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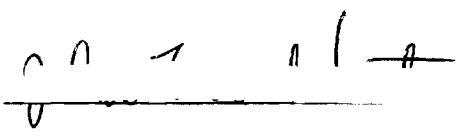

STUDIES OF ENERGY SHARING
IN NONLINEAR COUPLED OSCILLATOR SYSTEMS

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NONLINEAR COUPLED OSCILLATOR SYSTEMS

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SUMMARY

Systems of coupled oscillators find extensive use as mathematical models for a great variety of physical systems, from crystal lattices in solid state physics and molecular systems in chemical physics through canonical ensembles in statistical mechanics to systems of planets and galaxies in astrophysics.

In many cases the harmonic approximation, in which only linear forces are included in the model, suffices to provide a useful description of physical phenomena in terms of a system of coupled oscillators.

However, there are some important physical phenomena which cannot be adequately described unless nonlinear forces are also included in the coupled oscillator model. For example, in the description of a crystal lattice, nonlinear terms must be included in order to provide adequate theories of thermal conductivity, thermal expansion, and the propagation of sound waves. As another example, in chemical physics, the dissociation of a molecule is primarily a nonlinear phenomenon.

In principle it is always possible to obtain the general solution of the linear coupled oscillator system given by the harmonic approximation. An orthogonal transformation to normal mode variables, in which the equations of motion are completely decoupled, can always be made for physically realistic linear systems.

But if nonlinear forces are included in the coupled oscillator system, then the equations of motion cannot in general be completely decoupled by a linear transformation. If a normal mode transformation

which decouples only the linear portion of the system is applied, then the nonlinear terms couple the resulting "harmonic normal modes."

For many of the physical systems of interest, the linear and nonlinear couplings cooperate in a fashion such that the oscillator system is capable of complete sharing of energy among all of its harmonic normal modes. That is, the set of harmonic normal mode frequencies is "tuned," so that internal resonances are present in the harmonic normal mode system, and under certain conditions energy can be passed around among all of the modes.

The available perturbation methods, such as those of Krylov-Bogoliubov and Wigner-Brillouin, encounter a classic difficulty characterized by the appearance of terms having small divisors, when applied to nonlinear systems having tuned frequencies. Since these tuned systems share energy and are the systems of greatest interest in the present investigation, the ineffectiveness of these perturbation methods when applied to tuned systems is a serious shortcoming.

The primary aim of the present investigation is to develop a perturbation method which is effective in dealing with tuned nonlinear coupled oscillator systems.

In order to deal with the most extreme case of tuned harmonic normal mode frequencies, an exactly-tuned nonlinear one-dimensional coupled oscillator system has been chosen as a model to be analyzed.

This model is not meant to represent any particular physical system; rather, it has been designed to emphasize the feature of tuned frequencies in a reasonably simple nonlinear coupled oscillator system. It is felt that if a successful analytic method of solution can be developed for

this simple but extreme case, then considerable insight into the solution of more complicated physical three-dimensional nonlinear systems in which the tuning relations are perhaps more complicated will have been gained.

The system chosen for analysis in the present investigation is governed by a hamiltonian which is of the following form in harmonic normal mode variables:

$$H = \tfrac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + k^2\omega^2 q_k^2\right) + \alpha\tfrac{1}{3}\sum_{k,l,m}^{N} C_{klm}\, q_k q_l q_m . \qquad (1)$$

It can be shown that the cubic coupling energy terms, the portion of Equation (1) controlled by overall coupling parameter α, are of such a form that the frequencies $\omega_k = k\omega$ form an exactly-tuned set for this system.

Analysis of the system given by Equation (1) for an arbitrary number of oscillators, N, has been guided by the analysis of three simpler cases.

In the first case, some simplification of the analysis has been achieved by showing that the solution of a given tuned oscillator system is well-approximated by the solution to a corresponding "resonant" system. This "resonant" system is constructed by retaining only the slowly-varying coupling terms in the hamiltonian of the original system. The distinction between slowly- and rapidly-varying terms is most conveniently made in action-angle formalism.

In the second case, various methods of solution have been applied to obtain the general solution of a linear coupled oscillator system

which is somewhat similar to the nonlinear system of Equation (1), but which is much easier to solve. The harmonic normal mode hamiltonian of this linear system is of the form

$$H = \frac{1}{2}\sum_{k=1}^{N}(p_k^2 + \omega^2 q_k^2) + \alpha\frac{1}{2}\sum_{k,l}^{N} C_{kl} q_k q_l, \qquad (2)$$

where now the quadratic coupling energy terms are of such a form that the frequencies $\omega_k = \omega$ form an exactly-tuned set.

In the third case, the energy solution for a resonant exactly-tuned nonlinear two-oscillator system corresponding to the system given in Equation (1) has been obtained.

The results from these three preliminary investigations have guided the analysis of the N-oscillator nonlinear oscillator system. The resonant exactly-tuned nonlinear system corresponding to the system specified by Equation (1) has been found to possess N periodic solutions for N different special sets of initial conditions. Further, it has been found that the general solution of the N-oscillator resonant system can be expressed in terms of a perturbation expansion about a periodic solution.

The expansion parameter used in this method of general solution is a function only of ratios between initial conditions. To first order, the method of expansion about periodic solutions avoids the problem of the appearance of terms with small divisors. In higher-order approximations, terms of this sort may appear, but it seems likely that they can be eliminated by an appropriate choice of arbitrary constants in the formal solution.

To check the effectiveness of the method of expansion about periodic solutions developed here, results of this method have been compared with relatively "exact" numerical solutions of the equations of motion for various selected sets of initial conditions, using a high-speed digital computer.

This comparison indicates that the periodic solutions are valid zeroth-order terms for a perturbation expansion, and that reasonably good first-order approximations to the exact general solutions of small systems can be obtained. It is, however, crucial to expand about an appropriate periodic solution for a given set of initial conditions, in order to keep the expansion parameters small. Some difficulties have been encountered using the choice of arbitrary constants which results in computational simplicity of the second-order calculation; but it is felt that the second-order and higher calculations can in principle be successfully conducted, with another choice of arbitrary constants.

The principal conclusion which can be drawn from this investigation is that weakly-coupled energy-sharing nonlinear oscillator systems are amenable to analysis and that these systems have much in common with energy-sharing linear coupled oscillator systems. It is hoped that this result will encourage the development of more practical methods of analysis for such systems.

In the case of physically realistic systems, there are rapidly-varying as well as slowly-varying terms in the couplings, and the frequencies are not likely to be exactly tuned. Thus the analysis of the resonant exactly-tuned nonlinear system summarized above is not a complete analysis which is immediately applicable to physical problems. But

it should be the more difficult part of the analysis; it is expected that the effects of nonresonant coupling terms and detuning of the frequencies can be dealt with by standard perturbation methods, using the solution of the resonant exactly-tuned system as a zeroth-order solution.

Such perturbations of the resonant exactly-tuned system will not be considered analytically in this study. However, some computer studies both of the effect of nonresonant coupling terms and of detuning have been included in this investigation.

In addition to its primary aim of developing a perturbation method for resonant exactly-tuned nonlinear coupled oscillator systems, this investigation has had several related secondary aims.

First, an investigation in some detail has been made of the relationship between energy sharing and tuning of the uncoupled harmonic normal mode frequencies. A system need not be exactly tuned in order to be capable of complete energy sharing. A criterion for the tuning of a few simple nonlinear systems has been developed early in the investigation. Contact is made here with computer studies by Fermi, Pasta and Ulam, and by Jackson.

Second, some investigation of the constants of the motion for a nonlinear system has been made. Since knowledge of N constants of the motion is equivalent to knowledge of the general solution of a system, and since the general solution has been obtained, it would appear that at least the resonant exactly-tuned nonlinear systems investigated in this study possess N constants of the motion. However, the form of these constants of the motion has not yet been found. At best, one additional

constant of the motion other than the hamiltonian has been identified for nonlinear systems.

Finally, the approach to equilibrium has been studied briefly. A computer study of linear and nonlinear systems of five oscillators indicates that both quickly reach an equilibrium configuration after being started with all the energy in the lowest mode. The single-oscillator distribution densities of both systems are in reasonable agreement with the Boltzmann distribution of equilibrium statistical mechanics. Thus four oscillators evidently form an adequate heat bath for the fifth, in the canonical ensemble theory of equilibrium statistical mechanics.

CHAPTER I

INTRODUCTION

In the first two sections of this chapter, some historical remarks will be made and an outline of the content of the remainder of this thesis will be given. The third section will serve as a review of the solution of a general linear coupled oscillator system in normal modes. The final section will introduce some of the physical systems and physical phenomena for which nonlinear coupled oscillator systems are appropriate mathematical models.

Historical Introduction

Systems of coupled oscillators have for nearly three centuries been valuable to physicists as mathematical models for various physical systems. Newton (1686) first used a one-dimensional lattice of equal massed connected by linear springs, that is, a harmonic coupled oscillator system, as a model for the propagation of sound waves in air. This model enjoyed some success in theoretically predicting the speed of sound in air (Lamb 1931).

Many of the vital problems and techniques of modern theoretical physics and pure mathematics were first discovered in connection with such harmonic coupled oscillator systems: eigenvalues and eigenvectors, Fourier expansions, expansions in series of orthogonal functions, partial differential equations, wave propagation, and atomic theory of solids and crystal structure (Rayleigh 1926; Brillouin 1946).

Any conservative system of coupled particles in which only forces which are linear functions of displacements act can in principle be solved. A transformation can be made to a normal mode representation, in which the equations of motion are completely decoupled. The general particle solution is then a linear superposition of normal mode solutions.

(It should be emphasized that the general particle solution of the linear system is available "in principle." For example, the normal modes of a one-dimensional random A-B alloy are not easily calculated (Dean and Bacon, 1962). In general, the only solvable linear systems are periodic with respect to the distribution of the particle masses.)

For the linear system, there is no sharing of energy between the completely decoupled normal mode oscillators. Thus N constants of the motion are known to exist for any linear system having N degrees of freedom, in the form of the N normal mode energies.

It is just this relative simplicity which limits the usefulness of the linear system as a mathematical model adequately representing actual physical systems. As will be discussed in detail later in this chapter, there are numerous applications in which the harmonic potentials and constant normal mode energies of the linear model fail to provide physically realistic results. Harmonic potentials are symmetrical, and a linear model of a crystal lattice cannot exhibit thermal expansion. Constant normal mode energies cause trouble in many applications to solid state, chemical, statistical mechanical and astrophysical systems in which sharing of energy among the normal modes is a vital feature.

With these shortcomings in mind, now consider a conservative system of coupled particles in which nonlinear as well as linear forces act. For

this system, it is no longer true that a mathematical formalism for obtaining the general solution is immediately available. There is no general theorem which guarantees that the equations of motion of a nonlinear system can be diagonalized by a linear transformation.

At best, an orthogonal transformation which decouples the linear portion of the system may be applied. The description which results from this transformation will be referred to as the "harmonic normal mode" description in this investigation. The nonlinear forces transform in such a manner as to provide couplings between the harmonic normal modes of the system.

The anharmonic potentials of the nonlinear coupled oscillator system are unsymmetrical, and so an appropriately constructed nonlinear model of a crystal lattice can exhibit thermal expansion. Since the nonlinear forces couple the harmonic normal modes of the linear portion of the system, it should be possible to have sharing of energy between the harmonic normal modes, at least under some conditions. Thus the nonlinearly coupled oscillator system should provide a realistic description of certain physical phenomena for which the linear system fails to be an adequate model.

As mentioned, the solution of the nonlinear system is not available in general. However, there are numerous approximation schemes for dealing with the effects of the inclusion of nonlinear terms. Many of these rely upon the fact that the nonlinear coupling forces are weak in comparison to the linear restoring forces which constitute the harmonic normal mode oscillators themselves.

It happens that the available approximation schemes are least

effective in describing nonlinear effects which involve complete sharing of energy between harmonic normal modes. In order to improve the approximate calculations for these energy-sharing situations, it would be of considerable aid to have a simple energy-sharing nonlinear model upon which detailed analysis could be made.

Purpose of Investigation

The primary aim of this investigation is to formulate and analyze such an energy-sharing nonlinear coupled oscillator system. It is hoped that this work will be found useful in the development of improved approximation calculations for more complicated physically realistic systems in which nonlinear energy-sharing effects are not yet adequately treated by the presently available approximation methods.

As an illustration from solid state physics, the Kronig-Penney model for electron band theory was the first simple model in terms of which electron wave functions in a crystal could be easily calculated. These results have been improved and approximations have been made based on the Kronig-Penney and succeeding models, to produce the modern rather refined electron band theory of crystals.

In analogy to the Kronig-Penney model, a simple model of an anharmonic crystal lattice, in terms of which detailed calculations can be made with relative ease, would be quite useful in the development of lattice theory. In particular, an energy-sharing nonlinear system is relevant to the full understanding of lattice thermal conductivity, as will be discussed later in this chapter.

The remainder of the present introductory chapter will review the normal mode solution of a linear system, and will provide a detailed discussion of some of the physical systems and physical phenomena for which

the energy-sharing nonlinear coupled oscillator system is a relevant model.

In Chapter II, the relationship between energy sharing among harmonic normal modes and tuning of the harmonic normal mode frequencies will be investigated in detail. Based on the findings in Chapter II, the definite nonlinear model to be analyzed will be formulated and discussed in Chapter III.

Chapter IV will discuss the effectiveness of some of the available perturbation methods in solving nonlinear coupled oscillator systems capable of various amounts of energy sharing. For the completely energy-sharing model formulated in Chapter III, these methods display serious shortcomings. This indicates the need for a new method of solution.

In order to guide the analysis of the energy-sharing nonlinear system, various methods of solving a similar artificially constructed energy-sharing linear coupled oscillator system will be considered in Chapter V.

In Chapter VI a method of analytic solution for the nonlinear model formulated in Chapter III will be developed. The experience gained from solving the linear system in Chapter V, plus the benefit of the energy solution of the two-oscillator nonlinear system, will be used to guide the development of this method.

The validity of this method of solution will then be checked in Chapter VII, by comparing the analytic results with the results of numerical integration, for certain selected sets of initial conditions.

Chapter VIII will briefly present the results of two computer investigations, one concerning constants of the motion and the other concerning the approach to equilibrium. Finally, some overall conclusions about the results of this thesis will be stated in Chapter IX.

Normal Modes for Linear Systems

Consider a collection of point masses m_i with forces acting between them so that they vibrate about stable equilibrium positions. In three dimensions, let the displacements from the equilibrium positions in the x, y, z directions be denoted x_1, x_2, x_3 for the first particle; x_4, x_5, x_6 for the second particle; and so on.

The potential energy is assumed to depend only upon the x_i. Therefore, it can be expanded in a Taylor series of the form

$$V(x_1, \ldots, x_N) = V_0 + \sum_{i=1}^{N} \left(\frac{\partial V}{\partial x_i}\right)_0 x_i + \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{2} \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_0 x_i x_j + \ldots, \tag{1}$$

where the zero subscript implies evaluation for the equilibrium configuration, and $N = 3n$, where n is the total number of particles in the system.

The force acting on mass m_k in the x_1 direction (where $i = 3k-2$, $3k-1$, or $3k$) is

$$F_i = -\frac{\partial V}{\partial x_i} = -\left(\frac{\partial V}{\partial x_i}\right)_0 - \sum_{j=1}^{N} \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_0 x_j + \ldots. \tag{2}$$

For simplicity, let $V_0 = 0$. By symmetry, $\left(\frac{\partial V}{\partial x_i}\right)_0 = 0$; that is, no force acts on any particle for the equilibrium configuration. Thus the potential energy may be written

$$V(x_1, \ldots, x_N) = \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} B_{ij} x_i x_j + \frac{1}{6} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} C_{ijk} x_i x_j x_k + \cdots, \tag{3}$$

where the force constants

$$B_{ij} \equiv \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_0, \quad C_{ijk} \equiv \left(\frac{\partial^3 V}{\partial x_i \partial x_j \partial x_k}\right)_0, \ldots \tag{4}$$

are elements of symmetric tensors of rank 2, 3, ...

The harmonic approximation consists of retaining only the quadratic term in the potential energy, so that only linear forces act in the system. In many cases this is a good approximation, for small displacements. However, there exist some physical phenomena, such as thermal expansion and thermal conductivity, in which the effect of some of the higher-order terms cannot be successfully neglected, no matter how small the (nonzero) displacements are. This is a consequence of the zero point energy (Leibfried and Ludwig 1961, p. 277).

Since the potential energy in the harmonic approximation is a symmetric quadratic form, and since the kinetic energy,

$$T = \frac{1}{2} \sum_{i=1}^{N} \frac{p_i^2}{m_i} = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{x}_i^2, \tag{5}$$

is also a symmetric quadratic form, the hamiltonian for the linear system can be written in the following matrix form (Rubin 1963):

$$H = \frac{1}{2} \dot{X}^T M \dot{X} + \frac{1}{2} X^T B X, \tag{6}$$

where X is a column vector with elements x_i, X^T is its row vector transpose, M is a diagonal matrix with elements $M_{kk} = m_k$, and matrix B has elements B_{ij} already introduced.

The equations of motion for the linear system are

$$M\ddot{X} = -BX. \tag{7}$$

To solve these equations in terms of normal modes, first let $Y = M^{\frac{1}{2}} X$, so that Equations (7) become

$$\ddot{Y} = -M^{-\frac{1}{2}} B M^{-\frac{1}{2}} Y. \tag{8}$$

Now define the normal mode co-ordinates, $Q = S^T Y$. Matrix S is the orthogonal matrix which diagonalizes real symmetric matrix $M^{-\frac{1}{2}} B M^{-\frac{1}{2}}$, so that the equations of motion finally become

$$\ddot{Q} = -S^{-1} M^{-\frac{1}{2}} B M^{-\frac{1}{2}} S Y = -\Omega^2 Q, \tag{9}$$

where the elements of the diagonal matrix Ω^2 are $\Omega^2_{jj} = \omega_j^2$.

The individual normal mode equations of motion are

$$\ddot{q}_k = -\omega_k^2 q_k, \quad k = 1, 2, \ldots, N, \tag{10}$$

so that the normal mode oscillators execute sinusoidal motions (so that the amplitudes never change from the original values). The general solution in particle variables is the linear superposition of normal mode components given by

$$X = M^{-\frac{1}{2}} S Q. \tag{11}$$

The normal mode transformation $Q = S^T M^{\frac{1}{2}} X$ can always be made, at least in principle, for systems of the harmonic approximation. Therefore N constants of the motion are in principle always available for any harmonic system, in the form of the normal mode energies,

$$E_k = \tfrac{1}{2}\dot{q}_k^2 + \tfrac{1}{2}\omega_k^2 q_k^2, \quad k = 1, 2, \ldots, N. \tag{12}$$

The above discussion has provided a brief review of the well-known situation for the harmonic approximation. Now it is of interest to consider the situation in which cubic and higher degree terms are retained in the expression for the potential energy.

Suppose the cubic term in the potential energy expression of Equation (3) is kept in addition to the quadratic term of the harmonic approximation. Then the expression for the hamiltonian,

$$H = \tfrac{1}{2}\sum_{i=1}^{N} m_i \dot{x}_i^2 + \tfrac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} B_{ij} x_i x_j + \tfrac{1}{6}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} C_{ijk} x_i x_j x_k, \tag{13}$$

cannot be written in the simple matrix notation used earlier, because of the presence of the third rank tensor C_{ijk}.

In particular, any orthogonal transformation,

$$x_i = m_i^{-\frac{1}{2}} \sum_{j=1}^{N} S_{ij} q_j, \tag{14}$$

which reduces the second term in Equation (13) to a diagonal form, will not in general have the same effect on the third term. That is, the

inclusion of a cubic term in the hamiltonian of a system of coupled particles makes it impossible in general to make a linear transformation to a set of variables in which motions are completely decoupled. There exist no normal mode variables, in the completely decoupled sense, for a nonlinear system.

However, the concept of normal modes is still of value in dealing with nonlinear systems. If the orthogonal transformation given by Equation (14) is applied to the system of Equation (13), then a description is obtained in which the linear portion of the system is decoupled, and the nonlinear portion provides couplings between these harmonic normal modes. It is for this reason that the study of the behavior of systems of particles coupled by nonlinear as well as linear forces is characterized as the study of nonlinear coupled oscillator systems.

The primary aim of this thesis is to formulate and analyze a nonlinear coupled oscillator model which is capable of complete energy sharing among its harmonic normal modes. In the final section of this introductory chapter, a number of physical systems for which such a model is relevant will be examined in detail.

Physical Systems of Interest

In the physics of the past fifty years, nonlinear coupled oscillator systems have been used as mathematical models to represent physical systems ranging from crystal lattices, in which some vibrations have periods as small as 10^{-12} seconds, to systems of galaxies, in which the periods may be as large as 10^{+14} seconds.

In this section, some of the physical systems for which a completely energy-sharing model is most appropriate will be discussed in some detail.

Emphasis will be given to those physical phenomena which are basically nonlinear.

Crystal Lattice Vibrations

Thermal effects in solids due to lattice vibrations may be studied by considering dielectric crystals, in which electron effects are negligible; or the thermal effects due to lattice vibrations and electrons in conducting crystals may be studied separately, and the effects of interactions considered afterward (Ziman 1960).

For the perfect harmonic crystal lattice with periodic boundary conditions, the traveling waves which transport energy through the crystal are just the normal modes of the crystal lattice (Klemens 1958).

For this perfect harmonic crystal, there exists a certain equilibrium distribution of energy among the normal modes, such that if there is no temperature gradient, then there is no net heat transported through the crystal. But there are also other distributions such that some net heat is transported through the crystal, even with no temperature gradient (Peierls 1956; Hemmer 1959). For the perfect harmonic lattice, a linear system, there is no exchange of energy between normal modes and the system cannot return to its equilibrium distribution, as would an actual physical system. This possibility of a steady-state heat flow in the absence of a temperature gradient leads to infinite thermal conductivity in the perfect harmonic model of a crystal lattice.

As reviewed by Klemens (1958), finite lattice thermal conductivity may be introduced into the crystal lattice model by three kinds of interaction processes, which allow the exchange of energy between the lattice waves: imperfections in the crystal lattice; anharmonic forces in addition

to the harmonic forces; and the finite boundaries of the crystal (of greatest importance at low temperatures).

Imperfections in the crystal lattice alter the normal modes of the system so that the traveling waves (normal modes of the perfect system) are no longer exactly the normal modes of the imperfect system. The result is an exchange of energy among the traveling waves, which is needed if a return to an equilibrium energy distribution is to be possible.

Anharmonic forces provide a mechanism for the sharing of energy among the harmonic normal modes which has already been introduced briefly, and which will be examined in detail in Chapter II.

The effect of finite boundaries of the crystal can be described in terms of the first two kinds of interaction processes, according to Klemens.

The three kinds of interaction processes mentioned here are of varying importance in different actual materials and at different temperatures. Qualitatively, the details of the physical effects which account for finite thermal conductivity have been reasonably well explained during the past fifty years. Early treatments by Debye (1914) and Peierls (1929) have provided the basic foundations.

Attention will now be centered upon the effects of anharmonic forces on lattice thermal conductivity. Peierls shows that only those nonlinear terms which (a) conserve energy and (b) either conserve quasi-momentum (normal terms) or produce a difference in quasi-momentum equal to an inverse lattice vector (umklapp terms) are of importance in the explanation of thermal conductivity. The umklapp terms contribute a term to the thermal

conductivity which varies inversely as the temperature for high temperatures and which increases exponentially at low temperatures. These qualitative results have been verified experimentally.

In contrast to the success of the qualitative theory of Peierls, a quantitative theory of lattice thermal conductivity is still lacking. As will be illustrated in the next chapter, the work of Peierls shows that only those nonlinear terms which cause energy sharing are significant in thermal conductivity due to anharmonicities. Thus it would be of considerable aid in the development of a quantitative theory of thermal conductivity to have a definite model of an energy-sharing nonlinear coupled oscillator system, in terms of which detailed calculations could be made. The results of analysis of such a simple nonlinear model would then be used to guide the development of approximate calculations in more complex physically realistic crystal models.

It must be noted that in general, the system used to represent a crystal lattice should be quantized. However, this thesis is intended to be somewhat general in its fields of applications, and so all investigations in the present work are conducted in the hamiltonian formalism of classical mechanics. There are some applications in which the classical formalism is entirely adequate, such as the application to systems of galaxies; also, the classical formalism provides a simplicity of notation and description.

For application to systems such as crystal lattices, quantization of the hamiltonian formalism is conveniently carried out by the introduction of Heisenberg operators. The co-ordinate and momentum variables of the classical formalism are replaced by time-dependent creation and

destruction operators (Messiah 1963). These operators obey a commutation relation, so that the description of lattice vibrations may be recast in the following quantum-mechanical terms: quanta of the lattice vibrations, called phonons, form a Bose-Einstein system of particles of energy $h\omega$, quasi-momentum $h\vec{k}$, and wave packet velocity $\frac{\partial \omega(\vec{k})}{\partial \vec{k}}$, where $\omega(\vec{k})$ is the angular frequency of a normal mode lattice vibration, and $\vec{k}$ is the wave vector.

There is no limit to the number of phonons in a normal mode. The state of the system is given by the number N_i of phonons in each normal mode, so that the hamiltonian for the quantum mechanical linear system is

$$H = \sum_{i=1}^{N} (N_i + \tfrac{1}{2}) \hbar \omega_i, \tag{15}$$

including the zero-point energy. Cubic nonlinear terms in the hamiltonian give couplings between normal modes in the form of three-phonon interactions, in which two phonons are destroyed and one created, or vice-versa. Quartic nonlinear terms give four-phonon interactions, and so on.

For weakly nonlinear systems, which are of interest in the study of thermal conductivity at fairly low temperatures, Peierls (1956) keeps only cubic anharmonic terms in the hamiltonian. In fact, Ziman (1960, p. 146) feels that nonlinear terms of higher degree may be neglected even at rather high temperatures, since four-phonon interactions can be produced by two consecutive three-phonon interactions, involving the concept of an intermediate state.

Essentially, the quantization of the oscillator systems of this investigation involves merely a redescription in another formalism. In

later work, when references are made to the application to crystal lattices, the phonon description may at times be used, although the investigation will be conducted in the classical formalism.

Anharmonic terms account for other physical effects observed in actual crystals, besides finite thermal conductivity. For example, a strictly harmonic crystal model cannot exhibit thermal expansion. In order to display thermal expansion, a model must have a potential energy which is asymmetric as a function of the separation of nearest-neighbor pairs of atoms. Kittel (1956, p. 152) assumes a potential energy of the form

$$V(r) = br^2 - cr^3 - dr^4, \tag{16}$$

where r is the displacement from the equilibrium separation distance of two atoms: $r = x_i - x_j$. Using the classical Boltzmann distribution function, the average value of r for small displacements is found by Kittel to be

$$\langle r \rangle = \frac{3}{4} kT \frac{c}{b^2}; \tag{17}$$

that is, the potential given by Equation (16) exhibits linear thermal expansion, assuming moderate temperatures. Note that only the cubic term cr^3 introduces asymmetry into Equation (16), so that the quartic term dr^4 does not contribute to $\langle r \rangle$.

Besides thermal conductivity and thermal expansion, anharmonic terms enter into the explanation of other crystal phenomena, such as high temperature lattice specific heat and temperature dependence of lattice

elastic constants (Peierls 1956; Ziman 1960). However, the principal lattice physical phenomenon of interest with regard to completely energy-sharing nonlinear coupled oscillator systems is lattice thermal conductivity.

Canonical Ensembles and Equilibrium Behavior

In equilibrium statistical mechanics, the complete description of the equilibrium behavior of each particle in a physical system of many (10^{23}) particles is impossible. Only the collective thermodynamic properties of the overall system are of interest.

In order to obtain thermodynamic properties, statistical mechanics employs the concept of the canonical ensemble, a collection of identical macroscopic systems having identifiable individual energies, having weak interactions which exchange energy among them. The overall ensemble has states which are specified by the state of each of the systems in the ensemble at a given time.

The permissible states of a canonical ensemble satisfy the conditions

$$\sum_i n_i = N \quad \text{and} \quad \sum_i n_i E_i = E, \tag{18}$$

where n_i is the number of systems in the i^{th} state, having energy E_i; N is the total number of systems and E is the total energy in the ensemble. The second condition in Equations (18) contains the implicit assumption that the total interaction energy is a canonical ensemble is negligible compared with the total energy of the ensemble.

In order to obtain thermodynamic properties of the ensemble without

further considering the nature of the interaction processes which couple the individual systems, several probability assumptions are introduced (Davidson 1962). One assumption is that an examination of the ensemble states at various times will give the probability for the distribution of states of one individual macroscopic system. Another assumption is that any one state of the ensemble which satisfies the two conditions given by Equations (18) is as probable as any other.

From the foregoing considerations, the canonical partition function for the ensemble is calculated. All of the thermodynamic properties of the system can then be calculated, using this partition function.

In the study of canonical ensembles, it would be of interest to have a definite mathematical model in which the interaction processes which couple the individual systems were actually specified. A detailed study of the operation of this model would aid the understanding of the relevance of the probability assumptions which have been introduced.

It is necessary that the model chosen be capable of complete energy sharing between all the individual systems, even with the constraint that the total interaction energy be arbitrarily small.

A system of nonlinear coupled oscillators, such as the system of harmonic normal mode oscillators coupled by nonlinear forces introduced earlier in this chapter, might provide useful in the construction of a representation of a canonical ensemble. As will be seen in Chapter II, if the model is so chosen that the harmonic normal mode frequencies form an exactly-tuned set relative to the nonlinear coupling, such a system is capable of complete energy sharing among the harmonic normal modes for arbitrarily small nonlinear coupling energies. However, it must be noted

that a fairly large number of oscillators should be used to represent each subsystem of the ensemble.

Ergodicity and Nonequilibrium Behavior

In the field of nonequilibrium statistical mechanics, there is a long-standing desire to exhibit a mathematical model in terms of which the mechanisms of the approach to equilibrium can be studied.

The work of Poincaré (1957), Fermi (1923), and others (ter Haar 1954) indicates that a weakly nonlinear system of coupled oscillators will be ergodic, a property which is thought to be necessary for an approach to equilibrium. Unhappily, acceptance of ergodicity rules out almost any hope of obtaining a general analytic solution to the equations of motion of a system.

With these anticipated analytic difficulties in mind, the group of Fermi, Pasta and Ulam (1955) used a computer to provide numerical solutions for the harmonic normal mode energies of some simple weakly nonlinear coupled oscillator systems, for specific sets of initial conditions, as a first probe into the behavior of such systems.

The systems studied were found not only to be nonergodic, but failed to share appreciable energy except among several adjacent harmonic normal modes.

The reason for the failure of these particular systems to share appreciable energy was indicated recently by Ford (1961) to be a lack of tuning of the harmonic normal mode frequencies. A detailed explanation of this situation will be presented in Chapter II of this thesis.

Existing perturbation theories applied to some new appropriately tuned nonlinear systems by Ford indicated that even some completely energy-

sharing weakly nonlinear coupled oscillator systems retain the property of nonergodicity.

This indication of nonergodicity has encouraged an effort to develop the mathematical analysis of such energy-sharing nonlinear coupled oscillator systems. These systems may be too simple to be physically realistic, but it is felt that an understanding of such systems is a necessary first step in the development of a theory of the approach to equilibrium. It may well be that more complex physically realistic systems will be susceptible to the type of analysis developed here, and yet will exhibit the properties of ergodicity indicated by Poincaré, Fermi, and others.

Celestial Systems and Constants of the Motion

In present-day astrophysics, there is a growing interest in the study of constants of the motion for collective dynamical systems, such as systems of galaxies.

For example, for axially symmetric potentials, two constants of the motion, the total energy and the total angular momentum, are always known. Much effort has been recently expended in the search for additional constants of the motion (Contopoulos 1963), but to date the results are not entirely satisfactory (Hénon and Heiles 1964).

The detailed specifications of intergalactic potentials seem to be even less developed than specifications of the interatomic potentials in crystal lattice dynamics. Nonlinear forces act in addition to linear forces, and dissociation is possible. But the nature and strength of the nonlinear couplings for physically interesting cases are not apparently well known.

It is well known in classical mechanics (Whittaker 1944) that if N

constants of the motion are known for any system having N degrees of freedom, then it is possible to formally obtain the general solution of the system. Whittaker provides a formal method for constructing the general solution. Equivalently, given the general solution of a system, it should be possible to construct the corresponding constants of the motion.

In the nonlinear celestial problem, the difficulty of determining all the constants of the motion is thus indicated by the difficulty of finding general solutions to the nonlinear equations of motion. Any new information about either constants of the motion or general solutions of energy-sharing nonlinear systems should be of interest.

The specification of the constants of the motion for many-body dynamical systems is a problem of great general interest in present-day physics (Bohm and Carmi 1964). General conclusions about the overall collective properties of a many-body system may be drawn from a knowledge of the constants of the motion. Basic symmetry properties of a physical system are generated by the constants of the motion (Goldstein 1959), so that a knowledge of the constants of the motion gives a very fundamental description of a system.

In Chapter VI of this thesis, a method of analytic solution will be developed for energy-sharing nonlinear coupled oscillator systems. Possession of a form of general solution should lead to the knowledge of the form of N constants of the motion for a given nonlinear system. However, this has not been accomplished in the present work. The sole result obtained is the identification of one additional constant of the motion for nonlinear systems, in a form which has been known for many years.

Other Systems of Interest

The formulation and analysis of the nonlinear coupled oscillator systems of this thesis are done in a general manner, and there are numerous applications to physical problems other than those mentioned in detail above.

The problem of stability of accelerator orbits involves nonlinear couplings due to minor misalignments and equipment misadjustments (Hagedorn 1956). The crucial cases correspond to the exactly-tuned nonlinear oscillator systems in which long-term sharing of energy occurs even with arbitrarily weak couplings.

A similar problem exists in the containment of plasmas, in which weak nonlinear perturbations cause instabilities which have proved difficult to analyze.

The dissociation of chemical molecules is basically a nonlinear phenomenon. Recent studies of unimolecular reactions (Bunker 1962; Thiele and Wilson 1961) indicate that the dissociations of simple structures such as carbon dioxide may be successfully explained in terms of nonlinear coupled oscillators.

In hydrodynamics, certain couplings are present which result in large-scale phenomena of deep-water ocean waves (Lamb 1932; Longuet-Higgins 1962) for which the nonlinear systems studied here are relevant models.

Finally, the action of finite-amplitude sound waves in crystals involves a nonlinear coupled oscillator model, needed to explain the appearance of higher harmonics due to the structure of the crystal.

It has been the aim of Chapter I to provide an introduction to the reasons for the study of energy-sharing nonlinear coupled oscillator

systems. The general effect of the addition of nonlinear terms on energy sharing has been indicated, and some physical systems for which the nonlinear system is a relevant model have been described.

In the following chapter, the relationship between tuning of the harmonic normal mode frequencies and sharing of energy between harmonic normal modes will be investigated. This will serve as a prelude for the actual formulation of a particular model, in Chapter III.

CHAPTER II

THE RELATIONSHIP BETWEEN ENERGY SHARING AND TUNING FOR COUPLED OSCILLATOR SYSTEMS

It has been demonstrated in some early computer studies of simple nonlinear coupled oscillator systems (Fermi, Ulam and Pasta 1955) that not every system having a complete set of nonlinear couplings among all its oscillators is capable of complete sharing of energy among all of these oscillators.

Therefore, before attempting to formulate a nonlinear coupled oscillator model which is capable of complete energy sharing, it is necessary to investigate the circumstances which control the sharing of energy in oscillator systems.

In the first section of this chapter, some simple one-dimensional systems of coupled oscillators will be introduced. In the second section, the relationship between energy sharing and tuning of the oscillator frequencies will be examined, with the aid of action-angle formalism. The last section will present some computer results which can be used to check the validity of the conclusions reached in the first part of the chapter.

Some Simple One-Dimensional Systems

For simplicity, most of the remaining portion of this thesis will employ one-dimensional models for purposes of illustrating characteristics and methods of solution for various coupled oscillator systems. These

one-dimensional models contain many but not always all of the essential features of the physical systems to which they are relevant.

This approach of simplification is taken in order to isolate the analysis of the more difficult aspects of the physical problems discussed in Chapter I from the many other aspects which are already well understood. It is felt that if some of these more difficult aspects can be better understood, then the overall methods of attack for actual physical systems can be improved.

In the first subsection, a simple one-dimensional perfect harmonic system will be introduced. There is no mechanism for sharing energy between the normal modes of this perfect harmonic system, so in the second and third subsections additional sets of forces will be included so as to couple the normal modes.

The nonlinear coupling introduced in the second subsection provides an interaction between harmonic normal modes in a manner which has already been described briefly in Chapter I. The additional linear coupling introduced in the third subsection serves the same purpose, although in an artificial manner. The linear coupled oscillator system of the third subsection will be used in this thesis as an easily solvable companion model, the analysis of which will be used to guide the analysis of the more difficult nonlinear coupled oscillator system of the second subsection.

Perfect Harmonic System

For a perfect harmonic system in which the normal modes are completely decoupled, consider a finite one-dimensional array of $N + 2$ equally-spaced identical point masses m, connected by nearest-neighbor linear massless springs having identical force constants c.

Let the dimensionless independent variable, "oscillator time," be defined as $t = (\frac{c}{m})^{\frac{1}{2}} t'$, where t' is actual time in seconds. The dot notation will be used to denote differentiation with respect to this normalized oscillator time t. Thus frequencies are also dimensionless in this normalized description.

Denote the longitudinal displacement of the k^{th} particle from its equilibrium position Q_k, and the corresponding momentum $P_k = m \dot{Q}_k$. Let the ends of the array be fixed, so that $Q_0 = Q_{N+1} = 0$ and there are N moving particles.

(The boundary conditions used here are rather arbitrary, and are chosen for convenience. In calculations of actual physical systems, other types of boundary conditions may be used, such as the periodic boundary conditions used in dealing with crystal lattices. In actual applications, for a sufficiently large number of particles, the particular type of boundary conditions assumed can have little effect on the overall behavior of the system (Born and Huang 1954).)

The hamiltonian for the system described is

$$H = \tfrac{1}{2} \sum_{k=1}^{N} P_k^2 + \tfrac{1}{2} \sum_{k=0}^{N} (Q_{k+1} - Q_k)^2, \qquad (1)$$

where $Q_0 = Q_{N+1} = 0$. Hamilton's equations combine to give the individual equations of the motion for the system,

$$\ddot{Q}_k = \dot{P}_k = (Q_{k-1} - 2Q_k + Q_{k+1}), \qquad (2)$$

where $k = 1,2,\ldots,N$, and $Q_0 = Q_{N+1} = 0$; or, in matrix form,

$$\begin{bmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \\ \ddot{Q}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & \cdots \\ 1 & -2 & 1 & \\ 0 & 1 & -2 & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ \vdots \end{bmatrix} \tag{3}$$

For the remainder of this thesis, unless otherwise noted, an individual equation of the Equation (2) variety will be assumed valid for subscripts in the range from 1 through N, without the explicit appearance of the notation $k = 1,\ldots,N$.

The transformation which diagonalizes Equation (3) is

$$Q_k = \left(\frac{2}{N+1}\right)^{\frac{1}{2}} \sum_{l=1}^{N} \sin \frac{kl\pi}{(N+1)} q_l, \tag{4}$$

and the normal mode equations of motion are, in matrix form,

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} -\omega_1^2 & 0 & 0 & \cdots \\ 0 & -\omega_2^2 & 0 & \\ 0 & 0 & -\omega_3^2 & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \end{bmatrix} \tag{5}$$

where the normal mode frequencies are given by

$$\omega_k = 2 \sin \frac{k\pi}{2(N+1)}. \tag{6}$$

The individual normal mode equations,

$$\ddot{q}_k = \dot{p}_k = -\omega_k^2 q_k, \tag{7}$$

have solutions of the form

$$q_k = A_k \cos(\omega_k t + \theta_k), \tag{8}$$

where A_k and θ_k are arbitrary constants to be fixed by the initial conditions. The normal mode hamiltonian is

$$H = \tfrac{1}{2} \sum_{k=1}^{N} (p_k^2 + \omega_k^2 q_k^2). \tag{9}$$

The individual normal mode energies,

$$E_k = \tfrac{1}{2}(p_k^2 + \omega_k^2 q_k^2), \tag{10}$$

form a set of N constants of the motion.

Nonlinear Couplings between Harmonic Normal Modes

The perfect harmonic system just introduced cannot share energy among its normal modes, because the normal modes are completely decoupled. However, as indicated in the first chapter, couplings between the normal modes can be obtained by introducing an additional set of nonlinear forces in the system. For simplicity, the added nonlinear terms will be restricted to cubic coupling energy terms in the hamiltonian. As mentioned in the discussion of various applications to physical systems in Chapter I, cubic nonlinearities may be sufficient to explain many of the physical phenomena which are of interest.

In keeping with the simplicity of the nearest-neighbor particle

couplings of the perfect harmonic system just considered, Equation (1), let the cubic energy coupling also be of nearest-neighbor form in the nonlinear system:

$$H = \frac{1}{2}\sum_{k=1}^{N} P_k^2 + \frac{1}{2}\sum_{k=0}^{N}(Q_{k+1}-Q_k)^2 - \alpha\frac{1}{3}\sum_{k=0}^{N}(Q_{k+1}-Q_k)^3, \qquad (11)$$

where α is a parameter which controls the overall strength of the added nonlinear coupling. The corresponding particle equations of motion are

$$\ddot{Q}_k = (Q_{k-1}-2Q_k+Q_{k+1}) - \alpha\left[(Q_{k+1}-Q_k)^2-(Q_k-Q_{k-1})^2\right]. \qquad (12)$$

For this nonlinearly coupled system, the luxury of a transformation to normal mode variables which completely decouple the system is not available. At best, the harmonic normal mode transformation which decouples only that linear portion of the system which remains when $\alpha = 0$ may be made.

When the harmonic normal mode transformation

$$Q_k = \left(\frac{2}{N+1}\right)^{\frac{1}{2}}\sum_{l=1}^{N}\sin\frac{kl\pi}{(N+1)}\,q_l \qquad (13)$$

is applied to the nonlinear system governed by the hamiltonian of Equation (11), the resulting hamiltonian is of the form

$$H = \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2+\omega_k^2 q_k^2\right) + \alpha\frac{1}{3}\sum_{i,j,k}^{N} C_{ijk}\,q_i q_j q_k, \qquad (14)$$

where the general explicit form of the third-rank tensor elements C_{ijk} will be exhibited later, in conjunction with the computer studies of Chapter VII.

The general nature of the coefficients C_{ijk} is such that there can be energy exchange between widely separated harmonic normal modes, as well as between adjacent ones. Thus the nonlinear coupling chosen provides long-range as well as short-range interactions. In the quantum-mechanical description, the cubic coupling energy terms represent three-phonon interactions.

Linear Couplings between Harmonic Normal Modes

It is also possible to provide couplings between the normal modes of the perfect harmonic system of the first subsection, by introducing an additional set of linear forces in that system. This is a rather artificial construction, not meant to have physical relevance, but instead designed to provide a formal system which is easy to solve and which has many similarities both of behavior and analysis to the nonlinear coupled oscillator system of the second subsection.

In order to obtain a simple result in harmonic normal mode variables, an additional set of linear forces may be introduced in particle variables by assuming a particle hamiltonian of the form

$$H = \tfrac{1}{2}\sum_{k=1}^{N} P_k^2 + \tfrac{1}{2}\sum_{k=0}^{N} (Q_{k+1}-Q_k)^2 - \alpha \sum_{k=1}^{N} \cos\frac{k\pi}{(N+1)}\, Q_k^2, \qquad (15)$$

so that the particle equations of motion are

$$\ddot{Q}_k = (Q_{k-1}-2Q_k+Q_{k+1}) - \alpha 2\cos\frac{k\pi}{(N+1)}\, Q_k. \qquad (16)$$

When the harmonic normal mode transformation of Equation (13) is applied to the linear system governed by the hamiltonian of Equation (15), the resulting hamiltonian is of the form

$$H = \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2\right) - \alpha\sum_{k=0}^{N} q_k q_{k+1}, \tag{17}$$

and the corresponding harmonic normal mode equations are

$$\ddot{q}_k = -\omega_k^2 q_k + \alpha\left(q_{k-1} + q_{k+1}\right). \tag{18}$$

Thus the added couplings of the particles to their equilibrium positions, the linear forces $\alpha\ 2\cos\frac{k\pi}{(N+1)}\,Q_k$ in Equations (16), result in a nearest-neighbor coupling of the harmonic normal modes, as shown by Equations (18).

The term "harmonic normal modes" was used in conjunction with the nonlinear system to denote the set of variables obtained when the transformation which decoupled the linear portion of the system was applied to the whole system. The result was a description in terms of nonlinear couplings between harmonic normal mode oscillators.

In the present subsection, the same terminology will be formally used, so that a description is obtained in which there are linear couplings between harmonic normal mode oscillators. The "harmonic normal modes" are just the normal modes of the perfect harmonic system of the first subsection. This whole description of the linear coupled system is rather artificial, since in principle another different transformation could have been applied to the linear coupled particle system, Equation (15), to

completely decouple it to begin with. However, as mentioned, the formal construction of the linear coupled oscillator system of this subsection will provide some useful analogies to nonlinear coupled oscillator systems.

In this section, models in which there are couplings between the normal modes of the perfect harmonic system have been constructed. However, there is no guarantee that the presence of these couplings will automatically make possible complete sharing of energy among the normal modes, even under the most favorable conditions. In the next section, the mechanism of energy sharing will be investigated in some detail for the coupled oscillator systems introduced in this section.

Relation between Energy Sharing and Tuning

There exists a general relation between the sharing of energy among harmonic normal modes and the tuning of the harmonic normal mode uncoupled frequencies, for all systems of coupled oscillators. This relation will be examined in the present section, along with the concept of tuning.

In the first subsection, tuning of the linear coupled oscillator system will be studied, taking advantage of the fact that the general solution of the linear system is relatively easily obtained. In the second subsection, the same problem of tuning of the linear system will be studied using action-angle formalism; these results will be compared with those of the first subsection. In the final subsection, the action-angle formalism will be used to study the tuning of the nonlinear coupled oscillator system.

Tuning of the Linear System

Since nearest-neighbor couplings between all of the harmonic normal modes are present in the linear system governed by the hamiltonian of Equation (17), it might be expected that appreciable energy sharing among all of the harmonic normal modes could occur, at least for some sets of initial conditions.

However, this is not necessarily true. The maximum attainable amount of energy sharing between harmonic normal modes depends crucially upon the relations which exist between the set of uncoupled normal mode frequencies and the coupling parameter α, for linear coupled oscillator systems.

In order to see this, consider a linear coupled oscillator system governed by a hamiltonian of the form

$$H = \tfrac{1}{2}\sum_{k=1}^{N}\left(p_k^2+\omega_k^2 q_k^2\right) - \alpha\sum_{k=0}^{N} q_k q_{k+1}, \tag{19}$$

in which the uncoupled harmonic normal mode frequencies ω_k are allowed to be free variables. This system differs from that of Equation (17) in that the frequencies in the latter are fixed, with values given by Equations (6) of this chapter. The corresponding harmonic normal mode equations of motion are

$$\ddot{q}_k = -\omega_k^2 q_k + \alpha\left(q_{k-1}+q_{k+1}\right), \tag{20}$$

which are the same as Equations (18) except that the frequencies ω_k are free variables.

Example for $N = 2$. The important features of the general solution

for N linear coupled oscillators may be illustrated by a solution for N = 2. Based on this solution, some general conclusions about solutions for N oscillators may be drawn.

The equations of motion for the two-oscillator linear system with variable frequencies are

$$\ddot{q}_1 = -\omega_1^2 q_1 + \alpha q_2 ; \tag{21a}$$

$$\ddot{q}_2 = -\omega_2^2 q_2 + \alpha q_1 . \tag{21b}$$

A solution of the form

$$q_k = A_k \cos(\Omega_k t + \theta_k) \tag{22}$$

such as might be tried as a generalization of the solution which is valid when $\alpha = 0$, is not valid when $\alpha \neq 0$ if $\omega_2 \neq \omega_1$. Instead, a general solution of the form

$$q_k = \sum_{l=1}^{N} A_{kl} \cos(\Omega_l t + \theta_l) \tag{23}$$

must be assumed.

Substitution of this assumed solution into the equations of motion, Equations (21), produces the following linear eigenvalue-eigenvector problem in Ω_l, $\{A_{kl}\}$ for $l = 1,2$:

$$\begin{bmatrix} (\omega_1^2 - \Omega_l^2) & -\alpha \\ -\alpha & (\omega_2^2 - \Omega_l^2) \end{bmatrix} \begin{bmatrix} A_{1l} \\ A_{2l} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{24}$$

The eigenvalues obtained by solving the secular equation are

$$\Omega_1^2 = \tfrac{1}{2}(\omega_1^2+\omega_2^2) - \tfrac{1}{2}\left[(\omega_2^2-\omega_1^2)^2+4\alpha^2\right]^{\frac{1}{2}}; \tag{25a}$$

$$\Omega_2^2 = \tfrac{1}{2}(\omega_1^2+\omega_2^2) + \tfrac{1}{2}\left[(\omega_2^2-\omega_1^2)^2+4\alpha^2\right]^{\frac{1}{2}}, \tag{25b}$$

and the eigenvector components are related by

$$A_{21} = \frac{\alpha}{(\omega_2^2-\Omega_1^2)} A_{11}; \tag{26a}$$

$$A_{12} = \frac{\alpha}{(\omega_1^2-\Omega_2^2)} A_{22}, \tag{26b}$$

where the A_{kk} are taken as arbitrary, to be fixed along with the θ_l by the initial conditions on the q_k and p_k.

Two extremes of "tuning" will be recognized. The system will be regarded as strongly detuned if $|\omega_2 - \omega_1| >> \alpha$, and will be regarded as strongly tuned if $|\omega_2 - \omega_1| << \alpha$.

Suppose the system is strongly detuned: $|\omega_2 - \omega_1| >> \alpha$, or, if the frequencies are taken to be roughly unity in order of magnitude, $|\omega_2^2 - \omega_1^2| >> \alpha$. Then the eigenvalues of Equations (25) are given approximately by

$$\Omega_1^2 \approx \omega_1^2 - \frac{\alpha^2}{(\omega_2^2-\omega_1^2)}; \tag{27a}$$

$$\Omega_2^2 \approx \omega_2^2 + \frac{\alpha^2}{(\omega_2^2-\omega_1^2)}, \tag{27b}$$

and the eigenvector components of Equations (26) are given approximately by

$$A_{12} \approx -\frac{\alpha}{(\omega_2^2-\omega_1^2)} A_{22} \ll A_{22}\,; \tag{28a}$$

$$A_{21} \approx +\frac{\alpha}{(\omega_2^2-\omega_1^2)} A_{11} \ll A_{11}\,. \tag{28b}$$

Let the initial conditions be $q_k(0) = A_k$, $p_k(0) = 0$, for $k = 1,2$. In the square matrix which relates the eigenvector components $\{A_{kl}\}$ to the initial conditions $\{A_k\}$, the off-diagonal elements are small for the strongly detuned case:

$$\begin{bmatrix} 1 & -\alpha/(\omega_2^2-\omega_1^2) \\ \alpha/(\omega_2^2-\omega_1^2) & 1 \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{22} \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}. \tag{29}$$

The result for the eigenvector components for the strongly detuned linear system is

$$A_{11} \approx A_1 + \frac{\alpha}{(\omega_2^2-\omega_1^2)} A_2\,; \quad A_{12} \approx -\frac{\alpha}{(\omega_2^2-\omega_1^2)} A_2\,; \tag{30a}$$

$$A_{22} \approx A_2 - \frac{\alpha}{(\omega_2^2-\omega_1^2)} A_1\,; \quad A_{21} \approx \frac{\alpha}{(\omega_2^2-\omega_1^2)} A_1\,, \tag{30b}$$

so that one term dominates in each solution for the q_k in Equations (23).

There can be no appreciable sharing of energy in this strongly detuned system, for any set of initial conditions. The time dependence of the energy of the first harmonic normal mode oscillator is typical:

$$E_1 = \tfrac{1}{2}\omega_1^2\left[A_1^2 + \mathcal{O}\left(\frac{\alpha}{(\omega_2^2-\omega_1^2)}\right) A_1 A_2 \cos(\Omega_2-\Omega_1)t\right]. \tag{31}$$

In contrast with this strongly detuned case of a coupled system which cannot share appreciable energy, consider the strongly tuned case. Suppose the uncoupled harmonic normal mode frequencies satisfy the relation $|\omega_2 - \omega_1| \ll \alpha$, so that $|\omega_2^2 - \omega_1^2| \ll \alpha$. Then the eigenvalues of Equations (25) are given approximately by

$$\Omega_1^2 \approx \tfrac{1}{2}(\omega_1^2+\omega_2^2) + \alpha ; \tag{32a}$$

$$\Omega_2^2 \approx \tfrac{1}{2}(\omega_1^2+\omega_2^2) - \alpha , \tag{32b}$$

and the eigenvector components of Equations (26) are given approximately by

$$A_{12} \approx - \frac{\alpha}{\frac{1}{2}(\omega_2^2-\omega_1^2)+\alpha} A_{22} ; \tag{33a}$$

$$A_{21} \approx + \frac{\alpha}{\frac{1}{2}(\omega_2^2-\omega_1^2)+\alpha} A_{11} . \tag{33b}$$

For convenience in describing the strongly tuned case, a detuning parameter, s, may be introduced. Denote the frequencies $\omega_1 = \omega$ and $\omega_2 = (1 + s)\omega$, so that the condition for strong tuning, $|\omega_2 - \omega_1| \ll \alpha$, becomes $s \ll \alpha$. The eigenvalues given by Equations (32) become

$$\Omega_1^2 \approx \omega^2(1+s+\alpha) ; \tag{34a}$$

$$\Omega_2^2 \approx \omega^2(1+s-\alpha) , \tag{34b}$$

and the eigenvectors given by Equations (33) become

$$A_{12} \approx -\left(1+\frac{s}{\alpha}\right) A_{22}; \tag{35a}$$

$$A_{21} \approx +\left(1+\frac{s}{\alpha}\right) A_{11}. \tag{35b}$$

In terms of the initial conditions used previously, the eigenvector components are

$$A_{11} \approx A_1 + (1+s)A_2; \qquad A_{12} \approx \left(1+2\frac{s}{\alpha}\right)A_1 - \left(1+\frac{s}{\alpha}\right)A_2; \tag{36a}$$

$$A_{22} \approx -(1+s)A_1 + A_2; \qquad A_{21} \approx \left(1+\frac{s}{\alpha}\right)A_1 + \left(1+2\frac{s}{\alpha}\right)A_2. \tag{36b}$$

In this strongly tuned system, each term in the general solution, Equations (23), is of roughly the same order of magnitude. Therefore there can be almost complete energy sharing for certain sets of initial conditions. Keeping only lower order terms, the energy of the first oscillator is typical:

$$E_1 \approx \omega^2(A_1^2+A_2^2) + \omega^2(A_1^2-A_2^2)\cos(\Omega_2-\Omega_1)t. \tag{37}$$

Examination of Equation (37) shows that complete sharing of energy is obtained either for the initial conditions $A_1 \neq 0$, $A_2 = 0$ or for the initial conditions $A_1 = 0$, $A_2 \neq 0$. At the other extreme, if the initial conditions are such that either $A_2 = A_1$ or $A_2 = -A_1$, then there is no appreciable energy sharing. Any intermediate amount of energy sharing is available for an appropriate intermediate choice of initial conditions.

General Case for N Oscillators. A generalization of the results

just obtained for two oscillators may easily be made, for the case of N linear coupled oscillators.

The N equations of motion, Equations (20), may be solved by assuming a general solution of the form given by Equation (23). Substitution of this assumed solution into the equations of motion produces the following linear eigenvalue-eigenvector problem in Ω_ℓ, $\{A_{k\ell}\}$ for $\ell = 1,2,\ldots,N$:

$$\begin{bmatrix} (\omega_1^2-\Omega_\ell^2) & -\alpha & 0 & \cdots \\ -\alpha & (\omega_2^2-\Omega_\ell^2) & -\alpha & \\ 0 & -\alpha & (\omega_3^2-\Omega_\ell^2) & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} A_{1\ell} \\ A_{2\ell} \\ A_{3\ell} \\ \vdots \end{bmatrix} = 0. \tag{38}$$

For the strongly detuned case, in which all adjacent uncoupled frequencies ω_k differ by much more than the coupling parameter α, the square matrix in Equation (38) is essentially diagonal. All but one of the elements of the main diagonal are of the same order of magnitude, since the coupled and uncoupled frequencies are approximately equal in the strongly detuned case:

$$\Omega_k^2 = \omega_k^2 + O(\alpha^2). \tag{39}$$

One element of the main diagonal, say the k^{th}, is much smaller than the rest. This leads to a situation in which one eigenvector element, A_{kk}, is much larger than the rest. Thus one term dominates in each solution for the q_k, in Equation (23), in much the same way as for

the $N = 2$ system. For the same reasons as for the two-oscillator system, the strongly detuned linear coupled system of N oscillators cannot share appreciable energy.

On the other hand, for the strongly tuned case in which all of the uncoupled frequencies are very nearly equal, the square matrix in Equation (38) is essentially tridiagonal, with all elements of the order of α. The eigenvector components A_k are then all approximately equal, for a given ℓ and $k = 1,2,\ldots,N$. This leads to the possibility of complete energy sharing for selected sets of initial conditions.

In this section, it has been shown by use of general analytic solutions that tuning of the uncoupled harmonic normal mode frequencies is necessary in order that a linear coupled oscillator system be capable of sharing appreciable energy, even under the most favorable conditions. In the case of the linear system considered here, the coupling between harmonic normal modes is such that a set of frequencies which are all equal constitutes a tuned set.

The problem of obtaining a general analytic solution when the frequencies are free variables is not as easily solved for the nonlinear coupled oscillator system as it is for the linear coupled oscillator system considered here. Therefore, the examination of the relation between energy sharing and tuning for the linear system will be conducted over again in action-angle formalism in the next subsection. The action-angle formalism will be just as useful for nonlinear systems as it is for linear systems. The validity of the results obtained in the next subsection may be checked by a comparison with the results just obtained.

Use of Action-Angle Formalism for Tuning

The mechanism for energy sharing in a coupled oscillator system, which involves a correlation between the nature of the couplings and the values of the uncoupled frequencies, may be summarized as "internal resonance" (Ford 1961). For a given set of couplings, a coupled oscillator system is capable of sharing energy only if internal resonances are present; that is, only if the uncoupled frequencies are "tuned" relative to the couplings.

A convenient formalism for the display of the general tuning relations available for a given set of couplings is that obtained by the use of action-angle variables. As will be seen, the action variables are closely related to the harmonic normal mode energies. Thus action-angle formalism gives direct information about energy sharing without the need of investigating the behavior of the position and momentum variables.

To illustrate the use of action-angle variables, consider the two-oscillator linear coupled system governed by the hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2) - \alpha q_1 q_2, \tag{40}$$

which is the hamiltonian of Equation (19) for the case $N = 2$.

Make a canonical transformation to action-angle variables according to

$$q_k = \left(\frac{2J_k}{\omega_k}\right)^{\frac{1}{2}} \cos\varphi_k; \qquad p_k = -(2\omega_k J_k)^{\frac{1}{2}} \sin\varphi_k, \tag{41}$$

so that the hamiltonian of Equation (40) becomes

$$H = \omega_1 J_1 + \omega_2 J_2 - \alpha \left(\frac{J_1}{\omega_1}\right)^{\frac{1}{2}} \left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}} \left[\cos(\varphi_1+\varphi_2) + \cos(\varphi_1-\varphi_2)\right]. \quad (42)$$

Note that the energy of the k^{th} harmonic normal mode (neglecting that part associated with the coupling term) is given by

$$E_k = \tfrac{1}{2}(p_k^2 + \omega_k^2 q_k^2) = \omega_k J_k, \quad (43)$$

so that the action variables and harmonic normal mode energies are practically the same thing.

The equations of motion in action-angle variables are

$$\dot{J}_1 = -\alpha \left(\frac{J_1}{\omega_1}\right)^{\frac{1}{2}} \left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}} \left[\sin(\varphi_1+\varphi_2) + \sin(\varphi_1-\varphi_2)\right]; \quad (44a)$$

$$\dot{J}_2 = -\alpha \left(\frac{J_1}{\omega_1}\right)^{\frac{1}{2}} \left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}} \left[\sin(\varphi_1+\varphi_2) - \sin(\varphi_1-\varphi_2)\right]; \quad (44b)$$

$$\dot{\varphi}_1 = \omega_1 + \alpha \tfrac{1}{2} \left(\frac{1}{\omega_1 J_1}\right)^{\frac{1}{2}} \left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}} \left[\cos(\varphi_1+\varphi_2) + \cos(\varphi_1-\varphi_2)\right]; \quad (44c)$$

$$\dot{\varphi}_2 = \omega_2 + \alpha \tfrac{1}{2} \left(\frac{1}{\omega_2 J_2}\right)^{\frac{1}{2}} \left(\frac{J_1}{\omega_1}\right)^{\frac{1}{2}} \left[\cos(\varphi_1+\varphi_2) + \cos(\varphi_1-\varphi_2)\right]. \quad (44d)$$

These equations may be solved iteratively with some success, if the coupling parameter is small ($\alpha \ll 1$), and if the uncoupled frequencies (which are free parameters) are of the order of unity, by the following method. (The exact solution for the linear system is available by simpler methods.)

For the zeroth-order solution, use the solution

$$J_{k0} = \text{constant}; \quad \varphi_{k0} = \omega_k t + \theta_k \tag{45}$$

obtained when $\alpha = 0$. Substitution of these expressions on the right sides of Equations (44) leads to the first-order energy solutions

$$J_{11} = \alpha \left(\frac{J_{10}}{\omega_1}\right)^{\frac{1}{2}} \left(\frac{J_{20}}{\omega_2}\right)^{\frac{1}{2}} \left[\frac{\cos(\varphi_{10}+\varphi_{20})}{(\omega_1+\omega_2)} + \frac{\cos(\varphi_{10}-\varphi_{20})}{(\omega_1-\omega_2)}\right] + \text{constant}; \tag{46a}$$

$$J_{21} = \alpha \left(\frac{J_{10}}{\omega_1}\right)^{\frac{1}{2}} \left(\frac{J_{20}}{\omega_2}\right)^{\frac{1}{2}} \left[\frac{\cos(\varphi_{10}+\varphi_{20})}{(\omega_1+\omega_2)} - \frac{\cos(\varphi_{10}-\varphi_{20})}{(\omega_1-\omega_2)}\right] + \text{constant}; \tag{46b}$$

The constants of integration in Equations (46) may be recognized as the zeroth-order solutions, J_{k0}. Factor these terms out to obtain

$$J_{11} = J_{10}\left\{1 + \alpha \left(\frac{J_{20}}{J_{10}}\right)^{\frac{1}{2}} \frac{1}{(\omega_1\omega_2)^{\frac{1}{2}}} \left[\frac{\cos(\varphi_{10}+\varphi_{20})}{(\omega_1+\omega_2)} + \frac{\cos(\varphi_{10}-\varphi_{20})}{(\omega_1-\omega_2)}\right]\right\}; \tag{47a}$$

$$J_{21} = J_{20}\left\{1 + \alpha \left(\frac{J_{10}}{J_{20}}\right)^{\frac{1}{2}} \frac{1}{(\omega_1\omega_2)^{\frac{1}{2}}} \left[\frac{\cos(\varphi_{10}+\varphi_{20})}{(\omega_1+\omega_2)} - \frac{\cos(\varphi_{10}-\varphi_{20})}{(\omega_1-\omega_2)}\right]\right\}. \tag{47b}$$

Examination of Equations (47) shows that there will be little energy sharing unless the denominators of the time-varying terms are of the order of α. For, in a given energy equation, if a denominator of a time-varying term is of the order of α, then it is of the same order as the numerator, and the time-varying term makes a zeroth-order contribution to the energy. Here the assumption is made that J_{10} and J_{20} are of the same order of magnitude. If they are not, higher order iterations must be made in order for the crude sort of argument being made here to be valid.

The rapidly-varying terms in Equations (47), those having arguments $(\varphi_1 + \varphi_2)$, cannot contribute significantly to energy sharing for any set of positive values of ω_1 and ω_2. The only significant terms for purposes of energy sharing are the slowly-varying terms having arguments $(\varphi_1 - \varphi_2)$ in Equations (47). The denominators of these terms can be of order α if the uncoupled frequencies satisfy the condition $|\omega_1 - \omega_2| \leq \alpha$. Therefore action-angle formalism produces the same condition for tuning of the linear coupled two-oscillator system as did the analytic procedure of the subsection preceding this one.

In this thesis, the slowly-varying terms will be called "resonant" terms and the rapidly-varying terms will be called "nonresonant" terms, because of the different roles which these two kinds of terms play in providing internal resonances in coupled oscillator systems. Internal resonances are the mechanism of appreciable energy sharing.

It should be possible to study the energy-sharing properties of a given system by retaining only the slowly-varying "resonant" terms at the very start. In this scheme, the solution of the resulting "resonant" system could be obtained, and the effect of the "nonresonant" terms could then be obtained using standard perturbation methods. The effectiveness of such a procedure will be studied throughout the remainder of this thesis.

In this subsection, a crude but relatively simple action-angle formalism which lays bare the relation between energy sharing and tuning in systems of coupled oscillators has been presented in terms of a linear system. This action-angle formalism has been used to obtain the same tuning condition developed in the previous subsection for the linear

system. A simplification of the analysis of energy sharing has been suggested, in which only those slowly-varying terms which are principally responsible for energy sharing are retained in the system from the very beginning.

In the following subsection, action-angle formalism will be applied to a nonlinear coupled oscillator system.

Tuning of the Nonlinear System

Since a large number of long and short range couplings between all of the harmonic normal modes are present in the cubic nonlinear system governed by the hamiltonian

$$H = \frac{1}{2}\sum_{k=1}^{N}(p_k^2+\omega_k^2 q_k^2) + \alpha\frac{1}{3}\sum_{i,j,k}^{N} C_{ijk}\, q_i q_j q_k \qquad (48)$$

previously introduced in Equation (14) of this chapter, it might be expected that appreciable energy sharing among all of the harmonic normal modes could occur, at least for some sets of initial conditions.

However, as was the case for the linear system just examined, this is not necessarily true. Action-angle formalism may be employed to demonstrate that energy sharing in the nonlinear system depends crucially upon the tuning of the uncoupled harmonic normal mode frequencies.

To illustrate, consider a two-oscillator nonlinear system with cubic coupling terms, of the type introduced earlier in this chapter.

Let the original particle hamiltonian be of the form

$$H = \frac{1}{2}(P_1^2+P_2^2) + \frac{1}{2}\left[Q_1^2+(Q_2-Q_1)^2+Q_2^2\right] - \alpha\frac{1}{3}\left[Q_1^3+(Q_2-Q_1)^3+Q_2^3\right], \qquad (49)$$

which is the hamiltonian of Equation (11) of this chapter, for $N = 2$. Application of the harmonic normal mode transformation of Equation (13) to this system results in the harmonic normal mode hamiltonian

$$H = \tfrac{1}{2}(p_1^2+p_2^2+\omega_1^2 q_1^2+\omega_2^2 q_2^2) - \alpha\tfrac{1}{\sqrt{2}}(q_1^2 q_2 - q_2^3), \tag{50}$$

which is Equation (14) for $N = 2$. The variables q_1 and q_2 represent the in-phase and out-of-phase modes

$$q_1 = \tfrac{1}{\sqrt{2}}(Q_1+Q_2) \quad \text{and} \quad q_2 = \tfrac{1}{\sqrt{2}}(Q_1-Q_2), \tag{51}$$

and the uncoupled harmonic normal mode frequencies, given by Equation (6), are $\omega_1 = 1$ and $\omega_2 = \sqrt{3}$.

As with the linearly coupled system of the previous subsections, it may be necessary to replace the frequencies just specified by a tuned pair, in order to obtain a system capable of appreciable energy sharing. In order to do this in a general manner, the frequencies in Equation (50) will be taken to be free variables, of the order of unity in magnitude.

Changing the values of the uncoupled harmonic normal mode frequencies in this manner implies a changing of the linear particle couplings in the particle hamiltonian of Equation (49) as well. The general effect of tuning on the particle couplings will be considered in Chapter III.

Action-angle formalism will now be used to determine what choice of frequencies $\{\omega_k\}$ constitutes a tuned set for this system. That is, action-angle formalism will be used to determine what internal resonances are available due to this particular cubic nonlinear coupling.

In action-angle variables, the hamiltonian given by Equation (50)

becomes

$$H = \omega_1 J_1 + \omega_2 J_2 - \alpha \tfrac{1}{2} \left\{ \frac{J_1 J_2^{\frac{1}{2}}}{\omega_1 \omega_2^{\frac{1}{2}}} \left[\cos(2\varphi_1 + \varphi_2) + \cos(2\varphi_1 - \varphi_2) + 2\cos\varphi_2 \right] + \frac{J_2^{\frac{3}{2}}}{\omega_2^{\frac{3}{2}}} \left[\cos 3\varphi_2 + 3\cos\varphi_2 \right] \right\}. \tag{52}$$

As discussed in the previous subsection, tuning of the system of Equation (52) may be investigated by considering the corresponding resonant system. The possible slowly-varying terms in Equation (52) may be identified by making the temporary formal replacement $\varphi_k = \omega_k t$, and checking to see for what choices of frequencies ω_1 and ω_2 the arguments of certain of the coupling terms vanish.

In the case of the hamiltonian of Equation (52), the only possible slowly-varying "resonant" term is that having the argument $(2\varphi_1 - \varphi_2)$. This term is resonant if the uncoupled frequencies satisfy the relation $\omega_2 \approx 2\omega_1$.

Elimination of all the other rapidly-varying "nonresonant" terms from the coupling in the hamiltonian of Equation (53) results in the resonant hamiltonian

$$H = \omega_1 J_1 + \omega_2 J_2 - \alpha \tfrac{1}{2} \frac{J_1 J_2^{\frac{1}{2}}}{\omega_1 \omega_2^{\frac{1}{2}}} \cos(2\varphi_1 - \varphi_2). \tag{53}$$

The corresponding harmonic normal mode equations of motion for the resonant system are

$$\dot{J}_1 = -\alpha\left(\frac{J_1}{\omega_1}\right)\left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}}\sin(2\varphi_1-\varphi_2); \tag{54a}$$

$$\dot{J}_2 = +\alpha\left(\frac{J_1}{\omega_1}\right)\left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}}\tfrac{1}{2}\sin(2\varphi_1-\varphi_2); \tag{54b}$$

$$\dot{\varphi}_1 = \omega_1 - \alpha\tfrac{1}{2}\left(\frac{1}{\omega_1}\right)\left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}}\cos(2\varphi_1-\varphi_2); \tag{54c}$$

$$\dot{\varphi}_2 = \omega_2 - \alpha\tfrac{1}{4}\left(\frac{J_1}{\omega_1}\right)\left(\frac{1}{\omega_2 J_2}\right)^{\frac{1}{2}}\cos(2\varphi_1-\varphi_2). \tag{54d}$$

The zeroth-order solution of Equations (54) is

$$J_{k0} = \text{constant}; \qquad \varphi_{k0} = \omega_k t + \theta_k, \tag{55}$$

obtained when $\alpha = 0$. Substitution of this zeroth-order solution into the right sides of Equations (54) gives the first-order energy solution

$$J_{11} = J_{10}\left\{1 + \alpha\frac{1}{\omega_1}\left(\frac{J_{20}}{\omega_2}\right)^{\frac{1}{2}}\frac{\cos(2\varphi_1-\varphi_2)}{(2\omega_1-\omega_2)}\right\}; \tag{56a}$$

$$J_{21} = J_{20}\left\{1 - \alpha\left(\frac{J_{10}}{\omega_1}\right)\left(\frac{1}{\omega_2 J_{20}}\right)^{\frac{1}{2}}\tfrac{1}{2}\frac{\cos(2\varphi_1-\varphi_2)}{(2\omega_1-\omega_2)}\right\}. \tag{56b}$$

Examination of Equations (56) shows that there will be appreciable energy sharing only if the uncoupled harmonic normal mode frequencies ω_1 and ω_2 satisfy the general tuning condition

$$|2\omega_1 - \omega_2| \ll \alpha\left(\frac{J}{\omega}\right)^{\frac{1}{2}}, \tag{57}$$

where $(\frac{J}{\omega})^{\frac{1}{2}}$ represents a sort of typical amplitude of the oscillations of the system. If this condition is satisfied, then the denominators of the time-varying terms in Equations (56) will be of the order of $\alpha(\frac{J}{\omega})^{\frac{1}{2}}$, cancelling factors of the same order in the numerators. The time-varying terms are then zeroth-order contributions to the energy expressions and there can be appreciable energy sharing.

The tuning condition of Equation (57) may be rewritten in a more convenient form as

$$|2\omega_1 - \omega_2| \ll \alpha A , \tag{58}$$

where $A = \overline{|Q_{k+1} - Q_k|}$ is the "typical amplitude of oscillations" for a specific set of initial conditions.

The reason for the appearance of αA rather than just α in the tuning condition given by Equation (58) can be seen from a comparison of the hamiltonians of linear and cubic nonlinear systems. The particle hamiltonian for the linear system is

$$H = \tfrac{1}{2}\sum_{k=1}^{N} P_k^2 + \tfrac{1}{2}\sum_{k=0}^{N}(Q_{k+1}-Q_k)^2 - \alpha\sum_{k=0}^{N} 2\cos\frac{k\pi}{(N+1)}Q_k^2, \tag{59}$$

so that the coupling strength, defined in this thesis as the ratio of the average total coupling energy to the average total particle energy, is just the coupling parameter, α. But the particle hamiltonian for the cubic nonlinear system is

$$H = \tfrac{1}{2}\sum_{k=1}^{N} P_k^2 + \tfrac{1}{2}\sum_{k=0}^{N}(Q_{k+1}-Q_k)^2 - \alpha\tfrac{1}{3}\sum_{k=0}^{N}(Q_{k+1}-Q_k)^3, \tag{60}$$

so that the coupling strength for the cubic nonlinear system involves the "typical amplitude" A as well as α:

$$C_S = \frac{\alpha \frac{1}{3}(N+1)\overline{|Q_{k+1}-Q_k|^3}}{\frac{1}{2}(N+1)\overline{|Q_{k+1}-Q_k|^2}} = \alpha c A, \tag{61}$$

where c is approximately unity.

The exact specification of the typical amplitude A is a rather tricky matter, even for the two-oscillator system. In the general case of N oscillators, it is possible to give some approximation to A if there is complete energy sharing; but if there is not complete energy sharing, then the magnitude of A depends upon how many modes are participating, which depends upon the tuning of the system, which depends upon the magnitude of A. This constitutes a vicious circle, since none of the properties mentioned are known to begin with.

For purposes of the present thesis, it is sufficient to be content with a very rough estimate for the typical amplitude: assume that there is fairly widespread energy sharing in the system, so that there is some semblance of equipartition of energy. Then for systems in which the total coupling energy is small compared to the total energy, a rough approximation is

$$\frac{1}{2}\sum_{k=0}^{N}\overline{|Q_{k+1}-Q_k|^2} \approx \frac{1}{2}H \tag{62}$$

where the bar denotes a time average. The typical amplitude of the oscillations is then very roughly

$$A \approx \left(\frac{H}{N+1}\right)^{\frac{1}{2}}. \tag{63}$$

For systems in which there is only limited energy sharing, and for systems in which the total coupling energy is not small, the estimate supplied by Equation (63) is a poor one and must be modified. The main point to be made is that there is some sort of an amplitude factor which is crucial in the tuning relations for nonlinear systems, whether it is easily specified in all cases or not.

For further discussions of tuning, it is convenient to define a normalized tuning parameter, s. In terms of this tuning parameter, the tuning condition first given in Equation (58) may finally be written

$$s \equiv \frac{|2\omega_1 - \omega_2|}{C_s} \ll 1. \tag{64}$$

In addition to identifying sets of well-tuned frequencies by the condition of Equation (64), action-angle formalism can also be used to predict approximately the sharpness of tuning when the frequencies are not so well tuned. The use of normalization in the specification of the tuning parameter, s, sidesteps the practical difficulties of obtaining the typical amplitude, A, for a given system.

For the present case of the two-oscillator system, the first-order $\mathring{\varphi}_k$ Equations (54c) and (54d) can be approximated by

$$\dot{\varphi}_1 = \omega_1 - \alpha \tfrac{1}{2}\left(\frac{1}{\omega_1}\right)\left(\frac{J_2}{\omega_2}\right)^{\frac{1}{2}} \cos(2\theta_1 - \theta_2); \tag{65a}$$

$$\dot{\varphi}_2 = \omega_2 - \alpha \tfrac{1}{4}\left(\frac{J_1}{\omega_1}\right)\left(\frac{1}{\omega_2 J_2}\right)^{\frac{1}{2}} \cos(2\theta_1 - \theta_2), \tag{65b}$$

which is good enough if $\omega_2 \approx 2\omega_1$ and if only an approximate second-order energy solution is desired. The result of integrating Equations (65) is

$$\varphi_1 = (\omega_1 + \theta(C_s))t; \tag{66a}$$

$$\varphi_2 = (\omega_2 + \theta(C_s))t. \tag{66b}$$

Substitution of the first-order expressions given by Equations (56) and (66) on the right sides of Equations (54a) and (54b) for the energies, produces upon integration time-varying terms which have denominators of the form

$$(2\omega_1 - \omega_2) + \theta(C_s), \tag{67}$$

rather than denominators of the form $(2\omega_1 - \omega_2)$ as appeared in the first-order expressions for the energy, Equations (56).

This rather crude second-order result gives somewhat more information about the sharpness of the internal resonances than did the first-order result. If the condition given by Equation (64) is satisfied, then there can be complete energy sharing. If the frequencies are such that $|2\omega_1 - \omega_2| = \theta(C_s)$, then energy sharing should be cut about in half; in terms of the tuning parameter, this is the case $s = 1$.

A few simple calculations for other values of s lead to the normalized energy sharing curve presented in Figure 1. This is a universal curve, valid for all systems of two coupled oscillators. Of course, the definitions of the tuning parameters, s, are different for different types of couplings between oscillators.

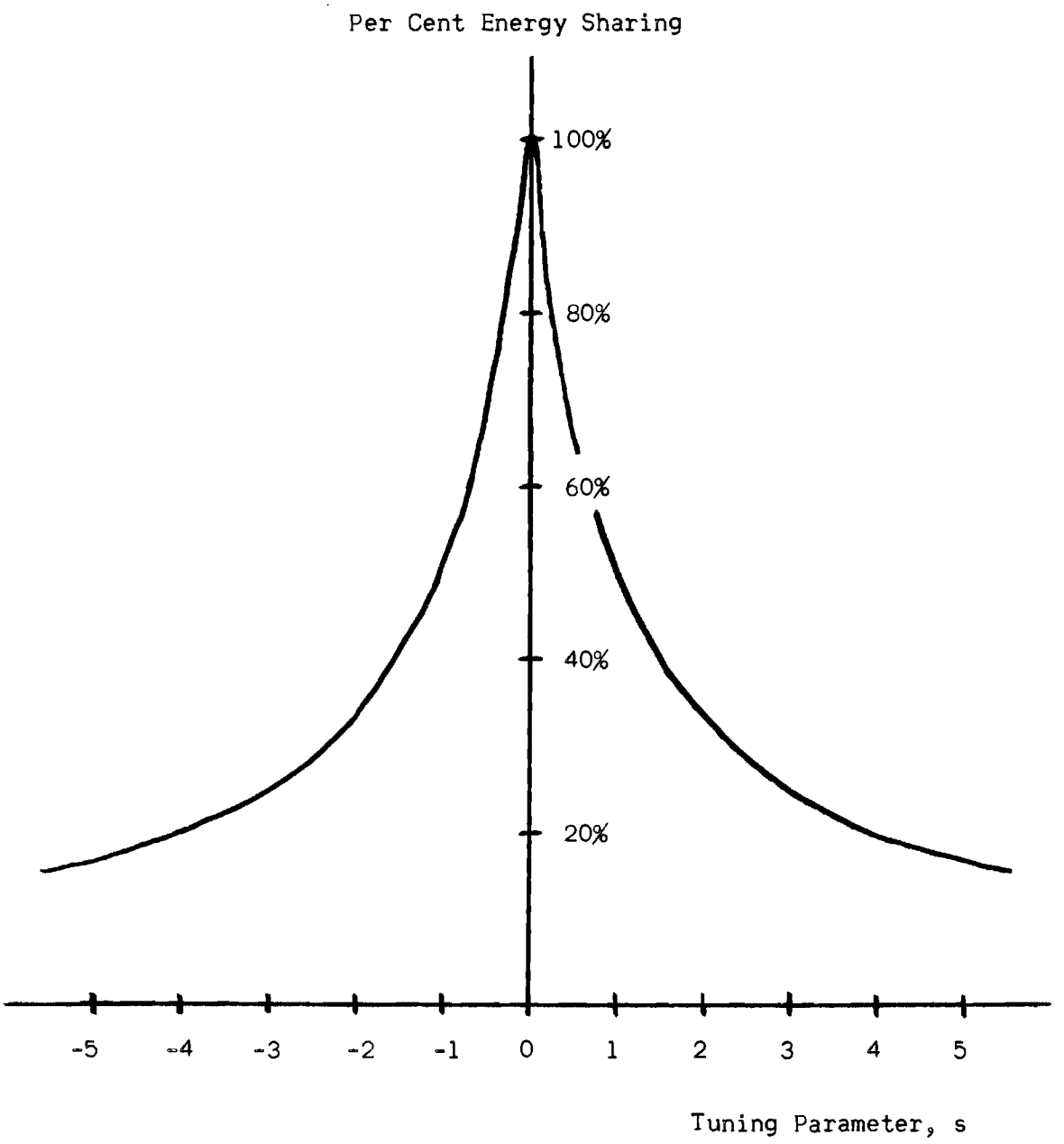

Figure 1. Predicted Sharpness of Tuning for Two Coupled Oscillators.

This resonance situation for the sharing of energy between two coupled oscillators is similar to the resonance situation encountered in a series RLC electrical circuit. In the electrical circuit, there is a tuning mechanism by which the values of the capacitance C and the inductance L may be varied to maximize the amount of power developed in the resistor R:

$$P = \frac{E^2 R}{R^2 + (\omega L - \frac{1}{\omega C})^2} . \tag{68}$$

Here the tuning parameter s corresponds to $(\omega L - \frac{1}{\omega C})^2$. The tuning curve for this electrical system is quite similar to the tuning curve of Figure 1 for two coupled oscillators.

The action-angle formalism for tuning, which has been illustrated in this subsection for a system of two nonlinear coupled oscillators, is easily generalized to deal with a system of N nonlinear coupled oscillators. For the particular cubic coupling used in systems of the type given by Equation (14), it is found that an exactly-tuned set of frequencies is given by the relation

$$\omega_k = k\omega = k \frac{2}{(N+1)} . \tag{69}$$

That is, an exactly-tuned set of frequencies for a cubic-coupled system is one in which the uncoupled harmonic normal mode frequencies are commensurable, having a one-to-one correspondence with the integers $1,2,3,\ldots$.

In this section, some rather basic studies of the relation between energy sharing and tuning for systems of coupled oscillators have been made. A method of finding the available internal resonances, which

constitute the mechanism for energy sharing, has been presented in terms of action-angle formalism. In addition to determination of a tuned set of frequencies for a given coupling, the method has provided an approximate indication of the sharpness of tuning, as a function of the coupling strength. For nonlinear systems, the coupling strength involves an average amplitude as well as the coupling parameter, α. This means that the tuning of a nonlinear system is in general a function of the level of amplitudes supplied by the initial conditions, in addition to the dependence upon the relations between the uncoupled frequencies.

In the next section, some relevant computer results will be discussed. The studies to be presented deal principally with nonlinear coupled oscillator systems in which the coupling energies are small. The results of the predictions supplied by the action-angle formalism of this section will be checked against some of the actual results for these various systems. However, it must be noted that the predictions which involve a calculation of the typical amplitude of oscillations cannot be taken too seriously in the next section, due to the crudeness of the approximations involved in calculations of the type given by Equation (63).

Computer Studies of Energy Sharing

It is of interest to check the general conclusions about the relation between energy sharing and tuning, which have been obtained in the preceding section, by comparing them with some actual computations for some appropriate nonlinear coupled oscillator systems.

In a sense, the computer is used in these studies as a research tool to provide "experimental" data, which may be used to establish the validity of certain theoretical results and to aid the development of

further theory.

In the first subsection, some of the results of an early computer study by Fermi, Ulam and Pasta will be presented and discussed. Some more recent work by Jackson will also be included. In the final subsection, some computer studies of the effect which tuning has on energy sharing will be discussed, for cubic nonlinear systems of two oscillators and five oscillators.

Computer Studies by Fermi, Pasta and Ulam (FPU) and Jackson

An early numerical investigation of the nature of sharing of energy between harmonic normal modes for broken quadratic, cubic, and quartic energy coupling terms was conducted by Fermi, Pasta and Ulam (1955) using one of the first high-speed digital computers built.

In a typical FPU calculation, the total energy of a system of 31 moving particles was placed initially in the first harmonic normal mode configuration. The spread of this energy among the other harmonic normal modes modes of a cubic nonlinear coupled system was then observed as a function of time.

The surprising result was that only the first few modes of the cubic-coupled system shared appreciable energy. Other couplings and other types of initial conditions were tried, always with the result of only limited energy sharing.

In the light of the discussions of the preceding section of this chapter, it appears that the lack of appreciable energy sharing may be attributed to a lack of tuning of the uncoupled harmonic normal mode frequencies in the models used by FPU.

This conjecture, which is the basis for the earlier discussions

in this chapter, was first advanced by Ford (1961) on the basis of perturbation theory. As a check on the validity of this conjecture and the conclusions developed from it in this chapter, some theoretical calculations of the extent of energy sharing expected in the cubic FPU system will be made. These predictions will then be compared with the actual FPU computer results.

The cubic-coupled FPU systems are just the simple one-dimensional systems with nearest-neighbor couplings specified by Equation (11) of this chapter. Therefore, when the transformation given by Equation (13) is applied, a system of nonlinear coupled oscillators governed by the hamiltonian of Equation (14) is obtained, having uncoupled harmonic normal mode frequencies

$$\omega_k = 2 \sin \frac{k\pi}{2(N+1)} \tag{70}$$

specified earlier by Equation (6).

It was concluded in Equation (68) that a tuned set of frequencies for the cubic nonlinear system is one in which the ω_k are commensurable, in the ratio $1,2,3,\ldots$. From Equation (70), it is seen that the frequencies for the FPU system come close to being tuned only on the low-frequency end, that is, for small k. Thus a qualitative prediction that only the lower modes can share appreciable energy can be made immediately.

A quantitative prediction of the extent of energy sharing can also be made, using a generalization of the tuning theory for two oscillators developed in the previous section. If the initial conditions are such that all of the energy is started in the first mode, then the most important couplings in the cubic nonlinear system are those which couple

the first mode to the higher modes. It is then appropriate to define $N-1$ tuning parameters, normalized as before, according to

$$s_k \equiv \frac{|k\omega_1 - \omega_k|}{C_s}, \quad k=2,\ldots,N. \tag{71}$$

Roughly speaking, if $s_k \ll 1$ then there will be appreciable participation of the k^{th} mode in the sharing of energy. For tuning parameters of the order of unity and larger, there will be partial participation in energy sharing, of an amount specified by the tuning curve given by Figure 1.

For a specific system, consider the first system of 31 moving particles for which FPU made numerical computations (FPU 1955, p. 12, Fig. 1). This is a cubic nonlinear system with coupling parameter $\alpha = 0.25$. The initial conditions used by FPU for this system (Jackson 1963b, p. 690) are

$$Q_k(0) = \sin\frac{k\pi}{(N+1)}; \qquad P_k(0) = 0, \tag{72}$$

corresponding to excitation of only the first harmonic normal mode.

The average amplitude of oscillations in this system for these initial conditions, the quantity A in Equation (70), may be approximated by Equation (63) as $A \approx 0.05$. A quick check in which this value is used in the expression for the particle hamiltonian, Equation (11), shows that the average total coupling energy is indeed small compared to the total energy of the system, so that Equation (65) is a good approximation for A from the point of view of weak coupling. That is, although the coupling

parameter, $\alpha \approx 0.25$, is fairly large, the coupling strength,

$$C_s \approx \alpha^{\frac{2}{3}} A \approx 0.008, \tag{73}$$

is quite small relative to unity, and this system is rather weakly coupled.

As pointed out earlier, in conjunction with the action-angle formalism for tuning, the quantity which is important in tuning considerations for the nonlinear system is not the coupling parameter alone, but instead is the quantity of Equation (73), referred to here as the "coupling strength," which involves the average amplitude as well as the coupling parameter.

In this thesis, a strongly coupled system will be regarded as one for which the average total coupling energy is a sizeable fraction of the total energy of the system; that is, a strongly-coupled system is one for which the coupling strength is greater than, say, 0.1.

Now that the coupling strength, Equation (73) has been calculated for the system of 31 particles, the tuning parameters may be calculated. The results are presented in the first part of Table 1. These results indicate that there will be appreciable energy sharing only among the first four or five of the 31 harmonic normal modes for this FPU system.

The actual numerical results obtained for this system by FPU are presented in Figure 2. The theoretical prediction of the extent of energy sharing in this system is essentially correct; in the actual system, very little energy reaches modes higher than the fifth.

In their next calculation (FPU 1955, p. 13, Fig. 2), FPU maintained all parameters of the first cubic-coupled N = 31 system unchanged,

except that the coupling parameter was increased to $\alpha = 1$, in an attempt to obtain more energy sharing.

Since the same initial conditions are used in this second calculation, the average amplitude A remains the same. The coupling strength is increased by a factor of four to $C_s \approx 0.03$. Therefore, even though the coupling parameter is now unity, this is still not a strongly coupled system in the sense defined in this thesis, because of the small average amplitude of the oscillations.

The tuning parameters for the $N = 31$ system having $\alpha = 1$ are presented in the second part of Table 1. On the basis of these calculations, there should be appreciable energy sharing only among the first six or seven modes. The actual numerical results of FPU, which are presented in Figure 3, indicate that this theoretical prediction is again essentially correct. Although this system is four times as strongly coupled as the first, there is still a lack of appreciable energy sharing among all the modes, due to a lack of tuning of the uncoupled frequencies.

The lack of tuning and hence energy sharing in the FPU systems considered here cannot be entirely remedied by increasing the coupling strength. For example, if the coupling strength is increased to unity, still only a little more than half of the frequencies will satisfy the tuning condition. But increasing the coupling strength even this much is fruitless, because the cubic-coupled system will dissociate due to breaking of the bonds between particles. Dissociation of FPU and other simple one-dimensional systems will be discussed further in Chapter III.

The approach taken by FPU in choosing initial conditions was to maintain uniform the amplitude of whatever mode j was initially excited,

Table 1. Tuning Parameters for N = 31 FPU Systems

First System: $\alpha = 0.25$

Mode, k	Tuning Parameter, s_k
2	0.03
3	0.11
4	0.29
5	0.57
6	1.01
7	1.62

Second System: $\alpha = 1.0$

Mode, k	Tuning Parameter, s_k
2	0.01
3	0.03
4	0.07
5	0.14
6	0.25
7	0.40
8	0.60
9	0.86
10	1.19

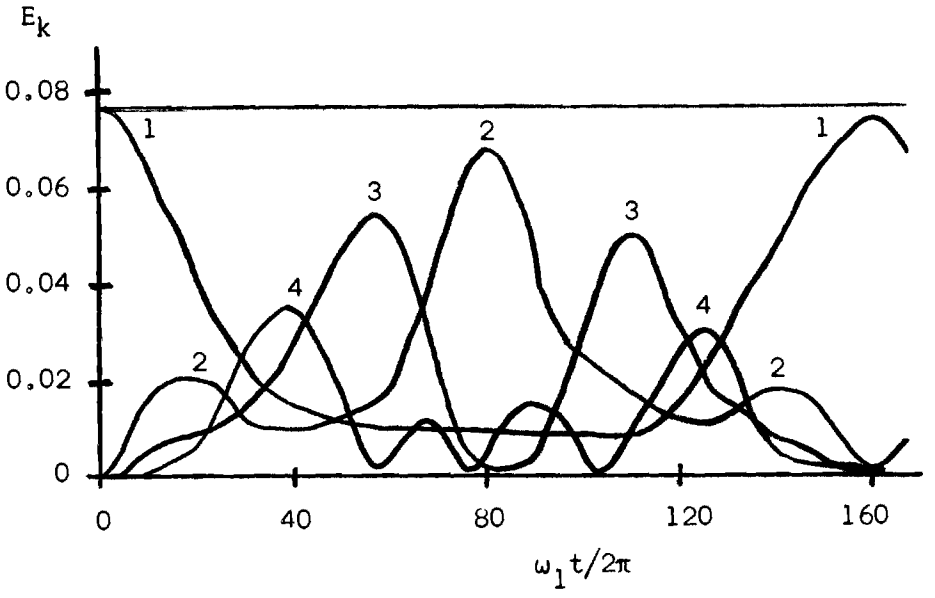

Figure 2. Mode Energies E_k versus Time for N = 31 FPU System, $\alpha = 0.25$. (Energies in modes higher than 5 are negligible. Time is measured as the number of uncoupled periods of the first mode.)

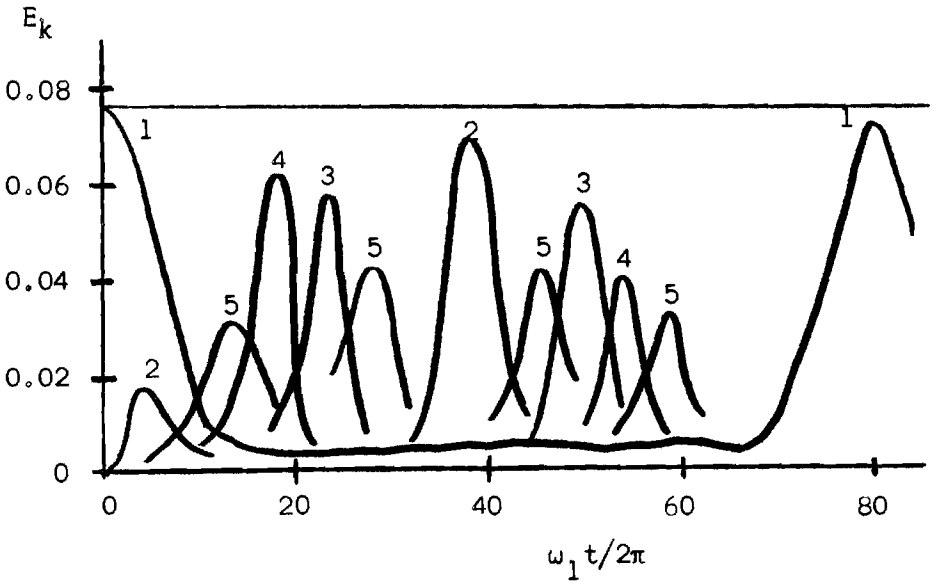

Figure 3. Mode Energies E_k versus Time for N = 31 FPU System, $\alpha = 1.00$. (Energies in modes higher than 7 are negligible. Time is measured as the number of uncoupled periods of the first mode.)

according to

$$q_k(0) = \left(\frac{N+1}{2}\right)^{\frac{1}{2}} \delta_{kj}. \tag{74}$$

Since the uncoupled frequencies ω_j are small for small j, this means the total energy of the system,

$$H = E_j(0) = \tfrac{1}{2}\omega_j^2\left(\frac{N+1}{2}\right), \tag{75}$$

was small whenever energy was initially placed in a low mode, such as the first. The average amplitude of oscillations,

$$A \approx \left(\frac{H}{N+1}\right)^{\frac{1}{2}} = \tfrac{1}{2}\omega_j, \tag{76}$$

was thus small for the cases presented here, leading to small coupling strengths and relatively weakly-coupled systems.

A better way to control the coupling strength would be to maintain uniform a certain total energy H, rather than to maintain uniform an amplitude q_j. If H is chosen equal to $(N+1)$ units of energy no matter what modes are excited, then

$$A = \left(\frac{H}{N+1}\right)^{\frac{1}{2}} = 1, \tag{77}$$

and control of the coupling strength is returned to the coupling parameter, α:

$$C_s = \alpha c A \approx \alpha. \tag{78}$$

This specification of a uniform total energy which is proportional

to the number of particles in the system maintains the average energy per particle uniform. It is therefore a physically more realistic approach. However, in practice, when dealing with artificial one-dimensional systems of the type considered here, this approach cannot always be used for systems having a large number of particles, because of dissociation. A smaller uniform total energy must be used in these cases.

In connection with the computer studies of FPU-type systems, some recent work by Jackson is of interest. In a paper which compares the predictions of a perturbation method for FPU systems with actual computer results, Jackson (1963b) provides computations for some smaller systems of three and eight moving particles, with several values of coupling parameters. It is of interest to compare some predictions of the extent of energy sharing for these small systems with Jackson's actual results.

For the three-oscillator system, Jackson uses initial conditions of the same type as FPU, with all energy initially in the first harmonic normal mode, so that the average amplitude of oscillations is $A \approx 0.38$ for this system.

In the first case, the coupling parameter is taken to be $\alpha = 0.25$, so that the coupling strength is $C_s \approx 0.06$. The tuning parameters defined in Equation (70) are $s_2 \approx 1.8$ and $s_3 \approx 7$, so that about 35 per cent of complete energy exchange between the first and second oscillators is expected on the basis of the tuning curve given in Figure 1 in this

chapter; participation of the third oscillator in energy sharing is expected to be negligible. Jackson's computer results show that there is actually only about 19 per cent energy exchange between the first two oscillators; participation of the third is less than two per cent.

In the second case, $\alpha = 0.50$, so that the coupling strength is $C_s \approx 0.13$. The tuning parameters are $s_2 \approx 0.9$ and $s_3 \approx 3.5$, which means that about 52 per cent energy sharing among the first two oscillators, and about 20 per cent participation of the third, are expected. The actual amounts are about 39 and seven per cent, respectively.

In the third case of the three-oscillator system, $\alpha = 0.75$, so that $C_s \approx 0.19$. The tuning parameters are $s_2 \approx 0.6$ and $s_3 \approx 2.3$, so the first two oscillators are expected from Figure 1 to share about 65 per cent, and the third about 23 per cent, of the maximum possible. Jackson's calculations show that the actual amounts are about 54 per cent and 17 per cent, respectively.

These comparisons indicate that the theoretical predictions of energy sharing run high by as much as a factor of two for cases in which little energy is shared, but are rather accurate for cases in which appreciable energy is shared. In all cases, the theory mildly overestimates the extent of energy sharing, for the three-oscillator system.

As a final comparison, consider the eight-oscillator system for which Jackson has made calculations. For this system the average amplitude of oscillations is $A \approx 0.17$, the coupling parameter is $\alpha = 0.25$, and the coupling strength is $C_s \approx 0.03$. The first three tuning parameters are $s_2 \approx 0.35$, $s_3 \approx 1.42$, and $s_4 \approx 3.5$, so that about 75 per cent of

the maximum possible energy sharing between the first two oscillators is expected; participation of the third should be about 40 per cent, and the fourth less than 20 per cent. The actual computer results indicate about 56, ten, and less than two per cent, respectively. The theoretical predictions are not as accurate as for the smaller system, but the essential features of the predictions are correct.

In this section, the conclusions obtained in the previous section, concerning the relation between tuning and energy sharing, have been tested by comparison with actual computer results. The conclusions have generally been found to be valid, for cases in which the frequencies have been fixed and the coupling strengths varied. In the next section, the validity of the theoretical predictions of the extent of energy sharing will be further checked, by considering cases in which the coupling strengths are fixed and the frequencies are varied.

Computer Studies of the Effects of Changing the Frequencies

It is not possible to obtain appreciable sharing of energy between all of the harmonic normal modes of an untuned system such as the FPU cubic-coupled nonlinear system, even for coupling strengths so large that the system dissociates, as has been indicated in the preceding subsection.

Many of the physical applications of coupled oscillator systems require a mathematical model in which complete energy sharing is possible, and yet in which the average total coupling energy is small compared to the total energy of the system. Such a model can be obtained rather simply from the FPU systems of this chapter, by changing the set of uncoupled frequencies from the untuned FPU set to a tuned set. This

formulation of a tuned model will be undertaken in Chapter III.

Since a change in the frequencies is necessary in the formulation of a model which is capable of sharing energy, some computer studies of the actual effect which changing the frequencies has on energy sharing will be conducted in this subsection. Results of theoretical predictions based on action-angle formalism will be compared with these computer results.

Two-Oscillator System. A computer study of energy sharing as a function of ω_2, with fixed $\omega_1 = 1$, was made for the $N = 2$ system governed by the hamiltonian given in Equation (50) of this chapter. The series of results is presented in Figure 4. The initial conditions are the same for all the curves: all initial positions and momenta are zero except $p_1 = \sqrt{2}$, so that the hamiltonian is maintained uniform at a magnitude of unity. The coupling parameter is $\alpha = 0.1$.

The values assigned to ω_2 range from the FPU value of $\omega_2 = \sqrt{3}$ to the exactly-tuned value of $\omega_2 = \omega$. It is evident that little energy is exchanged between oscillators in the FPU system with this coupling strength ($C_s \approx 0.038$, and $s_2 = 7$), due to a lack of tuning. On the other hand, in the exactly-tuned system there is almost 100 per cent energy sharing (there is a small amount of energy in the coupling between oscillators).

Figure 5 presents these same results as a tuning curve, which is compared with the theoretical tuning curve (Figure 1) developed earlier for the two-oscillator system using action-angle formalism. Examination of Figure 5 shows that the actual tuning curve for two oscillators is a bit sharper than the predicted tuning curve. For cases in which not

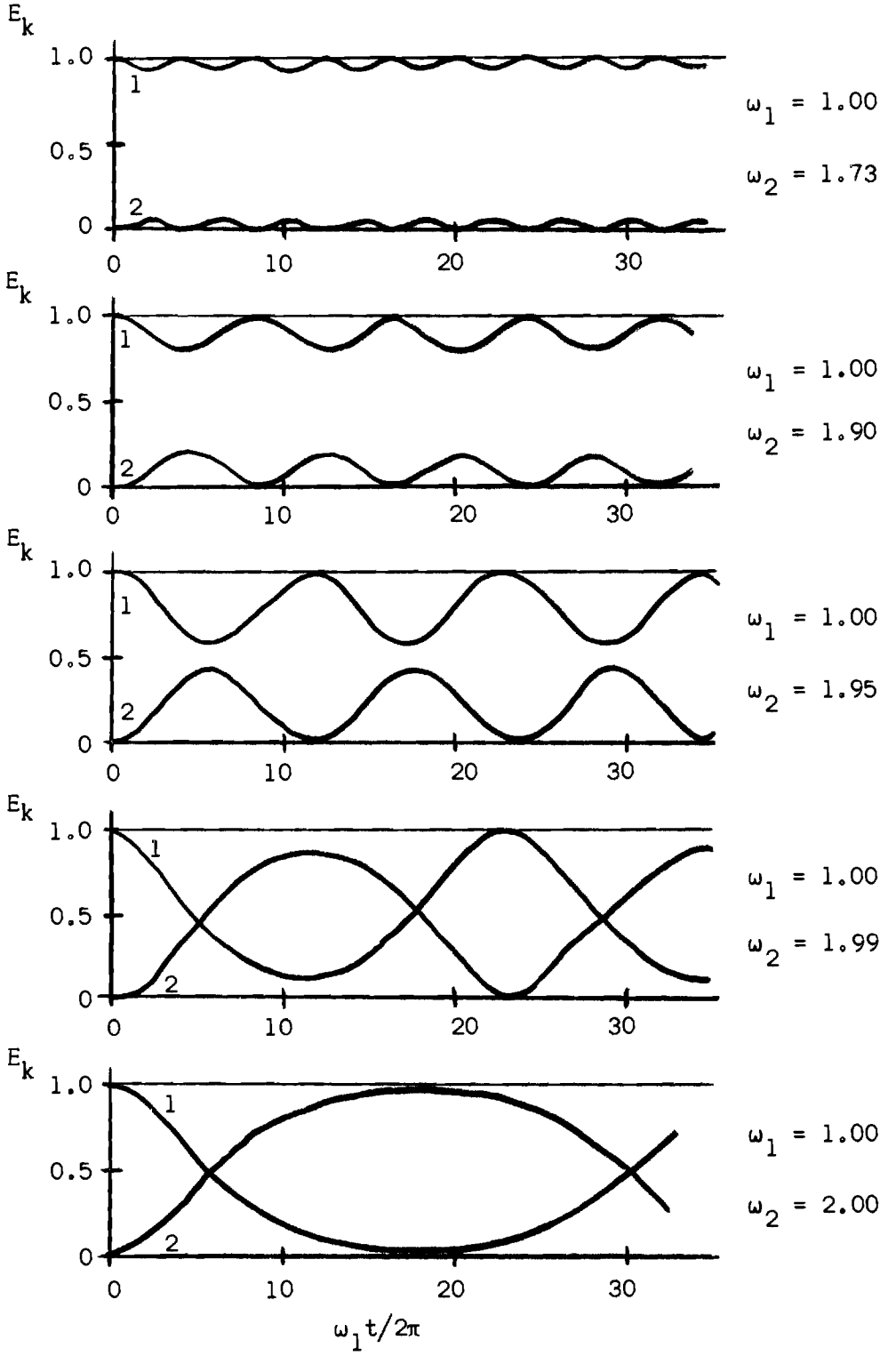

Figure 4. Energy Exchange for Two Nonlinear Coupled Oscillators as a Function of Frequencies.

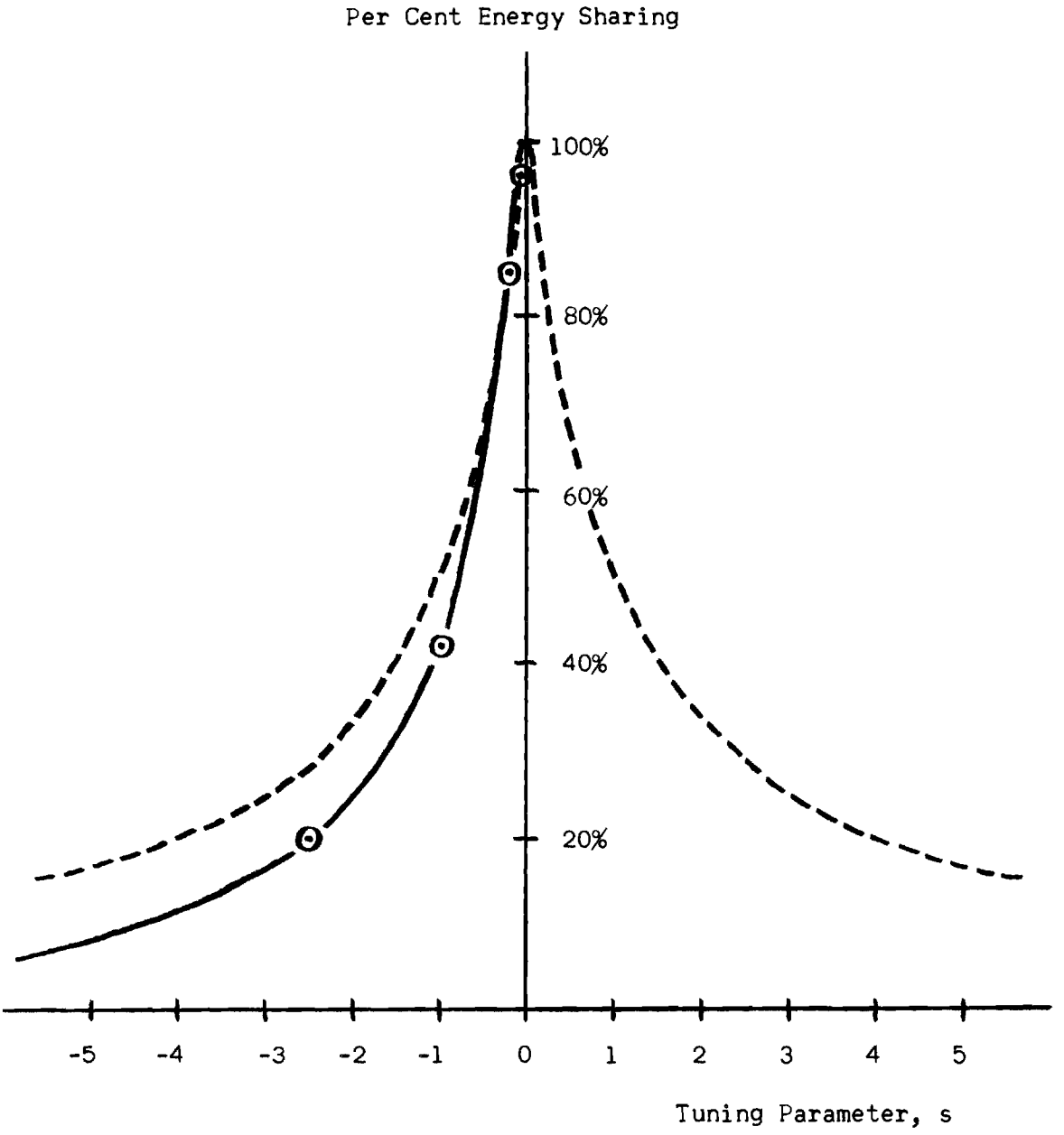

Figure 5. Actual Sharpness of Tuning for Two Coupled Oscillators. (Solid curve represents computer results; dashed curve gives predicted results.)

much energy is shared, more energy sharing is predicted than actually occurs. But the half-widths of the two curves agree, and in general the correspondence is good.

Five-Oscillator System. In order to check the effect which changing the frequencies from the FPU set to an exactly-tuned set has on energy sharing in a somewhat larger system, computer studies were made of the time behavior of the harmonic normal mode energies as a function of various sets of frequencies, for a cubic-coupled system of five oscillators.

There are many ways to "tune" a system having more than two degrees of freedom. The procedure used here is just one orderly means of varying the amount of tuning continuously from the relatively untuned case supplied by the FPU frequencies to the exactly-tuned case specified for the cubic-coupled system by Equation (69).

A variable untuned set of frequencies may be defined by

$$\omega_k = \sum_{\ell=1}^{k} \left(\omega_1 - [\ell-1]\Delta\omega\right), \tag{79}$$

where $\Delta\omega$ is a fixed frequency decrement. Summing the series which appears in this definition leads to

$$\omega_k = k\left(\omega_1 - \frac{(k-1)}{2}\Delta\omega\right), \tag{80}$$

so that the detuning is greater for the higher modes:

$$S_k \equiv \frac{|k\omega_1 - \omega_k|}{C_5} = \tfrac{1}{2}k(k-1)\frac{\Delta\omega}{C_5}, \quad k=2,\ldots,N. \tag{81}$$

Frequency decrement $\Delta\omega$ has been varied from zero, which provides

commensurable frequencies $\omega_k = k\omega_1$, to

$$\Delta\omega = \frac{1}{(N+1)}\omega_1, \tag{82}$$

which for $N = 5$ gives detuning which exceeds that of the FPU frequencies. Figure 6 presents graphically the effect of using various frequency decrements between the two extreme cases mentioned here.

The coupling parameter for the five-oscillator system used in this study is $\alpha = 0.1$, and the total energy is $H = 3$. Thus the average amplitude of oscillations is $A \approx 0.71$, and the coupling strength is $C_s \approx 0.05$.

Table 2 summarizes the tuning parameters for the various choices of decrements chosen for study. In calculating these parameters, ω_1 is chosen in each case so that $\omega_6 = 2$, as in Figure 6.

Table 2. Tuning Parameters for $N = 5$ Tuned Systems

Mode, k	Tuning Parameter, s_k, for Decrement $\Delta\omega$					
	0.095	0.057	(FPU)	0.029	0.009	0.000
2	2.1	1.2	0.85	0.61	0.21	0.00
3	6.4	3.6	2.95	1.83	0.64	0.00
4	13	7.2	7.2	3.66	1.27	0.00
5	21	12	14	6.1	2.10	0.00

Inspection of the detuning parameters in Table 2 indicates that appreciable energy sharing should be expected only for the two systems having frequency decrements of zero and 0.009 -- that is, for the exactly-tuned system and for the next best one. At the other extreme, very little energy should be shared between modes of the system having decrement

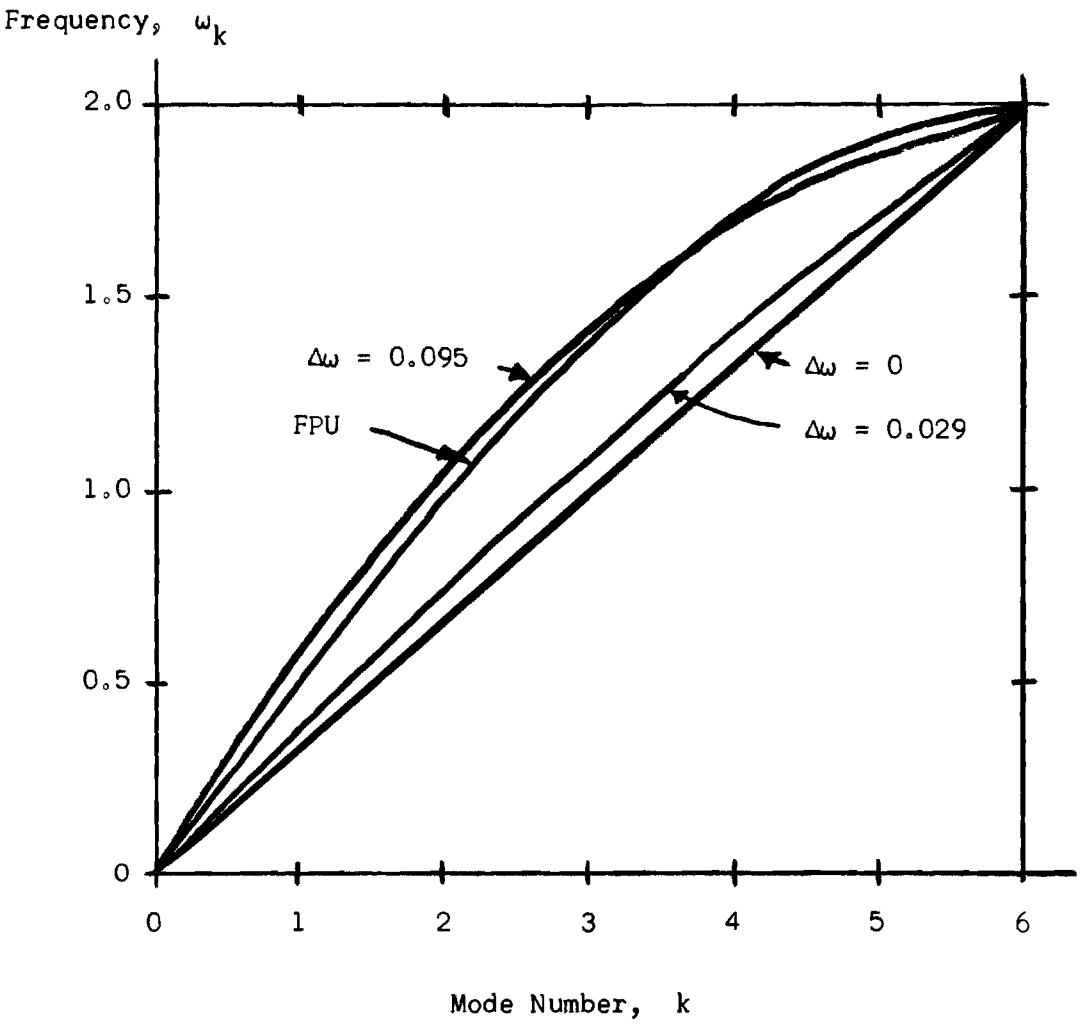

Figure 6. Frequencies versus Mode Numbers, for Various Decrements $\Delta\omega$: Five-Oscillator Cubic-Coupled System.

$\Delta\omega = 0.095$, except perhaps for about a 35 per cent exchange between the first two modes.

The actual numerical results are presented in Figures 7, 8, 9, 10 and 11 for the values of frequency decrement given in Table 2 (except for the case $\Delta\omega = 0.057$, for which the tuning parameters are much the same as those of the FPU system). Qualitatively, these computer results show that the systems for which $\Delta\omega = 0$ and $\Delta\omega = 0.009$, the two best tuned systems, share appreciable energy among all modes. The next system, for $\Delta\omega = 0.029$, shows a noticeable reduction in the participation of the fourth and fifth modes in energy sharing. The FPU system shares little energy except between the first two modes, and the most severely untuned system, that for which $\Delta\omega = 0.095$, shows negligible energy sharing; the exchange between even the first two oscillators amounts to less than ten per cent.

The theoretical predictions of the extent of energy sharing for these systems are essentially correct. There is perhaps more energy sharing than expected in those systems which do share energy, and less energy sharing than expected in the systems which do not share appreciable energy.

The computer studies of this subsection indicate that the reason for the failure of the FPU systems to share appreciable energy among all modes is due to a lack of tuning of the uncoupled harmonic normal mode frequencies. Actually, this conclusion might have been reached somewhat earlier, on the basis of the work of Peierls (1956) on crystal lattices. Peierls observes that in a crystal lattice two longitudinal phonons cannot combine to produce a third longitudinal phonon, or vice-versa, because

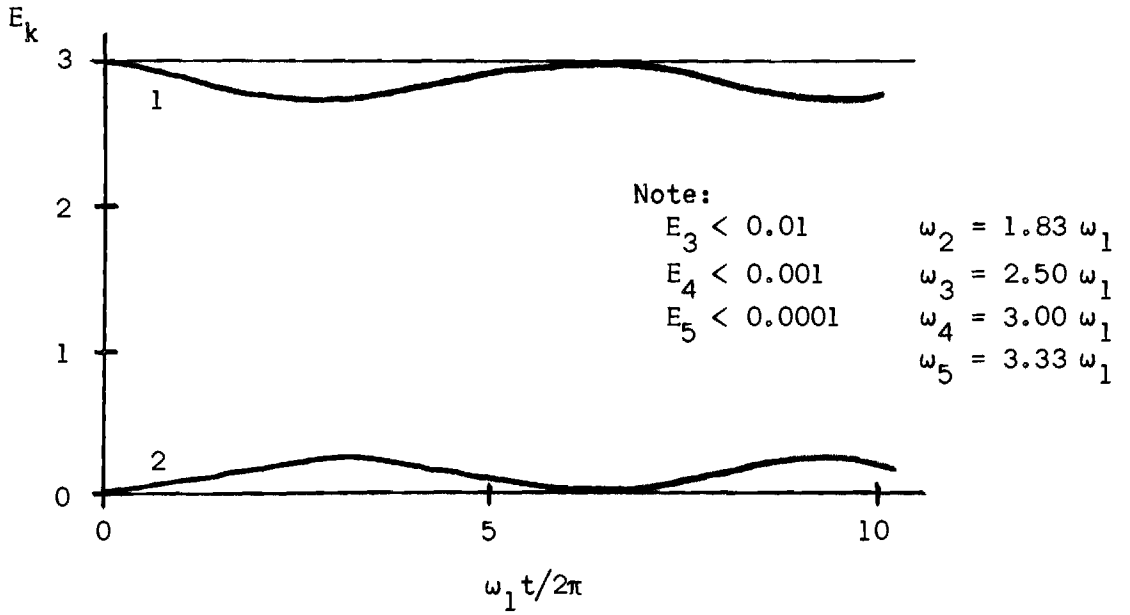

Figure 7. Energy Sharing for $\Delta\omega = 0.095$. $(\omega_1 = 0.57)$

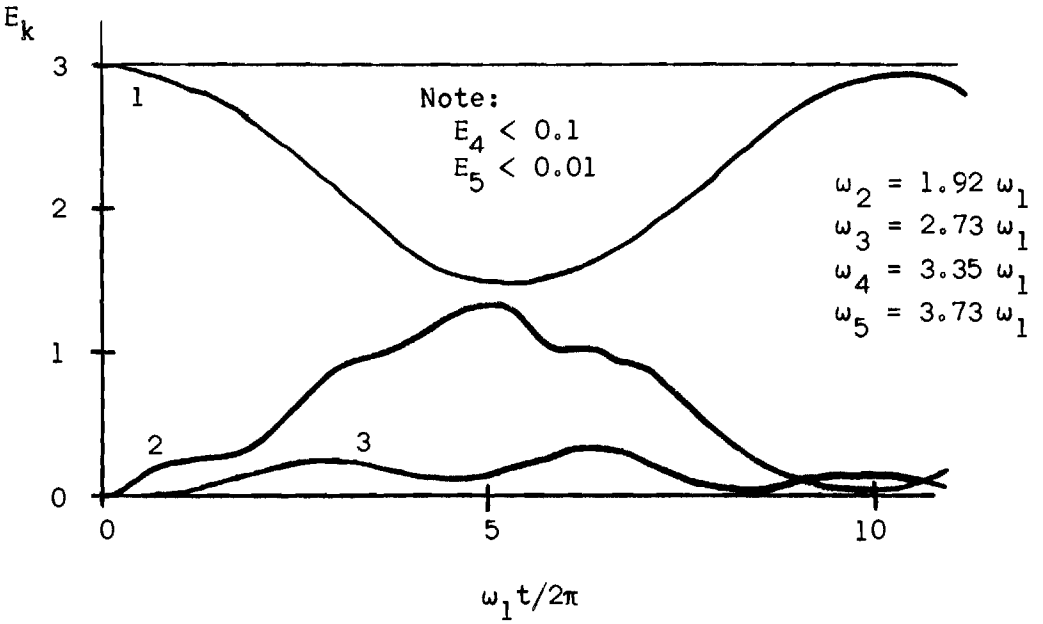

Figure 8. Energy Sharing for FPU Frequencies. $(\omega_1 = 0.52)$

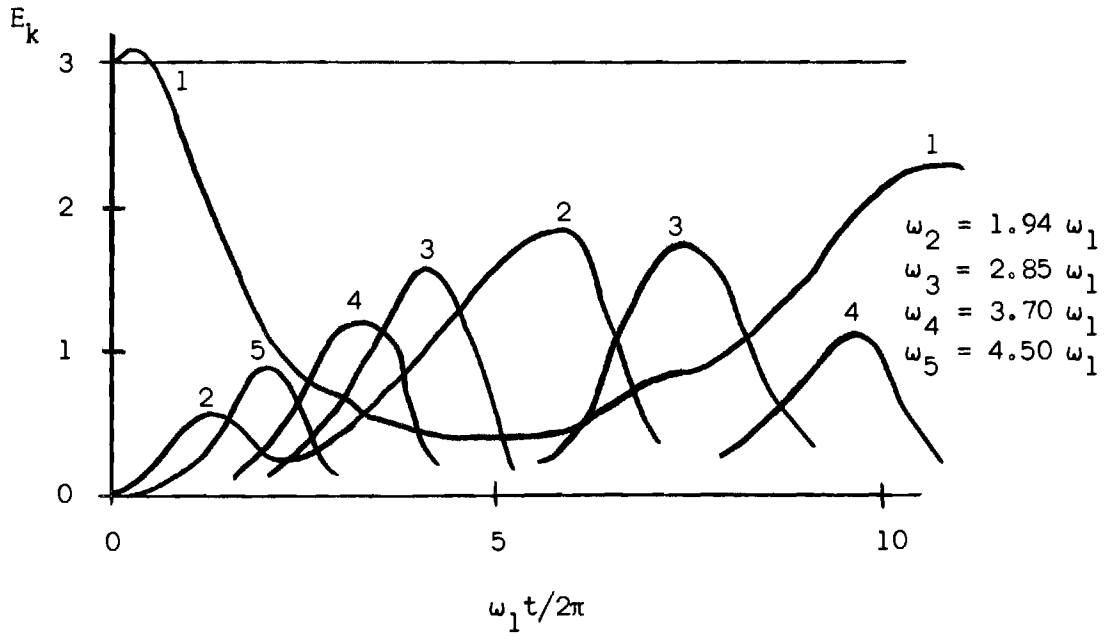

Figure 9. Energy Sharing for $\Delta\omega = 0.029$. ($\omega_1 = 0.38$)

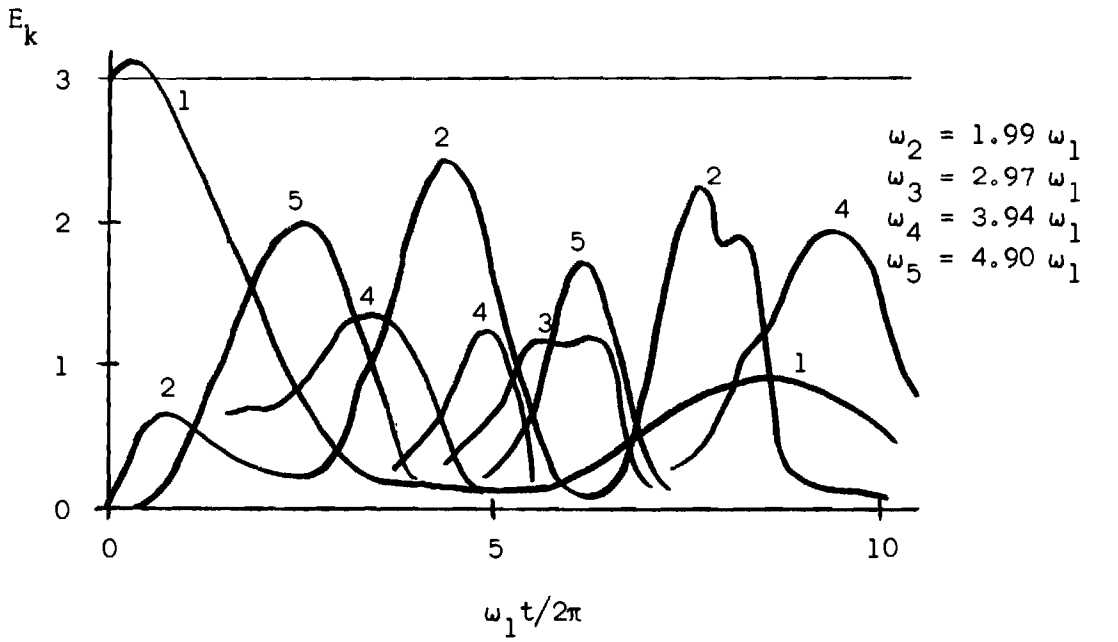

Figure 10. Energy Sharing for $\Delta\omega = 0.009$. ($\omega_1 = 0.34$)

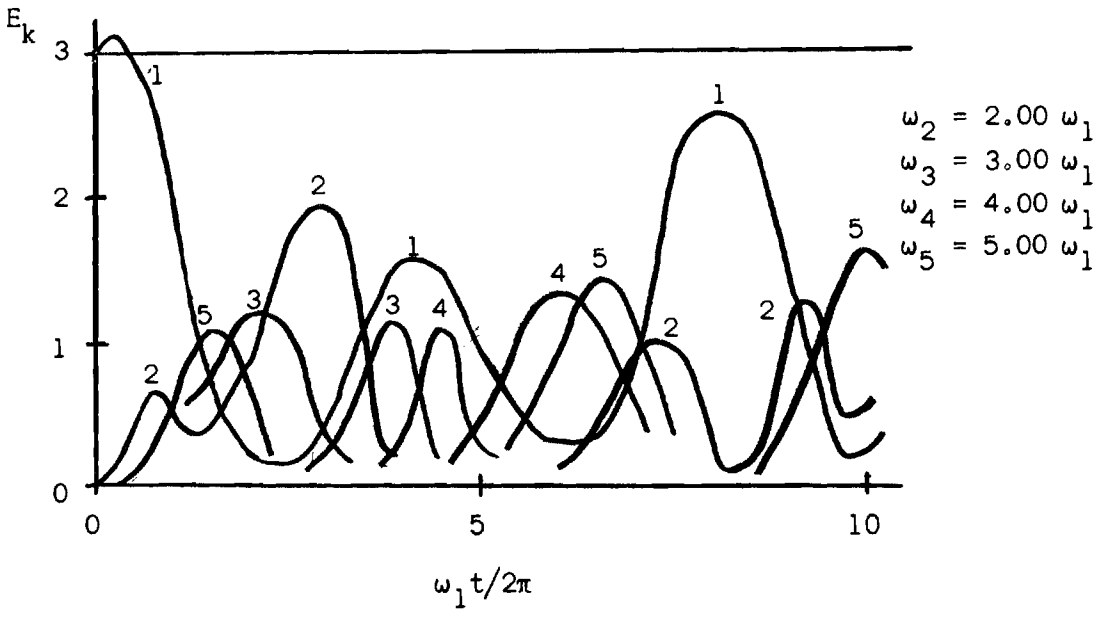

Figure 11. Energy Sharing for $\Delta\omega = 0$. ($\omega_1 = 0.33$)

such a phonon interaction does not satisfy conditions of conservation of energy and quasi-momentum. The longitudinal phonons correspond to the phonons of a one-dimensional systems such as the FPU system, and the cubic coupling of the FPU system corresponds to three-phonon interactions in the quantum-mechanical description.

The computer results of this subsection also demonstrate explicitly that a well-tuned oscillator system is capable of sharing energy among all of its modes. This will be of importance in Chapter III, where an energy-sharing model is to be formulated.

In this chapter, some important results have been obtained with regard to the relationship between energy sharing and tuning of an oscillator system. An action-angle formalism which is fairly reliable in predicting the extent of energy sharing in a given system has been developed and verified against established results. It has been shown that if certain tuning conditions on the set of uncoupled harmonic normal mode frequencies are satisfied for a given type of coupling, then a coupled oscillator system is capable of sharing appreciable energy among all of its modes. The crucial parameter in tuning has been shown to be not the coupling parameter alone, but instead involves also the typical amplitude of the oscillations of the system, for cubic-coupled nonlinear systems.

The mechanism of energy sharing has been shown to be the presence of internal resonances. The importance of such resonances is well known in physics, for example in celestial mechanics. In the solar system, Jupiter has a strong effect on the other bodies because of its great mass. It is observed that there are few or no asteroids having orbits the periods

of which are in the commensurable ratio of 1:2 with the period of Jupiter. Presumably this is because considerable energy is exchanged due to slowly-varying coupling terms, moving the asteroids out of these particular orbits. Similarly, gaps are observed in Saturn's rings, where particles would have periods putting them in resonance with the inner satellites of Saturn (Danby 1962).

The fact that the frequencies may be slightly detuned from the exact relation $\omega_2 = 2\omega_1$ without materially impairing the resonance phenomenon, is also noted in astronomical observations. The gaps in the distribution of the asteroids do not occur only at the exact 1:2 ratio of the periods of Jupiter and the mean motions of the asteroids; there is a range of values in the neighborhood of this ratio (Contopoulos 1963, p. 9).

In the following chapter, the information gained in this chapter will be put to use in the formulation of an appropriate nonlinear coupled oscillator model which is capable of sharing appreciable energy, and which is simple enough to be analytically tractable.

CHAPTER III

THE MODEL: A NONLINEAR COUPLED OSCILLATOR SYSTEM CAPABLE OF COMPLETE ENERGY SHARING

According to the discussions of Chapter II, not all nonlinear coupled oscillator systems are capable of sharing appreciable energy among all modes. In order to share energy completely, an oscillator system must have frequencies which satisfy certain tuning conditions, which in general depend upon the coupling strength for the particular system.

The primary aim of this investigation is to formulate and analyze a simple but representative nonlinear coupled oscillator system which is capable of complete energy sharing. The experience gained in Chapter II will be of considerable value in the choice of such a model.

In the first section of this chapter, the particular nonlinear coupled oscillator system which has been chosen for extensive further analysis in this thesis will be formulated. This model is capable of complete energy sharing for any strength of coupling. In the second section, the significance of the corresponding couplings in particle co-ordinates for this model will be discussed, since the model is formulated in harmonic normal mode variables. The final section will briefly review some other nonlinear models which are capable of complete energy sharing.

Formulation of the Model

As discussed in the introductory remarks of Chapter II, for purposes of simplicity one-dimensional models will be used in this thesis for purposes of illustrating characteristics and methods of analysis for various coupled oscillator systems.

In the spirit of this desire for simplicity, a very simple nonlinear coupled oscillator system which is capable of complete energy sharing will be chosen for further study.

The particular one-dimensional model which will be given extensive further analysis in this thesis is the exactly-tuned cubic-coupled nonlinear oscillator system governed by a hamiltonian of the form

$$H = \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2\right) + \alpha\frac{1}{3}\sum_{i,j,k}^{N} C_{ijk}\, q_i q_j q_k . \qquad (1)$$

This hamiltonian is the same as that appearing in Equation (14) of Chapter II, except that the relatively untuned set of "FPU-type" uncoupled harmonic normal mode frequencies appearing there has been replaced by the exactly-tuned set

$$\omega_k = k\omega = k\,\frac{2}{(N+1)}, \qquad (2)$$

given in Chapter II by Equation (69). For the sake of maintaining some generality in the analysis of this system, the explicit forms of the coupling coefficients, C_{ijk} in Equation (1), will not be specified until the computer studies of Chapter VII.

The model specified by Equation (1) is not meant to represent any

particular physical system. Rather, it has been designed to emphasize the feature of tuned frequencies in a moderately simple system. Because the frequencies of this model are exactly tuned, this system is capable of sharing appreciable energy among all of its modes for an arbitrarily small total coupling energy, according to the tuning studies of Chapter II. That is, there is zero detuning of the frequencies of this system, so that the system is capable of substantial energy sharing for any strength of coupling.

As will be discussed in Chapter IV, the available approximation schemes for solving nonlinear coupled oscillator systems which are capable of complete energy sharing are least effective when strong internal resonances are present in the systems. The model chosen here exhibits the strongest possible internal resonances, due to its exact tuning. It is felt that if a successful analytic method of solution can be developed for this simple but extreme case, then this will give considerable insight into the solution of more complicated three-dimensional nonlinear physically realistic systems, in which the tuning relations are more general.

Much of the remainder of this thesis will be devoted to analysis of the model formulated here. However, it is intended that the general methods of analysis developed in terms of this model will find application to other types of nonlinear coupled oscillator systems as well.

Since the system defined in Equation (1) is specified in terms of harmonic normal mode variables, some discussion of the corresponding particle couplings is relevant. This discussion is given in the following section.

Couplings in Particle Co-ordinates

A set of nearest-neighbor linear coupling forces between particles in the perfect harmonic system specified by Equation (2) in Chapter II produces the set of harmonic normal mode frequencies, "FPU" frequencies, given by

$$\omega_k = 2 \sin \frac{k\pi}{2(N+1)}, \tag{3}$$

which is Equation (6) of Chapter II.

This set of frequencies is not very well tuned, and does not promote appreciable energy sharing. When the set of frequencies given by Equation (3) is replaced by the exactly-tuned set given in Equation (2) of this chapter, some corresponding modifications of the particle model are necessary.

There are several cases. In the case of a crystal lattice model, the masses of all of the particles are to be maintained equal, which requires in particle co-ordinates that the set of nearest-neighbor linear coupling forces between particles be replaced by another set of linear forces. These new linear forces are likely to be longer in range. In another case, that of a chemical molecule particle model, varying the harmonic normal mode frequencies corresponds to varying the masses of the particles and maintaining the same linear nearest-neighbor couplings.

The crystal lattice model is of particular interest. Although the one-dimensional system specified by Equation (1) is physically artificial, information about its particle couplings may lend insight into the actual particle couplings present in physically realistic three-dimensional systems.

The details of the tuning procedure for a crystal lattice particle model are as follows. Starting with a simple FPU-type nearest-neighbor coupled system governed by

$$H = \tfrac{1}{2}\sum_{k=1}^{N} P_k^2 + \tfrac{1}{2}\sum_{k=0}^{N}(Q_{k+1}-Q_k)^2 - \alpha\tfrac{1}{3}\sum_{k=0}^{N}(Q_{k+1}-Q_k)^3, \tag{4}$$

a transformation to the harmonic normal mode representation is made, according to

$$Q_k = \sum_{l=1}^{N} \theta_{kl} q_l, \qquad \theta_{kl} = \left(\frac{2}{N+1}\right)^{\frac{1}{2}} \sin\frac{kl\pi}{(N+1)}, \tag{5}$$

so that the hamiltonian becomes

$$H = \tfrac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + \omega_k^2 q_k^2\right) - \alpha\tfrac{1}{3}\sum_{i,j,k}^{N} C_{ijk} q_i q_j q_k, \tag{6}$$

in which the harmonic normal mode frequencies are given by

$$\omega_k = 2\sin\frac{k\pi}{2(N+1)}. \tag{7}$$

This set of frequencies is not tuned very well, so it is replaced by the exactly-tuned set

$$\omega_k = 2\frac{k}{(N+1)}. \tag{8}$$

The problem is to determine a set of linear couplings between particles, which corresponds to this tuned set of frequencies.

One (but by no means the only) solution is supplied by application of the inverse of the transformation given by Equation (5) to the tuned hamiltonian, Equation (6). Application of the transformation

$$q_\ell = \sum_{k=1}^{N} \Theta_{\ell k} Q_k, \quad \Theta_{\ell k} = \left(\frac{2}{N+1}\right)^{\frac{1}{2}} \sin \frac{\ell k \pi}{(N+1)}, \tag{9}$$

to the hamiltonian of Equation (6) gives a particle hamiltonian of the form

$$H = \tfrac{1}{2} \sum_{k=1}^{N} P_k^2 + \tfrac{1}{2} \sum_{i,j}^{N} B_{ij} Q_i Q_j - \alpha \tfrac{1}{3} \sum_{k=0}^{N} (Q_{k+1} - Q_k)^3, \tag{10}$$

in which the coefficients B_{ij} are of a form such that the coupling forces are fairly long in range, as will be seen. Note that the cubic coupling term, the nonlinear portion of the original hamiltonian, has not been altered by tuning of the harmonic normal mode frequencies.

The calculation of the coefficients B_{ij} in Equation (10) have been performed for a five-particle system. The results are most conveniently presented in terms of the equations of motion of the nearest-neighbor "FPU" particle system and the new "tuned" particle system corresponding to Equation (10):

FPU System

$$\begin{bmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \\ \ddot{Q}_3 \\ \ddot{Q}_4 \\ \ddot{Q}_5 \end{bmatrix} = \begin{bmatrix} -2.00 & 1.00 & 0.00 & 0.00 & 0.00 \\ 1.00 & -2.00 & 1.00 & 0.00 & 0.00 \\ 0.00 & 1.00 & -2.00 & 1.00 & 0.00 \\ 0.00 & 0.00 & 1.00 & -2.00 & 1.00 \\ 0.00 & 0.00 & 0.00 & 1.00 & -2.00 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} \tag{11}$$

Tuned System

$$\begin{bmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \\ \ddot{Q}_3 \\ \ddot{Q}_4 \\ \ddot{Q}_5 \end{bmatrix} = \begin{bmatrix} -1.63 & 1.03 & -0.21 & 0.07 & -0.03 \\ 1.03 & -1.84 & 1.11 & -0.24 & 0.07 \\ -0.21 & 1.11 & -1.87 & 1.11 & -0.21 \\ 0.07 & -0.24 & 1.11 & -1.84 & 1.03 \\ -0.03 & 0.07 & -0.21 & 1.03 & -1.63 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{bmatrix} \qquad (12)$$

Note that the elements of the force-coefficient matrix for the tuned system alternate in sign, and decrease in magnitude fairly rapidly off the main diagonal. This result is reminiscent of a one-dimensional ionic crystal in which long-range forces act. The first row equation of motion, for example, may be interpreted as

$$\begin{aligned} \ddot{Q}_1 = {} & \left.\begin{aligned} & -0.76 \ (Q_1 - Q_0) \\ & -1.03 \ (Q_1 - Q_2) \end{aligned}\right\} \text{nearest-neighbor forces} \\ & +0.21 \ (Q_1 - Q_3) \quad \text{second-neighbor force} \\ & -0.07 \ (Q_1 - Q_4) \quad \text{third-neighbor force} \\ & +0.03 \ (Q_1 - Q_5) \quad \text{fourth-neighbor force} \\ & -0.01 \ (Q_1 - Q_6) \quad \text{fifth-neighbor force} \end{aligned} \qquad (13)$$

with similar interpretations for the other equations of motion, given by the other rows of Equation (12).

The combination of long-range interaction forces and fixed-end boundary conditions for a system of few particles, such as the example of the five-particle system, causes the end terms of the main diagonal to be smaller in magnitude than the central terms. In a system of many particles, boundary conditions should have less effect, and the main

diagonal terms should approach a magnitude of 2.00 as in the FPU system. Such a trend is observed in similar computations for N = 10 and N = 15.

Studies of these slightly larger systems indicate that the couplings are fairly long range in this type of system. For example, the ratio of tenth-neighbor forces to nearest-neighbor forces is about 0.005. Thus the forces decrease at a rate which is slightly greater than r^{-2}, where r is the interaction distance.

Three comments may be made with regard to the effect which tuning the harmonic normal mode frequencies has on the linear couplings between particles.

First, it is possible that these tuned systems might have been constructed in a direct fashion from basic physical considerations. If a one-dimensional model of an ionic crystal (alternating signs) were to be constructed, with long-range interaction forces diminishing as r^{-2}, then a particle system with linear couplings very similar to those obtained here by hindsight would have been chosen for investigation. A transformation to harmonic normal modes would then have produced a set of frequencies not too different from the tuned set chosen here to produce strong resonances. The model constructed in the direct manner described would have shared considerable energy and would have been physically more realistic than, say, the FPU systems. It may be that once a better experimental knowledge of the force coefficients present in an actual crystal lattice is available, the insight into the relationship between long-range couplings and tuned frequencies (hence energy sharing) which has been gained here will be of use in the "direct" construction

of three-dimensional mathematical models for crystal lattices.

Second, the introduction of long-range linear particle interactions as prescribed by hindsight produces an exactly commensurable set of harmonic normal mode frequencies in the one-dimensional model. This implies a dispersionless medium, because the group velocity,

$$v_G = \frac{\partial \omega(k)}{\partial k}, \tag{14}$$

is the same for all modes (phonons). This emphasizes the artificial nature of the one-dimensional model chosen here. Such a lack of dispersion is generally found only in continuous media, such as for example the uniform vibrating string. A system of discrete masses, even if infinite, generally displays dispersion. But the present result implies that the special set of long-range couplings found here is equivalent to a continuity or smearing-out of the discrete masses actually involved.

Third, the introduction of a particle system with long-range linear interactions is not necessarily the only means by which a tuned harmonic normal mode system can be obtained. There are other particle couplings which, when transformed, produce tuned frequencies. The long-range interaction interpretation was introduced only because it seems appropriate in an application to crystal lattice vibrations. It may well be true that long-range interactions are not important in crystal lattice phenomena; the work of Peierls (1956) indicates, for example, that the existence of transverse as well as longitudinal phonons in three-dimensional lattices provides the necessary internal resonances via interactions involving both longitudinal and transverse phonons, without the need for any tuning of the frequencies.

Bearing in mind these acknowledged shortcomings, it is nevertheless of some interest to use some typical values of physical parameters in the one-dimensional cubic-coupled model, in order to determine the order of magnitude of such characteristics as linear thermal expansion and the strength of coupling at various temperatures for this model. These characteristics are associated with the presence of the cubic anharmonic term in the potential, and have little to do with whether the harmonic normal mode frequencies are tuned or not.

The exact details of interatomic potentials are complicated. The work of Peierls (1929, 1956) and Ziman (1960) suggests that only cubic anharmonic terms are of importance in the explanation of many nonlinear physical crystal phenomena, except perhaps at very high temperatures.

The particle model discussed in this section may be represented as a one-dimensional lattice of mass points separated by lattice constant ℓ_o. If this model is "tuned," then there will be long-range forces as well as nearest-neighbor forces; but the nearest-neighbor forces are the strongest, and will be the only ones considered in what follows.

A typical nearest-neighbor interaction between particles consists of linear and quadratic forces, so that the corresponding pair-potential well consists of quadratic and cubic terms, as illustrated in Figure 12. The dissociation energy per particle is characterized by D in Figure 12.

Dissociation is possible (although extremely improbable in large systems) in any system in which the total energy of the system, the hamiltonian H, exceeds D. Here the probability of dissociation is related to the probability of energy distributions in which only a few

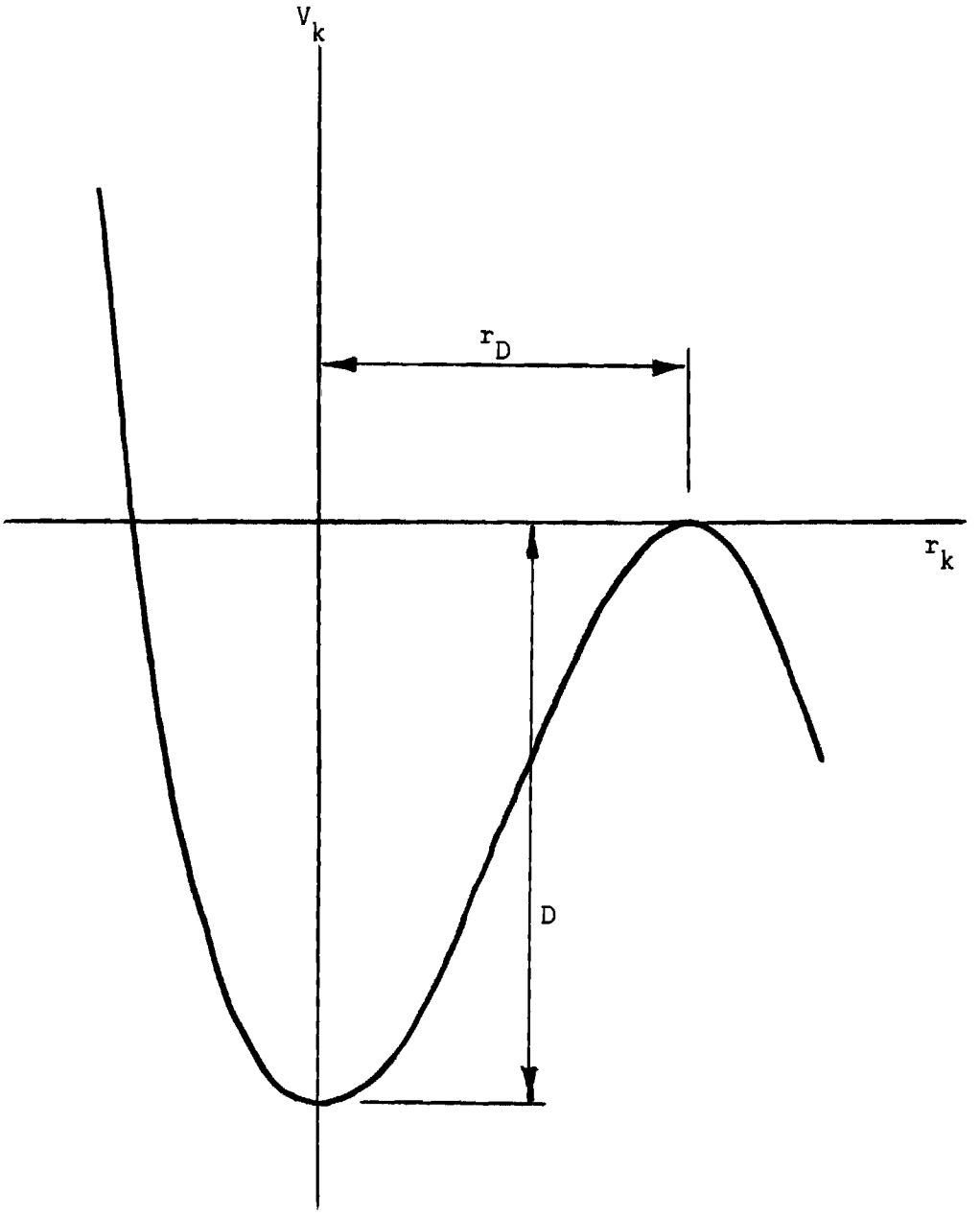

Figure 12. Typical Nearest-Neighbor Pair-Potential Well.

particles have a large fraction of the total energy of the system. Thus dissociation is much more likely in a system which is capable of complete energy sharing than in one which is not, because the distribution of energy among the particles is continually changing in an energy-sharing system.

The above remarks are more relevant to small systems than to large ones. For large systems, dissociation is not likely until the total energy per particle, H/N, is of the order of D.

The strength of the cubic interaction in this model may be specified in terms of the typical dissociation energy D, and the typical dissociation distance r_D, which will be assumed here to be one-half the lattice constant ℓ_o for this model. The cubic potential wells are given by

$$V(r) = br^2 - cr^3 - D, \tag{15}$$

where r is the pair separation $(Q_{k+1} - Q_k)$, with the two conditions

$$V(r_D) = 0 \quad \text{and} \quad \left.\frac{dV(r)}{dr}\right|_{r_D} = 0. \tag{16}$$

For a typical lattice, assume the values

$$D = 1 \times 10^{-18} \text{ joule}, \qquad \ell_o = 4 \times 10^{-10} \text{ meter}. \tag{17}$$

Then the force coefficients in Equation (15) are

$$b = 12\,\frac{D}{\ell_o^2} = 7.5 \times 10^{1} \text{ joule/meter}^2 \tag{18}$$

$$\text{and} \quad c = 16\,\frac{D}{\ell_o^3} = 2.5 \times 10^{11} \text{ joule/meter}^3.$$

The typical Debye temperature for a crystal is $\Theta_D \approx 200^\circ$ K. Assuming classical statistics above this temperature,

$$\langle E \rangle \approx b\langle r^2 \rangle - c\langle r^3 \rangle \approx kT. \tag{19}$$

At room temperature, $T \approx 300^\circ$ K or $kT \approx 4 \times 10^{-21}$ joule, the average interaction distance is

$$\langle r \rangle \approx \frac{\langle r^3 \rangle}{\langle r^2 \rangle} \approx 0.7 \times 10^{-11} \text{ meter}, \tag{20}$$

as calculated from Equation (19). Thus for this model the ratio of the cubic coupling energy to the quadratic coupling energy is about 0.04 at room temperature. This system is weakly coupled, according to the criterion set up in Chapter II. The temperature corresponding to the ratio 0.1, which was arbitrarily set in Chapter II as the upper bound defining systems with weak coupling strength, is

$$T \approx 4000^\circ \text{ K} . \tag{21}$$

The linear thermal expansion associated with this model may be calculated from the expression

$$\langle r \rangle \approx \tfrac{3}{4} kT \frac{c}{b^2} \approx 4.5 \times 10^{-16} T \tag{22}$$

for the average value of r at moderate temperatures, where T is the absolute temperature. This expression is derived by Kittel (1956, p. 152), and has already been introduced as Equation (17) in Chapter I. The corresponding expression for linear thermal expansion β is

$$l_T = l_0(1+\beta T) = l_0 + \langle r \rangle, \tag{23}$$

so that for the parameters used in this model,

$$\beta = \frac{\langle r \rangle}{\ell_0 T} \approx 1 \times 10^{-6} \frac{1}{^\circ K} . \qquad (24)$$

The actual values of β for most materials range from 1×10^{-6} to 30×10^{-6} per degree Kelvin, so that the quantity of thermal expansion exhibited by this model is of approximately the correct order.

In this section, the couplings between particles for the exactly-tuned nonlinear model specified in Equation (1) of this chapter have been investigated. A one-dimensional ionic crystal has been suggested as one possible interpretation. Using some typical crystal lattice parameters, the strength of coupling and thermal expansion exhibited by the cubic-coupled particle system have been calculated. For moderate temperatures, this crystal lattice model is weakly coupled, in the sense defined in Chapter II, and possesses a reasonable thermal expansion property.

In the following section, brief mention will be made of some other nonlinear systems which are capable of complete energy sharing.

Other Energy-Sharing Nonlinear Models

In addition to the original simple systems studied by the group of Fermi, Pasta and Ulam, there have been several other types of nonlinear systems which have received recent attention. Most of these are capable of complete energy sharing among all modes.

Jackson (1963b) has made a preliminary investigation of the effect of imperfections on energy sharing in the FPU cubic-coupled systems. The coupling parameters were made different for different coupling terms in an $N = 3$ system, so as to enhance the participation of certain

modes. The result was a small but not significant change in the extent of energy sharing.

Zabusky (1962) has studied in detail the finite string which is obtained from the cubic-coupled FPU system by taking the limit as $N \rightarrow \infty$. The linear spring constants, inverses of the particle masses, and coupling parameter are all taken to be proportional to N in Zabusky's model, so that the frequency spectrum remains discrete. A method of analysis which provides a solution which is good up to a certain breakdown time is presented. In a more recent paper (Kruskal and Zabusky 1964) the method is improved by the inclusion of higher derivatives.

Northcote and Potts (1963) have formulated a hard sphere model (one-dimensional) which is calculationally very simple and yet which displays complete energy sharing. This is accomplished by adding a very highly nonlinear interaction term to the basic linear system, in such a manner as to produce elastic collisions when two particles approach within a given distance of each other.

Perhaps the most fruitful approach is that taken by Prigogine (1962) and co-workers, in which the infinite string is obtained by keeping the linear spring constants, particle masses, and coupling parameter constant and taking the limit as $N \rightarrow \infty$. In this case there are infinitely many discrete frequencies, making available myriad internal resonances. The same sort of approach is taken by Peierls (1956), with regard to crystal lattice studies.

In this chapter, the exactly-tuned nonlinear coupled oscillator system which will be studied extensively in the remainder of this thesis has been formulated. Some discussion of the corresponding particle

couplings for this model has been given, and some of the other nonlinear energy-sharing systems which appear in the literature have been mentioned briefly.

In the next chapter, some of the available methods of solution of nonlinear coupled oscillator systems will be applied to the model formulated in this chapter.

CHAPTER IV

AVAILABLE PERTURBATION METHODS FOR NONLINEAR COUPLED OSCILLATOR SYSTEMS

Some presently available perturbation methods which are generally useful in the solution of systems of oscillators coupled by nonlinear forces will be illustrated in this chapter. These methods are found to be adequate in dealing with untuned systems, but encounter difficulties characterized by the appearance of "small divisors" when applied to tuned systems. This is a serious shortcoming in an investigation of energy-sharing oscillator systems, which are necessarily tuned systems. The problem of small divisors is most severe for an exactly-tuned model such as that formulated in the last chapter, as will be seen.

The problem of small divisors has already been hinted at in the action-angle formalism which was presented in Chapter II. In the first section of the present chapter, the manner in which small divisors appear in a simple series solution in position-momentum variables will be illustrated in detail. The second section will present the Krylov-Bogoliubov method of dealing with the small divisors problem. This method is prohibitively complicated for nonlinear systems of many coupled oscillators, and so in the last section a rather simple yet effective scheme, adapted from the Wigner-Brillouin perturbation method in quantum mechanics, will be illustrated.

Series Solution in Secular Terms

The fact that the nonlinear forces which couple the oscillators are weak in many physically meaningful cases, such as those discussed in Chapter I, leads naturally to the application of perturbation methods to nonlinear coupled oscillator systems. In most of the presently available perturbation schemes, the small coupling parameter α is used as the expansion parameter for power series representations of positions and momenta.

The principal difficulty encountered by such schemes is the appearance of terms with small divisors in the power series expansions. A certain number of such terms appear automatically in some higher order for any coupled oscillator system; additional terms with small divisors appear if the system is tuned.

These terms with small divisors are actually contributions to the terms which are of zeroth order in the expansion parameter α. As will be seen in the last two sections of this chapter, the main task of the established perturbation methods is to choose the zeroth-order term correctly, so that terms with small divisors do not appear in higher orders.

In this section, a simple-minded approach to the solution of the equations of motion of a nonlinear coupled oscillator system with arbitrary frequencies will be taken. The results of this sort of approach will point out very clearly the need for more refined methods of solution.

For simplicity, the following method and others to be presented in this chapter will be discussed in terms of their application to a two-oscillator nonlinear system. These discussions of two-oscillator

examples will contain all the essential features of applications to more general systems.

The nonlinear system chosen for the examples is governed by the hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1^2 q_1^2 + \omega_2^2 q_2^2) - \alpha \frac{1}{\sqrt{2}}(q_1^2 q_2 - q_2^3), \tag{1}$$

which is Equation (51) of Chapter II. The frequencies ω_1 and ω_2 will be taken as free parameters in Equation (1), unless otherwise specified. The equations of motion are

$$\ddot{q}_1 = -\omega_1^2 q_1 + \alpha\sqrt{2}\, q_1 q_2 ; \tag{2a}$$

$$\ddot{q}_2 = -\omega_2^2 q_2 + \alpha \frac{1}{\sqrt{2}}(q_1^2 - 3q_2^2). \tag{2b}$$

Perhaps the simplest approach to take in solving Equations (2) is the following. The solution to these equations when there is no coupling between the oscillators ($\alpha = 0$) is of the form

$$q_k = A_k \cos(\omega_k t + \theta_k), \tag{3}$$

where the A_k and the θ_k are constants to be fixed by the initial conditions. To find the general solution of Equations (2) when $\alpha \neq 0$, assume a solution of the form

$$q_k = A_k \cos(\omega_k t + \theta_k) + \sum_{l=1}^{\infty} \alpha^l q_{kl}. \tag{4}$$

This assumed form allows for no modification of the general solution

in zeroth order, due to the nonzero coupling. (Also, the power series expansion in α does not explicitly show that the correct expansion parameter for the cubic nonlinear system is the coupling strength $C_s \approx \alpha A$, where A is the average amplitude of the oscillations for a given set of initial conditions, rather than α alone. But this is not particularly important.)

To second order in α, writing $\tau_k = \omega_k t + \theta_k$, let

$$q_1 = A_1 \cos\tau_1 + \alpha q_{11} + \alpha^2 q_{12}; \tag{5a}$$

$$q_2 = A_2 \cos\tau_2 + \alpha q_{21} + \alpha^2 q_{22}. \tag{5b}$$

Substitution of Equations (5) into Equations (2) yields

$$(\omega_1^2-\omega_1^2)A_1\cos\tau_1 + \alpha\{\ddot{q}_{11}+\omega_1^2 q_{11}-\sqrt{2}q_{10}q_{20}\} + \alpha^2\{\ddot{q}_{12}+\omega_1^2 q_{12}-\sqrt{2}(q_{10}q_{21}-q_{11}q_{20})\} = 0; \tag{6a}$$

$$(\omega_2^2-\omega_2^2)A_2\cos\tau_2 + \alpha\{\ddot{q}_{21}+\omega_2^2 q_{21}-\tfrac{1}{\sqrt{2}}(q_{10}^2-3q_{20}^2)\} + \alpha^2\{\ddot{q}_{22}+\omega_2^2 q_{22}-\sqrt{2}(q_{10}q_{11}-3q_{20}q_{21})\} = 0, \tag{6b}$$

where $q_{10} = A_1 \cos\tau_1$ and $q_{20} = A_2 \cos\tau_2$.

Equate to zero the coefficients of the respective powers of α in Equations (6). The zeroth-order equations are identically satisfied. The first-order equations are

$$\ddot{q}_{11}+\omega_1^2 q_{11} = \tfrac{1}{\sqrt{2}}A_1A_2\cos(\tau_2+\tau_1) + \tfrac{1}{\sqrt{2}}A_1A_2\cos(\tau_2-\tau_1); \tag{7a}$$

$$\ddot{q}_{21} + \omega_2^2 q_{21} = \frac{1}{2\sqrt{2}}(A_1^2 - 3A_2^2) + \frac{1}{2\sqrt{2}} A_1^2 \cos 2\tau_1 - \frac{3}{2\sqrt{2}} \cos 2\tau_2 . \tag{7b}$$

The solution of Equation (7) for all values of ω_1 and ω_2 except the exactly-tuned case $\omega_2 = 2\omega_1$ (discussed later in this section) is

$$q_{11} = \frac{A_1 A_2 \cos(\tau_2 - \tau_1)}{\sqrt{2}[\omega_1^2 - (\omega_2 + \omega_1)^2]} + \frac{A_1 A_2 \cos(\tau_2 - \tau_1)}{\sqrt{2}[\omega_1^2 - (\omega_2 - \omega_1)^2]} ; \tag{8a}$$

$$q_{21} = \frac{(A_1^2 - 3A_2^2)}{2\sqrt{2}\,\omega_2^2} + \frac{A_1^2 \cos 2\tau_1}{2\sqrt{2}[\omega_2^2 - 4\omega_1^2]} - \frac{3A_2^2 \cos 2\tau_2}{2\sqrt{2}[\omega_2^2 - 4\omega_2^2]} . \tag{8b}$$

Substitution of this first-order solution into the second-order equations from Equations (6) yields

$$\begin{aligned} \ddot{q}_{12} + \omega_1^2 q_{12} = {} & \frac{(A_1^2 - 3A_2^2) A_1 \cos\tau_1}{2\omega_2^2} + \frac{A_1^3 \cos\tau_1}{4[\omega_2^2 - 4\omega_1^2]} \\ & + \frac{A_1^3 \cos 3\tau_1}{4[\omega_2^2 - 4\omega_1^2]} - \frac{3A_1 A_2^2 \cos(\tau_1 + 2\tau_2)}{4[\omega_2^2 - 4\omega_2^2]} \\ & - \frac{3A_1 A_2^2 \cos(\tau_1 - 2\tau_2)}{4[\omega_2^2 - 4\omega_2^2]} + \frac{A_1 A_2^2 \cos\tau_1}{2[\omega_1^2 - (\omega_1 + \omega_2)^2]} \\ & + \frac{A_1 A_2^2 \cos(\tau_1 + 2\tau_2)}{2[\omega_1^2 - (\omega_1 + \omega_2)^2]} + \frac{A_1 A_2^2 \cos\tau_1}{2[\omega_1^2 - (\omega_1 - \omega_2)^2]} \\ & + \frac{A_1 A_2^2 \cos(\tau_1 - 2\tau_2)}{2[\omega_1^2 - (\omega_1 - \omega_2)^2]} ; \end{aligned} \tag{9a}$$

$$\ddot{q}_{22}+\omega_2^2 q_{22}=\frac{A_1^2A_2\cos\tau_2}{2[\omega_1^2-(\omega_1+\omega_2)^2]}+\frac{A_1^2A_2\cos(2\tau_1+\tau_2)}{2[\omega_1^2-(\omega_1+\omega_2)^2]} \quad (9b)$$

$$+\frac{A_1^2A_2\cos\tau_2}{2[\omega_1^2-(\omega_1-\omega_2)^2]}+\frac{A_1^2A_2\cos(2\tau_1-\tau_2)}{2[\omega_1^2-(\omega_1-\omega_2)^2]}$$

$$-\frac{3(A_1^2-3A_2^2)A_2\cos\tau_2}{2\omega_2^2}-\frac{3A_1^2A_2\cos(2\tau_1-\tau_2)}{4[\omega_2^2-4\omega_1^2]}$$

$$-\frac{3A_1^2A_2\cos(2\tau_1+\tau_2)}{4[\omega_2^2-4\omega_1^2]}+\frac{9A_2^3\cos\tau_2}{4[\omega_2^2-4\omega_2^2]}+\frac{9A_2^3\cos 3\tau_2}{4[\omega_2^2-4\omega_2^2]}.$$

Integration of second-order Equations (9) will produce terms which are proportional to $t \sin\tau_k$ in q_{k2}, arising from the $\cos\tau_k$ terms. Such terms are called secular terms, from early celestial perturbation theory. A truncated series expansion in which secular terms appear is inadmissible as a form for the general solution q_k, for the secular terms are unbounded and aperiodic, whereas the q_k are known to be bounded and multiply periodic.

For the particular nonlinear system considered here, secular terms appear in second and higher even orders of α for all values of ω_1 and ω_2. Additional secular terms will appear in first order if the system is exactly tuned, that is, if $\omega_2 = 2\omega_1$. This may be seen from Equations (8), where for $\omega_2 = 2\omega_1$,

$$\cos(\tau_2-\tau_1)=\cos\tau_1 \text{ in Equation (8a)}, \quad (10a)$$

$$\text{and} \quad \cos 2\tau_1=\cos\tau_2 \text{ in Equation (8b)}. \quad (10b)$$

Other similar tuning effects will produce additional secular terms in various higher orders. For example, if $\omega_2 = \omega_1$, then there

will be additional secular terms produced by the integration of second-order Equations (9).

The appearance of secular terms as such is not crucial; the more important general effect is the appearance of terms with small divisors in the solution.

To see this, consider the example of a single ordinary differential equation of the form

$$\ddot{q} + \omega^2 q = \alpha A^2 \cos(\omega+s)t, \tag{11}$$

which is typical of Equations (7) or (9). Here A represents a sort of typical amplitude of the oscillations, and s is a tuning parameter such as was used in Chapter II. The general solution of Equation (11) with initial conditions $q(0) = A$ and $\dot{q}(0) = 0$, for example, is

$$q = A\left[1 - \frac{\alpha A}{[\omega^2-(\omega+s)^2]}\right]\cos\omega t + \frac{\alpha A^2}{[\omega^2-(\omega+s)^2]}\cos(\omega+s)t, \tag{12}$$

which is valid of $s \neq 0$. That portion of the right side which is controlled by α will be important only if

$$[\omega^2-(\omega+s)^2] \lessapprox \alpha A, \tag{13}$$

which is equivalent to the tuning condition given by Equation (64) in Chapter II.

This shows that small divisors appear and make the coupling, characterized by the right side of Equation (11), important in zeroth order in the coupling parameter α for any tuned system, that is, for

any system satisfying the condition given in Equation (13). The coupling parameter alone is not adequate for the right side of Equation (13); the "typical amplitude" of the oscillations must also be included.

Secular terms are just a special form of terms with small divisors, which appear in the solution, Equation (12), in the limit as the tuning parameter approaches zero:

$$\lim_{s\to 0} q = A\cos\omega t - \alpha A^2 \cos\omega t \lim_{s\to 0}\left[\frac{1-\cos st}{[\omega^2-(\omega+s)^2]}\right] + \alpha A^2 \sin\omega t \lim_{s\to 0}\left[\frac{\sin st}{[\omega^2-(\omega+s)^2]}\right] \qquad (14)$$

$$= A\cos\omega t + \frac{\alpha A^2}{2\omega} t \sin\omega t.$$

The last part of Equation (14) is the solution which would be obtained from Equation (11) if tuning parameter s were set equal to zero to begin with.

Thus although the appearance of secular terms in the solutions of exactly-tuned systems is rather spectacularly inadmissible, it is only a special case of a more general inadmissibility of terms which have small divisors, a situation which occurs any time the present "simple-minded" approach is applied to tuned coupled oscillator systems.

In this section, it has been shown how small divisors appear when a perturbation expansion is tried in which the zeroth-order term of the general solution is assumed to be exactly the solution of the uncoupled system. Small divisors appear in second and higher even orders for all choices of the frequencies, in the example illustrated, in the severe form of secular terms. Additional small divisors appear in all orders

for tuned systems, in the form of secular terms if the systems are exactly tuned.

A truncated series solution containing secular terms is of no use as a general solution valid for all time. Only if the series could be summed to eliminate secular terms, would this form be of much value. But the summation of the series is a prohibitively difficult task. Therefore, it is necessary to find a better method of solution.

In particular, it is necessary to develop a better means of determining the zeroth-order terms about which the perturbation expansion is made. The last two sections of this chapter present some available methods which do this.

Method of Krylov-Bogoliubov (KB)

Krylov and Bogoliubov (1947) observe that the result of summing the series of secular terms which arise in the method just described, is in general to alter the amplitude and frequency of the zeroth-order term in the trial solution, Equation (4) of the previous section.

In order to provide for amplitude and frequency modulation, KB suggest that a power series solution of the form

$$q_k = A_k \cos\tau_k + \sum_{l=1}^{\infty} \alpha^l q_{kl}(A_j, \tau_j) \tag{15}$$

should be assumed, where the amplitudes and frequencies are functions of time specified by the differential equations

$$\dot{A}_k = \sum_{l=1}^{\infty} \alpha^l A_{kl}(A_j, \tau_j); \tag{16a}$$

$$\dot{\tau}_k = \omega_k + \sum_{\ell=1}^{\infty} \alpha^\ell \omega_{k\ell}(A_j, \tau_j), \tag{16b}$$

and the $q_{k\ell}$ are periodic functions of each of the τ_j.

The solution proceeds formally as follows. The assumed series solutions, Equations (15), are substituted into the equations of motion, Equation (2) in the case of the two-oscillator nonlinear system. The coefficient of each power of α is equated to zero, which leads to a system of differential equations which can formally be solved for the $q_{k\ell}$ as functions of the A_ℓ and τ_ℓ. The $A_{k\ell}$ and $\omega_{k\ell}$ of Equations (16) are determined as functions of the A_ℓ and τ_ℓ at each step so as to eliminate secular terms.

Formally, it is then possible to solve the resulting system of coupled differential equations, Equations (16), to obtain the A_k and the τ_k as functions of time.

However, for nonlinear systems, the solution of the systems of coupled nonlinear differential equations, Equations (16), may be just as difficult as solution of the original problem. Therefore, the method of KB in its full generality is not very practical as a general method for the solution of many-particle nonlinear oscillator systems.

At best, a restricted version of the KB method, in which only frequency modulation is employed, may be used to give qualitative results for a fairly sharply-tuned system. As an example, Ford (1961) uses this method to analyze the $N = 31$ FPU system, obtaining most of the essential features of the exact solution.

In this section, the Krylov-Bogoliubov perturbation method and its

application to a nonlinear coupled oscillator system have been discussed briefly. The formalism involved in this application is in general prohibitively difficult, although it must be mentioned that the KB method has been quite successful in extensive applications in other fields of physics and engineering. As an alternative to the KB scheme, another more recent method will be presented in the following section.

Method of Wigner-Brillouin (WB)

A much simpler method than that of Krylov-Bogoliubov for the elimination of secular terms in the solutions of nonlinear coupled oscillator systems is a classical version of the Wigner-Brillouin perturbation method in quantum mechanics (Brueckner 1959). An adaptation of this method was first applied to fairly sharply-tuned FPU nonlinear systems by Jackson (1963a, 1963b) with considerable success.

The WB method involves only frequency shifts to eliminate secular terms. The computations involve generally algebraic rather than differential equations, which is a considerable advantage in nonlinear cases. The results of the method can be placed in a form such that the relationship between tuning and energy sharing is easily discussed without the need for a complete algebraic solution.

As an illustration, the WB method will be applied to the two-oscillator nonlinear system of Equations (1) and (2). This example will display all the essential features of more general applications.

To second order in α, assume a general solution of the form

$$q_1 = A_1 \cos\tau_1 + \alpha q_{11} + \alpha^2 q_{12}; \qquad (17a)$$

$$q_2 = A_2 \cos\tau_2 + \alpha q_{21} + \alpha^2 q_{22}, \tag{17b}$$

where $\tau_1 = \Omega_1 t + \theta_1$, $\tau_2 = \Omega_2 t + \theta_2$, and where Ω_1 and Ω_2 are unknown frequencies to be determined.

Substitution of Equations (17) into the equations of motion, Equations (2), yields

$$(\omega_1^2 - \Omega_1^2) A_1 \cos\tau_1 + \alpha\{\ddot{q}_{11} + \omega_1^2 q_{11} - \sqrt{2} q_{10} q_{20} + \alpha^2\{\ddot{q}_{12} + \omega_1^2 q_{12} - \sqrt{2}(q_{10} q_{21} - q_{11} q_{20})\} = 0; \tag{18a}$$

$$(\omega_2^2 - \Omega_2^2) A_2 \cos\tau_2 + \alpha\{\ddot{q}_{21} + \omega_2^2 q_{21} - \frac{1}{\sqrt{2}}(q_{10}^2 - 3q_{20}^2)\} + \alpha^2\{\ddot{q}_{22} + \omega_2^2 q_{22} - \sqrt{2}(q_{10} q_{11} - 3q_{20} q_{21})\} = 0, \tag{18b}$$

where $q_{10} = A_1 \cos\tau_1$ and $q_{20} = A_2 \cos\tau_2$. Equations (18) are identical with Equations (6) except for the replacement of ω_k by Ω_k in the zeroth-order coefficients and in the τ_k.

Equate to zero the coefficients of the respective powers of α in Equations (18). The first-order and the second-order equations are exactly the same as those obtained for the two-oscillator system in the first section of this chapter, except that Ω_k replaces ω_k in the τ_k.

Integration of the first-order equations, Equations (7), without specification of whether the system is tuned or not, results formally in

$$q_{11} = \frac{A_1 A_2 \cos(\tau_2 + \tau_1)}{\sqrt{2}[\omega_1^2 - (\Omega_2 + \Omega_1)^2]} + \frac{A_1 A_2 \cos(\tau_2 - \tau_1)}{\sqrt{2}[\omega_1^2 - (\Omega_2 - \Omega_1)^2]}; \tag{19a}$$

$$q_{21} = \frac{(A_1^2 - 3A_2^2)}{2\sqrt{2}\,\omega_2^2} + \frac{A_1^2 \cos 2\tau_1}{2\sqrt{2}[\omega_1^2 - 4\Omega_1^2]} - \frac{3A_2^2 \cos 2\tau_2}{2\sqrt{2}[\omega_2^2 - 4\Omega_2^2]}. \quad (19b)$$

If the system is tuned, then $\omega_2 \approx 2\omega_1$ and some of the denominators in Equations (19) may be small; this situation will be considered later in this section.

Substitution of this first-order solution into the second-order equations from Equations (18) yields

$$\ddot{q}_{12} + \omega_1^2 q_{12} = \frac{(A_1^2 - 3A_2^2)A_1 \cos\tau_1}{2\omega_2^2} + \frac{A_1^3 \cos\tau_1}{4[\omega_2^2 - 4\Omega_1^2]} \quad (20a)$$

$$+ \frac{A_1^3 \cos 3\tau_1}{4[\omega_2^2 - 4\Omega_1^2]} - \frac{3A_1A_2^2 \cos(\tau_1 + 2\tau_2)}{4[\omega_2^2 - 4\Omega_2^2]}$$

$$- \frac{3A_1A_2^2 \cos(\tau_1 - 2\tau_2)}{4[\omega_2^2 - 4\Omega_2^2]} + \frac{A_1A_2^2 \cos\tau_1}{2[\omega_1^2 - (\Omega_1 + \Omega_2)^2]}$$

$$+ \frac{A_1A_2^2 \cos(\tau_1 + 2\tau_2)}{2[\omega_1^2 - (\Omega_1 + \Omega_2)^2]} + \frac{A_1A_2^2 \cos\tau_1}{2[\omega_1^2 - (\Omega_1 - \Omega_2)^2]} + \frac{A_1A_2^2 \cos(\tau_1 - 2\tau_2)}{2[\omega_1^2 - (\Omega_1 - \Omega_2)^2]};$$

$$\ddot{q}_{22} + \omega_2^2 q_{22} = \frac{A_1^2A_2 \cos\tau_2}{2[\omega_1^2 - (\Omega_1 + \Omega_2)^2]} + \frac{A_1^2A_2 \cos(2\tau_1 + \tau_2)}{2[\omega_1^2 - (\Omega_1 + \Omega_2)^2]} \quad (20b)$$

$$+ \frac{A_1^2A_2 \cos\tau_2}{2[\omega_1^2 - (\Omega_1 - \Omega_2)^2]} + \frac{A_1^2A_2 \cos(2\tau_1 - \tau_2)}{2[\omega_1^2 - (\Omega_1 - \Omega_2)^2]}$$

$$- \frac{3(A_1^2 - 3A_2^2)A_2 \cos\tau_2}{2\omega_2^2} - \frac{3A_1^2A_2 \cos(2\tau_1 - \tau_2)}{4[\omega_2^2 - 4\Omega_1^2]}$$

$$- \frac{3A_1^2A_2 \cos(2\tau_1 + \tau_2)}{4[\omega_2^2 - 4\Omega_1^2]} + \frac{9A_2^3 \cos\tau_2}{4[\omega_2^2 - 4\Omega_2^2]} + \frac{9A_2^3 \cos 3\tau_2}{4[\omega_2^2 - 4\Omega_2^2]}.$$

The terms involving $\cos \tau_1$ in Equation (20a) and the terms involving $\cos \tau_2$ in Equation (20b) must be eliminated, since they will lead to secular terms upon integration of Equations (20).

The principal features of the WB method is that the zeroth-order terms in Equations (18) are set up specifically for the purpose of eliminating from higher orders those terms which produce secular terms upon integration. The $\cos \tau_1$ terms of Equation (20a) are grouped with the $\cos \tau_1$ zeroth-order term in Equation (18a); the $\cos \tau_2$ terms of Equation (20b) are grouped with the $\cos \tau_2$ zeroth-order term in Equation (18b).

The frequencies Ω_1 and Ω_2 are determined by equating the coefficients of the zeroth-order terms in Equations (18) to zero. The result is

$$\Omega_1^2 = \omega_1^2 + \alpha^2 \left\{ \frac{(A_1^2 - 3A_2^2)}{2\omega_2^2} + \frac{A_1^2}{4[\omega_2^2 - 4\Omega_1^2]} + \frac{A_2^2}{2[\omega_1^2 - (\Omega_1 + \Omega_2)^2]} + \frac{A_2^2}{2[\omega_1^2 - (\Omega_1 - \Omega_2)^2]} \right\}; \tag{21a}$$

$$\Omega_2^2 = \omega_2^2 + \alpha^2 \left\{ \frac{A_1^2}{2[\omega_1^2 - (\Omega_1 + \Omega_2)^2]} + \frac{A_1^2}{2[\omega_1^2 - (\Omega_1 + \Omega_2)^2]} - \frac{3(A_1^2 - 3A_2^2)}{2\omega_2^2} + \frac{9A_2^2}{4[\omega_2^2 - 4\Omega_2^2]} \right\}. \tag{21b}$$

Now that those terms which, upon integration, produce secular terms have been eliminated from the second-order Equations (20), the latter equations may be integrated.

The general solution of the two-oscillator nonlinear system by the WB method, correct to order a^2, is then

$$q_1 = A_1\cos\tau_1 - \alpha\left\{\frac{A_1A_2\cos(\tau_1+\tau_2)}{\sqrt{2}\left[\omega_1^2-(\Omega_1+\Omega_2)^2\right]}+\frac{A_1A_2\cos(\tau_1-\tau_2)}{\sqrt{2}\left[\omega_1^2-(\Omega_1-\Omega_2)^2\right]}\right\} \tag{22a}$$

$$+\alpha^2\left\{\frac{A_1^3\cos 3\tau_1}{4\left[\omega_2^2-4\Omega_1^2\right]\left[\omega_1^2-9\Omega_1^2\right]}-\frac{3A_1A_2^2\cos(\tau_1+2\tau_2)}{4\left[\omega_2^2-4\Omega_2^2\right]\left[\omega_1^2-(\Omega_1+2\Omega_2)^2\right]}\right.$$

$$-\frac{3A_1A_2^2\cos(\tau_1-2\tau_2)}{4\left[\omega_2^2-4\Omega_2^2\right]\left[\omega_1^2-(\Omega_1-2\Omega_2)^2\right]}$$

$$+\frac{A_1A_2^2\cos(\tau_1+2\tau_2)}{2\left[\omega_1^2+(\Omega_1+\Omega_2)^2\right]\left[\omega_1^2-(\Omega_1-2\Omega_2)^2\right]}$$

$$\left.+\frac{A_1A_2^2\cos(\tau_1-2\tau_2)}{2\left[\omega_1^2-(\Omega_1-\Omega_2)^2\right]\left[\omega_1^2-(\Omega_1-2\Omega_2)^2\right]}\right\};$$

$$q_2 = A_2\cos\tau_2 - \alpha\left\{\frac{(A_1^2-3A_2^2)}{2\sqrt{2}\,\omega_2^2}+\frac{A_1^2\cos 2\tau_1}{2\sqrt{2}\left[\omega_2^2-4\Omega_1^2\right]}\right. \tag{22b}$$

$$\left.-\frac{3A_2^2\cos 2\tau_2}{2\sqrt{2}\left[\omega_2^2-4\Omega_2^2\right]}\right\}$$

$$+\alpha^2\left\{\frac{A_1^2A_2\cos(2\tau_1+\tau_2)}{2\left[\omega_1^2-(\Omega_1+\Omega_2)^2\right]\left[\omega_2^2-(2\Omega_1+\Omega_2)^2\right.}\right.$$

$$+\frac{A_1^2A_2\cos(2\tau_1-\tau_2)}{2\left[\omega_1^2-(\Omega_1-\Omega_2)^2\right]\left[\omega_2^2-(2\Omega_1-\Omega_2)^2\right]}$$

$$-\frac{3A_1^2A_2\cos(2\tau_1-\tau_2)}{4\left[\omega_2^2-4\Omega_1^2\right]\left[\omega_2^2-(2\Omega_1-\Omega_2)^2\right]}$$

$$\left.-\frac{3A_1^2A_2\cos(2\tau_1+\tau_2)}{4\left[\omega_1^2-4\Omega_1^2\right]\left[\omega_2^2-(2\Omega_1+\Omega_2)^2\right]}+\frac{9A_2^2\cos 3\tau_2}{4\left[\omega_2^2-4\Omega_2^2\right]\left[\omega_2^2-9\Omega_2^2\right]}\right\}.$$

The overall accomplishment of the WB perturbation scheme is a conversion of the nonlinear differential equations problem of solving Equations (18) into the nonlinear algebraic problem of solving Equations (21) for the perturbed frequencies Ω_1 and Ω_2. In fact, Equations (22) may be used to discuss energy sharing immediately, without need for an exact solution of Equations (21) for Ω_1 and Ω_2.

An examination of Equations (21) shows that a good approximate solution to order α^2 is obtained by setting $\Omega_1 = \omega_1$ and $\Omega_2 = \omega_2$ in all the denominators on the right side. Provided that none of these denominators is less than αA in magnitude, where A is a typical amplitude, Equations (22) show that for small αA, $q_1 = A_1 \cos \tau_1$ and $q_2 = A_2 \cos \tau_2$. In this case the oscillators move almost as if uncoupled (except for the small frequency shift) and no appreciable exchange of energy between oscillators occurs.

To study the effect of tuning the frequencies using this formalism, examine the approximation in Equation (21a) for Ω_1:

$$\Omega_1^2 = \omega_1^2 + \alpha^2 \left\{ \frac{(A_1^2 + 3A_2^2)}{2\omega_2^2} + \frac{A_1^2}{4[\omega_2^2 - 4\omega_1^2]} + \frac{A_2^2}{2[\omega_1^2 - (\omega_2 + \omega_1)^2]} + \frac{A_2^2}{2[\omega_1^2 - (\omega_2 - \omega_1)^2]} \right\}. \tag{23}$$

Small denominators will occur only if $|2\omega_1 - \omega_2| \leq \alpha A$. The same conclusion is reached from an examination of Equation (21b).

If $|2\omega_1 - \omega_2| \approx \alpha A$, the threshold of tuning, then the exact solution of Equations (21) is of the form

$$\Omega_1 = \omega_1 + \alpha A + \cdots ; \tag{24a}$$

$$\Omega_2 = \omega_2 + \alpha A + \cdots . \tag{24b}$$

As a consequence, some of the terms of order αA in the general solution, Equations (22), have denominators of order αA and become zeroth-order terms: for example,

$$\frac{\alpha A_1 A_2 \cos(\tau_2 - \tau_1)}{\sqrt{2}\,[\omega_1^2 - (\Omega_2 - \Omega_1)^2]} \approx A \cos(\tau_2 - \tau_1). \tag{25}$$

This modification to the zeroth-order term means that appreciable energy sharing will occur in this system.

If $2\omega_1$ is not of the order of ω_2, then from Equations (21) the frequency shifts are of order $(\alpha A)^2$:

$$\Omega_1 = \omega_1 + (\alpha A)^2 + \cdots ; \tag{26a}$$

$$\Omega_2 = \omega_2 + (\alpha A)^2 + \cdots . \tag{26b}$$

An examination of the general solution, Equations (22), now shows that no first-order terms can have denominators of order αA. Among the second-order terms, denominators of order $(\alpha A)^2$ can occur only if $\omega_1 - \omega_2 \lessapprox (\alpha A)^2$, a sharper tuning condition than that provided in first order. Thus this system should exhibit energy sharing independent of α for the frequencies $\omega_2 = \omega_1$, as well as when $\omega_2 = 2\omega_1$. The amount of energy sharing when $\omega_2 = \omega_1$ may not be appreciable, however, since this is a higher-order resonance.

To order α^2, the two sets of tuned frequencies discussed above are the only two which can be found for this system. All the general conclusions obtained in Chapter II using action-angle formalism are included in the results of this WB formalism. In fact, the WB method has provided more extensive results and displays considerable mathematical simplicity in doing it.

Until now in this discussion, the WB method, which is the best perturbation method for nonlinear coupled oscillator systems now available, has been used only to eliminate those secular terms which arise in second and higher even orders of α for any choice of ω_1 and ω_2. The tuned sets of frequencies which make possible appreciable energy sharing have been identified. But the WB method has not yet been used to deal with an exactly-tuned system, such as the model formulated in Chapter III as the system of prime interest in the remainder of this thesis.

In order to analyze the exactly-tuned system using the WB method, consider the solution of Equations (2) when $\omega_2 = 2\omega_1$. Substitution of the trial solution, Equations (17), into Equations (2) results in Equations (18), as before. The first-order equations in α are Equations (7):

$$\ddot{q}_{11} + \omega_1^2 q_{11} = \frac{1}{\sqrt{2}} A_1 A_2 \cos(\tau_2 + \tau_1) + \frac{1}{\sqrt{2}} A_1 A_2 \cos(\tau_2 - \tau_1); \tag{7a}$$

$$\ddot{q}_{21} + \omega_2^2 q_{21} = \frac{1}{2\sqrt{2}} (A_1^2 - 3A_2^2) + \frac{1}{2\sqrt{2}} A_1^2 \cos 2\tau_1 - \frac{3A_2^2}{2\sqrt{2}} \cos 2\tau_2. \tag{7b}$$

Since $(\tau_2 - \tau_1) = \tau_1 + \alpha At$ and $2\tau_1 = \tau_2 + \alpha At$, integration of the

cos $(\tau_2 - \tau_1)$ term in Equation (7a) and integration of the cos $2\tau_1$ term in Equation (7b) will lead to terms having small divisors.

It is not particularly clear how these terms should be treated in general in the WB formalism. They can be conveniently moved down to zeroth order only if $\tau_2 = 2\tau_1$ is true exactly. This relation will be found to be valid only for certain restricted sets of initial conditions, which give rise to periodic solutions, as will be discussed in Chapter VI.

The WB perturbation scheme provides a simple and effective method of dealing with the problem of the small divisors which automatically arise in the solutions of untuned oscillator systems. This formalism also provides a surprisingly good analysis of tuned systems, so long as they are not too sharply tuned. Jackson (1963b) makes this observation in the case of strongly-coupled FPU systems.

However, the manner in which the WB method should be used to deal with exactly-tuned systems, such as are of principal interest in this part of the thesis, is not entirely clear in general. Apparently, contributions to zeroth order arise in all orders for an exactly-tuned system.

It has been indicated in this chapter that the better presently available perturbation methods, such as the method of Wigner-Brillouin as adapted by Jackson, are best suited to deal with systems which are other than exactly tuned systems, the rest of this thesis will be an effort to develop a different sort of perturbation scheme designed basically for exactly-tuned systems. If insight into the solution of an exactly-tuned system can be gained, then the solution of any system tuned to any degree should be available in principle, using the exactly-tuned system solution as a zeroth-order solution and employing conventional perturbation

methods with the tuning parameters playing the role of perturbation parameters.

This chapter has established the need for a better method of solution for exactly-tuned nonlinear coupled oscillator systems. In the next chapter, various methods will be used to solve the comparatively easy exactly-tuned linear coupled oscillator system introduced in Chapter II. The experience gained in the analysis of the linear system in Chapter V will be used as a guide in the analysis of the nonlinear coupled oscillator system in Chapter VI.

CHAPTER V

VARIOUS METHODS OF ANALYTIC SOLUTION FOR LINEAR COUPLED OSCILLATOR SYSTEMS

In Equation (17) of Chapter II, a rather artificially constructed system having linear couplings between adjacent harmonic normal modes was introduced. This system was constructed in order to provide a system which is easy to solve and yet which has many similarities both of behavior and analysis to a nonlinear coupled oscillator system.

In this chapter, an exactly-tuned linear coupled oscillator system of this sort will be solved by several different methods. The experience gained in solving the linear system in this chapter will be used as a guide in the analysis of more difficult exactly-tuned nonlinear systems in Chapter VI.

In the first section of this chapter, a solution of the linear system in its true normal modes will be illustrated, primarily for reference. In the second section, a solution of an $N = 2$ linear system by means of integration of the action-angle equations of motion will be presented. The third section will introduce the use of a trial solution for solving linear systems; the solutions of a given system and the resonant system which corresponds to it will be compared, and a prescription for the momenta of a resonant system will be presented. The final section of this chapter will illustrate a method of general solution by means of expansions about periodic solutions for the linear system.

Solution in Normal Modes

Principally for purposes of comparison with the other solutions to be obtained later in this chapter, the solution of the linear system in terms of its true normal modes will be obtained in this section. This method is of no value in the corresponding analysis of a nonlinear system.

The example chosen for solution in this chapter is the two-oscillator linear system governed by

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + \omega^2 q_1^2 + \omega^2 q_2^2) - \alpha q_1 q_2, \tag{1}$$

in which the equal frequencies, $\omega_1 = \omega_2 = \omega$, represent an exactly-tuned set for this linear coupling, as was indicated in Chapter II. The corresponding equations of motion are

$$\ddot{q}_1 + \omega^2 q_1 = \alpha q_2; \tag{2a}$$

$$\ddot{q}_2 + \omega^2 q_2 = \alpha q_1. \tag{2b}$$

Except as noted in the second section, the discussions of methods of solution as illustrated by two-oscillator examples in this chapter will contain all the essential features of applications to larger linear coupled oscillator systems.

For any relatively simple system of linearly coupled particles, a true normal mode solution can be obtained by the direct application to the particle hamiltonian of a transformation which completely decouples the system.

The same result may be achieved in less direct fashion by first applying a "harmonic normal mode" transformation to the particle hamiltonian to decouple that part of the system which remains when $\alpha = 0$, and by then following this with a second transformation which completes the decoupling of the entire system, including the coupling term controlled by α.

In the present case, the latter two-step procedure is the more convenient, because the first step has already been made, in that Equations (2) are formulated in harmonic normal mode variables.

The second step, the transformation which completes the decoupling of Equations (2), is

$$q_1 = \frac{1}{\sqrt{2}}(y_1 + y_2); \tag{3a}$$

$$q_2 = \frac{1}{\sqrt{2}}(y_1 - y_2), \tag{3b}$$

and the true normal mode equations of motion are

$$\ddot{y}_1 + (\omega^2 - \alpha) y_1 = 0; \tag{4a}$$

$$\ddot{y}_2 + (\omega^2 + \alpha) y_2 = 0, \tag{4b}$$

which are completely decoupled. The general harmonic normal mode solution is therefore

$$q_1 = C_1 \cos(\Omega_1 t + \theta_1) + C_2 \cos(\Omega_2 t + \theta_2); \tag{5a}$$

$$q_2 = C_1 \cos(\Omega_1 t + \theta_1) - C_2 \cos(\Omega_2 t + \theta_2), \tag{5b}$$

where $\Omega_1 = (\omega^2 - \alpha)^{\frac{1}{2}}$, $\Omega_2 = (\omega^2 + \alpha)^{\frac{1}{2}}$, and where C_1, C_2, θ_1, and θ_2 are constants to be determined by the initial conditions. Note that the effect of the coupling in this tuned case is to split the harmonic normal mode frequencies, so that $\Omega_1 < \omega$ and $\Omega_2 > \omega$. This is typical of larger linear systems, in which the frequencies are always split in symmetric pairs. If the number of oscillators, N, is odd, then one frequency remains unshifted while the others split.

In this section, the general solution of the $N = 2$ linear system has been obtained by transforming to the true normal modes. This solution will be compared to solutions obtained by other methods in the remainder of this chapter.

Solution in Action-Angle Variables

As mentioned in the discussion of celestial systems in Chapter I, a general solution of any system having N degrees of freedom can be constructed if N constants of the motion are known.

Such a method of solution for the $N = 2$ linear system will be illustrated in this section. The system actually solved here is the "resonant" system which corresponds to the "original" system introduced and solved in the first section of this chapter. A comparison of the solutions to the resonant and original systems will be delayed until the third section of this chapter.

The method presented in this section has direct application to the nonlinear $N = 2$ system as well. However, it should be noted that use of this particular form of the constants of the motion is not as easily generalized for systems of more than two oscillators as are most

of the other methods illustrated in this chapter.

As shown in Chapter II, the canonical transformation

$$q_k = \left(\frac{2J_k}{\omega_k}\right)^{\frac{1}{2}} \cos\varphi_k, \quad p_k = -(2\omega_k J_k)^{\frac{1}{2}} \sin\varphi_k \tag{6}$$

may be applied to Equation (1) to produce action-angle hamiltonian

$$H = \omega J_1 + \omega J_2 - \frac{\alpha}{\omega} J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} \left[\cos(\varphi_1+\varphi_2) + \cos(\varphi_1-\varphi_2)\right] \tag{7}$$

and corresponding equations of motion which have already been presented as Equations (44) of Chapter II, except for the present restriction that $\omega_1 = \omega_2 = \omega$.

In Chapter II, action-angle formalism was used to show that only the slowly-varying "resonant" terms in Equations (44) are important in energy sharing. The rapidly-varying "nonresonant" terms should at most serve to perturb the solution of the resonant system in a manner which can be adequately treated by usual perturbation procedures.

Therefore, it should be possible to obtain a good approximation to an actual energy-sharing situation by retaining only the slowly-varying terms in the hamiltonian of Equation (7), and solving the resulting "resonant" system.

When the rapidly-varying coupling term is dropped from Equation (7), the resonant hamiltonian is

$$H = \omega J_1 + \omega J_2 - \frac{\alpha}{\omega} J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} \cos(\varphi_1-\varphi_2), \tag{8}$$

with corresponding equations of motion

$$\dot{J}_1 = -\frac{\alpha}{\omega} J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} \sin(\varphi_1 - \varphi_2); \tag{9a}$$

$$\dot{J}_2 = +\frac{\alpha}{\omega} J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} \sin(\varphi_1 - \varphi_2); \tag{9b}$$

$$\dot{\varphi}_1 = \omega + \frac{\alpha}{\omega} \tfrac{1}{2} J_1^{-\frac{1}{2}} J_2^{\frac{1}{2}} \cos(\varphi_1 - \varphi_2); \tag{9c}$$

$$\dot{\varphi}_2 = \omega + \frac{\alpha}{\omega} \tfrac{1}{2} J_1^{\frac{1}{2}} J_2^{-\frac{1}{2}} \cos(\varphi_1 - \varphi_2). \tag{9d}$$

The hamiltonian is one constant of the motion, and Equations (9a) and (9b) immediately provide a second constant of the motion of the form

$$K = J_1 + J_2 . \tag{10}$$

In the present case it is convenient to employ the two known constants of the motion in the form of the total uncoupled oscillator energy K_1, and a quantity K_2 which is proportional to the total coupling energy:

$$K_1 = \omega J_1 + \omega J_2 ; \tag{11a}$$

$$K_2 = \omega J_1^{\frac{1}{2}} J_2^{\frac{1}{2}} \cos(\varphi_1 - \varphi_2). \tag{11b}$$

To obtain an equation of motion in a single variable, say J_2, square both sides of Equation (9b) to obtain

$$\dot{J}_2^2 = \left(\frac{\alpha}{\omega}\right)^2 [J_1 J_2 - J_1 J_2 \cos^2(\varphi_1 - \varphi_2)], \tag{12}$$

and use constants of the motion K_1 and K_2 to eliminate variables J_1, φ_1 and φ_2:

$$\dot{J}_2^2 = \left(\frac{\alpha}{\omega}\right)^2 \left[-J_2^2 + \left(\frac{K_1}{\omega}\right) J_2 - \left(\frac{K_2}{\omega}\right)^2\right] \tag{13}$$

$$= \left(\frac{\alpha}{\omega}\right)^2 \left[(J_2 - \lambda_2)(\mu_2 - J_2)\right],$$

where λ_2 and μ_2 are roots of the quadratic equation in J_2 in the first part of Equation (13), which satisfy the relation $\lambda_2 \leqq J_2 \leqq \mu_2$.

Equation (13) may be written

$$\frac{\alpha}{\omega} dt = \frac{dJ_2}{\left[(J_2 - \lambda_2)(\mu_2 - J_2)\right]^{\frac{1}{2}}} \tag{14}$$

so that integration of the right side produces an inverse sine function. This results in a general energy solution

$$E_1 = \omega J_1 = (K - \omega\lambda_2) - \omega(\mu_2 - \lambda_2) \sin^2 \tfrac{1}{2} \frac{\alpha}{\omega} t; \tag{15a}$$

$$E_2 = \omega J_2 = \quad \omega\lambda_2 \quad + \omega(\mu_2 - \lambda_2) \sin^2 \tfrac{1}{2} \frac{\alpha}{\omega} t. \tag{15b}$$

Substitution of the expressions for J_1 and J_2 into Equations (9c) and (9d), use of K_2, and integration of the resulting $\dot{\phi}_1$ and $\dot{\phi}_2$ equations yields

$$\varphi_1 = \omega t + \tan^{-1}\left[\left(\frac{\lambda_2}{\mu_2}\right)^{\frac{1}{2}} \tan \tfrac{1}{2} \frac{\alpha}{\omega} t\right]; \tag{16a}$$

$$\varphi_2 = \omega t + \tan^{-1}\left[\left(\frac{\mu_2}{\lambda_2}\right)^{\frac{1}{2}} \tan \tfrac{1}{2} \frac{\alpha}{\omega} t\right]. \tag{16b}$$

The action-angle solution given by Equations (15) and (16) is equivalent to the form

$$q_1 = \left(\frac{2}{\omega}\right)^{\frac{1}{2}}\left[\mu_2^{\frac{1}{2}}\cos\tfrac{1}{2}\tfrac{\alpha}{\omega}t\cos\omega t - \lambda_2^{\frac{1}{2}}\sin\tfrac{1}{2}\tfrac{\alpha}{\omega}t\sin\omega t\right]; \tag{17a}$$

$$q_2 = \left(\frac{2}{\omega}\right)^{\frac{1}{2}}\left[\lambda_2^{\frac{1}{2}}\cos\tfrac{1}{2}\tfrac{\alpha}{\omega}t\cos\omega t - \mu_2^{\frac{1}{2}}\sin\tfrac{1}{2}\tfrac{\alpha}{\omega}t\sin\omega t\right]. \tag{17b}$$

When initial conditions are introduced and the roots λ_2 and μ_2 are evaluated in terms of them, Equations (17) are found to be equivalent to the form

$$q_1 = C_1\cos(\Omega_1 t + \theta_1) + C_2\cos(\Omega_2 t + \theta_2); \tag{18a}$$

$$q_2 = C_1\cos(\Omega_1 t + \theta_1) - C_2\cos(\Omega_2 t + \theta_2), \tag{18b}$$

where $\Omega_1 = \omega - \frac{1}{2}\frac{\alpha}{\omega}$, $\Omega_2 = \omega + \frac{1}{2}\frac{\alpha}{\omega}$, and where C_1, C_2, θ_1, and θ_2 are constants to be determined by the initial conditions.

In this section, a method for obtaining the general solution of a resonant linear system by means of integration of the action-angle equations of motion, using known constants of the motion, has been illustrated for $N = 2$.

In the next section, a simpler method for obtaining the general solution of a resonant system by the use of suitable trial solutions will be illustrated. A comparison of the solutions of the original and the corresponding resonant systems will be made.

Use of Trial Solutions

Due to the relative simplicity of the linear system, the same

conclusions concerning the solution of the resonant linear system may be reached in a more straightforward and convincing fashion by employing trial solutions for q_1 and q_2, rather than by doing general integrations in action-angle variables.

This method is possible mainly by hindsight, for linear systems, in which the general solution is known to be a linear superposition of N individual true normal mode solutions. Unlike the action-angle formalism of the preceding section, it will not be particularly useful in the analysis of nonlinear systems.

The method of using trial solutions is presented in this section for two reasons. In the first subsection, the solutions of "original" and corresponding "resonant" systems will be compared. In the second subsection, a prescription for the form of the momenta in an exactly-tuned resonant system will be developed. These results, rather than the method used to obtain them, will be of use as guides in the later analysis of nonlinear systems.

Comparison of Solutions of Original and Resonant Systems

In this subsection, the general solutions of the original system, Equation (1) of this chapter, and the corresponding resonant system, specified in action-angle variables by Equation (8) of this chapter, will be obtained using appropriately-chosen trial solutions. These two solutions will then be compared.

Consider first the original system given by Equations (1) and (2). Let the initial conditions be

$$q_k = A_k \quad \text{and} \quad \dot{q}_k(0) = 0. \tag{19}$$

By hindsight, based on knowledge of the existence of N normal modes for an N-oscillator linear system, assume a trial general solution of the form

$$q_1 = C_1 \cos(\Omega_1 t + \theta_1) + C_2 \cos(\Omega_2 t + \theta_2); \tag{20a}$$

$$q_2 = C_1 \cos(\Omega_1 t + \theta_1) - C_2 \cos(\Omega_2 t + \theta_2). \tag{20b}$$

Substitution of this trial solution into Equations (2) and application of the position initial conditions of Equations (19) yields the general solution

$$q_1 = \tfrac{1}{2}(A_1 + A_2)\cos(\omega^2 - \alpha)^{\frac{1}{2}}t + \tfrac{1}{2}(A_1 - A_2)\cos(\omega^2 + \alpha)^{\frac{1}{2}}t; \tag{21a}$$

$$q_2 = \tfrac{1}{2}(A_1 + A_2)\cos(\omega^2 - \alpha)^{\frac{1}{2}}t - \tfrac{1}{2}(A_1 - A_2)\cos(\omega^2 + \alpha)^{\frac{1}{2}}t. \tag{21b}$$

For comparison, consider the corresponding resonant system. The result of retaining only the slowly-varying resonant terms in the hamiltonian of the original system by way of Equations (7) and (8) of this chapter, is a position-momentum hamiltonian

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + \omega^2 q_1^2 + \omega^2 q_2^2) - \alpha\tfrac{1}{2}\left(q_1 q_2 + \frac{1}{\omega^2} p_1 p_2\right) \tag{22}$$

in which the coupling energy now involves momenta as well as positions. This artificial appearance of momenta in the coupling is of no physical consequence. It is only the result of a formal approximation made in action-angle variables to simplify analysis of the energy-sharing properties of the system. However, it is important to note that in dealing

with actual physical systems by this "corresponding resonant system" approach, it is necessary to provide for the effect of the nonresonant terms, which when included prevent the appearance of momentum terms in the coupling. That is, a physical coupling depending upon positions alone cannot be represented by resonant coupling terms only.

As would be expected from the symmetric form of the resonant hamiltonian, Equation (22), with respect to momenta and positions, the four first-order hamilton equations of motion display a certain symmetry:

$$\dot{q}_1 = p_1 - \frac{\alpha}{\omega^2}\frac{1}{2}p_2 ; \tag{23a}$$

$$\dot{p}_1 = -\omega^2 q_1 + \alpha\frac{1}{2}q_2 ; \tag{23b}$$

$$\dot{q}_2 = p_2 - \frac{\alpha}{\omega^2}\frac{1}{2}p_1 ; \tag{23c}$$

$$\dot{p}_2 = -\omega^2 q_2 + \alpha\frac{1}{2}q_1 . \tag{23d}$$

Equations (23) can be combined to form two second-order equations of motion involving only the q_k, as before:

$$\ddot{q}_1 + \left(\omega^2 + \frac{\alpha^2}{\omega^2}\frac{1}{4}\right) q_1 = \alpha q_2 ; \tag{24a}$$

$$\ddot{q}_2 + \left(\omega^2 + \frac{\alpha^2}{\omega^2}\frac{1}{4}\right) q_2 = \alpha q_1 . \tag{24b}$$

A comparison of Equations (24) for the resonant system with Equations (2) for the original system shows that the only formal effect of dropping nonresonant terms in the coupling is a shift of the frequencies by the order of α^2, so far as the equations of motion of the linear system

are concerned.

However, the effect of dropping nonresonant terms on energy sharing is slightly greater than Equations (24) would indicate. In the original system, $p_k = \dot{q}_k$; whereas in the corresponding resonant system, $p_k = \dot{q}_k + O(\alpha)$, as is shown by first-order Equations (23b) and (23d). Therefore, the energy expressions

$$E_k = \tfrac{1}{2}(p_k^2 + \omega_k^2 q_k^2) \tag{25}$$

differ in order α for the two systems. This small difference is displayed as a small additional rapidly-varying energy which appears in the energy solutions to the original system, but does not appear in the energy solutions to the corresponding resonant system. The reason for this is that by its very definition, the resonant system contains no rapidly-varying contributions to the energies.

It follows from the above discussion that the energy solution to the original system can be closely approximated by the solution to the corresponding resonant system, if p_k in the energy expression for the resonant system is replaced by $\dot{q}_k$.

To solve Equations (24), assume a trial general solution of the form given by Equations (20), to obtain expressions of the form

$$q_1 = C_1 \cos\left[\left(\omega - \tfrac{1}{2}\tfrac{\alpha}{\omega}\right)t + \theta_1\right] + C_2 \cos\left[\left(\omega + \tfrac{1}{2}\tfrac{\alpha}{\omega}\right)t + \theta_2\right]; \tag{26a}$$

$$q_2 = C_1 \cos\left[\left(\omega - \tfrac{1}{2}\tfrac{\alpha}{\omega}\right)t + \theta_1\right] - C_2 \cos\left[\left(\omega + \tfrac{1}{2}\tfrac{\alpha}{\omega}\right)t + \theta_2\right]. \tag{26b}$$

Application of the position initial conditions given in Equations (19) results in the general solution

$$q_1 = \tfrac{1}{2}(A_1+A_2)\cos(\omega-\tfrac{1}{2}\tfrac{\alpha}{\omega})t + \tfrac{1}{2}(A_1+A_2)\cos(\omega+\tfrac{1}{2}\tfrac{\alpha}{\omega})t; \quad (27a)$$

$$q_2 = \tfrac{1}{2}(A_1+A_2)\cos(\omega-\tfrac{1}{2}\tfrac{\alpha}{\omega})t - \tfrac{1}{2}(A_1-A_2)\cos(\omega+\tfrac{1}{2}\tfrac{\alpha}{\omega})t. \quad (27b)$$

For the particular set of position initial conditions applied, in which the initial momenta are zero, the amplitudes of the general solution to the resonant system, Equations (27), are the same as the amplitudes of the solution to the original system, Equations (21). For the more general initial conditions

$$q_k(0) = A_k \quad \text{and} \quad \dot{q}_k(0) = B_k, \quad (28)$$

the amplitudes of Equations (27) will differ in second order in α from the amplitudes of Equations (21), because of the order α^2 difference in the frequencies which appear in the amplitudes of the velocities, the $\dot{q}_k$. However, as far as the energy sharing solution is concerned, this difference is relatively trivial.

In this subsection, the general solutions of the original and resonant systems have been compared in detail. To zeroth order in coupling parameter α, these two solutions are the same. There are second-order shifts in the frequencies, caused by the inclusion of non-resonant terms; and there are differences of second order in α in the magnitudes of the amplitude coefficients of the two solutions. The first-order differences in the momenta of the two systems are easily compensated for in the energy solutions.

This general result, that the solution of the resonant system closely approximates the solution of the corresponding original system

for a linear coupled exactly-tuned oscillator model, encourages the attempt to analyze nonlinear systems in terms of the corresponding resonant nonlinear systems. With this in mind, a prescription for the form of the momenta in an exactly-tuned resonant system will be developed in the next subsection, in terms of the linear system. This result will be especially useful in dealing with nonlinear systems.

Prescription for Momenta

As noted in the preceding subsection, the momenta p_k are not exactly equal to the velocities $\dot{q}_k$ in a resonant system, because of the appearance of momenta in the coupling term of a resonant hamiltonian.

Expressions for the momenta of the $N = 2$ resonant linear system of this chapter can be obtained by integration of the $\dot{p}_k$ equations, Equations (23b) and (23d), after substitution of the position expressions, Equations (26):

$$p_1 = -\omega C_1 \sin\left[\left(\omega - \tfrac{1}{2}\frac{\alpha}{\omega}\right)t + \theta_1\right] - \omega C_2 \sin\left[\left(\omega + \tfrac{1}{2}\frac{\alpha}{\omega}\right)t + \theta_2\right]; \tag{29a}$$

$$p_2 = -\omega C_1 \sin\left[\left(\omega - \tfrac{1}{2}\frac{\alpha}{\omega}\right)t + \theta_1\right] + \omega C_2 \sin\left[\left(\omega + \tfrac{1}{2}\frac{\alpha}{\omega}\right)t + \theta_2\right]. \tag{29b}$$

Note that these momentum expressions contain only terms which are of zeroth order in α. This is a necessary feature of a resonant system, in which only zeroth-order variations in the energies have been retained.

This result suggests a prescription for the form of a trial solution to be assumed for a resonant linear system. As before, assume the

general position solution to be

$$q_1 = C_1 \cos(\Omega_1 t + \theta_1) + C_2 \cos(\Omega_2 t + \theta_2); \tag{30a}$$

$$q_2 = C_3 \cos(\Omega_1 t + \theta_1) + C_4 \cos(\Omega_2 t + \theta_2). \tag{30b}$$

Prescribe the accompanying form for the momenta to be

$$p_1 = -\omega C_1 \sin(\Omega_1 t + \theta_1) - \omega C_2 \sin(\Omega_2 t + \theta_2); \tag{31a}$$

$$p_2 = -\omega C_3 \sin(\Omega_1 t + \theta_1) - \omega C_4 \sin(\Omega_2 t + \theta_2). \tag{31b}$$

That is, obtain the trial form for the momenta of a resonant system from the trial form for the positions by formally replacing cos by sin, and by multiplying each term by the negative of its uncoupled harmonic normal mode frequency, so that there are only terms of zeroth order in α in the amplitudes of the p_k.

When such a trial solution, Equations (30) and (31), is substituted into the first-order equations of motion of the resonant system, Equations (23), the algebraic equations arising from corresponding $\dot{q}_k$ and $\dot{p}_k$ equations are found to be the same. For example, substitution of Equations (30a) and (31a) into Equations (23a) and (23b) results in either case in

$$\Omega_1 C_1 = \omega_1 C_1 - \frac{\alpha}{\omega} \frac{1}{2} C_3; \tag{32a}$$

$$\Omega_2 C_2 = \omega_2 C_2 - \frac{\alpha}{\omega} \frac{1}{2} C_4. \tag{32b}$$

The success of this prescription for a trial solution of a

resonant system means that one can work with N first-order differential equations rather than with N second-order differential equations, in an exactly-tuned resonant system.

This observation is of considerable value in dealing with nonlinear exactly-tuned resonant systems, where the same prescription is found to be valid for the same reasons. Second-order differential equations for the resonant nonlinear system are somewhat more involved than are the first-order equations. Knowledge of the appropriate form of the p_k for use in the first-order equations provides a considerable simplification in the analysis of nonlinear systems.

Two illustrations of the use of trial solutions for exactly-tuned linear systems have been given in this section. The first example indicates that the solution of a given linear system may be rather well approximated by the solution of the corresponding resonant system. The second example gives a general prescription for the form of the momentum trial solution which is appropriate in the analysis of resonant exactly-tuned coupled oscillator systems.

The experience gained in terms of linear coupled systems thus far in this chapter will be of considerable aid in guiding the analysis of nonlinear systems. However, none of the methods yet presented are entirely adequate in dealing with nonlinear systems of more than two degrees of freedom.

The method which is illustrated in the next section is the one which will ultimately find general use in the analysis of nonlinear systems in Chapter VI. It is introduced in this chapter mainly to provide later comparison with an application of the same sort of method to a nonlinear system in Chapter VI.

Expansions about Periodic Solutions

The last method of solution for linear systems to be illustrated in this chapter is the method of making a perturbation expansion about a known periodic solution of the system. This method ultimately will be applied with some success in the analysis of nonlinear systems in Chapter VI.

For systems of linear differential equations, there is a theorem which assures that linear superposition is valid. This is not true in general for nonlinear systems. However, it is of interest to investigate for the nonlinear system whether there are any cases at all in which some form of linear superposition does hold, even if only approximately. The method of expansions about periodic solutions provides an appropriate formalism for such an investigation.

The solution of the $N = 2$ resonant exactly-tuned linear system, Equations (9) of this chapter, will be illustrated. The prescription for the form of momentum trial solutions developed for resonant systems in the preceding section will be used. Since the $\mathring{q}_k$ and $\mathring{p}_k$ first-order equations produce equivalent algebraic relations, only one set, the $\mathring{p}_k$ Equations (9b) and (9d), need be used:

$$\dot{p}_1 = -\omega^2 q_1 + \alpha \tfrac{1}{2} q_2 ; \tag{33a}$$

$$\dot{p}_2 = -\omega^2 q_2 + \alpha \tfrac{1}{2} q_1 . \tag{33b}$$

To obtain a "periodic solution" of this system, assume a solution of the periodic form

$$q_1 = C_1 \cos(\Omega t+\theta); \qquad p_1 = -\omega C_1 \sin(\Omega t+\theta); \tag{34a}$$

$$q_2 = C_2 \cos(\Omega t+\theta); \qquad p_2 = -\omega C_2 \sin(\Omega t+\theta). \tag{34b}$$

Here it has been assumed that the arguments of q_1 and q_2 are of the same frequency and have the same phase angle (except perhaps for a factor of 180°). Thus there are only three arbitrary constants in Equations (34): C_1, C_2, and θ.

Substitution of this trial solution, Equations (34), into the equations of motion, Equations (33), yields

$$\Omega C_1 = \omega C_1 - \frac{\alpha}{\omega} C_2; \tag{35a}$$

$$\Omega C_2 = \omega C_2 - \frac{\alpha}{\omega} C_1, \tag{35b}$$

the algebraic solution of which is

$$\Omega = \omega \mp \tfrac{1}{2}\frac{\alpha}{\omega}; \qquad C_2 = \pm C_1. \tag{36}$$

The equations of motion have placed another constraint upon the ratio of the amplitudes of the periodic solution, so that there is now just one arbitrary amplitude, say C_1, along with the one arbitrary phase angle, θ.

If position initial conditions (these assure that $\theta = 0$, and are the only appropriate type for periodic solutions for $N > 2$) of the type

$$q_k(0) = A_k \quad \text{and} \quad \dot{q}_k(0) = 0 \tag{37}$$

are specified, the solution of Equations (33) becomes

$$q_1 = A_1 \cos\left(\omega \mp \tfrac{1}{2}\frac{\alpha}{\omega}\right)t; \tag{38a}$$

$$q_2 = \pm A_1 \cos\left(\omega \mp \tfrac{1}{2}\frac{\alpha}{\omega}\right)t. \tag{38b}$$

Equations (38) show that a periodic solution of the form assumed in Equations (34) can occur only for two special sets of initial conditions of the type given in Equation (37): $A_2 = +A_1$ and $A_2 = -A_1$. This conclusion is confirmed by an examination of the general solution of this system, given by Equations (27) in this chapter. The general solution reduces to single periodic terms only for these specific sets of initial conditions.

Having identified the existence and form of the periodic solutions of this linear system, the general solution may now be obtained as a perturbation expansion about a periodic solution.

For this purpose, assume a trial general solution of the form

$$q_k = \sum_{l=-\infty}^{+\infty} \beta^{|l|} C_{kl} \cos(\Omega + l\Omega_e)t; \tag{39a}$$

$$p_k = -\omega \sum_{l=-\infty}^{+\infty} \beta^{|l|} C_{kl} \sin(\Omega + l\Omega_e)t, \tag{39b}$$

the zeroth-order terms of which are the trial periodic solutions, Equations (34). That is, the expansion parameter β is a function of the initial conditions only; for the system of Equations (33), if the initial

conditions are $A_2 = \pm A_1$, then $\beta = 0$ and Equations (39) reduce to Equations (34). For simplicity in Equations (39), the phase angles which are needed for general initial conditions are taken as zero, which is compatible with position initial conditions of the type given by Equations (37).

Justification for the form of the above simplified trial general solution will not be attempted here. This matter will be discussed in Chapter VI in conjunction with the perturbation expansion about a periodic solution for a nonlinear system. The main purpose of the present application to a linear system is to provide an illustration which later may be compared with the application to a nonlinear system in the next chapter.

To see how the by now familiar general solution of the linear system emerges from this perturbation expansion about a periodic solution, keep only terms through order β^1 in the trial general solution, Equations (39):

$$q_1 = C_{10}\cos\Omega t + \beta\left[C_{11}\cos(\Omega+\Omega_e)t + C_{1-1}\cos(\Omega-\Omega_e)t\right]; \tag{40a}$$

$$q_2 = C_{20}\cos\Omega t + \beta\left[C_{21}\cos(\Omega+\Omega_e)t + C_{2-1}\cos(\Omega-\Omega_e)t\right], \tag{40b}$$

with corresponding expressions for the momenta.

Substitution of these order β trial solutions, Equations (40), into the equations of motion, Equations (33), yields the same solution

in zeroth order as appears in Equations (36). Equating coefficients of first order in β yields

$$(\Omega+\Omega_e)C_{11} = \omega C_{11} - \frac{\alpha}{\omega}C_{21}; \tag{42a}$$

$$(\Omega-\Omega_e)C_{1-1} = \omega C_{1-1} - \frac{\alpha}{\omega}C_{2-1}; \tag{42b}$$

$$(\Omega+\Omega_e)C_{21} = \omega C_{21} - \frac{\alpha}{\omega}C_{11}; \tag{42c}$$

$$(\Omega-\Omega_e)C_{2-1} = \omega C_{2-1} - \frac{\alpha}{\omega}C_{1-1}. \tag{42d}$$

The solution of Equations (42a) and (42c) is

$$\Omega_e = \pm\frac{\alpha}{\omega}; \qquad C_{21} = \mp C_{11}, \tag{43}$$

which leads to a contradiction in Equations (42b) and (42d) unless the following amplitudes vanish:

$$C_{1-1} \equiv 0; \qquad C_{2-1} \equiv 0. \tag{44}$$

In a somewhat similar fashion, if the trial general solution is taken to higher order in β for the linear system, then the coefficients of the $(\Omega + \Omega_c)$ terms vanish for $|\ell| > 1$. The net result is that there are only two nonvanishing terms in each of the infinite-sum general solution expressions, Equations (39), for the linear system. That is, the general solution as an expansion about a periodic solution for the linear system, correct to all orders of β, is

$$q_1 = C_{10}\left[\cos(\omega \mp \tfrac{1}{2}\tfrac{\alpha}{\omega})t + \beta\frac{C_{11}}{C_{10}}\cos(\omega \pm \tfrac{1}{2}\tfrac{\alpha}{\omega})t\right]; \tag{45a}$$

$$q_2 = \pm C_{10}\left[\cos(\omega \mp \tfrac{1}{2}\tfrac{\alpha}{\omega})t \mp \beta \frac{C_{11}}{C_{10}} \cos(\omega \pm \tfrac{1}{2}\tfrac{\alpha}{\omega})t\right], \quad (45b)$$

with corresponding expressions for the momenta.

Suppose an expansion about the periodic solution for which $A_2 = +A_1$ is desired. Choose the upper set of signs in Equations (45) so that the appropriate periodic solution is obtained when $\beta = 0$. Application of the initial conditions, Equations (37), results in

$$C_{10} = \tfrac{1}{2}(A_1 + A_2); \qquad \beta = \frac{A_1 - A_2}{A_1 + A_2}, \quad (46)$$

where $C_{11} = C_{10}$ has been arbitrarily assumed in order to avoid a redundancy between β and C_{11}. The general solution may then be written as

$$q_1 = \tfrac{1}{2}(A_1 + A_2)\left[\cos(\omega - \tfrac{1}{2}\tfrac{\alpha}{\omega})t + \left(\frac{A_1 - A_2}{A_1 + A_2}\right)\cos(\omega + \tfrac{1}{2}\tfrac{\alpha}{\omega})t\right]; \quad (47a)$$

$$q_2 = \tfrac{1}{2}(A_1 + A_2)\left[\cos(\omega - \tfrac{1}{2}\tfrac{\alpha}{\omega})t - \left(\frac{A_1 - A_2}{A_1 + A_2}\right)\cos(\omega + \tfrac{1}{2}\tfrac{\alpha}{\omega})t\right]. \quad (47b)$$

The above solution, Equations (47), is the correct general solution, which reduces to an in-phase periodic solution when $A_2 = +A_1$. A similar expansion could have been made about the out-of-phase periodic solution.

The periodic solutions obtained here are the same as the true normal modes of the resonant linear system, which were identified at the beginning of this chapter.

The validity of the concept of linear superposition for the linear system is demonstrated by the appearance of only one other term,

the other normal mode solution, when a perturbation expansion is made about a given periodic (normal mode) solution. The degree to which linear superposition applies in a nonlinear system will be tested in this same formalism, by making an expansion about a periodic solution and investigating the terms which arise. These other terms are not expected to be the other periodic solutions, but the matter will be studied.

In this chapter, a linear coupled oscillator system has been subjected to analysis by a variety of methods. The solution has been obtained by way of normal mode analysis, integration of the action-angle equations of motion using constants of the motion, use of trial solutions, and perturbation expansions about periodic solutions. Much of the experience thus gained will be of significant aid as a guide in the analysis of nonlinear coupled oscillator systems.

In particular, the methods of integration of the action-angle equations and expansions about periodic solutions will be directly applied to systems of two and N oscillators, respectively. Analogy to the result for the linear system will be used to argue that the solution of the resonant system is a good approximation to the solution of the original system for nonlinear energy-sharing systems, at least weakly-coupled ones. And the prescription for the form of the momenta in a resonant exactly-tuned coupled oscillator system will be used in the analysis of nonlinear systems in Chapter VI.

Thus the function of the present chapter has been to provide experience which is needed in the analysis of nonlinear coupled oscillator systems in Chapter VI.

CHAPTER VI

A METHOD OF ANALYTIC SOLUTION FOR NONLINEAR COUPLED OSCILLATOR SYSTEMS

The analysis of exactly-tuned linear coupled oscillator systems presented in the preceding chapter will be used in this chapter as a guide in developing an analysis of exactly-tuned nonlinear coupled oscillator systems.

The main problem in the analysis of an exactly-tuned nonlinear system by a perturbation method, to which this thesis is limited, is the specification of an appropriate zeroth-order trial general solution. The general solution of a nonlinear system is not expected to be a simple linear superposition of individual solutions, because true decoupled normal modes do not exist in general for a nonlinear system. Available perturbation theories, such as the Wigner-Brillouin scheme discussed in Chapter IV, encounter certain difficulties in the construction of the complete zeroth-order term in the trial solution for an exactly-tuned nonlinear system.

Some simplification of the analysis can be achieved by considering only the "resonant" systems corresponding to the given exactly-tuned systems for which solutions are desired. That is, systems obtained by retaining only the slowly-varying coupling terms of given systems will be more simply analyzed. The energy solution of a resonant system is expected to be a good approximation to the energy solution of the corresponding original system in the nonlinear case, by analogy to the

linear case discussed in Chapter V, and by the discussion in action-angle formalism presented in Chapter II. More generally, the position and momentum solutions of the resonant system are expected to be good approximations to the solutions of the actual system, if the differences between the momenta of the two systems are compensated for.

In the first section of this chapter, the exact energy solution for a two-oscillator resonant exactly-tuned nonlinear system will be obtained, by use of action-angle variables, in order to gain insight into an appropriate form of position-momentum trial solution. The difficulties involved in directly obtaining the position-momentum general solutions from this energy solution will be discussed. An expansion of the exact energy solution will be discussed. An expansion of the exact energy solution in trigonometric series will then be made, as a guide in obtaining trigonometric series expansions for the positions and momenta.

In the second section, a method of obtaining a general position-momentum solution for a two-oscillator nonlinear system as a perturbation expansion about a periodic solution will be developed. The trigonometric series expansion of the energy solution, obtained in the first section, will be used as a guide.

In the final section of this chapter, which is to be considered the most important portion of the thesis, a method of obtaining a general position-momentum solution for a nonlinear coupled oscillator system having N degrees of freedom in one dimension will be developed, as a perturbation expansion about a periodic solution. Both the two preceding sections in this chapter and the experience gained in Chapter V will be used to develop this method.

To provide a check on the effectiveness of this method of perturbation expansions about periodic solutions in the general case of N oscillators, it will be necessary to resort to the use of a computer to compare the results of this method with the results of a numerical integration of the equations of motion, for certain specified sets of initial conditions. This will be carried out later, in Chapter VII.

Energy Solution for Two-Oscillator Nonlinear System

Consider the exactly-tuned cubic nonlinear system governed by

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + \omega^2 q_1^2 + 4\omega^2 q_2^2) - \alpha \tfrac{1}{\sqrt{2}}(q_1^2 q_2 - q_2^3), \quad (1)$$

which is the $N = 2$ case of the model formulated in Chapter III.

As discussed in conjunction with Equation (52) of Chapter II, the energy solution of the system given by Equation (1) may be well approximated by the solution of the corresponding "resonant" system in which only the slowly-varying "resonant" coupling terms are retained in the hamiltonian. The identification of these resonant terms is made most conveniently in action-angle formalism.

In action-angle variables, the hamiltonian given by Equation (1) becomes

$$H = \omega J_1 + \omega J_2 - \alpha \tfrac{1}{2\sqrt{2}} \tfrac{1}{\omega^{3/2}} \{ J_1 J_2^{\frac{1}{2}} [\cos(2\varphi_1 + \varphi_2) + \cos(2\varphi_1 - \varphi_2) + 2\cos\varphi_2] + \tfrac{1}{2} J_2^{\frac{3}{2}} [\cos 3\varphi_2 + 3\cos\varphi_2] \}. \quad (2)$$

Elimination of all rapidly-varying nonresonant terms from Equation (2)

results in the resonant hamiltonian

$$H = \omega J_1 + 2\omega J_2 - \alpha \frac{1}{2\sqrt{2}} \frac{1}{\omega^{3/2}} J_1 J_2^{\frac{1}{2}} \cos(2\varphi_1 - \varphi_2). \quad (3)$$

The resonant hamiltonian given in action-angle variables in Equation (3) corresponds to a position-momentum hamiltonian of the form

$$H = \tfrac{1}{2}(p_1^2 + p_2^2 + \omega^2 q_1^2 + 4\omega^2 q_2^2) - \alpha \frac{1}{4\sqrt{2}} (q_1^2 q_2 - \frac{1}{\omega^2} p_1^2 q_2 + \frac{1}{\omega^2} p_1 q_1 p_2). \quad (4)$$

Note that the q_2^3 term in the coupling of the original hamiltonian, Equation (1), contains no resonant portion whatever, and has been completely dropped in forming the resonant hamiltonian given by Equation (4).

One might regard the dropping of nonresonant terms as equivalent to the addition of certain mixed position-momentum coupling terms, which serves to reveal the underlying q, p symmetry of a resonant coupling. These symmetrizing portions of the coupling were formerly cancelled out by the presence of the nonresonant coupling terms in the original hamiltonian, Equation (1).

In the first subsection, it will be shown that an exact energy solution of the two-oscillator resonant nonlinear system can be obtained by integration of the equations of motion in action-angle variables, because two constants of the motion are readily available for this system.

The second subsection will present the formal method by which the companion φ_k solutions can be obtained, and will discuss the difficulties involved in trying to obtain the complete position-momentum

solution from the complete action-angle solution.

The third subsection will develop a trigonometric series expansion of the elliptic function exact energy solution obtained in the first subsection, in an effort to salvage some information about the appropriate form for a position-momentum solution from the energy solution.

Energy Solution in Action-Angle Variables

The method of solution of an exactly-tuned resonant nonlinear system of two oscillators by integration of the action-angle equations of motion will closely follow the method illustrated in Chapter V for the linear system. This sort of analysis was first carried out over forty years ago, by co-workers of Whittaker.

The system governed by the hamiltonian of Equation (3) has action-angle equations of motion of the form

$$\dot{J}_1 = -\alpha \frac{1}{\sqrt{2}} \omega^{-\frac{3}{2}} J_1 J_2^{\frac{1}{2}} \sin(2\varphi_1 - \varphi_2); \tag{5a}$$

$$\dot{J}_2 = +\alpha \frac{1}{\sqrt{2}} \frac{1}{2} \omega^{-\frac{3}{2}} J_1 J_2^{\frac{1}{2}} \sin(2\varphi_1 - \varphi_2); \tag{5b}$$

$$\dot{\varphi}_1 = \omega - \alpha \frac{1}{2\sqrt{2}} \omega^{-\frac{3}{2}} J_2^{\frac{1}{2}} \cos(2\varphi_1 - \varphi_2); \tag{5c}$$

$$\dot{\varphi}_2 = 2\omega - \alpha \frac{1}{2\sqrt{2}} \frac{1}{2} \omega^{-\frac{3}{2}} J_1 J_2^{-\frac{1}{2}} \cos(2\varphi_1 - \varphi_2). \tag{5d}$$

Two constants of the motion, the total uncoupled oscillator energy K_1 and a quantity K_2 which is proportional to the total coupling energy, are immediately available for this system:

$$K_1 = \omega J_1 + 2\omega J_2 ; \tag{6a}$$

$$K_2 = \omega J_1 J_2^{\frac{1}{2}} \cos(2\varphi_1 - \varphi_2). \tag{6b}$$

By squaring Equation (5b) and using constants of the motion K_1 and K_2, an equation in the single variable J_2 may be obtained:

$$\dot{J}_2^2 = \tfrac{1}{2}\left(\frac{\alpha}{\omega^{3/2}}\right)^2 \left[J_2^3 - \left(\frac{K_1}{\omega}\right)J_2^2 + \tfrac{1}{4}\left(\frac{K_1}{\omega}\right)^2 - \tfrac{1}{4}\left(\frac{K_2}{\omega}\right)^2\right]. \tag{7}$$

This may be written in the form

$$\frac{1}{\sqrt{2}}\frac{\alpha}{\omega^{3/2}}\,dt = \frac{dJ_2}{\left[(J_2-\lambda_2)(\mu_2-J_2)(\nu_2-J_2)\right]^{\frac{1}{2}}}, \tag{8}$$

where λ_2, μ_2, and ν_2 are the roots of the cubic equation in J_2 on the right side of Equation (7). These roots satisfy the relation $\lambda_2 \leqq J_2 \leqq \mu_2 \leqq \nu_2$.

The integration of Equation (8) has been performed by Baker and Ross (1921, 1922) in a more general treatment of systems coupled as in Equation (3) in which the frequencies are taken to be free parameters. The present exactly-tuned system is just a special case of their more general study.

Baker and Ross deal with the above cubic in J_2 as a quartic, the fourth root of which is $J_2 \to \infty$. Using standard transformations for quartics (Jeffreys 1950) to reduce the integral of Equation (8) to a standard form, Baker and Ross obtain an exact energy solution in terms of elliptic functions $\mathrm{sn}(u, k)$:

$$E_1 = \omega J_1 = \frac{1 - 2\ell - k(2m-1)\,sn(u,k)}{1 + k\,sn(u,k)}\,; \qquad (9a)$$

$$E_2 = 2\omega J_2 = \frac{2\ell + 2mk\,sn(u,k)}{1 + k\,sn(u,k)}\,, \qquad (9b)$$

where $k, \ell,$ and m are functions of the roots λ_2, μ_2, and ν_2 which are defined in the Baker-Ross paper.

The Baker-Ross form of solution is not the most convenient for purposes of the present study. A more appropriate form of solution, which is directly analogous to the form of solution for the linear system given in Chapter V by Equations (15), may be obtained by a somewhat more straightforward integration of Equation (8).

Use a standard transformation for a cubic (Jeffreys 1950), rather than for a quartic, to reduce the integral of Equation (8) to a standard form: let $J_2 = \xi_2^2 + \lambda_2$, so that

$$\frac{1}{2\sqrt{2}}\,\frac{\alpha}{\omega^{3/2}}\,t = \int \frac{d\xi_2}{\{[(\mu_2-\lambda_2)-\xi_2^2][(\nu_2-\lambda_2)-\xi_2^2]\}^{\frac{1}{2}}} \qquad (10)$$

$$= n_2^{\frac{1}{2}} \int \frac{d\xi_2}{\{[1-z_2^2][1-k_2^2 z_2^2]\}^{\frac{1}{2}}}$$

where $m_2 = \frac{1}{(\mu_2 - \lambda_2)}$, $n_2 = \frac{1}{(\nu_2 - \lambda_2)}$, $k_2 = \left(\frac{n_2}{m_2}\right)^{\frac{1}{2}}$, and $z = m_2^{\frac{1}{2}}\xi_2$.

The last integral in Equation (10) is the standard form for an elliptic integral of the first kind, so that $z_2 = sn(u_2, k_2)$, in analogy to an inverse sine integral; the time variable is

$$u_2 = n_2^{-\frac{1}{2}}\,\frac{1}{2\sqrt{2}}\,\frac{\alpha}{\omega^{3/2}}\,t. \qquad (11)$$

The form of energy solution obtained by this method is

$$E_1 = \omega J_1 = (K_1 - 2\omega\lambda_2) - 2\omega(\mu_2 - \lambda_2)sn^2(v_2, k_2); \quad (12a)$$

$$E_2 = 2\omega J_2 = \quad 2\omega\lambda_2 + 2\omega(\mu_2 - \lambda_2)sn^2(v_2, k_2). \quad (12b)$$

A comparison between this form and Equations (15) of Chapter V shows that the only essential difference between the general energy solutions of the linear and nonlinear systems is the replacement of trigonometric "sin" functions in the linear solution by elliptic "sn" functions in the nonlinear solution.

This result, along with the existence of a second constant of the motion for the resonant nonlinear system, encourages a belief that the characters of solutions of linear and nonlinear coupled oscillator systems are not greatly different. It seems likely that the difficulties which have troubled analyses of tuned nonlinear systems may be difficulties inherent in the particular mathematical formalisms involved, rather than difficulties due to any lack of underlying simplicity for nonlinear systems. The latter observation agrees with that of Baker (1921), who points out that seemingly overwhelming difficulties may be encountered simply because of a wrong choice of an infinite series representation, for example.

In this subsection, the exact energy solution has been obtained for the two-oscillator nonlinear system. In the case of the two-oscillator linear system for which the complete solution was obtained in Chapter V, the next step was integration of the $\dot{\phi}_k$ equations, after expressions for the J_k had been obtained. The following subsection

will point out the formal difficulties involved in following the same procedure to obtain the general position-momentum solution for the nonlinear system.

Integration of the $\dot{\phi}_k$ Equations

The integration of the $\dot{\phi}_k$ equations for the nonlinear system is not as easy as it was for the linear system in Chapter V, specifically because of the appearance of elliptic functions rather than trigonometric functions in the expressions for the J_k.

For example, consider the integration of the first equation, Equation (5c) of this chapter:

$$\dot{\varphi}_1 = \omega - \alpha \frac{1}{2\sqrt{2}} \omega^{-\frac{3}{2}} J_2^{\frac{1}{2}} \cos(2\varphi_1 - \varphi_2) \qquad (13)$$
$$= \omega - \alpha \frac{1}{2\sqrt{2}} \omega^{-\frac{5}{2}} \frac{K_2}{J_1},$$

where the second constant of the motion, K_2 from Equation (6b), has been used to eliminate φ_1 and φ_2 on the right side. Substitution of the solution for J_1 from Equation (12a) into Equation (13) and integration yields

$$\varphi_1 = \omega t - \alpha \frac{1}{2\sqrt{2}} \omega^{-\frac{5}{2}} K_2 \int_0^t \frac{dt}{a + b\,sn^2(u,k)} \qquad (14)$$

where $a = (\frac{K_1}{\omega} - 2\lambda_2)$ and $b = -2(\mu_2 - \lambda_2)$.

Reference to a standard work on elliptic functions (Hancock 1910) reveals that the integral encountered in Equation (14) may be evaluated in terms of elliptic functions and theta functions:

$$\int_0^t \frac{dt}{a+b\,sn^2(u,k)} = \frac{t}{U}\frac{1}{a}\int_0^U \frac{du}{1-c^2 sn^2(u,k)} \qquad (15)$$

$$= \left[1 + \frac{sn(a,k)}{cn(a,k)dn(a,k)}\frac{\Theta'(a)}{\Theta(a)}\right]\frac{t}{a}$$

$$+ \frac{1}{2}\frac{sn(a,k)}{cn(a,k)dn(a,k)} \log\frac{\Theta(u-a)}{\Theta(u+a)}\frac{t}{ua},$$

where $c^2 = \frac{b}{a}$.

The integration of the $\dot{\varphi}_k$ equations was done by Baker and Ross (1921), using a slightly different form of solutions for J_1 and J_2, results in equally complicated expressions involving theta functions for φ_1 and φ_2. Baker and Ross venture no further than to numerically approximate the action-angle variables for a given set of initial conditions, in order to obtain an approximation to the position-momentum solution of a given nonlinear system.

The relative complexity of the general action-angle solution discourages any attempt to obtain the general position-momentum solution for the nonlinear system considered. Formally, it is possible to express the various elliptic and theta functions in trigonometric series, take the various inverse tangents and other functions involved, and finally obtain a trigonometric series representation of the q,p solution from the J, φ solution. However, this procedure has been attempted and appears to be prohibitively difficult in the general case.

Apparently, a more practical approach is to deal only with the relatively simple energy solutions. In the next subsection, the energy solutions obtained in the first subsection will be studied in more detail, in an attempt to obtain some information from them with regard to the

general form of the position-momentum solution.

Series Expansion of Energy Solution

In order to avoid the mathematical difficulties of dealing with the complete J, φ solution, and yet in order to gain some insight into the actual form of the q,p solution for the two-oscillator nonlinear system, a more detailed investigation will be made of the relatively simple energy solution obtained earlier.

The elliptic functions which appear in the energy solution will be expanded in their trigonometric series representations. Two special limiting cases for the elliptic functions will be considered, in an attempt to find some situations in which special solutions are easier to find than the general solution.

Referring to Equation (12), the general solution for the energy of the first oscillator is

$$E_1 = (K_1 - 2\omega\lambda_2) - 2\omega(\mu_2 - \lambda_2)\, sn^2(u,k); \tag{16}$$

the energy of the second oscillator is just $E_2 = K_1 - E_1$.

A standard trigonometric series representation of the elliptic function $sn(u,k)$ is the following (Hancock 1910, p. 486):

$$sn(u,k) = \frac{2\pi}{K}\left(\frac{q}{k^2}\right)^{\frac{1}{2}}\left[\frac{1}{1-q}\sin\frac{\pi}{2K}u + \frac{q}{1-q^3}\sin 3\frac{\pi}{2K}u + \frac{q^2}{1-q^5}\sin 5\frac{\pi}{2K}u + \cdots\right], \tag{17}$$

where k is the modulus of elliptic function $sn(u,k)$, $0 \leq k \leq 1$; K is the complete elliptic integral of the first kind, with modulus k; and

q is defined by

$$q = e^{-\pi \frac{K'}{K}}, \tag{18}$$

where K' is the complete elliptic integral of the first kind, with modulus $k' = (1 - k^2)^{\frac{1}{2}}$.

For simplicity in the construction of later expressions, it is desirable to obtain a form for $sn(u,k)$ which involves only argument u and modulus k. To do this starting from Equation (17), it is necessary to express K and q in terms of modulus k.

Modulus k may be related to q by the expression (Hancock 1910, p. 480)

$$k^{\frac{1}{2}} = 2q^{\frac{1}{4}}\left[\frac{(1+q^2)(1+q^4)(1+q^6)\cdots}{(1+q)(1+q^3)(1+q^5)\cdots}\right], \tag{19}$$

from which an expression for q in terms of k may be obtained:

$$q = \tfrac{1}{16}k^2\left(1 + \tfrac{1}{2}k^2 + \tfrac{21}{64}k^4 + \cdots\right). \tag{20}$$

Complete elliptic integral K may be written (Hancock 1910, p. 476) as

$$K = \tfrac{\pi}{2}(1 + 2q + 2q^4 + 2q^9 + \cdots)^2, \tag{21}$$

which, with the use of Equation (19), becomes

$$K = \tfrac{\pi}{2}\left(1 + \tfrac{1}{4}k^2 + \tfrac{9}{64}k^4 + \tfrac{25}{256}k^6 + \cdots\right). \tag{22}$$

Substitution of Equations (20) and (22) for q and K on the right side of Equation (17) leads to

$$\begin{aligned} sn(u,k) = &\left(1+\tfrac{1}{16}k^2+\tfrac{7}{256}k^4+\cdots\right)\sin\tfrac{\pi}{2K}u \\ &+\left(\tfrac{1}{16}k^2+\tfrac{1}{32}k^4+\cdots\right)\sin 3\tfrac{\pi}{2K}u \\ &+\left(\tfrac{1}{256}k^4+\cdots\right)\sin 5\tfrac{\pi}{2K}u+\cdots, \end{aligned} \tag{23}$$

where the arguments of the trigonometric functions are odd integer multiples of

$$\tfrac{\pi}{2K}u = \left(1-\tfrac{1}{4}k^2-\tfrac{5}{64}k^4-\tfrac{11}{256}k^6+\cdots\right)u \tag{24}$$

Equation (23) shows that the elliptic function $sn(u,k)$ may be expressed as a trigonometric series in sines of odd multiples of an argument which is proportional to u. The coefficients of the expansion and of u may be expressed as power series in even powers of modulus k.

The two limiting cases of $sn(u,k)$ occur for $k = 0$ and for $k = 1$. If $k = 0$, then $sn(u,k) = \sin u$, as Equations (23) and (24) show. At the other extreme, if $k = 1$, then $sn(u,k) = \tanh u$, a bounded function which has an infinite period. The latter limit is more easily established using a different series representation from that of Equation (23), although Equation (24) does indicate that the period of $sn(u,k)$ increases as k increases to unity.

A trigonometric series expansion for $sn^2(u,k)$, which is the function which appears in the typical energy solution given by Equation (15), may be obtained by squaring both sides of Equation (23). The

typical energy solution for the two-oscillator nonlinear system may therefore be written as a trigonometric series:

$$E_1 = (K_1 - 2\omega\lambda_2) - 2\omega(\mu_2 - \lambda_2)\left\{\tfrac{1}{2}\left(1 + \tfrac{1}{8}k^2 + \tfrac{1}{16}k^4\right)\right. \qquad (25)$$
$$-\tfrac{1}{2}\left(1 \quad -\tfrac{3}{256}k^4\right)\cos\tfrac{\pi}{K}u$$
$$-\tfrac{1}{16}\left(\quad k^2 + \tfrac{1}{2}k^4\right)\cos 2\tfrac{\pi}{K}u$$
$$-\tfrac{1}{256}\left(\quad \tfrac{3}{2}k^4\right)\cos 3\tfrac{\pi}{K}u$$
$$\left. + \cdots \right\}$$

where $\frac{1}{2}u$ is given by Equation (24).

The elliptic function modulus k is a function only of the initial conditions applied to the system. In the integration of Equation (8) earlier in this chapter, modulus k_2 was introduced in Equation (10) as

$$k_2 = \left(\frac{\mu_2 - \lambda_2}{\nu_2 - \lambda_2}\right)^{\frac{1}{2}} \qquad (26)$$

where λ_2, μ_2, and ν_2 are roots of the cubic in J_2 which appears in Equation (7), and are therefore functions of the initial conditions only, through constants of the motion K_1 and K_2.

It follows from Equations (25) and (26) that if the initial conditions are such that $(\mu_2 - \lambda_2) = 0$, then the energy solution is constant in time:

$$E_1 = (K_1 - 2\omega\lambda_2). \qquad (27)$$

This result indicates the existence of periodic solutions for the nonlinear system, for certain special sets of initial conditions.

Furthermore, by solving the cubic of Equation (7), it is found that if $\lambda_2 \approx \mu_2$, then $\lambda_2 \approx \mu_2 \ll \nu_2$, so that for sets of initial conditions in the neighborhood of the set for which $\lambda_2 = \mu_2$, k_2 is small. This indicates that an energy solution of the form

$$E_1 = (K_1 - 2\omega\lambda_2) - 2\omega(\mu_2 - \lambda_2)\sin^2 U \qquad (28)$$

will be a good approximation to the exact solution in the region in which $\lambda_2 \approx \mu_2$. But this is just the form of the solution to the linear two-oscillator system, Equations (15) of Chapter V, which further demonstrates the similarities previously noted between linear and nonlinear coupled oscillator systems.

The general form of the energy solution, Equation (25), implies that the general form of the position-momentum solution will be similar in many respects to that obtained for the linear system. This encourages an attempt to develop a method of perturbation expansions about periodic solutions as a scheme for obtaining general position-momentum solutions for resonant nonlinear systems.

In this section, a trigonometric series expansion for the exact energy solution of the resonant exactly-tuned nonlinear two-oscillator system has been obtained. This expression will be used to guide the development of an appropriate form of trial general solution for the positions and momenta of an N-oscillator resonant nonlinear system in the remainder of this chapter. In the next section, some consideration will be given to the position-momentum solution of the two-oscillator nonlinear system, as a final preliminary to the analysis of the general situation.

Two-Oscillator General Solution

The trigonometric series expansion, Equation (25) of this chapter, for the energy solution of the two-oscillator resonant exactly-tuned nonlinear system described by Equations (5), provides evidence of the existence of periodic solutions for special sets of initial conditions for this system. It also indicates that expansions about periodic solutions may provide a workable scheme for obtaining the general position-momentum solution of the system.

In this section the existence of periodic solutions and the details of expansions about them will be investigated for the two-oscillator nonlinear system. This is the last preliminary example to be worked out prior to the more general analysis of the N-oscillator system, to be given in the last section of this chapter.

The resonant exactly-tuned two-oscillator nonlinear system governed by the hamiltonian given in Equation (4) of this chapter has the following first-order equations of motion:

$$\dot{q}_1 = p_1 - \alpha \frac{1}{\sqrt{2}}\left(-2\frac{1}{\omega^2}p_1 q_2 + \frac{1}{\omega^2}q_1 p_2\right); \tag{29a}$$

$$\dot{q}_2 = p_2 - \alpha \frac{1}{\sqrt{2}}\left(\frac{1}{\omega^2}p_1 q_1\right); \tag{29b}$$

$$\dot{p}_1 = -\omega^2 q_1 + \alpha \frac{1}{\sqrt{2}}\left(2q_1 q_2 + \frac{1}{\omega^2}p_1 p_2\right); \tag{30a}$$

$$\dot{p}_2 = -4\omega^2 q_2 + \alpha \frac{1}{\sqrt{2}}\left(q_1^2 - \frac{1}{\omega^2}p_1^2\right). \tag{30b}$$

In order to find the periodic solutions of this system, assume a trial position solution of the form

$$q_1 = C_1 \cos(\Omega t + \theta); \tag{31a}$$

$$q_2 = C_2 \cos 2(\Omega t + \theta), \tag{31b}$$

where $\Omega = \omega + \frac{a}{\omega} b_o$.

Equations (31) represent a generalization of the form of the periodic solution found to exist for the linear system in Chapter V. Here it is recognized that in the exactly-tuned nonlinear system the uncoupled frequency of the k^{th} oscillator is $k\omega$, rather than just ω as in the linear system.

The trial momentum solution corresponding to Equations (31) is

$$p_1 = -\omega C_1 \sin(\Omega t + \theta); \tag{32a}$$

$$p_2 = -2\omega C_2 \sin 2(\Omega t + \theta), \tag{32b}$$

in accordance with the prescription for momenta developed in Chapter V for resonant exactly-tuned linear systems. This prescription is valid for any resonant system in which only slowly-varying terms are retained in the coupling, because it is the only possible choice consistent with Equations (31) which assures that there are no rapidly-varying energy terms in

$$E_k = \tfrac{1}{2}(p_k^2 + k^2\omega^2 q_k^2). \tag{33}$$

The result of substitution of Equations (31) and (32) into either Equations (29) or Equations (30) is the same:

$$\Omega = \omega \mp \kappa \frac{1}{4\sqrt{2}} C_1; \qquad C_2 = \pm \tfrac{1}{8} C_1. \tag{34}$$

A comparison of Equations (34) of this chapter and Equations (36) of Chapter V shows that the two periodic solutions which have been shown to exist for this nonlinear two-oscillator system are completely analogous to the periodic solutions of the linear system. The primary difference is that the part of the frequencies which is of order α is proportional to the amplitude in the nonlinear system, whereas it is not in the linear system. Minor differences in the numerical factors are basically due to the difference in the uncoupled frequencies of the two systems.

In order to make a perturbation expansion about a periodic solution, it is necessary to assume some form of trial solution. One rather general form of trial solution which results in an energy expression of the form of Equation (25) of this chapter is

$$\begin{aligned} q_k = {} & [a_{k00}\beta^0 + a_{k01}\beta^1 + a_{k02}\beta^2 + \cdots] \cos k\tau \\ & + [\quad a_{k-11}\beta^1 + a_{k-12}\beta^2 + \cdots] \cos(k\tau - \tau_e) \\ & + [\quad a_{k11}\beta^1 + a_{k12}\beta^2 + \cdots] \cos(k\tau + \tau_e) \\ & + [\quad\quad a_{k-22}\beta^2 + \cdots] \cos(k\tau - 2\tau_e) \\ & + [\quad\quad a_{k22}\beta^2 + \cdots] \cos(k\tau + 2\tau_e) + \cdots, \end{aligned} \tag{35}$$

where

$$\tau = (\omega + b_0\beta^0 + b_1\beta^1 + b_2\beta^2 + \cdots)t + \theta \tag{36}$$

and

$$\tau_e = (c_0\beta^0 + c_1\beta^1 + c_2\beta^2 + \cdots)t + \theta_e \tag{37}$$

with companion momentum expressions obtained from the q_k by the prescription of multiplying the q_k expressions by $-k\omega$ and replacing "cos" by "sin." Expansion parameter β is to be a function only of the initial conditions, so that if $\beta = 0$ then the general solution given by Equations (35) reduces to the periodic solution given by Equations (31).

The expression for the oscillator energy E_k corresponding to the trial solution of the form given in Equations (35) is

$$E_k = \tfrac{1}{2}k^2\omega^2\{[a_{k00}^2 + 2a_{k00}a_{k01}\beta^1 + \cdots] + [a_{k00}(a_{k-11}+a_{k11})\beta^1 + \cdots]\cos(\Omega_e t+\theta_e) + [a_{k00}(a_{k-22}+a_{k22})\beta^1 + \cdots]\cos 2(\Omega_e t+\theta_e) + \cdots. \quad (38)$$

Thus the frequency denoted Ω_e is the fundamental frequency of energy exchange between oscillators. Comparison of Equation (38) with Equation (25) verifies that the trial solution of the form given in Equations (35) does indeed lead to an energy solution of the same form ($\Omega_e t = \frac{\pi}{k} u$) as that obtained by integration of the equations of motion in action-angle variables. However, the roles of expansion parameters β and k^2 in Equations (38) and (25) are clearly different.

Substitution of the trial solution of Equations (35) into the equations of motion, Equations (29) or (30), results in the solution

$$q_1 = a_{100}\Big\{\Big[\beta^0 + \frac{a_{101}}{a_{100}}\beta^1 + \frac{a_{102}}{a_{100}}\beta^2 + \cdots\Big]\cos\tau \qquad (39a)$$

$$+\Big[\frac{a_{111}}{a_{100}}\beta^1 + \frac{a_{112}}{a_{100}}\beta^2 + \cdots\Big]\cos(\tau-\tau_e)$$

$$+\Big[(2+\sqrt{3})\frac{a_{111}}{a_{100}}\beta^1 + (2+\sqrt{3})\frac{a_{112}}{a_{100}}\beta^2 + \cdots\Big]\cos(\tau+\tau_e)$$

$$+\Big[\tfrac{1}{6}(3+\sqrt{3})\Big(\frac{a_{111}}{a_{100}}\Big)^2\beta^2 + \cdots\Big]\cos(\tau-2\tau_e)$$

$$+\Big[\tfrac{1}{6}(9+5\sqrt{3})\Big(\frac{a_{111}}{a_{100}}\Big)^2\beta^2 + \cdots\Big]\cos(\tau+2\tau_e) + \cdots\Big\};$$

$$q_2 = +\frac{1}{2\sqrt{2}}a_{100}\Big\{\Big[\beta^0 + \frac{a_{101}}{a_{100}}\beta^1 + \Big\{\frac{a_{102}}{a_{100}} + 6(2+\sqrt{3})\Big(\frac{a_{111}}{a_{100}}\Big)^2\Big\}\beta^2 + \cdots\Big]\cos 2\tau \qquad (39b)$$

$$+\Big[-(1-\sqrt{3})\frac{a_{111}}{a_{100}}\beta^1 - (1-\sqrt{3})\frac{a_{112}}{a_{100}}\beta^2 + \cdots\Big]\cos(2\tau-\tau_e)$$

$$+\Big[-(5+3\sqrt{3})\frac{a_{111}}{a_{100}}\beta^1 - (5+3\sqrt{3})\frac{a_{112}}{a_{100}}\beta^2 + \cdots\Big]\cos(2\tau+\tau_e)$$

$$+\Big[\tfrac{1}{3}(\sqrt{3})\Big(\frac{a_{111}}{a_{100}}\Big)^2\beta^2 + \cdots\Big]\cos(2\tau-2\tau_e)$$

$$+\Big[-\tfrac{1}{3}(12+7\sqrt{3})\Big(\frac{a_{111}}{a_{100}}\Big)^2\beta^2 + \cdots\Big]\cos(2\tau+2\tau_e) + \cdots\Big\},$$

$$\text{where } \tau = \Big(\omega - \alpha\frac{1}{4\sqrt{2}}a_{100}\Big[\beta^0 + \frac{a_{101}}{a_{100}}\beta^1 + \Big(\frac{a_{102}}{a_{100}} - 4(2+\sqrt{3})\Big(\frac{a_{111}}{a_{100}}\Big)^2\Big)\beta^2 + \cdots\Big]\Big)t + \theta \qquad (40)$$

$$\text{and } \tau_e = \alpha\frac{\sqrt{3}}{2\sqrt{2}}a_{100}\Big[\beta^0 + \frac{a_{101}}{a_{100}}\beta^1 + \cdots\Big]t + \theta_e. \qquad (41)$$

This expansion was made about the periodic solution obtained by taking upper signs in Equation (34). Note that τ_e is typically determined to one less order in β than the rest of the solution.

Four of the arbitrary coefficients in the solution given by Equations (39), say a_{100}, $\frac{a_{111}}{a_{100}}\beta$, θ, and θ_e, can be fixed in terms of the initial conditions

$$q_k(0) = A_k; \qquad \dot{q}_k(0) = B_k. \tag{42}$$

The other coefficients, such as $\frac{a_{101}}{a_{100}}$, $\frac{a_{102}}{a_{100}}$, and $\frac{a_{112}}{a_{100}}$, remain arbitrary in this solution.

One way in which these arbitrary coefficients might be specified is in such a manner as to provide agreement with the elliptic function expansions of the energy solution given in Equation (24) of this chapter. However, this is neither practical in all orders of β nor particularly desirable, since an energy solution is not available for nonlinear systems of more than two oscillators.

A more satisfactory arrangement is to choose $\frac{a_{101}}{a_{100}}$, $\frac{a_{112}}{a_{100}}$, and similar terms on even-numbered "diagonals" of the solution given by Equations (39) zero, and to choose $\frac{a_{102}}{a_{100}}$ and similar terms on odd-numbered interior "diagonals" so that the higher-order corrections to the frequencies Ω and Ω_e vanish. For the system being considered at present, this means choose

$$\frac{a_{102}}{a_{100}} = 4(2+\sqrt{3})\left(\frac{a_{111}}{a_{100}}\right)^2. \tag{43}$$

Equations (40) and (41) show that with this choice and with

$$\frac{a_{101}}{a_{100}} \equiv 0, \qquad \frac{a_{112}}{a_{100}} \equiv 0, \tag{44}$$

there are no higher-order corrections to the frequencies Ω and Ω_e, at least through order β^2. In the next section of this chapter, it will be shown that the choice described results in no higher-order frequency corrections of any order in the expansion parameter β.

As a result of the preceding discussion, the form of trial solution finally decided upon for the two-oscillator nonlinear system is

$$\begin{aligned} q_k = & \quad A_{k00}\cos k\tau \\ & +\beta\left[\, A_{k10}\cos(k\tau+\tau_e) + A_{k-10}\cos(k\tau-\tau_e)\right] \\ & +\beta^2\left[\, A_{k11}\cos(k\tau+\tau_e+\tau_e) + A_{k1-1}\cos(k\tau+\tau_e-\tau_e) \right. \\ & \left. + A_{k-11}\cos(k\tau-\tau_e+\tau_e) + A_{k-1-1}\cos(k\tau-\tau_e-\tau_e)\right] + \cdots, \end{aligned} \tag{45}$$

where $\tau = (\omega + \frac{a}{\omega} b)t + \theta$, and $\tau_e = \frac{a}{\omega} ct + \theta_e$.

Equation (45) has been written in a manner which is easily generalized for systems of more than two oscillators. Note that the second and third terms of order β^2 are really cos $k\tau$ terms, and were written on the first line of the previous trial solution form, given by Equation (35).

The trial solution given by Equation (45) is essentially the same as the trial solution used in Chapter V for an expansion about a periodic solution for the linear system (Equations (39) of Chapter V). This form of trial solution was introduced without justification for the linear system, principally to provide a comparison between solutions of the linear

and nonlinear systems.

The comparison shows that whereas the expansion about a periodic solution for the linear system truncates automatically at order β^1, so that there are just N total terms in an expression for a given q_k, the expansion about a periodic solution for the nonlinear system continues indefinitely and has an infinite number of higher harmonics.

Furthermore, any semblance of linear superposition for the nonlinear system is very weak, even for arbitrarily small β (near a periodic solution), because both the $\cos(k\tau - \tau_e)$ and the $\cos(k\tau + \tau_e)$ terms appear to first order in β; there are $(2N-1)$ rather than N terms in an expression for a given q_k, to order β. The only similarity to linear superposition stems from the fact that the coefficients of the $\cos(k\tau - \tau_e)$ terms are somewhat smaller than the coefficients of the $\cos(k\tau + \tau_e)$ terms, as may be seen from Equations (39) for the case $N = 2$. Thus there are N dominant terms in an expression for a given q_k, to order β, for the nonlinear system. However, these dominant terms of order β still do not correspond to excitation of the other periodic solutions of the system, as was the case for the linear system in which the periodic solutions were the true normal modes.

In addition to the appearance of higher harmonics in the solution of the nonlinear system, the other major difference between the solutions of linear and nonlinear coupled oscillator systems is the dependence upon amplitude shown by the frequency components which are of order α, in the nonlinear system. This leads to an inverse proportionality between the period of energy exchange and the typical amplitude of the oscillations of the nonlinear system; in the linear system, the period of energy exchange

is independent of the amplitudes of the oscillations.

In this section, a workable form of trial general solution for the positions and momenta of a two-oscillator resonant exactly-tuned nonlinear system has been developed. In the next section, a generalization of this method for nonlinear systems of more than two oscillators will be made.

Many-Oscillator General Solution

The preceding discussions of this chapter have served to make plausible a certain form of trial solution for a nonlinear system of two oscillators. The remainder of this chapter will be devoted to the development of a more general form of trial solution, which is appropriate for nonlinear systems of many oscillators. Particular attention will be given to methods of dealing with the algebraic computations which are involved.

For a typical resonant exactly-tuned nonlinear system of N oscillators, consider the model formulated in Chapter III. If only resonant terms are retained in the coupling energy of this model, then the hamiltonian is of the form

$$H = \tfrac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + k^2\omega^2 q_k^2\right) + \alpha \sum_{j,l,m} C_{jlm} z_j z_l z_m, \qquad (46)$$

where z is a general symbol standing for either q or p, and where the C_{jlm} are numerical coefficients.

Hamilton's equations of motion for this system are

$$\dot{q}_k = p_k + \alpha \sum_{\ell \leq m}^{N} CEM_{k\ell m} z_\ell z_m ; \qquad (47a)$$

$$\dot{p}_k = -k^2\omega^2 q_k - \alpha \sum_{\ell \leq m}^{N} CEM'_{k\ell m} z_\ell z_m , \qquad (47b)$$

where $z_\ell z_m$ is of the form $q_\ell p_m$ or $p_\ell q_m$ in Equation (47a), or is of the form $q_\ell q_m$ or $p_\ell p_m$ in Equation (47b). The coefficients in the equations of motion, CEM and CEM', are not the same in the two Equations (47a) and (47b); in order to maintain some generality of analysis, these coefficients will not be explicitly specified in this chapter. Some definite values will be used only in Chapters VII and VIII, in conjunction with the numerical studies of these systems.

Since the coupling term of the hamiltonian given by Equation (46) is "resonant," the only $CEM_{k\ell m}$ and $CEM'_{k\ell m}$ which appear in Equations (47) are those for which either $m - \ell = k$ or $m + \ell = k$.

In the first subsection, a method of finding N stable periodic solutions for an N-oscillator nonlinear system will be presented. Having this indication of the existence at least in many cases of N distinct periodic solutions, the second subsection will discuss the details of obtaining a perturbation expansion about a periodic solution.

Periodic Solutions

In order to search for periodic solutions, assume a trial solution of the form

$$q_k = C_k \cos k(\Omega t + \theta) ; \qquad (48a)$$

$$p_k = -k\omega C_k \sin k(\Omega t + \theta) , \qquad (48b)$$

where $\Omega = \omega + \frac{a}{\omega} b$. This form is assumed for the case $\omega_1 = \omega_2/2 = \omega_3/3 = \ldots$. Substitute this trial solution into Equations (47a) and (47b). The algebraic relations obtained in this manner from Equation (47a) will be the same as those obtained from Equation (47b).

The result of the above procedure is a system of N nonlinear equations in $N + 1$ unknowns, which are the N coefficients C_k and the frequency component b. This may be regarded as a nonlinear eigenvalue-eigenvector problem. It is convenient to normalize the eigenvector at first, in order to provide the $(N+1)^{th}$ condition, and to later determine the norm to agree with the magnitude of the hamiltonian desired for a given numerical solution.

Since the eigenvalue-eigenvector problem set forth here is nonlinear, the usual matrix methods do not apply. In fact, there is no general theorem which guarantees that there even exist N eigenvalues, distinct or repeated, and N corresponding eigenvectors.

However, two distinct eigenvalues and corresponding eigenvectors were found for the two-oscillator nonlinear system studied earlier in this chapter, and these nonlinear periodic solutions were entirely analogous to the periodic solutions of the linear two-oscillator system. It may be assumed that the same analogy holds also between linear and nonlinear systems of more than two oscillators. On the basis of such an analogy, the N-oscillator nonlinear system would be expected to possess N distinct periodic solutions.

Assume that an analogy between linear and nonlinear periodic solutions holds. One method of solving the nonlinear eigenvalue-eigenvector problem for periodic solutions of the nonlinear system is to supply an approximate solution to the N equations, using the assumed analogy to

the periodic solutions of the linear system, and then to improve this approximate solution by an iterative procedure. This procedure is repeated N times, for each periodic solution. The analogy to the linear system implies that the approximate periodic solutions of the nonlinear system involve sets of initial conditions for which a "vibrating string" diagram of the system has zero, one, two, ..., N-1 modes. Also, the frequencies associated with these periodic solutions should show the same splitting in symmetric pairs observed for the linear system in the early part of Chapter V, although values for the frequencies are not included in the initial periodic solution approximations.

To provide an iterative scheme for improving the approximate periodic solution, the problem may be linearized and standard matrix techniques employed. The nonlinear eigenvalue-eigenvector equations are of the form

$$b_k = \frac{1}{k^2} \sum_{l \leq m}^{N} CEM_{klm} C_l C_m / C_k, \tag{49}$$

where b_k is the approximation to frequency component b which is provided by the k^{th} of Equations (49). (The initial approximate solution provides values only for the C_i.) There are $M \leq N$ of the Equations (49), for the M values of k for which $C_k \neq 0$.

Refinements of the approximations to the C_i are made by defining new values $\bar{C}_i$,

$$\bar{C}_i = C_i + \Delta C_i, \tag{50}$$

and making a Taylor series expansion of the b_k to first order,

$$b_k(\bar{C}_i) \approx b_k(C_i) + \sum_{j=1}^{N} \frac{\partial b_k(C_i)}{\partial C_j} \Delta C_j, \tag{51}$$

so that the corrections ΔC_j may be determined:

$$\Delta C = BPR^{-1} \Delta B \tag{52}$$

where ΔC is a column vector with M elements ΔC_i; BPR is an M by M matrix with elements

$$BPR_{kj} = \frac{\partial b_k(C_i)}{\partial C_j}, \tag{53}$$

and ΔB is a column vector with M elements $\Delta B_k = b_k(\bar{C}_i) - b_k(C_i)$. The value of $b_k(\bar{C}_i)$ is approximated by

$$b_k(\bar{C}_i) \approx b = \frac{1}{M} \sum_{k=1}^{M} b_k(C_i), \tag{54}$$

since b is a next-iteration value, compared with the $b_k(C_i)$.

Given a set of approximate relations for the C_i, one iteration consists of a calculation of the $b_k(C_i)$, b, and $\frac{\partial b_k(C_i)}{\partial C_j}$, from which vector ΔB and matrix BPR^{-1} are constructed. The corrections ΔC_j from Equation (52) are then calculated, and the $\bar{C}_i$ are constructed from the relation given in Equation (50). These $\bar{C}_i$ are then the approximate C_i for the next iteration. Iterations are discontinued when successive calculations for the C_i and b agree within some

specified amount.

Such an iteration procedure would be tedious if performed by hand, but is ideally suited for a high-speed digital computer. The rate of convergence of the method could be improved by taking more terms in the Taylor expansion, which would require the calculation of higher derivatives. But there are many of these, and a more complicated method such as the method of steepest descents for the solution of ΔC is required; so the simple method outlined here is probably adequate in most cases.

The actual results of the computations for small systems ($N < 10$) is reserved for Chapter VII. However, it is important to note before continuing the present development of a method of expansions about periodic solutions, that the actual numerical computations in Chapter VII provide some evidence that N distinct periodic solutions exist for each small nonlinear N-oscillator system which has been investigated.

In the nonlinear system, there is the possibility of the existence of singular "unstable" periodic solutions in addition to the N "stable" solutions being sought. For example, a perfectly valid singular periodic solution for the two-oscillator nonlinear system specified by Equations (29) of this chapter is

$$q_1 \equiv 0, \qquad p_1 \equiv 0 ; \tag{55a}$$

$$q_2 = C_2 \cos 2(\omega t + \theta), \quad p_2 = -2\omega C_2 \sin 2(\omega t + \theta), \tag{55b}$$

in addition to the other two solutions given by Equations (34).

The energies corresponding to the periodic solution given by

Equations (55) are

$$E_1 \equiv 0, \quad E_2 \equiv 2\omega^2 C_2^2. \tag{56}$$

The solution given by Equations (55) is referred to here as "unstable" in the sense that it is not suitable as a zeroth-order solution about which a general solution can be obtained in a perturbation expansion. That is, it is not possible to obtain an energy solution which is confined to an arbitrarily small neighborhood of the energy solution given by Equations (56), by making a sufficiently small change in the initial conditions from the set which produces the unstable periodic solution. As can be shown in a numerical solution using the computer, or from the elliptic function solution, any deviation of the initial conditions from the set

$$\begin{aligned} q_1(0) &= 0, \quad & p_1(0) &= 0; \\ q_2(0) &= C_2, \quad & p_2(0) &= D_2 \end{aligned} \tag{57}$$

results in complete energy exchange between the two oscillators (although the period of this energy exchange may be very long compared to the periods of the uncoupled oscillators).

The procedure outlined in the present discussion for finding periodic solutions is quite unlikely to specify unstable periodic solutions. The iteration procedure cannot converge to an unstable solution, simply because the nature of the instability is such as to drive successive iterations away from one of these singular solutions.

Moreover, the beginning approximations are much closer to stable

periodic solutions, obtained by analogy with those of the linear system, than they are to unstable periodic solutions. As can be seen by a comparison of Equations (29) of this chapter and Equations (23) of Chapter V, there is no possibility for the existence of unstable periodic solutions for the linear system.

In this subsection, a method of obtaining N stable and distinct periodic solutions for a resonant exactly-tuned nonlinear system of N oscillators has been described. Some evidence for the general existence of N such solutions has been presented.

Although the method has been illustrated here in terms of an application to a cubic nonlinear system, it should be generally applicable to other types of resonant exactly-tuned nonlinear systems as well.

In the following subsection, a method of making perturbation expansions about the periodic solutions identified in the present subsection will be developed.

Expansion about Periodic Solution

Since there is some evidence for the existence of N periodic solutions, a general method will now be developed for obtaining the general solution of a resonant exactly-tuned nonlinear coupled oscillator system in the neighborhood of a periodic solution, as a perturbation solution about the periodic solution.

This subsection is to be considered the most important portion of the thesis. Nearly all of the material before this has been presented as a preliminary to this analysis of the general case.

The expansions about periodic solutions for nonlinear systems do not automatically truncate at some low order, as do those for the

linear system. Any definite indication of the accuracy of the approximate solution obtained in some region of a periodic solution by artificial truncations of the expansions developed here must await a computer study in Chapter VII.

A form of trial general solution will be assumed which is a generalization of the last form of solution, Equation (45), developed for the two-oscillator nonlinear system earlier in this chapter. The generalization consists of recognizing that for an N-oscillator system, there must be $N-1$ expansion parameters, β_j.

With the aid of certain conventions to be defined, the assumed trial solution may be written rather concisely:

$$q_k = A_0 \sum_{j_1} \sum_{j_2} \cdots \sum_{j_T=-N1}^{+N1} \left[\beta_{j_1} \beta_{j_2} \cdots \beta_{j_T} A_{k j_1 j_2 \cdots j_T} \cdot \cos\left(k\tau + \sum_{i=1}^{T} \tau_{j_i}\right) \right], \tag{58a}$$

$$p_k = -k\omega A_0 \sum_{j_1} \sum_{j_2} \cdots \sum_{j_T=-N1}^{+N1} \left[\beta_{j_1} \beta_{j_2} \cdots \beta_{j_T} A_{k j_1 j_2 \cdots j_T} \cdot \sin\left(k\tau + \sum_{i=1}^{T} \tau_{j_i}\right) \right], \tag{58b}$$

for $k = 1,2,\ldots,N$, where $N1 = N-1$, and where

$$\tau = \left(\omega + \frac{\alpha}{\omega} b A_0\right) t + \theta; \tag{59}$$

and

$$\tau_{j_i} = \left(\frac{\alpha}{\omega} c_{j_i} A_0\right) t + \theta_{j_i}. \tag{60}$$

The following conventions are assumed in writing the trial solution in this form: if $j_i = 0$, then $\beta_o = 1$ and $\tau_o = 0$; if $j_i < 0$, then $\beta_{ji} = \beta_{-ji}$ and $\tau_{ji} = -\tau_{ji}$. Also, if $j_i = 0$ and there is another subscript $j_\ell \neq 0$ for which $\ell > i$, then all the coefficients $A_{kj_1j_2\cdots j_T}$ containing this combination are zero. In practice, the series is truncated to order T as shown, although in principle, $T \to \infty$. The overall amplitude of the general solution is given by A_o, the introduction of which will be discussed presently.

The solution given by Equations (58) has many terms if it is not truncated to low order. That is, if the expansion is truncated to order β^T, there are roughly $N(2N-1)^T$ terms to be calculated. For example, for $N = 8$, a truncation to the fairly low order β^3 still produces about 27,000 terms, an excessive number for even a modestly large high-speed computer to handle efficiently.

On the other hand, the solution given by Equations (58) suffers in accuracy if it is truncated to too low an order. For example, for the two-oscillator resonant nonlinear system discussed earlier in this chapter, there exists an extreme case of initial conditions for which the actual period of energy exchange is infinite (Baker 1921). Many terms are needed to accurately represent solutions near this extreme. Systems of more than two oscillators are not expected to possess solutions having infinite periods, but there will in general be cases in which many terms in the solution given by Equations (58) are needed for an accurate approximation to the exact solution of resonant exactly-tuned systems of N oscillators.

A sensible compromise between truncation in low order to minimize

the number of terms to be calculated, and truncation in high order to maximize the accuracy of the series approximation, is to carry the complete solution out to a reasonable order, such as $T = 2$ for $N < 10$, and then to include only those higher-order terms which make a significant contribution to the solution, if needed. For a certain set of initial conditions and an expansion about a certain periodic solution, some of the β_j will be smaller than others, so that many of the higher-order terms in β_j are not significant and may be neglected. This sort of compromise meets both demands of minimizing the number of terms and yet maximizing the accuracy of the approximation.

For $T = 2$, the assumed trial solution given concisely by Equations (58) is of the following explicit form:

$$
\begin{aligned}
q_k = A_0 \{ & A_{k00} \cos k\tau \\
& + \sum_{j_1=1}^{N1} \beta_{j_1} [A_{kj_10} \cos(k\tau + \tau_{j_1}) \\
& \qquad + A_{k-j_10} \cos(k\tau - \tau_{j_1})] \\
& + \sum_{j_2=j_1}^{N1} \sum_{j_1=1}^{N1} \beta_{j_2}\beta_{j_1} [A_{kj_1j_2} \cos(k\tau + \tau_{j_1} + \tau_{j_2}) \\
& \qquad + A_{kj_1-j_2} \cos(k\tau + \tau_{j_1} - \tau_{j_2}) \\
& \qquad + A_{k-j_1j_2} \cos(k\tau - \tau_{j_1} + \tau_{j_2}) \\
& \qquad + A_{k-j_1-j_2} \cos(k\tau - \tau_{j_1} - \tau_{j_2})] \}
\end{aligned}
\tag{61}
$$

with corresponding momentum expressions obtained by the same prescription used in Equations (58b), and where τ and τ_{ji} are defined by Equations (59) and (60).

To obtain the solution of the cubic-coupled nonlinear system

formulated as the model to be analyzed in this thesis, substitute the trial solutions given by Equations (61) into the equations of motion, Equations (47), and equate coefficients of similar combinations of β_{ji}. Equations (47a) and (47b) provide identical systems of algebraic relations involving the amplitude components $A_{k\ell m}$ and the frequency components b and cj, for all orders of β.

Zeroth-Order Terms. The zeroth-order algebraic equations in this expansion about a periodic solution are exactly the same as the original equations for the periodic solution alone, Equations (49). They will be written here with two minor changes, namely the notational replacement of C_k by A_{k00}, and the explicit inclusion of the norm of the eigenvector $\{A_{k00}\}$, indicated here by A_0:

$$b_k A_0 = \frac{1}{k^2} \sum_{\ell \equiv m}^{N} CEM_{k\ell m} A_0 A_{\ell 00} A_{m00} / A_{k00}. \qquad (62)$$

As discussed earlier in connection with periodic solutions of the N-oscillator nonlinear system, the value of the norm A_0 is first set at unity for purposes of the eigenvector problem, and is then adjusted to a different amplitude to provide a total system energy which agrees with the magnitude of the hamiltonian desired in a particular situation.

The role which overall amplitude A_0 plays in the general solution given by Equations (58) is quite similar.

The fact that the zeroth-order equations of the expansion about a periodic solution are the same as the equations of the periodic solution itself, means that the zeroth-order terms for the perturbation expansion presented here have been chosen correctly to all orders in the expansion

parameters β_{ji}. This appears to be an advantage over many of the existing perturbation methods which use α as an expansion parameter. The problem of the appearance of small divisors, which arises in the latter methods and causes contributions in zeroth order from supposedly higher-order terms, should not arise in the perturbation scheme now being described.

The nonlinear eigenvalue-eigenvector solution scheme described in the previous subsection may be used to solve the zeroth-order equations of the expansion.

Note from Equations (62) that the periodic-solution frequency component b is determined by the zeroth-order equations, but that the energy-sharing frequency components c_j are not determined until first order, even though the c_j are zeroth-order terms. This is to be expected, since the construction of the perturbation expansion has absolutely no effect on the zeroth-order periodic solution.

First-Order Terms. The first-order algebraic equations in an expansion about one of the N possible zeroth-order solutions are of the general forms

$$(b+c_{j_1})A_0A_{kj_10} = \frac{1}{k^2}\sum_{l\leq m}^{N} CEM_{klm}\,A_0\,(A_{l00}A_{mj_10} + A_{l\pm j_10}A_{m00}); \tag{63a}$$

$$(b-c_{j_1})A_0A_{kj_10} = \frac{1}{k^2}\sum_{l\leq m}^{N} CEM_{klm}\,A_0\,(A_{l00}A_{m-j_10} + A_{l\mp j_10}A_{m00}), \tag{63b}$$

where the upper-lower signs on the subscripts follow a certain convention: for a given term, if $m + \ell = k$ in the coefficient $CEM_{k\ell m}$, then use upper signs on the subscripts; if $m - \ell = k$, then use lower signs. As mentioned earlier, these are the only two cases for the resonant system. The basis for this convention becomes clear if the rather lengthy result of substitution of Equations (61) into Equations (47) is carried out in detail, observing the manner in which the various trigonometric terms combine.

The quantities A_{k00} and b in Equations (63) are known from the solution of the zeroth-order Equations (62). The quantities to be determined from Equations (63) are therefore the $A_{k\ell 0}$ and the c_j, so that Equations (63) represent a system of $2N$ equations in $2N+1$ unknowns.

An examination of Equations (63) shows that because the zeroth-order solutions are known for an expansion about a given periodic solution, this system of equations forms a linear eigenvalue-eigenvector problem. As before, the eigenvector solutions may be normalized, with the norms to later be determined by the initial conditions.

The linear eigenvalue-eigenvector problem encountered here is not quite as simple computationally as the one usually encountered in connection with simple physical systems. When Equations (63) are written in $2N$ by $2N$ matrix formulation in the usual form

$$\begin{bmatrix} (b-c_{j_1}) & M_{12} & M_{13} & \cdots \\ M_{21} & (b-c_{j_1}) & M_{23} & \\ M_{31} & M_{32} & (b-c_{j_1}) & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} A_{1j_10} \\ A_{1-j_10} \\ A_{2j_10} \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \tag{64}$$

matrix M is found to be unsymmetric, primarily because of the presence of the factors $1/k^2$ in Equations (63).

In the usual simple physical problem, matrix M is symmetric. In this case, there is a theorem which guarantees the existence of 2N real eigenvalues, and there are well-established transformation methods for diagonalizing matrix M and thus determining its eigenvalues.

But in the case of a nonsymmetric matrix M, 2N real eigenvalues do not necessarily exist, and methods of finding eigenvalues if they do exist are not so well established.

One very unsophisticated method of determining the eigenvalues of nonsymmetric matrix M is to determine the roots of its secular equation by evaluation of the determinant of M. Such a method may be implemented in two steps, for efficiency. Since M is known to have two repeated roots of zero, which are trivial since they duplicate the results of the zeroth-order solution, the first step is to find the general location of the nonzero roots of interest. This may be done by an orderly evaluation of the secular determinant over the domain in which the roots are expected. The second step is to use a general rootfinder method to home in on the roots by an iterative procedure, refining the approximate values of the roots obtained in the first step.

The straightforward method indicated is practical if a high-speed digital computer is used to evaluate the determinants and to conduct the iterative procedure for improving approximations to the roots. Standard methods for finding the real or complex roots of arbitrary functions are readily available. The procedure chosen for the present problem refines given approximations to roots by a method of variation of the approximation,

followed by second-order interpolation to an improved approximation.

A discussion of actual computer results for these eigenvalues will not be given. It suffices to state that in every case in which a calculation was attempted, for small nonlinear systems of $N < 10$ oscillators, the values of all $2N$ real eigenvalues have been successfully determined, using the procedure outlined above. Of these $2N$ eigenvalues, two are always zero, and the remaining $2(N-1)$ occur in c_j, $-c_j$ pairs, as may be seen from the symmetry of Equations (63). Therefore, there are just $N-1$ eigenvalues of interest. For definiteness, let these be all the positive ones. These $N-1$ eigenvalues c_j give the frequency components of energy sharing associated with the $N-1$ expansion parameters β_j.

Having obtained the $N-1$ eigenvalues of interest, the corresponding $N-1$ normalized eigenvectors may be found by any standard method of solution of $2N$ homogeneous linear algebraic equations in $2N$ unknowns. In the actual computations, it was found convenient to use a Gauss-Jordan procedure, programmed on the computer.

Computation of these eigenvectors completes the solution through first order in the expansion parameters β_j. The frequency components b and c_j have now been determined by the zeroth- and first-order equations, and there are no higher-order corrections to arguments τ and τ_j to any order in β_j. It might therefore appear that extending the expansion to order β^2 and higher can have no effect upon the frequencies calculated for a given system by this method.

However, this is not true; the actual frequency corrections given in Equations (59) and (60) are proportional to the zeroth-order norm A_0

as well as to b and c_j, respectively. The zeroth-order norm A_0 and the $N-1$ norms of the $N-1$ first-order eigenvectors, together with phase angle θ and the $N-1$ phase angles θ_j, are the $2N$ arbitrary constants of the general solution which are fixed by the $2N$ initial conditions

$$q_k(0) = A_k, \qquad \dot{q}_k(0) = B_k. \tag{65}$$

Components of the zeroth- and first-order eigenvectors appear in the higher-order equations, and so for a given set of initial conditions, inclusion of higher-order terms in the expansion about a periodic solution affects the values of the N norms. It is this sort of effect of higher-order terms on the zeroth-order norm A_0 which alters the frequency components in Equations (59) and (60) when higher-order terms are included, for a given set of initial conditions.

Although the frequencies are thus altered by the inclusion of higher-order terms, the shift is uniform for all of the components. Each of the components τ_j which control the periods of the various energy-sharing harmonics are proportional to the amplitude A_0, so that inclusion of higher-order terms can have no effect upon the ratios of the periods of the various energy-sharing processes specified in the first-order solution. The effect of higher-order terms is restricted to an expansion or contraction of the overall energy-sharing time scale, in addition to the inclusion of higher harmonics.

Second-Order Terms. The second-order algebraic equations in an expansion about one of the N possible zeroth-order solutions are of the general form

$$(b+c_{j_1}+c_{j_2})A_0 A_{kj_1j_2} = \frac{1}{k^2}\sum_{l\leq m}^{N} CEM_{klm} A_0 (A_{l00} A_{mj_1j_2} + A_{l\pm j_1 0} A_{mj_2 0} + A_{l\pm j_1 \pm j_2} A_{m00}); \quad (66a)$$

$$(b+c_{j_1}-c_{j_2})A_0 A_{kj_1-j_2} = \frac{1}{k^2}\sum_{l\leq m}^{N} CEM_{klm} A_0 (A_{l00} A_{mj_1-j_2} + A_{l\pm j_1 0} A_{m-j_2 0} + A_{l\pm j_1 \mp j_2} A_{m00}); \quad (66b)$$

$$(b-c_{j_1}+c_{j_2})A_0 A_{k-j_1j_2} = \frac{1}{k^2}\sum_{l\leq m}^{N} CEM_{klm} A_0 (A_{l00} A_{m-j_1j_2} + A_{l\mp j_1 0} A_{mj_2 0} + A_{l\mp j_1 \pm j_2} A_{m00}); \quad (66c)$$

$$(b-c_{j_1}-c_{j_2})A_0 A_{k-j_1-j_2} = \frac{1}{k^2}\sum_{l\leq m}^{N} CEM_{klm} A_0 (A_{l00} A_{m-j_1-j_2} + A_{l\mp j_1 0} A_{m-j_2 0} + A_{l\mp j_1 \mp j_2} A_{m00}), \quad (66d)$$

where the convention for upper and lower signs introduced in conjunction with Equations (63) is used. All of the terms in Equations (66) are known from the zeroth- and first-order solutions, except for the amplitude components of the form A_{klm} in which all three subscripts are nonzero.

Equations (66) together form a system of 4N linear nonhomogeneous algebraic equations in 4N unknowns. There are $(N-1)^2$ such systems of equations, since j_1 and j_2 each range from 1 to $N-1$. These systems of equations are easily solved by any standard method for solving sets of

linear nonhomogeneous equations.

Higher-Order Terms. Second-order Equations (66) may be written in a more concise form with the aid of an added convention:

$$(b \oplus c_{j_1} \oplus c_{j_2}) A_0 A_{k \oplus j_1 \oplus j_2} \tag{67}$$
$$= \frac{1}{k^2} \sum_{l \in m}^{N} CEM_{klm} A_0 (A_{l00} A_{m \oplus j_1 \oplus j_2} + A_{l \pm j_1 0} A_{m \oplus j_2 0} + A_{l \pm j_1 \pm j_2} A_{m00}),$$

where the added convention is to change all the circled signs of subscripts j_1 if the circled sign of c_{j_1} is negative, and to leave them unchanged otherwise; and similarly for j_2. The upper-lower sign convention introduced with Equations (63) still holds.

Using the above convention to write one equation which represents eight, the third-order equations in an expansion about one of the N possible zeroth-order solutions are of the form

$$(b \oplus c_{j_1} \oplus c_{j_2} \oplus c_{j_3}) A_0 A_{k \oplus j_1 \oplus j_2 \oplus j_3} \tag{68}$$
$$= \frac{1}{k^2} \sum_{l \in m}^{N} CEM_{klm} A_0 (A_{l000} A_{m \oplus j_1 \oplus j_2 \oplus j_3} + A_{l \pm j_1 00} A_{m \oplus j_2 \oplus j_3 0} + A_{l \pm j_1 \pm j_2 0} A_{m \oplus j_3 0} + A_{l \pm j_1 \pm j_2 \pm j_3} A_{m00}).$$

Equations (68) form $(N-1)^3$ systems of $8N$ linear nonhomogeneous algebraic equations in $8N$ unknowns, which are the amplitude

coefficients $A_{k\ell mn}$ which have four nonzero subscripts. All the other terms which appear in Equations (68) are known from lower-order calculations.

The obvious generalization of Equation (68) to a case of order T results in $(N-1)^T$ systems of $2^T N$ equations in $2^T N$ unknowns. These higher-order $(T \equiv 3)$ equations are easy to solve by standard methods, but there are so many equations and unknowns for even small systems that the task becomes prohibitively lengthy.

As mentioned before, if these higher-order terms are to be used, some method of choosing only those higher-order terms which are most significant must be used, in order to reduce the number of terms to be calculated. This can be done only for a given system, that is, for a given set of initial conditions and an expansion about a particular periodic solution.

Expansion Parameters. Assume a simple set of all position or all momentum initial conditions are applied to the system, so that all the phase angles in the general expansion, Equations (59) and (60) are either zero or $\frac{\pi}{2}$, so that there are only N initial conditions of importance.

The eigenvector of the zeroth-order solution and the $N-1$ eigenvectors of the $N-1$ first-order solutions have all been normalized to unity to begin with. The components of these eigenvectors are multiplied by A_0 and $A_0\beta_j$, respectively, in the solutions given by Equations (58) or (61).

When the N initial conditions are applied to the system, the values of the N norms must be changed from unity to another set of values, in order that the solutions given by Equations (58) or (61)

agree in amplitude with the initial conditions at $t = 0$.

Thus it is natural to identify the parameters A_0 and the $A_0\beta_j$ with the norms of the zeroth- and first-order eigenvectors. This means that the expansion parameters β_j will represent the ratios of the norms of the j^{th} eigenvectors from first order to the norm of the eigenvector from zeroth order. If the initial conditions are those which produce the periodic solution about which a given expansion has been made, then the norms of the first-order eigenvectors will be zero and the expansion parameters β_j will be zero, as is required.

With the expansion parameters β_j normalized in this manner, a useful expansion about a given periodic solution for a given set of initial conditions can be recognized as one for which all of the expansion parameters β_j are less than unity; the smaller the better. There is only one overall amplitude in the entire solution, and that is A_0. Thus it is not the level of the amplitudes of the initial conditions which determines how good a given expansion about a periodic solution will be. It is only the ratios between the initial condition amplitudes which is significant.

No matter how high an order of truncation is to be used, approximate values for A_0 and the $N-1$ expansion parameters β_j may be obtained after the zeroth- and first-order solutions have been obtained, for a given set of initial conditions, by use of a matrix inversion. For example, if the initial conditions are

$$q_k(0) = A_k \; ; \quad \dot{q}_k(0) = 0, \tag{69}$$

then from the zeroth- and first-order portions of Equations (58) it

follows that

$$\begin{bmatrix} A_{100} & (A_{1\text{-}10}+A_{110}) & (A_{1\text{-}20}+A_{120}) & \cdots \\ A_{200} & (A_{2\text{-}10}+A_{210}) & (A_{2\text{-}20}+A_{220}) & \\ A_{300} & (A_{3\text{-}10}+A_{310}) & (A_{3\text{-}20}+A_{320}) & \\ \vdots & & & \ddots \end{bmatrix} \begin{bmatrix} A_0 \\ A_0\beta_1 \\ A_0\beta_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \end{bmatrix} . \tag{70}$$

A general expression for A_0 and the β_j as a function of the initial conditions A_k may be obtained by inversion of the square matrix, since its components are fixed for an expansion about a given periodic solution.

This evaluation of the β_j gives a useful preliminary estimate of which periodic solution is best expanded about, for a given set of initial conditions. The values obtained in this approximation are not expected to change drastically due to the inclusion of higher-order terms, at least for most sets of initial conditions.

It is anticipated that in order to obtain general solutions for all sets of initial conditions for a given nonlinear coupled oscillator system, it will be necessary to make expansions about all N of the available periodic solutions. The best that can be hoped is that general solutions for nearly all sets of initial conditions can be well approximated in this manner, without going above, say, second order in the expansion parameters.

In this chapter, a method for obtaining a representation of the general solution of a resonant exactly-tuned cubic-coupled nonlinear N-oscillator system has been developed, in which the general solution is supplied as a set of perturbation expansions about periodic solutions

of the system.

First, as a guide in addition to the experience gained in the analysis of linear systems, the energy solution of a nonlinear two-oscillator system has been obtained. A method for finding periodic solutions and making perturbation expansions about them has been developed for the nonlinear two-oscillator system. Finally, a generalization of this method, appropriate for nonlinear many-oscillator systems, has been obtained. This general method has been illustrated in terms of a cubic nonlinear system, but it is expected to find application to other types of nonlinear systems as well.

Since there is no other readily available analytical method for solution of resonant exactly-tuned nonlinear many-oscillator systems, the effectiveness of this method in determining approximations to the general solutions of these systems for various initial conditions is not known. It is necessary to test the results of the method developed in this chapter against the results of numerical integrations of the equations of motion, for particular sets of initial conditions.

This may be done in a practical manner by the use of a high-speed digital computer. A stringent test of the validity of one of these approximate solutions, obtained by truncation of the exact infinite series form, is to compare the results of the approximation with those of numerical solution for an extreme case. For the two-oscillator nonlinear system, such an extreme case would be that for which the period of energy exchange is infinite, as indicated by the elliptic function solution. Computed comparisons of this and other types will be presented in the following chapter.

CHAPTER VII

COMPARISONS BETWEEN ANALYTIC SOLUTIONS AND NUMERICAL SOLUTIONS FOR SOME NONLINEAR COUPLED OSCILLATOR SYSTEMS

The analytic method of solution for resonant exactly-tuned nonlinear coupled oscillator systems, which has been developed in Chapter VI in the form of perturbation expansions about periodic solutions, provides a set of infinite series expressions for the positions and momenta, and hence for the energy of a given system. In principle, there is no limit to the number of terms which can be calculated in these expressions; but in practice, it is necessary to truncate the infinite series expressions to some reasonable order.

At this stage of the investigation, it is not known how effective a set of truncated perturbation expansions will be in approximating the exact general solution of a nonlinear system, for all possible sets of initial conditions. Therefore, some computer studies will be described in this chapter in which some analytic approximations, truncated to first and second order for small systems, are compared with corresponding numerical solutions of the equations of motion for some cubic-coupled systems of two, three and five oscillators.

The first-order approximations are found to be reasonably effective, even for some rather extreme sets of initial conditions. However, some difficulties are encountered in the calculation of second-order approximations as formulated in Chapter VI. It is likely that a different

choice of arbitrary constants in the perturbation expansions will have to be made in order to obtain second-order approximations which consistently represent improvements over the corresponding first-order approximations.

The type of system in terms of which computer studies will be discussed in this chapter is the exactly-tuned cubic-coupled nonlinear system which was formulated in Chapter III. The explicit expressions for the coupling coefficients of this system, which have not been given so far in the interest of generality, will now be supplied. The hamiltonian introduced in Equation (1) of Chapter III, corresponding to FPU-type nearest-neighbor couplings between particles but having the FPU frequencies replaced by a set of commensurable frequencies, may be written as

$$
\begin{aligned}
H = {} & \tfrac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + k^2\omega^2 q_k^2\right) \\
& -\alpha\left(\frac{2}{N+1}\right)^{\frac{1}{2}}\Bigg\{\sum_{j>k=1}^{N}\left[\sin\frac{k\pi}{N+1}+\sin\frac{j\pi}{N+1}\right]q_j q_k q_{j+k} \\
& \quad +\sum_{j=1}^{N}\left[\sin\frac{j\pi}{N+1}\right]q_j q_j q_{2j} \\
& \quad +\sum_{j>k=1}^{N}\left[\sin\frac{k\pi}{N+1}-\sin\frac{j\pi}{N+1}\right]q_j q_k q_{j-k} \\
& \quad +\sum_{j>k=1}^{N}\left[-\sin\frac{k\pi}{N+1}-\sin\frac{j\pi}{N+1}\right]q_j q_k q_{2N+2-j-k} \\
& \quad +\sum_{j=1}^{N}\left[-\sin\frac{j\pi}{N+1}\right]q_j q_j q_{2N+2-2j}\Bigg\},
\end{aligned}
\tag{1}
$$

with the convention that $q_i = 0$ if $i \leq 0$ or if $i \geq N + 1$.

The hamiltonian given by Equation (1) contains both resonant and nonresonant coupling terms. In order to construct a resonant system corresponding to this given system, so that the methods of analysis developed in Chapter VI can be used, it is necessary to eliminate the nonresonant terms from these equations.

To do this, a transformation to action-angle variables is made, and the rapidly-varying terms are dropped from the coupling, in the manner discussed in Chapter II. The inverse transformation back to position-momentum variables is then made. The resulting hamiltonian is

$$H = \frac{1}{2}\sum_{k=1}^{N}\left(p_k^2 + k^2\omega^2 q_k^2\right) \tag{2}$$

$$-\alpha\left(\frac{2}{N+1}\right)^{\frac{1}{2}}\frac{1}{4}\Bigg\{\sum_{j>k=1}^{N}\left[\sin\frac{k\pi}{N+1}+\sin\frac{j\pi}{N+1}\right]\cdot\left[q_j q_k q_{j+k}+\frac{q_j p_k p_{j+k}}{\omega_k\omega_{j+k}}+\frac{p_j q_k p_{j+k}}{\omega_j\omega_{j+k}}-\frac{p_j p_k q_{j+k}}{\omega_j\omega_k}\right]$$

$$+\sum_{j=1}^{N}\left[\sin\frac{j\pi}{N+1}\right]\cdot\left[q_j q_j q_{2j}+2\frac{q_j p_j p_{2j}}{\omega_j\omega_{2j}}-\frac{p_j p_j q_{2j}}{\omega_j\omega_j}\right]$$

$$+\sum_{j>k=1}^{N}\left[\sin\frac{k\pi}{N+1}-\sin\frac{j\pi}{N+1}\right]\cdot\left[q_k q_{j-k} q_j+\frac{q_k p_{j-k} p_j}{\omega_{j-k}\omega_j}+\frac{p_j q_{j-k} p_k}{\omega_k\omega_j}-\frac{p_k p_{j-k} q_j}{\omega_k\omega_{j-k}}\right]\Bigg\}.$$

A comparison of Equations (1) and (2) shows that two distinct types of nonresonant coupling terms have been dropped from the given hamiltonian in order to construct the corresponding resonant hamiltonian. First, those terms for which neither relation $j + \ell - m = 0$ nor relation $j - \ell - m = 0$ is true, where j, ℓ and m are the subscripts of

the three positions appearing in the coupling terms, have been dropped. Second, the nonresonant portions of those terms which remain have been eliminated. This second elimination of terms in action-angle variables is better interpreted in position-momentum variables as an addition of certain supplementary mixed terms which complete the q,p symmetry of those coupling terms which remain after the first step has been taken.

In the physical application to crystal lattice vibrations, Peierls (1956) shows that only those coupling terms which produce three-phonon interactions in which energy is conserved are important in explaining lattice conductivity due to anharmonicities. That is, only couplings for which either $\omega_j + \omega_\ell - \omega_m = 0$ or $\omega_j - \omega_\ell - \omega_m = 0$ are important, for the reason of "internal resonances" discussed in Chapter II of this thesis. These phonon-energy-conserving couplings are the terms which remain after the first elimination described above, exclusive of the second "symmetrization" elimination.

Peierls further classifies the important coupling terms as "normal" terms and "umklapp" terms. Normal terms are those for which phonon quasi-momentum as well as phonon energy is conserved. Umklapp terms are energy-conserving terms for which the difference in quasi-momentum is equal to a vector in an inverse lattice. Peierls shows that the umklapp terms are the more important of the two types, so far as lattice conductivity due to anharmonicities is concerned.

The work of Peierls is concerned with physically realistic three-dimensional crystals, in which there are one longitudinal mode and two transverse modes of vibration for one atom in a unit cell. The dispersion curves have three branches, and although the frequencies associated

with each branch are of the FPU type, interactions which conserve energy can be obtained by involving both longitudinal and transverse modes. Both normal and umklapp coupling terms are present in such a situation.

In contrast, only normal terms are present in the one-dimensional system used as a model in this study. In Equation (1), the first three coupling terms are normal terms; the last two would be umklapp terms if they conserved phonon energy, but they do not. That is, the last two terms are nonresonant terms, and are not important in energy sharing in the one-dimensional model.

Thus the model used in this study will not exhibit the proper lattice thermal conductivity due to anharmonicities. This is not surprising, since the present model with its commensurable frequencies represents only the lower portion of the longitudinal branch of the three-dimensional system.

The equations of motion which correspond to the "original" system given by Equation (1) are

$$\dot{p}_k + k^2\omega^2 q_k \tag{3a}$$

$$= \alpha\left(\frac{2}{N+1}\right)^{\frac{1}{2}} 2 \left\{ \sum_{r=1}^{N-k} \left[\sin\frac{(k+r)\pi}{N+1} - \sin\frac{r\pi}{N+1} - \sin\frac{k\pi}{N+1}\right] q_r q_{k+r} \right.$$

$$+ \sum_{r=1}^{k-1} \left[\left(-1+\cos\frac{r\pi}{N+1}\right)\sin\frac{(k-r)\pi}{N+1}\right] q_r q_{k-r}$$

$$\left. + \sum_{r=N-k}^{N} \left[\left(-1+\cos\frac{r\pi}{N+1}\right)\sin\frac{(k+r)\pi}{N+1}\right] q_r q_{2N+2-r-k} \right\};$$

$$\dot{q}_k - p_k = 0. \tag{3b}$$

These 2N first-order equations may, of course, be conveniently combined into N second-order equations, since momentum and velocity are identical in the "original" system.

The equations of motion which correspond to the "resonant" system given by Equation (2) are

$$\dot{p}_k + k^2\omega^2 q_k \tag{4a}$$

$$= \alpha\left(\frac{2}{N+1}\right)^{\frac{1}{2}}\frac{1}{2}\left\{\sum_{r=1}^{N-k}\left[\sin\frac{(k+r)\pi}{N+1} - \sin\frac{r\pi}{N+1} - \sin\frac{k\pi}{N+1}\right]\right.$$
$$\cdot\left[q_r q_{k+r} + \frac{p_r p_{k+r}}{\omega_r \omega_{k+r}}\right]$$
$$+\sum_{r=1}^{k-1}\left[\left(-1+\cos\frac{r\pi}{N+1}\right)\sin\frac{(k-r)\pi}{N+1}\right]$$
$$\left.\cdot\left[q_r q_{k-r} - \frac{p_r p_{k-r}}{\omega_r \omega_{k-r}}\right]\right\};$$

$$\dot{q}_k - p_k \tag{4b}$$

$$= -\alpha\left(\frac{2}{N+1}\right)^{\frac{1}{2}}\frac{1}{2}\left\{\sum_{r=1}^{N-k}\left[\sin\frac{(k+r)\pi}{N+1} - \sin\frac{r\pi}{N+1} - \sin\frac{k\pi}{N+1}\right]\right.$$
$$\cdot\left[\frac{q_r p_{k+r}}{\omega_k \omega_{k+r}} - \frac{p_r q_{k+r}}{\omega_k \omega_r}\right]$$
$$+\sum_{r=1}^{k-1}\left[\left(-1+\cos\frac{r\pi}{N+1}\right)\sin\frac{(k-r)\pi}{N+1}\right.$$
$$\left.\left.\cdot\left[\frac{q_r p_{k-r}}{\omega_k \omega_{k-r}} + \frac{p_r q_{k-r}}{\omega_k \omega_r}\right]\right\}.\right.$$

The technique used in the computer studies described in this chapter is to first calculate analytic approximations to the solutions of the equations of motion of the resonant system, Equations (4), by the methods described in Chapter VI; these same equations are then numerically integrated, for the same set of initial conditions, and the two results are compared.

In the interest of accuracy and efficiency, the first computer program is one which constructs and punches out on cards Equations (4). These cards are then used to insert the equations of motion into two other programs: one which calculates an approximate solution by methods already outlined in Chapter VI, and one which integrates the equations of motion by a standard fourth-order Runge-Kutta method, to provide the "exact" numerical solution. The latter is indeed not mathematically exact, but it is numerically correct to considerably more significant figures than are needed for the graphs presented in this study. Several checks, such as calculation of the hamiltonian and reversal of integration back to the initial conditions, give assurance of the accuracy of the numerical solutions.

In the first section of this chapter, the results of calculations of sets of initial conditions for periodic solutions of resonant exactly-tuned systems of two, three and five nonlinear coupled oscillators will be given. The second section will compare some first-order approximate solutions with corresponding numerical solutions, for several typical sets of initial conditions. In the third section, a few of the corresponding second-order expansions will be exhibited, and some of the difficulties involved in using the formulation described in Chapter VI will be discussed.

It has been indicated previously in this thesis that the couplings between oscillators in any physically realistic model must include non-resonant as well as resonant terms. The systems such as those given by Equations (2) and (4) of this chapter, for which a method of solution was developed in Chapter VI, contain only resonant coupling terms; it is felt

that the effect of including nonresonant terms (as in Equations (1) and (3)) can be adequately treated by standard perturbation methods, using the solution of the resonant system as a zeroth-order solution. Although no such attempt will be made in the present investigation to deal analytically with nonresonant terms, the fourth and final section of this chapter will present some computer studies of the effect of inclusion of nonresonant terms in the couplings of systems considered earlier in this chapter.

Existence of Periodic Solutions

In Chapter VI it was indicated that resonant exactly-tuned nonlinear coupled systems of N oscillators may be expected to possess N periodic solutions, on the basis of an assumed analogy to corresponding linear coupled oscillator systems.

Although there is no general theorem which guarantees the existence of N periodic solutions for the nonlinear system, and although the assumed analogy to linear systems cannot always be relied upon, it has been possible to calculate N distinct stable periodic solutions to systems of the type given by Equation (4) in every case which has been attempted, including a system of eight oscillators. The results are presented in Table 3.

The values of initial conditions for these periodic solutions were calculated by first obtaining approximate sets of values by analogy to the periodic solutions (true normal modes) of a corresponding linear system; these approximations were then refined by an iterative procedure, as outlined in Chapter VI. As may be seen from Table 3, the periodic solutions thus obtained are completely analogous to those of the linear system with

Table 3. Periodic Solutions for Nonlinear Systems

Two Coupled Oscillators

	PS1	PS2
b	0.11785	-0.11785
A_{100}	0.94281	0.94281
A_{200}	0.33333	-0.33333

Three Coupled Oscillators

	PS1	PS2	PS3
b	0.07834	0.00000	-0.07834
A_{100}	0.91364	0.97922	0.91364
A_{200}	0.36959	0.00000	-0.36959
A_{300}	0.16932	-0.20280	0.16932

Five Coupled Oscillators

	PS1	PS2	PS3	PS4	PS5
b	0.03452	0.01136	0.00000	-0.01136	-0.03452
A_{100}	0.87863	0.96204	0.98649	0.96204	0.87863
A_{200}	0.39600	0.23664	0.00000	-0.23664	-0.39600
A_{300}	0.22181	-0.01892	-0.13217	-0.01892	0.22181
A_{400}	0.12908	-0.09962	0.00000	0.09962	-0.12908
A_{500}	0.07299	-0.09056	0.09675	-0.09056	0.07299

Eight Coupled Oscillators

	PS1	PS2	PS3	PS4
b	0.01374	0.00588	0.00147	0.00103
A_{100}	0.85064	0.92450	0.79320	0.89935
A_{200}	0.40462	0.34535	0.34781	-0.12924
A_{300}	0.24855	0.12421	-0.20178	0.25545
A_{400}	0.16608	0.11760	0.36583	-0.04854
A_{500}	0.11419	-0.04289	0.09074	-0.06043
A_{600}	0.07866	-0.06093	0.04067	-0.20273
A_{700}	0.05340	-0.05661	-0.24619	0.10059
A_{800}	0.03529	-0.04138	-0.06944	0.22800

regard to frequency splitting, and have the same number of nodes with the exception of PS3 and PS6 for $N = 8$. (For $N = 8$ in Table 3, the four periodic solutions not given may be obtained from the four which are given by changing the signs of the frequency components b, and by changing the signs of the even-numbered coefficients, as in the case for $N = 2$.)

The results exhibited in Table 3, obtained as described, have been checked by numerical integration of the equations of motion using the values given for the A_{k00} as the initial conditions. In every case, the solutions given in Table 3 have been verified to be stable periodic solutions. That is, for these sets of initial conditions the energies in each of the harmonic normal modes remain constant at their initial values, to within the error expected in the numerical integrations and other calculations.

Since in every case N distinct stable periodic solutions were found, no special effort was made to determine whether more than N such solutions exist for the nonlinear systems studied. However, the fact that only two stable periodic solutions exist for the two-oscillator nonlinear system (for which the exact solution is known), plus the rather close correspondence of the nonlinear periodic solutions to those of the linear cases, may give some indication that there may be no more than N such stable periodic solutions for nonlinear systems of the type studied.

In this section, the existence of periodic solutions for some resonant exactly-tuned cubic coupled oscillator systems has been demonstrated. The next section will present some results of perturbation

expansions about these periodic solutions, in which the expansions are truncated to first order in the expansion parameters.

First-Order Approximate General Solutions

According to the discussions presented in Chapter VI, it should be possible to obtain approximations to the general solution of a resonant exactly-tuned nonlinear system, for all sets of initial conditions, as a set of perturbation expansions about the periodic solutions displayed in the previous section. These expansions must be truncated to some fairly low order, and it is not known how good an approximation to a given general solution is obtained by such a method.

The most severe test is obtained by truncating the expansions to first order in the expansion parameters, and then comparing these first-order approximations with exact numerical solutions for extreme initial conditions, that is, for initial conditions which are as far away from the periodic solution initial conditions as possible.

A guide to which initial conditions are most extreme is given by the two-oscillator nonlinear system, for which the exact energy solution is known. For $N = 2$, these are the initial conditions in which one of the oscillators starts with all of the energy, and the other has none. In fact, for this two-oscillator resonant exactly-tuned system, the period of energy exchange is infinite for these extreme initial conditions; this was noted in Chapter VI.

For systems of more than two oscillators, the same sort of initial conditions in which one oscillator starts with all the energy are expected to be the most extreme relative to the periodic-solution initial conditions. An inspection of Table 3 supports this conjecture. However, for

systems of more than two oscillators, no cases in which the period of energy exchange in infinite are expected to occur, because of the greater spread of energy throughout the larger systems. The two-oscillator system is probably a special case in this respect.

In the following subsections, computer comparisons of first-order approximations and exact solutions will be given in graph form, for various selected sets of initial conditions including the extreme ones, for systems of two, three and five oscillators coupled as in Equation (4). Rather than plotting position or momentum versus time, graphs of energy versus time will be presented. This sort of presentation best displays the differences in long-term behavior of the various solutions; these long-term characteristics are the most difficult to approximate by the present method.

Two-Oscillator Nonlinear System

In Figure 13, the numerical and first-order approximate solutions are compared for the two-oscillator nonlinear system, for a rather extreme set of initial conditions. The long-term energy-sharing period is approximated to within 10%, and the amplitude to within 7%. This is a good approximation, considering that the set of initial conditions is close to that set for which the period of energy exchange is infinite in this two-oscillator system. Several other less extreme sets of initial conditions have been used, with the expected result that the first-order approximations were better than those for the case illustrated.

Three-Oscillator Nonlinear System

As in the case of the two-oscillator system, the most extreme set of initial conditions (in the sense of being far from any periodic solution)

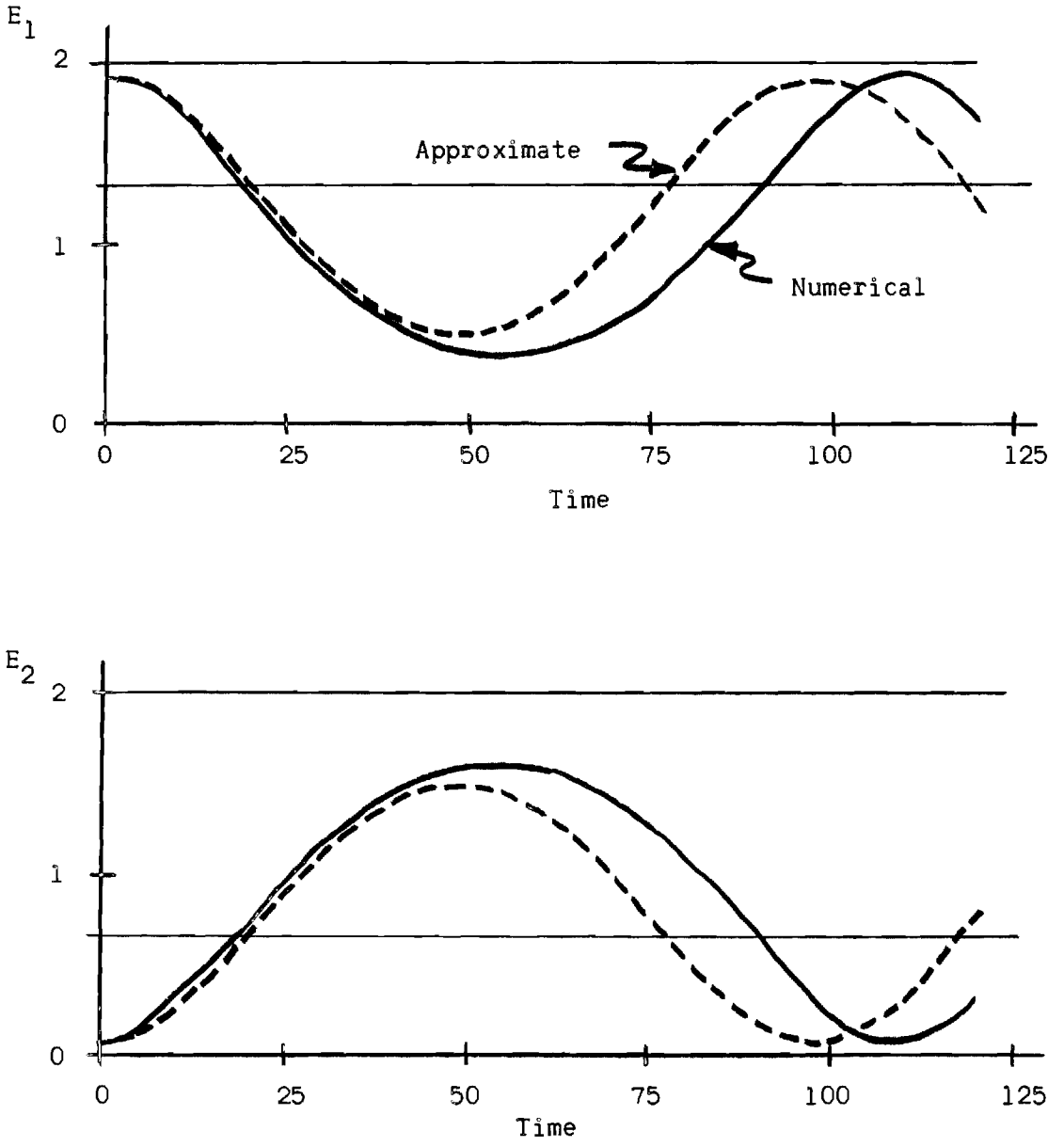

Figure 13. Comparison of Numerical and First-Order Approximate Solutions, for N = 2: Extreme Initial Conditions.

for a system of more than two oscillators should be of the class in which all the energy is initially given to only one oscillator.

Figure 14 presents results of a first-order approximation for a three-oscillator system started from "intermediate" initial conditions in which each oscillator initially has one-third of the total energy. In this example, the approximation to the main component of oscillation of E_1 is within about 3% of the correct value given by the numerical solution; approximations to the other frequency components are probably about as good. The amplitudes are approximated to within probably 10% of their correct values.

The shortcomings of graphical presentations of approximations to multi-component periodic data must be borne in mind while examining illustrations such as Figure 14. In the case of E_1, in which one periodic component clearly dominates, a graphical presentation adequately displays the differences in amplitude and frequency of two slightly different periodic functions. But these differences are not so clear in cases such as E_2 and E_3, in which there are several components with roughly the same amplitudes but different frequencies. In these latter cases, small errors in the determination of the approximate frequencies of the individual components of E_2 and E_3 result in overall approximations to E_2 and E_3 which appear progressively worse as time increases.

Thus in terms of a graphical presentation, the length of time over which an approximate solution and an "exact" numerical solution agree may be taken as a measure of how good the approximation is.

A better way to compare an approximate solution to a numerical

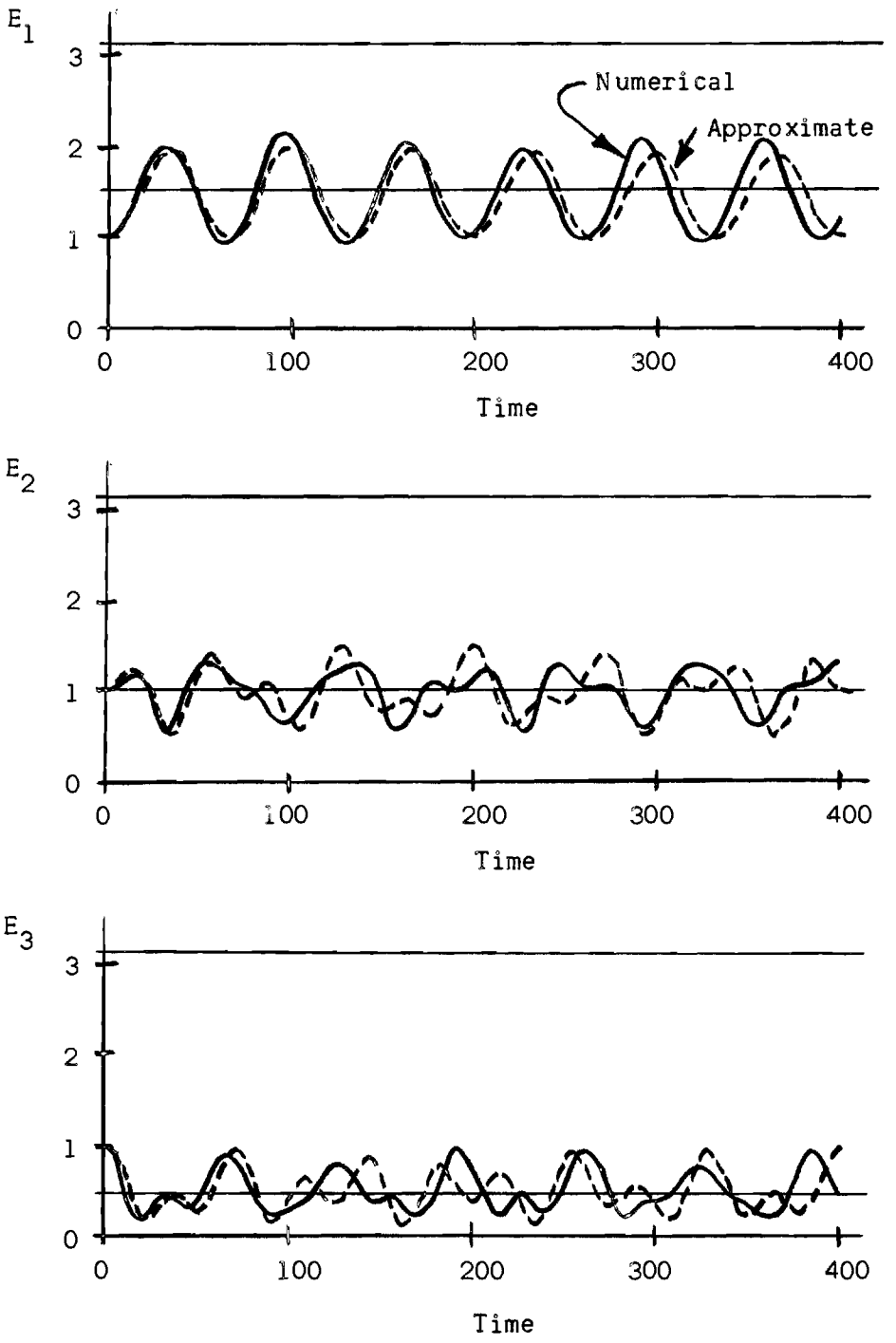

Figure 14. Comparison of Numerical and First-Order Approximate Solutions, for N = 3: Intermediate Initial Conditions.

solution would be to analyze each into its harmonic components and compare these frequency spectra. However, this general problem of harmonic analysis of functions having many components with frequencies which are not commensurable (unlike those of a sawtooth or similar function) is not a trivial one, and no harmonic analysis of solutions will be done in the present investigation.

Figure 15 compares numerical and first-order solutions of a three-oscillator system for an extreme set of initial conditions in which all of the energy is initially given to the first oscillator (Case 1). Figure 16 (Case 2) is similar, with nearly all the energy being given initially to the third oscillator. The energies are rather simple combinations of only a few trigonometric components in these cases. The approximation illustrated in Figure 15 is good to about 7% in frequencies and amplitudes. The approximation shown in Figure 16 is not as good; the errors are about 16% in frequencies and perhaps 50% in amplitudes.

In each of the figures in this section, solid lines such as those as $E_1 = 2.16$, $E_2 = 0.00$ and $E_3 = 0.84$ in Figures 15 and 16 have been drawn to indicate the energies corresponding to the periodic solutions about which the perturbation expansions have been made. From this it may be seen that the set of initial conditions used in Figure 16 are more "extreme" than those used in Figure 15; the results of the first-order approximation calculations bear this out.

Five-Oscillator Nonlinear System

The relative success of the first-order expansion approximations for two and three oscillators encourages some similar calculations for a slightly larger system. A five-oscillator system has been chosen as a

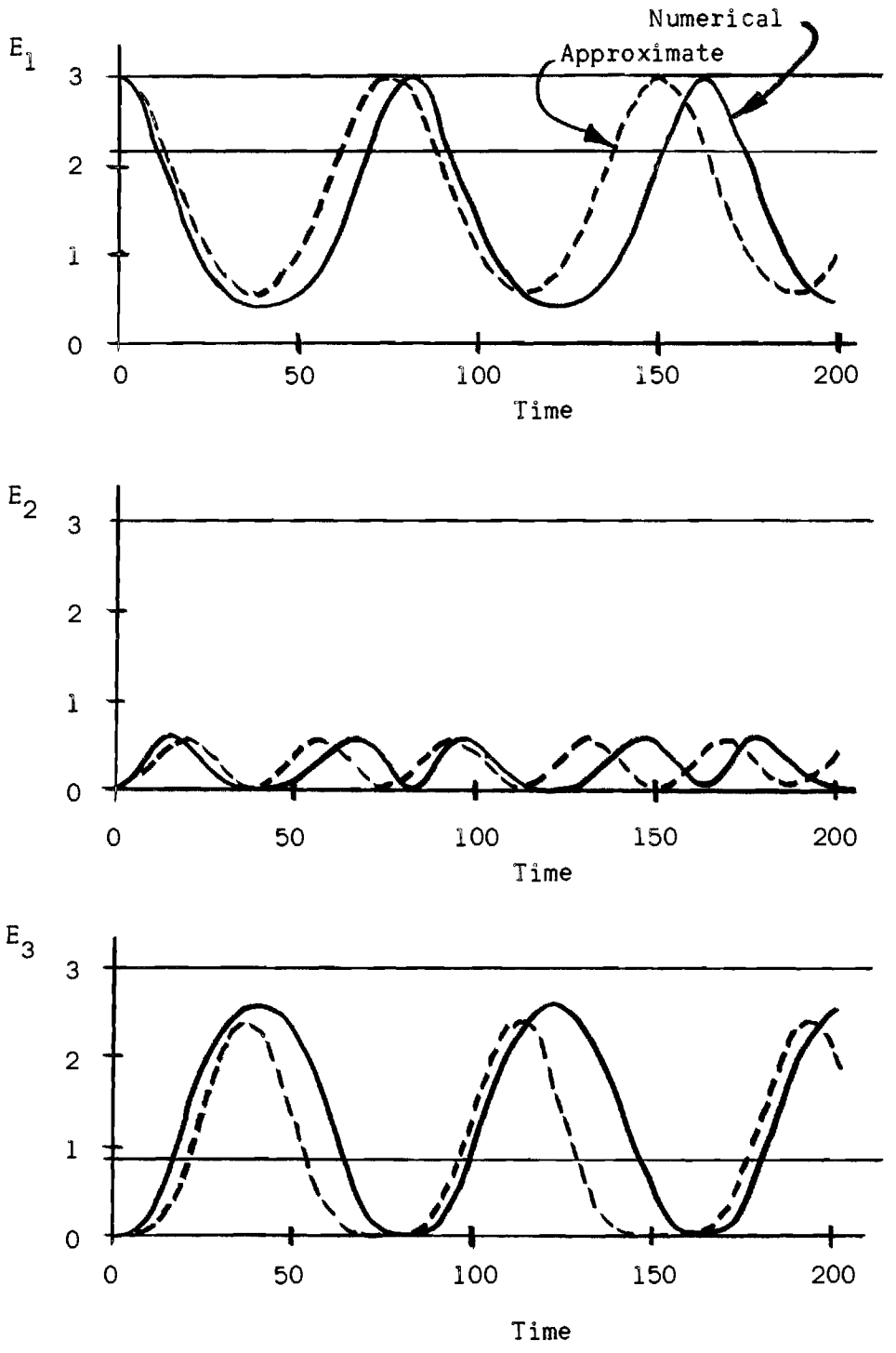

Figure 15. Comparison of Numerical and First-Order Approximate Solutions, for N = 3: Extreme Initial Conditions, Case 1.

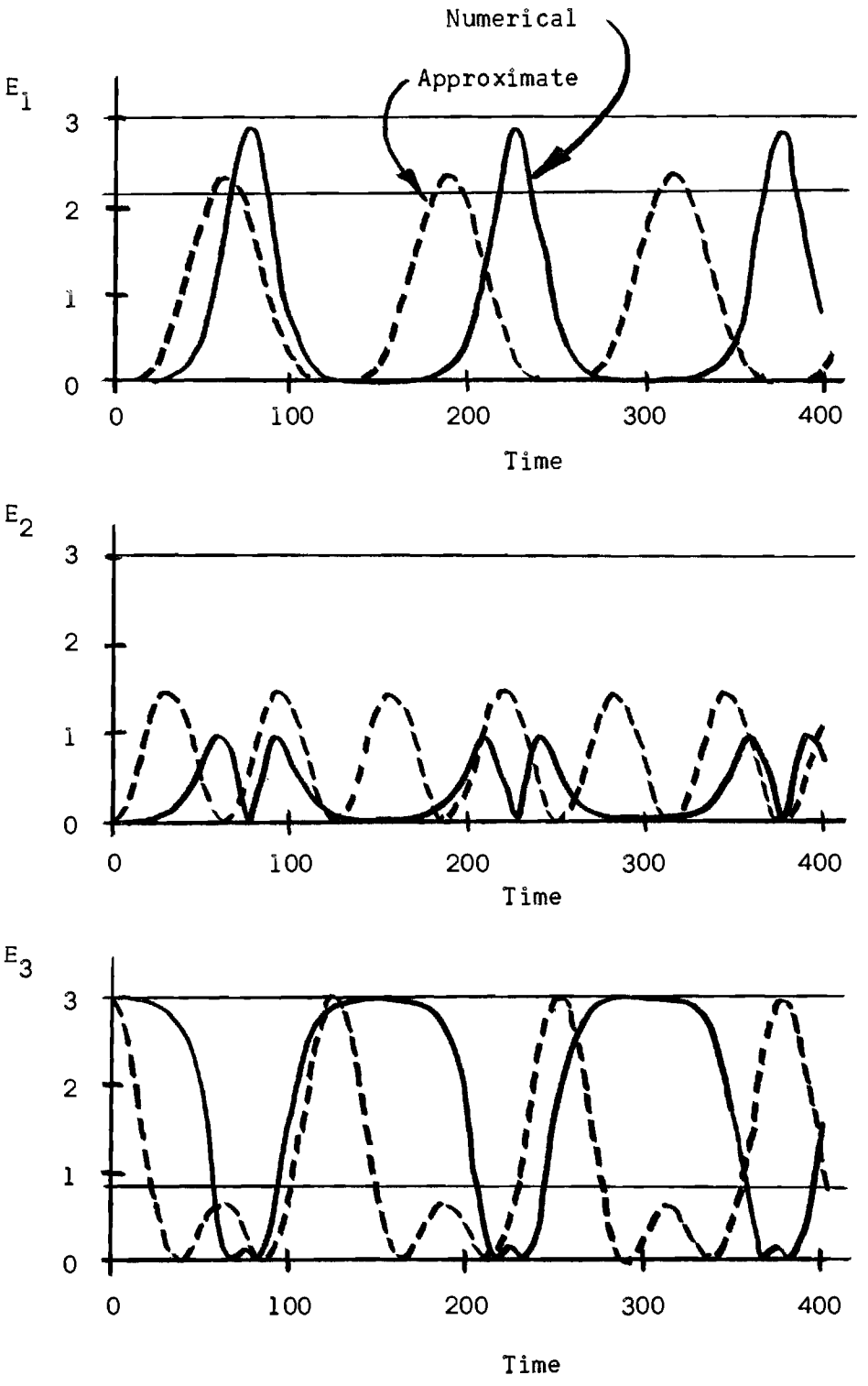

Figure 16. Comparison of Numerical and First-Order Approximate Solutions, for N = 3: Extreme Initial Conditions, Case 2.

system which is probably large enough not to be a special case (as are systems of two and three oscillators), and yet which is small enough to be computationally feasible.

In Figure 17, numerical and first-order approximate results are compared for a five-oscillator system started from intermediate initial conditions in which each oscillator has one-fifth of the energy. The approximation is good to probably 10% in the frequencies and amplitudes. The two graphs which are given are typical.

It may be seen from the energy plots in Figure 17 that the fundamental frequency content is present in the first-order approximation, but that higher-harmonic terms are also needed and do not appear. In addition, the energy-sharing time scale needs to be contracted, in order to obtain a better approximation.

These conclusions are confirmed by an examination of the other graphs presented in this section, and by the results for a five-oscillator system started from extreme initial conditions, as shown in Figure 18. For this last first-order case, all the energy has been initially given to the first oscillator. The three graphs which are given are typical. The fundamental frequency is well approximated, probably to within 2%, but there is a considerable need for higher-harmonic terms.

In this section, some first-order perturbation expansion approximations have been compared with the corresponding "exact" numerical solutions for various selected sets of initial conditions. As anticipated from the discussions of Chapter VI, these approximations are best when the initial conditions are close to those of a periodic solution; that is, when the expansion parameters are the smallest.

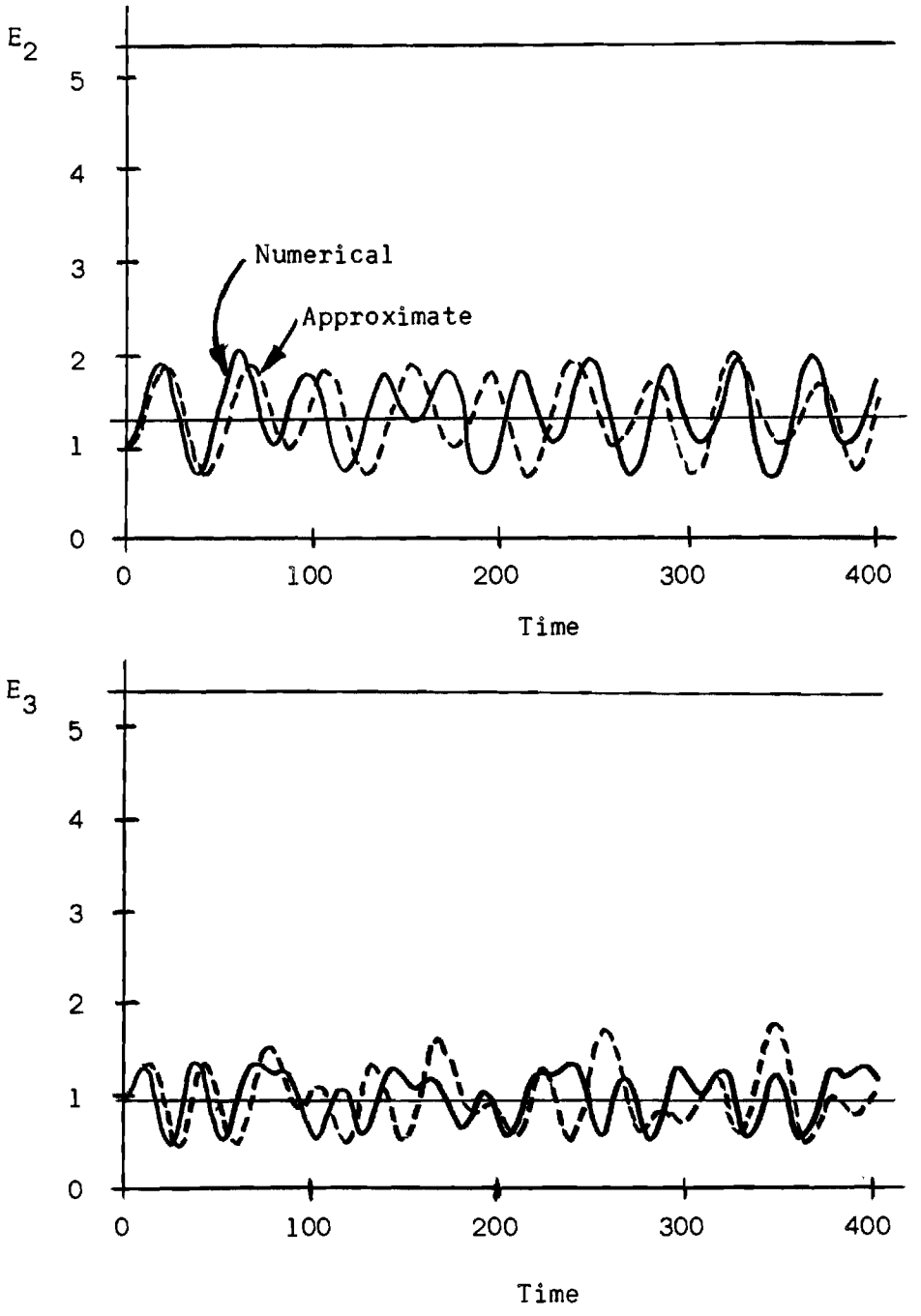

Figure 17. Comparison of Numerical and First-Order Approximate Solutions, for N = 5: Intermediate Initial Conditions.

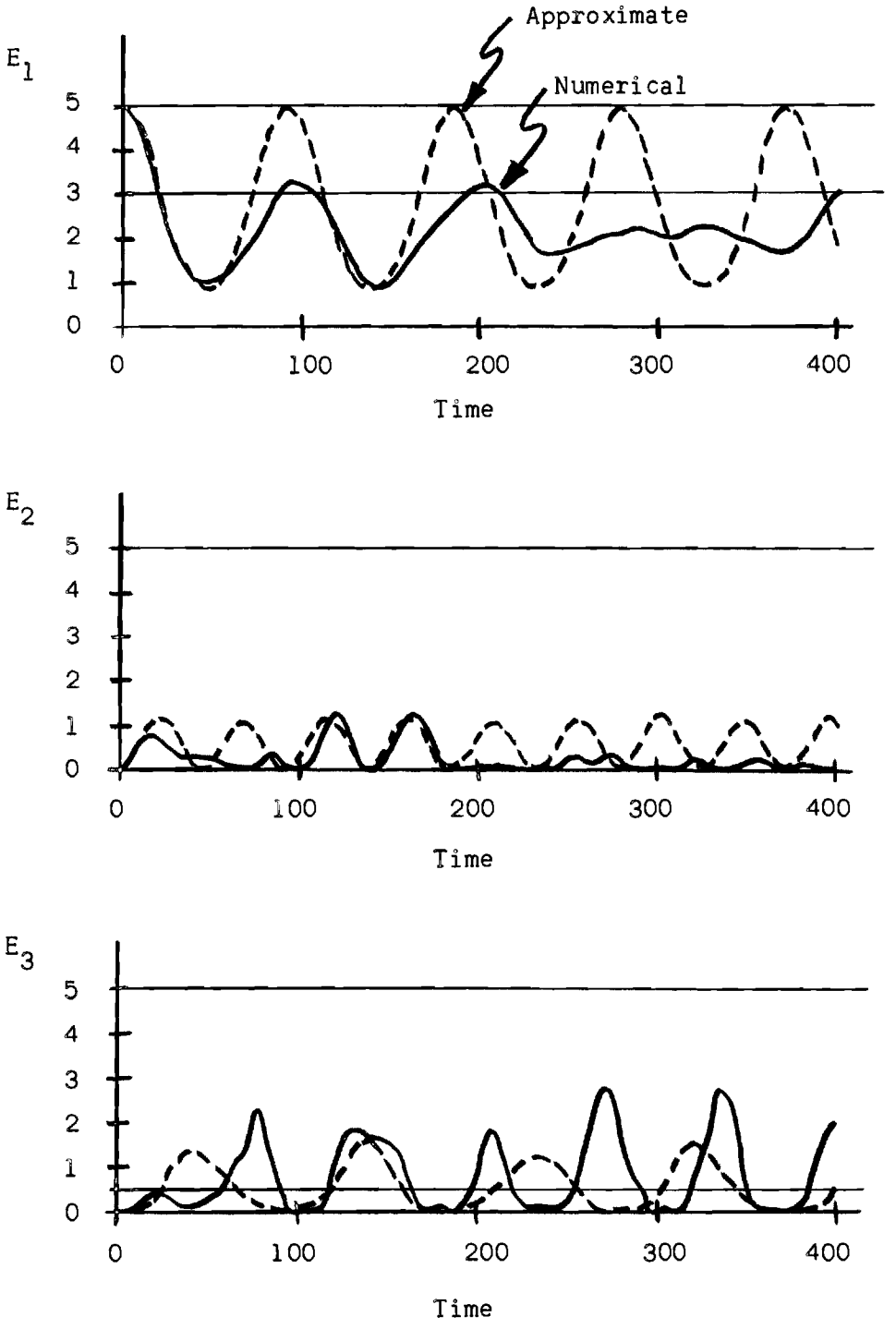

Figure 18. Comparison of Numerical and First-Order Approximate Solutions, for N = 5: Extreme Initial Conditions.

For the systems investigated, the first-order truncation of the expansions gives fairly good approximations to the general solution for most sets of initial conditions. These approximations are better for smaller systems than for large ones, according to the trend observed here. However, for the most "extreme" sets of initial conditions, especially, it would be useful to be able to improve the first-order results so as to reduce the error of the approximations below, say, 10%. The difficulties which have been encountered in trying to carry out the necessary second-order calculations as formulated in Chapter VI will be described in the following section, and some alterations which may remedy these difficulties will be discussed.

Second-Order Approximate General Solutions

The general form of a perturbation expansion about a periodic solution for a resonant nonlinear many-oscillator system, which was developed in Chapter VI, contains a large number of arbitrary constants. In order to make the calculation of second-order terms computationally feasible, a choice of certain of these arbitrary constants such that there are no second-order corrections to the frequencies Ω and Ω_j was made in Chapter VI. For lack of a better choice, the remaining arbitrary constants were chosen zero.

In this formulation, the only numerical computations involved are inversions of matrices, which can be handled in a relatively routine manner by presently available techniques on presently available computers. In any other formulation in which the second-order corrections to the frequencies are nonzero, it is necessary to solve eigenvalue-eigenvector problems involving unsymmetric matrices, a task which does not lend itself

so well to computation at present, as has been seen in the computation of the periodic solution frequencies and amplitudes.

Thus the particular formulation of the calculation of second-order terms set forth in Chapter VI was chosen for its computational simplicity. In that formulation it was expected that the improvement in the approximation to the frequencies would come about implicitly by means of the second-order refinement in the value of amplitude A_0, to which all of the frequencies are proportional.

Several difficulties have been encountered in carrying out the calculations as set forth in Chapter VI. First and most important is that for expansions about periodic solutions in which some of the energy-sharing frequencies (determined in the first-order calculation and unaltered in second order) satisfy the relation $\Omega_j + \Omega_i = 0$, the matrices from which the coefficients of second-order terms of the form $\cos(k\Omega t + \Omega_j t + \Omega_i t) = \cos(k\Omega t)$ are to be calculated are singular, so that these coefficients cannot be determined. There does not appear to be a simple reduction of the order of these matrices, which would remove the redundancy which causes the singularity.

A second difficulty, of the same nature and only slightly less severe, arises in conjunction with the calculation of the second-order terms of the form $\cos(k\Omega t + \Omega_j t - \Omega_j t) = \cos(k\Omega t)$ and those of the form $\cos(k\Omega t - \Omega_j t + \Omega_j t) = \cos(k\Omega t)$. For the two-oscillator system, the matrix used in the calculation of the coefficients of these terms is singular. In the case of the three- and five-oscillator systems, these matrices are not singular, but the coefficients calculated from them are quite large, which is numerically about the same thing.

It was noted in Chapter VI that second-order terms of the type mentioned above are actually zeroth-order terms. It should also have been noted there that the appearance of zeroth-order terms in higher orders is dangerous to convergence; this is the sort of difficulty which causes trouble in the application of the Wigner-Brillouin perturbation method to exactly-tuned nonlinear systems.

At this point it seems clear that the entire formulation of the calculation of second-order terms should be reconsidered; that is, a different choice of arbitrary constants should be made so that zeroth-order terms do not appear in higher orders. However, since this would probably involve a considerable reorganization of the computer programs for the second-order calculations, it will not be carried out in the present investigation.

Instead, a partial remedy of the difficulties will be made by doing some calculations in which the second-order $\cos(k\Omega t)$ terms are simply omitted from the second-order expansions. In several cases, the second-order approximations so obtained are better than the corresponding first-order approximations. In some other cases, new difficulties in obtaining convergence of the expansion become apparent.

In Figure 19, numerical and second-order approximate solutions are compared for a two-oscillator system, using the same initial conditions as in Figure 13. The result is a slightly worse approximation, if anything. The frequency is in error by about 12%, and the amplitude by about 10%, compared with 10% and 7% respectively for the first-order approximation. However, this is not too surprising, since the deletion of the $\cos(k\Omega t + \Omega_j t - \Omega_j t)$ and $\cos(k\Omega t - \Omega_j t + \Omega_j t)$ terms leaves only

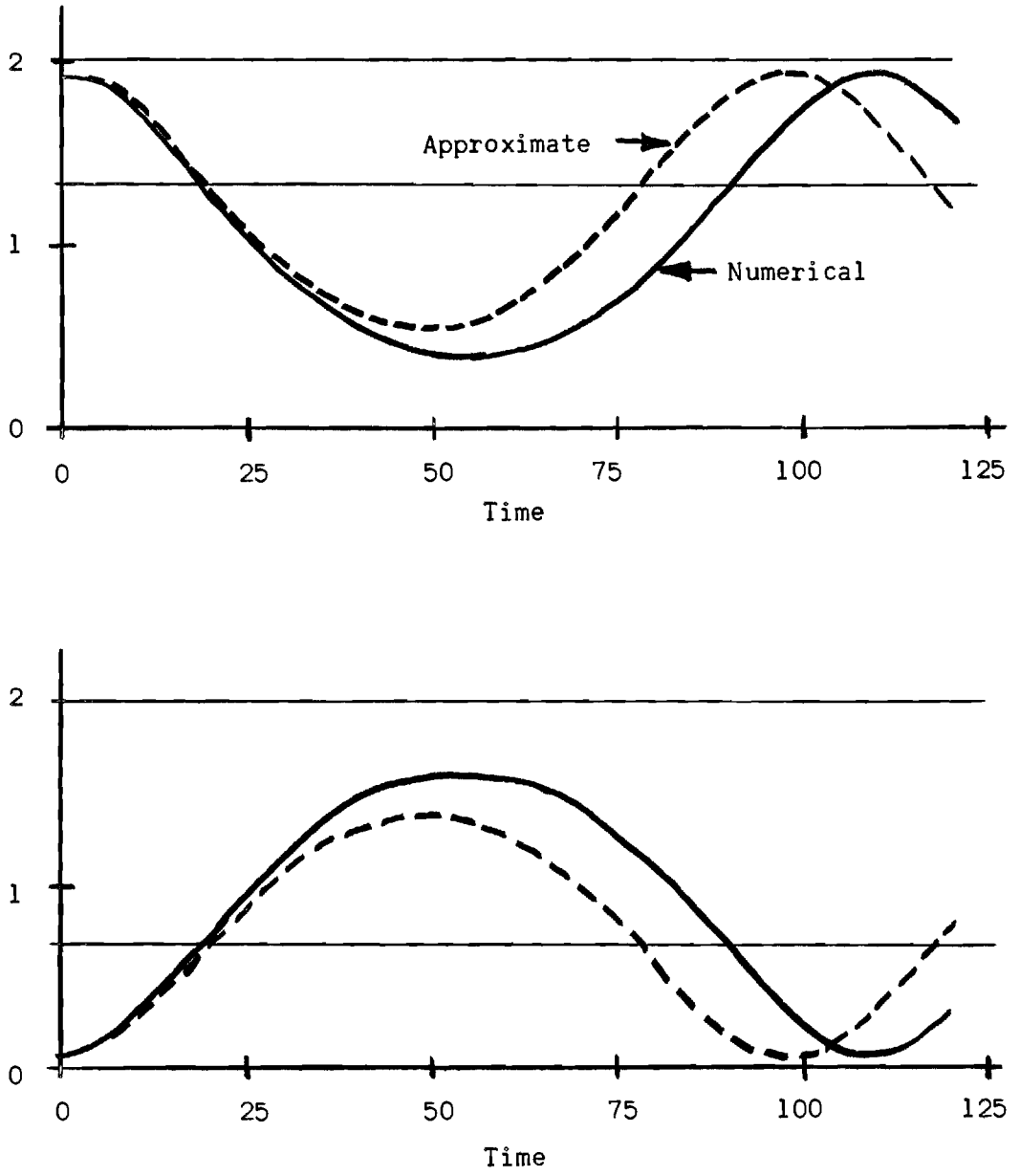

Figure 19. Comparison of Numerical and Second-Order Approximate Solutions, for N = 2: Extreme Initial Conditions, No Higher-Order cos(kΩt) Terms in Approximate Solution.

two second-order correction terms, so that the second-order expansion has been considerably mutilated.

Figure 20 presents the result of calculating a second-order expansion in which all terms are included, just as set forth in Chapter VI, for a three-oscillator system started with intermediate initial conditions. A comparison of this result with Figure 14 shows that the second-order approximation calculated in this manner is considerably worse than the corresponding first-order approximation.

In contrast, Figure 21 presents the results of a second-order approximation calculation for the same system in which the second-order $\cos(k\Omega t)$ terms have been deleted. A comparison of Figure 21 with Figure 14 shows that this second-order calculation represents an improvement over the first-order approximation. The approximation to the frequencies has not improved, but the approximation to the amplitudes of E_2 and E_3 is better.

Figures 22 and 23 present the results of calculating second-order approximations for the five-oscillator system started from intermediate initial conditions. In the calculation given by Figure 22, the second-order $\cos(k\Omega t)$ terms have been included; in that given by Figure 23, they have been deleted. The graphs are very similar; there is only a small difference in the two sets of numbers from which they were plotted. Both approximations are much worse than the corresponding first-order approximation given by Figure 17.

This brings to light a further difficulty in obtaining "good" second-order approximations; it is related to the first two. The second-order expansion for this case of five oscillators contains several rather

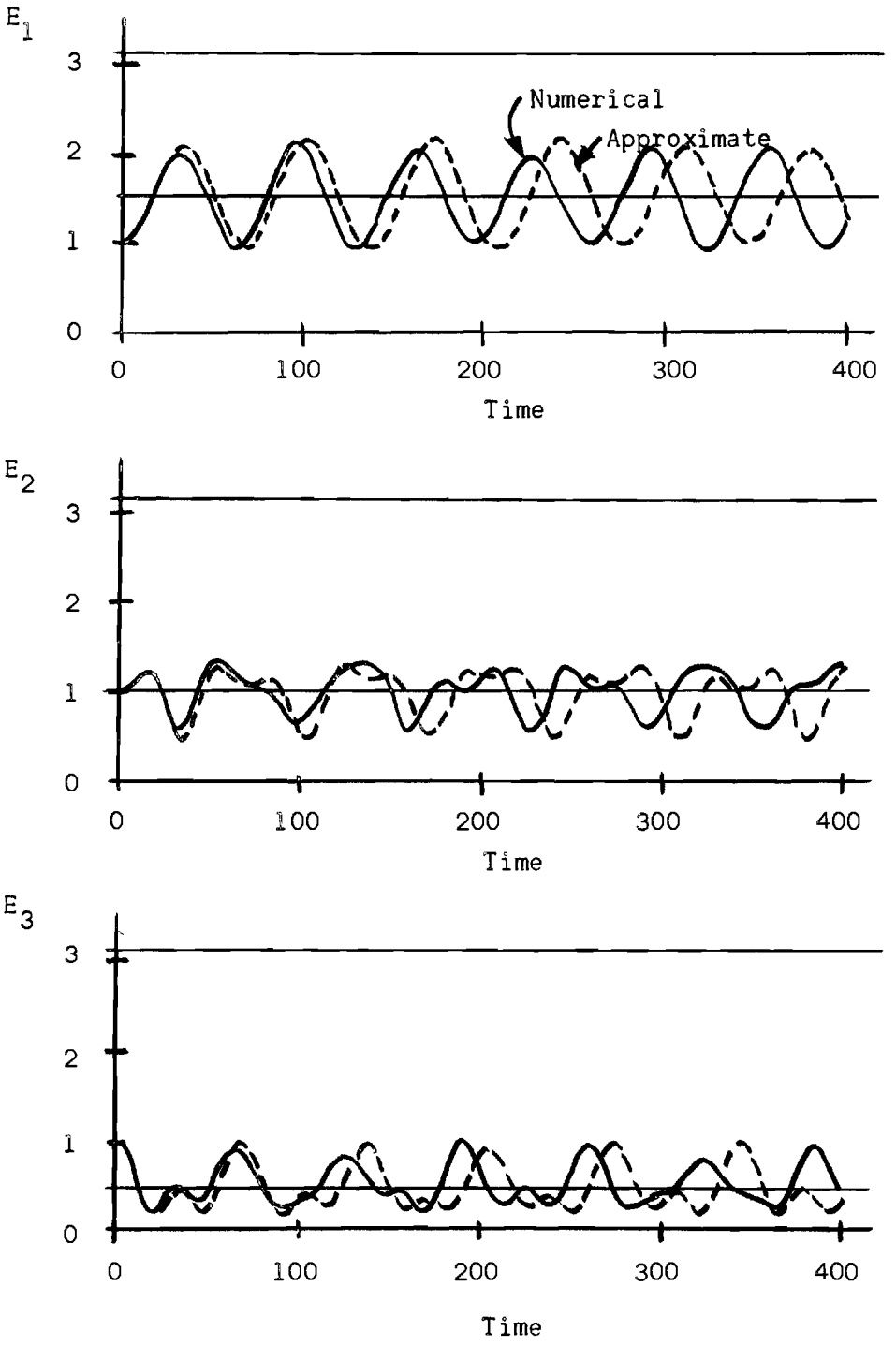

Figure 20. Comparison of Numerical and Second-Order Approximate Solutions for N = 3: Intermediate Initial Conditions, All Terms Included in Approximate Solution.

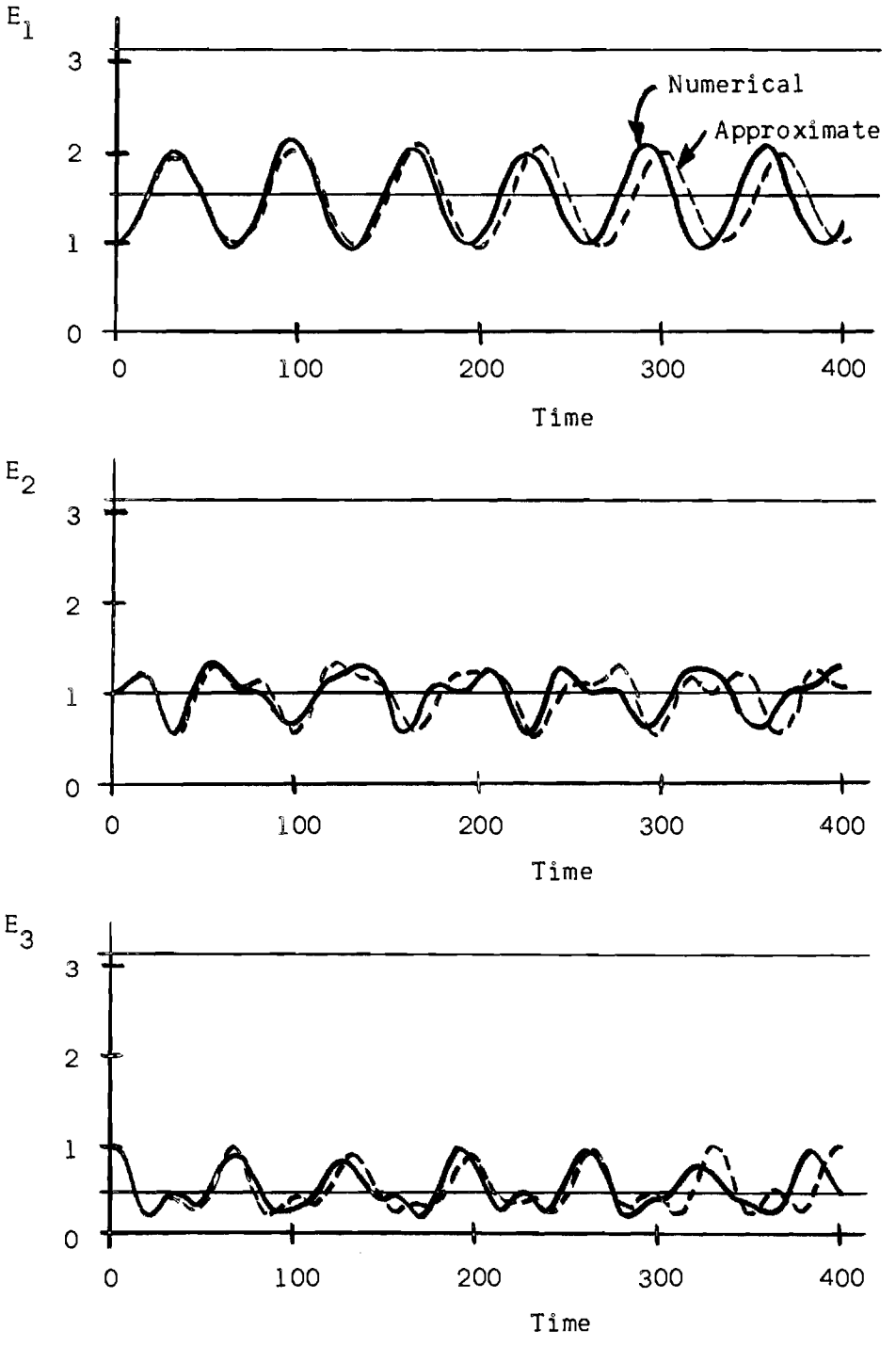

Figure 21. Comparison of Numerical and Second-Order Approximate Solutions, for N = 3: Intermediate Initial Conditions, No Higher-Order $\cos(k\Omega t)$ Terms in Approximate Solution.

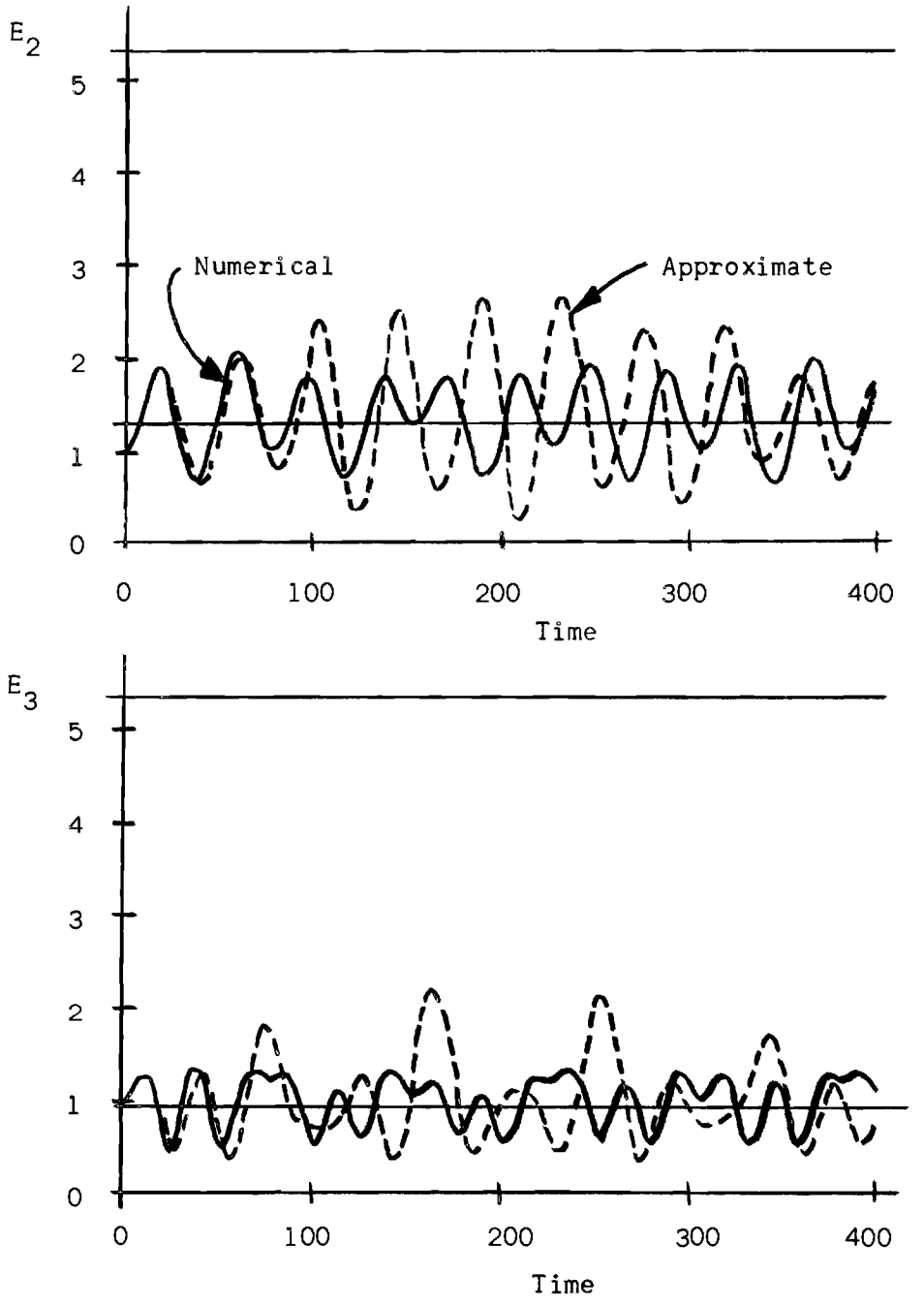

Figure 22. Comparison of Numerical and Second-Order Approximate Solutions, for N = 5: Intermediate Initial Conditions, All Terms Included in Approximate Solution.

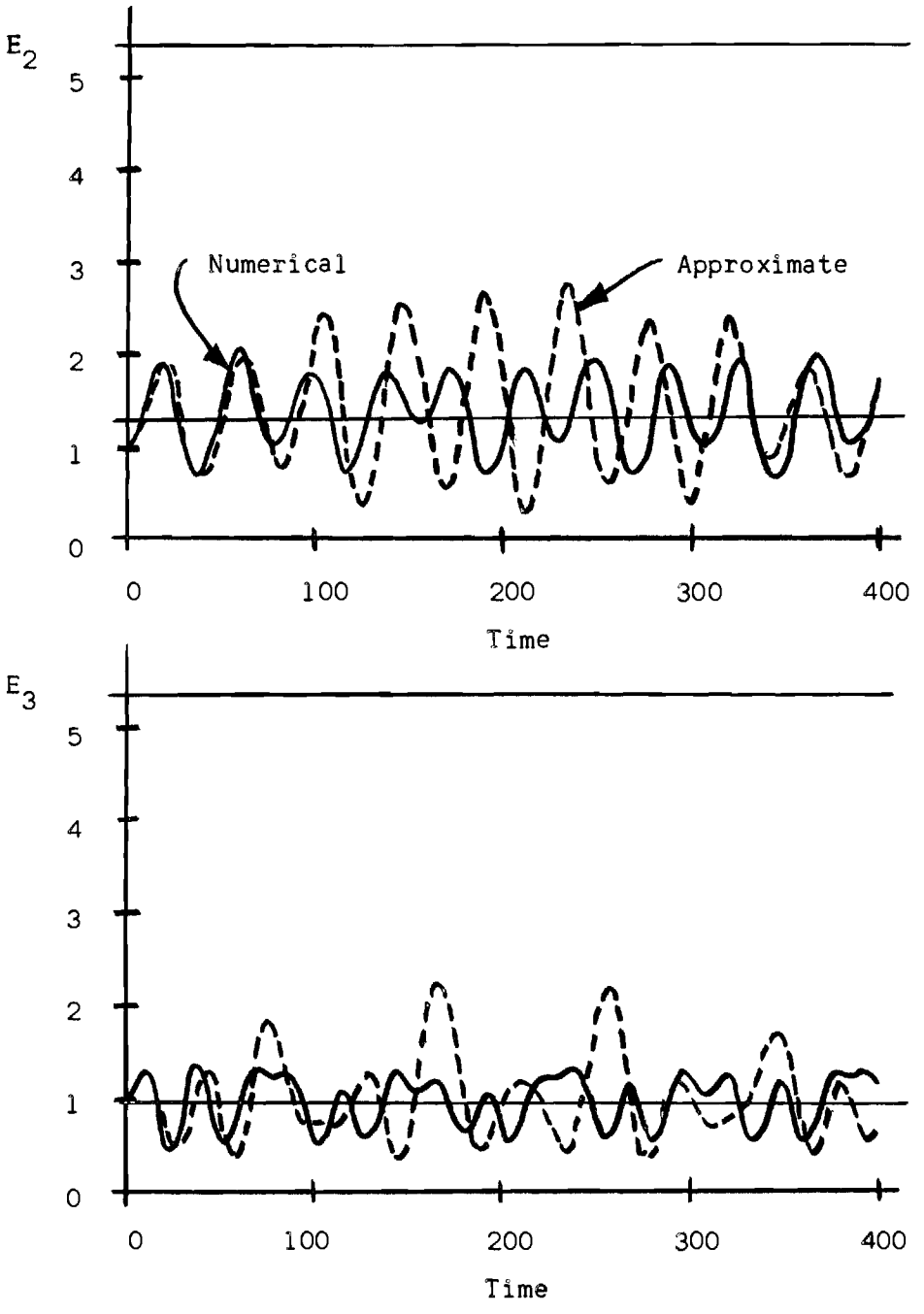

Figure 23. Comparison of Numerical and Second-Order Approximate Solutions, for N = 5: Intermediate Initial Conditions, No Higher-Order cos(kΩt) Terms in Approximate Solution.

large terms of the form $\cos(k\Omega t + \Omega_j t + \Omega_i t)$ where $\Omega_j + \Omega_i$ is quite small, so that these second-order terms are very nearly of the same frequencies as the zeroth-order terms. This is the same problem of the appearance of higher-order terms with large amplitudes and frequencies close to those of the zeroth-order solution, which was characterized in previous perturbation schemes as the problem of the appearance of terms with small divisors. As in other perturbation schemes and as in the case of the higher-order $\cos(k\Omega t)$ terms in the present scheme, this problem will have to be dealt with by the appropriate choice of arbitrary constants.

The last two figures in this section, Figures 24 and 25, present the results of calculating second-order approximate solutions in which no second-order $\cos(k\Omega t)$ terms are included, for systems of three and five oscillators started from initial conditions in which the first oscillator has all of the energy. These two figures correspond to Figures 15 and 18, respectively. Comparing Figures 24 and 15, there is little if any net improvement due to the inclusion of second-order terms for the three-oscillator system. The approximations to the frequencies are worse, but the shapes of the waveforms are better due to the inclusion of higher harmonics. A comparison of Figures 25 and 18, however, shows a definite improvement of the second-order approximate solution over the first-order solution.

In this section, it has been demonstrated that there exist certain shortcomings in the procedure for calculating second-order terms as set forth in Chapter VI. In particular, difficulties arise when second-order terms having frequencies which are close to those of the periodic solution about which the expansion is being made appear in the solution.

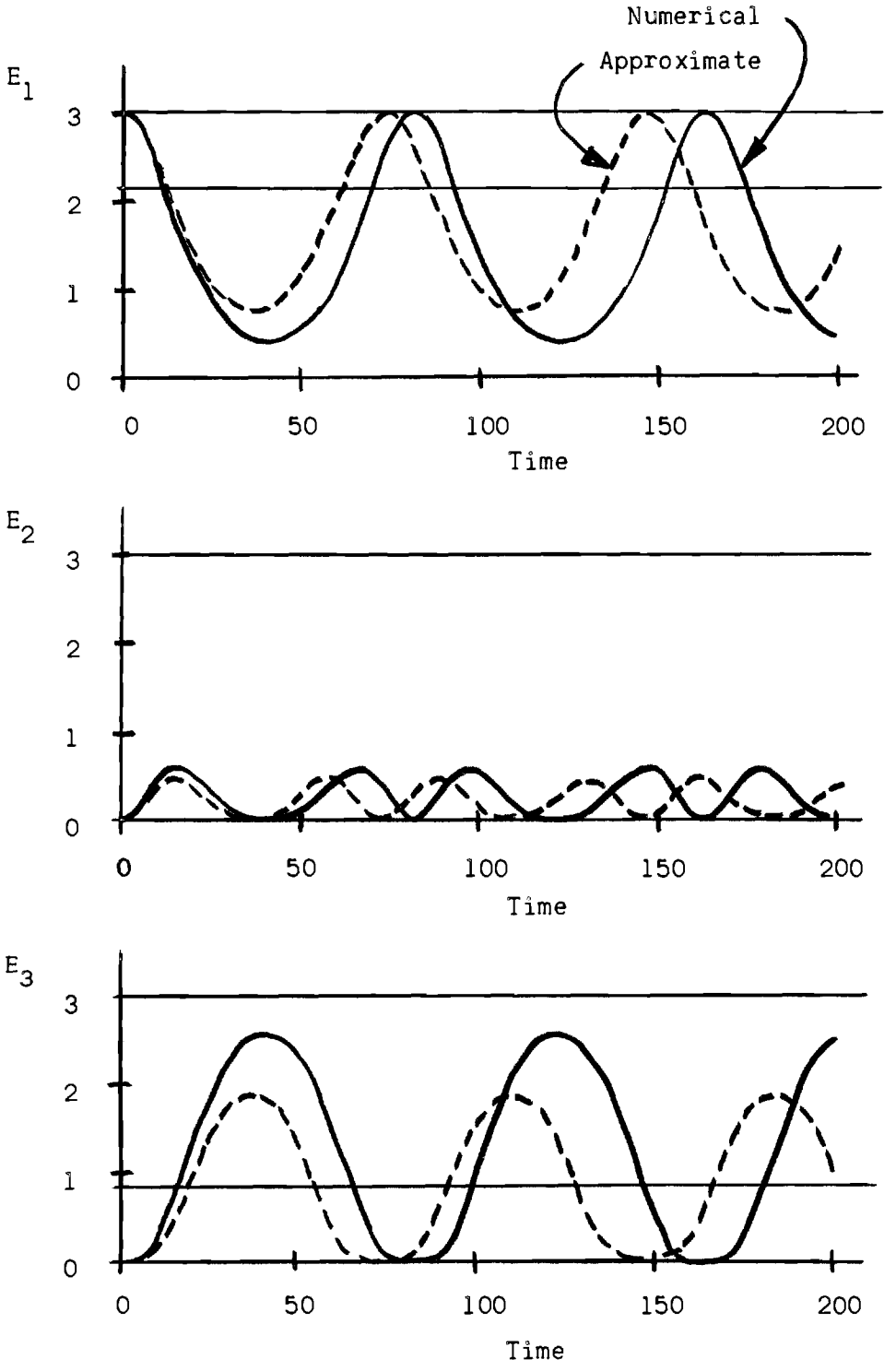

Figure 24. Comparison of Numerical and Second-Order Approximate Solutions, for N = 3: Extreme Initial Conditions, Case 1; No Higher-Order $\cos(k\Omega t)$ Terms in Approximate Solution.

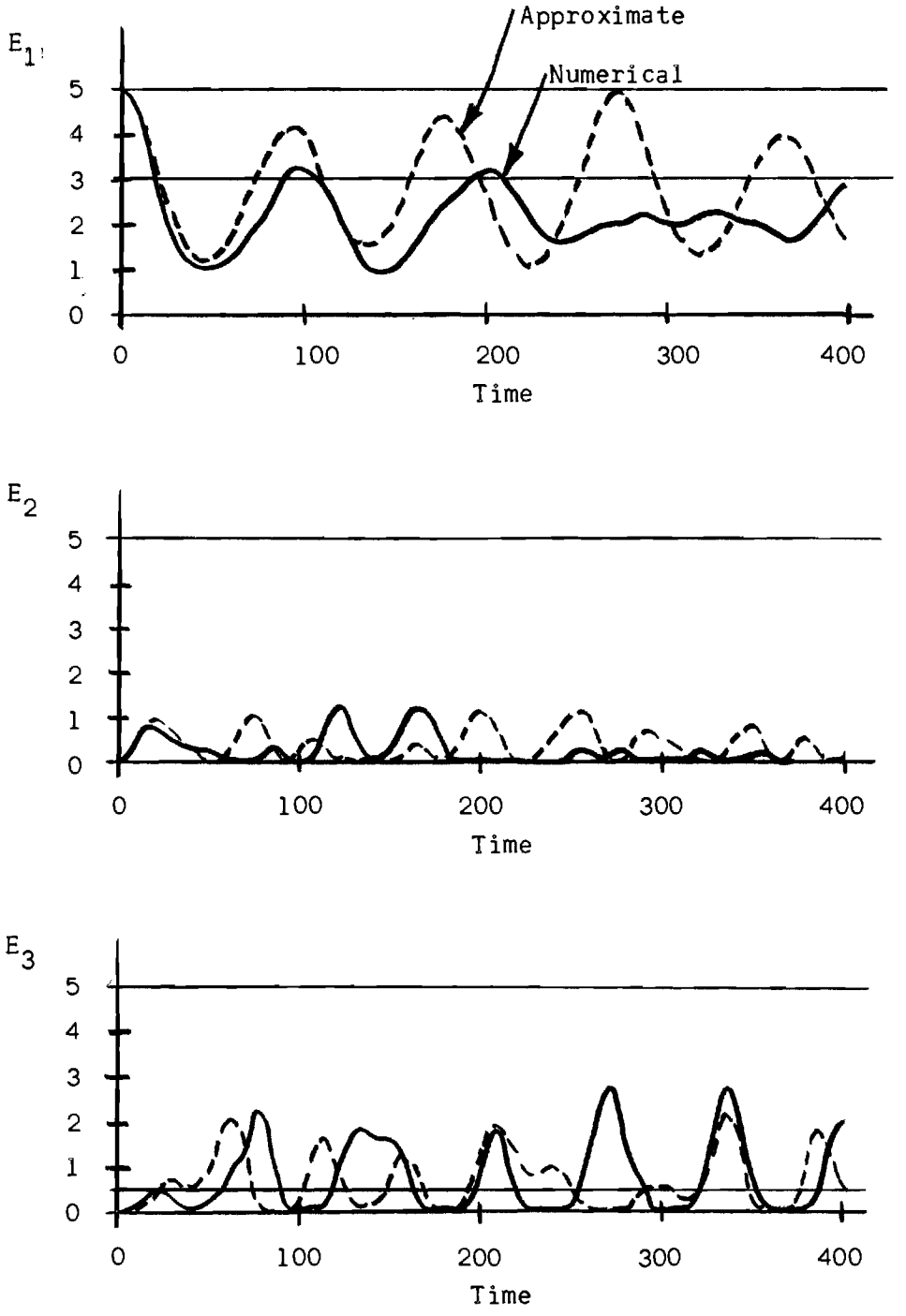

Figure 25. Comparison of Numerical and Second-Order Approximate Solutions, for N = 5: Extreme Initial Conditions, No Higher Order cos(kΩt) Terms in Approximate Solution.

In order to overcome these difficulties, some better method of choosing the many arbitrary constants which appear in the formal solution must be found. It may be possible to develop a variational method which will choose these constants so as to maximize the rate of convergence of the perturbation expansion in a given instance. In any event, it seems likely that a choice of arbitrary constants for which the higher-order frequency corrections are nonzero will be made, so that the problem of numerical computation of higher-order solutions will become more difficult to carry out than it was in the present investigation.

The net result of the computations which have been presented in the first three sections of this chapter is that the general method developed in Chapter VI is an effective one, at least in dealing with small nonlinear systems of the type studied here. The first-order approximations obtained here are considerably better than the approximations which can be obtained for these exactly-tuned nonlinear systems by any of the other existing perturbation methods. At present, there are certain difficulties in the calculation of higher-order terms in the approximation; but these difficulties seem familiar, and it is likely that they can be overcome.

Now that some evidence has been given for the effectiveness of a method for approximating the general solutions of resonant exactly-tuned nonlinear coupled oscillator systems, it is of interest to know something about the behavior of similar exactly-tuned systems in which nonresonant as well as resonant coupling terms are included. That is, it is of interest to investigate the solution of systems of the type given by Equations (1) and (3) of this chapter. This will be done in the final

section of this chapter.

Effect of Nonresonant Coupling Terms

If a good approximation to the general solution of a resonant exactly-tuned nonlinear coupled oscillator system can be obtained, an important basic step has been taken, but the result is still far from any application to a physical system. Physical systems contain nonresonant as well as resonant coupling terms, and physical systems are not generally exactly tuned.

It is anticipated that both the effects due to nonresonant coupling terms and detuning of the frequencies can be adequately treated by the use of standard perturbation techniques, in which the solution of the basic resonant exactly-tuned system is used as the zeroth-order term.

Neither of these effects will be treated analytically in this thesis. It is more appropriate to reserve these analyses for the particular physical problems which are of interest. Any general inclusion of nonresonant terms or general detuning for systems of more than two or three oscillators becomes quite complicated; in an application to a certain physical problem, a welcome limitation in the generality of the analysis of the effects of nonresonant terms and detuning is obtained.

However, since some selected computer studies of the actual effects of detuning have already been presented, in Chapter II, some computer studies of the actual effect of the inclusion of nonresonant coupling terms will also be included in this thesis, in the present section.

In the first subsection, the effect of nonresonant coupling terms on periodic solutions will be investigated. In the second subsection, the qualitative effect of nonresonant coupling terms on completely

energy-sharing solutions for systems of three and five oscillators will be discussed.

Nonresonant Terms and Periodic Solutions

It may be noted that although the so-called nonresonant terms identified in Chapter II as rapidly-varying coupling terms do not contribute to appreciable energy sharing in a system of coupled oscillators, they do cause a small amount of energy exchange among oscillators, depending upon the coupling strength for a given system.

Therefore it is not clear what the effect of the inclusion of nonresonant terms in the couplings, as in the systems given by Equations (1) and (3) of this chapter, will be on the general solution, especially relative to the solution of the corresponding resonant system, Equations (2) and (4). A preliminary study for a two-oscillator linear coupled system (Chapter V) indicates that the effect of including nonresonant terms may be no more serious than to shift the frequencies slightly, and to introduce some small rapidly-varying terms in the energy solutions, due to the difference between the momenta of the "resonant" and "given" systems.

As a preliminary study, the effect of nonresonant couplings on the existence of periodic solutions, given in this chapter in Table 3, will be investigated. At this point in the thesis, it is not known whether or not the inclusion of nonresonant coupling terms will invalidate these solutions.

Figure 26 presents typical results of calculations in which the appropriate periodic solution initial conditions from Table 3 were applied to systems of three and five oscillators having both resonant and nonresonant

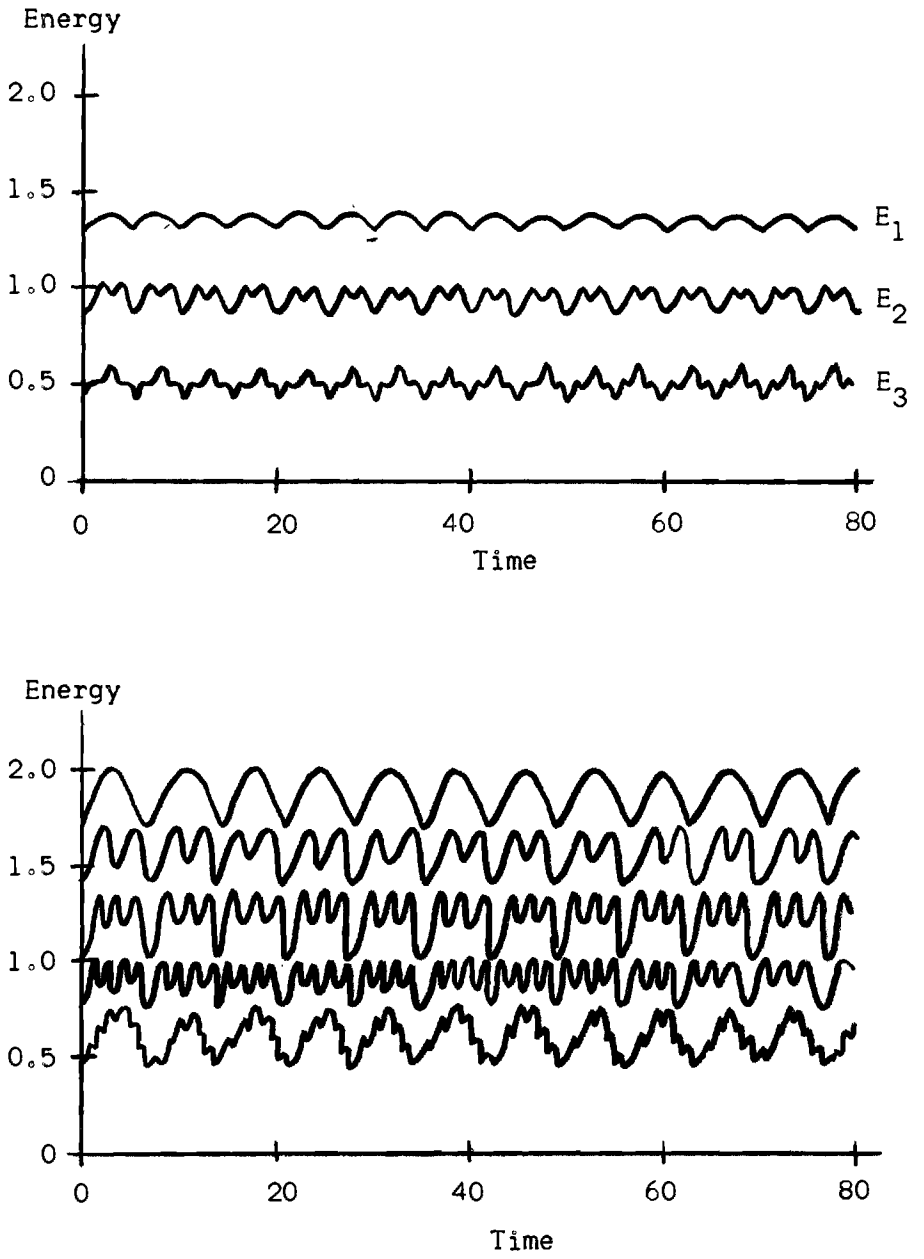

Figure 26. Typical Periodic Solutions for Nonlinear Systems in which Nonresonant Coupling Terms are Included: for $N = 3$ and $N = 5$.

couplings, as specified by Equations (1) and (3). The solutions shown were obtained by numerical integration of the equations of motion. The results given in Figure 26 are typical of the results obtained for each of N sets of initial conditions.

These results demonstrate that the periodic solutions continue to exist in the presence of nonresonant coupling terms. The principal effect is an inclusion of some small rapidly-varying energy terms. The long-term magnitudes of the energies, and hence the amplitudes of the positions and momenta, do not vary beyond the amount expected due to numerical inaccuracies.

This indicates that the periodic solutions are stable, in the sense that their existence is not affected by the inclusion of nonresonant coupling terms. This is important evidence to support the conjecture that the effects due to nonresonant coupling terms can be adequately treated by standard perturbation methods, using the solution of the resonant system as the zeroth-order term. If periodic solutions are stable in the sense used here, then expansions about periodic solutions are likely to be stable also.

The present subsection has dealt with periodic solutions. In the following section, the effect of nonresonant terms in the opposite extreme case, completely energy-sharing solutions, will be investigated.

Nonresonant Terms and Completely Energy-Sharing Solutions

Completely energy-sharing solutions may generally be obtained for coupled oscillator systems in which all the oscillators are coupled, by starting the system with one oscillator having all of the energy.

In such a situation, nonresonant terms may play a significant role

in the character of energy sharing, that is, in the order in which the various oscillators participate in the sharing of energy. All of the oscillators in a resonant system are not coupled to all of the others; many of the "given" couplings are nonresonant and have been eliminated.

In any cubic-coupled oscillator system, it is necessary for two oscillators to be energized in order to have a transfer of energy to a third oscillator. If a system is resonant, so that it is missing some of its couplings, and if all of the energy is started off with one oscillator, then it is possible that some of the oscillators never receive any energy, if they are not coupled to two others which have become energized. At the very least, the spread of energy throughout a resonant system is inhibited if the complete energy-sharing type of initial conditions are used.

The inclusion of nonresonant coupling terms causes a small amount of energy to be spread among a number of oscillators, in addition to the large transfers of energy among a few oscillators due to resonant couplings. These additional oscillators are thus brought into participation in energy sharing earlier than would have otherwise been the case. There is a more homogeneous spread of energy in a system in which nonresonant as well as resonant coupling terms are included, although the resonant couplings still control the large-scale transfers of energy.

Even after the initial energy has spread out in the system, nonresonant coupling terms continue to increase the participation of all of the oscillators in the sharing of energy. The same mechanism described above acts significantly whenever the energy of an individual oscillator becomes very small.

In addition to the effects just described, nonresonant terms may be expected to cause shifts in the energy-sharing frequencies, as was the case for the two-oscillator linear system discussed in Chapter V.

As evidence which may substantiate the heuristic discussion just presented, several computer studies will be presented. It will be seen that although the preceding discussion may not be complete, nonresonant coupling terms do have a definite effect on the character of energy sharing for the complete energy-sharing cases. This is because the terms identified in this study as "nonresonant" terms actually contain some higher-order resonant portions. This is most easily demonstrated in the canonical formalism of Birkhoff (1927); the action-angle formalism used in this thesis identifies only the lowest-order resonant terms, whereas the Birkhoff method identifies all orders of resonant coupling terms. Further consideration of this matter is beyond the scope of the present investigation.

The two-oscillator resonant exactly-tuned system has an infinite period of energy exchange for initial conditions in which one of the oscillators is started with all the energy. This may be verified from the exact energy solution given in Chapter VI. But the inclusion of nonresonant coupling terms in addition to the resonant coupling term causes the period to become finite. That is, the perturbation of the nonresonant couplings prevents the occurrence of the special situation in which the period is infinite. The exact solution for the $N = 2$ system of the type given in Equations (1) and (3), in which nonresonant terms appear, has been given in Chapter II, in the bottom plot of Figure 4.

The three-oscillator resonant exactly-tuned system with completely

energy-sharing initial conditions likewise has its energy-sharing characteristics noticeably changed by the inclusion of nonresonant coupling terms, although the change is not so drastic as for the two-oscillator system. Figure 27 gives a comparison of the exact solutions, obtained by numerical integration of the equations of motion first of the resonant type, given by Equation (4), and second of the resonant-plus-nonresonant type, given by Equations (2).

The differences noted in Figure 27 for three oscillators are better displayed for five oscillators, for which Figure 28 provides some typical samples. The calculations presented in Figure 28 last about ten times as long as those for Figure 27. The principal effect of inclusion of the nonresonant coupling terms seems to be a shift from a rather oscillatory energy solution to one in which energy is shared in a more random manner. There ia also more uniform participation of all of the oscillators in the system in which the nonresonant terms are included.

In this chapter, a number of computer studies have been made to test the effectiveness of the method developed in Chapter VI for obtaining approximations to the general solutions of resonant exactly-tuned nonlinear coupled oscillator systems. This method has been found to be quite effective in principle, although the expected practical limitations on the order to which the approximation can be conveniently taken are indeed present.

Also in this chapter, the effect of the inclusion of nonresonant coupling terms has been studied. It appears that the existence of periodic solutions is not invalidated by the inclusion of nonresonant terms. It seems likely that standard perturbation methods may be used to deal with

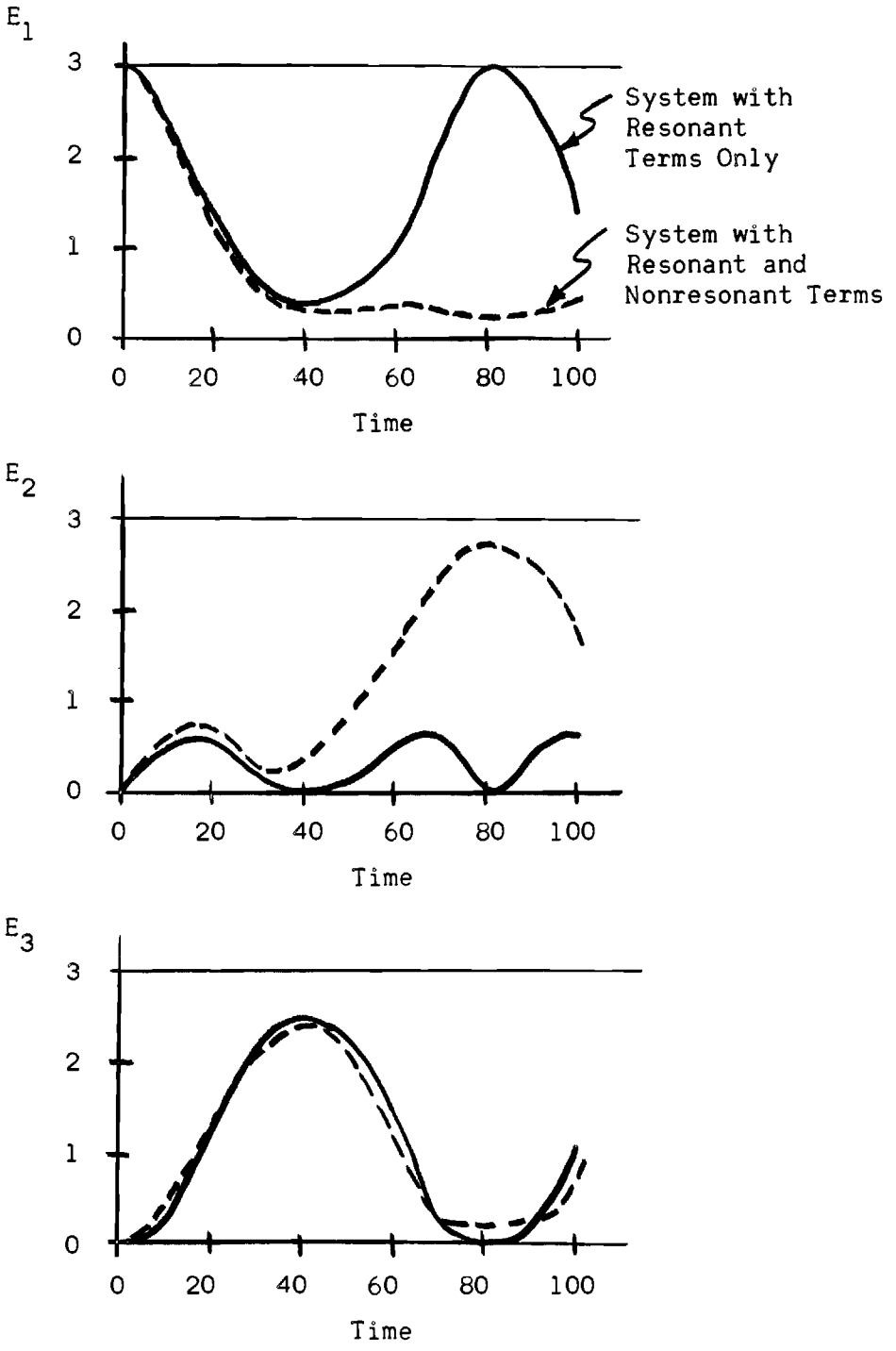

Figure 27. Effect of Inclusion of Nonresonant Coupling Terms on Completely Energy-Sharing Solution, for N = 3.

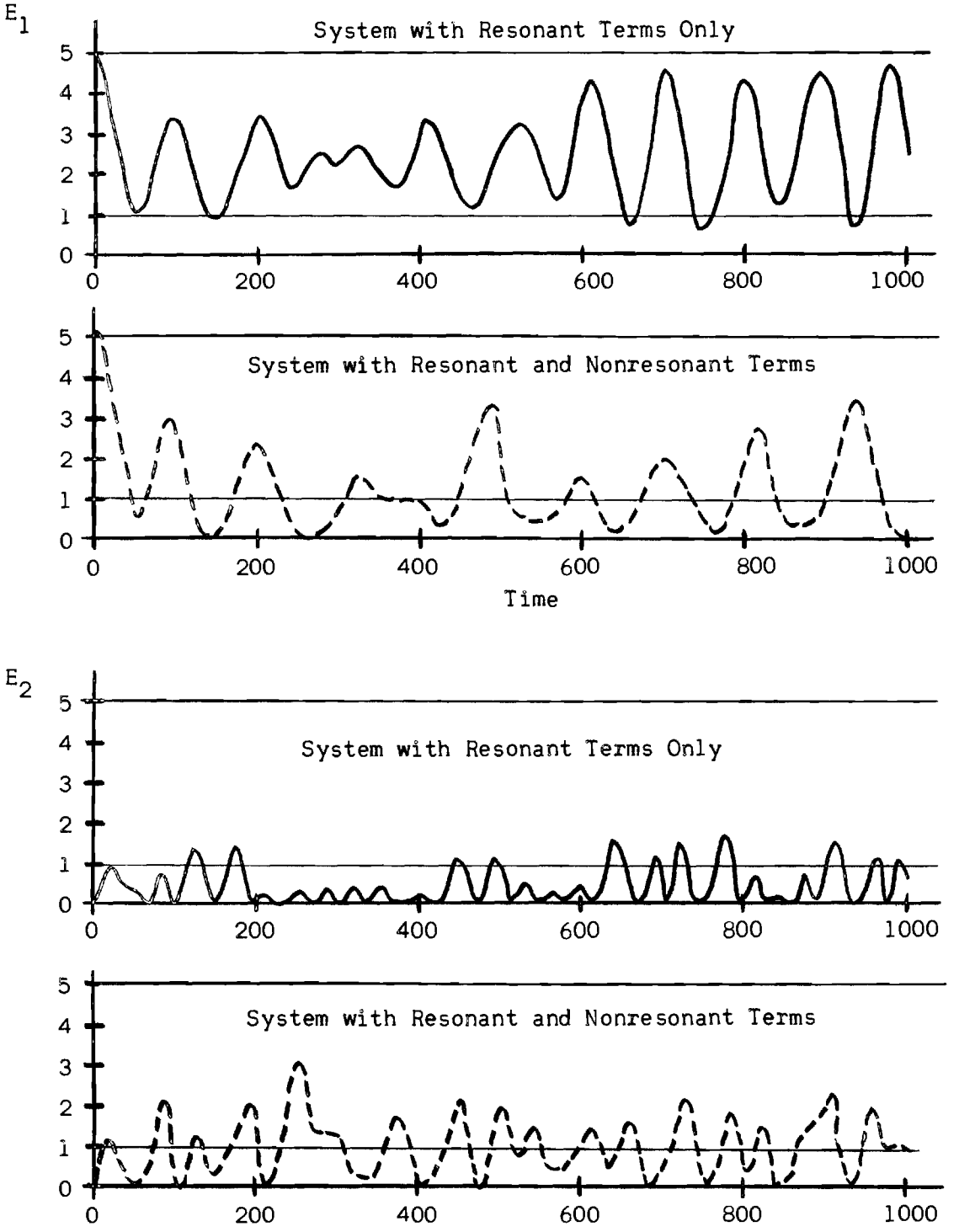

Figure 28. Typical Effect of Inclusion of Nonresonant Coupling Terms on Completely Energy-Sharing Solution, for $N = 5$.

nonresonant terms, as discussed earlier. Nonresonant terms do seem to assume a role of some importance in altering the manner in which energy spreads and is shared in a nonlinear coupled oscillator system.

The development of a method of analysis for nonlinear coupled oscillator systems terminates with the end of this chapter. In the following chapter, some studies of constants of the motion for nonlinear systems and the approach to equilibrium for coupled oscillator systems will be briefly described. A final chapter which summarizes the conclusions drawn in this study will then be given.

CHAPTER VIII

COMPUTER STUDIES OF CONSTANTS OF THE MOTION AND THE APPROACH TO EQUILIBRIUM

A brief account of two computer studies of topics which are related to the preceding investigations of nonlinear coupled oscillator systems will be given in this chapter. In the first section, studies of the constants of the motion will be described. In the second section, some studies of the approach to equilibrium of nonlinear and linear coupled oscillator systems will be discussed.

Constants of the Motion for Nonlinear Systems

It has been demonstrated in Chapter VII that the analytical method of obtaining approximations to the general solutions of resonant exactly-tuned nonlinear coupled oscillator systems, developed in Chapter VI, is fairly effective, at least for small systems of the type considered.

As discussed in Chapter V, there are N constants of the motion for a linear coupled N-oscillator system, in the form of the N true normal mode energies. These constants of the motion are analytic in the position and momentum variables. Since an approximation to the general solution of a nonlinear system is available from Chapter VI, and since certain analogies between linear and nonlinear systems have been noted previously in this thesis, there is encouragement for a belief in the existence of N analytic constants of the motion for a nonlinear coupled N-oscillator system. In particular, the analogy between N periodic

solutions for the nonlinear system and N true normal mode solutions for the linear system encourages this belief.

However, the identification of N analytic constants of the motion for nonlinear systems has not been accomplished in the present work. The only additional constant of the motion other than the hamiltonian which has been identified is the total coupling energy in a resonant exactly-tuned N-oscillator system.

In Chapter VI, in conjunction with the integration of the action-angle equations of motion of the two-oscillator resonant exactly-tuned nonlinear system, the total coupling energy was identified as a second constant of the motion. This was shown directly from the action equations of motion, Equations (5a) and (5b) of Chapter VI, in which the relation

$$\dot{J}_1 + 2\dot{J}_2 = 0 \tag{1}$$

directly provides a second constant of the motion,

$$J_1 + 2J_2 = K, \tag{2}$$

which, when multiplied by frequency, ω, is equal to the total energy of the individual oscillators.

The difference between the quantity ωK from Equation (2) and the hamiltonian H is the total coupling energy,

$$K_2 = H - (\omega J_1 + 2\omega J_2), \tag{3}$$

which may be identified, as an alternative to K, as the second constant of the motion for the two-oscillator system.

For resonant exactly-tuned nonlinear coupled systems of more than two oscillators, it is easily shown in the same manner that the sum

$$J_1 + 2J_2 + 3J_3 + \cdots + NJ_N \tag{4}$$

is a constant, so that the total coupling energy can be identified as the second constant of the motion for such systems of any number of oscillators.

As an alternative to the simple method just described for identifying the total coupling energy as a second constant of the motion, it can also easily be shown for these systems of N oscillators that the total coupling energy has zero Poisson bracket with the hamiltonian, which is an equivalent requirement for a constant of the motion. A quick check with the action-angle equations of motion shows that the total coupling energy has zero Poisson bracket with the sum given in Equation (4); this establishes the result.

Probably the only computer study of interest in connection with this second constant of the motion involves the question of whether the inclusion of nonresonant as well as resonant coupling terms invalidates the total resonant coupling energy as a second constant.

To answer this question, the total coupling energy and the resonant portion of the total coupling energy have been computed for the five-oscillator nonlinear system with completely energy-sharing initial conditions, with nonresonant as well as resonant couplings (Equations (1) and (3) of Chapter VII). Typical plots of energy versus time have already been given in Figure 28 in Chapter VII, for this case.

A typical comparison of the time behavior of the resonant coupling

energy with that of the total coupling energy (resonant plus nonresonant terms) is presented in Figure 29. Examination of this figure shows that the resonant portion of the coupling energy remains essentially constant, in contrast to the rapid and wide variations of the total coupling energy.

The expression for the resonant portion of the coupling energy used in Figure 29 does not include a correction for the effect of the inclusion of nonresonant terms, which must be made if a true constant of the motion is to be obtained. However, it is expected that this correction can be made by standard perturbation methods (Whittaker 1944, Chapter 16). The result indicated in Figure 29 should be sufficient to indicate that the resonant portion of the coupling energy remains a valid second constant of the motion even when nonresonant coupling terms are included in the system.

The Approach to Equilibrium

Since some calculations of the behavior of a system of five nonlinear coupled oscillators in which all of the energy is initially given to one oscillator have already been made, some discussion of the nature of the approach to equilibrium may be given in this section.

Specifically, five units of energy were initially given to the first oscillator of an exactly-tuned five-oscillator system in which both resonant and nonresonant nonlinear coupling forces were present. This system is the one given by Equations (1) and (3) of Chapter VII. The spread of this energy through the system was studied for a fairly long time (over 60 uncoupled periods of the first oscillator, or over

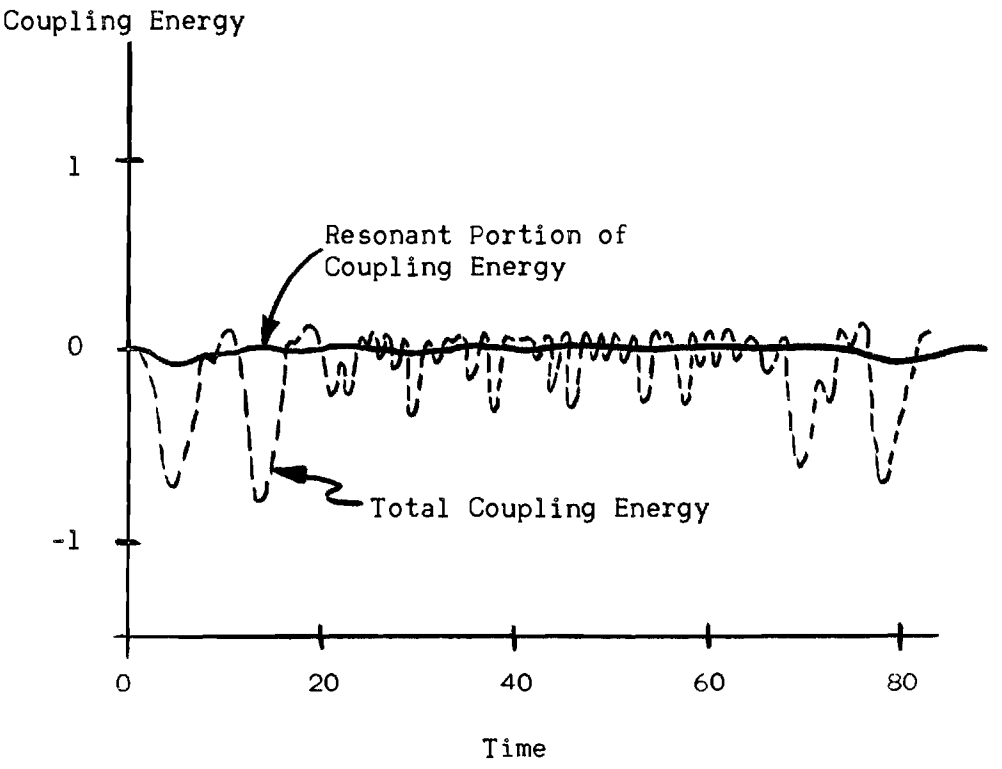

Figure 29. Coupling Energies for N = 5 Nonlinear System.

300 uncoupled periods of the fifth).

For comparison, five units of energy were also initially given to the first oscillator of an exactly-tuned five-oscillator system in which both resonant and nonresonant linear coupling forces were present. This system is the one given by Equations (17) and (18) of Chapter II. The spread of energy through this system was studied for over 150 uncoupled periods of the uncoupled oscillators, all of which have the same frequencies.

Typical comparisons of the results for these nonlinear and linear systems are presented in Figures 30 and 31. These curves present at least two interesting features. First, both systems appear to reach a steady state after the elapse of only a small number of uncoupled periods. Second, the energy curves for the nonlinear and linear systems show a rather striking similarity. Clearly there must be differences in the time scales of these curves for various individual oscillators, because of differences in the uncoupled frequencies. But considering the fact that totally different couplings are involved, the similarities seem more impressive than the differences.

In view of this similarity, it is interesting to compare the single-oscillator energy distribution densities $f(E)$ for the nonlinear and linear coupled systems. Here $f(E)$ is defined to be the fractional amount of time which a single oscillator spends in the energy interval $(E, E+dE)$.

First, the $f(E)$ data points were computed using $\Delta E = 0.3$ for each individual oscillator. Then, since all of the individual $f(E)$ plots thus obtained were more or less the same, for presentation the

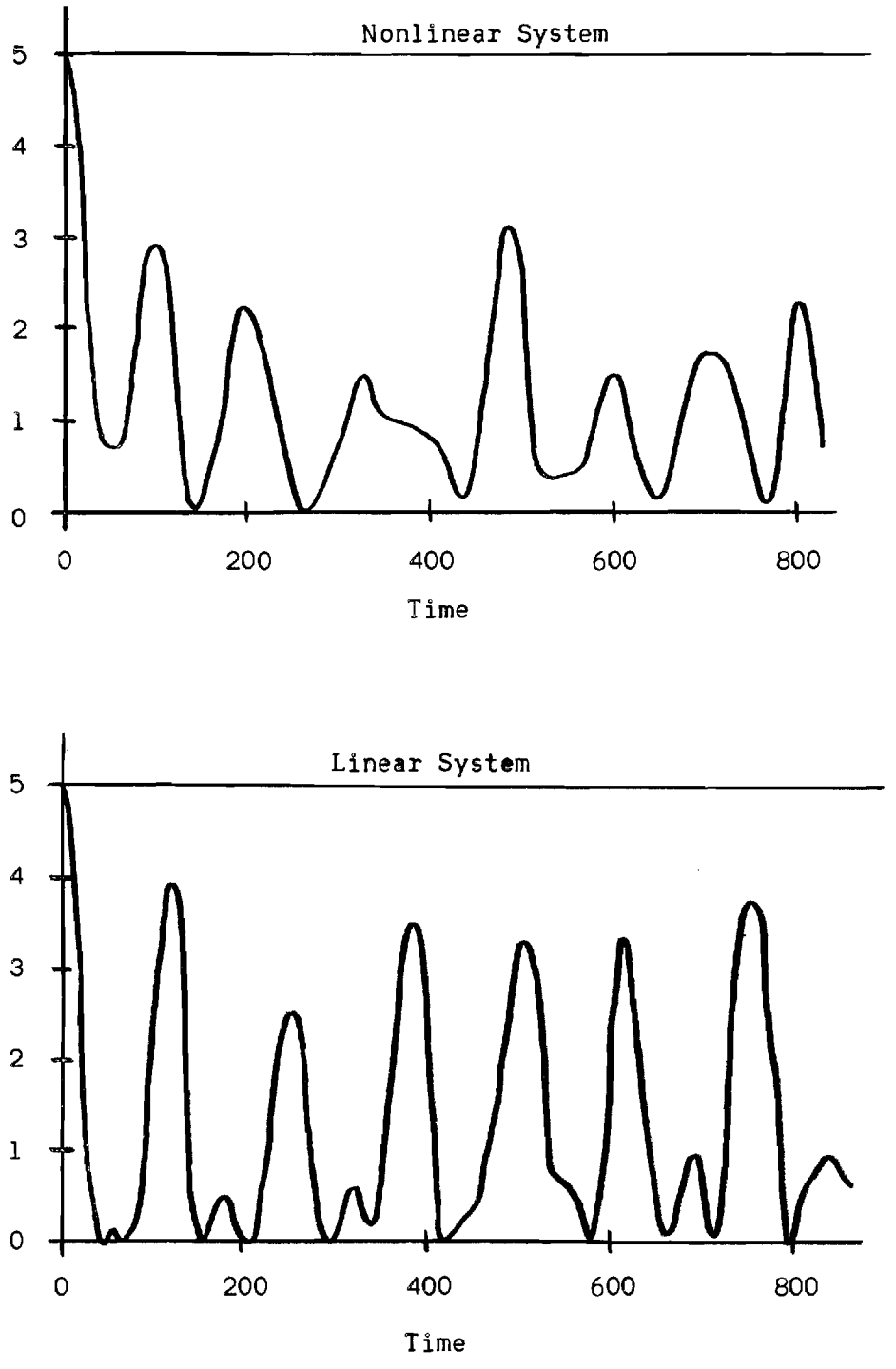

Figure 30. Comparison of Plots of E_1 versus Time for Nonlinear and Linear $N = 5$ Systems.

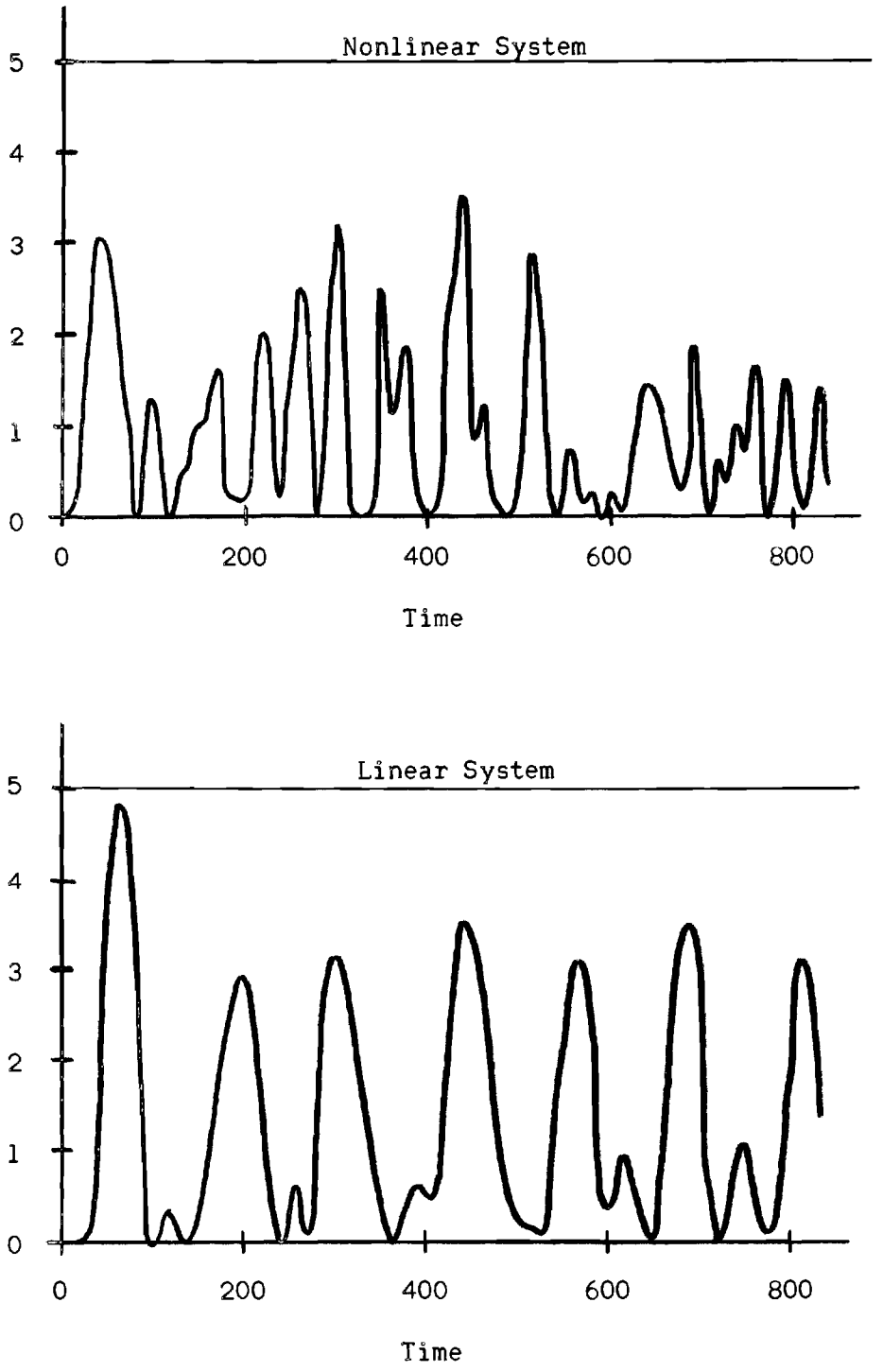

Figure 31. Comparison of Plots of E_5 Versus Time for Nonlinear and Linear $N = 5$ Systems.

averages of the $f(E)$ data over all five oscillators were taken. These data points are plotted as histograms for the nonlinear and linear systems in Figure 32.

According to equilibrium statistical mechanics, the single-oscillator energy distribution density for a system at temperature T is given by

$$f(E) = \frac{1}{kT} e^{-\frac{E}{kT}}, \tag{5}$$

where $kT = E$, the average energy of a single oscillator. For these particular five-oscillator systems, this average energy is unity. Thus according to equilibrium statistical mechanics, $f(E) = e^{-E}$. The solid curves in the plots of Figure 32 represent e^{-E}.

Thus in Figure 32 the histograms represent the time average $f(E)$, while the solid curve represents the phase average $f(E)$. It is interesting to note the close agreement of the time averages and phase averages for both the nonlinear and the linear coupled oscillator systems. It thus appears that for these systems, four oscillators form an adequate heat bath for the fifth.

In this chapter, brief discussions of constants of the motion and the approach to equilibrium for coupled oscillator systems have been given. The work on constants of the motion appears to show promise of some later development. The studies of the approach to equilibrium for the nonlinear and linear coupled oscillator systems serve to further emphasize the similarities and analogies between nonlinear systems and linear systems. The results for the five-oscillator systems given in

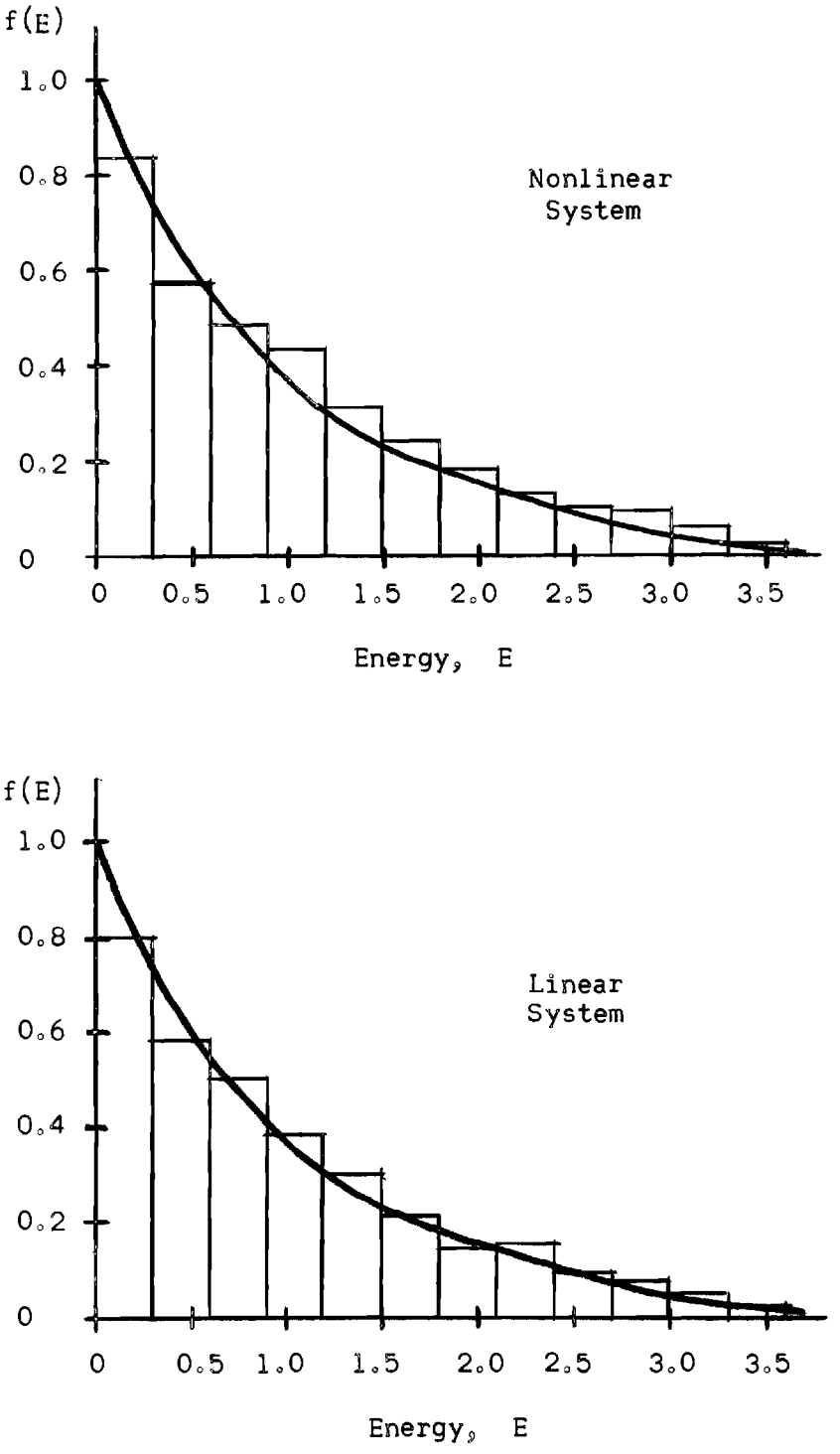

Figure 32. Energy Distribution Densities for Nonlinear and Linear Five-Oscillator Systems.

the last section of this chapter provide some indication that five oscillators are enough to adequately represent many of the essential properties of larger systems.

In the final chapter of study, some conclusions will be drawn concerning the work which has been done.

CHAPTER IX

CONCLUSIONS

The principal conclusion which can be drawn from this investigation is that weakly-coupled energy-sharing nonlinear coupled oscillator systems are amenable to analysis and that these systems have much in common with energy-sharing linear coupled oscillator systems. A method of analysis of these nonlinear systems has been presented, and the effectiveness of this method has been tested by comparisons with selected numerical solutions. There are indications that this method of analysis is capable of providing good first-order approximations to the general solutions of the nonlinear systems considered. Although certain difficulties have been encountered in the calculation of second-order terms, it seems likely that the results of the first-order approximations can be improved by making higher-order approximations, calculated in an appropriate manner.

There are several disadvantages inherent in the method presented in this thesis, aside from difficulties in the calculation of second-order terms. The calculation of enough terms to give good approximations to the behavior of nonlinear systems of more than about ten oscillators, for all sets of initial conditions, is prohibitively lengthy even using the presently available high-speed digital computers. Also, the analysis of the effects of detuning and of nonresonant coupling terms, both of which must be considered when dealing with physically realistic situations, is likely to be quite complicated in the form suggested here, even for as many as

five oscillators.

The existing perturbation methods of Krylov-Bogoliubov and Wigner-Brillouin, the latter as applied by Jackson, are quite successful when applied to all but the most sharply tuned systems. The method presented in this thesis provides a moderately successful analysis of exactly-tuned systems, and an indication has been given of the manner in which detuned systems can be dealt with, starting from the exactly-tuned case. Therefore, the Krylov-Bogoliubov and Wigner-Brillouin methods for untuned systems together with the present method for tuned systems should provide methods of solution which are appropriate for all extremes of tuning for nonlinear coupled oscillator systems.

It is hoped that the principal result of these studies, a demonstration that exactly-tuned weakly-coupled nonlinear oscillator systems are indeed susceptible to some form of analysis, will encourage the development of more practical methods of analysis in the future.

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VITA

John Forrest Waters was born on October 6, 1936, in Washington, D. C., the son of Nathaniel and Edna Waters. He graduated from Montgomery Blair High School in Silver Spring, Maryland in 1954. He then enrolled in undergraduate school at the Georgia Institute of Technology in Atlanta, Georgia in the Co-operative Program with the U. S. Naval Ordnance Laboratory in White Oak, Maryland. He graduated in 1959 with a B. S. degree in Electrical Engineering.

John F. Waters then returned to undergraduate school in physics at Georgia Tech, and was granted the B. S. degree in Physics in 1961. He continued his studies in the Graduate Division, and received the M. S. degree in Physics in 1962 from Georgia Tech. His work for the Ph. D. in physics was completed in 1964.

From 1961 to 1964, Mr. Waters was a Graduate Instructor in physics at the Georgia Institute of Technology. In 1960, he married the former Miss Nancy Patterson of Kingsport, Tennessee. They have two children. Mr. Waters is a member of Sigma Pi Sigma and the American Physical Society.