# Zeta Functions of the Dirac Operator on Quantum Graphs

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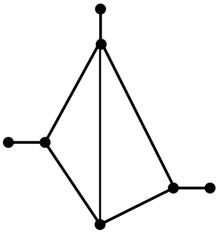
# Metric Graphs

$\Gamma = \{V, \mathcal{B}, L\}$



Quantum Graph: Metric Graph + Differential Operator

# Dirac Operator

$$\mathcal{D} = -\mathrm{i}\alpha \frac{\mathrm{d}}{\mathrm{d}x_b} + m\beta$$

$\alpha$ and $\beta$ are 4x4 matrices that satisfy $\alpha^2 = \beta^2 = \mathrm{I}$ and $\alpha\beta + \beta\alpha = 0$

Vertex Conditions: $\mathbb{A}\boldsymbol{\psi}^+ + \mathbb{B}\boldsymbol{\psi}^- = \mathbf{0}$

$$\begin{aligned}\boldsymbol{\psi}^+ &= (\psi_1^1(0), \psi_2^1(0), \ldots, \psi_2^B(0), \psi_1^1(L_1), \psi_2^1(L_1), \ldots, \psi_1^B(L_B), \psi_2^B(L_B))^T \\ \boldsymbol{\psi}^- &= (-\psi_4^1(0), \psi_3^1(0), \ldots, \psi_3^B(0), \psi_4^1(L_1), -\psi_3^1(L_1), \ldots, \psi_4^B(L_B), -\psi_3^B(L_B))^T\end{aligned}$$

The operator is self-adjoint if and only if $\mathbb{A}$ and $\mathbb{B}$ are $4B$ x $4B$ matrices that satisfy

$$\mathrm{rank}(\mathbb{A}, \mathbb{B}) = 4B \text{ and } \mathbb{A}\mathbb{B}^\dagger = \mathbb{B}\mathbb{A}^\dagger.$$

[1] J. Bolte and J. M. Harrison, Spectral statistics for the Dirac operator on graphs, *J. Phys. A: Math. Gen.* **36:2747** (2003).

# Solutions

$$\mathcal{D} = -\mathrm{i}\alpha \frac{\mathrm{d}}{\mathrm{d}x_b} + m\beta$$

$$\alpha = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} \mathrm{I} & 0 \\ 0 & -\mathrm{I} \end{pmatrix}$$

Solutions to $\mathcal{D}\psi_k = E(k)\psi_k$ are of the form

$$\psi^b(x_b) = \mu^b_\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ \mathrm{i}\gamma(k) \end{pmatrix} \mathrm{e}^{\mathrm{i}kx_b} + \mu^b_\beta \begin{pmatrix} 0 \\ 1 \\ -\mathrm{i}\gamma(k) \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}kx_b} + \hat{\mu}^b_\alpha \begin{pmatrix} 1 \\ 0 \\ 0 \\ -\mathrm{i}\gamma(k) \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx_b} + \hat{\mu}^b_\beta \begin{pmatrix} 0 \\ 1 \\ \mathrm{i}\gamma(k) \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx_b}$$

where

$$\gamma(k) := \frac{E(k) - m}{k} \qquad E(k) := \sqrt{k^2 + m^2}$$

# Solutions

Solutions to $\mathcal{D}\psi_k = E(k)\psi_k$ are of the form

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where

$$\gamma(k) := \frac{E(k) - m}{k} \qquad E(k) := \sqrt{k^2 + m^2}$$

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Solutions to $\mathcal{D}\psi_k = -E(k)\psi_k$ are of the form

$$\psi^b(x_b) = \mu_\alpha^b \begin{pmatrix} \mathrm{i}\gamma(k) \\ 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}kx_b} + \mu_\beta^b \begin{pmatrix} 0 \\ -\mathrm{i}\gamma(k) \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}kx_b} + \hat{\mu}_\alpha^b \begin{pmatrix} -\mathrm{i}\gamma(k) \\ 0 \\ 0 \\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx_b} + \hat{\mu}_\beta^b \begin{pmatrix} 0 \\ \mathrm{i}\gamma(k) \\ 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}kx_b}$$

# Secular Equation

For positive solutions:

$$\det\left(\mathbb{A}+\gamma(k)\mathbb{B}\begin{pmatrix}\cot kL & -\csc kL\\ -\csc kL & \cot kL\end{pmatrix}\right)=0$$

For negative solutions:

$$\det\left(\gamma(k)\mathbb{A}-\mathbb{B}\begin{pmatrix}\cot kL & -\csc kL\\ -\csc kL & \cot kL\end{pmatrix}\right)=0$$

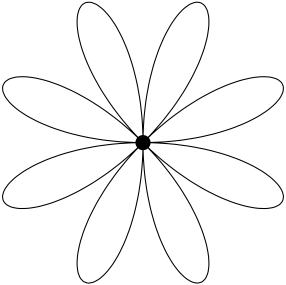
# Spectral Zeta Function

Given the set of roots $\{\ldots < k_{-2} < k_{-1} < k_1 < k_2 < \ldots\}$ of the secular equation, the spectral zeta function is defined as

$$\zeta(s) = 2 \sum_{j=-\infty}^{\infty}{}' \, E(k_j)^{-s}$$
$$= 2 \sum_{j=-\infty}^{\infty}{}' \, k_j^{-s}$$

in the massless case.

# Rose Graph



# Vertex Conditions

$$u_o^b \boldsymbol{v}^b(0) = u_t^b \boldsymbol{v}^b(L_b) = \eta \qquad \text{for all bonds } b$$

$$\sum_{b=1}^{B} u_o^b \boldsymbol{w}^b(0) = \sum_{b=1}^{B} u_t^b \boldsymbol{w}^b(L_b)$$

where

$$\boldsymbol{v}^b(x_b) = \begin{pmatrix} \psi_1^b(x_b) \\ \psi_2^b(x_b) \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{w}^b(x_b) = \begin{pmatrix} -\psi_4^b(x_b) \\ \psi_3^b(x_b) \end{pmatrix}$$

Secular Equation:

$$\sum_{b=1}^{B} \frac{\cos\theta_b - \cos kL_b}{\sin kL_b} = 0 \qquad \text{where } \cos\theta_b = \frac{1}{2}\mathrm{tr}(u_o^b (u_t^b)^{-1})$$

# Spectral Zeta Function

$$f(z) = z \sum_{b=1}^{B} \frac{\cos\theta_b - \cos zL_b}{\sin zL_b}$$

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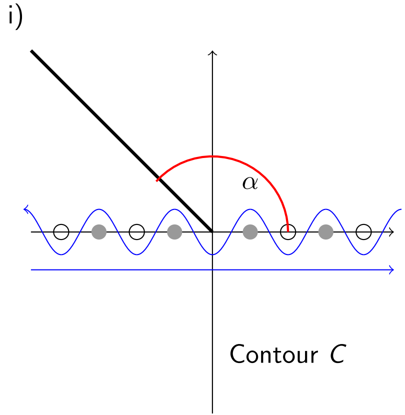
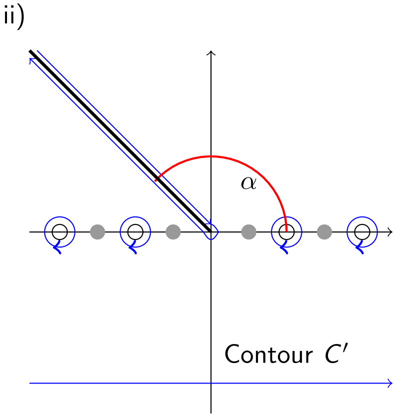
$$\begin{aligned}
\zeta(s) &= 2 \sum_{j=-\infty}^{\infty}{}' \, k_j^{-s} \\
&= \frac{1}{\mathrm{i}\pi} \int_C z^{-s} \frac{f'(z)}{f(z)} \, \mathrm{d}z \\
&= \frac{1}{\mathrm{i}\pi} \int_C z^{-s} \frac{\mathrm{d}}{\mathrm{d}z} \log f(z) \, \mathrm{d}z
\end{aligned}$$

where $C$ is any contour that encloses the zeros of $f$ (while avoiding its poles).

[2] J. Harrison and K. Kirsten, Zeta functions of quantum graphs, *J. Phys. A: Math. Theor.* **44** (2011).

# Contours




The shaded circles are the zeros of $f$ and the empty circles are the poles.

# Spectral Zeta Function – Rose Graph

$$\zeta(s) = \zeta_p(s) + \zeta_l(s) + \zeta_b(s)$$

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$$\begin{aligned}
\zeta_p(s) &= 2\sum_{b=1}^{B}\left(\sum_{n=-\infty}^{-1}\left(\frac{n\pi}{L_b}\right)^{-s} + \sum_{n=1}^{\infty}\left(\frac{n\pi}{L_b}\right)^{-s}\right) \\
&= 2(e^{-\mathrm{i}\pi s}+1)\zeta_R(s)\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-s}
\end{aligned}$$

$$\zeta_l(s) = 0 \quad \text{if } \operatorname{Re}(s) > 0$$

# Spectral Zeta Function – Rose Graph

$$\zeta_b(s) = e^{\mathrm{i}(\pi-\alpha)s}\frac{2\sin\pi s}{\pi}\int_0^\infty u^{-s}\frac{\mathrm{d}}{\mathrm{d}u}\log f(ue^{\mathrm{i}\alpha})\,du$$

This converges for $0 < \mathrm{Re}(s) < 2$.

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$$\zeta_b(s) = e^{\mathrm{i}(\pi-\alpha)s}\frac{2\sin\pi s}{\pi}\int_0^\infty u^{-s}\frac{\mathrm{d}}{\mathrm{d}u}\log\left(ue^{\mathrm{i}\alpha}\hat{f}(u)\right)\,du$$

$$\hat{f}(u) = \sum_{b=1}^{B}\frac{\cos\theta_b - \cos L_b e^{\mathrm{i}\alpha}u}{\sin L_b e^{\mathrm{i}\alpha}u}$$

# Spectral Zeta Function – Rose Graph

**Theorem**

$$\zeta(s) = e^{\mathrm{i}(\pi-\alpha)s}\frac{2\sin s\pi}{\pi}\left[\int_0^1 u^{-s}\frac{\mathrm{d}}{\mathrm{d}u}\log\left(ue^{\mathrm{i}\alpha}\hat{f}(u)\right)\,\mathrm{d}u + \frac{1}{s} + \int_1^\infty u^{-s}\frac{\mathrm{d}}{\mathrm{d}u}\log\hat{f}(u)\,\mathrm{d}u\right] + 2(e^{-\mathrm{i}\pi s}+1)\zeta_R(s)\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-s}$$

*where* $\mathrm{Re}(s) < 2$ *and*

$$\hat{f}(u) = \sum_{b=1}^{B}\frac{\cos\theta_b - \cos L_b e^{\mathrm{i}\alpha}u}{\sin L_b e^{\mathrm{i}\alpha}u}.$$

# Spectral Determinant – Rose Graph

$$\det{}'(\mathcal{D}) = \prod_{j=-\infty}^{\infty}{}' \, k_j^2$$

$$= \exp(-\zeta'(0))$$

$$= \frac{(2\pi)^{2B}(-1)^{B+1}}{B^2} \left( \sum_{b=1}^{B} \frac{\cos\theta_b - 1}{L_b} \right)^2 \prod_{b=1}^{B} \left( \frac{L_b}{\pi} \right)^2$$

# Spectral Zeta Function – General Graph Without Mass

**Theorem**

$$\zeta(s) = e^{\mathrm{i}(\pi-\alpha)s}\frac{2\sin s\pi}{\pi}\left[\int_0^1 u^{-s}\frac{\mathrm{d}}{\mathrm{d}u}\log\left((ue^{\mathrm{i}\alpha})^{4B-1}\hat{f}(u)\right)\,\mathrm{d}u + \frac{4B-1}{s} + \int_1^\infty u^{-s}\frac{\mathrm{d}}{\mathrm{d}u}\log\hat{f}(u)\,\mathrm{d}u\right] + 2(e^{-\mathrm{i}\pi s}+1)\zeta_R(s)\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-s}$$

*where* $\mathrm{Re}(s) < M$ *and*

$$\hat{f}(u) = \det\left(\mathbb{A} + \mathbb{B}\begin{pmatrix}\cot ue^{\mathrm{i}\alpha}L & -\csc ue^{\mathrm{i}\alpha}L \\ -\csc ue^{\mathrm{i}\alpha}L & \cot ue^{\mathrm{i}\alpha}L\end{pmatrix}\right).$$

# Spectral Determinant – General Graph Without Mass

$$\det{}'(\mathcal{D}) = \frac{{c_0}^2(-1)^B}{\det(\mathbb{A} - \mathrm{i}\mathbb{B})^2} \prod_{b=1}^{B} (2L_b)^2$$

$$c_0 = f(0) \neq 0$$

# Spectral Zeta Function – General Graph With Mass

Given the set of roots $\{k_1, k_2, \ldots\}$ of the positive energy secular equation and the set of roots $\{\tilde{k_1}, \tilde{k_2}, \ldots\}$ to the negative energy secular equation, the spectral zeta function is defined as

$$\begin{aligned}
\zeta(s) &= 2\sum_{j=1}^{\infty}{}' \, E(k_j)^{-s} + 2\sum_{j=1}^{\infty}{}' \left(-E(\tilde{k}_j)\right)^{-s} \\
&= 2\sum_{j=1}^{\infty} \left(\sqrt{k_j^2 + m^2}\right)^{-s} + 2\sum_{j=1}^{\infty} \left(-\sqrt{\tilde{k}_j{}^2 + m^2}\right)^{-s} \\
&= \zeta^+(s) + \zeta^-(s).
\end{aligned}$$

# General Graph With Mass

Positive Eigenvalues:

$$f(z) = \det\left(\mathbb{A} + \gamma(z)\mathbb{B}\begin{pmatrix} \cot zL & -\csc zL \\ -\csc zL & \cot zL \end{pmatrix}\right)$$

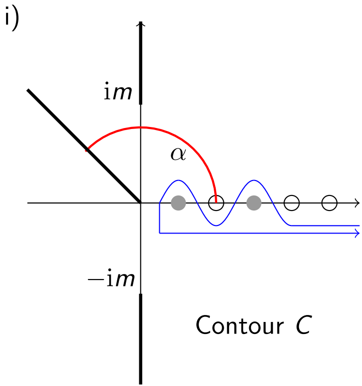
$$\zeta^+(s) = \frac{1}{\mathrm{i}\pi}\int_C (z^2+m^2)^{-s/2}\frac{\mathrm{d}}{\mathrm{d}z}\log f(z)\mathrm{d}z$$

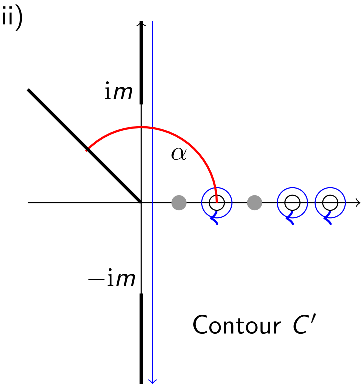
Negative Eigenvalues:

$$g(z) = \det\left(\gamma(z)\mathbb{A} - \mathbb{B}\begin{pmatrix} \cot zL & -\csc zL \\ -\csc zL & \cot zL \end{pmatrix}\right)$$

$$\zeta^-(s) = \frac{(-1)^{-s}}{\mathrm{i}\pi}\int_C (z^2+m^2)^{-s/2}\frac{\mathrm{d}}{\mathrm{d}z}\log g(z)\mathrm{d}z$$

# Contour




The shaded circles are the zeros of $f/g$ and the empty circles are the poles.

# Spectral Zeta Function – General Graph With Mass

$$\zeta_p^+(s) = 2\sum_{b=1}^{B}\sum_{n=1}^{\infty}\left(\left(\frac{n\pi}{L_b}\right)^2 + m^2\right)^{-s/2}$$
$$= 2\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-s} E\left(\frac{s}{2}, \left(\frac{mL_b}{\pi}\right)^2\right)$$

$$\zeta_b^+(s) = \frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)\int_m^{\infty}(t^2 - m^2)^{-s/2}\frac{\mathrm{d}}{\mathrm{d}t}f(\mathrm{i}t)\mathrm{d}t$$

which converges for $-1 < \mathrm{Re}(s) < 1$.

# Spectral Zeta Function – General Graph With Mass

## Theorem

$$\zeta(s) = 2(1+(-1)^{-s})\sum_{b=1}^{B}\left(\frac{\pi}{L_b}\right)^{-s} E\left(\frac{s}{2}, \left(\frac{mL_b}{\pi}\right)^2\right)$$
$$+\frac{2}{\pi}\sin\left(\frac{\pi s}{2}\right)\left[\int_m^\infty (t^2-m^2)^{-s/2}\frac{\mathrm{d}}{\mathrm{d}t}\log\hat{f}(t)\,\mathrm{d}t\right.$$
$$\left.+(-1)^{-s}\int_m^\infty (t^2-m^2)^{-s/2}\frac{\mathrm{d}}{\mathrm{d}t}\log\hat{g}(t)\,\mathrm{d}t\right]$$

*where* $-1 < \mathrm{Re}(s) < 1$ *and*

$$\hat{f}(t) = \det\left(\mathbb{A} + \hat{\gamma}(t)\mathbb{B}\begin{pmatrix}\coth tL & -\operatorname{csch} tL \\ -\operatorname{csch} tL & \coth tL\end{pmatrix}\right)$$
$$\hat{g}(t) = \det\left(\hat{\gamma}(t)\mathbb{A} - \mathbb{B}\begin{pmatrix}\coth tL & -\operatorname{csch} tL \\ -\operatorname{csch} tL & \coth tL\end{pmatrix}\right), \ \textit{and}$$
$$\hat{\gamma}(t) = \frac{\sqrt{t^2-m^2}+\mathrm{i}m}{t}.$$

# Summary

- We found a formulation of the spectral zeta function of the Dirac operator using a contour integral technique.
- In the case of zero mass, we analytically continued our expression to a domain including $s = 0$ and calculated the zeta-regularized spectral determinant.
- We did this first for a rose graph without mass, and then for a general graph with and without mass.

# References

[1] J. Bolte and J. M. Harrison, Spectral statistics for the Dirac operator on graphs, *J. Phys. A: Math. Gen.* 36(11):2747-2769 (2003).

[2] J. Harrison and K. Kirsten, Zeta functions of quantum graphs, *J. Phys. A: Math. Theor.* 44(33):235301, 29 (2011).

[3] J. Harrison, T. Weyand, and K. Kirsten, Zeta functions of the Dirac Operator on quantum graphs, *J. Math. Phys.* 57(10):102301, 17 (2016).


THANK YOU