AN INVESTIGATION OF SEVERAL SOLUTIONS OF THE EQUATIONS OF COMPRESSIBLE FLUID FLOW

A THESIS

Presented to

the Faculty of the Division of Graduate Studies

Georgia Institute of Technology

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Aeronautical Engineering

by

Rex Burnham Powell

September 1950

AN INVESTIGATION OF SEVERAL SOLUTIONS OF THE EQUATIONS OF COMPRESSIBLE FLUID FLOW

Approved:

Date Approved by Chairman October 13, 1950

ACKNOWLEDGMENTS

The author wishes to thank Professor H. W. S. LaVier for his co-operation and helpful criticisms throughout the preparation of this thesis.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

$x, y, z,$	coordinates in the physical flow field
q	velocity of the compressible flow
θ	angle the velocity q makes with the positive x axis
$u, v, w,$	components of the velocity q
ρ	density
p	pressure
γ	ratio of the specific heat at constant pressure to the specific heat at constant volume
Φ	velocity potential
Ψ	stream function
i	$\sqrt{-1}$
Z	complex variable $x + iy$
a	velocity of sound
M	Mach number
C	slope of the straight line isentrope
W	velocity of the incompressible flow
$U, V,$	components of the velocity W
F	complex function $\Phi + i\Psi$
ξ, η	coordinates in the ζ plane
λ	correction factor, $\frac{1}{4}(W_1/a_o)^2$
μ	Mach angle
Γ	characteristic line
C_p	pressure coefficient
c_p	specific heat at constant pressure

c_v	specific heat at constant volume
ϕ	perturbation velocity potential
C_l	coefficient of lift
β	shock wave angle

Subscripts

x, θ, q, etc.	derivative with respect to the subscript
o	stagnation condition
1	free stream condition
L	limiting value
c	critical value

Superscripts

$\bar{\ }$	conjugate of the function
*	dimensionless velocity
'	perturbation velocity

AN INVESTIGATION OF SEVERAL SOLUTIONS OF THE EQUATIONS OF COMPRESSIBLE FLUID FLOW

INTRODUCTION

The equations of motion of a compressible fluid, as will be seen later, are non-linear differential equations. Since there is no general method for solving such equations, the solution depends on the equation and the boundary conditions. In the supersonic case, where the non-linear equation of the velocity potential is of the hyperbolic type, the problem may be solved by the method of characteristics. In the subsonic case however, the characteristics do not exist, therefore some other approach must be made. One such approach is to require that the perturbation or disturbance velocities, due to the presence of a body in a uniform stream, be small. The equations then become linear and the solution may be had by ordinary means. In all pratical cases the solutions based on this method are approximate. However, in many cases the results are accurate enough for engineering work.

Chaplygin[1] and Molenbroek[2] obtained a solution to the

---

[1]Chaplygin, S. H., "On Gas Jets," National Advisory Committee for Aeronautics Technical Memorandum No. 1063, 1944.

[2]Molenbroek, P.,"Uber einige Bewegungen eines Gases mit Annahame eines Geschwindigkeitspotential," Archiv. d. Math. u. Phys.,(2), 9:157, 1890.

subsonic problem by transforming the equations from the physical plane to the "hodograph" plane in which q, the magnitude of the velocity, and θ, the angle between the velocity and the positive x axis, are used as the independent variables. The equations of motion then become linear. The difficulty of this method is to obtain a set of boundary conditions in the hodograph plane which will give a complete solution.

FUNDAMENTAL EQUATIONS

Introduction. Solution of the problem of flow from infinity of a compressible fluid past an arbitrary body requires that the relation between the stream function and the velocity potential be known. In addition, the equations of the conservation of mass and momentum and certain thermodynamic relations are useful in the analysis of compressible fluid dynamics. These relations are presented in the following section.

Conservation of Mass. Consider the control volume V, with volume $\delta x \delta y \delta z$ (figure 1), in a flow of velocity q with components u, v, and w, in the x, y, and z directions respectively. The mass flow through the side normal to the x axis and nearest the origin is

$$\rho u \, \delta z \, \delta y \quad ,$$

and the flow through the other side normal to the x axis is

$$\left[(\rho u) + (\rho u)_x \, \delta x \right] \delta z \, \delta y .$$

Where ρ is the mass density of the fluid. Therefore the total mass increase per unit time parallel to the x axis is

$$\rho u \, \delta z \, \delta y - \rho u \, \delta z \, \delta y - (\rho u)_x \, \delta x \, \delta y \, \delta z$$
$$- (\rho u)_x \, \delta x \, \delta y \, \delta z .$$

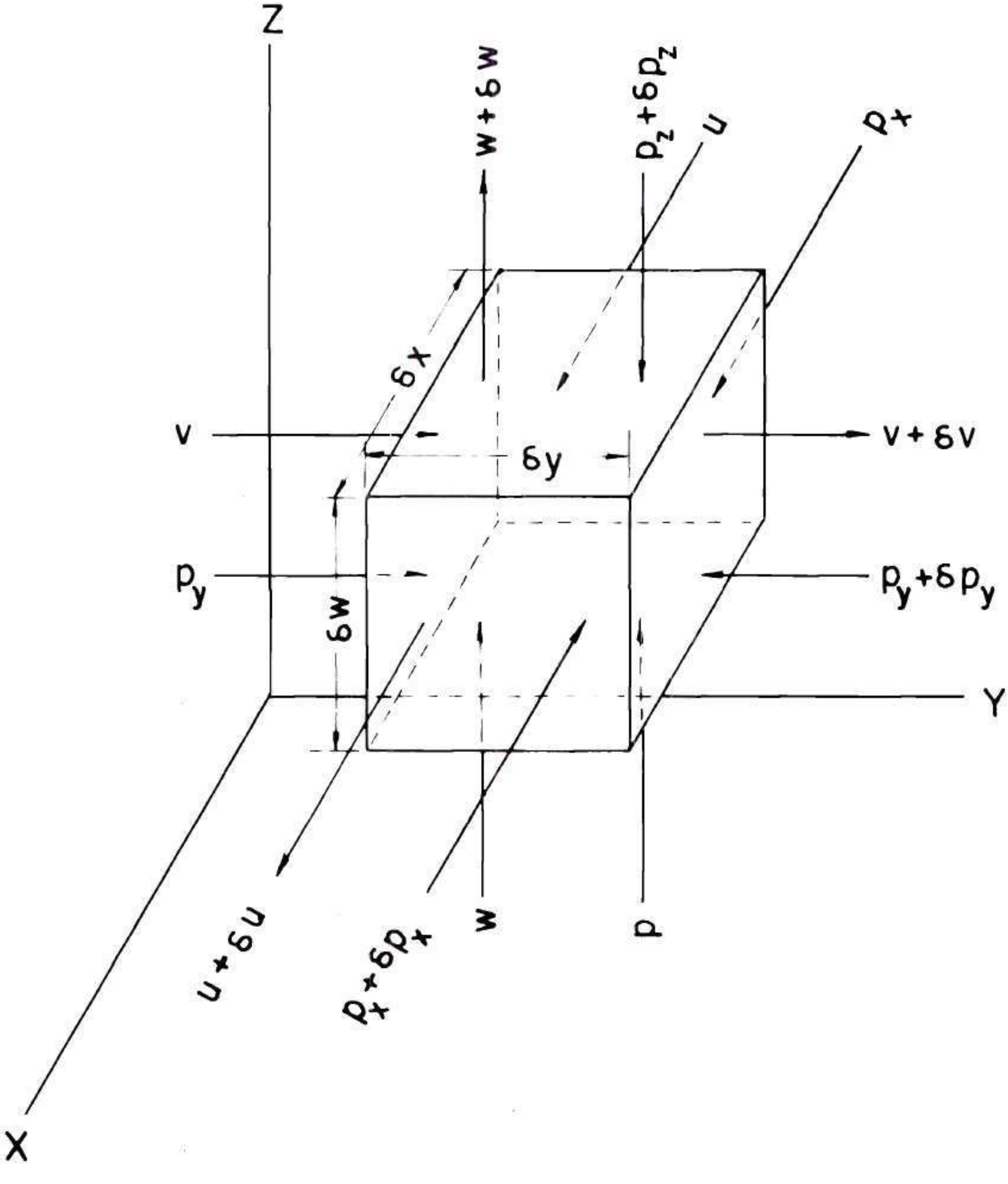


FIGURE 1, VELOCITIES AND FORCES ON A FLUID ELEMENT

Likewise the mass increase per unit time parallel to the y and z axis is

$$-(\rho v)_y \,\delta x\, \delta z\, \delta y \quad \text{, and} \quad -(\rho w)_z \,\delta x\, \delta z\, \delta y$$

respectively. Then the total increase for the control volume v per unit time is

$$-\left[(\rho u)_x + (\rho v)_y + (\rho w)_z\right] \delta x\, \delta y\, \delta z .$$

But the total increase per unit time is also

$$\rho_t \,\delta x\, \delta y\, \delta z \quad .$$

Therefore the continuity equation becomes

$$(\rho u)_x + (\rho v)_y + (\rho w)_z + \rho_t = 0 \quad ,$$

which for a steady two dimensional flow reduces to

$$(\rho u)_x + (\rho v)_y = 0 \quad . \qquad (1)$$

<u>Conservation of Momentum.</u> Consider again the volume V in the moving stream. The acceleration in the x direction is

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_y \frac{dy}{dt} + u_z \frac{dz}{dt} + u_t$$

$$= u u_x + v u_y + w u_z + u_t$$

and

$$\frac{dv}{dt} = uv_x + vv_y + wv_z + v_t$$

$$\frac{dw}{dt} = uw_x + vw_y + ww_z + w_t$$

in the y and z directions. In the absence of external forces, the only force acting on the volume V is the pressure force. From figure 1 the resultant force in the x direction is

$$-p_x \, \delta x \, \delta y \, \delta z \quad ,$$

where p is the pressure per unit area. In the y and z directions are

$$-p_y \, \delta x \, \delta y \, \delta z$$

$$-p_z \, \delta x \, \delta y \, \delta z \; .$$

Then applying Newton's second law of motion,

$$\rho \, \delta x \, \delta y \, \delta z \left(uu_x + vu_y + wu_z + u_t \right) = -p_x \, \delta x \, \delta y \, \delta z ,$$

or

$$uu_x + vu_y + wu_z + u_t = -\frac{1}{\rho} p_x$$

$$uv_x + vv_y + wv_z + v_t = -\frac{1}{\rho} p_y$$

$$uw_x + vw_y + ww_z + w_t = -\frac{1}{\rho} p_z$$

$$\cdot$$

These are the well known Eulerian equations of motion. For a steady two dimensional flow they are simply

$$u u_x + v u_y = -\frac{1}{\rho} p_x$$
$$u v_x + v v_y = -\frac{1}{\rho} p_y \quad . \tag{2}$$

Integration of equation (2) gives Bernoulli's equation

$$\frac{u^2}{2} + \frac{p}{\rho} = \text{const.} \tag{3}$$

which will be used later in the above one dimensional form.

Thermodynamic Relations. When considering the Bernoulli equation in one dimensional form it is sometime useful to write it in the form

$$\frac{u^2}{2} + c_p T = \text{const.}$$

where c_p is the specific heat at constant pressure. The above expression is true since the volume remains constant in one dimensional flow. From the Charles-Boyle gas law

$$T = \frac{p}{\rho(c_p - c_v)}$$

and the Bernoulli equation becomes

$$\frac{u^2}{2} + \frac{c_p}{c_p - c_v} \frac{p}{\rho} = \text{const}$$

or

$$\frac{u^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \text{const} , \tag{4}$$

where c_v is the sepcific heat at constant volume and γ is c_p/c_v.

Now if the requirement that the process be isentropic is introduced, that is

$$p = \text{const} \cdot \rho^{\gamma}$$

or

$$\frac{p}{p_o} = \left(\frac{\rho}{\rho_o}\right)^{\gamma} ,$$

where the subscript o denotes the conditions at rest, ρ becomes

$$\rho = \rho_o \left(\frac{p}{p_o}\right)^{\frac{1}{\gamma}} .$$

Now substituting this expression for ρ in equation (4) gives the following relation which becomes useful in the discussion of the characteristics method.

$$\frac{u^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho_o} \left(\frac{p}{p_o}\right)^{-\frac{1}{\gamma}}$$

$$\frac{u^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_o}{\rho_o} \left(\frac{p}{p_o}\right)^{(\gamma - 1)/\gamma} = \text{const} . \tag{5}$$

Writing equation (4) between any point and the stagnation point gives

$$\frac{u^2}{2} + \frac{\gamma}{\gamma-1}\,\frac{p}{\rho} = \frac{p_0}{\rho_0}\,\frac{\gamma}{\gamma-1} \,. \tag{6}$$

If the speed of sound, a, is given by

$$a^2 = \gamma p/\rho \,,$$

and the Mach number, M, is defined as the ratio of the velocity to the local speed of sound, then equation (6) becomes

$$M^2 a^2 + \frac{2a^2}{\gamma-1} = \frac{2a_0^2}{\gamma-1}$$

$$\frac{\gamma-1}{2}M^2 + 1 = \left(\frac{a_0}{a}\right)^2 \,.$$

Since $a_0^2/a^2 = T_0/T$, it follows that

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2 \,.$$

The isentropic law may also be written in the form[3]

$$\frac{p_0}{p} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma-1)} \,.$$

Therefore the following relations for isentropic flow may be obtained:

$$\frac{T_0}{T} = 1 + \frac{\gamma-1}{2}M^2$$

$$\frac{p_0}{p} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{\gamma/(\gamma-1)}$$

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma-1}{2}M^2\right)^{1/(\gamma-1)} \,.$$

---

[3]Liepmann, H. W., and A. E. Puckett, Aerodynamics of a Compressible Fluid, (New York: John Wiley and Sons, 1947), p26.

<u>Velocity Potential and Stream Function.</u> If the condition of irrotationality is satisfied, that is

$$v_x - u_y = 0 \quad ,$$

(two dimensional flow) then there exists a function Φ, such that

$$\Phi_x = u \quad , \qquad \Phi_y = v$$

which satisfies the condition of irrotationality. This function Φ is the so-called velocity potential.

Considering equation (1) it is clear that the equation is satisfied if there exists a function Ψ , such that

$$\Psi_y = \rho u$$
$$\Psi_x = -\rho v$$

.

This function Ψ is the stream function.

Writing the expression for Ψ in a slightly different form so that Φ and Ψ are in the same units and combining the two, equations (8) are obtained.

$$\frac{\rho_0}{\rho}\Psi_y = \Phi_x = u$$
$$-\frac{\rho_0}{\rho}\Psi_x = \Phi_y = v \tag{8}$$

These are the relations between the velocity potential and the stream function for two dimensional flow.

Investigation of equations (8) will show that they are non-linear in the dependent variables. It is this fact which presents the difficulty since there is as yet no general treatment known for such equations. In the case of the flow velocity everywhere greater than the local speed of sound the solution may be handled by the method of characteristics. Since, as will be shown later, the equations of the characteristic lines do not exist in the case of completely subsonic flow, the solution is accomplished by the use of approximations which linearize the equations, or by transforming the equations into the hodograph plane where they become exact linear equations. In the hodograph plane the velocity q and the direction of the velocity, θ, are taken to be the independent variables.

THE HODOGRAPH TRANSFORMATION

Introduction. The fundamental equations of the stream function and velocity potential (equations 8) are transformed into exact linear equations with q and θ as the independent variables. These equations are then simplified by the introduction of the Mach number and the differential operator $d\omega$. Solution of the final simplified equations is then carried out by the use of the approximations of Chaplygin and Karman-Tsien.

In order to apply these solutions to a specific problem, a point transformation is found which transforms a known incompressible flow pattern into the desired compressible flow pattern.

Basic Equations. As mentioned before, the non-linear equations (8) may be transformed into exact linear equations by the introduction of two new variables: q, the absolute magnitude of the velocity, and θ, the angle the velocity vector makes with the positive x axis. These are refered to as the hodograph variables.

To show this, let Φ and Ψ be functions of the complex variable $\Phi + i\frac{\rho_0}{\rho}\Psi$, then

$$d\Phi + i\frac{\rho_0}{\rho}d\Psi = (\Phi_x dx + \Phi_y dy) + i\frac{\rho_0}{\rho}(\Psi_x dx + \Psi_y dy) .$$

Substituting equations (8) and simplifying gives,

$$d\Phi + i\frac{\rho_0}{\rho}d\Psi = (u\,dx + v\,dy) + i\frac{\rho_0}{\rho}\left(-\frac{\rho}{\rho_0}v\,dx + \frac{\rho}{\rho_0}u\,dy\right)$$

$$= (u - iv)\,dx + (v + iu)\,dy$$

$$= (u - iv)(dx + i\,dy) .$$

Now introducing the new variables q and θ,

$$d\Phi + i\frac{\rho_0}{\rho}d\Psi = q e^{-i\theta} dZ$$

or

$$dZ = \left(d\Phi + i\frac{\rho_0}{\rho}d\Psi\right)\frac{1}{q}e^{i\theta} \qquad (9)$$

where Z is the complex variable $x + iy$.

Now considering q and θ the independent variables and assuming ρ a single valued function of q only, then from equation (9) it follows that

$$Z_\theta = \frac{1}{q}e^{i\theta}\left(\Phi_\theta + i\frac{\rho_0}{\rho}\Psi_\theta\right)$$

and

$$Z_q = \frac{1}{q}e^{i\theta}\left(\Phi_q + i\frac{\rho_0}{\rho}\Psi_q\right) .$$

Also

$$Z_{\theta q} = e^{i\theta}\left[-\frac{1}{q^2}\Phi_\theta + i\frac{d\left(\frac{\rho_0}{\rho q}\right)}{dq}\Psi_\theta\right]$$

$$+ e^{i\theta}\left[\frac{1}{q}\Phi_{\theta q} + i\frac{\rho_0}{\rho q}\Psi_{\theta q}\right]$$

and

$$Z_{q\theta} = \frac{1}{q}\left[ie^{i\theta}\Phi_q - e^{i\theta}\frac{\rho_0}{\rho}\Psi_q\right] + \frac{e^{i\theta}}{q}\left[\Phi_{q\theta} + i\frac{\rho_0}{\rho}\Psi_{q\theta}\right].$$

Since by continuity, $Z_{\theta q}$ and $Z_{q\theta}$ are equal, then

$$-\frac{1}{q^2}\Phi_\theta + i\,\frac{d\left(\frac{\rho_0}{\rho q}\right)}{dq}\Psi_\theta = \frac{i}{q}\Phi_q - \frac{\rho_0}{\rho q}\Psi_q \quad .$$

Equating real and imaginaries

$$\Phi_q = q\,\frac{d\left(\frac{\rho_0}{\rho q}\right)}{dq}\Psi_\theta$$

$$\Phi_\theta = \frac{\rho_0}{\rho}\,q\,\Psi_q \quad . \tag{10}$$

Equations (10) are the hodograph equations derived by the method first used by Molenbroek[4]. These equations are linear in the dependent variables.

Although equations (10) are linear, the solution is rather involved. Bers and Gelbert[5] developed a new Σ-monogenic function theory, analogous to the analytic function theory, which may be used to obtain a rigorous solution to equations of the form

---

[4] Molenbroek, P., op. cit.

[5] Bers, Lipman, and Gelbert, Abe, "On a Class of Differential Equations in Mechanics of Continua," <u>Quarterly of Applied Mathematics</u>, 1:168, July, 1943.

$$\Phi_\theta = F_1(q)\,\Psi_q$$

$$\Phi_q = -F_2(q)\,\Psi_\theta$$

But for many problems approximations may be made which greatly simplify the solution and yet do not effect the results enough to make them invalid for engineering use.

Simplification of Basic Equations. First consider Euler's equation for the case of one dimensional flow.

$$q\,dq + \frac{dp}{\rho} = 0$$

This may be written as

$$q\,dq + \frac{d\rho}{\rho}\,\frac{dp}{d\rho} = 0,$$

and since the sonic velocity, a, is

$$a = \sqrt{\frac{dp}{d\rho}}\;,$$

then

$$q\,dq + \frac{d\rho}{\rho}\,a^2 = 0$$

or

$$\frac{d\rho}{dq} = -\frac{q}{a^2}\,\rho = -\frac{q^2}{a^2}\,\frac{\rho}{q}$$

$$= -M^2\,\frac{\rho}{q} \qquad (11)$$

where M is the Mach number.

Now turning to equation (10) and performing the indicated differentiation, the result is

$$\Phi_q = q \quad -\frac{1}{q^2}\frac{\rho_o}{\rho} + \frac{1}{q}\frac{d(\rho_o/\rho)}{dq} \quad \Psi_\theta$$

$$= -\frac{1}{q}\frac{\rho_o}{\rho} - \frac{\rho_o}{\rho^2}\frac{d\rho}{dq} \quad \Psi_\theta \quad .$$

Substituting relation (11)

$$\Phi_q = -\frac{1}{q}\frac{\rho_o}{\rho} + \frac{\rho_o}{\rho}\frac{M^2}{q} \quad \Psi_\theta$$

$$\Phi_q = -(1-M^2)\frac{\rho_o}{\rho}\frac{1}{q}\Psi_\theta \quad .$$

Then the hodograph equations (10) become

$$\Phi_q = -\frac{(1-M^2)}{q}\frac{\rho_o}{\rho}\Psi_\theta$$

$$\Phi_\theta = \frac{\rho_o}{\rho} q \Psi_q \quad . \tag{12}$$

These equations may be simplified further by the introduction of a new variable, ω. Written in the differential form, ω is defined by

$$\partial\omega = \sqrt{1-M^2}\;\frac{\partial q}{q} \quad . \tag{13}$$

With the use of equation (13), equations (12) become

$$\Phi_\omega = -\frac{\rho_o}{\rho}\sqrt{1-M^2}\;\Psi_\theta$$

$$\Phi_\theta = \frac{\rho_o}{\rho}\sqrt{1-M^2}\;\Psi_\omega \quad . \tag{14}$$

These are the final simplified hodograph equations. As mentioned before approximations may be made which greatly simplify the solution of these equations. A discussion of these approximations is now in order.

<u>The Chaplygin Approximation.</u> Chaplygin in his discussion on gas jets[6] noticed that the factor $\frac{\rho_0}{\rho}\sqrt{1-M^2}$ differed little from unity for moderate subsonic speeds.

To show the full meaning of this, let us look at the factor, which by using the isentropic relation may be written as

$$\frac{\rho_0}{\rho}\sqrt{1-M^2} = \sqrt{1-M^2}\left(1+\frac{\gamma-1}{2}M^2\right)^{\frac{1}{\gamma-1}} \tag{15}$$

where γ is the ratio of the specific heats of the gas. Expanding equation (15),

$$\begin{aligned}\frac{\rho_0}{\rho}\sqrt{1-M^2} &= \left(1+\frac{M^2}{2}+(2-\gamma)\frac{M^4}{8}+\cdots\right)\\ &\quad\left(1-\frac{M^2}{2}-\frac{M^4}{8}-\cdots\right)\\ &=\left(1+\frac{M^2}{2}+(2-\gamma)\frac{M^4}{8}-\frac{M^2}{2}-\frac{M^4}{4}\right.\\ &\quad\left.-(2-\gamma)\frac{M^6}{16}-\frac{M^4}{8}+\cdots\right)\\ &=\left(1-\frac{\gamma+1}{8}M^4+\cdots\right)\end{aligned}$$

---

[6]Chaplygin, S. A., op. cit.

Using the value of γ for air (1.4), the above series becomes

$$\frac{\rho_0}{\rho}\sqrt{1-M^2} = 1 - 0.3\,M^4 + \cdots \quad . \tag{16}$$

Then replacing $\frac{\rho_0}{\rho}\sqrt{1-M^2}$ by one in equations (14) the hodograph equations become simply the Cauchy-Riemann relations.

$$\Phi_\omega = -\Psi_\theta$$

$$\Phi_\theta = \Psi_\omega \tag{17}$$

Hence by inspection of equation (16) it can be seen that by using $\frac{\rho_0}{\rho}\sqrt{1-M^2}$ as unity, one is in reality neglecting all powers of M higher than the first. That is, the incompressible and moderate subsonic compressible flow differ only by factors proportional to M^4.

The Chaplygin approximation may also be interpreted as the use of a hypothetical gas having a γ value of -1. This can be shown by the use of equation (15). From this equation it is seen that for $\frac{\rho_0}{\rho}\sqrt{1-M^2}$ to equal one when M is less than one, γ must equal -1. Then by using the isentropic relation

$$\frac{p}{\rho^\gamma} = \text{const}$$

and letting $\gamma = -1$, the relation becomes

$$p\rho = \text{const} \; .$$

That is the isentrope is a straight line with the equation

$$p_2 - p_3 = C\left(\frac{1}{\rho_2} - \frac{1}{\rho_3}\right) \tag{18}$$

where C is the slope and the subscripts 2 and3 denote two different points on the curve. With the correct choice of C the line given by equation (18) may be made tangent to the true isentrope at some arbitrary point. A graphical representation of this is shown in figure 2.

Chaplygin used the straight line isentrope tangent to the true isentrope at the point corresponding to the stagnation condition of the fluid. That is, C must equal the slope of the true isentrope at the point ρ_o and p_o. Therefore

$$C = \left(\frac{dp}{dv}\right)_o = \left(\frac{dp}{d\rho}\,\frac{d\rho}{dv}\right)_o = a_o^2\,\frac{d\rho}{d\left(\frac{1}{\rho}\right)}$$
$$= -a_o^2\rho_o^2 \; . \tag{19}$$

Now substituting (19) in (18), the equation of the straight line isentrope becomes

$$p_o - p = -a_o^2\rho_o^2\left(\frac{1}{\rho_o} - \frac{1}{\rho}\right) \; . \tag{20}$$

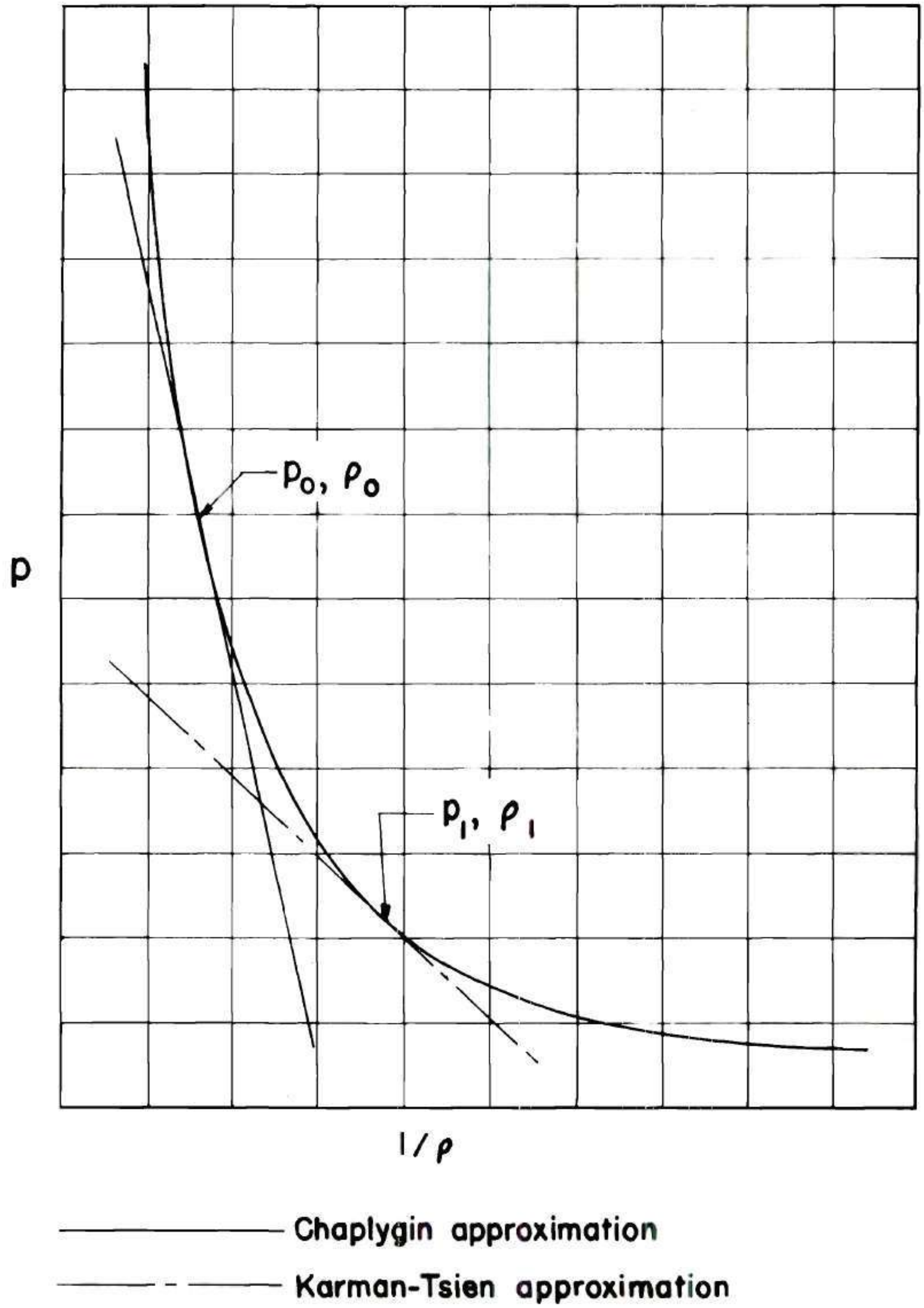


FIGURE 2, APPROXIMATIONS TO THE TRUE ISENTROPE

The Karman-Tsien Approximation. Like the Chaplygin approximation, the Karman-Tsien approximation is a straight line tangent to the true isentrope. Since the Chaplygin method is useful only for speeds up to about one-half the speed of sound, Karman suggested that if the point of tangency were shifted from the stagnation condition to the undisturbed condition (ρ_1 ,p_1) the useful region of the straight line isentrope approximation could be greatly extended. This idea was developed jointly with Tsien[7] and later become the Karman-Tsien method used extensively in aeronautical work for the solution of problems involving compressible flow at subsonic speeds.

Because of the $\gamma = -1$ inferred by the Karman-Tsien approximation, this method becomes less accurate near the critical Mach number; that is the Mach number at which the local Mach number first reaches one somewhere on the body immersed in the stream. This will be brought out later in the application of this method to the problem of subsonic compressible flow about a circular cylinder.

Since the Chaplygin condition, given by the preceeding section, holds true for the Karman-Tsien method, equations (17) may be used for this method also.

---

[7]Tsien, H.S., "Two Dimensional Subsonic Flow of Compressible Fluids," Journal of the Aeronautical Sciences, 6:399, August, 1939.

Keeping in mind the fact that the straight line given by equation (18) is now tangent to the true isentrope at ρ_1, p_1 (figure 2), the slope now becomes

$$C = -a_1^2 \rho_1^2 \quad .$$

Then equation (18) becomes

$$p_1 - p = -a_1^2 \rho_1^2 \left(\frac{1}{\rho_1} - \frac{1}{\rho} \right) \quad . \tag{21}$$

Now returning to equations (17),

$$\begin{aligned} \Phi_\omega &= -\Psi_\theta \\ \Phi_\theta &= \Psi_\omega \end{aligned} \tag{17}$$

it is seen that the complex variable $\Phi + i\Psi$ is an analytic function of $\omega - i\theta$. But for the convenience of calculation and for the purpose of showing the relation retween the boundary conditions for incompressible and compressible flow, the physical and hodograph planes. They are defined by equations (22).

$$\begin{aligned} U &= W \cos\theta \\ V &= W \sin\theta \\ W &= a_0 e^{\omega} \quad . \end{aligned} \tag{22}$$

Where U and V are the components of the incompressible velocity W.

For incompressible flow, $q \to W$, $M \to 0$, and $\frac{\rho_0}{\rho} \to 1$. Then equations (12) become

$$\Phi_W = -\frac{1}{W}\Psi_\theta$$

$$\Phi_\theta = W\,\Psi_W \quad . \qquad (23)$$

These may be written in terms of the variables of equations (22) as

$$\Phi_U U_W = -\frac{1}{W}\Psi_{-V}\,(-V_\theta)$$

$$\Phi_{-V}(-V_\theta) = W\,\Psi_U U_W \quad .$$

Differentiating equations (22) and substituting in the above,

$$\Phi_U \cos\theta = -\frac{1}{W}\Psi_{-V}\,(-W\cos\theta)$$

$$\Phi_{-V}(-W\cos\theta) = W\,\Psi_U \cos\theta$$

or

$$\Phi_U = \Psi_{-V}$$

$$\Phi_{-V} = -\Psi_U \quad .$$

(24)

Let us now look at the corresponding hodograph equations for the Karman-Tsien method and for the incompressible condition (equations 17 and 23 respectively). Equations (23) may be simplified by defining a new variable Ω , in the same way that the basic hodograph equations were simplified by the variable ω. This variable Ω is given by

$$\Omega = \log W$$

or

$$\partial \Omega = \frac{\partial W}{W} \quad . \tag{25}$$

Substituting equation (25) in (23) the result is

$$\Phi_W = -\frac{d\Omega}{dW} \Psi_\theta$$

$$\Phi_\theta = \frac{dW}{d\Omega} \Psi_W \quad .$$

Then it follows

$$\Phi_\Omega = -\Psi_\theta$$

$$\Phi_\theta = \Psi_\Omega \quad . \tag{26}$$

Now by inspection of equations (17) and (26) it is seen that if $\Omega=\omega$ the compressible flow, approximated by the Karman-Tsien method, and the incompressible flow will satisfy the same equations. Therefore the relation between W and q becomes

$$\frac{dW}{W} = \sqrt{1-M^2} \quad \frac{dq}{q}$$

$$\int \frac{dW}{W} = \int \sqrt{1-M^2} \quad \frac{dq}{q} \quad . \tag{27}$$

Bernoulli's equation is given as

$$\frac{q^2}{2} + \frac{\gamma}{\gamma-1}\frac{p}{\rho} = \frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{q_1^2}{2}$$

or

$$\frac{q^2}{2} + \frac{a^2}{\gamma-1} = \frac{a_1^2}{\gamma-1} + \frac{q_1^2}{2} \quad .$$

Applying the Karman-Tsien approximation, $\gamma = -1$,

$$\frac{q^2}{2} + \frac{a^2}{-2} = \frac{a_1^2}{-2} + \frac{q_1^2}{2} \quad .$$

Therefore

$$q^2 - a^2 = -a_1^2 + q_1^2 \quad ,$$

and writing this between the stagnation condition and the local condition,

$$-a_o^2 = q^2 - a^2 . \tag{28}$$

Substituting (28) in (27) gives

$$\int \frac{dW}{W} = \int \sqrt{1 - \frac{q^2}{q^2 + a_o^2}}\ \frac{dq}{q} = \int \sqrt{\frac{a_o^2}{q^2 + a_o^2}}\ \frac{dq}{q}$$

$$\log W = \int \frac{a_o^2}{q\sqrt{q^2 + a_o^2}}\, dq .$$

Integrating,

$$\log W = -\log\left(\frac{a_o + \sqrt{q^2 + a_o^2}}{q}\right) + K$$

then

$$W = K\left(\frac{q}{a_o + \sqrt{q^2 + a_o^2}}\right),$$

where K is the constant of integration. To evalute K consider the case when the perturbation velocity due to any disturbance is small; that is $q \ll a_o$. Then $q \to W$, and

$$W = K\left(\frac{W}{2a_o}\right)$$

$$K = 2a_o\,.$$

Then the final relation between q and W is

$$W = \frac{2a_o q}{a_o + \sqrt{q^2 + a_o^2}} \qquad (29)$$

and solving for q,

$$q = \frac{4a_o^2 W}{4a_o^2 - W^2}\,. \qquad (30)$$

Now using the approximation of Chaplygin and also of Karman-Tsien that

$$\frac{\rho_o}{\rho}\sqrt{1-M^2} = 1\,,$$

we may find the density ratio for the Karman-Tsien method. Solving for $\frac{\rho_o}{\rho}$ and substituting equation (28) gives

$$\left(\frac{\rho_o}{\rho}\right)^2 = \frac{1}{1-M^2} = \frac{q^2 + a_o^2}{a_o^2}$$

$$= \frac{16a_o^2 W^2}{(4a_o^2 - W^2)^2} + 1$$

$$\left(\frac{\rho_0}{\rho}\right)^2 = \frac{(4a_0^2 - W^2)^2 + 16\,a_0^2 W^2}{(4a_0^2 - W^2)^2}$$

$$= \frac{(4a_0^2 + W^2)^2}{(4a_0^2 - W^2)^2}$$

or

$$\frac{\rho_0}{\rho} = \frac{4a_0^2 + W^2}{4a_0^2 - W^2} \quad . \qquad (31)$$

Equations (24, 29, 30) and (31) are the basic equations of the Karman-Tsien method. They give the relation between the incompressible velocity W and the compressible velocity q, both in the physical plane. Equations (24) are the Cauchy-Riemann equations and therefore the complex potential

$$F = \Phi + i\Psi$$

must be an analytic function of

$$\overline{W} = U - iV \quad ,$$

where $\overline{W}$ is the conjugate of W and F is the incompressible complex potential chosen to give the correct complex potential in the compressible flow when transformed, and U and V are defined by equations (22). That is,

$$\Phi + i\Psi = F(U - iV) = F(\overline{W})$$

and

$$\Phi - i\Psi = \overline{F}(U + iV) = \overline{F}(W) \; . \qquad (32)$$

Now in order to use equation (32), the function F must be found which gives the relation between U and V, and x and y; or the transformation from the hodograph plane to the physical plane. This is necessary since the hodograph plane is merely a method of representing the non-linear equations as linear ones; hence the physical picture of the flow is lost when the flow is represented in the hodograph plane.

The function F is now found by writing

$$d\Phi = \Phi_x \, dx + \Phi_y \, dy$$
$$d\Psi = \Psi_x \, dx + \Psi_y \, dy$$

Then by using the definitions given by equations (8), these become

$$d\Phi = q\cos\theta \, dx + q\sin\theta \, dy$$
$$d\Psi = -q\frac{\rho}{\rho_0}\sin\theta \, dx + q\frac{\rho}{\rho_0}\cos\theta \, dy \quad .$$

Using equations (22, 30,) and (31) and solving for dx and dy,

$$dx = \frac{U}{qW}\, d\Phi - \frac{V}{qW}\frac{\rho_0}{\rho}\, d\Psi$$

$$dy = \frac{V}{qW}\, d\Phi + \frac{U}{qW}\frac{\rho_0}{\rho}\, d\Psi$$

$$dx = \frac{U}{W}\, d\Phi\left(\frac{4a_0^2 - W^2}{4a_0^2 W}\right) - \frac{V}{W}\, d\Psi\left(\frac{4a_0^2 + W^2}{4a_0^2 W}\right)$$

$$dy = \frac{V}{W}\, d\Phi\left(\frac{4a_0^2 - W^2}{4a_0^2 W}\right) + \frac{U}{W}\, d\Psi\left(\frac{4a_0^2 + W^2}{4a_0^2 W}\right) .$$

Then

$$dx = \frac{V}{W^2}\, d\Phi \left(1 - \frac{W^2}{4a_o^2}\right) - \frac{V}{W^2}\, d\Psi \left(1 + \frac{W^2}{4a_o^2}\right)$$

$$dy = \frac{V}{W^2}\, d\Phi \left(1 - \frac{W^2}{4a_o^2}\right) + \frac{U}{W^2}\, d\Psi \left(1 + \frac{W^2}{4a_o^2}\right) \quad .$$

Now using relations (32),

$$dZ = dx + i\,dy = \frac{\left(1 - \frac{W^2}{4a_o^2}\right)}{W^2} d\Phi\,(U + iV)$$

$$+ i\left(\frac{1 + \frac{W^2}{4a_o^2}}{W^2}\right) d\Psi\,(U + iV)$$

$$dZ = \frac{U + iV}{W^2}\,(d\Phi + i\,d\Psi) - \frac{W^2}{4a_o^2}\,(d\Phi - i\,d\Psi)$$

$$dZ = \frac{d\bar{F}}{\bar{W}} - \frac{W\,d\bar{F}}{4a_o^2} \quad . \tag{33}$$

This is the relation between the compressible hodograph plane and the physical incompressible plane.

Therefore, by knowing the analytic function $F(\bar{W})$, which is the incompressible condition, the velocity q in the compressible condition may be found by using equation (30). Also the coordinates of the point at which the velocity occurs may be found by equation (33), the density ratio by equation (31), and the pressure by equation (20). Using this method however, it is impossible to predict whether or not the chosen function

$F(\overline{W})$ will give the boundary shape and flow pattern required. This is the main difficulty of all the hodograph methods.

Transformation from the Incompressible Flow to the Compressible Flow. However, by using a method demonstrated by Bateman[8], the two flow fields are related by using equation (33), and the final shape of the body in the physical compressible plane may be pre-determined. In order to arrive at this transformation, it is convenient to let ξ and η be the coordinates in the complex plane ζ , in which the incompressible velocity components U and V are plotted.

The approximate shape of the resulting boundary may be found by starting with the function

$$F(\overline{W}) = \Phi + i\Psi = h(\zeta)W_1$$

which is the complex potential of the incompressible flow in the ζ plane. Then

$$dF = W_1 \frac{dh}{d\zeta}\, d\zeta = \overline{W} d\zeta \quad . \qquad (34)$$

---

[8] Bateman, H., "The Lift and Drag Functions for an Elastic Fluid in Two Dimensional Irrotational Flow," Proceedings of The National Academy of Science, 24:246-51, 1938.

Likewise,

$$d\bar{F} = W_1 \frac{d\bar{h}}{d\bar{\zeta}}\, d\bar{\zeta} = W\, d\bar{\zeta} \qquad , \qquad (35)$$

where the bar denotes the conjugate of the respective functions. Substituting (34) and (35) in (33), dz becomes

$$dZ = d\zeta - \frac{W_1}{4a_0^2} \frac{d\bar{h}}{d\zeta}\, d\zeta\, W_1$$

$$= d\zeta - \lambda \left(\frac{d\bar{h}}{d\zeta}\right)^2 d\zeta$$

where $\lambda = \frac{1}{4}\left(\frac{W_1}{a_0}\right)^2$. Therefore,

$$Z = \zeta - \lambda \int \left(\frac{d\bar{h}}{d\zeta}\right)^2 d\zeta \quad . \qquad (36)$$

Now it becomes clear that for moderate subsonic speeds (for which this method was developed) the incompressible flow and the compressible flow differ only by the factor λ . The correction factor λ may be obtained in terms of the free stream Mach number, M_1, by use of equation (29),

$$\lambda = \frac{1}{4}\left(\frac{W_1}{a_0}\right)^2$$

$$\lambda = \frac{q_1^2}{\left(a_0^2 + \sqrt{q_1^2 + a_0^2}\,\right)^2} \quad .$$

Then by equation (28)

$$\lambda = \frac{q_1^2}{\left(\sqrt{a_1^2 - q_1^2} + a_1^2\,\right)^2},$$

$$\lambda = \frac{M_1^{\,2}}{(\,1-M_1^2 \;+1\,)^2} \quad . \tag{37}$$

Values of λ for various Mach numbers are given in table I and are shown graphically in figure 3.

In order to make pratical application of the Karman-Tsien method, it is now necessary to obtain an expression for the pressure coefficient, C_p, in terms of W and λ . By definition

$$C_p = \frac{p - p_1}{\frac{1}{2}\rho_1 q_1^{\,2}}$$

and by equation (21),

$$C_p = \frac{2a_1^2}{q_1^{\,2}}\left(1 - \frac{\rho_1}{\rho}\right)$$

$$C_p = \frac{2a_1^{\,2}}{q_1^2} \quad 1 - \frac{4a_0^2 - W_1^{\,2}}{4a_0^2 + W_1^{\,2}} \cdot \frac{4a_0^2 + W^2}{4a_0^2 - W^2}$$

$$= \frac{(1+\lambda)^2}{2\lambda}\left[1 - \frac{1-\lambda}{1+\lambda} \cdot \frac{1+\lambda(W/W_1)^2}{1-\lambda(W/W_1)^2}\right]$$

$$= \frac{(1+\lambda)^2}{2\lambda}\left\{\frac{2\lambda - 2\lambda(W/W_1)^2}{(1+\lambda)\left[1-\lambda(W/W_1)^2\right]}\right\}$$

$$= (1+\lambda)\frac{1-(W/W_1)^2}{1-\lambda(W/W_1)^2} \quad . \tag{38}$$

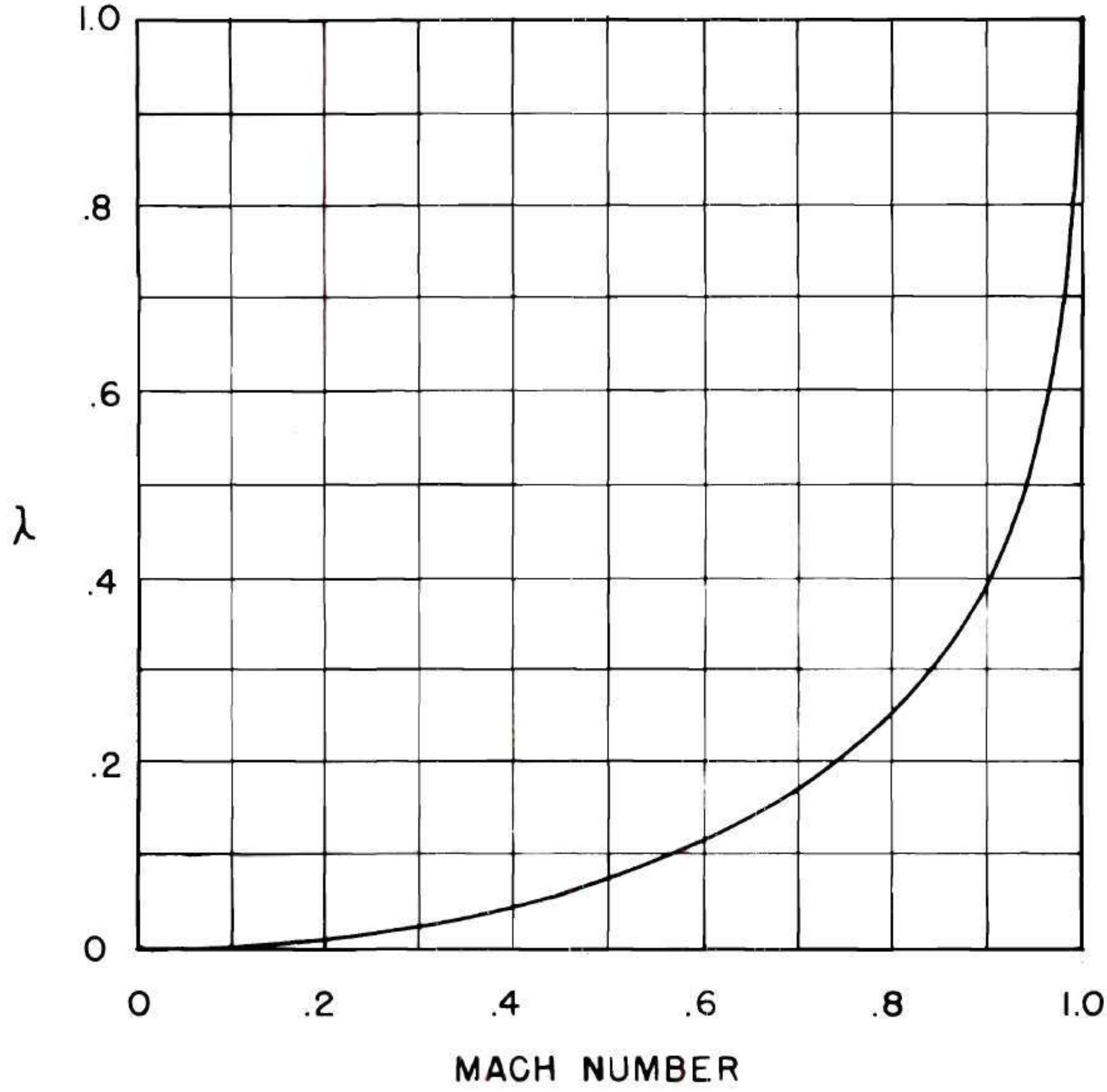


FIGURE 3, VARIATION OF λ WITH FREE STREAM MACH NUMBER

TABLE I

VALUES OF λ FOR VARIOUS FREE STREAM MACH NUMBERS

M_1	λ	M_1	λ
.05	.001	.55	.090
.10	.003	.60	.111
.15	.006	.65	.137
.20	.010	.70	.167
.25	.016	.75	.204
.30	.024	.80	.250
.35	.033	.85	.310
.40	.044	.90	.393
.45	.057	.95	.525
.50	.072	1.00	1.000

This method is applied to the problem of compressible flow about a circular cylinder and is given in the appendix. Comparisions are made with various experimental data available on the problem in an effort to show the value of this method.

THE METHOD OF CHARACTERISTICS

Basic Equations and Definitions. In the preceeding section it was shown how the non-linear equations of compressible fluid flow could be solved by transforming them into a plane in which q, the magnitude of the velocity vector, and θ, the angle of inclination of the vector q, were used as the independent variables. The outstanding advantage of this method, with the approximations introduced, is that it is easily applied to the case of completely subsonic flow. However, in the case of completely supersonic flow the problem is conveniently solved by use of the idea of "characteristic lines". These lines do not exist for the subsonic case and therefore the method is useful only for the supersonic case.

First, the continuity and Eulerian equations are combined with use of the condition of irrotational flow into a differential equation of the Monge[9] variety. Assuming a steady two dimensional isentropic flow which is also irrotational, the continuity equation is

$$(\rho u)_x + (\rho v)_y = 0 \quad , \qquad (39)$$

and the Eulerian equations (equations 3) may be written in the form

[9] Bateman, H., Partial Differential Equations of Mathematical Physics, (New York: The MacMillan Co., 1932), p 501.

$$u u_x + v u_y = -\frac{1}{\rho} p_x \tag{40}$$

$$u v_x + v v_y = -\frac{1}{\rho} p_y \quad . \tag{41}$$

Differentiating equation (39) and using the relation

$$\rho_x = \rho_p \, p_x = \frac{1}{a^2} p_x$$

gives

$$p_x \frac{u}{a^2} + u_x \rho + p_y \frac{v}{a^2} + v_y \rho = 0 \quad . \tag{42}$$

Sloving equations (40) and (41) for ($p_x u$ $p_y v$),

$$p_x u + p_y v = -\rho u v (u_y + v_x) - \rho (u^2 u_x + v^2 v_y$$

and substituting in equation (42), produces

$$u_x + v_y - \frac{1}{a^2} \quad u v (u_y + v_x) + u^2 u_x + v^2 v_y \quad = 0$$

$$u_x \left(1 - \frac{u^2}{a^2}\right) + v_y \left(1 - \frac{v^2}{a^2}\right) - \frac{uv}{a^2}\left(u_y + v_x\right) = 0 \quad . \tag{43}$$

Now, by the hypothesis of irrotational flow equation (6) is satisfied. That is

$$u_y - v_x = 0 \; . \tag{6}$$

Therefore, the potential function Φ exists such that

$$\Phi_x = u \quad \Phi_y = v \quad .$$

Using these equations, equation (43) becomes

$$\left(1-\frac{u^2}{a^2}\right)\Phi_{xx}+\left(1-\frac{v^2}{a^2}\right)\Phi_{yy}-\frac{2uv}{a^2}\Phi_{xy}=0\ . \qquad (44)$$

This is a partial differential equation with the solution given by the Monge equations. Three types of equation (44) exist: elliptic, parabolic, and hyperbolic. The type depends on the value of the equation

$$\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{v^2}{a^2}\right)-\frac{u^2v^2}{a^4}=1-\frac{u^2+v^2}{a^2}$$
$$=1-\frac{q^2}{a^2}\ , \qquad (45)$$

since $q^2 = u^2 + v^2$. If equation (45) is less, equal, or greater than zero the equation is of the hyperbolic, parabolic, or elliptic type respectively. That is, equation (44) is of the hyperbolic type for completely supersonic flow, parabolic for sonic flow, and elliptic for subsonic flow.

Since equation (44) is a second order partial differential equation, the solution will not be unique. However, at some point the two solutions will be coincident. At this point the solutions are discontinuous and the locus of these "discontinuities or singularities" is called the characteristic of equation (44).

Equations of The Characteristic Lines. Consider a two dimensional flow around a corner of small deviation, such that an isentropic expansion takes place. This disturbance, caused by the deviation, is propagated throughout the fluid in a straight line called the Mach line, and is inclined at the Mach angle to the original flow direction. Since the flow is expanded, the direction and magnitude of the velocity must change. Consequently the value of the potential must change also. In the region upstream of the disturbance the potential is $\Phi = \Phi_1$; in the region downstream, $\Phi = \Phi_2$. But since the flow is continuous, along the Mach line Φ_1 must equal Φ_2 and the first derivatives must have the same absolute value. That is, the two solutions of equation (44) are coincident along the Mach line and it can be concluded that the Mach line is the characteristic line of equation (44). Therefore it is possible to determine the equations of the characteristic lines on the basis of the physical meaning of the lines.

In order to arrive at an expression for the characteristic lines, consider again the case of an infinitely small deviation causing a disturbance in the form of a plane Mach wave and inclined at the angle μ. Where

$$\mu = \sin^{-1}\frac{1}{M} \quad .$$

If the solutions of equation (44) are represented by the lines l_1 and l_2 (figure 4), these curves will intersect at some point P on the Mach line. Tangents drawn to the lines l_1 and l_2 at the point P will be at the angle μ to the velocity vector q which is inclined at an angle θ to the positive x axis. Then

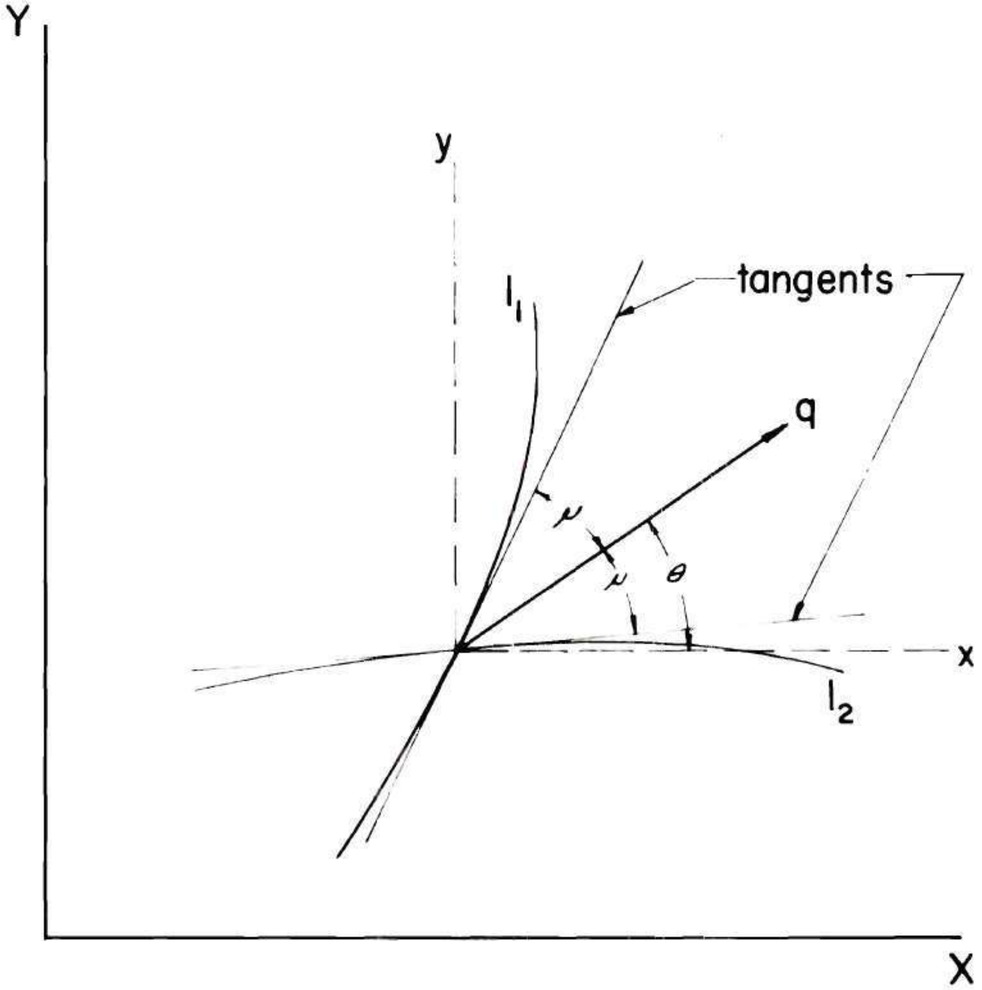


FIGURE 4, SOLUTIONS REPRESENTED BY THE CHARACTERISTICS LINES if dl is an element of length along the line l, by geometry of figure 4 the following equations may be deduced.

$$\left(\frac{dx}{dy}\right)_{l_1} = \tan(\mu+\theta)$$

$$\left(\frac{dx}{dy}\right)_{l_1} = \tan(\mu-\theta) \quad . \qquad (46)$$

Also by inspection

$$\cos\theta = \frac{u}{q} \qquad \sin\theta = \frac{v}{q}$$

$$\cos(\mu+\theta) = \frac{dx}{dl} \quad \sin(\mu+\theta) = \frac{dy}{dl}$$

$$\sin\mu = \frac{1}{M} = \frac{a}{q} \quad .$$

Expanding sin $(\mu+\theta)$ and substituting the above relations gives

$$\frac{dx}{dl} = (\cos\mu)\frac{u}{q} - \frac{a}{q}\frac{v}{q}$$

$$\frac{dy}{dl} = \frac{au}{q^2} + (\cos\mu)\frac{v}{q} \quad .$$

Combining and eliminating cos μ ,

$$\frac{dx}{dl} = \left(\frac{dy}{dl} - \frac{a}{q}\frac{u}{q}\right)\frac{u}{v} - \frac{a}{q}\frac{v}{q}$$

or

$$\frac{dx}{dl} - \frac{dy}{dl}\frac{u}{v} = -\frac{a}{q^2}\left(\frac{u^2}{v} + v\right) \quad .$$

Multiplying by v and dividing by a,

$$\frac{v}{a}\frac{dx}{dl} - \frac{dy}{dl}\frac{u}{a} = -1 \quad . \tag{47}$$

Also

$$(dx)^2 + (dy)^2 = (dl)^2 ,$$

or

$$\left(\frac{dx}{dl}\right)^2 + \left(\frac{dy}{dl}\right)^2 = 1 \quad . \tag{48}$$

Now equating equation (48) to the square of equation (47),

$$\left(\frac{dx}{dl}\right)^2 + \left(\frac{dy}{dl}\right)^2 = \left(\frac{u}{a}\right)^2\left(\frac{dy}{dl}\right)^2 - \frac{2uv}{a^2}\frac{dx\,dy}{(dl)^2} + \left(\frac{v}{a}\right)^2\left(\frac{dx}{dl}\right)^2,$$

or

$$\left(1-\frac{v^2}{a^2}\right)\left(\frac{dx}{dl}\right)^2 + \left(1-\frac{u^2}{a^2}\right)\left(\frac{dy}{dl}\right)^2 + \frac{2uv}{a^2}\frac{dx\,dy}{(dl)^2} = 0 \quad ,$$

which becomes finally

$$\left(1-\frac{u^2}{a^2}\right)\left(\frac{dy}{dx}\right)^2 + \frac{2uv}{a^2}\frac{dy}{dx} + \left(1-\frac{v^2}{a^2}\right) = 0 \quad . \tag{49}$$

This is the equation of the characteristic lines in Cartesian coordinates.

Letting

$$H = \left(1-\frac{u^2}{a^2}\right), \quad L = \left(1-\frac{v^2}{a^2}\right), \quad K = -\frac{uv}{a^2} \tag{50}$$

equation (49) becomes a quadratic with the solutions

$$\Gamma_1 = \frac{K}{H} - \frac{1}{H}\sqrt{K^2 - HL}$$

$$\Gamma_2 = \frac{K}{H} + \frac{1}{H}\sqrt{K^2 - HL}$$

(51)

where

$$\Gamma = \frac{dy}{dx} \quad .$$

Therefore it is now evident that if the flow is subsonic (i.e., equation 44 is elliptic) the value of $\sqrt{K^2 - HL}$ is imaginary and the lines do not exist.

<u>Characteristic Lines in the Hodograph Plane</u>. Using equations (50), equation (44) may be written as

$$1 + u_x + L v_y + 2K v_x = 0 \quad . \qquad (52)$$

The change along the characteristic lines of the velocity components is

$$du = u_x\,dx + u_y\,dy = (u_x + \Gamma dy)\,dx$$

$$dv = v_x\,dx + v_y\,dy = (v_x + \Gamma v_y)\,dx$$

or

$$u_x = \frac{du}{dx} - \Gamma u_y$$

$$v_x = \frac{dv}{dx} - \Gamma v_y \quad .$$

By the hypothesis of irrotational flow u_y v_x, and by combining u_x and v_x,

$$u_x = \frac{du}{dx} - \lambda \frac{dv}{dx} + \lambda^2 v_y \quad . \tag{53}$$

Substituting equation (53) into (52),

$$H\left(\frac{du}{dx} - \lambda \frac{dv}{dx} + \lambda^2 v_y\right) + L v_y + 2K\left(\frac{dv}{dx} - \lambda v_y\right),$$

which becomes

$$H\frac{du}{dx} + (2K - \lambda H)\frac{dv}{dx} + \left(H\lambda^2 - 2K\lambda + L\right) v_y = 0 \, . \tag{54}$$

Now λ is defined as the solution of the quadratic equation

$$H\lambda^2 - 2K\lambda + L = 0 \quad ,$$

and solving the two solutions (equations 51) together

$$2K - H\lambda_1 = H\lambda_2 \quad ,$$

which when combined with equation (54) gives the two characteristics

$$\begin{aligned} du + \lambda_1 \, dv = 0 \quad & (a.) \\ du + \lambda_2 \, dv = 0 \quad & (b.) \; . \end{aligned} \tag{55}$$

The equations for du and dv in terms of the coordinates θ and q may be had by inspection of figure 5. In figure 5a, q changes in magnitude only. Therefore,

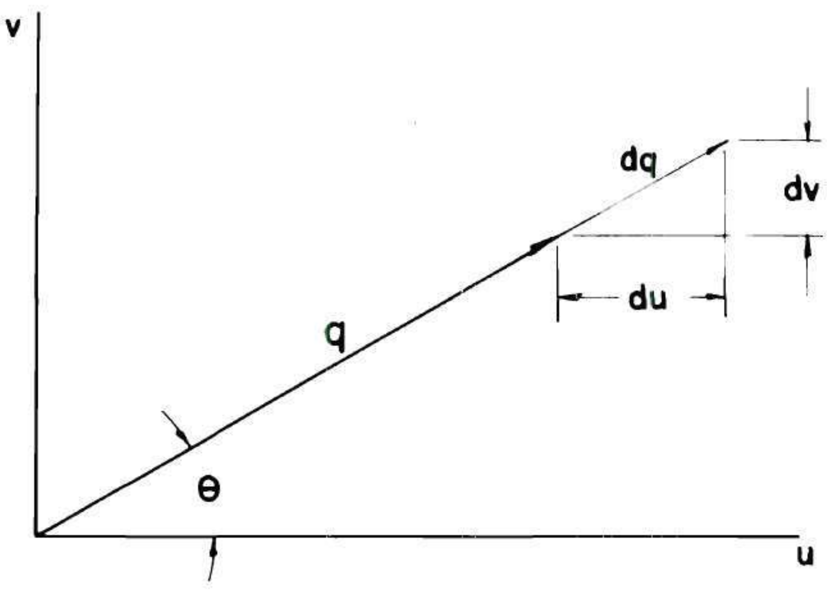


Figure 5a

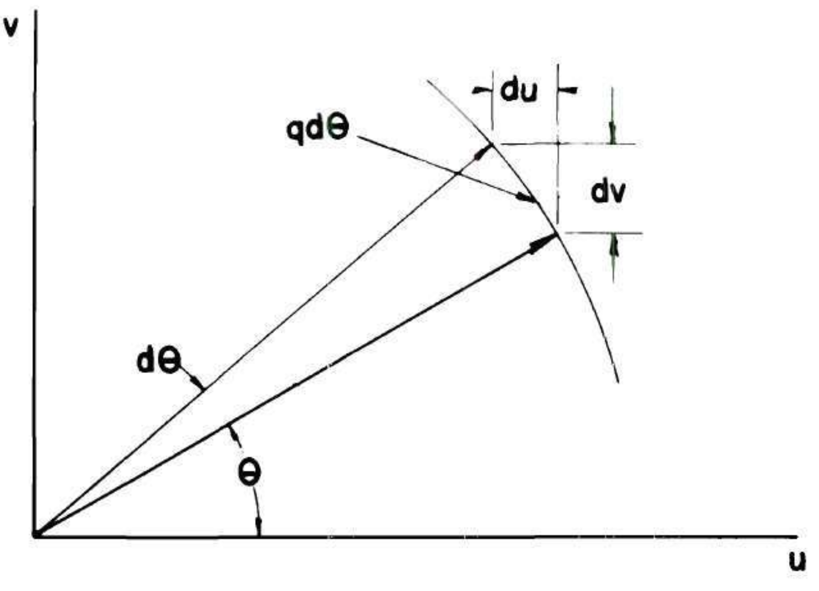


Figure 5b

FIGURE 5, POSSIBLE VARIATIONS OF q

$$du = -q\,d\theta\,\sin\theta$$

$$dv = q\,d\theta\,\cos\theta \quad .$$

In figure 5b, q changes in direction only. Therefore,

$$du = -q\,d\theta\,\sin\theta$$

$$dv = q\,d\theta\,\cos\theta \; .$$

Combining the two changes gives

$$du = dq\cos\theta - q\sin\theta\,d\theta$$

$$dv = dq\sin\theta + q\cos\theta\,d\theta \qquad (56)$$

Substituting equations (56) and (46) in (55), the following expressions are obtained: (equation 55a first)

$$dq\cos\theta - q\sin\theta\,d\theta + \lambda_1\,dq\sin\theta + \lambda_1\,q\cos\theta\,d\theta = 0$$

$$\frac{dq}{d\theta} = -q\,\frac{(\lambda_1 - \tan\theta)}{(1 + \lambda_1\tan\theta)} \quad .$$

By equations (46)

$$\frac{dq}{d\theta} = -q\,\frac{\dfrac{\tan\theta - \tan\mu}{1 + \tan\theta\tan\mu} - \tan\theta}{1 + \dfrac{\tan^2\theta - \tan\mu\tan\theta}{1 + \tan\theta\tan\mu}} \quad .$$

Substituting the relations for the functions of μ and θ obtained previously,

$$\frac{dq}{d\theta} = -q\left[\frac{\frac{v}{u} - \frac{1}{\sqrt{M^2-1}}}{1 + \frac{v}{u\sqrt{M^2-1}}} - \frac{v}{u}\right] \div \left[1 + \frac{\left(\frac{v}{u}\right)^2 - \frac{v}{u\sqrt{M^2-1}}}{1 + \frac{v}{u\sqrt{M^2-1}}}\right] .$$

Simplifying,

$$\frac{dq}{d\theta} = -q\left[\frac{-q^2}{u^2\sqrt{M^2-1} + uv}\right] \div \left[\frac{q^2\sqrt{M^2-1}}{u^2\sqrt{M^2-1} + uv}\right]$$

$$\frac{dq}{d\theta} = -q\,\frac{-q^2}{q^2\sqrt{M^2-1}}$$

$$\frac{dq}{d\theta}\sqrt{M^2-1} = q \quad .$$

Likewise from equation 55b

$$\frac{dq}{d\theta}\sqrt{M^2-1} = -q \quad .$$

Therefore the two equations may be written as

$$\frac{dq}{d\theta}\sqrt{M^2-1} = \pm q$$

$$\pm d\theta = \frac{dq}{q}\sqrt{M^2-1} \quad . \tag{57}$$

In order to obtain dq/q in terms of M so that equation (57) may be integrated, consider the energy equation in the form

$$\frac{q^2}{2} + \frac{a^2}{\gamma-1} = \frac{a_0^2}{\gamma-1} \quad .$$

Upon substituting the relation,

$$\frac{a_0^2}{a^2} = 1 + \frac{\gamma-1}{2} M^2$$

for a^2 it becomes

$$\frac{q^2}{2} + \frac{a_0^2}{(\gamma-1)\left(1+\frac{\gamma-1}{2}M^2\right)} = \frac{a_0^2}{\gamma-1} \quad .$$

Simplifying,

$$\frac{q^2}{2} = \frac{a_0^2}{\gamma-1}\left[\frac{\frac{\gamma-1}{2}M^2}{1+\frac{\gamma-1}{2}M^2}\right] \quad ,$$

or

$$\frac{q^2}{2}\left[\frac{2}{(\gamma-1)M^2} + 1\right] = \text{Const.}$$

Differentiating with respect to q gives,

$$\left[\frac{2}{(\gamma-1)M^2} + 1\right] q - q^2\left[\frac{2}{(\gamma-1)M^3}\frac{dM}{dq}\right] = 0$$

$$\frac{dM}{dq} = \frac{q\left[1+\frac{2}{(\gamma-1)M^2}\right](\gamma-1)M^3}{2q^2} \quad .$$

Then

$$\frac{dM}{M} = \frac{dq}{q}\left[\frac{(\gamma-1)M^2+2}{2}\right]$$

$$\frac{dM}{M} = \frac{dq}{q}\left[1+\frac{\gamma-1}{2}M^2\right] . \tag{58}$$

Now with equation (58), (57) becomes

$$\pm d\theta = \frac{\sqrt{M^2-1}}{M\left(1+\frac{\gamma-1}{2}M^2\right)}\,dM$$

or

$$\pm d\theta = \frac{M^2-1}{M\sqrt{M^2-1}\left(1+\frac{\gamma-1}{2}M^2\right)}\,dM .$$

Substituting

$$M^2 = \left(\frac{\gamma+1}{2}-\frac{\gamma-1}{2}\right)M^2 ,$$

$d\theta$ becomes

$$\pm d\theta = \frac{\frac{\gamma+1}{2}M^2 - \frac{\gamma-1}{2}M^2 - 1}{M\sqrt{M^2-1}\left(1+\frac{\gamma-1}{2}M^2\right)}$$

$$\pm d\theta = \frac{\gamma+1}{2}\,\frac{M\,dM}{\sqrt{M^2-1}\left(1+\frac{\gamma-1}{2}M^2\right)} - \frac{dM}{M\sqrt{M^2-1}} . \tag{59}$$

Making the following substitution in order to facilate integration,

$$\sin\phi = \frac{\sqrt{M^2-1}}{M}$$

$$\sec\phi = M$$

$$dM = \sec\phi\tan\phi = \frac{\sin\phi}{\cos^2\phi}\,d\phi\,,$$

equation (59) becomes

$$\pm\theta = \frac{\gamma+1}{2}\int \frac{\frac{\sin\phi}{\cos^2\phi\,d\phi}}{\sin\phi\;\;2+\frac{(\gamma-1)\sec^2\phi}{2}} - \int\frac{dM}{M\sqrt{M^2-1}}$$

$$\pm\theta = \gamma+1\int\frac{d\phi}{2\cos^2\phi+(\gamma-1)} - \int\frac{dM}{M\sqrt{M^2-1}}\;.$$

Now performing the integration,

$$\pm\theta = (\gamma+1)\left[\frac{1}{\sqrt{\gamma^2-1}}\tan^{-1}\frac{\sqrt{\gamma^2-1}}{\gamma+1}\sqrt{M^2-1}\right] - \cos^{-1}\frac{1}{M} + \text{const.}$$

$$\pm\theta = \sqrt{\frac{\gamma+1}{\gamma-1}}\;\tan^{-1}\sqrt{\frac{\gamma-1}{\gamma+1}}\;\sqrt{M^2-1} + \text{const}\;. \tag{60}$$

Equation (60) gives the variation of the Mach number along the characteristic line as a function of the direction of the velocity along the line. With the value of the Mach number it is possible to determine all the flow quantities by use of the energy equation. This method, although set up for the case of an expansion, may also be used in the case of a compression as long as the disturbances remain small and no finite discontinuities such as a shock occur.

In order to fully understand the geometrical meaning of equation (60), it is convenient to introduce two quantities: (1) q_1, the limiting velocity or the maximum velocity attainable by isentropically expanding the gas through a nozzle, (2) q_c, the critical velocity or the velocity at which the Mach number is unity. Then by writing (60) in terms of q_c and q_1 it becomes evident that they are the limits of the characteristic lines.

To obtain the necessary relations for q_c and q_1, consider flow from a tank at a pressure p_o and a density ρ_o, being expanded isentropically through a nozzle. At the critical velocity $q = a = q_c$. Then the energy equation (equation 4) may be written as

$$q_c^2 + \frac{2q_c^2}{\gamma-1} = \frac{2\gamma}{\gamma-1}\frac{p_o}{\rho_o}$$

$$q_c^2\left(1+\frac{2}{\gamma-1}\right) = \frac{2\gamma}{\gamma-1}\frac{p_o}{\rho_o}$$

$$q_c^2 = \frac{2\gamma}{\gamma+1}\frac{p_o}{\rho_o} \qquad (61)$$

The velocity at any point, given by (equation 5)

$$q^2 = \frac{2\gamma}{\gamma-1}\frac{p_o}{\rho_o}\left[1-\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}}\right]$$

attains its maximum value when p= 0. Therefore the limiting velocity becomes

$$q_L^2 = \frac{2\gamma}{\gamma-1}\ \frac{p_o}{\rho_o} \tag{62}$$

and inserting equation (61),

$$q_L^2 = \frac{\gamma+1}{\gamma-1} q_c^2 \ . \tag{63}$$

These relations (61 and 62) are constant since p_o/ρ_o is constant in an isentropic process. Now using (62) in equation (5), the result is

$$\frac{q^2}{q_L^2} = 1-\left(\frac{p}{p_o}\right)^{\frac{\gamma-1}{\gamma}} .$$

Since the constant may be replaced by

$$\frac{\gamma}{\gamma-1}\ \frac{p_o}{\rho_o} \ .$$

With the isentropic flow relation

$$\left(\frac{p_o}{p}\right)^{\frac{\gamma-1}{\gamma}} = 1 + \frac{\gamma-1}{2} M^2$$

$q^2/q_l{}^2$ becomes

$$\frac{q^2}{q_L^2} = 1 - \frac{2}{2+(\gamma-1)M^2}$$

$$\frac{q^2}{q_L^2} = \frac{2+(\gamma-1)M^2-2}{2+(\gamma-1)M^2}$$

or

$$\frac{q_L^2}{q^2} = \frac{2+(\gamma-1)M^2}{(\gamma-1)M^2} \quad . \tag{64}$$

Equation (60) may be written in terms of $\sin^{-1}$ instead of $\cos^{-1}$ and $\tan^{-1}$ giving the following equation of the characteristic lines.

$$\pm\theta = \frac{\pi}{2} - \sin^{-1}\sqrt{1-\frac{1}{M^2}} - \sqrt{\frac{\gamma+1}{\gamma-1}}\ \sin^{-1}\sqrt{\frac{(\gamma-1)(M^2-1)}{2+(\gamma-1)M^2}} + \text{const}.$$

Now inserting q_c and q_l as given by relations (63) and (64), the equation of the characteristics becomes

$$\pm\theta = \frac{\pi}{2} - \sin^{-1}\sqrt{1-\frac{1}{M^2}} - \frac{q_L}{q_c}\ \sin^{-1}\left(\frac{q}{q_L}\right)\sqrt{1-\frac{1}{M^2}} + \text{const} . \tag{65}$$

It is clear that[10]

$$0 \leq \left| \sin^{-1}\sqrt{1-\frac{1}{M^2}} \right| \leq 1$$

$$0 \leq \left| \sin^{-1}\sqrt{1-\frac{1}{M^2}}\,\frac{q}{q_c} \right| \leq 1 \quad .$$

---

[10] These relations also bear out the fact that the characteristics do not exist for the case of subsonic flow. If M were less than one, the value of the radical would no longer be a real number. Therefore the equations become meaningless.

Therefore when the $\sin^{-1}$ terms are equal to their maximum value of unity, q must be equal to q_1, since $p = 0$ by definition of the limiting velocity and $M \to \infty$. Likewise, when the $\sin^{-1}$ terms are equal to their minimum value of zero, M must equal to unity and also by definition, q must equal q_c. Then equation (60) represents a curve which is confined to the region between q_1 and q_c.

When equation (60) is plotted in the hodograph plane the result is an epicycloid with its origin on the circle, $q = q_c$, (figure 6) and tangent to the circle $q = q_1$. There will be a corresponding curve for every value of the constant which determines the position of the origin of the curve on the $q = q_c$ circle.

These curves are constructed by first constructing the circles $q = q_c$ and $q = q_1$ in the hodograph plane. Then by fixing the value of the constant the origin in determined. A circle with diameter $(q_1 - q_c)$ is made tangent to the $q = q_c$ circle at the origin and by rolling the $(q_1 - q_c)$ circle around the $q = q_c$ circle, two epicycloids are generated. The two possible directions of roll correspond to the plus and minus values of θ in equation (60). The minus representing the expansion, and the positive the compression. By repeating this process for various values of the constant two families of curves are generated, there being two curves for every constant chosen. These curves are shown in figure 7.

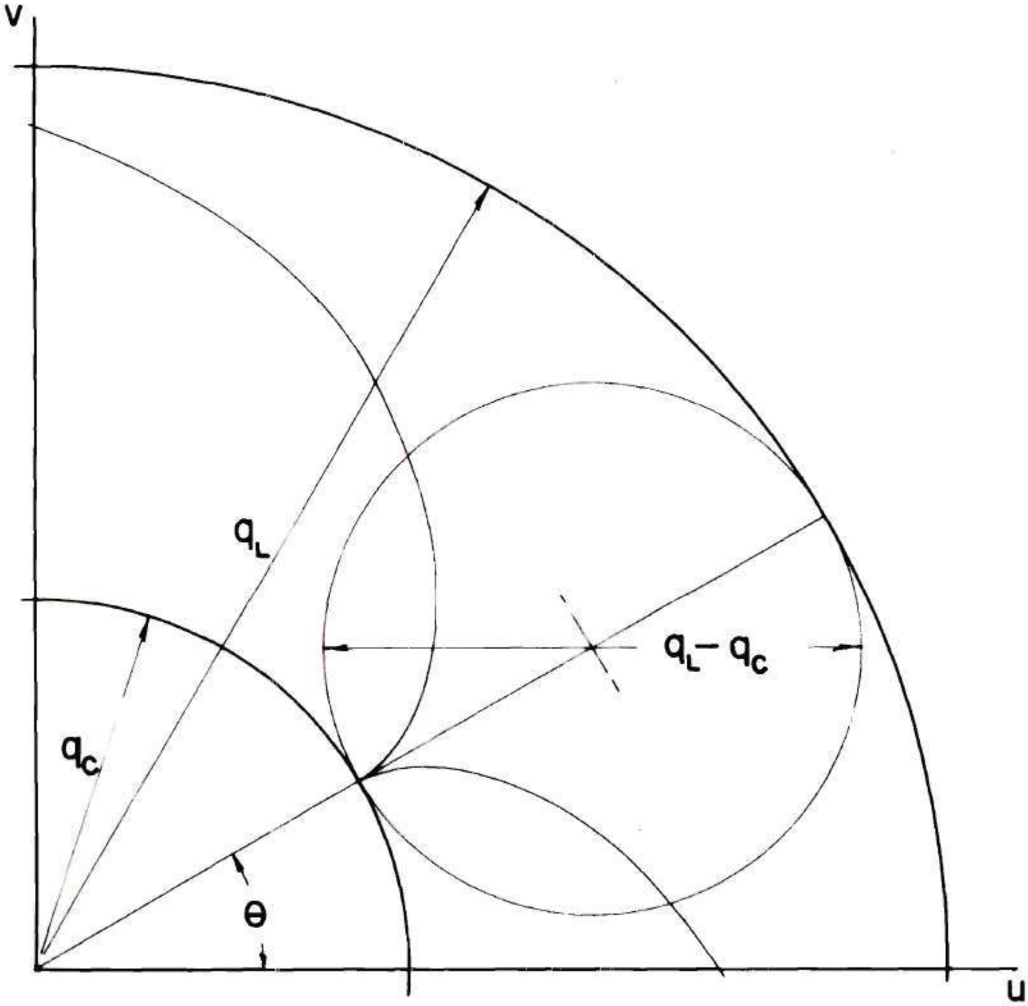


FIGURE 6, CONSTRUCTION OF CHARACTERISTIC DIAGRAM

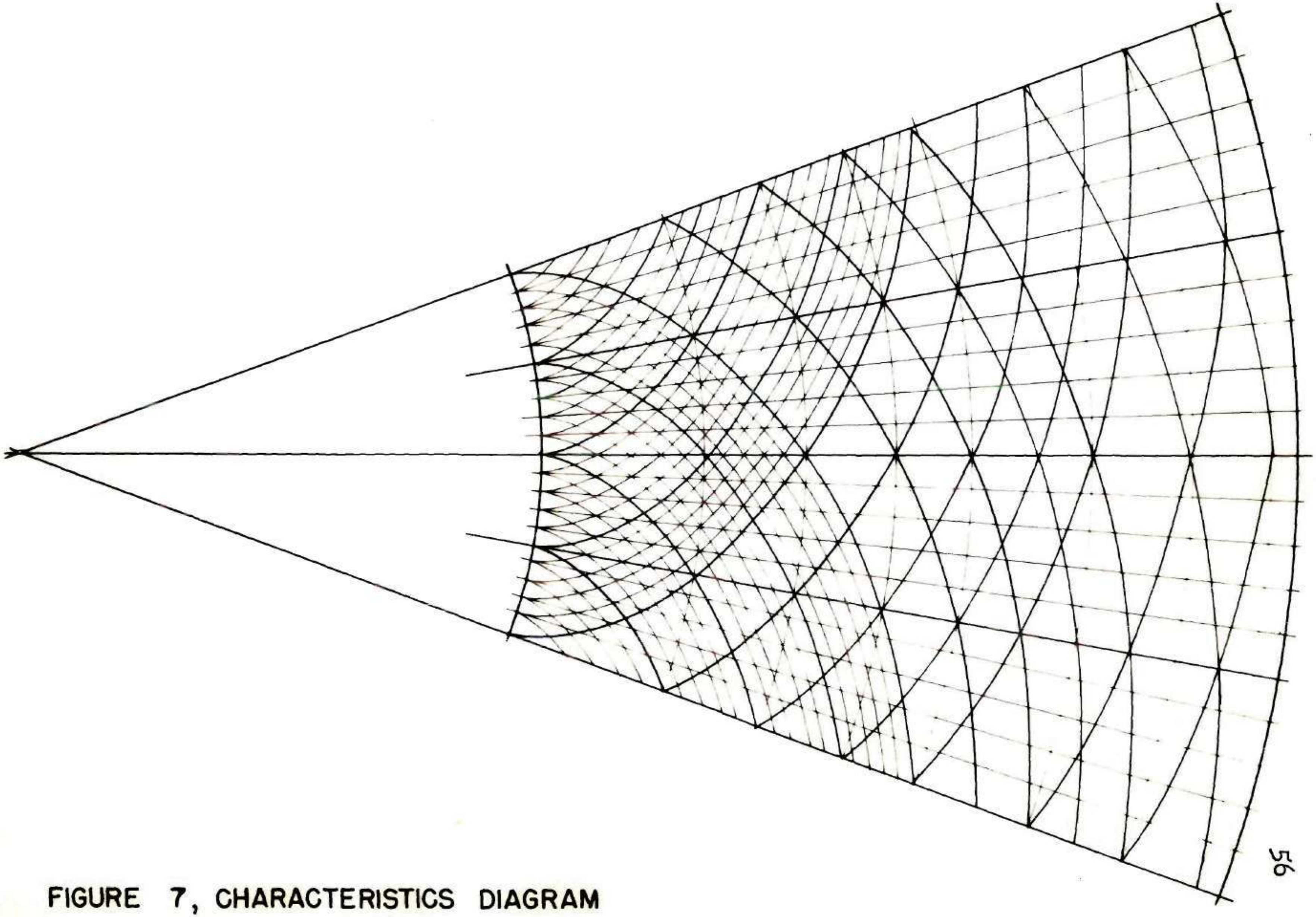


FIGURE 7, CHARACTERISTICS DIAGRAM

For every value of q in the region q_c to q_1 there are two curves which are symmetrical about the velocity vector passing through the point P, and are inclined at the angle μ to the vector. In order to determine the epicycloid curve for a given characteristic family it is necessary to return to equation (55).

From equation (55a), the first characteristic is,

$$\left(\frac{du}{dv}\right)_2 = -\Gamma_1 = -\left(\frac{dy}{dx}\right)_1 \qquad ,$$

and

$$\left(\frac{dy}{dx}\right)_1 \left(\frac{dv}{du}\right)_2 = -1 \qquad .$$

Likewise, from equation (55b)

$$\left(\frac{dy}{dx}\right)_2 \left(\frac{dv}{du}\right)_1 = -1 \qquad .$$

The above relations require that the tangent to the epicycloid of one family must be perpendicular to the characteristic of the other. Therefore, the tangent to the characteristic Γ_1 in the physical (xy) plane must be parallel to the normal to epicycloid 2 at the point P. From this condition it is possible to determine the correct family of epicycloids for each family of characteristics.

The epicycloid curves (figure 7) may be used to graphically determine the characteristics in the physical plane and also for the numerical calculations. A practical example of such calcilations is shown in the appendix.

OBLIQUE SHOCK FLOW

Introduction. In the section on the method of characteristics it was stated that the method could be applied to the case of a compression as well as an expansion, as long as the disturbances remain small. When these disturbances are small, the physical changes of the flow take place along the Mach lines and the enthalpy remains constant; that is, the process is isentropic. However, when the disturbances become larger and the Mach lines converge to form a finite discontinuity (figure 8) or shock wave, the enthalpy no longer remains constant and the process is not isentropic. Therefore, it is in order to arrive at a solution to such a flow condition. It is important to note that although the method of characteristics is not applicable through a shock, it is still of great importance since the flow on either side of the shock wave is isentropic. The following section is devoted to the solution of oblique shock flow.

Oblique Shock Equations. Assume now that there is a finite disturbance in a steady uniform stream, moving at a supersonic velocity, so that a discontinuity or shock wave is produced (figure 9). Let q be the velocity, with components u and v, and the subscripts 1 and 2 denote conditions in front and to the rear of the shock wave S, respectively. Writing the continuity equation for the flow normal to S gives

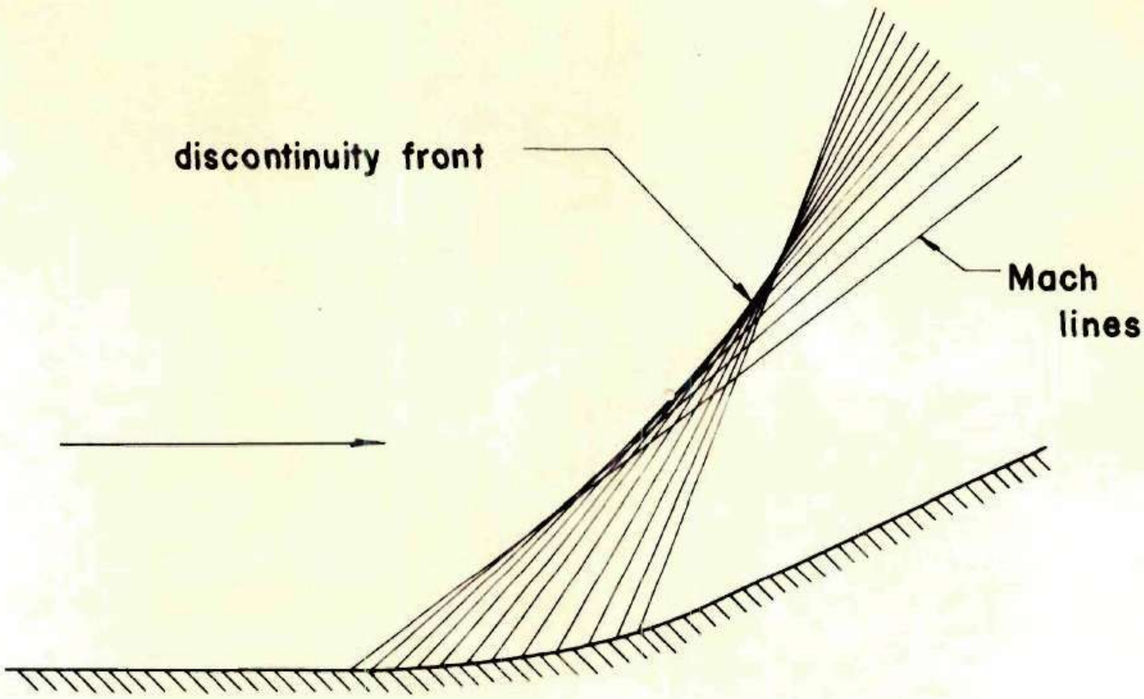


FIGURE 8, COMPRESSION OF A SUPERSONIC FLOW

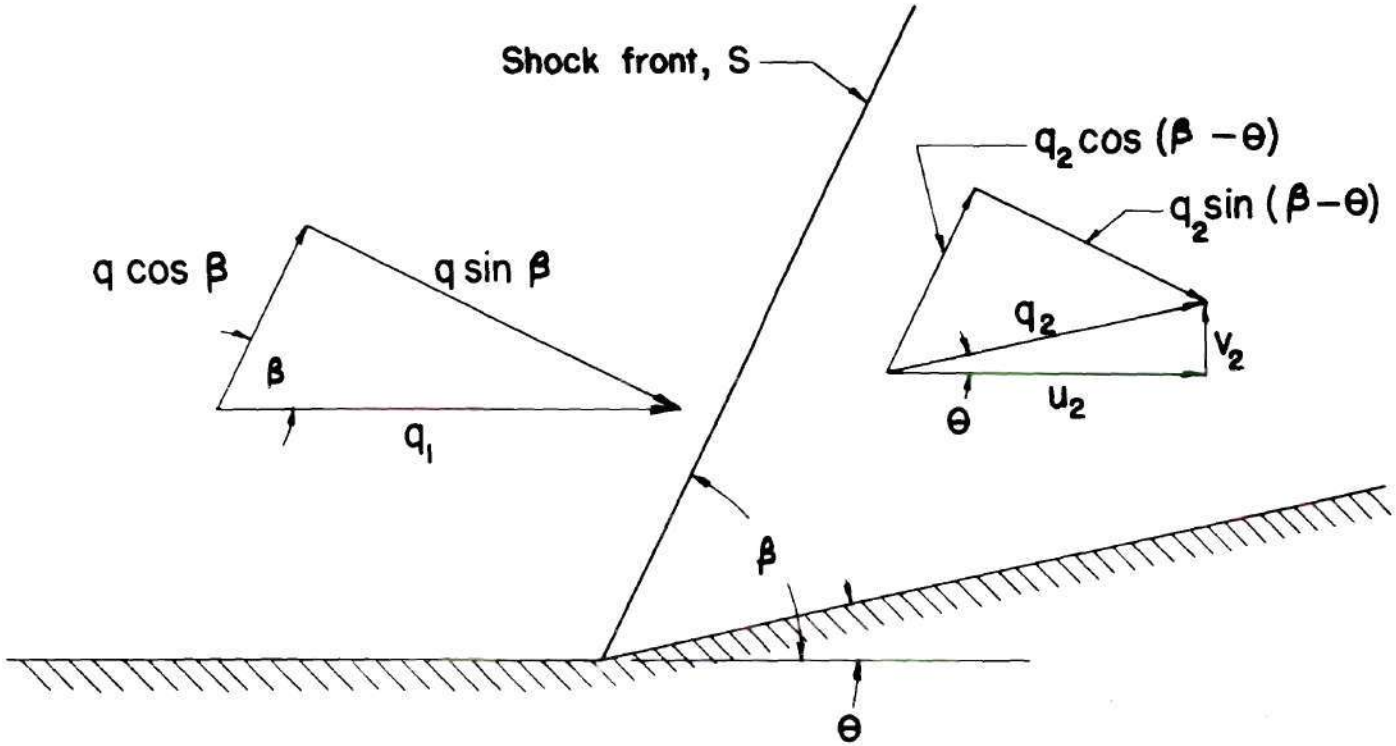


FIGURE 9, OBLIQUE SHOCK FLOW

$$\rho_1(q_1 \sin\beta) = \rho_2 q_2 \sin(\beta-\theta) \, . \tag{66}$$

The momentum equation normal to S is

$$p_1 + \rho_1 (q_1 \sin\beta)^2 = \rho_2 \left[q_2 \sin(\beta-\theta) \right]^2 + p_2 \, , \tag{67}$$

and parallel to S

$$\rho_1 q_1^2 \sin\beta \cos\beta = \rho_2 q_2^2 \sin(\beta-\theta) \cos(\beta-\theta) \, , \tag{68}$$

since the pressure gradient parallel to S is zero. By use of equation (66) it can be seen from equation (68) that the velocities parallel to S are equal. Or,

$$q_1 \cos\beta = q_2 \cos(\beta-\theta) \, .$$

Writing equation (4) between conditions 1 and 2,

$$\frac{u_1^2}{2} + \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} = \frac{(u_2^2 + v_2^2)}{2} + \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} \, , \tag{69}$$

which in terms of q_1 and q_2 becomes

$$\frac{(q_1 \sin\beta)^2}{2} + \frac{\gamma}{\gamma-1} \frac{p_1}{\rho_1} = \frac{q_2 \sin(\beta-\theta)}{2} + \frac{\gamma}{\gamma-1} \frac{p_2}{\rho_2} \, . \tag{70}$$

Dividing equation (68) by (66) gives

$$q_1 \cos\beta = q_2 \cos(\beta-\theta) \, ,$$

which upon expanding and substituting for sin θ and cosθ from figure 9, becomes

$$q_1 \cos\beta = q_2 \cos\beta \cos\theta + q_2 \sin\beta \sin\theta$$

$$= u_2 \cos\beta + v_2 \sin\beta \, .$$

Therefore,

$$\tan\beta = \frac{u_1 - u_2}{v_2}$$

or

$$v_2 = \frac{(u_1 - u_2)\cos\beta}{\sin\beta} \, . \qquad (71)$$

Equations (66) through (71) are the equations of oblique shock flow. Inspection of these equations shows that when any two of the parameters are known, all flow quantities are completely determined.

Solution of the Equations: Equations of Rankine and Hugoniot. Solution of the equations above depends on the parameters which are of most interest. One useful relation is the relation of the pressure to the density.

In order to arrive at this relation, first divide (67) by (66). The result is

$$q_1 \sin\beta + \frac{p_1}{\rho_1 q_1 \sin\beta} = q_2 \sin(\beta - \theta) + \frac{p_2}{\rho_2 q_2 \sin(\beta - \theta)} \, .$$

or

$$q_1 \sin\beta - q_2 \sin(\beta-\theta) = \frac{p_2}{\rho_2 q_2 \sin(\beta-\theta)} - \frac{p_1}{\rho_1 q_1 \sin\beta} .$$

Now multiplying by $(q_1 \sin\beta + q_2 \sin(\beta - \theta))$,

$$q_1 \sin^2\beta + \left[q_2 \sin(\beta-\theta)\right]^2$$

$$= \frac{q_1 p_2 \sin\beta}{\rho_2 q_2 \sin(\beta-\theta)} + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{q_2 p_1 \sin(\beta-\theta)}{\rho_1 q_1 \sin\beta} ,$$

and by use of equation (66) this becomes

$$(q_1 \sin\beta)^2 + q_2 \sin(\beta-\theta)^2 = \frac{p_2}{\rho_1} + \frac{p_2}{\rho_2} - \frac{p_1}{\rho_1} - \frac{p_1}{\rho_2}$$

$$= (p_1 - p_2)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) . \tag{72}$$

Substituting (72) in (70),

$$(p_1 - p_2)\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = \frac{2\gamma}{\gamma-1}\left(\frac{p_1}{\rho_1} - \frac{p_2}{\rho_2}\right)$$

and dividing by p_1 and collecting the terms containing p_2/p_1, this equation becomes

$$\frac{p_2}{p_1}\left[\frac{1}{\rho_1} + \frac{1}{\rho_2}\left(\frac{2\gamma}{\gamma-1} - 1\right)\right] = \frac{1}{p_1}\left(\frac{2\gamma}{\gamma-1} - 1\right) - \frac{1}{\rho_2} .$$

Which when solved for p_2/p_1 gives

$$\frac{p_2}{p_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{\rho_2}{\rho_1} - 1}{\frac{\gamma+1}{\gamma-1} - \frac{\rho_2}{\rho_1}} , \tag{73}$$

and when solved for ρ_2/ρ_1 gives

$$\frac{\rho_2}{\rho_1} = \frac{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1}{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}} , \tag{74}$$

Equations (73) and (74) are the equations of Rankine and Hugoniot and give ratio of the pressure or density before and after the shock in terms of the other.

Shock Polars. Since two of the parameters must be known before all flow quantities are determined, the solution of oblique shock flow lends itself nicely to graphical means. Therefore there are a number of curves, plotted by holding one parameter constant, which are very useful for the solution of oblique shock waves.

The first of these curves is a plot of the wave angle, β versus the deflection angle θ, with M_1 held constant. By use of equation (66), (67) may be written as

$$p_2 - p_1 = \rho_1 q_1^2 \sin^2\beta - \frac{\rho_1^2 q_1^2 \sin^2\beta}{\rho_2} .$$

$$p_2 - p_1 = \rho_1 q_1^{\,2} \sin^2\beta \left(1 - \frac{\rho_1}{\rho_2}\right) ,$$

or

$$q_1^{\,2} \sin^2\beta = \frac{p_2 - p_1}{\rho_1\left(1 - \frac{\rho_1}{\rho_2}\right)} . \tag{75}$$

Substitution of equation (74) in (75) produces,

$$q_1^{\,2} \sin^2\beta = \frac{(\gamma+1)p_2 + (\gamma-1)p_1}{2\rho_1}$$

or

$$q_1^{\,2} \sin^2\beta = \frac{\gamma p_1}{\rho_1} + \frac{\gamma+1}{2}\,\frac{(p_2 - p_1)}{\rho_1} . \tag{76}$$

From equation (66)

$$\frac{\rho_1}{\rho_2} = \frac{q_2 \sin(\beta-\theta)}{q_1 \sin\beta} = \frac{u_2 \sin(\beta-\theta)}{u_1 \sin\beta \cos\theta} . \tag{77}$$

But from equation (71)

$$\tan\beta = \frac{u_1 - u_2}{v_2} ,$$

and from figure 9

$$\cot\theta = \frac{u_2}{v_2} .$$

Therefore,

$$\frac{\tan\beta}{\cot\theta} = \frac{u_1 - u_2}{u_2}$$

and

$$u_2 = \frac{u_1 \cot\theta}{\tan\beta + \cot\theta} \quad .$$

Substituting this expression in (77),

$$\frac{\rho_1}{\rho_2} = \frac{\cot\theta \, \sin(\beta-\theta)}{\sin\beta \cos\theta \, (\tan\beta + \cot\theta)}$$

$$= \cot\beta \, \frac{\sin(\beta-\theta)}{\cos(\beta-\theta)} \quad .$$

Then

$$1 - \frac{\rho_1}{\rho_2} = \frac{\sin\theta}{\sin\beta \cos(\beta-\theta)} \quad . \qquad (78)$$

Now substituting (78) in (75),

$$\frac{p_2 - p_1}{\rho_1} = q_1^2 \, \frac{\sin\beta \sin\theta}{\cos(\beta-\theta)} \quad , \qquad (79)$$

and combining (79) and (76),

$$q_1^2 \sin^2\beta = a_1^2 + \frac{\gamma+1}{2} q_1^2 \, \frac{\sin\beta \sin\theta}{\cos(\beta-\theta)} \quad .$$

Introducing the Mach number this becomes,

$$M_1^2 \sin^2 \beta = 1 + \frac{\gamma+1}{2} M_1^2 \frac{\sin\beta \sin\theta}{\cos(\beta-\theta)} \quad . \tag{80}$$

Equation (80) gives the variation of β with θ. These curves are shown in figure 10 for various Mach numbers.

Another useful curve may be obtained by substituting equation (75) in (76). This gives

$$u_1^2 \sin^2\beta = a_1^2 + \frac{\gamma+1}{2} u_1^2 \sin^2\beta \left(1 - \frac{\rho_1}{\rho_2} \right) \quad . \tag{81}$$

A third curve may be obtained by combining equation (81) with (74), producing

$$M_1^2 \sin^2\beta = 1 + \frac{\gamma+1}{2} M_1^2 \sin^2\beta \left[1 - \frac{\frac{\gamma+1}{\gamma-1} + \frac{p_2}{p_1}}{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1} \right] ,$$

or

$$M_1^2 \sin^2\beta = 1 + \frac{\gamma+1}{2} M_1^2 \sin^2\beta \left[\frac{\left(\frac{\gamma+1}{\gamma-1} - 1\right)\left(\frac{p_2}{p_1} - 1\right)}{\frac{\gamma+1}{\gamma-1}\frac{p_2}{p_1} + 1} \right] . \tag{82}$$

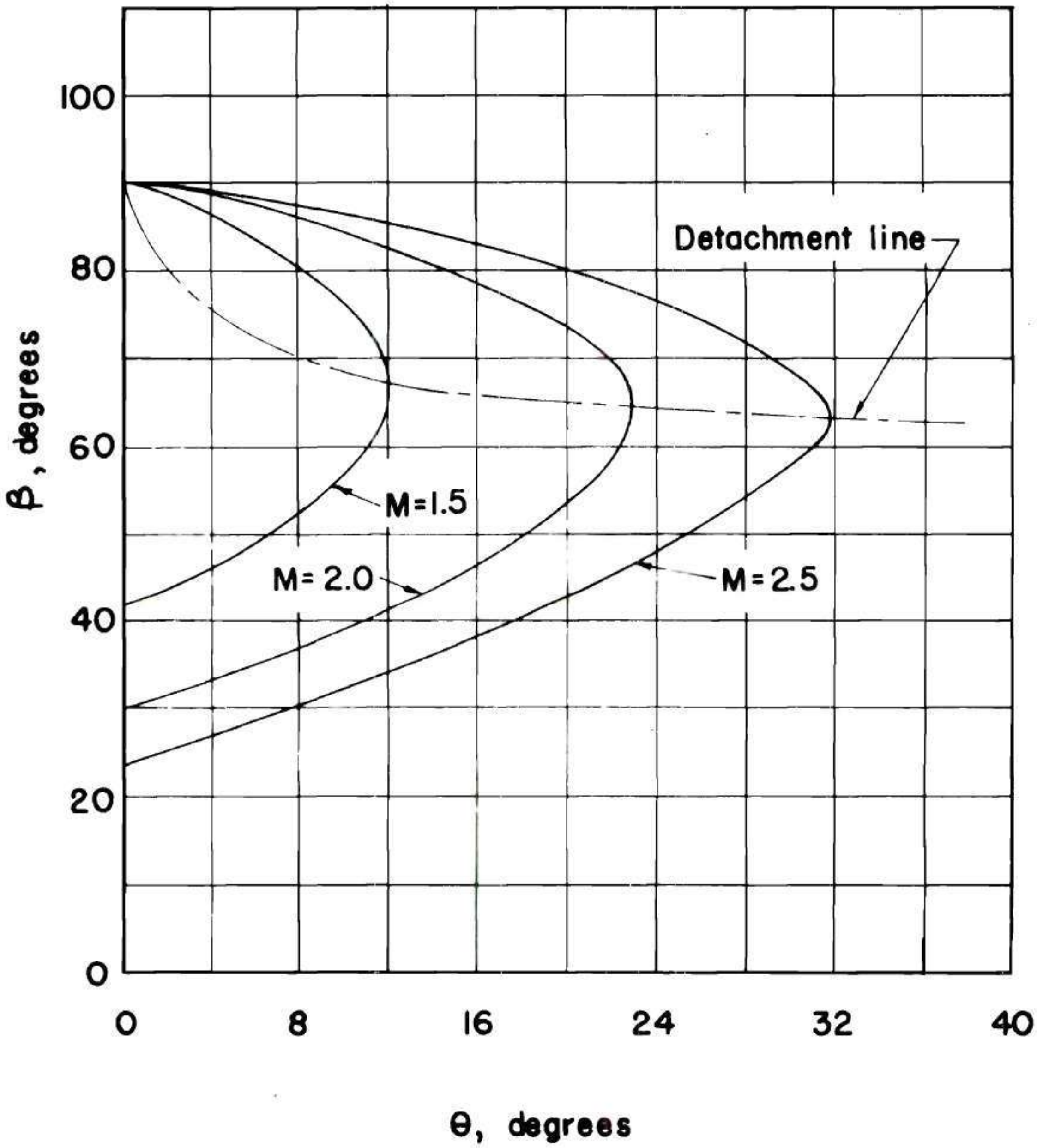


FIGURE 10, VARIATION OF β WITH θ

Equation (81) permits a graph of β versus ρ_2/ρ_1 and (82) gives β versus p_2/p_1 for various free stream Mach numbers. These equations are shown in figures 11 and 12 respectively. In addition to the graphs of equations (80), (81), and (82), a relation between p_2/p_1 and θ may be obtained by using equations (82) and (80) in parametric form. These curves are shown in figure 13.

The arrangements given above of the oblique shock equations are only four of the many possible combinations. The ones given are essentially the same equations obtained by Laitone[11] although they are derived and plotted in a different manner; the preference of the two groups depending on the value of the parameters given.

This section would not be complete without presenting the classical shock polar first introduced by Busemann[12]. The equation of this polar may be obtained by first substituting equation (66) in (67). This gives,

$$p_1 + \rho_1 q_1^2 \sin^2\beta = p_2 + \rho_1 u_1 \sin\beta \sqrt{(u_2^2 + v_2^2)}\ \sin(\beta - \theta) .$$

---

[11]Laitone, E. V., "Exact and Approximate Solutions of Two Dimensional Oblique Shock Flow," Journal of the Aeronautical Sciences, 14:25, January, 1947.

[12]Busemann, A., "Gasdynamik," Handbuch der Experimentalphysik, Vol. IV, Akademische Verlagsgesllschaft, Leipzig, 1931.

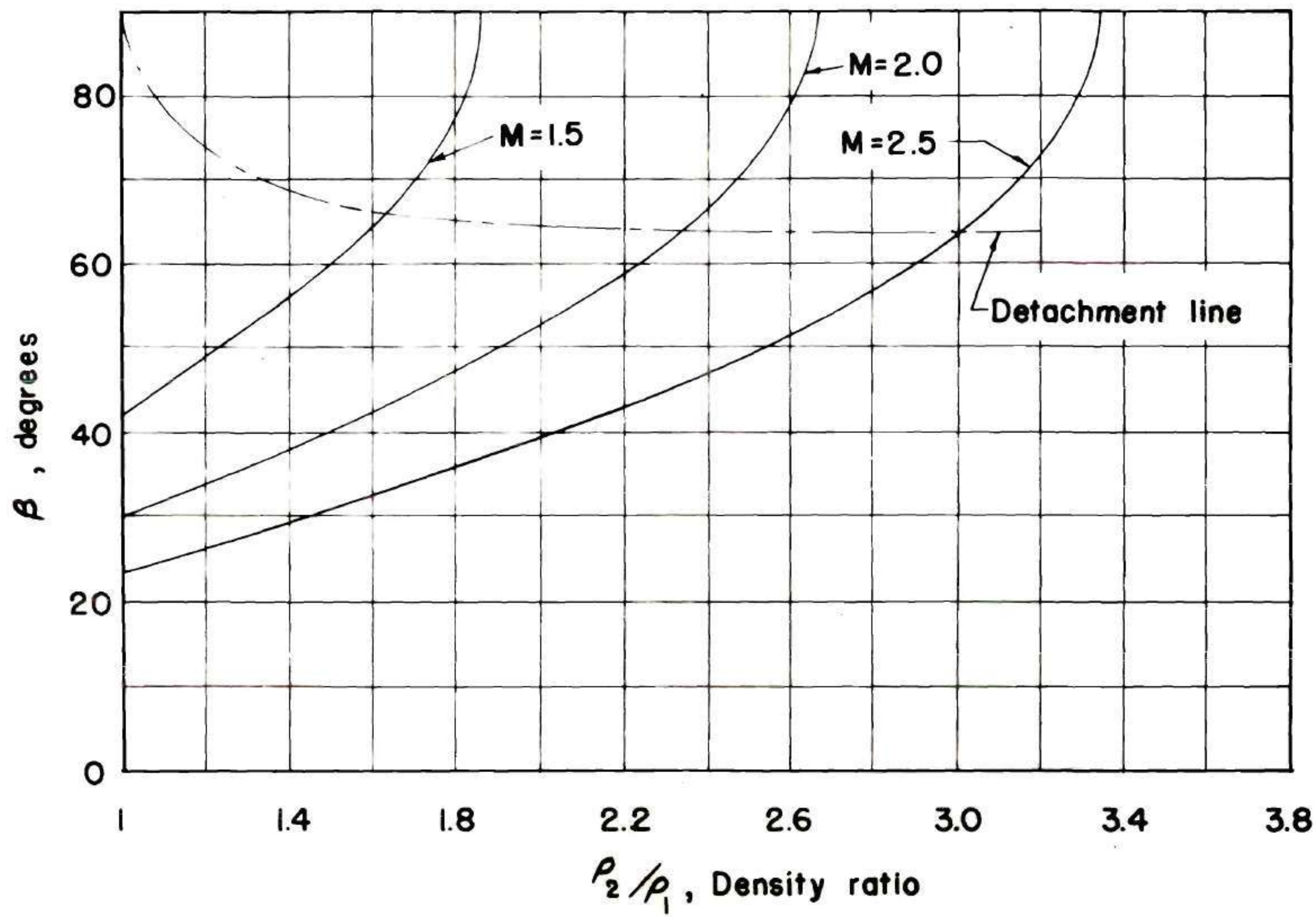


FIGURE 11, VARIATION OF DENSITY THROUGH SHOCK WAVES

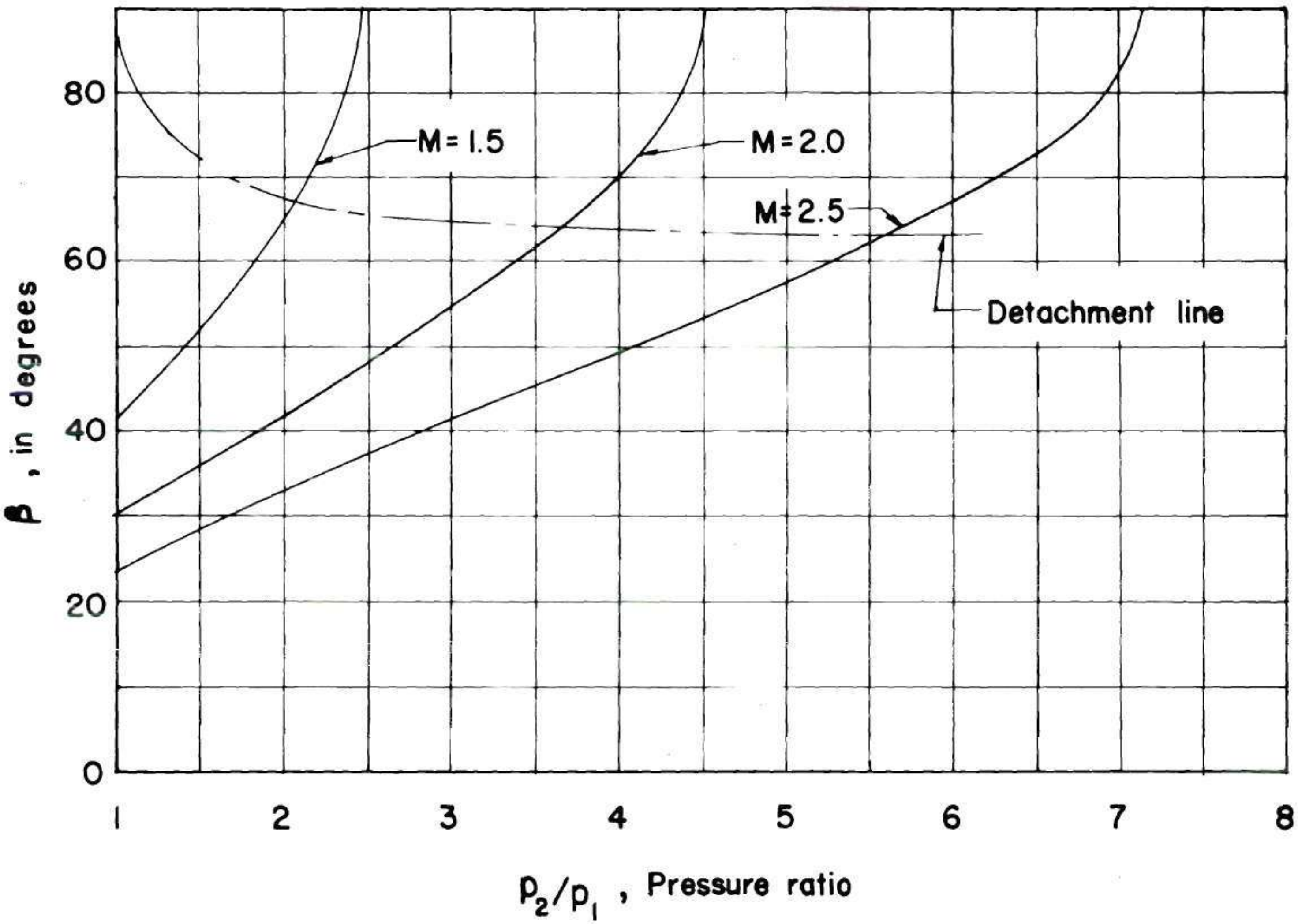


FIGURE 12, VARIATION OF PRESSURE THROUGH SHOCK WAVES

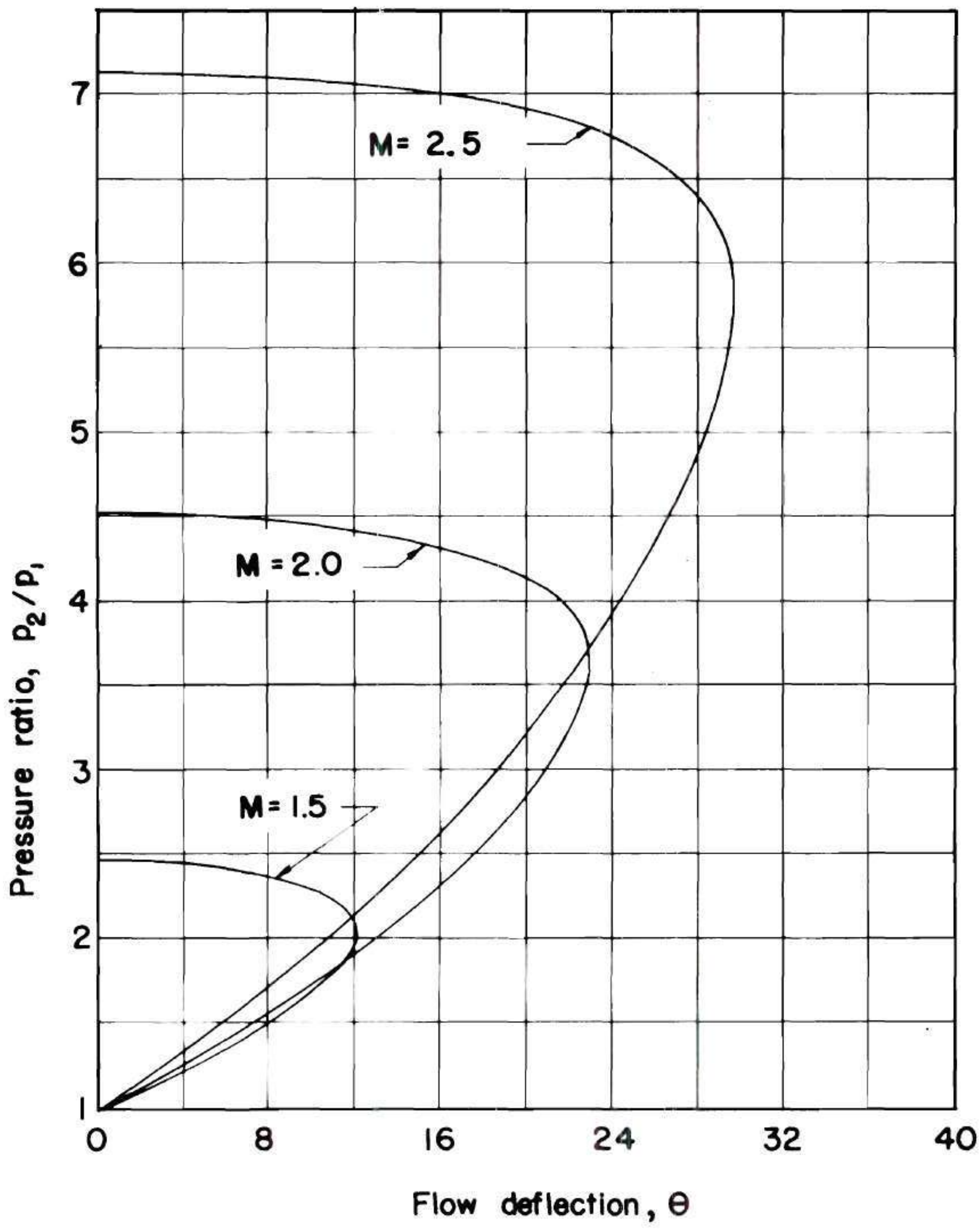


FIGURE 13, VARIATION OF PRESSURE WITH θ

Expanding sin (β - θ) and substituting from figure 9 for sin θ and cos θ gives,

$$\sin(\beta-\theta) = \frac{u_2}{\sqrt{u_2^2+v_2^2}}\ \sin\beta - \frac{v_2}{\sqrt{u_2^2+v_2^2}}\ \cos\beta .$$

Substituting the above in equation (83),

$$p_1 - p_2 = \rho_1 u_1 \sin\beta\ \left[(u_2-u_1)\sin\beta - v_2\cos\beta\right]$$

and then substituting equation (71) for v_2, equation (83) becomes

$$p_1 - p_2 = \rho_1 u_1\ \left[u_2\sin^2\beta - u_1\sin^2\beta - (u_1-u_2)\cos^2\beta\right]$$

$$p_1 - p_2 = \rho_1 u_1 \left(u_2\sin^2\beta + u_2\cos^2\beta - u_1\sin^2\beta - u_1\cos^2\beta\right),$$

or

$$p_2 = p_1 + \rho_1 u_1 (u_1 - u_2) \quad . \tag{84}$$

Writing equation (66) using the expression for sin (β -θ),

$$\rho_1 u_1 \sin\beta = \rho_2 \left(u_2\sin\beta - v_2\cos\beta\right).$$

Now with equation (71) this becomes

$$\rho_1 u_1 (u_1 - u_2) = \rho_2 u_2 (u_1 - u_2) - v_2^2 \rho_2 \quad . \tag{85}$$

Dividing equation (84) by (85) gives

$$\frac{p_2}{\rho_2 (u_2 u_1 - u_2^2 - v_2^2)} = \frac{p_1}{\rho_1 u_1 (u_1 - u_2)} + 1$$

or

$$\frac{p_1}{\rho_1} = \frac{p_2}{\rho_2} \frac{u_1(u_1 - u_2)}{(u_2 u_1 - u_2^2 - v_2^2)} - u_1(u_1 - u_2). \tag{86}$$

Now by writing equation (4) between condition 1 and the point where $q/a = 1$,

$$\frac{1}{2} q_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} = \frac{a^2}{2} + \frac{a^2}{\gamma - 1}$$

and calling this value of a, a critical, the above equation becomes

$$\frac{1}{2} q_1^2 + \frac{\gamma}{\gamma - 1} \frac{p_1}{\rho_1} = \frac{\gamma + 1}{2(\gamma - 1)} a_c^2 . \tag{87}$$

Then equation (69), with use of (87), becomes

$$\frac{u_2^2 + v_2^2}{2} + \frac{\gamma}{\gamma - 1} \frac{p_2}{\rho_2} = \frac{\gamma + 1}{2(\gamma - 1)} a_c^2 . \tag{88}$$

Now, by using the three equations, (86, 87, 88), p_1, p_2, ρ_1, and ρ_2 may be eliminated, leaving the equation of the shock polar.

In order to eliminate these quantities, first substitute equation (86) in (87), giving

$$\frac{p_2}{p_1} =$$

$$\frac{\left[(\gamma+1)\,a_c^2-(\gamma-1)\,u_2^2\right](u_1u_2-u_2^2-v_2^2)}{2\gamma u_1(u_1-u_2)}+u_2u_1-u_2^2-v_2^2 .$$

(89)

Then by substituting (89) in (88) the following equation is obtained.

$$v_2^{\,2}\left[\frac{(\gamma-1)}{2\gamma}-\frac{(\gamma+1)\,a_c^2-(\gamma-1)\,u_2^2}{2\gamma u_1(u_1-u_2)}-1\right]=$$

$$\frac{\left[(\gamma+1)\,a_c^2+(\gamma+1)\,u_2u_1\right](u_1-u_2)}{2\gamma u_1} .$$

Which when solved for v_2^2 gives

$$v_2^2=(u_1-u_2)^2\,\frac{u_1u_2-a_c^2}{\frac{2}{\gamma+1}\,u_1^2-u_1u_2+a_c^2} .$$

(90)

Now introducing the dimensionless velocities,

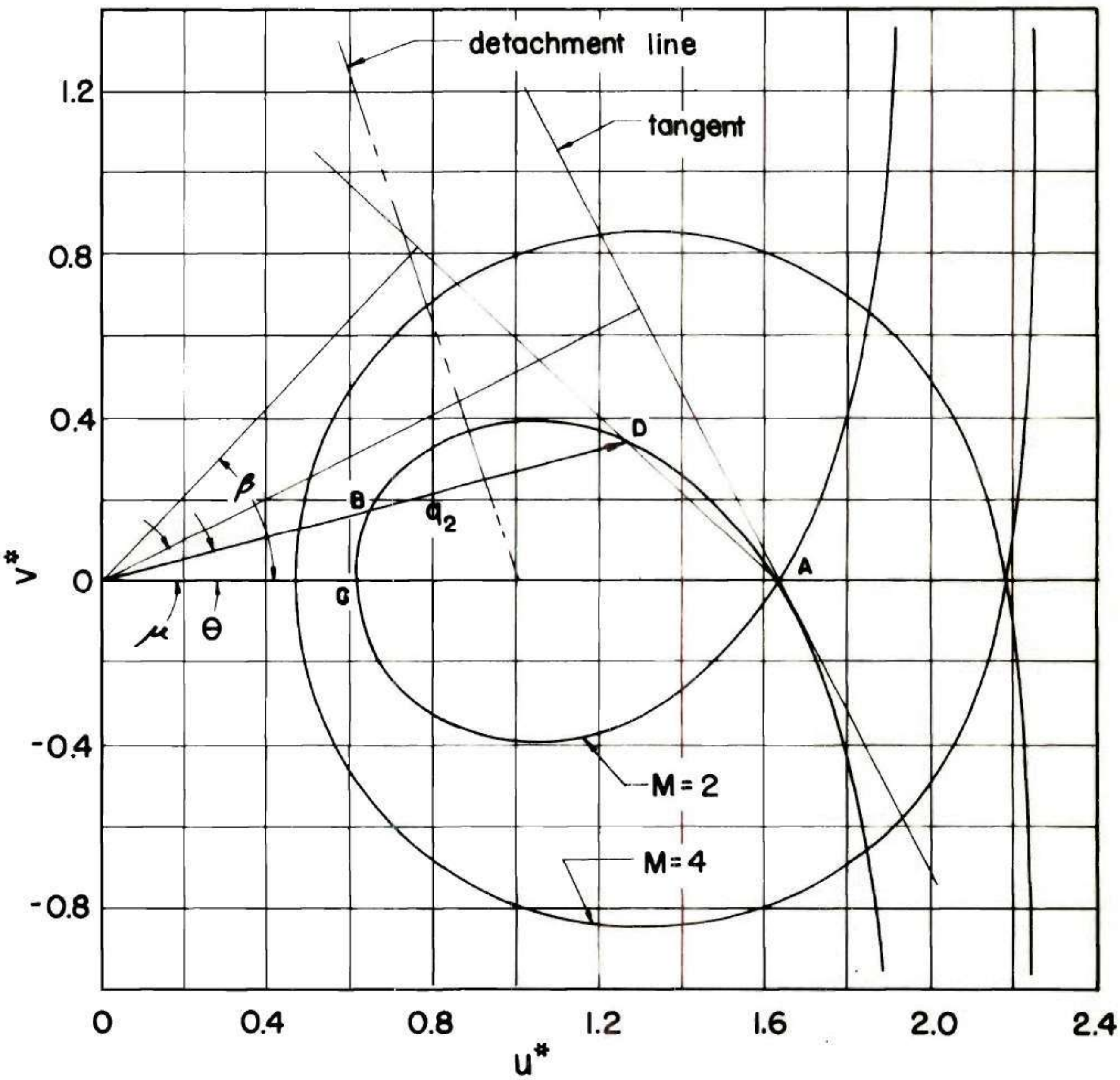
$$u^{*}=\frac{u}{a_c}\quad , \qquad v^{*}=\frac{v}{a_c}$$

equation (90) becomes finally, the equation of the shock polar in the hodograph plane (equation 91).

$$V_2^{*2} = (u_1^* - u_2^*)^2 \frac{u_1^* u_2^* - 1}{\frac{2}{\gamma+1} u_1^{*2} - u_1^* u_2^* + 1} \quad . \tag{91}$$

When equation (91) is plotted for a given Mach number the result is the curve shown in figure 14, and has a double point at u_1 equal u_2. Of the complete curve, only the part corresponding to $(u_2)^2 + (v_2)^2 \leq q_1$ represents actual shock transitions. With use of this part of the curve it is possible to determine graphically the flow quantities in the following way:

The velocity vector q_2, when drawn from the origin will have its tip on the curve (point P, figure 14). It will be inclined at the angle θ to the u* axis and a line drawn perpendicular to line AP through the origin will be at the angle β to the u^* axis. When the tip of q_2 coincides with point A, $q_1 = q_2$. This corresponds to the flow passing through a Mach wave when θ equals zero. As θ increases there are two solutions to equation (91) (point P and B) until q_2 becomes tangent to the curve. When two solutions exist, point P corresponds to the weak shock transition which actually occurs in the physical problem, and point B represents the strong shock transition which very seldom occurs. When θ increases beyond the value at which q_2 is tangent to the curve, the shock be-


detachment line
tangent
1.2
0.8
0.4
0
-0.4
-0.8
v^*
D
β
B
q_2
A
C
μ
θ
M=2
M=4
0
0.4
0.8
1.2
1.6
2.0
2.4
u^*

FI

comes detached; point C corresponding to the normal shock condition.

The velocities before and after, normal to the shock are represented by the vectors DA and PD respectively, and the velocities parallel to the shock by OD.

THE METHOD OF SMALL PERTURBATIONS

Assumptions. The approximations mentioned in the hodograph discussion were based on the hypothesis that the free stream Mach number was lower enough that the compressible flow pattern differed little from the incompressible pattern. Now if the assumption is made that the perturbation or disturbance velocities due to the introduction of a body into the stream (which may have a large Mach number) are vanishingly small, the equations may be linearized and solved.

In order to develop an approximate theory on the above assumption, the velocities at any point may be written as

$$u = u_1 + u'$$
$$v = v'. \tag{92}$$

Where u' and v' are the perturbation velocities. Now if the assumption that

$$u'/u_1 \ll 1 \tag{93}$$

is made, then the higher orders of u'/u_1 may be neglected.

The assumption is also made that the derivatives of u' and v' are small. Here however, there must be some reference to give meaning to the word small. In the case of the velocities, this reference was the undisturbed velocity u_1. In the case of the derivatives, reference will be made to some characteristic length, such as the thickness, since the rate of change of the velocities is dependent on the shape of the body introduced into the stream. This length will be denoted by τ. Now the assumption is made that

$$u_x = u'/\tau = (u'/u_1)(u_1/\tau) \ll 1 \qquad (94)$$

Equations (93) and (94) are the assumptions upon which the approximate theory will be based; the degree of approximation depending on the magnitude of the equations (93) and (94).

Approximate Equations. Using equations (92), equation (43) may be written as

$$\left[\frac{a^2}{u_1^2} - \frac{(u_1+u')^2}{u_1^{\,2}}\right](u_1+u')_x + \left[\frac{a^2}{u_1} - \frac{v'^2}{u_1^2}\right] v_y'$$

$$- \frac{(u_1+u')v'}{u_1}\left[(u_1+u')_y + v_x'\right] = 0 .$$

Which by equations (93) and (94), becomes

$$\left(\frac{a^2}{u_1^{\,2}} - 1\right) u_x + \frac{a^2}{u_1} v_y = 0$$

or

$$\left(1 - \frac{u_1^2}{a^2}\right)\phi_{xx} + \phi_{yy} = 0 . \qquad (95)$$

Where   is defined as

$$\phi_x = u' \quad \phi_y = v' .$$

The speed of sound, a, is given by equation (4) as

$$a^2 = a_1^2 - \frac{\gamma - 1}{2} \quad (u_1 + u')^2 + u'^2 - u_1^2 \quad ,$$

which becomes

$$a^2 = a_1^2 - (\gamma - 1)\, u_1 u' \quad . \tag{96}$$

Substituting equation (96) in (95) gives

$$\left(1 - \frac{u_1^2}{a_1^2 - (\gamma-1)u_1 u'}\right)\phi_{xx} + \phi_{yy} = 0$$

or

$$\left[1 - \frac{u_1^2}{a^2}\,\frac{1}{1-(\gamma-1)\dfrac{u_1^2}{a_1^2}\dfrac{u_1'}{u_1}}\right]\phi_{xx} + \phi_{yy} = 0$$

$$\left(1 - \frac{u_1}{a_1}\right)^2 \phi_{xx} + \phi_{yy} = 0 \quad .$$

In terms of the Mach number this becomes

$$(1 - M^2)\, \phi_{xx} + \phi_{yy} = 0 \quad . \tag{97}$$

Equation (97) is the equation of the potential function based on the assumptions of equations (93) and (94). This equation is linear and may be handled by ordinary methods.

Pressure Coefficient. By the above approximations and by the definition of μ, equations (51) become

$$\Gamma_1 = \frac{1}{\sqrt{M^2-1}} = \tan \mu_1$$

$$\Gamma_2 = \frac{1}{\sqrt{M^2-1}} = \tan \mu_1 \quad . \tag{98}$$

Then equations (57) become

$$\frac{du}{u_1} = \pm\, d\theta \tan \mu_1 \quad .$$

The pressure coefficient is defined as

$$C_p = \frac{\Delta p}{\frac{1}{2}\rho_1 u_1^2} = \frac{\rho u du}{\frac{1}{2}\rho_1 u_1^2} \quad .$$

Or

$$C_p = \pm \frac{2 \tan \mu_1 \Delta\theta}{\rho_1 u_1^2}\, \rho u \quad .$$

By use of the approximations, $\rho_1 u_1 \cong \rho u$, and with equations (98) the pressure coeficient becomes

$$C_p = \pm \frac{2\Delta\theta}{\sqrt{M^2-1}} \quad . \tag{99}$$

Where $\Delta\theta$ is the flow deviation in radians, relative to the free stream. Equation (99) is the so-called "linear theory" used extensively in airfoil work.

CONCLUSIONS

It is important to remember that every method presented deals exclusively with the solution of the non-linear equations (8). They were first solved approximately by the hodograph transformation, later by the method of characteristics, and finally by the method of small perturbations. On the basis of the hypotheses and assumptions of each method, they may be summarized by the following statements.

1. The hodograph transformation is based on the hypothesis that the flow is subsonic and isentropic. By applying the Chaplygin or Karman-Tsien approximation to the transformed equations, the problem of subsonic compressible flow about an arbitrary body may be readily solved. The solutions based on these approximations agree very closely with experimental results up until the critical Mach number is reached, then the conditions of the hypothesis are no longer met.

2. The method of characteristics is based on the hypothesis that the flow is completely supersonic and isentropic. Although the latter may be violated, as in the case of weak shocks, without greatly affecting the accuracy of the final results, the former must be fulfilled since the characteristic lines do not exist for the case of subsonic flow.

The outstanding feature of this method is that it permits a semi-graphical analysis of the flow. It also gives an exact solution for an isentropic compression or expansion and a very close approximation for small compressions when the flow is no longer isentropic.

3. The method of small perturbation is based on the assumptions that the flow is isentropic and that the perturbation velocities perpendicular to the direction of the free stream due to the presence of a body are small enough to be neglected. With the above assumptions the non-linear equations become approximate linear equations. These equations may be used to give a good first approximation, but in no case is the solution exact.

4. The problem of two dimensional shock flow may be solved exactly by use of the conservation of mass and momentum equations and Bernoulli's equation. The resulting system of equations involves five parameters. Therefore, the flow is completely determined when any two of the parameters are given.

5. No solution is, in itself, complete for all problems. Therefore, the solution to be used depends on the problem and the accuracy required.

BIBLIOGRAPHY

Bateman, H., Partial Differential Equations of Mathematical Physics. New York: The MacMillan Co., 1932. 522pp.

_______,"The Lift and Drag Functions for an Elastic Fluid in Two Dimensional Irrotational Flow," Proceedings of The National Academy of Science, 24: 246,1938.

Bergman, Stefan, "On Two Dimensional Flows of Compressible Fluids," National Advisory Committee for Aeronautics Technical Note No. 972, August 1945, 81pp.

_______,"Methods for Determination and Computation of Flow Patterns of a Compressible Fluid," National Advisory Committee for Aeronautics Technical Note No. 1018, September 1946, 71pp.

Bers, Lipman, and Abe Gelbart, "On a Class of Differential Equations in Mechanics of Continua," Quarterly of Applied Mathematics, 1:168,July, 1943

Chaplygin, S. H., "On Gas Jets," National Advisory Committee for Aeronautics Technical Memorandum, No. 1063, 1944, 112pp.

Cherry, T. M., "Flow of a Compressible Fluid About a Cylinder," Proceedings of The Royal Society, A192:45, 1947.

Courant, R., and K. O. Friedrichs, Supersonic Flow and Shock Waves, New York: Interscience Publishers, Inc., 1947 680pp.

Craggs, J. W., "The Breakdown of the Hodograph Transformation for Irrotational Compressible Flow in Two Dimensions," Proceedings of The Cambridge Philisophical Society, 44:360, July, 1948.

Emmons, H. W., "The Numerical Solution of Compressible Fluid Flow Problems," National Advisory Committee for Aeronautics Technical Note No. 932, May 1944, 32pp.

Ferri, Antonio, "Application of the Method of Characteristics to Supersonic Rotational Flow," National Advisory Committee for Aeronautics Technical Note No. 1135, September, 1946. 25pp.

_______,Elements of Aerodynamics of Supersonic Flows. New York: The MacMillan Co., 1949, 434pp.

Garrick, I. E., and Carl Kaplan, "On the Flow of a Compressible Fluid by the Hodograph Method,"
Part I, "Unification and Extension of Present Day Results," National Advisory Committee for Aeronautics Report No. 789, 1944, 24pp.
Part II, "Fundamental Set of Particular Flow Solutions of the Chaplygin Differential Equation," National Advisory Committee for Aeronautics Report No. 790, 1944, 21pp.

Greene, L. M., "The Attenuation Method for Compressible Flow Systems," Journal of the Aeronautical Sciences, 12:345, 1945.

Imai, Isao, and Hidernori Hasimoto, "Application of the WKB Method to the Flow of a Compressible Fluid."
Part I, Journal of Mathematics and Physics, 28:173, 1949

Kaplan, Carl, "Two Dimensional Subsonic Compressible Flow Past Elliptic Cylinders," National Advisory Committee for Aeronautics Report No. 624, 1938, 8pp.

_______, "The Flow of a Compressible Fluid Past a Curved Surface," National Advisory Committee for Aeronautics Report No. 768, 1943, 23pp.

Karman, Th. Von, "Compressibility Effects in Aerodynamics," Journal of the Aeronautical Sciences, 8:337, 1941.

Kiebel, I.A., "Exact Solutions of Equations of Gas Dynamics," National Advisory Committee for Aeronautics Technical Memorandum No. 1260, 1950, 12pp.

Laitone, E. V., "Exact and Approximate Solutions of Two Dimensional Oblique Shock Flow," Journal of the Aeronautical Sciences, 14:25, 1947.

Liepmann, H. W., and A. E. Puckett, Aerodynamics of a Compressible Fluid. New:York: John Wiley and Sons, 1947 262pp.

Lighthill, M. J., "The Hodograph Transformation in Transonic Flow," Proceedings of The Royal Society, A191:323, 1947

Meyer, R.E., "The Method of Characteristics for Problems of Compressible Flow Involving Two Independent Variables," Part II, Quarterly Journal of Applied Mathematics, 1:451, 1948.

Molenbroek, P., "Uber einige Bewegungen eines Gases mit Annahme eines Geschwindigkeitspotentials," Archiv d. Math. u. Phys. (2), 9:157, 1890.

Orlin, J. W., N. J. Linder, and J. G. Bitterly, "Application of the Analogy Between Water Flow with a Free Surface and Two Dimensional Compressible Gas Flow," National Advisory Committee for Aeronautics Technical Note No. 1185, 1947, 20pp.

Perl, W., "Calculation of Compressible Flows Past Aerodynamic Shapes by use of the Streamline Curvature," National Advisory Committee for Aeronautics Technical Note No. 1328, 1947, 38pp.

Rockett, J. A., and W. D. Hayes, "The Method of Characteristics in Compressible Flow,"
Part IC, "Two DimensionalFlow with Large Entropy Changes," Monograph II, U. S. Air Force Technical Report, No. 102-AC 49/6-100, December 1949, 32pp.

Sears, W. R., "The Linear Perturbation Theory for Irrotational Flows," Journal of Mathematics and Physics, 28:268, 1949.

Sibert, H. W., "Approximations Involved in the Linear Differential Equations for Compressible Flow," Journal of the Aeronautical Sciences, 14:680, 1947.

Streeter, V. L., Fluid Dynamics. New York: McGraw Hill Book Company, Inc., 1948, 263pp.

Theodorsen, Theodore, "Extension of the Chaplygin Proofs on the Existence of Compressible Flow Solutions to the Supersonic Region," National Advisory Committee for Aeronautics Technical Note No. 1028, 1946, 8pp.

Tsien, H. S., "Two Dimensional Subsonic Flow of Compressible Fluids," Journal of the Aeronautical Sciences, 6:399, 1939.

_______, and L. Lees, "The Glauert- Prandtl Approximation for Subsonic Flows of a Compressible Fluid," Journal of the Aeronautical Sciences, 12:173, 1945.

_______, "The Limiting Line in Mixed Subsonic and Supersonic Flow of Compressible Fluids," National Advisory Committee for Aeronautics Technical Note No. 961, 1944, 29pp.

Wang, Chi-Teh, and G. V. R. Rao, "A Study of the Non-Linear Characteristics of Compressible Flow Equations by Means of Variational Methods," Journal of the Aeronautical Sciences, 17:343, 1950.

_______, "Variational Methods in the Theory of Compressible Fluids," Journal of the Aeronautical Sciences, 15:675, 1950.

APPENDIX

APPLICATION OF THE KARMAN-TSIEN METHOD TO THE FLOW OVER CIRCULAR CYLINDERS

As mentioned in the discussion on the Karman-Tsien method, when transforming the incompressible flow field to the compressible field, the resulting shape of the body may be pre-determined by equation (36). However the correction factor λ is correct only for one Mach number. Therefore a single correction may be used accurately only in two cases: (1) when the Mach number of the undisturbed stream is low or, (2) when the perturbation velocities due to the body are small when compared to the free stream velocity. Most practical cases will lie somewhere between the two above cases. Then by using only a single correction, the transformed shape will be slightly distorted.

Also it is interesting to note that the correction factor, λ, increases very rapidly near $M = 1$ (figure 3), or in effect, the distortion becomes increasingly large as the free stream Mach number approaches the value of the critical Mach number unless the perturbation velocities are very small.

Kaplan[13] has shown that the ratio of the maximum velocity over the body to the free stream velocity is

---

[13] Kaplan, C.,"Two Dimensional Subsonic Compressible Flow Past Elliptic Cylinders," National Advisory Committee for Aeronautics Technical Report No. 624, 1938, p5.

$$\frac{q_{MAX}}{q_1} = \left[\frac{2}{\gamma+1} \frac{1}{M^2} + \frac{\gamma-1}{\gamma+1} \right]^{\frac{1}{2}} . \tag{100}$$

Since for the circular cylinder q_{max} occurs at $\theta = 90^\circ$, the ratio q_{max}/q becomes

$$\frac{q_{MAX}}{q_1} = \frac{W_{MAX}}{W} \frac{1-\lambda}{1-\lambda\,(W/W_1)^2}$$

$$= \frac{2(1-\lambda)}{1-4\lambda} . \tag{101}$$

The velocity at any point on the cylinder in an incompressible flow being given by

$$W = 2W_1 \sin\theta ,$$

where θ is the angle subtented by the point. Zero corresponding to the leading edge. Substituting (101) in (100), and solving for M_1 produces,

$$\frac{2-2\lambda}{1-4\lambda} = \left[\frac{2}{\gamma+1} \frac{1}{M_c^2} + \frac{\gamma-1}{\gamma+1} \right]^{\frac{1}{2}}$$

$$M_c = 0.402 .$$

<u>Pressure Distribution at M 0.41.</u> This method will now be applied to the problem of the circular cylinder at a Mach number of 0.41.

Imai and Hasimato[14] found that if an ellipse of thickness ratio 0.9165 was transformed, the resultant shape was very nearly a perfect circle. This ellipse is close to a circle itself and therefore the distortion is very small. For the purpose of this problem this distortion will be neglected.

First the correction factor λ is found from figure 3 to be 0.046. Then using the relation for the velocity at any point on the cylinder in the incompressible case cited above, and using equation (38), the pressure coefficient is

$$C_p = 1.046 \frac{1-(2\sin\theta)^2}{1-.046(2\sin\theta)^2} \quad . \qquad (102)$$

The values are then computed for various stations. These are shown in table II and figure 15.

It is interesting to note that the theoretical curve compares very favorably with the experimental curve obtained by Orlin[15] although it is in the region of the critical Mach number. This is due to the fact that at the point at which $M = 1$ first occurs, the perturbation velocities become very small compared to M_1. Therefore it appears that the results

---

[14] Imai, I., and Hasimoto, H. "Application of the WKB Method to the Flow of a Compressible Fluid," Journal of Mathematics and Physics, 28:173, 1950.

[15] Orlin, J. W., N. J. Linder, and J.G. Bitterly, "Application of the Analogy Between Water Flow with a Free Surface and Two Dimensional Compressible Gas Flow," National Advisory Committee for Aeronautics Technical Note No.1185, 1947.

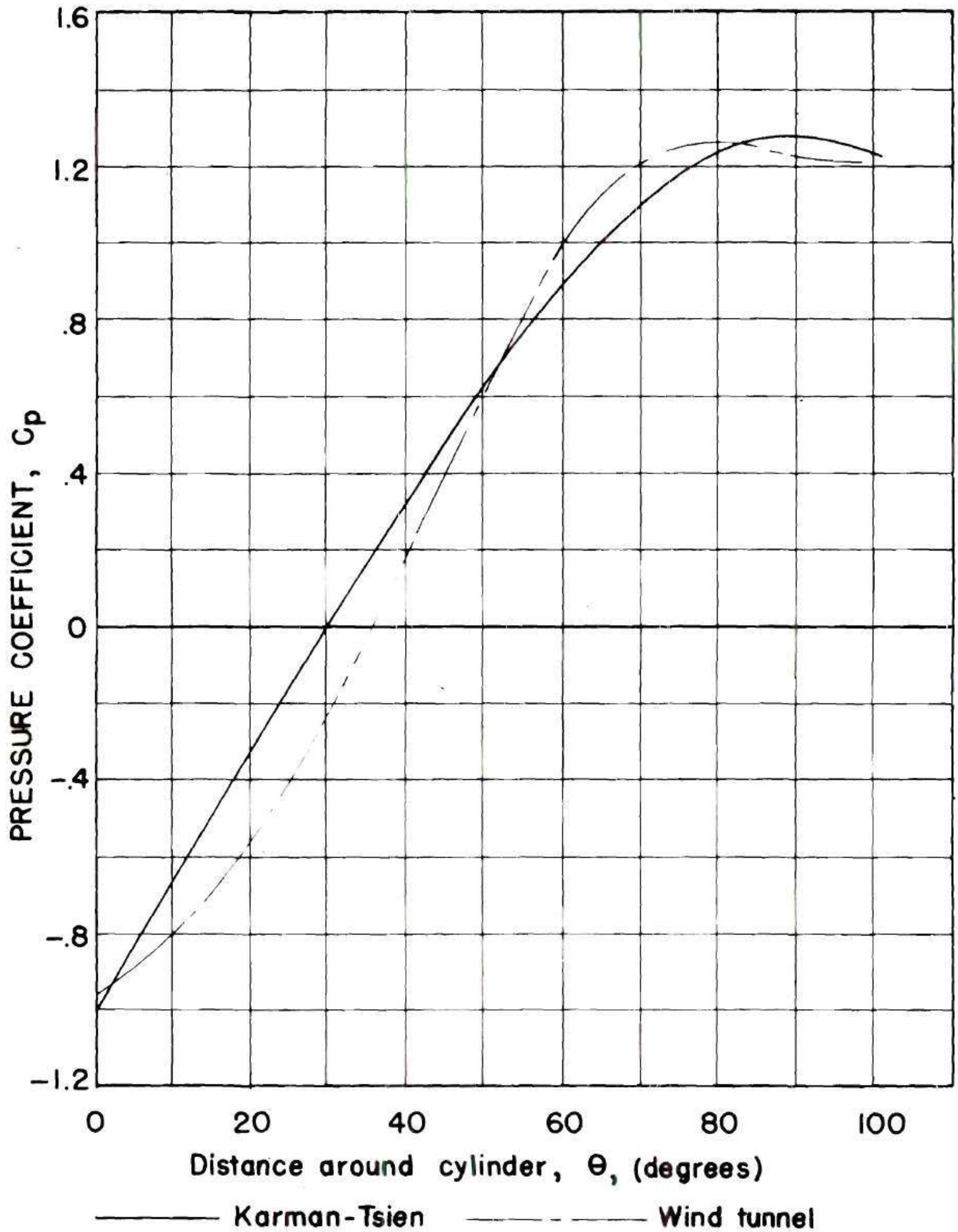


FIGURE 15, PRESSURE DISTRIBUTION, CIRCULAR CYLINDER

M = 0.41

TABLE II

Pressure distribution about circular cylinder

$M = 0.41$

θ	$1-2\sin^2\theta$	$1-\lambda(2\sin\theta)^2$	C_p
0	1.000	1.000	1.046
10	0.652	0.994	0.686
20	0.312	0.978	0.334
30	0	0.954	0
40	-0.286	0.924	-0.324
50	-0.532	0.892	-0.623
60	-0.732	0.862	-0.888
70	-0.880	0.837	-1.099
80	-0.920	0.821	-1.235
90	-1.000	0.816	-1.281

are not to much in error due to the neglect of the distortion caused by the transformation.

Pressure Distribution at M 0.50. By repeating the above process for a circular cylinder at a Mach number of 0.50, figure 16 and table III were obtained. The purpose of this pressure distribution is to show the effect of the critical Mach number. Examination of figure 16 will show that at the high values of θ, where the critical velocity is reached, the theoretical curve becomes more in error when compared to experimental results. This is due to the fact that the hypothesis, upon which this method is based, are no longer applicable to the problem.

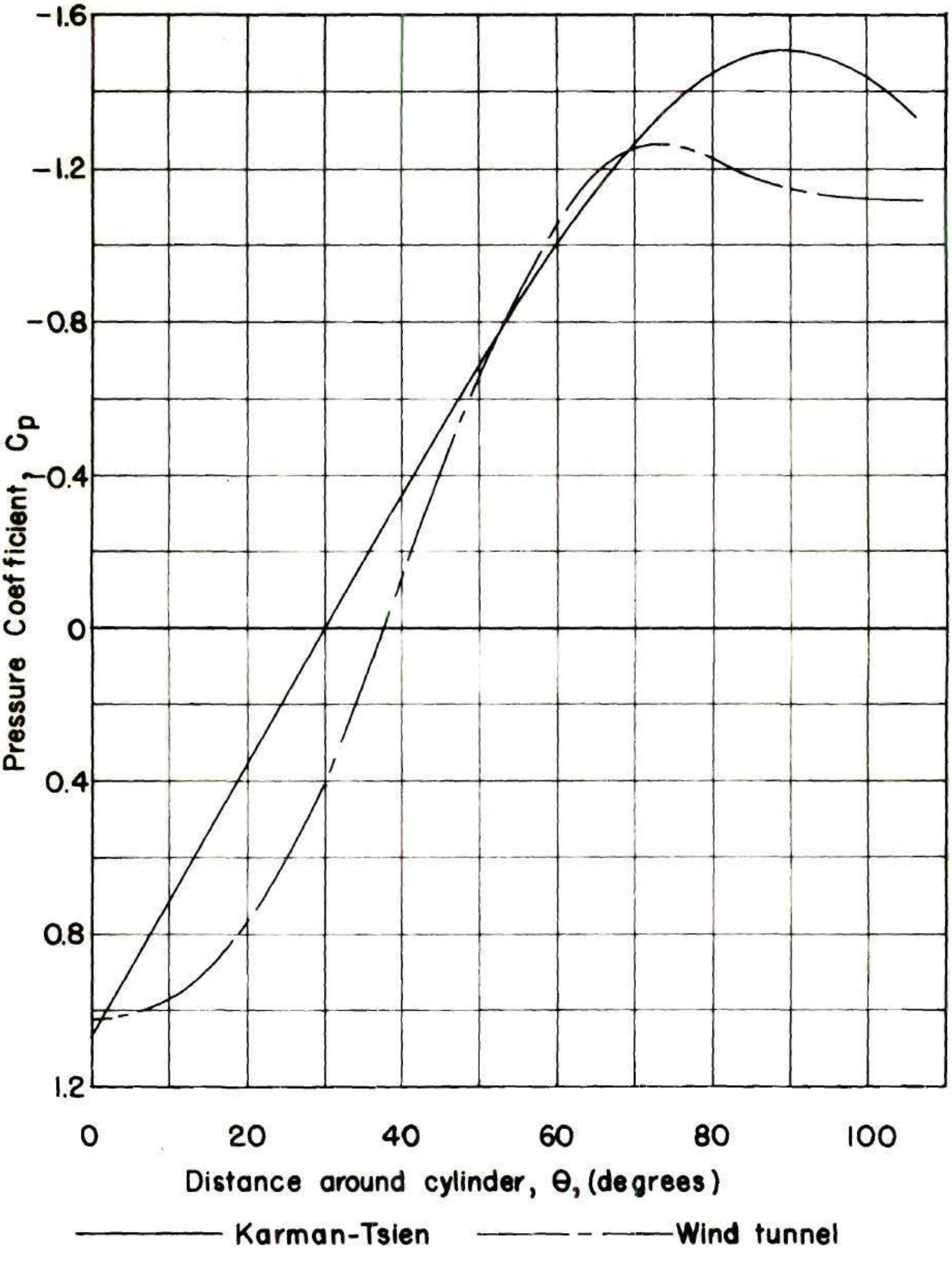


FIGURE 16, PRESSURE DISTRIBUTION, CIRCULAR CYLINDER

M = 0.5

TABLE III

Pressure distribution about circular cylinder

$M = 0.50$

θ	$1-2\sin^2\theta$	$1-\lambda(2\sin\theta)^2$	C_p
0	1.000	1.000	1.072
10	0.652	0.991	0.705
20	0.312	0.966	0.346
30	0	0.928	0
40	-0.286	0.881	-0.348
50	-0.532	0.831	-0.686
60	-0.732	0.784	-1.001
70	-0.880	0.746	-1.265
80	-0.970	0.721	-1.442
90	-1.000	0.712	-1.505

APPLICATION OF THE CHARACTERISTICS AND SHOCK DIAGRAMS

In order to show the principle of the application of the characteristics and shock diagrams, they will be employed to solve the basic problem of the velocity and pressure distribution about a typical diamond shaped supersonic airfoil at an angle of attack of two degrees and at a Mach number of two. The geometrical properties of the airfoil are shown in figure 17a.

<u>Velocity Distribution</u>. The characteristics diagram, figure 7, is used for this problem in the following manner.

1. From equation (64), q_1/q_L is found to be 0.667 since $M = 2$. Therefore the characteristics diagram is oriented so that an epicycloid curve coincides with the point $q_1/q_L = 0.667$ on the $\theta = 0^\circ$ line (figure 17b). Considering now the region 2 and assuming that no finite discontinuities occur, q_2/q_L may be found by constructing a line at an angle $\theta = 5^\circ$ to the $\theta = 0^\circ$ line. The line OA then represents q_2. From this construction, q_2/q_L is found to be 0.630.

2. Repeat this process for regions 3 and 4, noting that the angle of the region 3 is measured relative to the free stream. The velocities q_3 and q_4 are represented by the OB and OC respectively. The values are: $q_3/q_L = 0.721$, and $q_4/q_L = 0.651$.

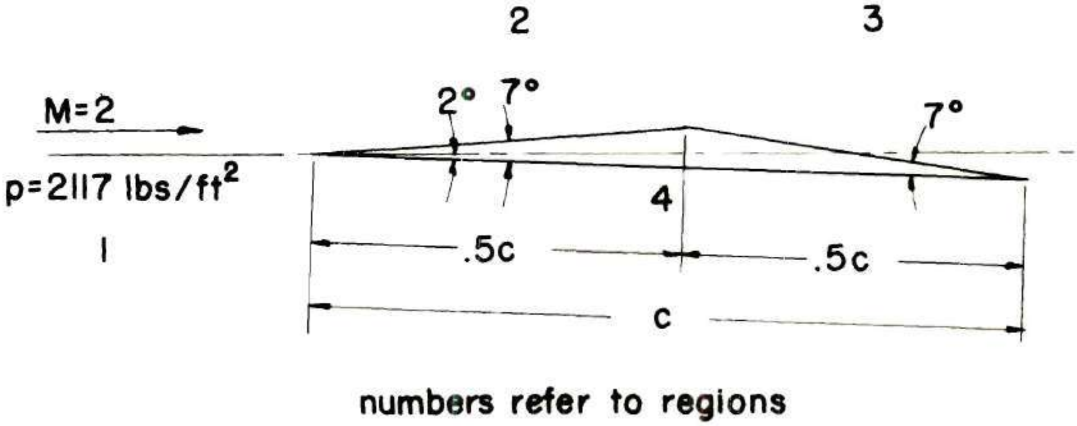


Fig. 17a

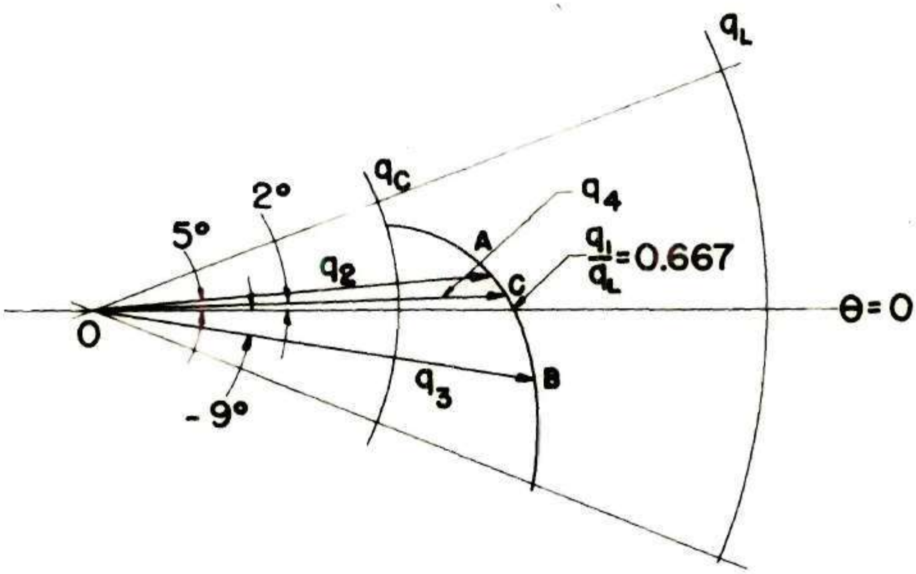


Fig. 17b

FIGURE 17, GEOMETRICAL CHARACTERISTICS OF DIAMOND AIRFOIL AND EPICYCLOID CURVE

3. Then using equation (64), the respective Mach numbers are

$$M_2 = 1.82$$
$$M_3 = 2.33$$
$$M_4 = 1.92 \quad .$$

Pressure Distribution. Using the definition of the pressure coefficient,

$$C_p = \frac{p - p_1}{\frac{1}{2}\rho q^2} \quad ,$$

and writing it in terms of the Mach number,

$$C_p = \frac{p - p_1}{\frac{\gamma p M^2}{2}}$$

the pressure distribution may now be found using figure 13 and the isentropic relations.

Pressures in regions 2 and 4 are found from figure 13 to be

$$p_2/p_1 = 1.27$$
$$p_4/p_1 = 1.09 \quad .$$

From the isentropic relation,

$$p_3/p_1 = \left[\frac{1 + \frac{\gamma - 1}{2} M_1^2}{1 + \frac{\gamma - 1}{2} M_3^2}\right]^{\frac{\gamma}{\gamma - 1}}$$

$$p_3/p_1 = 0.594 \quad .$$

The isentropic relation was used in this case since the flow deviation angle, θ, is minus and an expansion occurs. Taking $p_1 = 2117$ lbs/ sq ft (standard air), then

$$p_2 = 2731 \text{ psf} \qquad p_2 - p_1 = 614$$
$$p_3 = -1257 \text{ psf} \qquad p_3 - p_1 = -860$$
$$p_4 = 2308 \text{ psf} \qquad p_4 - p_1 = 191 \quad .$$

Finally substituting these in the equation for the pressure coefficient gives

$$c_{p2} = 0.104$$
$$c_{p3} = -0.145$$
$$c_{p4} = 0.032$$

As a comparison, the pressure coefficients will also be found by equation (99). Substituting the values of $\Delta\theta$ given below in equation (99)

$$\Delta\theta_2 \quad 5^\circ = 0.087 \text{ radians}$$
$$\Delta\theta_3 = -9^\circ = -0.157 \quad \text{"}$$
$$\Delta\theta_4 = 2^\circ = 0.035 \quad \text{"}$$

gives

$$c_{p2} = 0.101$$
$$c_{p3} = -0.181$$
$$c_{p4} = 0.040 \quad .$$

The distributions found by both methods are shown in figure 18. By integrating these pressure diagrams the lift coefficient, C_l, may be obtained. They are

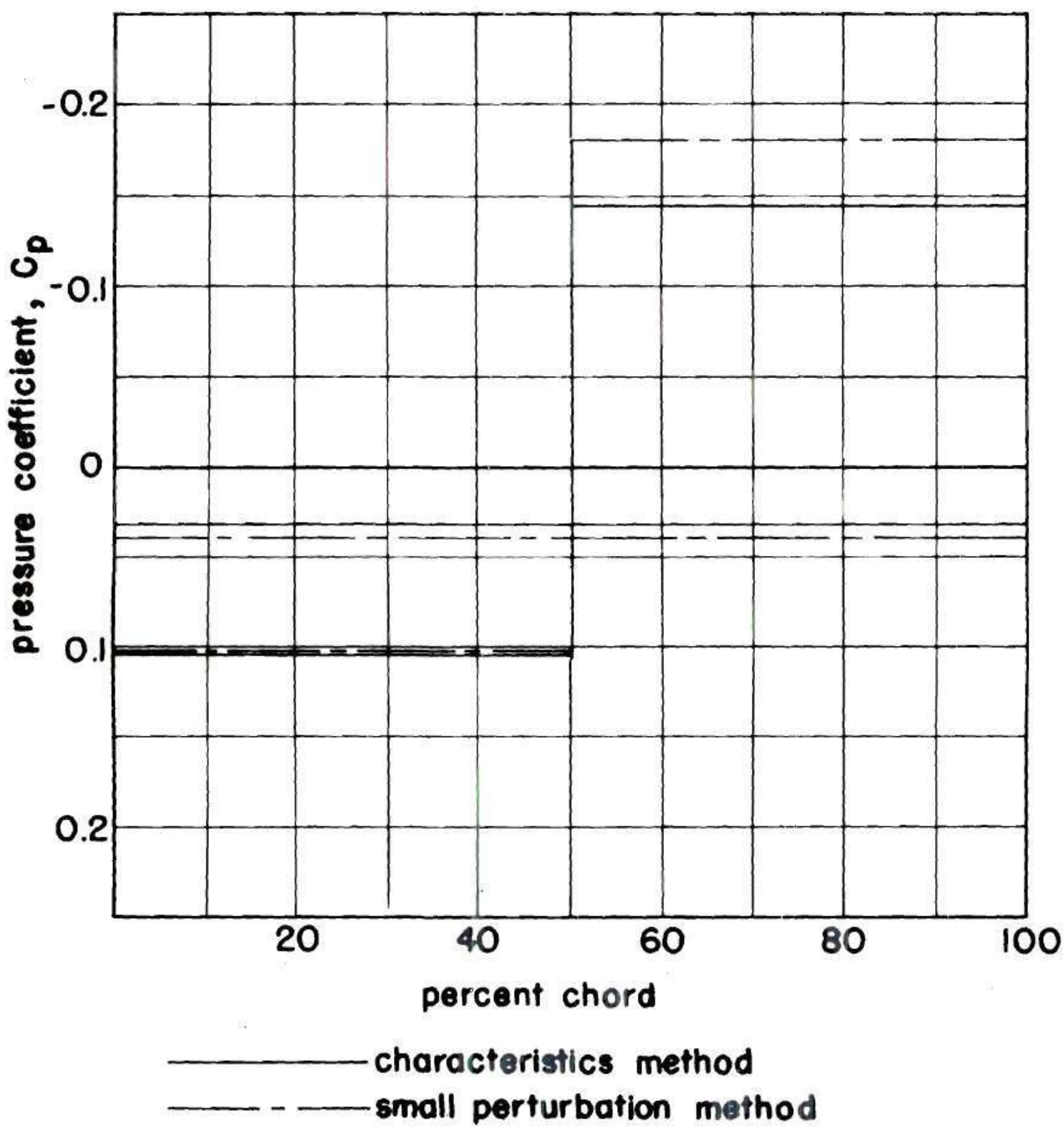


FIGURE 18, PRESSURE DISTRIBUTION - DIAMOND AIRFOIL

$M = 2$ $\alpha = 2°$

Characteristics and shock diagram method, $C_1 = 0.050$

Small perturbation method, $C_1 = 0.076$

When compared with the experimental value[16], $C_1 = 0.045$, some idea is obtained as to the degree of approximation involved in both methods.

---

[16] Ferri, A., Elements of Aerodynamics of Supersonic Flows, (New York: The MacMillan Co., 1949), p147.