\mathcal{S}_{n,p}^{sep,k}$ has positive dimensional simplices. So $n > 2k$, and in particular there are $M_{k,1}$ and $M_{k+1,1}$-bounding spheres in $\mathcal{S}_{n,p}^{sep,k}$. Choose a sphere v to be an $M_{k,1}$-bounding sphere. Observe that every k-separating sphere is disjoint from a $M_{k,1}$-bounding sphere, and the $M_{k,1}$-bounding spheres are exactly the Out_n orbit of v. Let $a_1, \ldots, a_n$ be a maximal collection of disjoint nonseparating spheres of M_n with $a_1, \ldots, a_k$ engulfed by v. Consider as a generating set for Out_n the transpositions and inversions of $a_1, \ldots, a_n$ and the transposition diffeomorphism t corresponding to $a_1 \mapsto a_1 a_{k+1}^{-1}$. Observe that every inversion fixes v. Observe that a transposition ϕ either fixes v, in the case it swaps spheres on the same side of $M_n - v$, or $\phi(v)$ and v are contained in a common $M_{k+1,1}$-bounding sphere that is k-separating, as in Figure 4.5a. Finally, v and $t(v)$ are contained in a common $M_{k+1,1}$-bounding sphere as in Figure 4.5b. The connectivity then follows by Putnam's Lemma.

Consider the case with $p > 0$. If $p = 1$ then to have dimension 2 simplices $n \geq k + 2$, and $M_{2,2}$-bounding spheres are k-separating. If $p > 1$ then to have dimension 2 simplices $n + p \geq 5$. If $p = 1$ then $n \geq 4$ so that $M_{2,2}$-bounding spheres are disjoint from a



(a) Transpositions move $M_{k,1}$-bounding spheres distance 2.

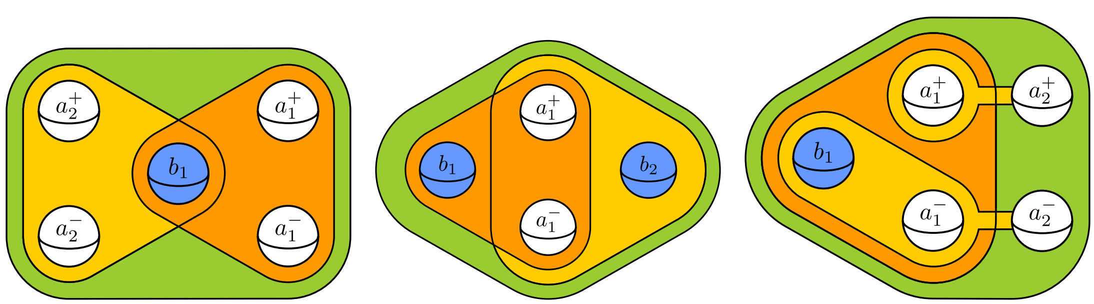
(b) Transvections move $M_{k,1}$-bounding spheres distance 2.

Figure 4.5: Nontrivial Out_n generator actions on the base sphere v move v at most distance 2 in $\mathcal{S}_{n,p}^{sep,k}$.

$M_{k,1}$-bounding sphere and must be k-separating. If $p > 1$ then $n \geq 4$ so that $M_{1,3}$ and $M_{2,2}$-bounding spheres are disjoint from a $M_{1,2}$ or $M_{k,1}$-bounding sphere and must be k-separating.

Choose a sphere v to be an $M_{1,2}$-bounding sphere. Observe that every k-separating sphere is disjoint from a $M_{1,2}$-bounding sphere, and the $M_{1,2}$-bounding spheres are exactly the $\mathrm{Out}_{n,p}$ orbit of v. Let $b_1, \ldots, b_s$ be the bounding spheres and let a_1 be a nonseparating sphere engulfed by v and $a_2, \ldots, a_n$ disjoint nonseparating spheres disjoint from v and a_1. Consider as a generating set for $\mathrm{Out}_{n,p}$ diffeomorphisms corresponding to transpositions of $a_1, \ldots, a_n$, transpositions of $b_1, \ldots, b_s$, t the transvection $a_1 \mapsto a_1 a_2^{-1}$, and u the b_1 push corresponding to conjugation $b_1 \mapsto a_1 b_1 a_1^{-1}$. Observe first that u leaves v fixed. Observe that ϕ a transposition of $a_1, \ldots, a_n$ either leaves v if it fixes a_1, or swaps a_1, and then $\phi(v)$ and v are engulfed by an $M_{2,2}$-bounding sphere as in Figure 4.6a. Observe that ψ a transposition of $b_1, \ldots, b_s$ either leaves v if it fixes b_1, or swaps b_1, and then $\psi(v)$ and v are engulfed by an $M_{1,3}$-bounding sphere as in Figure 4.6b. Finally, v and $t(v)$ are engulfed by an $M_{2,2}$-bounding sphere as in Figure 4.6c. The connectivity then follows by Putnam's Lemma. □

Lemma 4.7. *Let $n \geq 3$ and $\phi \in \mathrm{Aut}\left(\mathcal{S}_n^{sep,k}\right)$. For $n/2 > j \geq k$, if x is a $M_{j,1}$-bounding spheres engulfing a sphere y, then $\phi(x)$ is a $M_{j,1}$-bounding spheres engulfing the sphere $\phi(y)$.*



(a) a-Transposition (b) b-Transposition (c) Transvection

Figure 4.6: Nontrivial $\mathrm{Out}_{n,p}$ generator actions on the base sphere v move v at most distance 2 in $\mathcal{S}_{n,p}^{sep,k}$.

Proof. Suppose that x bounds an $M_{j,1}$ in $\mathcal{S}_n^{sep,k}$. Consider the subcomplex $\mathcal{E}_x$ spanned by spheres engulfed by x and the subcomplex $\mathcal{F}_x$ spanned by spheres disjoint but not engulfed by x. The link of x is a join $\mathcal{E}_x * \mathcal{F}_x$ and $\mathcal{E}_x \cong \mathcal{S}_{j,1}^{sep,k}$ and $\mathcal{F}_x \cong \mathcal{S}_{n-j,1}^{sep,k}$. Then according to Lemma 4.6, $\mathcal{E}_x$ and $\mathcal{F}_x$ have simplices with maximal dimension $j-k$ and $n-j-k$, respectively. So the link of $\phi(x)$ must have the same structure and $\phi(x)$ must be $M_{j,1}$-bounding. Note that since

$$\phi\left(\mathcal{E}_x\right) = \mathcal{E}_{\phi(x)}$$

any sphere y engulfed by x has $\phi(y)$ engulfed by $\phi(x)$. $\square$

We hope to extend automorphisms of $\mathcal{S}_n^{sep,k}$ to automorphisms of $\mathcal{S}_n^{sep,k-1}$ by a combinatorial characterization of $M_{k-1,1}$-bounding spheres in $\mathcal{S}_n^{sep,k}$.

This is the direct analog of handle pairs of Brendle-Margalit [5].

Definition 4.8. If x is an $M_{k,1}$-bounding sphere engulfed by $M_{k+1,1}$-bounding sphere y we say that a pair v, w of $M_{k,1}$-bounding spheres *carve* x from y if

(1) Each pair of v, w, y intersects, but v, w, y are all disjoint from x.

$$\begin{tikzcd}
v \arrow[r, dash] & x \arrow[r, dash] & y \\
w \arrow[ur, dash] & &
\end{tikzcd}$$

(2) The $M_{k,1}$-bounding sphere x is the unique sphere engulfed by y and disjoint from both v and w.

(3) There is more than one $M_{k,1}$-bounding sphere engulfed by y and disjoint from v but not w.

(4) There is more than one $M_{k,1}$-bounding sphere engulfed by y and disjoint from w but not v.

It follows immediately from this combinatorial definition and Lemma 4.7 that carving is characteristic. The following Lemma shows additionally that there is a unique nonseparating sphere that was "carved away" from y.

Lemma 4.9. *Let $\phi \in \operatorname{Aut} \mathcal{S}_n^{sep,k}$. If v, w carve x from y, then*

(1) $\phi(v), \phi(w)$ carve $\phi(x)$ from $\phi(y)$

(2) One of the spheres v or w contains a disk or annulus p with $\partial s \subset y$ whose image in y^{in}/y is homotopic to a nonseparating sphere.

(3) There is an arc α with endpoints on y such that v and w separate α from x.

(4) x is the unique $M_{k,1}$-bounding sphere engulfed by y and disjoint from p and α.

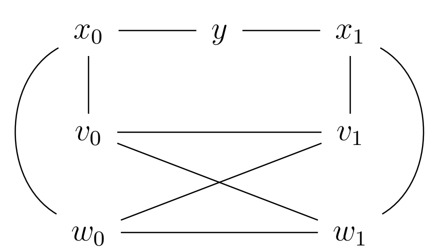
Proof. (1) follows from Lemma 4.7 and the combinatorial definition of carving.
(2) Fix representatives for v, w, x, and y that intersect minimally and transversely. Then $w \cap y^{in}$ is a collection of disks and annuli with boundary on y. No component disk or annulus of $w \cap y^{in}$ can be separating, or else there would be at most $M_{k,1}$ in y^{in} disjoint from w. Similarly $v \cap y^{in}$ is a collection of disks and annuli, no one of that separates y^{in}. Let β be any nontrivial loop in $y^{in} - x^{in}$ and based at a point on x. Then β must intersect either v or w, or else the pushes of any nonseparating sphere of x about α would yield infinitely many $M_{k,1}$-bounding spheres engulfed by y and disjoint from v and w, contrary to the hypothesis. Since no such β exists, there must a component p of $v \cap y^{in}$ or $w \cap y^{in}$ whose image in the quotient y^{in}/y is homotopic to a nonseparating sphere.
(3) Let a be a nonseparating sphere engulfed by y and exgulfed by x and disjoint from the nonseparating component p as above. If $v \cap y^{in}$ or $w \cap y^{in}$ have a component that intersects a, then as v and w are separating there must be an arc intersecting a with endpoints on y that they separate from x. Suppose that v and w are disjoint from a. If there is a loop γ based at a that winds through a noseparating sphere engulfed by x and is disjoint from v and w, then the pushes of a along γ leave v and w unchanged, but the images of x give infintely many

$M_{k,1}$-bounding spheres engulfed by y and disjoint from v and w, in contradiction with the definiton of carving. Then v and w must separate a from x in y, and there is an arc α with end points on y that intersects a once and so must be separated from x by v and w.

(4) Assume to the contrary there is some x' engulfed by y, distinct from x, and disjoint from p and α. Then there must be some nonseparating sphere a engulfed x' but not x. Note that $y^{in} - x^{in} \cong M_{1,2}$ and consider the components of $a \cap (y^{in} - x^{in})$. If $a \cap (y^{in} - x^{in})$ has a nonseparating disk, it must intersect α. If $a \cap (y^{in} - x^{in})$ contains a nontrivial arc, it must intersect p. So a must be engulfed by x. $\square$

Definition 4.10. Define an $M_{k-1,1}$-sharing pair $\{x_0, x_1\}$ to be a pair of $M_{k,1}$-bounding spheres $x_0, x_1 \in \mathcal{S}_n^{sep,k}$ such that:

(1) There are $M_{k,1}$-bounding spheres x_2, x_3 and a $M_{k+1,1}$-bounding sphere y such that the induced subgraph of $\mathcal{S}_n^{sep,k}$ on $y, x_0, x_1, v_0, v_1, w_0, w_1$ is exactly



and

(2) For $i = 0, 1$ the spheres v_i, w_i carve x_i from y_i.

(3) For $z_0 \in \{v_0, w_0\}$ and $z_1 \in \{v_1, w_1\}$, there is no $M_{k,1}$-bounding sphere engulfed by y and disjoint from both z_0 and z_1.

Lemma 4.11. *The spheres of an $M_{k-1,1}$-sharing pair uniquely engulf a $M_{k-1,1}$-bounding sphere in M_n.*

Proof. Let $\{x_0, x_1\}$ be a sharing pair with y, v_0, w_0, v_1, w_1 as above. Let p_i and α_i be as specified in Lemma 4.9, so that, without loss of generality, p_i is a component of $v_i \cap y^{in}$

that is nonseparating in y. And α_i is a loop with endpoints on y that v_i and w_i separate from x_i. But as v_0 and v_1 are disjoint, so are p_0 and p_1.

Since there is no $M_{k,1}$-bounding sphere disjoint from both v_0 and v_1, it must be no $M_{k-1,1}$-bounding sphere is disjoint from both p_0 and p_1 or from both α_0 and α_1.

Then α_1 must intersect x_0, and there are $k-1$ disjoint nonseparting spheres $a_1, \ldots, a_{k-1}$ engulfed by x and disjoint from α_1.

Consider the images in $y^{in}/y \cong M_{k+1,1}$. Then the images of p_0 and s_1 are distinct nonseparating spheres in y^{in}/y. Further since there is no $M_{k,1}$-bounding sphere disjoint from both, forgetting the basepoint y/y in y^{in}/y gives distinct, disjoint spheres $\overline{s_0}$ and $\overline{s_1}$ of y^{in}. Let Σ be a system of $n-k-1$ disjoint spheres exgulfed by y.

The let z be the unique sphere separating $a_1, \ldots, a_k$ from $\overline{s_0}$, $\overline{s_1}$, and Σ. Then z is $M_{k-1,1}$-bounding and uniquely engulfed by both x_0 and x_1. □

Lemma 4.12. *Sharing pairs are characteristic.*

If $\{x_0, x_1\}$ *is an* $M_{k-1,1}$*-sharing pair with* $x_0, x_1 \in \mathcal{S}_n^{sep,k}$ *and* $\phi \in \operatorname{Aut}\mathcal{S}_n^{sep,k}$*, then* $\{\phi(x_0), \phi(x_1)\}$ *is an* $M_{k-1,1}$*-sharing pair.*

Proof. Let $y, x_0, v_0, w_0, x_1, v_1, w_1$ be as in Definition 4.10. By Lemma 4.7 $\phi(x_0), \ldots, \phi(x_3)$ are $M_{k,1}$-bounding and $\phi(y)$ is $M_{k+1,1}$-bounding. The ϕ image of the induced subgraph on $y, x_0, v_0, w_0, x_1, v_1, w_1$ is an isomorphic graph. Property (2) is preserved by Lemma 4.9. Property (3) of Definition 4.10 is preserved by its combinatorial definition. □

Remark 9. Every $M_{k-1,1}$-bounding sphere x admits an $M_{k-1,1}$-sharing pair in $\mathcal{S}_n^{sep,k}$ engulfing x, provided $n \geq 3k$. By a change of coordinates the, arrangement in Figure 4.7 shows a possible pair sharing x.

We call three spheres an $M_{k-1,1}$-sharing triple if the spheres pairwise form sharing pairs and all engulf a common $M_{k-1,1}$-bounding sphere.

Lemma 4.13. *Sharing triples are characteristic.*

If $\{x_0, x_1, x_2\}$ *is an* $M_{k-1,1}$*-sharing triple with* $x_0, x_1, x_2 \in \mathcal{S}_n^{sep,k}$ *and* $\phi \in \operatorname{Aut}\mathcal{S}_n^{sep,k}$*, then*

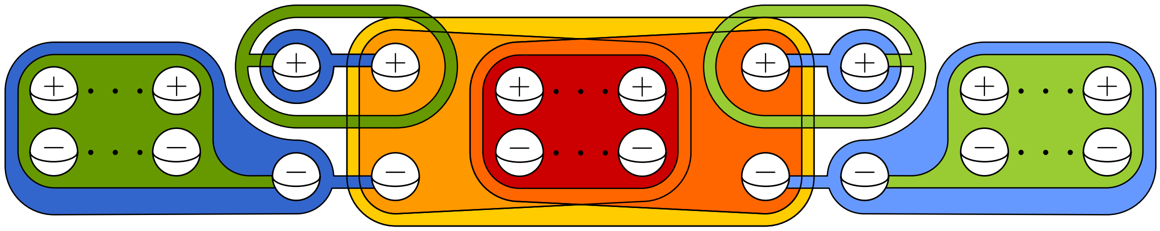


Figure 4.7: A sharing pair and the requisite carvings. The pair $\{x_0, x_1\}$ is shown bounding dark and light orange, respectively. The pair shares the $M_{k-1,1}$-bounding sphere x shown bounding red. The pair is engulfed by $M_{k+1,1}$-bounding sphere y shown in yellow. Dark orange x_0 is carved by v_0 and w_0 shown in dark green and blue. Light orange x_1 is carved by v_1 and w_1 shown in light green and blue.

$\{\phi(x_0), \phi(x_1), \phi(x_2)\}$ *is an* $M_{k-1,1}$*-sharing triple.*

Proof. According to Lemma 4.12 if x_0, x_1, x_2 pairwise form sharing pairs, then so do $\phi(x_0), \phi(x_1), \phi(x_2)$. It remains only to see that $\phi(x_0), \phi(x_1), \phi(x_2)$ all engulf a common $M_{k-1,1}$-bounding sphere, rather that a distinct $M_{k-1,1}$-bounding sphere for each pair. We reduce the proof to showing $\phi(x_0), \phi(x_1), \phi(x_2)$ all engulf a common $M_{k-1,1}$-bounding sphere if and only if there is no $M_{k+1,1}$-bounding sphere y engulfing $\phi(x_0), \phi(x_1)$, and $\phi(x_2)$. Then if there were a y engulfing $\phi(x_0), \phi(x_1)$, and $\phi(x_2)$, we would have $\phi^{-1}(y)$ engulfs x_0, x_1, x_2, which would contradict that $\{x_0, x_1, x_2\}$ is a sharing triple.

Observe that, as in the proof of Lemma 4.12, since $\phi(x_0), \phi(x_1), \phi(x_2)$ are pairwise sharing pairs, there are three pairwise-disjoint $M_{1,1}$-bounding spheres z_0, z_1, z_2 such that z_i is uniquely engulfed by $\phi(x_i)$ and disjoint but not engulfed by $\phi(x_{i+1})$, for $i \in \mathbb{Z}/3$. The sphere shared by $\{\phi(x_i), \phi(x_{i+1})\}$ is in $\phi(x_i)^{in} - z_i^{in}$. If $\phi(x_0), \phi(x_1), \phi(x_2)$ all engulf a common $M_{k-1,1}$-bounding sphere x, then any sphere engulfing $\phi(x_0), \phi(x_1), \phi(x_2)$ contains x, z_0, z_1, and z_2 so must be $M_{j,1}$-bounding for $j \geq k+2$. If $\phi(x_0), \phi(x_1), \phi(x_2)$ do not engulf a common $M_{k-1,1}$-bounding sphere, then for $i \in \mathbb{Z}/3$ we have a distinct $M_{k-1,1}$-bounding sphere shared by $\{\phi(x_i), \phi(x_{i+1})\}$ and that engulfs z_{i-1}. But then $\phi(x_0), \phi(x_1), \phi(x_2)$ do all engulf a common $M_{k-2,1}$-bounding sphere x such that $\phi(x_i)$ engulfs x and z_i and z_{i+1}. Then the same $M_{k+1,1}$-bounding sphere y fits into the defining pentagon of Definition 4.10 for all three sharing pairs. $\square$

Let x be an $M_{k-1,1}$-bounding sphere. We will show that any two sharing pairs engulfing x are connected by a sequence of sharing triples. Let $\mathcal{P}_x$ be the *sharing pair* graph defined as follows. The vertices of $\mathcal{P}_x$ are $M_{k-1,1}$-sharing pairs in $\mathcal{S}_n^{sep,k}$ engulfing x, where $n \geq 3k$. Two vertices $\{x_0, x_1\}$ and $\{x_1, x_2\}$ of $\mathcal{P}_x$ are adjacent if the pairs have a common member and the three spheres form an $M_{k-1,1}$-sharing triple engulfing x.

Lemma 4.14. *The sharing pair graph $\mathcal{P}_x$ is connected for $n \geq 3k$ and $k \geq 2$.*

Proof. We appeal to Putman's Lemma 2.3. Fix an $M_{k-1,1}$-bounding sphere x and an $M_{k-1,1}$-sharing pair $v = \{x_0, x_1\}$ engulfing x with x_0 and x_1 having geometric intersection 1. Let $G \leq \mathrm{Out}_n$ be the subgroup fixing x^{in}, so that $G \cong \mathrm{Out}_{n-k+1,1}$.

Observe that if two x-sharing pairs $\{x_0, x_1\}$ and $\{x_1, x_2\}$ contain a common member x_1 and with all three spheres engulfed by a common $M_{k+1,1}$-bounding sphere y, then we can find a length 2 path in $\mathcal{P}_x$ by choosing x_3 with intersection 1 with y and engulfing an $M_{1,1}$-bounding sphere that is disjoint from y. Then $\{x_0, x_1, x_3\}$ and $\{x_1, x_2, x_3\}$ are sharing triples, since they are pairwise sharing pairs and all engulf x. So if y_0 is the $M_{k+1,1}$-bounding sphere engulfing $v = \{x_0, x_1\}$, and $\{x_0', x_1'\}$ is any sharing pair engulfed by y_0', then there is $g \in G$ such that $g(x_1) = x_1'$ and $g(y_0) = y_0'$. So the orbit $G \cdot v$ is at most distance 2 from any sharing pair vertex of $\mathcal{P}_x$. This completes the first criterion of Putnam's Lemma 2.3.

Fix a system of nonseparating spheres $a_0, \dots, a_{n-k}$ disjoint and not engulfed by x with a_0 but not a_{n-k} engulfed by x_0 and a_{n-k} but not a_0 engulfed by x_1. Then $G \cong \mathrm{Aut}\langle a_0, \dots, a_{n-k}, \rangle$ is generated by diffeomorphism classes corresponding to inversions of $a_0, \dots, a_{n-k}$ (though these always fix v), transpositions of $a_0, \dots, a_{n-k}$ the transvection $t' : a_0 \mapsto a_0 a_1^{-1}$.

Consider first the action of transpositions t on the sharing pair $v \in \mathcal{P}_x$. If neither a_0 nor a_{n-k} are swapped by t, then the sharing pair v is fixed. If both a_0 and a_{n-k} are swapped by t, then x_0 and x_1 are swapped, so that the sharing pair $v = \{x_0, x_1\}$ is still fixed. If exactly one of a_0 or a_{n-k} is swapped by t transposition then exactly one of x_0, x_1 are exchanged from the sharing pair, so that the transposition action moves the sharing pair v distance 1 in $\mathcal{P}_x$, as shown in Figure 4.8.

Finally consider the transvection action $t' : a_0 \mapsto a_0 a_1^{-1}$ on $\mathcal{P}_x$. Then $t'(x_0)$ (shown in yellow in Figure 4.9) intersects x_0 (shown in orange) twice and $t'(x_1) = x_1$ (shown in light green). Since $n - k \geq 4$ there is a nonseparating sphere a_2 disjoint and not engulfed from $x_0, x_1, t'(x_0)$. So there is an $M_{k,1}$ bounding sphere x_2 (shown in dark green) engulfing a_2 and such that $\{t'(x_0), x_1, x_2\}$ and $\{x_0, x_1, x_2\}$ are sharing triple— let x_2 be the image of x_1 under the transposition $(a_2 a_{n-k})$. Then we have a length 2 path of in $\mathcal{P}_x$ from $t'(v)$ to v:

$$\{t'(x_0), x_1\} \to \{x_1, x_2\} \to \{x_0, x_1\}.$$

It follows from Putman's Lemma 2.3 that $\mathcal{P}_x$ is connected. □

We define a map $\operatorname{Aut} \mathcal{S}_n^{sep,k} \to \operatorname{Aut} \mathcal{S}_n^{sep,k-1}$ as $\phi \mapsto \hat{\phi}$ by extending $\phi \in \operatorname{Aut} \mathcal{S}_n^{sep,k}$ to $M_{k-1,1}$-bounding spheres via $M_{k-1,1}$-sharing pairs. More explicitly, if $x \in S_n^{sep,k-1}$ is an $M_{k-1,1}$-bounding sphere there is an $M_{k-1,1}$-sharing pair $\{x_0, x_1\}$ that engulfs x uniquely. Then by Lemma 4.12, $\{\phi(x_0), \phi(x_1)\}$ is a sharing pair. We define $\hat{\phi}(x)$ as the $M_{k-1,1}$-bounding sphere engulfed by $\{\phi(x_0), \phi(x_1)\}$. By Lemma 4.14 any other choice $\{x_0', x_1'\}$ of x-sharing pair is connected by a sequence of sharing triples, which by Lemma 4.13 gives a sequence of sharing triples from $\{\phi(x_0), \phi(x_1)\}$ to $\{\phi(x_0'), \phi(x_1')\}$, so that both share the

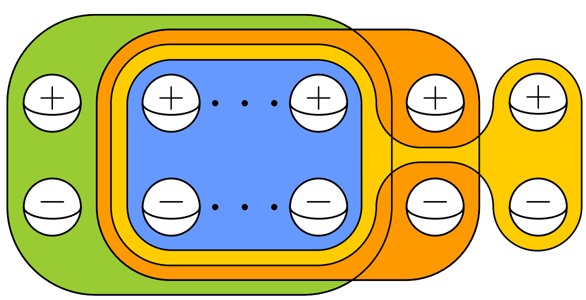


Figure 4.8: The sharing pair v is formed by the green x_1 and orange x_0 spheres. Transpositions move the sharing pair v either distance 0 in $\mathcal{P}_x$, by swapping orange and green, or distance 1, by, for example, swapping orange and yellow. Observe that the orange, yellow, and green $M_{k,1}$-bounding spheres form a sharing triple for the blue sphere x.

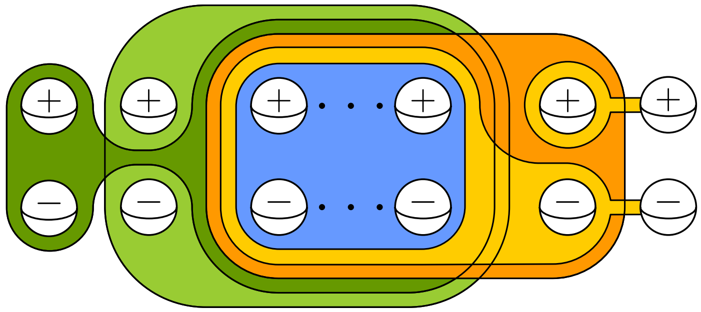


Figure 4.9: The chosen transvection moves v distance 2 in $\mathcal{P}_x$. Observe that x_0 orange, x_1 light green, x_2 dark green, and $t'(x_0)$ yellow can be organized into two sharing triples: orange with the greens and yellow with the greens.

same $M_{k-1,1}$-bounding sphere $\hat{\phi}(x)$, which is thus well defined.

Certainly $\hat{\phi}$ is simplicial. To see that observe that if x and x' are disjoint $M_{k-1,1}$-bounding spheres, then $n \geq 3k$ so there are disjoint $M_{k-1,1}$-sharing pairs that ϕ takes to disjoint sharing pairs. Then $\hat{\phi}(x)$ is disjoint from $\hat{\phi}(x')$. If $y \in \mathcal{S}_n^{sep,k}$ is disjoint from x, then y is $M_{j,1}$-bounding with $j \leq \frac{n}{2}$ so there is an x-sharing pair disjoint from y, with its ϕ-image disjoint from $\phi(y)$.

Lemma 4.15. *For $n \geq 3k$ and $k \geq 2$, the natural restriction map* $\operatorname{Aut} \mathcal{S}_n^{sep,k-1} \to \operatorname{Aut} \mathcal{S}_n^{sep,k}$ *is an isomorphism.*

Proof. We claim that the constructed map extension $\operatorname{Aut} \mathcal{S}_n^{sep,k} \to \operatorname{Aut} \mathcal{S}_n^{sep,k-1}$ given by $\phi \mapsto \hat{\phi}$ is the inverse homomorphism to the restriction $\operatorname{Aut} \mathcal{S}_n^{sep,k-1} \to \operatorname{Aut} \mathcal{S}_n^{sep,k}$ with $\psi \mapsto \psi|_k$. By definition the restriction of $\hat{\phi}$ to $\mathcal{S}_n^{sep,k}$ is ϕ. So the extension is injective and restriction is surjective. But restriction must also be injective, since if $\psi \in \mathcal{S}_n^{sep,k-1}$ restricts to the identity, then for any $M_{k-1,1}$-bounding sphere x there is an x-sharing pair $\{x_0, x_1\}$ that ψ fixes. But then $\psi(x) = x$ is the unique $M_{k-1,1}$-bounding sphere engulfed by $\{x_0, x_1\}$. □

Proof of Theorem 1.4. The proof is by induction on k using Lemma 4.15. Theorem 1.3 provides the base case. □

The Free Factor Complex

The free factor complex $\mathcal{FF}_n$ is the simplicial complex with a k-simplex given by conjugacy classes of length $k+1$ chains of proper free factors. Bestvina and Bridson announced that the free factor complex is a combinatorial model for $\mathrm{Out}\, F_n$, though as of this writing the result remains unpublished [38].

Theorem 1.5. *For $n \geq 3$, the natural map $\mathrm{Out}\, F_n \to \mathrm{Aut}\, \mathcal{FF}_n$ is an isomorphism.*

We proceed by proving first that the complex of coconnected spheres of M_n has automorphism group $\mathrm{Out}\, F_n$. The complex of coconnected spheres, that was introduced by Hatcher and Vogtmann to prove homological stability results for $\mathrm{Out}\, F_n$ in [28], and the complex of nonseparating spheres [30]. This complex is a fibration over the free factor complex.

Definition 4.16. If F_n can be expressed as the internal free product of subgroups $A, B \leqslant F_n$, then A and B are *free factors* of F_n. The free factor complex has as vertices the conjugacy classes of free factors of F_n. A collection of $A_1, \dots A_k$ of free factors spans a simplex if there is are free factors $A_1' < \cdots < A_k'$ such that A_i is conjugate to A_i'. We will frequently abuse notation and refer to both a free factor and its conjugacy class as a free factor, as dictated by context. The distinction is rarely relevant since the free factor complex is known to be flag [8].

Hatcher [8] characterized the free splitting complex as a complex of spheres in M_n. We define the following three simplicial complexes related to the free factor complex:

- Let $\mathcal{S}_n^{nonsep}$ be the simplicial complex with k-simplices specified by $k+1$ disjoint non-separating spheres in M_n.
- Let $\mathcal{S}_n^{coc}$ be the subcomplex of $\mathcal{S}_n^{nonsep}$ with simplices given by collections of spheres that are coconnected (i.e. have connected complement) in M_n.

· Let $\mathcal{SF}_n$ be the barycentric subdivision of the $(n-2)$-skeleton of $\mathcal{S}_n^{coc}$. Thus vertices of $\mathcal{SF}_n$ are coconnected sets of at most $n-1$ spheres, and simplices are given by chains of proper subsets.

For a simplex $\Sigma_0 \subset \cdots \subset \Sigma_k$ of $\mathcal{SF}_n$, we obtain a corresponding simplex of $\mathcal{FF}_n$ by the (conjugacy class of) free factors

$$\pi_1(M_n - \Sigma_k, x_0) \leqslant \cdots \leqslant \pi_1(M_n - \Sigma_0, x_0)$$

so we obtain a surjection of posets

$$\mathcal{SF}_n \to (\mathcal{FF}_n)^{op}.$$

A single nonseparating sphere of M_n corresponds to a rank $n-1$ free factor of F_n. The fiber over a rank k free factor corresponds to all choices of collections $n-k$ factors of rank $n-1$ any j of that intersect in a rank j free factor.

We begin by showing

Theorem 4.17. *For $n \geq 3$ the natural map*

$$\mathrm{Out}\, F_n \to \mathrm{Aut}\, \mathcal{SF}_n \cong \mathrm{Aut}\, \mathcal{S}_n^{coc} \cong \mathrm{Aut}\, \mathcal{S}_n^{nonsep}$$

is an isomorphism.

The result relies on the following theorem of Pandit [30].

Theorem 4.18. *For $n \geq 3$ we have $\mathrm{Aut}\, \mathcal{S}_n^{nonsep} \cong \mathrm{Out}(F_n)$.*

Our first goal is to show that $\mathrm{Aut}\, \mathcal{SF}_n \cong \mathrm{Aut}\, \mathcal{S}_n^{coc}$.

Let $M_{n,p}$ be the manifold M_n with interiors of p disjoint closed balls removed. We call n the *genus* of $M_{n,p}$. If Σ is a set of disjoint embedded spheres of $M_{n,p}$, we will denote by $M_{n,p}|\Sigma$ the manifold $M_{n,p}$ cut along Σ.

Lemma 4.19. *Automorphisms of $\mathcal{SF}_n$ preserve the cardinality of sets of spheres.*

Proof. We induct downward on the cardinality of sets of spheres. We claim as a base case that a set of spheres $\Sigma \in \mathcal{SF}_n^{(0)}$ has $n-1$ spheres if and only if it is adjacent to finitely many sets of spheres in $\mathcal{SF}_n$, namely, the proper subsets of Σ. If $\Sigma \in \mathcal{SF}_n^{(0)}$ has fewer than $n-1$ spheres, then $M_n|\Sigma$ has genus $k \geq 2$. The complex of coconnected nonseparating spheres of $M_n|\Sigma$ is isomorphic to $\mathcal{S}_k^{coc}$, which is infinite. Choose any nonseparating sphere a of $M_n|\Sigma$. Then $\Sigma \cup \{a\}$ is coconnected in M_n and adjacent to Σ in $\mathcal{SF}_n$.

Assume that automorphisms of $\mathcal{SF}_n$ preserve the size of sets of spheres with at least $k+1$ spheres. Let $A_k \subset \mathcal{SF}_n^{(0)}$ be the sets of spheres of $\mathcal{SF}_n$ with k or fewer spheres. A set of spheres $\Sigma \in A_k$ has k spheres if and only if $\operatorname{link}(\Sigma) \cap A_k$ is finite. By hypothesis automorphisms of $\mathcal{SF}_n$ preserve A_k and its complement, so must preserve the class of sets of k spheres. $\square$

We now prove the first isomorphism of Theorem 4.17.

Lemma 4.20. *For $n \geq 3$ we have $\operatorname{Aut}\mathcal{SF}_n \cong \operatorname{Aut}\mathcal{S}_n^{coc}$.*

Proof. As $\mathcal{SF}_n$ is the barycentric subdivision of the $n-2$ skeleton $\mathcal{S}_n^{coc(n-2)}$, there is a natural map

$$\Phi : \operatorname{Aut}\mathcal{S}_n^{coc} \to \operatorname{Aut}\mathcal{SF}_n.$$

We will construct the inverse. Let $\phi \in \operatorname{Aut}\mathcal{SF}_n$. The vertices of $\mathcal{SF}_n$ are the simplices of $\mathcal{S}_n^{coc}$ with dimension $n-2$ or less. Then ϕ induces a bijection ϕ_* of simplices of $\mathcal{S}_n^{coc(n-2)}$. By Lemma 4.19 we have ϕ_* preserves the dimension of simplices, so ϕ_* is an automorphism of $\mathcal{S}_n^{coc(n-2)}$.

It remains to see that ϕ_* also preserves $n-1$ simplices. To see this we will show that a collection of n disjoint separating spheres Σ form a simplex in $\mathcal{S}_n^{coc}$ if and only if

$$\mathcal{S}_n^{coc} \cap \left(\bigcap_{x \in \Sigma} \operatorname{link}(x) \right)$$

is finite. Note that if Σ is a coconnected set of n spheres, then $M_n|\Sigma$ is homeomorphic to $M_{0,2n}$. Then $\pi_2(M_n|\Sigma)$ is the free abelian group generated by any $2n-1$ of the balls, and an embedded sphere must be degree at most 1 over any generator. There are thus finitely many embedded spheres of $M_n|\Sigma$. Then $\bigcap_{x\in\Sigma}\operatorname{link}(x)$ contains finitely many vertices of $\mathcal{S}_n^{coc}$. Conversely suppose Σ is a non-coconnected set of n disjoint spheres. Then $M_n|\Sigma$ has a component M' with genus at least one and at least two boundary spheres. Choose a nonseparating sphere x of M', a boundary sphere y, and a loop α based at y intersecting x once. The push map of x along α produces a collection A of infinitely many spheres of M_n. Each $a\in A$ is nonseparating in $M'\subset M|\Sigma$, so $\{a,x\}$ is coconnected for any $x\in\Sigma$. Then $A\subset\bigcap_{x\in\Sigma}\operatorname{link}(x)$. Thus ϕ_* must also preserve $n-1$ simplices and gives a simplicial automorphism of $\mathcal{S}_n^{coc}$. Then $\phi\mapsto\phi_*$ gives the inverse homomorphism to Φ. □

Call a collection of m disjoint spheres $\Sigma\subset\mathcal{S}_n^{coc(0)}$ a *bounding m-tuple* (pair, triple, etc.) if Σ is not coconnected but every proper subset of Σ is. The genus of the bounding tuple is the smaller of the genera of the two components of $M_n|\Sigma$. The following lemma shows we can detect the genus combinatorially.

Lemma 4.21. *The link of a genus k bounding m-tuple of $\mathcal{S}_n^{coc}$ is isomorphic to the join $\mathcal{S}_k^{coc}*\mathcal{S}_{n-k-m+1}^{coc}$.*

Proof. Consider $\Sigma\subset\mathcal{S}_n^{coc(0)}$ a bounding m-tuple with genus k. Then $M_n|\Sigma$ has two components, $R_1\cong M_{k,m}$ and $R_2\cong M_{n-k-m+1,m}$. Let V_i be the complex of coconnected nonseparating spheres in R_i. So $V_1\cong\mathcal{S}_k^{coc}$ and $V_2\cong\mathcal{S}_{n-k-m+1}^{coc}$. We claim that $\operatorname{link}(\Sigma)$ is the join V_1*V_2. Certainly $\operatorname{link}(\Sigma)\subset V_1*V_2$. Consider sets of spheres Σ_i giving simplices of V_i. The $R_i|\Sigma_i$ are connected. $M_n|(\Sigma_1\cup\Sigma_2)$ is $R_1|\Sigma_1$ and $R_2|\Sigma_2$ glued along Σ, and hence connected. So $\Sigma_1\cup\Sigma_2$ must be coconnected in M_n and the join $\Sigma_1*\Sigma_2$ lies in $\operatorname{link}(\Sigma)$. □

We now prove the second isomorphism of Theorem 4.17.

Lemma 4.22. *For $n \geq 3$ we have* $\operatorname{Aut} \mathcal{S}_n^{coc} \cong \operatorname{Aut} \mathcal{S}_n^{nonsep}$.

Proof. Restriction gives a natural map

$$\Phi : \operatorname{Aut} \mathcal{S}_n^{nonsep} \to \operatorname{Aut} \mathcal{S}_n^{coc}.$$

We will construct the inverse. Observe that since $\mathcal{S}_n^{coc(0)} = \mathcal{S}_n^{nonsep(0)}$ any $\phi \in \operatorname{Aut} \mathcal{S}_n^{coc}$ induces a set map ϕ_* of $\mathcal{S}_n^{nonsep(0)}$. If ϕ_* is a simplicial automorphism, then $\phi \mapsto \phi_*$ is the inverse homomorphism to Φ. As $\mathcal{S}_n^{nonsep}$ is a flag complex (Lemma 3 of [11]), it will suffice to show that ϕ_* sends pairs of disjoint spheres to pairs of disjoint spheres. Disjoint nonseparating spheres form a bounding pair if and only if they are not adjacent in $\mathcal{S}_n^{coc}$. So it suffices to show that ϕ preserves bounding pairs of $\mathcal{S}_n^{coc}$. We will demonstrate this through the stronger result that ϕ preserves the set of genus k bounding m-tuples.

Case 1. Suppose Σ is a genus k bounding m-tuple with $m > 2$. Any $\Sigma' \subset \mathcal{S}_n^{coc(0)}$ is a bounding m-tuple if and only if Σ' does not span a simplex in $\mathcal{S}_n^{coc}$, but every proper subset of Σ' does. Hence if $\phi \in \operatorname{Aut} \mathcal{S}_n^{coc}$, then $\phi(\Sigma)$ is a bounding m-tuple. By Lemma 4.21, $\operatorname{link}(\Sigma)$ is isomorphic to $\mathcal{S}_k^{coc} * \mathcal{S}_{n-k-m+1}^{coc}$. We can determine k by the maximal simplex dimension on the sides of the join. Then $\phi(\Sigma)$ is also genus k.

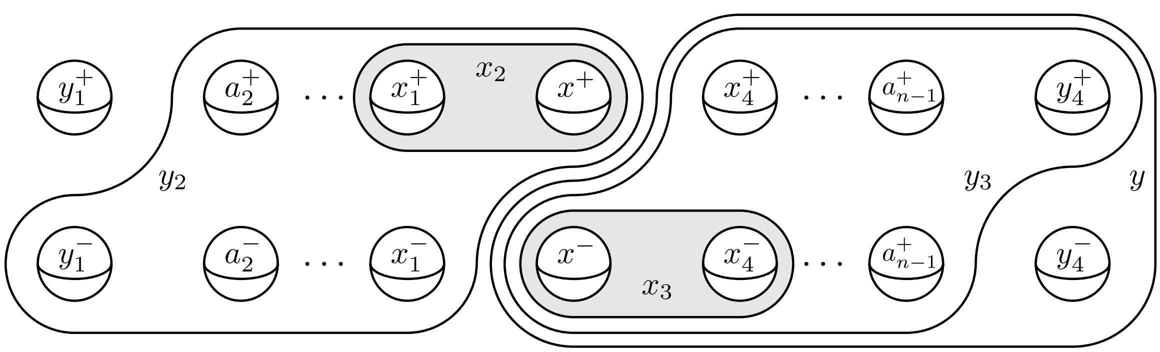


Figure 4.10: The manifold $M_n|\{a_i\}_{i=1}^n$ is a 3-sphere with $2n$ balls removed. We obtain M_n by again identifying the spheres with $+$ and $-$ labels via a vertical reflection. The spheres $\Sigma' = \{x_i, y_i\}_{i=1}^4$ are such that $M_n|\Sigma'$ contains x and y in disjoint copies of $M_{0,4}$. The $M_{0,4}$ containing x (identify x^+ and x^-) is shaded. The $M_{0,4}$ containing y is the exterior of y_2 and y_3.

Case 2. Suppose $\Sigma = \{x, y\}$ has $m = 2$ spheres. Choose a collection Σ' of disjoint non-separating spheres such that there are two separate components of $M_n|\Sigma'$ homeomorphic to $M_{0,4}$ and containing x and y respectively. We can construct Σ' as follows. $M_n|\Sigma$ has two components, homeomorphic to $M_{k,2}$ and $M_{n-k-1,2}$. So we have a set of spheres $\{a_i\}_{i=1}^n$ coconnected in M_n disjoint from y with $a_{k+1} = x$. Choose x_2, x_3, y_2, y_3 as shown in figure 4.10 and relabel $a_1 = y_1$, $x_1 = a_k$, $x_4 = a_{k+2}$, $y_4 = a_n$. Then $\{x_1, \ldots, x_4\}$ (resp. $\{y_1, \ldots, y_4\}$ are the boundary spheres of a component of $M|\Sigma'$ homeomorphic to $M_{0,4}$ and containing x (resp. y). Further $\{x, x_1, x_2\}$ and $\{x, x_3, x_4\}$ are genus 0 bounding triples. Let $\Sigma' = \{x_i, y_i\}_{i=1}^4$.

By Case 1 we have that $\{\phi(x_1), \ldots, \phi(x_4)\}$ is a genus 0 bounding 4-tuple and $\{\phi(x), \phi(x_1), \phi(x_2)\}$ and $\{\phi(x), \phi(x_3), \phi(x_4)\}$ are genus 0 bounding triples. So $\{\phi(x_1), \ldots, \phi(x_4)\}$ define a component of $M|\Sigma'$ homeomorphic to $M_{0,4}$ and containing $\phi(x)$.

If $\{x_1, \ldots, x_4\} \neq \{y_1, \ldots, y_4\}$ then $\phi(x)$ and $\phi(y)$ lie in disjoint $M_{0,4}$ homeomorphic components of $M|\phi(\Sigma')$. Then $\phi(x)$ and $\phi(y)$ are are disjoint. They are also not adjacent in $\mathcal{S}_n^{coc}$, so they are bounding a pair.

Suppose $\{x_1, \ldots, x_4\} = \{y_1, \ldots, y_4\}$. Then $n = 3$ and $M_3|\{x_i\}_{i=1}^4$ is homeomorphic to two copies of $M_{0,4}$. As x, y form a bounding pair, the bounding triples must be $\{x, x_1, x_2\}$, $\{x, x_3, x_4\}$, $\{y, x_1, x_2\}$, and $\{y, x_3, x_4\}$. Then the ϕ image of these triples are bounding triples giving $\phi(x)$ and $\phi(y)$ contained in disjoint $M_{0,4}$. Then $\phi(x)$ and $\phi(y)$ are disjoint and must form a bounding pair. □

Dyer and Formanek gave an algebraic proof of the fact that automorphisms of automorphisms of free groups are simply automorphisms of free groups [39]. Vogtmann and Bridson gave a more recent geometric proof [25].

Theorem 4.23. *The natural maps*

$$\mathrm{Aut}\, F_n \to \mathrm{Aut}\,\mathrm{Aut}\, F_n$$

and

$$\operatorname{Out} F_n \to \operatorname{Aut} \operatorname{Out} F_n$$

are isomorphisms for $n \geq 3$*. In particular* $\operatorname{Out} \operatorname{Out} F_n = 1$ *and the center of* $\operatorname{Out} F_n$ *is trivial.*

Lemma 4.24. *Automorphisms of the free factor complex preserve the rank of free factors.*

Let $\phi \in \operatorname{Aut} \mathcal{FF}_n$ *and let* A *be a free factor of* F_n*. Then* A *and* $\phi(A)$ *have the same rank.*

Proof. Suppose that A is a free factor of $\mathcal{FF}_n$. Then the link of A is a join

$$\operatorname{link} A \cong \text{span}\{B \in \mathcal{FF}_n{}^{(0)} \mid B < A\} * \text{span}\{B \in \mathcal{FF}_n{}^{(0)} \mid A < B\}$$

between the sub- and super-factors of A. So if A is rank k then $\{B \in \mathcal{FF}_n{}^{(0)} \mid B < A\}$ spans a dimension $k-2$ subcomplex isomorphic to $\mathcal{FF}_k$ and similarly $\{B \in \mathcal{FF}_n{}^{(0)} \mid A < B\}$ spans a dimension $n-k-2$ subcomplex isomorphic to $\mathcal{FF}_{n-k}$. If $\phi \in \operatorname{Aut} \mathcal{FF}_n$ then ϕ preserves this join and $\phi(A)$ must be either rank k or $n-k$.

Let $\mathcal{G}$ be the subcomplex of $\mathcal{FF}_n$ spanned by rank 1 and rank $n-1$ free factors. Then $\mathcal{G}$ is a connected and bipartite with the rank 1 and rank $n-1$ factors forming the parts of the bipartition. Then if $A, A' \in \mathcal{G}^{(0)}$ are free factors the same rank, the free factors $\phi(A), \phi(A')$ must have the same rank for any $\phi \in \operatorname{Aut} \mathcal{FF}_n$. Assume to the contrary that there is $\phi \in \operatorname{Aut} \mathcal{FF}_n$ an automorphism and a rank $n-1$ free factor A with $\phi(A)$ rank 1. Then ϕ must swap the bipartition of $\mathcal{G}$. In fact, there is a map $\operatorname{Aut} \mathcal{FF}_n \to \mathbb{Z}/2$ taken by sending $\phi' \in \operatorname{Aut} \mathcal{FF}_n$ to the generator of $\mathbb{Z}/2$ if ϕ' swaps the bipartition on $\mathcal{G}$. Then if $G = \langle \phi, \operatorname{Out} F_n \rangle \leq \operatorname{Aut} \mathcal{FF}_n$ is the subgroup generated by $\operatorname{Out} F_n$ and the automorphism ϕ we have an exact sequence

$$1 \longrightarrow \operatorname{Out} F_n \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

But by Theorem 4.23 we know $\mathrm{Out}\, F_n$ is centerless and $\mathrm{Out}\,\mathrm{Out}\, F_n = 1$, so we may apply Lemma 2.10 to conclude that $G \cong \mathrm{Out}\, F_n \times \mathbb{Z}/2$. Then there is an order two automorphism t of $\mathcal{FF}_n$ that swaps the bipartition of $\mathcal{G}$ and commutes with $\mathrm{Out}\, F_n$. Then there is rank one free factor $\langle b \rangle$ and free factor A with rank $n-1$ such that $t(\langle b \rangle) = A$. Let A have free basis $a_1, \ldots, a_{n-1}$ with $b = a_{n-2}$ if $b \in A$. Then the transvection ϕ'' with $b \mapsto ba_1$ and $a_i \mapsto a_i$ is an outer automorphism acting on the free factors with

$$A = \phi''(A) = \phi'' t(\langle b \rangle) = t\phi''(\langle b \rangle) = t(\langle ba_1 \rangle)$$

but this contradicts that t is injective. It must be that for any automorphism $\phi \in \mathcal{FF}_n$ the rank of A and $\phi(A)$ are the same if A is rank 1 or rank $n-1$.

But then if A is a rank k free factor, the link of A is a join with sides of dimension $k-2$ and $n-k-2$, and any rank $n-1$ free factor B containing A is on the dimension $n-k-2$ side. So the link of $\phi(A)$ is isomorphic to $\mathcal{FF}_k * \mathcal{FF}_{n-k}$ with $\phi(B)$ the rank $n-1$ free factor on the dimension $n-k-2$ side. But then $\phi(A)$ must be rank k as well. $\square$

Theorem 1.5. *For $n \geq 3$, the natural map $\mathrm{Out}\, F_n \to \mathrm{Aut}\, \mathcal{FF}_n$ is an isomorphism.*

Proof. Let A be a free factor of rank $n-1$. Recall the bijection between the conjugacy classes of rank $n-1$ free factors and homotopy classes of nonseparating spheres of M_n. There is a unique nonseparating sphere z in M_n specifying the conjugacy class of a free factor A_x in $\pi_1(M_n - x, q)$. So if $\phi \in \mathrm{Aut}\, \mathcal{FF}_n$ we define a map of spheres by defining $\hat{\phi}(x)$ to be the sphere specifying the free factor $\phi(A_x)$ for any nonseparating sphere x. If Σ is a coconnected set of k spheres then we have the conjugacy class representatives of free factors A_x for $x \in \Sigma$ such that $\bigcap_{x \in \Sigma'} A_{x'}$ is a free factor of rank $n - |\Sigma'|$ for any $\Sigma' \subset \Sigma$. Coversely, if $A_1, \ldots, A_\ell$ are rank $n-1$ free factors bijecting to nonseparating spheres $x_1, \ldots, x_\ell$ such that $\bigcap_{j \in s} A_j$ a rank $n - |s|$ free factor for any $s \subset \{1, \ldots, \ell\}$ then it must be that

$$F_n = \bigcap_{j=1}^{\ell} A_j * \langle a_1 \rangle * \cdots * \langle a_\ell \rangle$$

for some $a_j \in A_j$ so that $\{x_j\}_{j=1}^{\ell}$ must be coconnected nonseparating spheres.

Then by Lemma 4.24 this property is preserved by ϕ, so that we have a well defined simplicial map $\hat{\phi} : \mathcal{SF}_n \to \mathcal{SF}_n$ with $\hat{\phi}(\Sigma) = \{\hat{\phi}(x) \mid x \in \Sigma\}$. This gives us homomorphisms

$$\operatorname{Out} F_n \hookrightarrow \operatorname{Aut} \mathcal{FF}_n \xrightarrow{\hat{}} \operatorname{Aut} \mathcal{SF}_n$$

$$\phi \longmapsto \hat{\phi}$$

If $\hat{\phi}$ fixes every coconnected set of sphere then ϕ must fix every free factor, so these homomorphisms are in fact isomorphisms. $\square$

CHAPTER 5

COMPLEX OF STRONGLY SEPARATING CURVES

The content of this chapter is joint work with Alan McLeay at the University of Glasgow.

The complex of strongly separating spheres $\mathcal{C}^{ss}S_{g,p} \subset \mathcal{C}S_{g,p}$ is the induced subcomplex whose vertices are separating curves in the genus g, n-punctured surface $S_{g,p}$ such that both components have complexity $3g' + p' - 3 > 0$. In particular

$$\mathcal{C}^{ss}S_{g,p} = \mathcal{C}^{sep}S_{g,p}$$

is the complex of separatng curves whenever $p \leq 1$, and otherwise $\mathcal{C}^{ss}S_{g,p} \subset \mathcal{C}^{sep}S_{g,p}$ is the subcomplex discarding curves bounding pairs of pants.

In [40, 41] Bowditch demonstrates that if $\operatorname{Aut} \mathcal{C}^{ss}S_{g,p}$ is the mapping class group, then every quasi-isometry of the Weil-Petersson metric on Teichmüller space associated to $S_{g,p}$ is induced by a mapping class of $S_{g,p}$. In [12] Bowditch completes the reduction by showing that $\operatorname{Aut} \mathcal{C}^{ss}S_{g,p}$ is the mapping class group in all but finitely many cases.

Theorem 5.1. *If $g + p \geq 7$, then the natural map*

$$\mathrm{MCG}^{\pm}(S_{g,p}) \to \operatorname{Aut}(C^{ss}(S_{g,p}))$$

is an isomorphism.

Bowditch asks which of the remaining low complexity $g+p \leq 6$ cases have $\operatorname{Aut} \mathcal{C}^{ss}S_{g,p}$ given by the mapping class group. In this chapter we provide an independent proof of Bowditch's result that settles some of these low complexity cases and provide evidence that the remaining unknown cases of $\operatorname{Aut} \mathcal{C}^{ss}$ are themselves mapping class groups. The goal of this chapter is to prove the following theorem.

Theorem 5.2. *The natural map*

$$\mathrm{MCG}^{\pm}\left(S_{g,p}\right) \rightarrow \mathrm{Aut}\left(C^{ss}(S_{g,p})\right)$$

is an isomorphism for the green entries in the table below.

$g \setminus p$	0	1	2	3	4	5	6
0	×	×	×	×	×	×	×
1	×	×	×	×	?	?	[12]
2	×	×	×	?	✓	[12]	[12]
3	[42]	[43]	✓	✓	[12]	[12]	[12]
4	[42]	[43]	✓	[12]	[12]	[12]	[12]
5	[42]	[43]	[12]	[12]	[12]	[12]	[12]
6	[42]	[43]	[12]	[12]	[12]	[12]	[12]

We also show that in the unsettled cases of $\mathcal{C}^{ss}S_{1,4}$, $\mathcal{C}^{ss}S_{1,5}$, and $\mathcal{C}^{ss}S_{2,3}$, any automorphisms with respect the fibers of the point-forgetting projection arise from mapping class groups, and that if $\mathcal{C}^{ss}S_{1,4}$ is a mapping class combinatorial model, so is $\mathcal{C}^{ss}S_{2,3}$.

Point-Forgetting Projection

Notice that to have any edges at all in the graph $\mathcal{C}^{ss}S_{g,p}$ it must be that

$$3g + p \geq 7.$$

As we have seen in Chapter 3, the puncture forgetting map

$$\mathcal{C}^{ss}S_{g,p} \twoheadrightarrow \mathcal{C}^{sep}S_g$$

is a simplicial quotient map for $p \leq 2$. In particular $\mathcal{C}^{ss}S_{2,1}$ and $\mathcal{C}^{ss}S_{2,2}$ have quotients onto $\mathcal{C}^{sep}S_2$, which is disconnected. All of these disconnected complexes have automorphisms permuting the connected components, so that their automorphism groups contain an infinite symmetric group. We conjecture that the strongly separating curve complex $\mathcal{C}^{ss}S_{g,p}$ is rigid whenever it is connected.

Lemma 5.3. *Automorphisms of the strongly separating curve complex induce isomorphisms on region adjacency graphs.*

Let $\phi \in \operatorname{Aut}\mathcal{C}^{ss}S_{g,p}$ and let Δ be any simplex of $\mathcal{C}^{ss}S_{g,p}$. Then ϕ induces an isomorphism between the region adjacency graphs $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$.

Proof. The proof of 3.2 shows in particular that ϕ induces an incidence preserving bijection between the edges of $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$. But since every sphere of $\mathcal{C}^{ss}S_{g,p}$ is separating, every simplex $k-1$ simplex Δ gives an adjacency graph $\mathcal{G}_\Delta$ that is a tree with k edges and $k+1$ vertices. But then by Whitney's Theorem 2.1 an incidence preserving bijection between $\mathcal{G}_\Delta$ and $\mathcal{G}_{\phi(\Delta)}$ must be an isomorphism. □

Definition 5.4. We call a region adjacency graph *linear* if it is a tree with exactly 2 leaves. Recall that a leaf is a degree 1 vertex of a tree.

Lemma 5.5. *Automorphisms of the strongly separating curve complex preserve the topological type of curves.*

Let $\phi \in \operatorname{Aut}\mathcal{C}^{ss}S_{g,p}$ and supposed that $\operatorname{Aut}\mathcal{C}^{ss}S_{g,p}$ is connected. Then for any curve x in $\mathcal{C}^{ss}S_{g,p}$ there is a homoemorophism ψ of $S_{g,p}$ such that $\psi(x) = \phi(x)$.

Proof. We will consider adjacency graphs and consider several cases depending on the genus of the surface. In each case we will utilize a combinatorial characterization of the curve in terms of the region adjacency graph of a maximal simplex and apply Lemma 5.3.

Case 1. Suppose that $g = 0$.

Let x be a separating curve of $S_{0,p}$. As in Figure 5.1 we may choose a maximal simplex Δ of $\mathcal{C}^{ss}S_{g,p}$ containing x so that the region adjacency graph $\mathcal{G}_\Delta$ is linear. Then the curve x bounds an $S_{0,p'}$ if and only if the edge e_x is distance $p'-3$ from a leaf of $\mathcal{G}_\Delta$. The result then follows from Lemma 5.3.

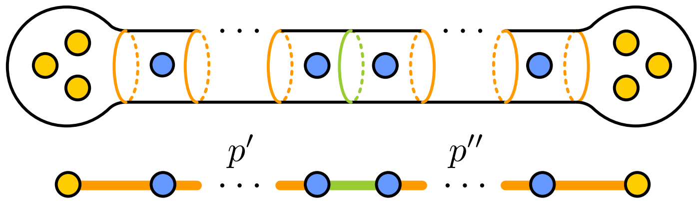


Figure 5.1: A maximal simplex and the region adjacency graph.

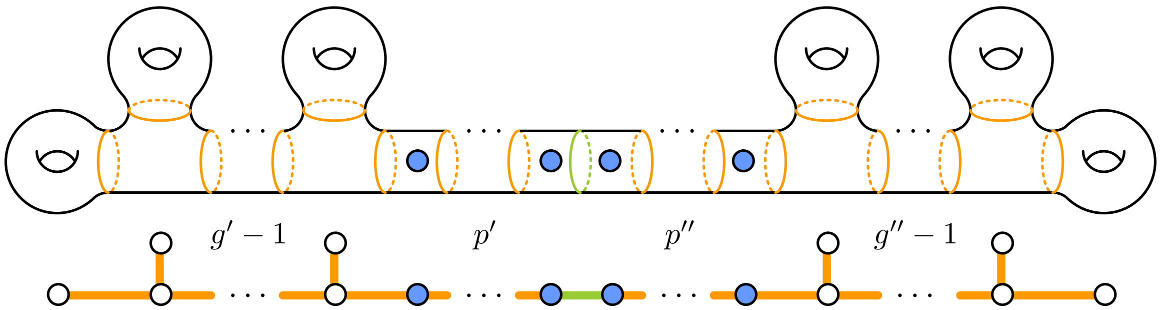


Figure 5.2: A maximal simplex and the region adjacency graph.

Case 2. Suppose that $g \geq 2$.

Suppose that x is a separating curve with sides $S_{g',p'}$ and $S_{g'',p''}$ with both $g', g'' > 0$. As in Figure 5.2 we may choose a maximal simplex Δ of $\mathcal{C}^{ss}S_{g,p}$ containing x that has no curves bounding a 3-punctured disk. Note that the adjacency graph $\mathcal{G}_\Delta$ of a maximal simplex Δ is a tree with vertices at most degree 3. Then Δ has a curve bounding a 3-punctured disk if and only if with the tree $\mathcal{G}_\Delta$ has exactly g leaves and $g+p-2$ non-leaves. Then e_x is a cut edge of the tree $\mathcal{G}_\Delta$ separating $\mathcal{G}_\Delta$ into two trees with with g' and g'' leaves, respectively, and $g'+p'-2$ and $g''+p''-2$ non-leaves, respectively. Then by Lemma 5.3 $\phi(x)$ is such a cut edge in the isomorphic tree $\mathcal{G}_{\phi(\Delta)}$ so that the sides of $\phi(x)$ must be an $S_{g',p'}$ and an $S_{g'',p''}$.

If instead x has an $S_{0,p'}$ side, then any maximal simplex Δ containing x has a region adjacency graph $\mathcal{G}_\Delta$ with greater that g leaves. We may choose a simplex Δ as in Figure 5.3 containing x and with a a region adjacency graph $\mathcal{G}_\Delta$ that has $g+1$ leaves and x separates one leaf v representing a 3-punctured disk from the other leaves, and the component of the leaf v in the cut tree $\mathcal{G}_\Delta - e_x$ has $p'-3$ edges. Then by Lemma 5.3 $\phi(x)$ is such a cut edge in the isomorphic tree $\mathcal{G}_{\phi(\Delta)}$, and by the previous subcase the g leaves $\phi_*(w)$ of $\mathcal{G}_{\phi(\Delta)}$ for

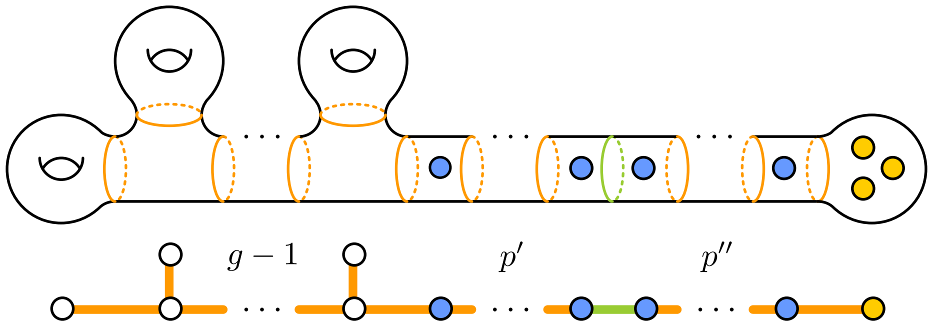


Figure 5.3: A maximal simplex and the region adjacency graph.

$w \neq v$ represent curves bounding $S_{1,1}$s so that $\phi(v)$ must represent a 3-punctured disk and its component in the cut tree $\mathcal{G}_{\phi(\Delta)} - e_{\phi(x)}$ specifies an $S_{0,p'}$ bounded by $\phi(x)$.

Case 4. Suppose that $g = 1$.

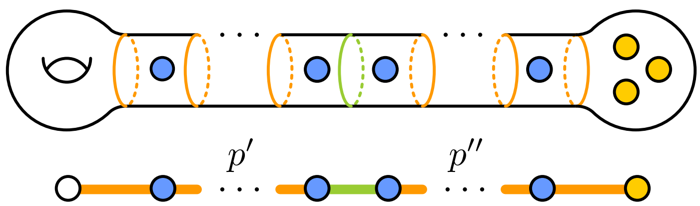


Figure 5.4: A maximal simplex and the region adjacency graph.

Let x be a strongly separating curve of $S_{1,p}$. As in Figure 5.4 we may choose a maximal simplex Δ of $\mathcal{C}^{ss}S_{g,p}$ containing x so that the region adjacency graph $\mathcal{G}_\Delta$ is linear. Then by Lemma 5.3 we have that ϕ induces an automorphism of $\mathcal{G}_\Delta$. Let x_0 and x_1 be the curves of Δ with x_0 bounding the $S_{1,1}$ and x_1 bounding the 3-punctured disk. We can be sure that $\phi(x_0)$ and $\phi(x_1)$ each bound either a 3-punctured disk or an $S_{1,1}$, since a leaf of $\mathcal{G}_\Delta$ must be either $S_{1,1}$ or $S_{0,4}$. A strongly separting curve x bounds a k-punctured disk if and only if e_x is distance $k-3$ from a leaf v in an adjacency region graph $\mathcal{G}_{\Delta'}$ for some maximal simplex of maximal dimension Δ' such that v represents a 3-punctured disk. It then suffices to show that $\phi(x_1)$ bounds 3-punctured disk.

Suppose that $p \geq 6$. Observe that a maximal dimension simplex has exactly one curve that bounds a 3-punctured disk, and simplices with multiple curves bounding 3-punctured disks may be maximal with respect to inclusion, but do not have maximal dimension. Ob-

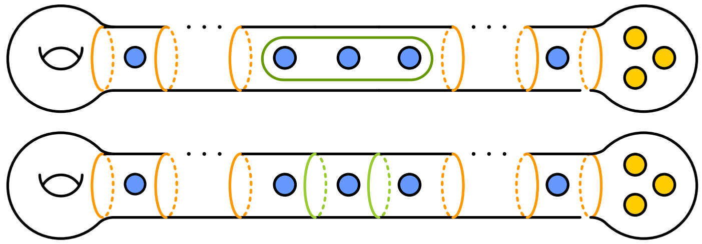


Figure 5.5: (Top) A maximal simplex of submaximal dimension has multiple curves bounding 3-punctured disks. (Bottom) Replacing curves bounding a 3-punctured disk with a pair of curves results in a simplex of higher dimension.

serve that if x bounds a 3-punctured disk then as in Figure 5.5 (top) we may choose a maximal (with respect to inclusion) simplex Δ of $\mathcal{C}^{ss}S_{1,p}$ such that the region adjacency graph $\mathcal{G}_\Delta$ has 3 leaves, and the edge e_x representing x in $\mathcal{G}_\Delta$ is incident to a degree 3 vertex. Then by replacing x with a pair y, y' of curves as in Figure 5.5 we would obtain a maximal simplex of maximal dimension

$$\Delta' = \Delta \cup \{x\} - \{y, y'\}.$$

This shows a separating curve x bounds a 3-punctured disk if and only if x is contained in a maximal simplex Δ of submaximal dimension such that $\Delta - \{x\}$ is contained in a maximal simplex of maximal dimension. But then if x bounds a 3-punctured disk, so does $\phi(x)$ for any $\phi \in \operatorname{Aut} \mathcal{C}^{ss}S_{1,p}$.

Suppose that $p = 4$ or 5. Note that in this case any two distinct curves of the same type intersect, so that $\mathcal{C}^{ss}S_{1,p}$ is colorable by curve type and in particular the "extremal curve" subcomplex $\mathcal{E}$ of curves bounding 3-punctured disks and p-punctured disks is connected and bipartitie. Assume to the contrary that there is x bounding a 3-punctured disk such that $\phi(x)$ does not represent a 3-punctured disk for some automorphism $\phi \in \mathcal{C}^{ss}S_{1,p}$. Then ϕ must swap the bipartition of $\mathcal{E}$ Let $G < \operatorname{Aut} \mathcal{C}^{ss}S_{1,p}$ be the subgroup generated by ϕ and $\mathrm{MCG}^{\pm}\, S_{1,p}$. We obtain a map $\rho_| : G \to \mathbb{Z}/2$ with $\rho_|(g)$ the generator of $\mathbb{Z}/2$ if g exchanges the bipartition of $\mathcal{E}$. Then $\mathrm{Aut}\,\mathcal{G}_\Delta \cong \mathbb{Z}/2$ so that we have an exact sequence

$$1 \longrightarrow \mathrm{MCG}^{\pm} S_{1,p} \longrightarrow G \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

Then by Corollary 11 it must be that $G \cong \mathbb{Z}/2 \times \mathrm{MCG}^{\pm} S_{1,p}$. Then there an automorphism $\phi' \in \mathrm{Aut}\,\mathcal{C}^{ss}S_{g,p}$ that is order 2 and commutes with $\mathrm{MCG}^{\pm} S_{1,p}$. If x_0 and x_1 are disjoint curves bounding a 3-punctured disk and a 4-punctured disk respectively, then there is a mapping class ψ such that $\psi(x_0) = x_0$ and $\psi(x_1) \neq x_1$, for example the half twist exchanging punctures on either side of x_1 along an arc disjoint from x_0. But this gives a contradiction:

$$x_1 = \phi'(x_0) = \phi'\psi(x_0) = \psi\phi'(x_0) = \psi(x_1) \neq x_1.$$

It can only be that every automorphism preserves the type of curve. □

Remark 10. Let q be a puncture of $S_{g,p}$. Observe that the we have the projection

$$\rho_q : \mathcal{C}(S_{g,p}, q) \longrightarrow \mathcal{C}S_{g,p-1}$$

restricts to the projection

$$\rho_{|_q} : \mathcal{C}^{ss}S_{g,p} \longrightarrow \mathcal{C}^{sep}S_{g,p-1}$$

since $\mathcal{C}^{ss}S_{g,p}$ does not contain any curves bounding 2-punctured disks. So if x is a separating curve of $S_{g,p-1}$ the fiber $\rho_{|_q}^{-1}(x) \subset \mathcal{C}^{ss}S_{g,p}$ is isomorphic to a subforsest of the Bass-Serre tree $\mathcal{T}_x$ given by the x splitting of $\pi_1(S_{g,p-1}, q)$, though $\rho_{|_q}^{-1}\rho_{|_q}(x) \subset \mathcal{C}^{ss}S_{g,p}$ is not connected if $\rho_{|_q}(x)$ is curve bounding a 2-punctured disk of $S_{g,p-1}$.

Definition 5.6. Let $\mathrm{Aut}(\mathcal{C}^{ss}S_{g,p}, q) < \mathrm{Aut}\,\mathcal{C}^{ss}S_{g,p}$ be the subgroup preserving the connected fibers of $\rho_{|_q}$, so that $\phi \in \mathrm{Aut}(\mathcal{C}^{ss}S_{g,p}, q)$ if

$$\phi\left(\rho_{|_q}^{-1}\rho_{|_q}(x)\right) = \rho_{|_q}^{-1}\rho_{|_q}(\phi(x))$$

for all x such that $\rho(x)$ does not bound a 2-punctured disk.

We recall the computation of the automorphism group of the separating curve complex due to Brendle, Margalit, and Kida in [43] and [42].

Theorem 5.7. *The natural map*

$$\mathrm{MCG}^{\pm} S_{g,p} \to \mathrm{Aut}\, \mathcal{C}^{sep} S_{g,p}$$

is an isomorphism if $g = 0, p \geq 5$ *or* $g = 1, p \geq 3$ *or* $g = 2, p \geq 2$ *or* $g \geq 3$.

We show that automorphisms preserving the fibers of point forgetting projection $\rho_{|_q}$ are in fact mapping classes.

Lemma 5.8. *The natural map*

$$\mathrm{MCG}^{\pm}(S_{g,p}, q) \to \mathrm{Aut}\, \mathcal{C}^{ss}(S_{g,p}, q)$$

is an isomorphism if $g = 0, p \geq 6$ *or* $g = 1, p \geq 4$ *or* $g = 2, p \geq 3$ *or* $g \geq 3$.

Proof. Consider action by the Birman exact sequence.

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(S_{g,p-1}, q) \arrow[r] \arrow[d] & \mathrm{MCG}^{\pm}(S_{g,p}, q) \arrow[r, "f_q"] \arrow[d, "\beta"] & \mathrm{MCG}^{\pm} S_{g,p-1} \arrow[r] \arrow[d, "\gamma"] & 1 \\
1 \arrow[r] & \pi_1(S_{g,p-1}, q) \arrow[r, "\alpha"] & \mathrm{Aut}\, \mathcal{C}^{ss}(S_{g,p}, q) \arrow[r, "\rho_{|_q}^*"] & \mathrm{Aut}\, \mathcal{C}^{sep} S_{g,p-1} \arrow[r] & 1
\end{tikzcd}$$

The diagram commutes by Lemma 3.7, so we need only consider the exactness of the second row. Certainly any point push must move some strongly separating curve, so α is injective. By Theorem 5.7 we have $f_q\gamma = \rho_{|_q}^*\beta$ is surjective, so that $\rho_{|_q}^*$ must be surjective. As in the proof of Lemma 3.7, image $\alpha \subset \ker \rho_{|_q}^*$, since maps point pushing q around loops of $S_{g,p-1}$ act on the fibers of $\rho_{|_q}$.

Let $\phi \in \ker \rho_{|_q}^*$. Suppose that $x \in \mathcal{C}^{sep} S_{g,p-1}$ is a curve that does not bound a 2-punctured disk. Then

$$x = (\rho_{|_q}^*\phi)(x) = \rho_q\phi(y)$$

for any $y \in \rho_{|_q}^{-1}(x)$. Then $\phi(\rho_{|_q}^{-1}(x)) = \rho_{|_q}^{-1}(x)$ so that ϕ is determined by its action on each $\rho_{|_q}^{-1}(x)$. But $\rho_{|_q}^{-1}(x)$ is $\pi_1(S_{g,p-1}, q)$ equivalently isomorphic to the Bass Serre tree $\mathcal{T}_x$, and since $\ker \rho_{|_q}^*$ acts on $\mathcal{T}_x$ it follows by Theorem 2.5 that $\ker \rho_{|_q}^*$ acts on $\mathcal{T}_x$ as $\pi_1(S_{g,p-1}, q)$. Then we may compose ϕ with a push map to assume that $\phi(y) = y$ for any strongly separating curve y in $S_{g,p}$ that bounds a 3-punctured disk containing q. Then if y in $S_{g,p}$ bounds a 3-punctured disk containing q. Let Σ be any collection of curves disjoint from y such that y bounds the only 3-punctured disk disjoint from every curve of Σ. No curve of Σ bounds a 3-punctured disk containing q, so ϕ fixes Σ, and it must be that $\phi(y) = y$. But then

$$\text{image } \alpha = \ker \rho_{|_q}^*.$$

Then the rows of the commutative diagram are exact and by the Five Lemma the map β is an isomorphism. $\square$

Definition 5.9. Let $\operatorname{Aut}^{\rho_|} \mathcal{C}^{ss} S_{g,p} < \operatorname{Aut} \mathcal{C}^{ss} S_{g,p}$ be the subgroup that permutes the connected fibers of the point forgetting map, so $\phi \in \operatorname{Aut}^{\rho_|} \mathcal{C}^{ss} S_{g,p}$ if there is a permutation σ of the punctures P such that

$$\phi\left(\rho_{|_q}^{-1}\rho_{|_q}(x)\right) = \rho_{|_{\sigma(q)}}^{-1}\rho_{|_{\sigma(q)}}(\phi(x))$$

for every puncture $q \in P$ and curve $x \in \mathcal{C}^{ss} S_{g,p}$ such that x does not bound a 3-punctured disk containing q.

Corollary 10. *The natural map*

$$\operatorname{MCG}^{\pm} S_{g,p} \to \operatorname{Aut}^{\rho_|} \mathcal{C}^{ss} S_{g,p}$$

is an isomorphism if $g = 0, p \geq 6$ *or* $g = 1, p \geq 4$ *or* $g = 2, p \geq 3$ *or* $g \geq 3$.

Proof. Let $\phi \in \operatorname{Aut}^{\rho_|} \mathcal{C}^{ss} S_{g,p}$ and let σ be the associated permutation of the punctures. Then there is $\psi \in \operatorname{MCG}^{\pm} S_{g,p}$ such that $\psi^{-1}(q) = \sigma(q)$ for every puncture $q \in P$. So $\psi\phi$ preserves ${\rho_|}_q^{-1} {\rho_|}_q(x)$ for every $x \in \mathcal{C}^{ss} S_{g,p}$ and some $q \in P$. Then $\psi\phi \in \operatorname{Aut} \mathcal{C}^{ss}(S_{g,p}, q)$ so by Lemma 5.8 it must be that ϕ is induced by a mapping class. □

Remark 11. We will use a similar technique to Chapter 3 to show that $\operatorname{Aut} \mathcal{C}^{ss} S_{g,p}$ always permutes the fibers of the puncture forgetting map $\rho_|$ if $S_{g,p}$ has sufficiently high complexity by showing $\operatorname{Aut} \mathcal{C}^{ss} S_{g,p}$ permutes the coloring of an arc complex associated to the fibers of $\rho_|$.

Definition 5.11. Define the strongly separating pointed arc complex $\mathcal{A}^{ssep} S_{g,p}$ to be the complex of homotopy classes of 3-punctured disks and pointed loops in $S_{g,p}$ whose regular neighborhood is bounded by strongly separating curves. Two loops or punctured disks are adjacent in $\mathcal{A}^{ssep} A S_{g,p}$ if their homotopy classes have disjoint representatives including endpoints.

Lemma 5.12. *Automorphisms of* $\operatorname{Aut} \mathcal{C}^{ss} S_{g,p}$ *induce automorphisms of* $\mathcal{A}^{ssep} S_{g,p}$.

There is an $\operatorname{MCG}^{\pm} S_{g,p}$ *equivariant homomorphism*

$$\operatorname{Aut} \mathcal{C}^{ss} S_{g,p} \to \operatorname{Aut} \mathcal{A}^{ssep} S_{g,p}.$$

Proof. The proof is similar to that of Lemma 3.13. Suppose that $\phi \in \operatorname{Aut} \mathcal{C}^{ss}$.

If x is a strongly separating loop, then the boundary of a regular neighborhood of x is two strongly curves y, y' that cobound a punctured annulus of $S_{g,p}$. The curves y, y' cobound a punctured annulus if and only if there is a maximal dimension simplex Δ of

$\mathcal{C}^{ss}S_{g,p}$ such that in the adjacency graph $\mathcal{G}_\Delta$, the edges corresponding to y and y' are incident to the same degree 2 vertex v_x. Lemma 5.3 guarantees that $\phi(y)$ and $\phi(y')$ cobound a regular neighborhood of a strongly separating loop that we define to be $\hat{\phi}(x)$.

If x is a 3-punctured disk, then Lemma 5.5 guarantees that its bounding curve y has $\phi(y)$ bound a 3-punctured disk $\hat{\phi}(x)$.

Then $\hat{\phi}$ is simplicial since loop or 3-punctured disk of $S_{g,p}$ are disjoint if and only if their above characterizing-curves span a simplex in $\mathcal{C}^{ss}S_{g,p}$. So we have a homomorphism by $\phi \mapsto \hat{\phi}$. $\square$

Lemma 5.13. *$\mathcal{A}^{ssep}S_{g,p}$ is uniquely colorable for $g \geq 3$ or $g \geq 2, p \geq 4$.*

Proof. The proof is similar to the proof of Lemma 3.12, though the arc complex has only strongly separating loops and 3-punctured disks, so the similar argument requires a higher complexity surface.

Again we argue that there are color forcing paths between nests, as in Example 11. But if V and V' are nests respectively parallel to curves that intersect, the argument in Example 11 would require at least 6 punctures. We consider the cases of different complexities separately. Since connected bipartite graphs are uniquely colorable we assume that $p \geq 3$.

Case 1. Assume that $g \geq 3$.

Fix a puncture order $\sigma : P \to \{1, \ldots, p\}$ and a separating curve x of S_g and let N be a σ-nest of $S_{g,p}$ parallel to x. Let $V = \{N_i\}_{i=1}^p$ be the corresponding clique of $\mathcal{C}^{ss}S_{g,p}$. Let z be a regular neighborhood of the spine of N, so that z is p-punctured disk whose bounding curve intersects each rib N_i of the nest with geometric intersection 2. Choose any collection Σ of nonseparating curves whose Dehn twists generate $\mathrm{MCG}^{\pm}(S_{g,p}, z)$, the mapping classes fixing z pointwise. Then let H be the generating set for $\mathrm{MCG}^{\pm}\, S_{g,p}$ given by twists about the curves of Σ and the half-twist about the vertabrae of the nest N.

Figure 5.6 shows a color forcing sequence between the ribs N_i and N_{i+1} and their image under the half-twist about a vertebra of N.

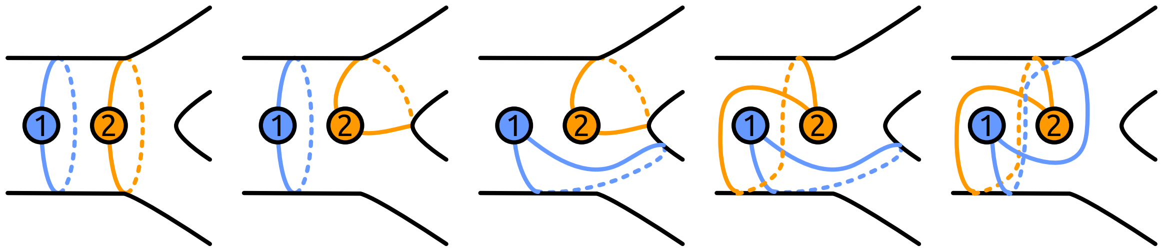


Figure 5.6: A color forcing sequence between two parallel loops and their braid.

If x, x' are disjoint strongly separating spheres then as in 11 these is a color forcing sequence between σ-nests parallel to x and x'. It must be that V forces a coloring on its orbit $\mathrm{MCG}^{\pm} S_{g,p} \cdot V$. Observe that any 3-punctured disk of $S_{g,p}$ is disjoint from $p-3$ loops in $\mathrm{MCG}^{\pm} S_{g,p} \cdot V$. Then a V also forces a coloring on every 3-punctured disk. A coloring on a loop bounding a k-punctured disk is determined by $p-k$ loops of $\mathrm{MCG}^{\pm} S_{g,p} \cdot V$ and loops bounding $k-1$-, $k-2$-, ..., and 4-punctured disks and a 3-punctured disk. We conclude by induction that V forces a coloring on $\mathcal{A}^{ssep} S_{g,p}$.

Case 2. Assume that $g = 2$ and $p \geq 4$.

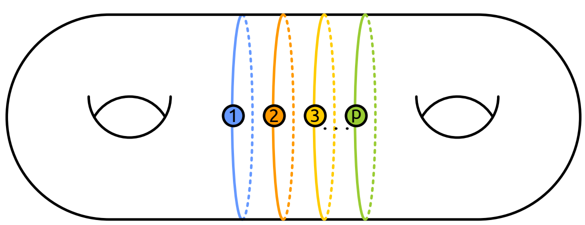


Figure 5.7: A base nest of strongly separating curves.

Fix a nest as in Figure 5.7. Let $V = \{N_i\}$ be the corresponding simplex of $\mathcal{A}^{ssep} S_{g,p}$. Certainly V requires p colors to color. Let H be the generating set for $\mathrm{MCG}\, S_{g,p}$ consisting of the braids about the vertebrae of N and the Dehn twists about the nonseparating curves shown in Figure 5.8.

We first show V forces a coloring on $T_{v_i} \cdot V$ for a half-twist T_{v_i} about the i^{th} vertebra v_i of nest N. Figure 5.9 (left) shows that $N_i, \ldots, N_{i+3}$ force a coloring on $v_i \cdot N_i, v_i \cdot N_{i+1}$. A similar arugment concludes $N_i, \ldots, N_{i+3}$ force a coloring on $v_{i+2} \cdot N_i, v_{i+2} \cdot N_{i+1}$. Figure 5.9 (right) shows that $N_i, \ldots, N_{i+3}$ force a coloring on $v_{i+1} \cdot N_i, v_{i+1} \cdot N_{i+1}$.

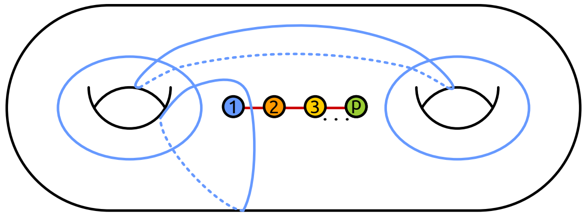


Figure 5.8: The mapping class group is generated by Dehn twists about these nonseparating curves and braiding about the vertebrae of the nest, shown in red.

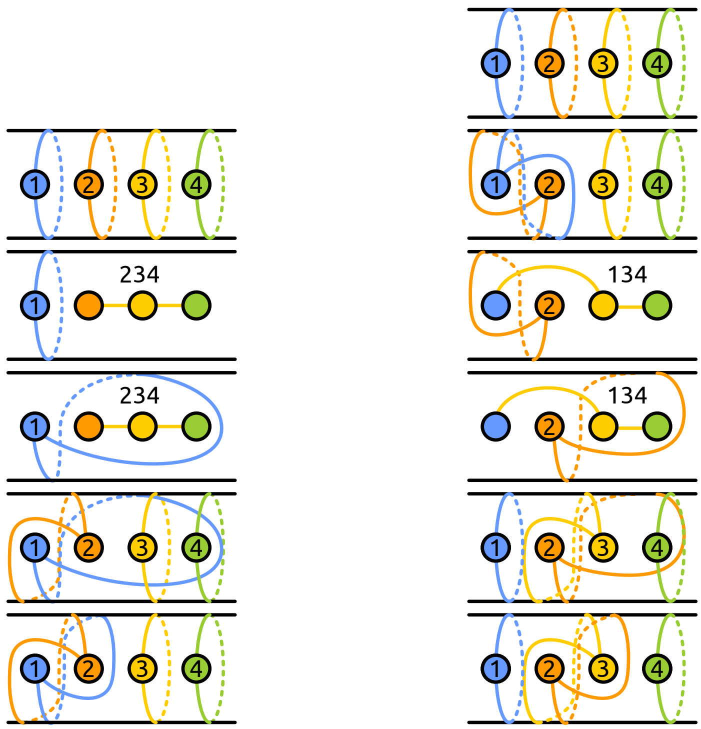


Figure 5.9: A nest forces a coloring on its image under braids about the vertebrae of the nest.

It remains only to be seen that V forces a coloring on its image under Dehn twists of H. Of the Dehn twists described by Figure 5.8 only one, say T_α, fixes fewer than $p-1$ loops of V. As in Example 11, V forces a coloring on loops parallel to punctured disks that are the regular neighborhoods of vertebrae and half-twists of vertebrae, that force colorings on $p-3$ size subsets of $T_\alpha \cdot V$. Since $p-3$ size subsets of $T_\alpha \cdot V$ cover $T_\alpha \cdot V$, we have that V forces a coloring on $T_\alpha \cdot V$. So V forces a coloring on $\mathcal{A}^{ssep}S_{2,p}$ by Lemma 2.4.

□

Remark 12. The unique coloring of $\mathcal{A}^{ssep}S_{g,p}$ fails for the lowest complexity cases. In

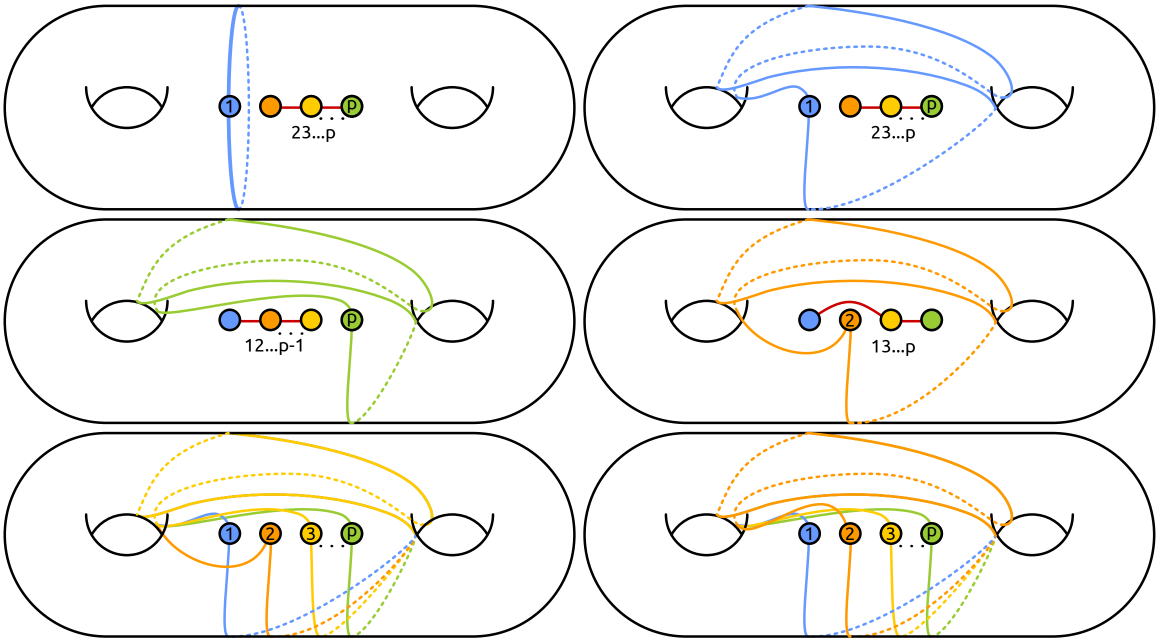


Figure 5.10: The loops of V force a coloring on their image $T_\alpha \cdot V$ under the Dehn twist T_α by considering color forcing sequence passing through loops contained in p-punctured disks of $S_{2,p}$.

particular the loops of $\mathcal{A}^{ssep}S_{2,3}$ and $\mathcal{A}^{ssep}S_{1,4}$ are disconnected, and $\mathcal{A}^{ssep}S_{1,5}$ may be colored by 3 topolocial curve types, rather than the 5 punctures.

Proof of Theorem 5.2. We consider only the cases

$$(g, p) \in \{(2, 4), (3, 2), (3, 3), (4, 2)\}$$

which are undecided in [12] By Lemma 5.12 any automorphism $\phi \in \operatorname{Aut} \mathcal{C}^{ss}S_{g,p}$ induces an automorphism ϕ_* of $\mathcal{A}^{ssep}S_{g,p}$, so by Lemma 5.13 there is some permutation σ of the punctures such that ϕ permutes the puncture-coloring of $\mathcal{A}^{ssep}S_{g,p}$ by σ. By composing ϕ with a mapping class permuting the punctures by σ^{-1}, we may assume that ϕ_* fixes the puncture-coloring of $\mathcal{A}^{ssep}S_{g,p}$.

Suppose that $x' \in \rho_q^{-1}\rho_q(x)$ for $x \in \mathcal{C}^{ss}S_{g,p}$ such that x does not bound a 3-punctured disk containing q. Then $\rho_q^{-1}\rho_q(x)$ is isomorphic to the Bass Serre tree $\mathcal{T}_x$ and in particular connected. Then there is a path $x = x_0, \dots, x_n = x'$ with x_i and x_{i+1} cobounding an annulus punctured by q. But then $\phi(x_0), \ldots, \phi(x_n)$ is a path of curves with $\phi(x_i)$ and $\phi(x_{i+1})$ cobounding annuli punctured by q. So $\phi(x') \in \rho_q^{-1}\rho_q(\phi(x))$. But then $\phi \in \mathrm{Aut}^{\rho|}\,\mathcal{C}^{ss}S_{g,p}$. From Corollary 10 we conclude that ϕ is induced by a mapping class. $\square$

The Low Complexity Cases

Remark 13. In [5] Brendle and Margalit demonstrate that automorphisms of subgraphs of the curve complex $\mathcal{C}S_g$ can often be extended to the full complex $\mathcal{C}S_g$ by *sharing pairs* where two curves bound subsurfaces that intersect to give a third curve. The action of automorphisms on the sharing pair is then used to extend the automorphism to the shared curve.

In light of Theorem 5.7 automorphisms of $\mathcal{C}^{ss}S_{g,p}$ need only be extended to curves bounding 2-punctured disks. Brendle and Margalit give a combinatorial characterization of sharing pairs in terms of five additional curves beyond the sharing pair. Their technique can also be used to verify that $\mathrm{Aut}\,\mathcal{C}^{ss}S_{g,p}$ is the mapping class group in all high genus cases, but in low complexity cases there simply is not enough room to use their techniques to demonstrate if sharing pairs are preserved. However, a similar technique allows us to give reductions between computations of $\mathrm{Aut}\,\mathcal{C}^{ss}S_{g,p}$ for different genus g and number of punctures p.

Definition 5.14. Let z be curve bounding a 2-punctured disk. When $p \geq 4$ define a *sharing pair* of $S_{g,p}$ for z to be a pair of curves $\{x, x'\}$ both of that bound 3-punctured disks containing z and such that x and x' have geometric intersection 2. If $p = 3$ we instead demand that x and x' have geometric intersection 4. If $p = 2$ we instead demand that x and x' bound $S_{1,3}$s containing z and have either geometric intersection 2, or x and x' have geometric intersection 4 and there is a curve x'' that bounds an $S_{1,1}$ on the same side as z of x and x'.

Let $\mathcal{P}'_z$ be the sharing pair graph defined as follows. Let the vertices of $\mathcal{P}'_zS_{g,p}$ be the sharing pairs for z. Two sharing pairs for z are adjacent if there is a curve y bounding an

$S_{1,1}$ that lies on the opposite side of z for each curve in each sharing pair.

Lemma 5.15. *Let z be a curve bounding a 2-punctured disk in $S_{g,p}$. The sharing pair graph $\mathcal{P}'_z$ is connected if $g \geq 1, p \geq 5$, or $g \geq 2, p \geq 3$, or $g \geq 3, p \geq 2$.*

Proof. We appeal to Putman's Lemma 2.3 using $\mathrm{MCG}\, S_{g,p}$ and considering cases based on the genus and number of punctures, since our definition of sharing pairs is surface dependent. Since in every case $\mathrm{MCG}\, S_{g,p}$ acts transitively on the sharing pairs of $S_{g,p}$ (except for $p = 4$), it suffices to choose a generating set H and sharing pair v for the 2-punctured disk z and show that for each $h \in H$ there is a sequence of sharing pairs $\{x_i, x'_i\}$ from $h \cdot v$ to v such that for each i there is an $S_1, 1$ disjoint from x_i, x'_i, x_{i+1}, and x'_{i+1}.

Case 1. Consider $g \geq 1$ and $p \geq 5$.

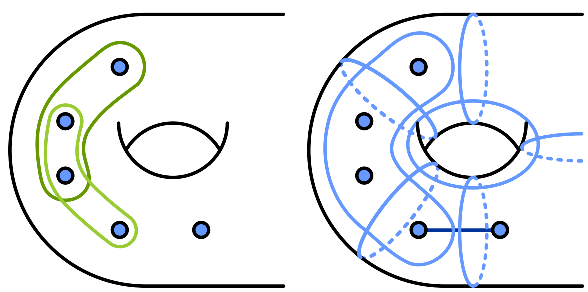


Figure 5.11: (Left) A sharing pair. (Right)Curves for twists generating MCG.

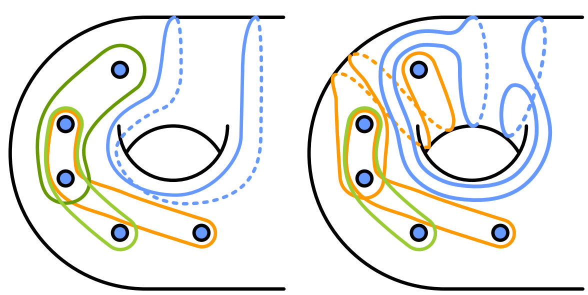


Figure 5.12: (Left) Braiding points about a minimally intersect arc β moves v disjointly from an $S_{1,1}$. (Right) A twist about a nonseparating arc intersecting one curve of the sharing pair. The resulting sharing pair is distance 2 from v.

Fix a sharing pair $v = x_0, x'_0$ as in Figure 5.11 left. Let H be the generating set for $\mathrm{MCG}(S_{g,p}, z)$ given by Dehn twists about the curves shown in Figure 5.11 right. In particular H may be taken to consist of Dehn twists about curves disjoint from the sharing pair

v, the twist T_a about a nonseparating curve a intersecting x_0 twice and x_0' zero times, the twist $T_{a'}$ about a curve a' intersecting x_0' twice and x_0 zero times, the half twist T_α such that T_α^2 is the Dehn twist about the boundary of the 4-punctured disk containing v (shown in green in Figure 5.11), and a half twist T_β swapping a puncture in x_0 with a puncture out of v about an arc β intersecting x_0 once and x_0' zero times. Then T_α swaps x_0 and x_0', but fixes v. In Figure 5.12 (left) we see that the half twist T_β has $T_\beta \cdot v = \{x_0', T_\beta x_0\}$ that is contained in a 5-punctured disk with v so that v and $T_\beta \cdot v$ are distance 1 in $\mathcal{P}_z'$. In Figure 5.12 (right) we see that $T_a \cdot v = \{x_0', T_a x_0\}$ that is contained in a 5-punctured disk with $T_\beta v$ so that v and $T_a \cdot v$ are distance 2 in $\mathcal{P}_z'$. The case of $T_{a'}$ is similar.

Case 2. Consider $p = 4$ and $g \geq 2$.

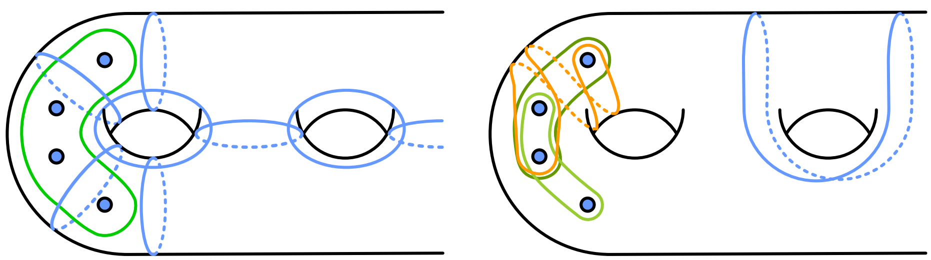


Figure 5.13: (Left) A half twist about the green curve and twists about the blue generate $\mathrm{MCG}\, S_{g,p}$. (Right) Any two of these are a sharing pair.

When there are 4 punctures we have two topological types of sharing pairs: those with intersection 2 and those with intersection 4. But as in Figure 5.13 any intersection 4 pair is contained with an intersection 2 pair in an $S_{1,4}$ disjoint from an $S_{1,1}$. So all of $\mathcal{P}_z'$ is connected to the orbit of v. We may generated $\mathrm{MCG}\, S_{g,p}$ by a half-twist exchanging the non-z punctures, but fixing v and Dehn twists about a set separating curves as in Figure 5.13. The twist of $x forthethe_0$ about a separating curve disjoint from x_0' and intersecting x_0 twice is also contained with x_0 and x_0' in the complement of an $S_{1,1}$.

Case 3. Consider $p = 3$ and $g \geq 2$.

The case is similar. By choosing a generating set H for $\mathrm{MCG}(S_{g,p}, z)$ consisting of Dehn twists about nonseparating curves that intersect the sharing pair minimally as in Figure 5.14, we can ensure that $h \cdot v$ is always distance 1 from v.

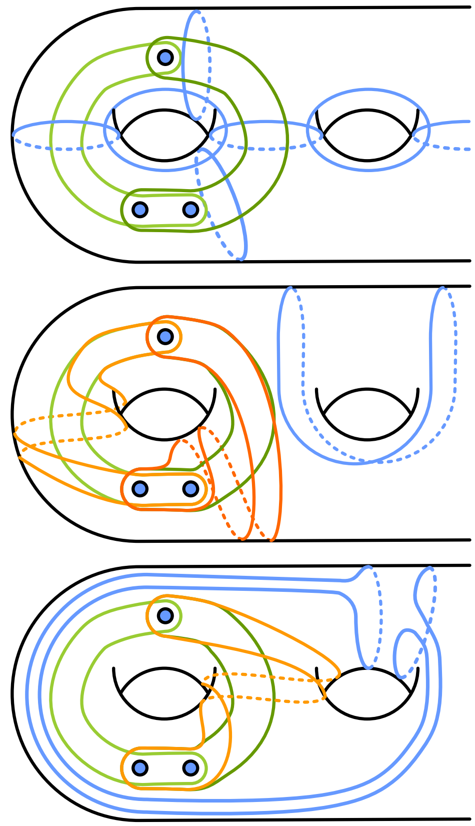


Figure 5.14: (Top) A sharing pair and a generating set. (Middle,Bottom) The images of sharing pair v under generators are contained with v in the complement of an $S_{1,1}$.

Case 4. Consider $p = 2$ and $g \geq 3$.

Figure 5.15 shows a base sharing pair v and a set of nonseparating curves whose Dehn twists H generate the mapping class group fixing z. Figure 5.16 shows a length 5 path in $\mathcal{P}'_z$ that contains the image of v under H. for the the

The case is similar to Case 2, but the paths in $\mathcal{P}'_z$ are longer. Figure 5.15 gives a base sharing pair v and a generating set. Figure 5.16 shows a path in $\mathcal{P}'_z$ containing the image of v under the generating set. □

Lemma 5.16. *If the natural map*

$$\mathrm{MCG}\, S_{g-1,p+1} \to \mathrm{Aut}\, \mathcal{C}^{ss} S_{g-1,p+1}$$

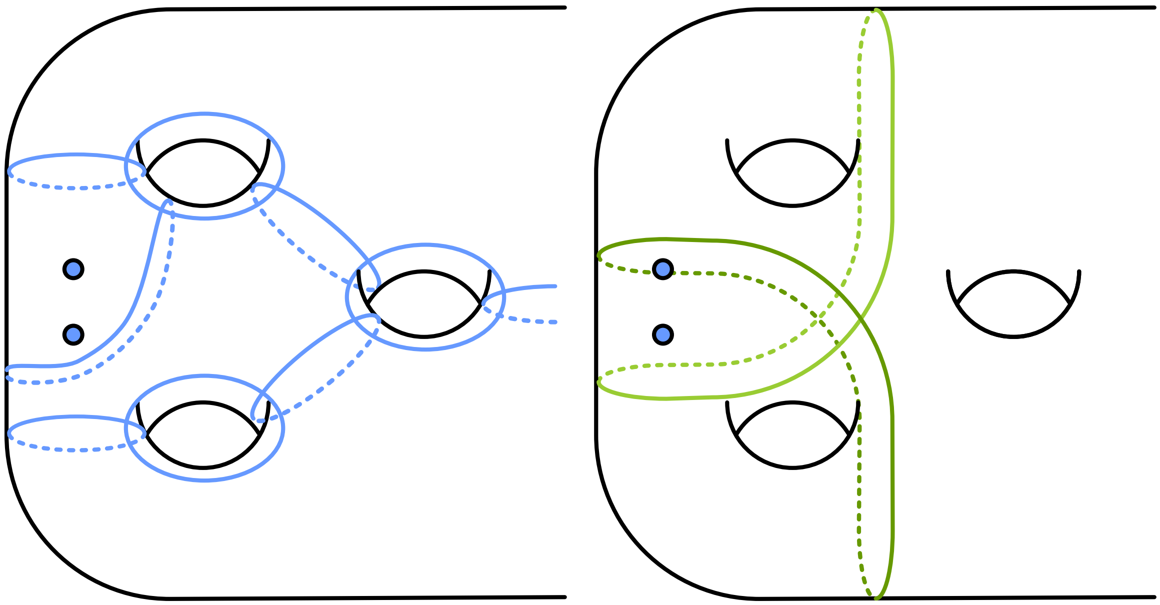


Figure 5.15: (Right) A set of nonseparating curves whose Dehn twists generate the mapping class group fixing z. (Left)A sharing pair.

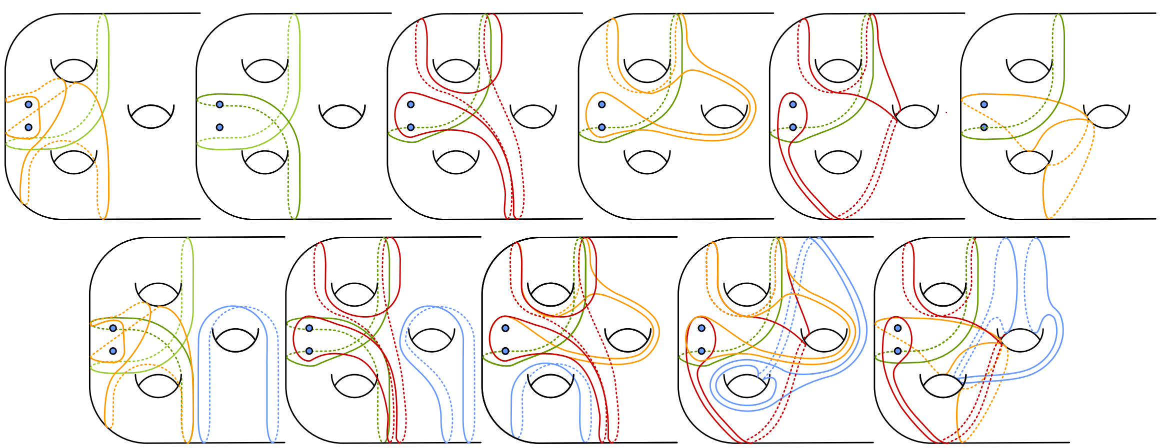


Figure 5.16: A length 5 path in $\mathcal{P}'_z$ that contains the image of v under H. In the top row there are sharing pairs, giving vertices of $\mathcal{P}'_z$. In the bottom row are $S_{1,1}$s disjoint from both sharing pairs above, giving edges of $\mathcal{P}'_z$.

is an isomorphism, then so is

$$\mathrm{MCG}\, S_{g,p} \to \mathrm{Aut}\, \mathcal{C}^{ss} S_{g,p}$$

provided $g \geq 1, p \geq 5$, *or* $g \geq 2, p \geq 3$, *or* $g \geq 3, p \geq 2$.

Proof. Assume that $\mathrm{Aut}\, \mathcal{C}^{ss} S_{g-1,p+1} \cong \mathrm{MCG}^p\, m S_{g-1,p+1}$. Let $\phi \in \mathcal{C}^{ss} S_{g,p}$. Let z be a curve bounding a 2-punctured disk of $S_{g,p}$. Let $\{x, x'\}$ be a sharing pair for z with $x, x' \in \operatorname{Aut} C_s s S_{g,p}$. Then there is a curve $y \in \operatorname{Aut} \mathcal{C}^{ss} S_{g,p}$ on the large side of x and x' and such that y bound an $S_{1,1}$ and an $S_{g-1,p+1}$ that contains x and x'. By Lemma 5.5 $\phi(y)$ also bounds an $S_{1,1}$, so without loss of generality we may compose ϕ with a mapping class so that ϕ fixes $y = \phi(y)$. Then considering y as if it were puncture for the large side of its complement, note that the link of y contains a subcomplex

$$\mathcal{L}_y \cong \mathcal{C}^{ss} S_{g-1,p+1}$$

Then by hypothesis ϕ acts on $\mathcal{L}_y$ as a mapping class, so there is $\psi \in \operatorname{MCG}^{\pm} S_{g,p}$ such that

$$\psi(y') = \phi(y')$$

for every $y' \in \mathcal{L}_y$. In particular the $\phi(x), \phi(x')$ are the homeomorphic image of x, x' so that $\phi(x), \phi(x')$ must be a sharing pair for a 2-punctured-disk-bounding curve we denote $\hat{\phi}(z)$. Further if $\{\hat{x}, \hat{x}'\}$ is another sharing pair for z, then there by Lemma 2.3, the sharing pair complex $\mathcal{P}'_z$ is connected so there is a sequence $\{x_0, x'_0\}, \ldots, \{x_\ell, x'_\ell\}$ of sharing pairs such that $x_i, x'_i, x_{i+1}, x'_{i+1}$ all share z and such that an $S_{1,1}$ bounding curve y' lies on their large sides. But then $\phi(x_i), \phi(x'_i), \phi(x_{i+1}), \phi(x'_{i+1})$ is the homeomorphic image $x_i, x'_i, x_{i+1}, x'_{i+1}$ so the sharing pairs $\phi(x_i), \phi(x'_i)$ and $\phi(x_{i+1}), \phi(x'_{i+1})$ must share the same 2-punctured disk.

It follows there is a well defined extension $\hat{\phi} \in \operatorname{Aut} \mathcal{C}^{sep} S_{g,p}$ of $\phi \in \operatorname{Aut} \mathcal{C}^{ss} S_{g,p}$ given by $z \mapsto \hat{\phi}(z)$ if z bounds a 2-punctured disk and $x \mapsto \phi(x)$ otherwise. By Theorem 5.7 it must be that $\hat{\phi}$ is induced by a mapping class. But then ϕ is induced by a mapping class. $\square$

Remark 14. According to Lemma 5.16 we can reduce considering $\mathcal{C}^{ss} S_{2,3}$ to considering $\mathcal{C}^{ss} S_{1,4}$.

The Four Punctured Torus

The strongly separating curve complex $\mathcal{C}^{ss}S_{1,4}$ may be considered the graph with vertices 3-punctured and 4-punctured disks of $S_{1,4}$ and an edge between a 3-punctured and a 4-punctured disk if the 3-punctured disk is contained in the 4-punctured disk. In particular note that $\mathcal{C}^{ss}S_{1,4}$ is bipartite.

Lemma 5.17. *$\mathcal{C}^{ss}S_{1,4}$ has no cycles smaller than octagons.*

Proof. Certainly if two 3-punctured disks are contained in a 4-punctured disk, then it is unique. So there are no 4-cycles. Suppoose to the contrary that

$$x_0, y_0, x_1, y_1, x_2, y_2$$

is a hexagon in $\mathcal{C}^{ss}S_{1,4}$ with x_i a 3-punctured disk and y_i a 4-punctured disk. Without loss of generality we may assume that x_0 has punctures p_0, p_1, and p_2 and that x_1 has p_0 and p_1 and that x_2 has p_0 and q, where q is either p_1 or p_2. If x_0, x_1, x_2 are not all contained in the same 4-punctured disk, then they contain

· a_0 arc from p_0 to p_1 in x_0

· a_0' arc from q to p_0 in x_0

· a_1 arc from p_1 to p_0 in x_1

· a_2 arc from p_0 to q in x_2

Such that $a_0a_1a_2a_0'$ is a nontrivial loop of the torus. We have a_0a_1 is contained in the 4-punctured disk y_0 so it is null-homotopic in the unpunctured torus. Similarly a_2a_0' is contained in the 4-punctured disk y_2 so it is null-homotopic in the unpunctured torus. But then $a_0a_1a_2a_0'$ is nullhomotopic. It must be that x_0, x_1, x_2 are contained in the same 4-punctured disk. □

Definition 5.18. Let $\mathcal{O} = (x_i, y_i)_{i\in 4}$ be an octogon of $\mathcal{C}^{ss}S_{1,4}$ with x_i a 3-punctured disk and y_i a 4-punctured disk. Let $P(x_i)$ be the punctures of x_i. The *point configuration* of the octogon $\mathcal{O}$ is the sequence of the punctures not in x_i, so $P(\mathcal{O}) = (P - P(x_i))_{i\in 4}$.

Lemma 5.19. *Up to relabeling the points, the point configuration $P(\mathcal{O})$ of an octagon $\mathcal{O}$ of $\mathcal{C}^{ss}S_{1,4}$ is one of the following: (p_0, p_1, p_2, p_3) or (p_0, p_1, p_0, p_1) or (p_0, p_0, p_1, p_1).*

Proof. Fix minimally intersecting representatives of the homotopy classes of octagon $(x_i, y_i)_{i\in 4}$.

Case 1 (p_0, p_0, p_1, p_1).

Suppose that x_i and x_{i+1} contain the same 3 points. We may assume that x_0 and x_1 both contain p_1, p_2, p_3. Consider the image of x_1 in the x_0 complementary region $S_{1,4} - x_0$. Since x_0 and x_2 have two points in common we may assume without loss of generality that $p_1, p_2 \in x_2$ Let a_1 be an arc from p_1 to p_2 in x_2 and let a_2 be an arc from p_2 to p_1 in x_1. If a_1 is not contained in y_0, we would have a_1a_2 a nontrivial loop in the torus, since a_2 is in y_0. But this contradicts that x_2 and x_1 lie in the 4-punctured disk y_1. So the arc a_1 must be in y_0. Assume to the contrary that $p_0 \notin P(x_2)$. Then there is an arc a_3 from $p_1 \to p_3$ in x_2, but by an argument similar to the above a_3 is in y_0. But if a_1 and a_3 are in y_0, then it must be x_2 is in y_0, a contradiction. We have that x_2 contains the points p_0, p_1, p_2. By a similar argument x_3 contains the same points.

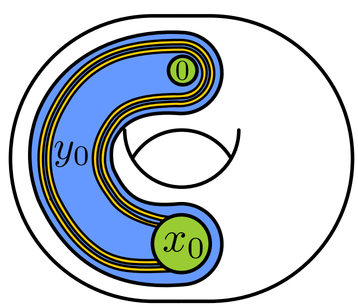


Figure 5.17: Two 3-punctured disks in a 4-punctured disk that share all three punctures. The 3-punctured disk x_1 has arcs outside of the 3-punctured disk x_0 parallel to the boundary of y_0.

Case 2 (p_0, p_1, p_0, p_1). Suppose that x_i and x_{i+2} contain the same points. We may assume that x_0 and x_2 contain p_1, p_2, p_3. As x_0 and x_2 are not contained in the same 4-disk there must be two points, say p_1 and p_3, and arcs a_0 in x_0 from p_1 to p_3 and a_2 in x_2 from p_3 to p_1 such that a_0a_2 is a nontrivial curve in the unpunctured torus. Assume to the contrary that p_1 and p_3 are both in x_1, then there is an arc a_1 in x_1 from p_3 to p_1. Since $a_0a_1 \subset y_1$, we have a_0a_1 is nullhomotopic in the unpunctured torus. But then $a_0a_2 = \overleftarrow{a}_1a_2$ is nontrivial in the unpunctured torus so not contained in y_2, a contradiction. It must be that either p_1 or p_3 not in x_1. We may assume that $p_3 \notin P(x_1)$.

A similar argument forces p_1 or $p_3 \notin P(x_3)$. Suppose that $p_3 \in P(x_3)$. Consider an arc $b_3 \subset x_3$ from p_2 to p_3. All arcs from p_1 to p_2 in x_0, x_1, and x_2 must lie in a common 4-curve by the above argument. But then the arcs from p_2 to p_3 in x_0 and x_2 cannot lie in a 4-disk, but this contradicts that p_2 to p_3 in x_3 lies in both. It must be that $p_3 \notin P(x_3)$. $\square$

Remark 15. All of these point configurations are realized by octagons of $\mathcal{C}S_{1,4}$ as we can see in Figures 5.21, 5.22, and 5.23. However we claim that on the basis of computational evidence that the octagon with point configuration (p_0, p_1, p_2, p_3) is unique up to homeomorphism of $S_{1,4}$, as discussed in Section 5.4.

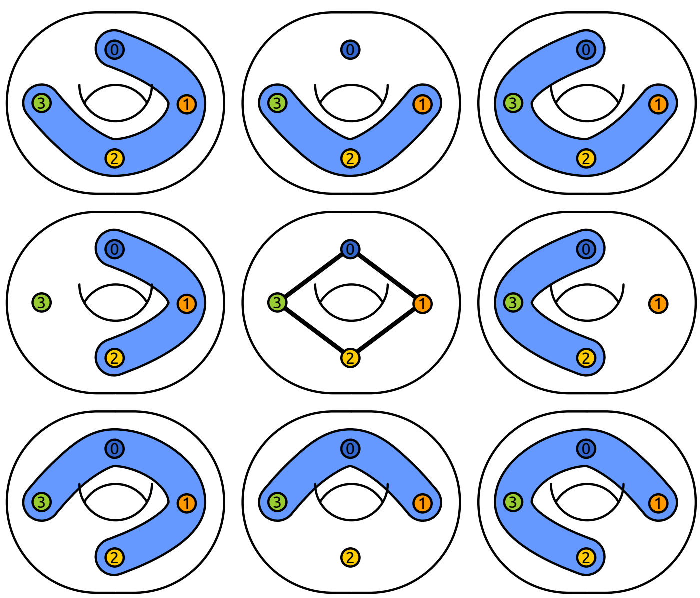


Figure 5.18: The standard octagon in $\mathcal{C}^{ss}S_{1,4}$. The octagon consists of the 8 punctured disks around the outside. Representing arcs are shown in the middle.

Conjecture 20. *Any octagon of $\mathcal{C}^{ss}S_{1,4}$ that has the point configuration (p_0, p_1, p_2, p_3) is homeomorphic to the standard octagon shown in Figure 5.18.*

Lemma 5.21. *If Conjecture 20 is true then automorphisms of $\mathcal{C}^{ss}S_{1,4}$ preserve the standard octagons.*

Proof. Assume that Conjecture 20 is true. We will show that an octagon has point configuration (p_0, p_1, p_2, p_3) if and only if the pairs of 4-punctured disks at distance four have infinitely many distinct length 4 paths in $\mathcal{C}^{ss}S_{1,4}$, and the pairs of 3-punctured disks at distance four have finitely many distinct length 4 paths in $\mathcal{C}^{ss}S_{1,4}$

Let $\mathcal{O} = (x_i, y_i)_{i \in 4}$ be a standard octagon. In light of Conjecture 20 there is an arc α between the two punctures $P(x_i) \cap P(x_{i+1})$ that is in $y_{i-1}, x_i, y_i, x_{i+1}, y_{i+1}$. Then the half twist T_α fixes $y_{i-1}, x_i, y_i, x_{i+1}, y_{i+1}$, but does not fix $x_{i+2}, y_{i+2}, x_{i-1}$ so that

$$y_{i-1}, T_\alpha^n x_i, T_\alpha^n y_i, T_\alpha^n x_{i+1}, y_{i+1}$$

is a distinct length 4 path of $\mathcal{C}^{ss}S_{1,4}$ for every n.

We further claim that there are finitely many paths of length 4 between x_0 and x_2, and similarly between x_1 and x_3. Let

$$x_0 \to y_{0,n} \to x_{1,n} \to y_{1,n} \to x_2$$

be a distinct path in $\mathcal{C}^{ss}S_{1,4}$ for every $n \in \mathbb{Z}$. Then there are two integers n, n' such that $P(x_{1,n}) = P(x_{1,n'})$, but then these paths would give an octagon with point configuration (p_0, p_1, p_2, p_1) or (p_0, p_3, p_2, p_3) which would contradict Lemma 5.19. It must be that there are exactly two paths of length 4 between x_0 and x_2 in $\mathcal{C}^{ss}S_{1,4}$. Similarly there are exactly two paths of length 4 between x_0 and x_2 in $\mathcal{C}^{ss}S_{1,4}$.

Suppose that $\mathcal{O} = (x_i, y_i)_{i \in 4}$ is an octagon with point configuration (p_0, p_0, p_1, p_1).

Assume to the contrary that there are infinitely many distinct paths

$$y_3 \to x_{0,n} \to y_{0,n} \to x_{0,n} \to y_1$$

for $n \in \mathbb{Z}$. Then since

$$y_3 \to x_{0,n} \to y_{0,n} \to x_{1,n} \to y_1 \to x_2 \to y_2 \to x_3$$

is an octagon so by Lemma 5.19, it must be that the point configurations are equal $P(x_{0,n}) = P(x_{1,n})$. But then since there are infinitely many length four paths y_3 to y_1, so there must be two integers n, n' such that $P(x_{0,n}) = P(x_{0,n'})$. But then those paths together give an octagon with point configuration (p_0, p_0, p_0, p_0) or (p_1, p_1, p_1, p_1) in contradiction to Lemma 5.19.

Suppose that $\mathcal{O} = (x_i, y_i)_{i \in 4}$ is an octagon with point configuration (p_0, p_1, p_0, p_1). Let α_i be an arc of x_i between p_2 and p_3. Then the loop $\alpha_i \overleftarrow{\alpha}_j$, for the reversed arc $\overleftarrow{\alpha}$, is contained in a 4-curve so it is trivial in the unpunctured torus. The x_2 is determined by one additional arc α' from p_1 to p_2, so that x_0 and x_2 do not fill the torus. There must be a nonseparating loop β of the torus based at p_0 which is disjoint from both. Let ψ_β be the point pushing map pushing p_0 along β. The loop β intersects the four curves, so Then

$$x_0 \to \psi_\beta^n y_0 \to \psi_\beta^n x_1 \to \psi_\beta^n y_1 \to x_2$$

for all $n \in \mathbb{Z}$ gives infinitely many distinct length 4 paths from x_0 to x_2 in $\mathcal{C}^{ss}S_{1,4}$. □

Definition 5.22. Let z be a 2-punctured disk. Define the octagon sharing pair graph $\mathcal{P}_z^o$ to be the graph defined as follows. The vertices of $\mathcal{P}_z^o$ are sharing pairs for z in $\mathcal{C}^{ss}S_{1,4}$. Define two sharing pairs for z to be adjacent in $\mathcal{P}_z^o$ if they are contained in two standard octagons $\mathcal{O}, \mathcal{O}'$ of $\mathcal{C}^{ss}S_{1,4}$ such that the subgraph $\mathcal{O} \cap \mathcal{O}'$ is two edges incident at a 4-punctured disk.

Lemma 5.23. *The sharing pair graph $\mathcal{P}_z^o$ is connected.*

Proof. We appeal to Putman's Lemma 2.3. Observe that any standard octagon $(x_i, y_i)_{i \in 4}$ can be uniquely represented by disjoint arcs $(a_i)_{i \in 4}$ with a_i in x_i and x_{i+1} from puncture p_{i+2} to p_{i+3} for $i \in \mathbb{Z}/4$. Let two sharing pairs $\{x_0, x_1\}$ and $\{x'_0, x'_1\}$ be adjacent in $\mathcal{P}^o_z$. Then without loss of generality $x_0 = x'_0$ and there are two octagons $(x_i, y_i)_{i \in 4}$ and $(x'_i, y'_i)_{i \in 4}$ with representing arcs $(a_i)_{i \in 4}$ and $(a'_i)_{i \in 4}$ respectively such that $a_i = a'_i$ for $i = 3, 0, 1$ and z is a regular neighborhood of a_0.

Fix a sharing pair $\{x_0, x_1\}$ and let $(a_i)_{i \in 4}$ be the arcs representing an octagon that contains x_0 and x_1. The pure mapping class subgroup fixing the 2-punctured disk z acts transitively on sharing pairs. By Putman's Lemma it suffices to see that there is a generating set H for that we can move from $\{x_0, x_1\}$ to $\{hx_0, hx_1\}$ in $\mathcal{P}_z$. Let $(\alpha_i)_{i \in 4}$ be the disjoint nonseparating curves such that α_i geometric intersection 1 with a_i. Let β be the nonseparating curve disjoint from every a_i. Let γ be the separating curve that the bounds the regular neighborhood of $a_0 a_1$. Take as a generating pure mapping class subgroup fixing the 2-punctured disk z the Dehn twists $H = \{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, T_\beta, T_\gamma\}$. But since each generated fixes either a_3 or a_1, we have that $\{x_0, x_1\}$ is adjacent to $\{hx_0, hx_1\}$ in $\mathcal{P}^o_z$ for all $h \in H$. $\square$

Proposition 24. *If Conjecture 20 is true then the natural map* $\operatorname{MCG}^{\pm} S_{1,4} \to \mathcal{C}^{ss} S_{1,4}$ *is an isomorphism.*

Proof. Assume that Conjecture 20 is true. Extend $\phi \in \operatorname{Aut} \mathcal{C}^{ss} S_{1,4}$ to $\hat{\phi} \in \operatorname{Aut} \mathcal{C}^{sep} S_{1,4}$ and apply Theorem 5.7. We need only define $\hat{\phi}$ on 2-punctured disks.

If z is a two punctured disk let $\{x_0, x_1\}$ be a sharing pair for z. Choose a standard octagon $\mathcal{O}$ containing $\{x_0, x_1\}$ as distance 2 3-punctured disks. Then by Lemma 5.21 $\phi(\mathcal{O})$ is a standard octagon and by Lemma 5.5 $\phi(x_0)$ and $\phi(x_1)$ are 3-punctured disks. So $\{\phi(x_0), \phi(x_1)\}$ must also be a sharing pair and we define $\hat{\phi}(z)$ as the 2-punctured disk shared by $\{\phi(x_0), \phi(x_1)\}$.

It remains only to see that $\hat{\phi}(z)$ is well defined. Suppose that $\{x'_0, x'_1\}$ is another sharing pair sharing z. By Lemma 5.23 there is sequence of standard octagons $(\mathcal{O})_{i=1}^{\ell}$ with $\{x_0, x_1\}$

in $\mathcal{O}_1$ and $\{x'_0, x'_1\}$ in $\mathcal{O}_\ell$ such that the subgraph $\mathcal{O}_i \cap \mathcal{O}'_{i+1}$ is two edges of $\mathcal{C}^{ss}S_{1,4}$ incident at a 4-punctured disk. That is $\mathcal{O}_i \cap \mathcal{O}'_{i+1}$ have in common 3-punctured disks $x_{0,i}$ and $x_{1,i}$ and a 4-punctured disk y_i adjacent to them in $\mathcal{C}^{ss}S_{1,4}$, and $\{x_{0,i}, x_{0,i+1}\}$ is a sharing pair for z for all i. But then $\{\phi(x_{0,i}), \phi(x_{0,i+1})\}$ are 3-punctured disks in the sequence of standard octagons $(\mathcal{O})_{i=1}^{\ell}$ $\{\phi(x_{0,i}), \phi(x_{0,i+1})\}$ and $\{\phi(x_{0,i+1}), \phi(x_{0,i+2})\}$ sharing pairs for the same 2-punctured disk. So $\hat{\phi}$ is well defined. $\square$

Computational Evidence

In this section we examine computational evidence for Conjecture 20, which says that any octagon of $\mathcal{C}^{ss}S_{1,4}$ with a point configuration that is a permutation of (p_0, p_1, p_2, p_3) is homeomorphic to the standard octagon shown in 5.18.

Computations in the curve complex can be performed by representing curves by their intersection numbers with the arcs of a triangulation of $S_{1,4}$. We take as our triangulation

$$\begin{tikzcd}
p_0 \arrow[r, "e_0"] & p_1 \arrow[r, "e_3"] & p_2 \arrow[r, "e_6"] & p_3 \arrow[r, "e_9"] & p_0 \\
p_0 \arrow[u, "e_1"] \arrow[r, "e_0"] & p_1 \arrow[ul, "e_2"] \arrow[u, "e_4"] \arrow[r, "e_3"] & p_2 \arrow[ul, "e_5"] \arrow[u, "e_7"] \arrow[r, "e_6"] & p_3 \arrow[ul, "e_8"] \arrow[u, "e_{10}"] \arrow[r, "e_9"] & p_0 \arrow[ul, "e_{11}"] \arrow[u, "e_1"']
\end{tikzcd}$$

with oriented edges $E = \{e_j\}_{j \in 12}$. With this convention any multicurve x is uniquely

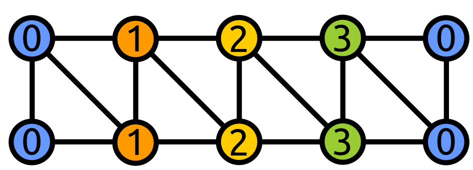


Figure 5.19: A traingulation of the surface $S_{1,4}$.

represented by a tuple in $\mathbb{Z}^E_{\leq 0}$ giving the geometric intersection number $|i|(e_j, x)$. We refer the reader to [44] for details on these *normal coordinates*. Normal coordinates uniquely determine a number of line segments at each angle of each triangle, so that any tuple $\mathbb{Z}^E_{\leq 0}$

is a multicurve if and only if it satisfies a triangle inequality for each triangle:

$$|i|(e_j, x) \leq |i|(e_k, x) + |i|(e_\ell, x)$$

if e_j, e_k, e_ℓ form a triangle and x a multicurve.

From the normal coordinates of a surface we can also compute a nonunique representation of the curve as a cyclically reduced word $w(x)$ on the alphabet $E = \{e_j\}_{j\in 12}$ by choosing a parametrization of x and listing the edges in the order that they are crossed. In these coordinates the Dehn twist $T_y(x)$ of x about y is easy to compute essentially by replacing subwords of x crossing y with an appropriate $w(y)$, and fast algorithms are known [45]. Hamidi-Tehrani [46] comptues geometric intersection number $|i|(x, y)$ by showing that

$$|i|(T_y^{n+1}x, e_j) - |i|(T_y^n x, e_j) = |i|(x, y)|i|(y, e_j)$$

for sufficiently large n.

Large finite subgraphs of $\mathcal{C}^{ss}S_{1,4}$ are made easier to compute by the fact that the mapping class group acts transitively on the edges of $\mathcal{C}^{ss}S_{1,4}$. We have computed a subgraph $\mathcal{C}^{ss}S_{1,4}$ by iteratively applying a generating set to the edge shown in 5.20. This is vastly more efficient than computing intersections between known curves to look for disjoint curves, as the curve complex is δ hyperblic so that finite subgraphs $\mathcal{C}$ of $\mathcal{C}^{ss}S_{1,4}$ are sparse.

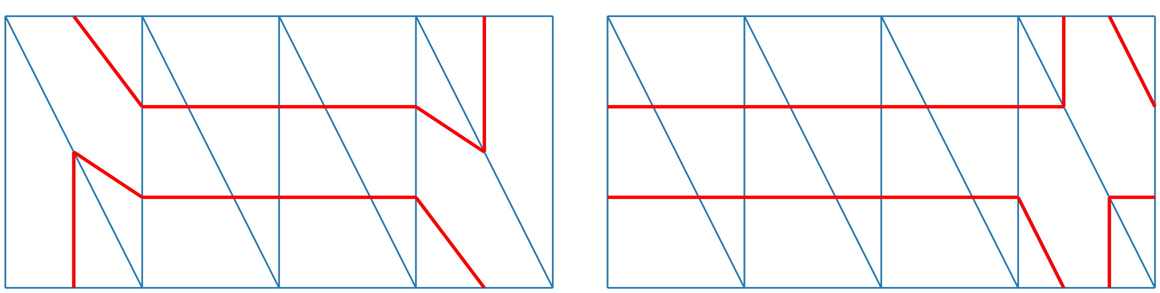


Figure 5.20: The curves (1,0,1,0,2,2,0,2,2,1,2,1) and (0,2,2,0,2,2,0,2,2,2,2,2) are disjoint so span an edge of $\mathcal{C}^{ss}S_{1,4}$.

Using this we have examined a subgraph $\mathcal{C}$ of $\mathcal{C}^{ss}S_{1,4}$ with 105278 edges. Using stan-

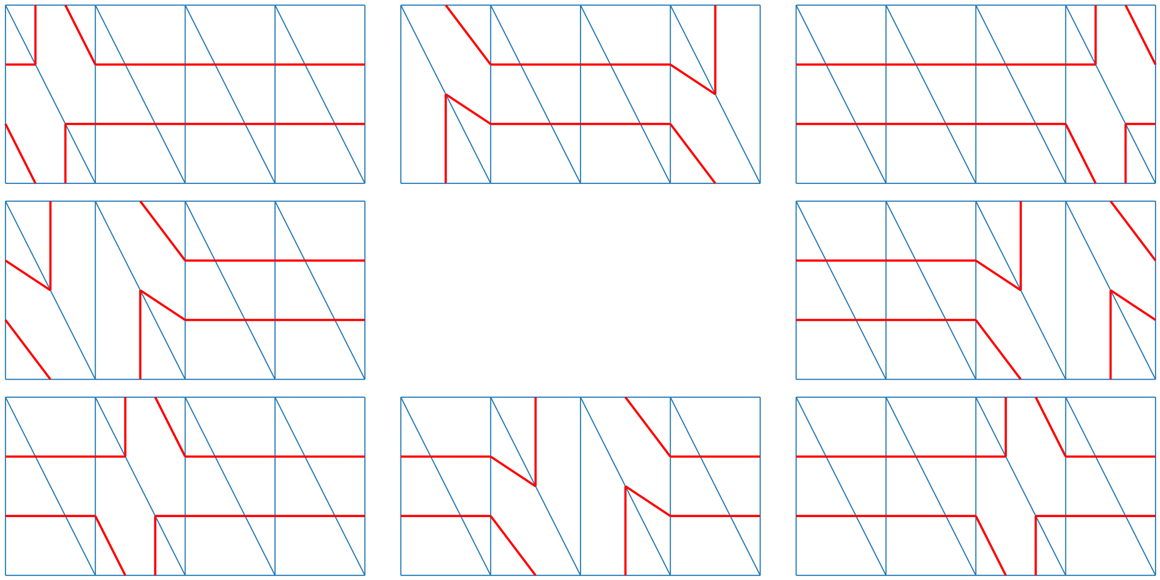


Figure 5.21: The standard octagon represented in normal coordinates.

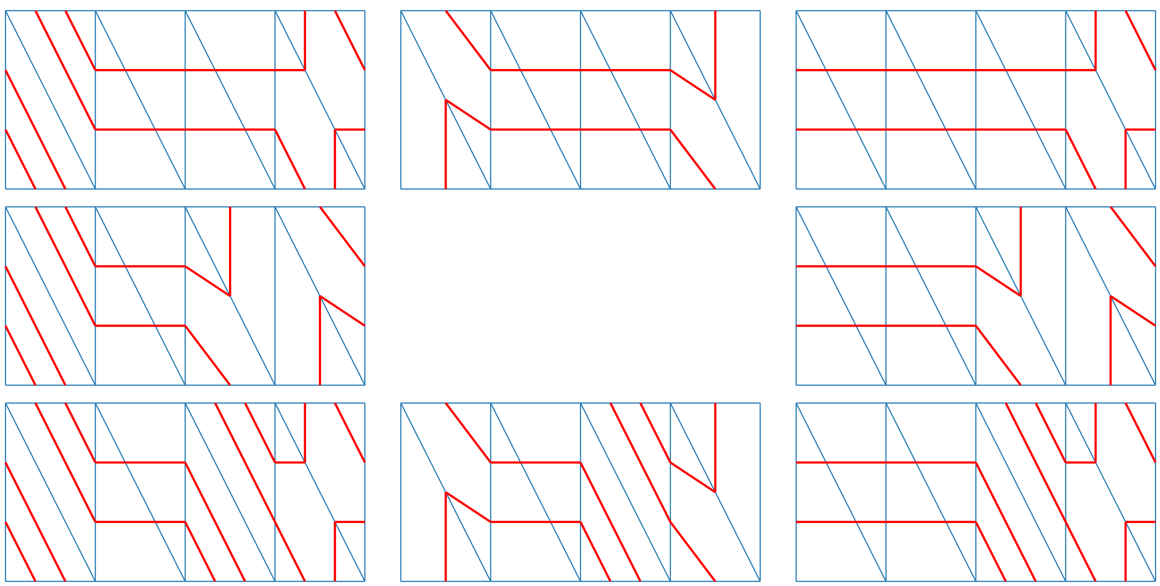


Figure 5.22: An octagon with point configuration (p_0, p_3, p_0, p_3) represented in normal coordinates.

dard graph algorithms we compute 5255 octagons based at the curve (1,0,1,0,2,2,0,2,2,1,2,1). Of these 918 octagons have point configurations given by permutations of (p_0, p_1, p_2, p_3). By computing the $\binom{8}{2}$ pairwise intersections of the 8 curves, we can verify that these are indeed homeomorphic to standard octagons, in support of Conjecture 20.

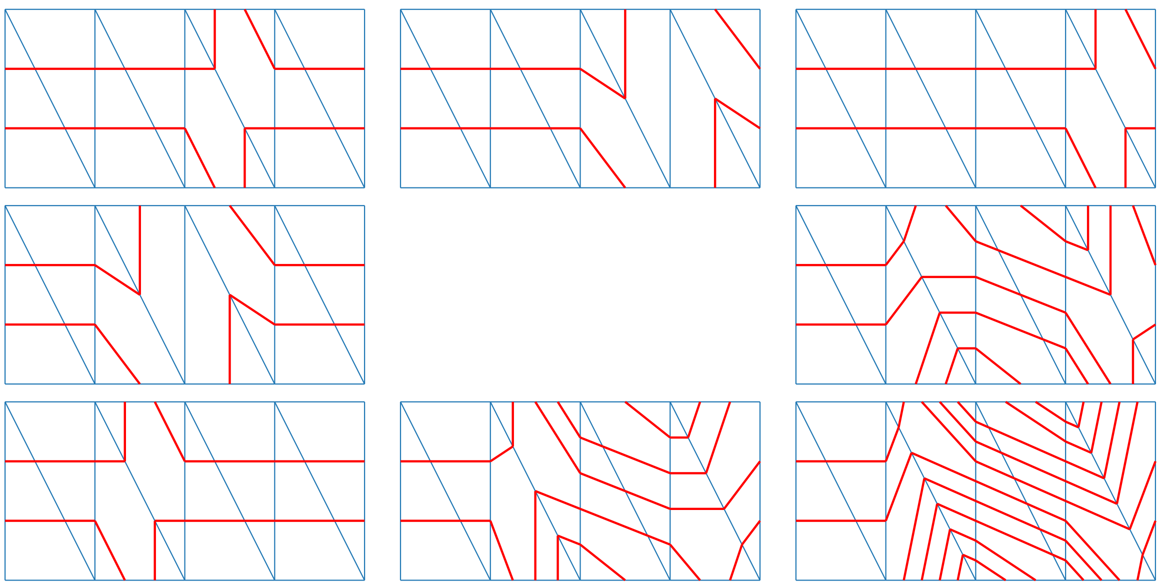


Figure 5.23: An octagon with point configuration (p_3, p_3, p_2, p_2) represented in normal coordinates. Note the top-left three curves and the bottom-right three curves differ by the Dehn twist $T_{e_6}^2$.

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