Density of matroid classes

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Density in graphs

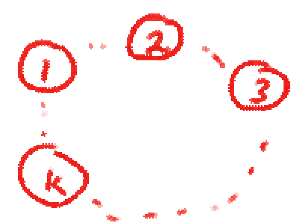
Let $\mathcal{G}$ be a set of graphs.

$$h(n) = \max\left(|E(G)| : G \in \mathcal{G} \text{ simple, } n\text{-vertex}\right)$$

Examples

planar graphs: $h(n) = 3n-6, \quad n \geq 3$

k-colourable graphs: $\displaystyle\lim_{n\to\infty} \frac{h(n)}{\binom{n}{2}} = 1 - \frac{1}{k}$



Density in matroids

Let $\mathcal{M}$ be a set of matroids.

$$h(r) = \max\left(\varepsilon(M) : M \in \mathcal{M} \text{ rank-}r \right).$$

number of points
in M

Density in matroids

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Examples

graphic matroids: $h(r) = \binom{r+1}{2}$

cographic matroids: $h(r) = 3r - 3, \quad r \geq 2$

GF(q)-representable: $h(r) = \dfrac{q^r - 1}{q - 1}$

Subgraphs and Restrictions

Erdös-Stone Theorem. [16 BR]

Let $\mathcal{G}$ be a subgraph-closed class. Then

$$\lim_{n \to \infty} \frac{h(n)}{\binom{n}{2}} = 1 - \frac{1}{k-1},$$

where $k = \min(\chi(G) : G \notin \mathcal{G})$.

Base Case: For all bipartite H and $\varepsilon > 0$, if G is a sufficiently large simple graph with

$$|E(G)| \geq \varepsilon \binom{n}{2},$$

then G has an H-subgraph.

Theorem [Furstenberg, Katznelson 23AR]

For any $m \geqslant 1$, $\varepsilon > 0$, and any prime-power q, if M is a simple $GF(q)$-representable matroid with sufficiently large rank and with $|M| \geqslant \varepsilon\, |PG(r(M)-1, q)|$, then M has an $AG(m-1, q)$ restriction.

Related to the Multidimensional Hales-Jewett Theorem.

Critical exponent

$$G(r-1, q, k) = PG(r-1, q) \setminus F,$$

F is a rank-$(r-k)$ flat.

Note: $G(r-1, q, 1) = AG(r-1, q)$.

For a simple $GF(q)$-representable matroid M,

$$c_q(M) = \min\left(k : G(r-1, q, k) \text{ contains } M\right).$$

Theorem [G., Nelson AR49]

Let $\mathcal{M}$ be a deletion-closed class of GF(q)-representable matroids. Then

$$\lim_{r \to \infty} \frac{h(r)}{|PG(r-1,q)|} = 1 - q^{1-c},$$

where $c = \min(c_q(M) : M \notin \mathcal{M})$.

Minors

Mader's Theorem [6 AR]

If $\mathcal{G}$ is a proper minor-closed class of graphs, then

$$h(n) \leq c \cdot n, \quad \text{for all } n \geq 0.$$

Kung's Theorem [29 AR]

If M is a rank-r matroid with no $(q+2)$-point line as a minor, then

$$\varepsilon(M) \leq \frac{q^r - 1}{q - 1}.$$

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Note. If q is a prime-power, then $M = PG(r-1, q)$ attains equality.

Kung's Theorem [29 AR]

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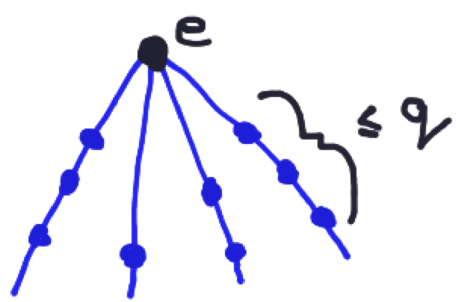
Proof.



$$\begin{aligned}\varepsilon(M) &\le 1 + q\,\varepsilon(M/e)\\ &\le 1 + q\left(\frac{q^{r-1}-1}{q-1}\right)\\ &= \frac{q^r-1}{q-1}\end{aligned}$$

$\blacksquare$

Growth-rate Theorem [G., Kabell, Kung, Whittle. 46 AR]

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(I) $h(r) \le c.r$,

(II) $h(r) \le c.r^2$ and $\mathcal{M}$ contains all graphic matroids, or

(III) there is a prime-power q such that $h(r) \le c \cdot q^r$ and $\mathcal{M}$ contains all GF(q)-representable matroids.

Growth-rate Theorem [G., Kabell, Kung, Whittle 46 AR]

Let $\mathcal{M}$ be a minor-closed class of matroids not containing all lines. Then either

(I) $h(r) \leq c \cdot r$, <u>linearly dense</u>

(II) $h(r) \leq c \cdot r^2$ and $\mathcal{M}$ contains all graphic matroids, or <u>quadratically dense</u>

(III) there is a prime-power q such that $h(r) \leq c \cdot q^r$ and $\mathcal{M}$ contains all GF(q)-representable matroids. <u>base-q exponentially dense</u>

Applications.

(i) If M is a rank-r $\mathbb{R}$-representable matroid with no n-point line as a minor, then $\varepsilon(M) \le c \cdot r^2$.

(ii) If M is a rank-r matroid representable over $GF(4)$ and $GF(8)$, then

$$\varepsilon(M) \le c \cdot 2^r.$$

Questions

(1) Can we refine the Growth-rate Theorem?

(2) What are the extremal matroids?

Base-q exponentially dense

Examples

(1) $\mathcal{M} = \{$ rank $\leq k$ truncations of $GF(q)$-rep. matroids$\}$.

$$h(n) = \frac{q^{n+k}-1}{q-1}, \quad n \geq 2.$$

(2) $\mathcal{M} = \{$ rank $\leq k$ projections of $GF(q)$-rep. matroids over $GF(q^2)$ $\}$

$$h(n) = \frac{q^{n+k}-1}{q-1} - \underbrace{q\cdot\left(\frac{q^{2k}-1}{q^2-1}\right)}_{\text{constant}}, \quad n \geq 2k$$

Base-q exponentially dense

Let $\mathcal{M}$ be a base-q exponentially dense minor-closed class of matroids.

Base-q exponentially dense

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Main Theorem [G., Nelson 49 AR]

For sufficiently large n,

$$h(n) = \frac{q^{n+k} - 1}{q - 1} - c,$$

where $k, c \geqslant 0$.

Base-q exponentially dense

Connectivity Lemma. For $q \geq 3$, $\mathcal{M}$ contains <u>round</u> extremal matroids of arbitrary rank.

round = not the union of two hyperplanes

= ∞ vertical connectivity

Base-q exponentially dense

Key Lemma. For $m \geq 1$ and $\varepsilon > 0$,

if $M \in \mathcal{M}$ with $|M| > \varepsilon \cdot q^{r(M)}$ and

M has sufficiently large rank, then

M has an $AG(m-1, q)$-restriction.

Base-q exponentially dense

Proof Sketch. Consider $M \in \mathcal{M}$ minor-minimal subject to:

(1) M has an $AG(m-1, q)$ restriction N

(2) M round

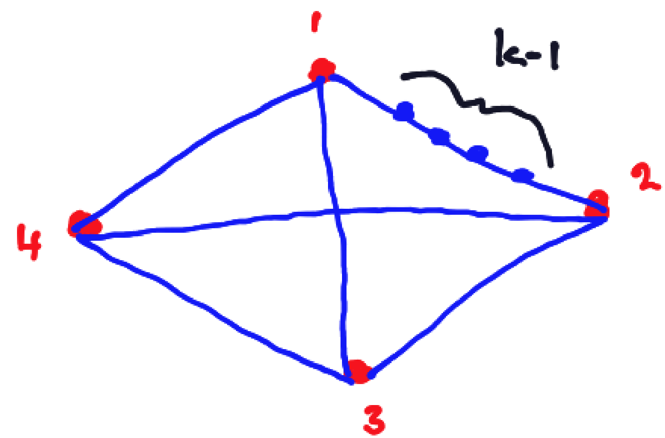
(3) $\varepsilon(M) > \dfrac{q^{r(M)+k} - 1}{q - 1}$.

Quadratically dense

Example Γ a group of size k.

$$\mathcal{M} = \{ \Gamma\text{-Dowling matroids} \}.$$

$$h(n) = n + (k-1)\binom{n}{2}.$$



Quadratically dense

Conjecture

$\mathcal{M} = \{ \mathbb{Q}\text{-rep. matroids with no } (k+3)\text{-point line as a minor} \}.$

For sufficiently large n,

$$h(n) = n + k\binom{n}{2}.$$

Quadratically dense

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$\mathcal{M} = \{\mathbb{Q}\text{-rep. matroids with no } (k+3)\text{-point line as a minor}\}$.

For sufficiently large n,

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Quadratically dense

Conjecture Let $\mathcal{M}$ be a quadratically dense class. Then

$$h(n) = an^2 + bn + c,$$

for sufficiently large n.

Linearly dense classes

How linear are they?

Linearly dense classes

Example.

$\mathcal{G} = \{\text{graphs with no } K_9\text{-minor}\}$

For $n \geq 13$,

$$h(n) = \begin{cases} 7n-27, & n \equiv 1 \pmod 5 \\ 7n-28, & n \not\equiv 1 \pmod 5 \end{cases}$$

[Song, Thomas 44 AR]

Linearly dense classes

Conjecture. Let $\mathcal{M}$ be a linearly dense minor-closed class of matroids. Then, for sufficiently large n,

$$h(n) = \begin{cases} an + b_0, & n \equiv 0 \pmod{k} \\ an + b_1, & n \equiv 1 \pmod{k} \\ \quad \vdots & \\ an + b_{k-1}, & n \equiv k-1 \pmod{k} \end{cases}$$

Thank you.