HARDY-SOBOLEV-MAZ'YA INEQUALITIES FOR FRACTIONAL INTEGRALS ON HALFSPACES AND CONVEX DOMAINS

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To Donna, Nicolas, Jackson, and Craig Jr.

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SUMMARY

This thesis will present new results involving Hardy and Hardy-Sobolev-Maz'ya inequalities for fractional integrals. There are two key ingredients to many of these results. The first is the conformal transformation between the upper halfspace and the unit ball. The second is the pseudosymmetric halfspace rearrangement, which is a type of rearrangment on the upper halfspace based on Carlen and Loss' concept of competing symmetries along with certain geometric considerations from the conformal transformation.

After reducing to one dimension, we can use the conformal transformation to prove a sharp Hardy inequality for general domains, as well as an improved fractional Hardy inequality over convex domains. Most importantly, the sharp constant is the same as that for the halfspace.

Two new Hardy-Sobolev-Maz'ya inequalities will also be established. The first will be a weighted inequality that has a strong relationship with the pseudosymmetric halfspace rearrangement. Then, the psuedosymmetric halfspace rearrangement will play a key part in proving the existence of the standard Hardy-Sobolev-Maz'ya inequality on the halfspace, as well as some results involving the existence of minimizers for that inequality.

CHAPTER I

INTRODUCTION

This work will present results involving three types of related integral inequalities: Hardy, Sobolev, and Hardy-Sobolev-Maz'ya. Hardy inequalities are a well known type of integral inequality, the study of which dates back to the early part of the 20th century. Much literature has been dedicated to the topic in the intervening years. Another well known integral inequality is the Sobolev inequality. Both of these inequalities involve the same integral, but they differ in the particular integral that is the lower bound. A third type of integral inequality is where the remainder of the Hardy inequality in bounded below by the Sobolev term. This is known as a Hardy-Sobolev-Maz'ya inequality.

In classical results, the common integral is a norm of the gradient. However, in recent years, equivalent results for what are referred to as fractional integrals have been discovered. These fractional integrals are related to the gradient through identities involving the Fourier transform. It is these fractional integrals for which this paper is concerned.

Integral inequalities are types of variational and optimization problems. After proving the existence of the inequality, one can still ask what is the best constant possible, also referred to as the sharp constant, and whether there exists a function, called an optimizer, in the target space so that the inequality is an equality. This paper shall prove results involving each of these.

The structure of the paper is as follows. In this Chapter, we shall present the main results of the thesis, followed by a brief discussion of applications and related areas. In the following Chapter, we will define the basic concepts necessary for a complete and thorough presentation of the topics contained herein, and present the historical development of the subject leading to the results proven in this paper.

Chapter 3 will present some additional definitions and tools that, while of interest in their own right, will be of much use in later Chapters. In particular, it will discuss properties and results related to conformal transformations, in particular the one between the unit ball and the upper halfspace, and it will discuss two types of rearrangements as well. One of the rearrangements is the well-known spherically symmetric decreasing rearrangement. The other is a new result, which we shall refer to as the pseudosymmetric halfspace rearrangement, that has the property that the resulting function can be written as a product of two functions, one known, and one having a known symmetry.

In Chapter 4, we prove several one-dimensional fractional Hardy inequalities, a new fractional Hardy inequality over general domains in $\mathbb{R}^n$, and an improved fractional Hardy inequality over convex domains in $\mathbb{R}^n$. The sharp constant for the latter inequality will be the same as that for the sharp fractional Hardy inequality on the upper halfspace. To establish the one-dimensional results, we make use of the conformal transformation, and to establish the n-dimensional results, we reduce the problem to one dimension.

Fractional Hardy-Sobolev-Maz'ya inequalities are the main topics of Chapters 5 and 6. In the former, we establish the existence of two such inequalities. The first is a weighted inequality that has a notable relationship with the pseudosymmetric halfspace rearrangement. The second is a standard Hardy-Sobolev-Maz'ya inequality for the halfspace. To prove this, it is shown that the minimizers are those obtained from the pseudosymmetric halfspace rearrangement. Then, the inequality is proven using, among other things, the weighted Hardy-Sobolev-Maz'ya inequality mentioned above. Finally, in Chapter 6, we prove some results regarding the sharp constant and existence of an optimizer.

1.1 Main Results

This paper presents two new important theorems answering open questions in the area of fractional Hardy and Sobolev inequalities, as well as several related results for each. The first is a Hardy inequality with for functions supported in either a general or convex domain. We first present some brief notation that shall be discussed in more detail later.

As in [18], let Ω be any domain in $\mathbb{R}^n$ with non-empty boundary. Fix a direction $w \in \mathbb{S}^{n-1}$ and define

$$d_{w,\Omega}(x) = \min\{|t| \;:\; x + tw \notin \Omega\},$$

$$\delta_{w,\Omega}(x) = \sup\{|t| : x + tw \in \Omega\},$$

and set

$$\frac{1}{M_\alpha(x)^\alpha} = \frac{\int_{\mathbb{S}^{n-1}} \left[\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)}\right]^\alpha \mathrm{d}w}{\int_{\mathbb{S}^{n-1}} |w_n|^\alpha \,\mathrm{d}w}.$$

We then have the following result.

THEOREM. *Let $1 < \alpha < 2$. For any $f \in C_c^\infty(\Omega)$,*

$$\int\limits_\Omega \int\limits_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \geq D_{n,2,\alpha} \int\limits_\Omega \frac{|f(x)|^2}{M_\alpha(x)^\alpha} \,\mathrm{d}x.$$

In particular if Ω is a convex region, then, for any $f \in C_c^\infty(\Omega)$,

$$\int\limits_\Omega \int\limits_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \geq D_{n,2,\alpha} \int\limits_\Omega |f(x)|^2 \left[\frac{1}{d_{\Omega(x)}} + \frac{1}{D_\Omega(x) - d_\Omega(x)}\right]^\alpha \mathrm{d}x,$$

When Ω is convex, the constant

$$D_{n,2,\alpha} = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \frac{\beta\left(\frac{1+\alpha}{2}, 1 - \frac{\alpha}{2}\right) - 2^\alpha}{\alpha 2^\alpha},$$

is the best possible.

Note that $d_\Omega(x)$ is the distance of $x \in \Omega$ to the boundary of Ω and $D_\Omega(x)$ is, essentially, the width of the smallest slab, consisting of two bounding hyperplanes containing Ω. This result is proven below in two parts in Theorems 4.2.2 and 4.2.3.

The second result is the existence of a fractional Hardy-Sobolev-Maz'ya inequality on the upper halfspace $\mathbb{H}^n$, proven in Theorem 5.2.1.

THEOREM. *Let $n \geq 2$, $1 < \alpha < 2$. There exists $M_{n,\alpha} > 0$ so that*

$$\int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y - D_{n,2,\alpha} \int_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha} \, \mathrm{d}x \geq M_{n,\alpha} \left(\int_{\mathbb{H}^n} |f(x)|^{2^*} \, \mathrm{d}x \right)^{2/2^*},$$

for all $f \in C_c^\infty(\mathbb{H}^n)$.

Here, note $D_{n,2,\alpha}$ is the same constant as in the Hardy inequality above, and $2^* = 2n/(n - \alpha)$.

1.2 Applications and Related Areas

The applications of the results in this paper are wide and varied. Of particular importance recently is research into fractional analogues of Brownian motion. In the study of stochastic processes, the classical Hardy inequality

$$\int_\Omega |\nabla f(x)|^p \, \mathrm{d}x \geq D_p^\Omega \int_\Omega \frac{|f(x)|^p}{d(x)^p} \, \mathrm{d}x,$$

for $n \geq 2$, $1 < p < \infty$, and Ω a Lipschitz domain, is related to the study of what is known as killed Brownian motion. The norm of the gradient in the Hardy inequality is known as the Dirichlet integral and represents a bilinear form that is associated with Brownian motion that is killed upon leaving Ω [17]. This process can be generalized by replacing the norm of the gradient with fractional integrals, as discussed in detail in this paper. As such, one then studies what are referred to as α-stable processes in $\mathbb{R}^n$ for $0 < \alpha < 2$, which are a particular Lévy process. The Dirichlet form for an α-stable process in $\mathbb{R}^n$ is the double integral

$$a_{n,\alpha} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y, \tag{1}$$

where $n \geq 1$, $0 < \alpha < 2$, and the "core" of functions that are studied under this form are smooth functions. [10].

If one wished to study the killed α-stable process in Ω, then the Dirichlet form is the same, but the core is limited to smooth functions with support in Ω. Alternatively, another way to study functions with boundary conditions would be to study the censored stable process. Loosely speaking, a censored stable process is a stable process with the jumps between Ω and its complement suppressed. It has the same core as the killed α-stable process, but its Dirichlet form is

$$a_{n,\alpha} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y, \tag{2}$$

It has been suggested that the censored stable process is a better generalization and more closely resembles the killed Brownian motion than the killed stable process. See [17],[9]. The reference [9] contains the construction of censored stable processes and a wealth of information about them. For the connection between Hardy inequalities and censored stable processes, we refer the reader to [17].

While Hardy inequalities for fractional integrals are of interest in their own right, one particular use is that they deliver spectral information on the generators of censored stable processes. The generator of a censored stable process is defined by the closure of the quadratic form defined by (2). In particular, the generator of the censored α-stable process in Ω is

$$\Delta_{\Omega}^{\alpha/2} u(x) = 2a_{n,\alpha} \lim_{\epsilon \to 0^+} \int_{\Omega \cap \{|x-y| > \epsilon\}} \frac{u(y) - u(x)}{|x - y|^{n+\alpha}} \, \mathrm{d}y, \tag{3}$$

referred to as the regional fractional Laplacian. When $\Omega = \mathbb{R}^n$, then (3) is called the fractional Laplacian, which is the generator for the integral operator defined in (1). The fractional Laplacian is the generalization of the usual Laplacian operator Δ, well-known in differential equations. Indeed, the fractional Laplacian is an example of a non-local integro-differential operator with applications to, among others, potential theory, magnetic fields, and a wide class of physical systems, including Lévy flights and stochastic interfaces. See, generally, [23],[8], and [41].

The results herein are also related to so-called ground state representations and mathematical physics. From Schrödinger's time independent equation

$$E[u(x)] = -\Delta u(x) + V(x)u(x),$$

we have the Hamiltonian operator $H = -\Delta + V$, where V is a potential. If we minimize the functional $E[u]$ over $u \in L^2(\mathbb{R}^n)$ such that $\|u\|_2 = 1$, then the minimal value E_0 is called the ground state energy, and any minimizer, if it exists, is called the ground state.

Consider the Hardy-Sobolev-Maz'ya inequality

$$\int\limits_{\mathbb{H}^n} |\nabla f(x)|^2 \, \mathrm{d}x - \frac{1}{4} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^2} \, \mathrm{d}x \geq M_n \left(\int\limits_{\mathbb{H}^n} |f(x)|^{\frac{2n}{n-2}} \, \mathrm{d}x \right)^{\frac{n-2}{n}}, \tag{4}$$

for $n \geq 3$, $M_n > 0$, and all f in the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the left side of (4). The left hand side of (4) can be represented by the operator $-\Delta - V$. Similarly, we can generalize this for fractional integrals by changing $-\Delta$ to $(-\Delta)^{\alpha/2}$ and letting $V(x) = x_n^{-\alpha}$. Thus, the sharp Hardy inequality for the halfspace is

$$\begin{aligned} \int\limits_{\mathbb{H}^n} \int\limits_{\mathbb{H}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y - D_{n,p,\alpha} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^p}{x_n^\alpha} \, \mathrm{d}x \\ \geq m_p \int\limits_{\mathbb{H}^n} \int\limits_{\mathbb{H}^n} \frac{|g(x) - g(y)|^p}{|x-y|^{n+\alpha}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}}, \end{aligned}$$

where $n \geq 1$, $2 \leq p < \infty$, $0 < \alpha < p$ with $\alpha \neq 1$, and $g(x) = x_n^{(1-\alpha)/p} f(x)$. If $p = 2$, then this is an equality with $m_p = 1$. In this case, the equation is the ground state representation. See [29],[23], and [24] for discussion and further examples. The fractional Hardy-Sobolev-Maz'ya inequality that is one of the main results of this paper is further application of this.

CHAPTER II

HISTORICAL BACKGROUND

2.1 Basic Definitions and Notation

In this section, we give precise definitions to be used throughout the paper. Many of the inequalities in this paper will be for functions with support in the upper halfspace of the n-dimensional real Euclidean space. The remaining results will be over general domains, convex domains, and bounded domains that are "sufficiently regular," that is, essentially, locally the graph of a Lipschitz continuous function. In what follows, unless otherwise specified, all sets shall be open.

Let $\mathbb{R}$ denote the set of real numbers, and let n be a positive integer. We denote the n-dimensional real Euclidean space by

$$\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}, i = 1, \dots, n\},$$

and we further denote the upper halfspace of $\mathbb{R}^n$ by

$$\mathbb{H}^n = \left\{x = (x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\right\}.$$

For any $\Omega \subseteq \mathbb{R}^n$, we say say that Ω is convex if, for all $x, y \in \Omega$ and all $t \in [0, 1]$, then $(1-t)x + ty \in \Omega$. In other words, a set Ω is convex if for every $x, y \in \Omega$, the line segment that connects x and y is contained in Ω.

DEFINITION 2.1.1. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, and let $\partial\Omega$ denote the boundary of Ω. Then Ω is called a a Lipschitz domain if for every point $x \in \partial\Omega$, there exists a radius $r > 0$ and a map $A_x : B_r(x) \to B_1(\mathbf{0})$ such that

1. A_x is a bijection;

2. A_x and A_x^{-1} are both Lipschitz continuous functions;

3. $A_x(\partial\Omega \cap B_r(p)) = \{x \in B_1(\mathbf{0}) : x_n = 0\}$; and

4. $A_x(\Omega \cap B_r(p)) = \{x \in B_1(\mathbf{0}) : x_n > 0\}$,

where $B_r(x)$ is the open ball of radius r about x, and $\mathbf{0}$ is the origin in $\mathbb{R}^n$.

For these Lipschitz domains, we will often be concerned with the distance to the boundary from an interior point. Let $\Omega \subseteq \mathbb{R}^n$ have a non-empty boundary. We denote the distance to the boundary of Ω by

$$d_\Omega(x) := \mathrm{dist}(x, \partial\Omega),$$

for all $x \in \mathbb{R}^n$, and we write simply $d(x)$ if no confusion arises. Further, we denote the inradius of Ω by

$$d_\Omega^0 := \sup_{x\in\Omega} d_\Omega(x),$$

which represents the radius of the largest circle that can be inscribed in Ω.

We also note here some values that will be used throughout the paper. Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$, where $\mathbb{S}^0$ refers to the interval $[0, 1]$. Then, the Lebesgue measure, or surface area in this case, of that sphere is

$$|\mathbb{S}^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)},$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}t,$$

is known as the Gamma function. Similarly, the measure of the unit ball is

$$|B_1(x)| = \frac{1}{n}|\mathbb{S}^{n-1}|,$$

for any $x \in \mathbb{R}^n$. Another value we will use is the Euler beta function, defined as

$$\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t,$$

for positive real numbers x, y. An often used identity for the beta function is

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Finally, we define the following commonly used function

$$1_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega \end{cases}.$$

This is known as the indicator, or characteristic, function on the set Ω.

2.2 Function Spaces

Now, all the inequalities discussed in this Chapter and those proven later are defined for smooth functions with compact support. However, this is the minimum space for which the inequalities can be defined. Other spaces of interest are L^p spaces and Sobolev spaces.

2.2.1 Common function spaces

We state herein the definitions for the most common function spaces that will be found in this paper. First, we denote by $C_c^\infty(\Omega)$ the the set of all infinitely differentiable functions of compact support whose support is contained in Ω, often referred to as smooth functions. Now, we state the definition for L^p spaces.

DEFINITION 2.2.1. Let $1 \le p < \infty$, and let f be a measurable function, with usual Lebesgue measure. Then, we define the space

$$L^p(\Omega) = \left\{ f : \|f\|_{p(\Omega)} = \left(\int_\Omega |f|^p \right)^{1/p} < \infty \right\}.$$

If $\Omega = \mathbb{R}^n$, or the context is otherwise clear, we write $\|f\|_p$ for the L^p-norm.

To be precise, the norm defined above requires that each f is an equivalence class. In particular, we identify two functions f and g if $f = g$ almost everywhere (that is, they differ only on a set of measure zero).

We can also define a broader class of functions known as locally p^{th}-power integrable functions. This space, however, is not a normed space.

DEFINITION 2.2.2. Let $1 \leq p < \infty$, and let f be a measurable function defined on all Ω. Then, we define the space

$$L^p_{loc}(\Omega) = \left\{ f : \|f\|_{p(K)} < \infty, \forall \text{ compact } K \subset \Omega \right\}.$$

2.2.2 Distributions

In general, we are often interested in spaces of functions known as Sobolev spaces, as it is often the case that the optimizer of an inequality, if it exists, lies in a Sobolev space. To define the Sobolev spaces, we must first understand what a distribution is, and these are defined with respect to the space of test functions.

Note that a multiindex refers to any a vector $a \in \mathbb{R}^n$ such that each a_j is a nonnegative integer. If $a = (a_1, \ldots, a_n)$ is a multiindex, and $f \in C_c^\infty(\Omega)$, then

$$D^a f = \left(\frac{\partial}{\partial x_1}\right)^{a_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{a_n} f.$$

DEFINITION 2.2.3. The space of test functions $D(\Omega)$ is the space of all functions in $C_c^\infty(\Omega)$ accompanied by the following notion of convergence. Let f_j be a sequence in $C_c^\infty(\Omega)$, and let $f \in C_c^\infty(\Omega)$. Then, if the following conditions hold:

1. there is a fixed, compact set $K \subset \Omega$ so that $\bigcup_k \operatorname{supp} f_k \subset K$, and
2. for each multiindex a, the sequence of partial derivatives $D^a f_j$ converges uniformly to $D^a f$,

then f_j converges to f in $D(\Omega)$.

The specification of how the test functions converge allows a topology to be defined on the space, so that $D(\Omega)$ can be shown to be a locally convex topological vector space. This classification allows us to define the space of distributions, which will be the dual space of $D(\Omega)$.

DEFINITION 2.2.4. A distribution on Ω is a linear functional $T : D(\Omega) \to \mathbb{C}$ such that

$$\lim_{j\to\infty} T(f_j) = T\left(\lim_{j\to\infty} f_j\right),$$

for any convergent sequence $\{f_j\}$ in $D(\Omega)$. The space of all distributions on Ω is denoted $D'(\Omega)$.

We want to associate with each distribution a function. We can do this easily, since integrals are a common type of linear functional. Let $f \in L^1_{loc}(\Omega)$ and $\phi \in D(\Omega)$. Then, if we let

$$T_f(\phi) := \int_\Omega f\phi,$$

it is easy to show that $T_f \in D'(\Omega)$. We then associate the distribution T_f with the function f. In fact, we say the distribution is the function. These functions are uniquely determined by the distribution, as was shown in [29].

THEOREM 2.2.5. *Let $f, g \in L^1_{loc}(\Omega)$. Suppose that the distributions defined by f, g are equal. That is, $\int_\Omega f\phi = \int_\Omega g\phi$, for all $\phi \in D(\Omega)$. Then $f(x) = g(x)$ almost everywhere.*

Although we associate functions with distributions, it turns out that not all distributions are functions, and the most significant example is what is referred to as the Dirac delta "function", most simply defined as

$$\delta(x) = \begin{cases} \infty, & \text{if } x = 0 \\ 0, & \text{if } x \neq 0 \end{cases}.$$

However, while distributions are not always associated with functions, distributions do have a clear relationship to measures. Indeed, the set of positive distributions (where the distibution is nonnegative when acting upon a nonnegative test function) is equivalent to the set of Borel measures [29].

At this point, with the above definition of a distribution, we can now define what it means to take the derivative of a non-differentiable function.

DEFINITION 2.2.6. Let $T \in D'(\Omega)$, and let $a = (a_1, \ldots, a_n)$ be a multiindex. We define the distributional, or weak, derivative $D^a T$ with respect to each $\phi \in D(\Omega)$ by

$$(D^a T)(\phi) = (-1)^{|a|} T(D^a \phi),$$

where $|a| = \sum_{i=1}^{n} a_i$.

If $a_i = 1$, $a_j = 0, \forall j \neq i$, then we write $\partial_i T$ for $D^a T$. We also denote the n-tuple of component functions $(\partial_1 T, \ldots, \partial_n T)$ by the symbol ∇T, known as the distributional gradient. Note that if f is a differentiable function in each variable, then by integration by parts,

$$\int_\Omega (\partial_i f)\phi = -\int_\Omega (\partial_i \phi) f.$$

Hence, if f is a distribution, then the distributional gradient denoted ∇f is defined to be the n-tuple of functions satisfying

$$\int_\Omega (\nabla f)\phi = -\int_\Omega (\nabla \phi) f,$$

for all $\phi \in D(\Omega)$. As one would hope, if f is differentiable, then its classical derivatives correspond with its distributional derivatives [29]. Still, while it can be shown that $D^a T$ is a distribution, it may not be a function if the distribution T is associated with a nondifferentiable function.

2.2.3 Sobolev spaces

With the definitions above, we can now define the Sobolev spaces.

DEFINITION 2.2.7. Let $1 \leq p < \infty$ and let m be a positive integer. Then, we define the Sobolev spaces

$$W^{m,p}(\Omega) = \left\{ f : \|f\|_{W^{m,p}(\Omega)} = \left(\sum_{|a| \leq m} \|D^a f\|_{p(\Omega)}^p \right)^{1/p} < \infty \right\},$$

where $D^a f$ is considered as a distributional derivative with multiindex $|a|$, and, in particular, $D^0 f = f$.

Under the given norm, these spaces are all Banach spaces, that is, complete vector spaces [2]. Probably the most important case is where $p = 2$. In that instance, we denote this space also by $H^m(\Omega)$ and assign the following inner product

$$\langle f, g\rangle_{H^m(\Omega)} = \sum_{|a|\leq m} \int_\Omega \overline{D^a f} D^a g.$$

Then, $H^m(\Omega)$ is a Hilbert space.

Throughout this paper, we will only be interested in the case $m = 1$. As a result, we can restate the Sobolev norm in terms of the distributional gradient, as well as define an additional space of interest.

DEFINITION 2.2.8. Let $1 \leq p < \infty$. Then, we define the Sobolev spaces

$$W^{1,p}(\Omega) = \left\{ f : \|f\|_{W^{1,p}(\Omega)} = \left(\|f\|_{p(\Omega)}^p + \|\nabla f\|_{p(\Omega)}^p \right)^{1/p} < \infty \right\},$$

and

$$\dot{W}^{1,p}(\Omega) = \left\{ f : \|f\|_{\dot{W}^{1,p}(\Omega)} = \|\nabla f\|_{p(\Omega)} < \infty \right\},$$

where ∇f is considered as a distributional gradient.

We are also interested in the following subsets of these spaces. In particular, these subspaces are used in the study of differential equations, and elsewhere, with "zero boundary" conditions on the boundary of Ω.

DEFINITION 2.2.9. Let $1 \leq p < \infty$. Then, we define the Sobolev space $W_0^{m,p}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{m,p}(\Omega)}$, and we define the Sobolev space $\dot{W}_0^{1,p}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{\dot{W}^{1,p}(\Omega)}$.

While it should be clear that the former is a Banach space, it is a result of the Hardy inequalities, stated below, that the latter is as well. For certain Ω, some of the spaces in this section coincide. One well-known example is from [2].

THEOREM 2.2.10. *Let $1 \leq p < \infty$ and let m be a positive integer. Then, $W^{m,p}(\mathbb{R}^n) = W_0^{m,p}(\mathbb{R}^n)$.*

2.3 *Classical Results*

2.3.1 Hardy inequalities

The original inequality studied by Godfrey Harold "G. H." Hardy [26] was the one-dimensional inequality

$$4\int_0^\infty |f'(x)|^2 u dx \geq \int_0^\infty |f(x)|^2 x^{-2}\,\mathrm{d}x.$$

In higher dimensions, there are several different types of Hardy inequalities. The primary types are weighted norm inequalities whereby the weight is the square of the distance to the origin or to the boundary of some Lipschitz domain Ω.

For general Lipschitz domains, the following Theorem is well known. For convex Ω, the sharp constant for $n = 2$ was proven in [32]. Later, in [31], the sharp constant, over convex Ω, was proven for all n.

THEOREM 2.3.1. *Let $n \geq 2$ and $1 < p < \infty$. If Ω is a Lipschitz domain, then there exists $D_p^\Omega > 0$ so that*

$$\int_\Omega |\nabla f(x)|^p\,\mathrm{d}x \geq D_p^\Omega \int_\Omega \frac{|f(x)|^p}{d(x)^p}\,\mathrm{d}x,$$

for all $f \in \dot{W}_0^{1,p}(\Omega)$. If Ω is convex, then $D_p^\Omega = \left(\frac{p-1}{p}\right)^p$.

Thus, the sharp constant D_p^Ω varies with the domain, unless the domain is convex. A special case of particular interest herein is when Ω is the upper halfspace.

THEOREM 2.3.2. *Let $1 < p < \infty$, then for all $f \in \dot{W}_0^{1,p}(\mathbb{H}^n)$,*

$$\int_{\mathbb{H}^n} |\nabla f(x)|^p\,\mathrm{d}x \geq \left(\frac{p-1}{p}\right)^p \int_{\mathbb{H}^n} \frac{|f(x)|^p}{x_n^p}\,\mathrm{d}x,$$

where $\left(\frac{p-1}{p}\right)^p$ is the best possible constant.

The optimal constant is also known for the Hardy inequality involving the distance to the origin. See, e.g., [24].

THEOREM 2.3.3. *Let $1 \leq p < n$, then for all $f \in \dot{W}_0^{1,p}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} |\nabla f(x)|^p \,\mathrm{d}x \geq \left(\frac{|n-p|}{p}\right)^p \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} \,\mathrm{d}x,$$

where $\left(\frac{|n-p|}{p}\right)^p$ is the best possible constant.

2.3.2 Sobolev inequality

Now, in addition to the Hardy inequalities above, where the L^2-norm of the gradient is bounded below by a constant times a Hardy term, we can also bound the gradient from below by another norm. In particular, a norm involving the critical Sobolev exponent $pn/(n-p)$. Thus, from [38] we get the following inequality, called a Sobolev inequality.

THEOREM 2.3.4. *Let $1 < p < n$, and let $f \in \dot{W}_0^{1,p}(\mathbb{R}^n)$. Then,*

$$\|\nabla f\|_p^p \geq S_{n,p} \|f\|^p_{\frac{pn}{n-p}},$$

where

$$S_{n,p} = \pi n^{2/p} \left(\frac{n-p}{p-1}\right)^{2-\frac{2}{p}} \left(\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(1+n/2)\Gamma(n)}\right)^{2/n}$$

is the best possible constant. Equality is attained by functions of the form

$$A\left(\gamma^2 + |x-a|^{\frac{p}{p-1}}\right)^{1-n/p},$$

where $A, \gamma \in \mathbb{R}$ are nonzero, and $a \in \mathbb{R}^n$.

This inequality is part of a class of inequalities that are used to prove Sobolev embedding theorems, which give inclusions between certain Sobolev spaces. They are named after the Russian mathematician Sergei Lvovich Sobolev. In particular, the following Sobolev embedding theorem follows automatically from Theorem 2.3.4.

COROLLARY 2.3.5. *Let $1 < p < n$, then $\dot{W}^{1,p}(\mathbb{R}^n) \subseteq L^{\frac{pn}{n-p}}(\mathbb{R}^n)$.*

2.3.3 Hardy-Sobolev-Maz'ya inequality

Consider now the remainder of the Hardy inequalities mentioned above. Then, one can ask whether such a remainnder can be bounded below by a Sobolev term. That is, can the L^P-norm of the gradient be bounded below by both a Hardy and a Sobolev term? Indeed, this is known as the Hardy-Sobolev-Maz'ya inequality, and it was first shown by Maz'ya in [34] for $p = 2$.

THEOREM 2.3.6. *Let $n \geq 3$. Then, there exists $M_n > 0$ so that*

$$\int\limits_{\mathbb{H}^n} |\nabla f(x)|^2 \, \mathrm{d}x - \frac{1}{4} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^2} \, \mathrm{d}x \geq M_n \left(\int\limits_{\mathbb{H}^n} |f(x)|^{\frac{2n}{n-2}} \, \mathrm{d}x \right)^{\frac{n-2}{n}}, \tag{5}$$

for all f in the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the left side of (5).

With certain limiting assumptions, a similar inequality was proven for certain bounded domains in [21] for $2 \leq p < n$, with the distance to supporting hyperplane x_n being replaced by the distance to the boundary $d(x)$. Also, a weighted sharp Hardy-Sobolev-Maz'ya type inequality has been shown in [3], where, instead of distance to the boundary, the Hardy term involves distance taken to the origin.

Now, as with any optimization problem, we next ask whether the exact value of the best possible constant is known, and whether there exists a nonzero function, known as an optimizer, so that equality is obtained. In the classical case, some of these results have been established. In particular, if $n \geq 4$, then it is known that $M_n < S_{n,2}$, although the exact value of M_n is unknown. See [39], [7]. If $n = 3$, however, then it was found in [7] that $M_3 = S_{3,2}$. While there is no function for which equality is obtained when $n = 3$ [7], such an optimizer does exist in the space specified in Theorem 2.3.6 when $n \geq 4$ [39]. Further improvements in the general case have been shown in [35], [22], and [21].

2.4 Fractional Integrals

2.4.1 Basic definitions and result

Herein, we seek to generalize the concept of the norm of a gradient. This is done through use of the Fourier transform, of which we use the following convention for defining herein:

$$\hat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i(k,x)} f(x)\, \mathrm{d}x,$$

for all $f \in L^1(\mathbb{R}^n)$, where $(k,x) = \sum_{k=1}^n k_i x_i$. The usual method for defining the Fourier transform applies for $f \in L^2(\mathbb{R}^n)$. If we let $f \in H^1(\mathbb{R}^n)$, then it is a known fact from Fourier analysis that $\widehat{\nabla f}(k) = 2\pi i \hat{f}(k)$, and therefore, by Plancherel's Theorem,

$$\|\nabla f\|_2^2 = \int_{\mathbb{R}^n} |2\pi k|^2 |\hat{f}(k)|^2\, \mathrm{d}k.$$

We use this expression to make the following generalization.

DEFINITION 2.4.1. Let $0 < \alpha < 2$. Then, for all $f \in L^2(\mathbb{R}^n)$, we define the following operator

$$(f, (-\Delta)^{\alpha/2} f) = \int_{\mathbb{R}^n} (2\pi|k|)^\alpha\, |\hat{f}(k)|^2\, \mathrm{d}k. \tag{6}$$

Note the relation of this operator to the fractional Laplacian, as discussed in the applications section in the previous Chapter. Since the L^2-norm of a Fourier transform is equal to the L^2-norm of the function itself, then the following result, a proof of which is given in [23], follows.

LEMMA 2.4.2. *Let $n \geq 1$ and $0 < \alpha < 2$. Then, for all $f \in L^2(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (2\pi|k|)^\alpha\, |\hat{f}(k)|^2\, \mathrm{d}k = a_{n,\alpha} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}}\, \mathrm{d}x\, \mathrm{d}y, \tag{7}$$

where

$$a_{n,\alpha} = 2^{\alpha-1}\pi^{-n/2} \frac{\Gamma(\frac{n+\alpha}{2})}{\left|\Gamma(-\frac{\alpha}{2})\right|}. \tag{8}$$

It should be noted that $a_{n,\alpha} \to 0$ as $\alpha \to 2$, which is consistent with the fact that the double integral on the right-hand side of (7) does not converge upon the L^2 norm of the gradient. Similarly, $a_{n,\alpha} \to 0$ as $\alpha \to 0$.

2.4.2 Fractional Sobolev spaces

Before we present the fractional analogues to the classical inequalities, we must first define analogous spaces to the Sobolev spaces above.

DEFINITION 2.4.3. Let $0 < \alpha < 2$. Then, we define the fractional Sobolev space

$$W^{\alpha/2,2}(\mathbb{R}^n) = \left\{ f : \|f\|_{W^{\alpha/2,2}(\mathbb{R}^n)} = \left\{ \int\limits_{\mathbb{R}^n} (1 + |2\pi k|^\alpha)\, |\hat{f}(k)|^2 \, \mathrm{d}k \right\}^{1/2} < \infty \right\}.$$

The Fourier transform is well defined, since by Plancherel's theorem

$$\|f\|^2_{W^{\alpha/2,2}(\mathbb{R}^n)} = \|\hat{f}\|_2^2 + (f, (-\Delta)^{\alpha/2} f) = \|f\|_2^2 + (f, (-\Delta)^{\alpha/2} f).$$

Similar to the Sobolev space $W^{m,p}(\mathbb{R}^n)$, the space $W^{\alpha/2,2}(\mathbb{R}^n)$ can also be made into a Hilbert space. To do so, we associate the following inner product

$$\langle f, g \rangle_{H^{\alpha/2}(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} (1 + |2\pi k|^\alpha)\, \overline{\hat{f}(k)} \hat{g}(k) \, \mathrm{d}k,$$

and denote the Hilbert space by $H^{\alpha/2}(\mathbb{R}^n)$.

We can also characterize the norm in terms of the double integral in (7). This will also allow us to generalize to subsets Ω contained in $\mathbb{R}^n$, as well as for the cases $p \geq 2$.

DEFINITION 2.4.4. Let $0 < \alpha < 2$, $1 \leq p < \infty$, and let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain. Then, we define the fractional Sobolev space

$$W^{\alpha/2,p}(\Omega) = \left\{ f : \|f\|_{W^{\alpha/2,p}(\Omega)} = \left(\|f\|_p^p + \int\limits_\Omega \int\limits_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \right)^{1/p} < \infty \right\}.$$

DEFINITION 2.4.5. Let Ω be the upper halfspace $\mathbb{H}^n$ with $0 < \alpha < 2$, or let Ω be a convex or bounded Lipschitz domain with $1 < \alpha < 2$. Then, for all $1 \leq p < \infty$, we define the fractional Sobolev space

$$\dot{W}^{\alpha/2,p}(\Omega) = \left\{ f : \|f\|_{\dot{W}^{\alpha/2,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \right)^{1/p} < \infty \right\}.$$

Now, as above for the classical Sobolev spaces, we are also interested in fractional integrals that have "zero boundary conditions" on the boundary of Ω.

DEFINITION 2.4.6. Let Ω be the upper halfspace $\mathbb{H}^n$ with $0 < \alpha < 2$, $\alpha \neq 1$, or let Ω be a convex or bounded Lipschitz domain with $1 < \alpha < 2$. We define the Sobolev space $W_0^{\alpha/2,2}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{W^{\alpha/2,2}(\Omega)}$, and we define the Sobolev space $\dot{W}_0^{\alpha/2,2}(\Omega)$ as the completion of $C_c^\infty(\Omega)$ with respect to $\|\cdot\|_{\dot{W}^{\alpha/2,2}(\Omega)}$.

Now, as will be seen below, Hardy inequalities exist only with respect to certain Ω and with respect to particular α. Most importantly herein, they exist when Ω is the upper halfspace $\mathbb{H}^n$ and $0 < \alpha < 2$, and when Ω is convex or any bounded Lipschitz domain and $1 < \alpha < 2$. While it is clear that $W_0^{\alpha/2,2}(\Omega)$ is a Banach space, it is a result of the Hardy inequalities, stated below, that $\dot{W}_0^{\alpha/2,2}(\Omega)$ is as well.

Finally, similar to above, we have from [2]

THEOREM 2.4.7. *Let $0 < \alpha < 2$. Then, $W^{\alpha/2,2}(\mathbb{R}^n) = W_0^{\alpha/2,2}(\mathbb{R}^n)$.*

2.4.3 Fractional Hardy inequalities

With these definitions and generalizations, we can state fractional analogues to the Hardy inequalities given above. The following is the analogue of Theorem 2.3.1 for bounded Lipschitz domains, as proven in [19].

THEOREM 2.4.8. *Let $n \geq 1$, $\alpha > 1$, and $0 < p < \infty$. If Ω is a bounded Lipschitz domain, then there exists $D^{\Omega}_{n,p,\alpha} > 0$ so that*

$$\int_{\Omega}\int_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \geq D^{\Omega}_{n,p,\alpha}\int_{\Omega}\frac{|f(x)|^p}{d(x)^{\alpha}}\,\mathrm{d}x,$$

for all $f \in \dot{W}_0^{\alpha/2,p}(\Omega)$.

This result is just an existence proof. Although represented here by $D^{\Omega}_{n,p,\alpha}$, the exact value of the sharp constant is unknown.

We now state two sharp Hardy inequalities with remainder. First, we state the following sharp Hardy inequality with remainder, an analogue of Theorem 2.3.3 involving the distance to the origin [24].

THEOREM 2.4.9. *Let $n \geq 1$, $0 < \alpha < \min\{p, n\}$, and $p \geq 2$. Then, letting $g(x) = |x|^{(n-\alpha)/p} f(x)$, for all $f \in \dot{W}_0^{\alpha/2,p}(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y - C_{n,p,\alpha}\int_{\mathbb{R}^n}\frac{|f(x)|^p}{|x|^{\alpha}}\,\mathrm{d}x$$
$$\geq m_p \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|g(x)-g(y)|^p}{|x-y|^{n+\alpha}}\frac{\mathrm{d}x}{|x|^{(n-\alpha)/2}}\frac{\mathrm{d}y}{|y|^{(n-\alpha)/2}},$$

where $0 < m_p \leq 1$ is given by

$$m_p = \min_{0<u<1/2}\left((1-u)^p - u^p + pu^{p-1}\right), \tag{9}$$

and the optimal constant in the Hardy inequality is

$$C_{n,p,\alpha} = 2\int_0^1 r^{\alpha-1}|1-r^{(n-\alpha)/p}|^p \Phi_{n,\alpha}\,\mathrm{d}r,$$

with

$$\Phi_{n,\alpha} = |\mathbb{S}^{n-2}|\int_{-1}^{1}\frac{(1-t^2)^{(n-3)/2}}{(1-2tr+r^2)^{(n+\alpha)/2}}\,\mathrm{d}t.$$

If $p = 2$, then this is an equality with $m_p = 1$.

A similar result has also been stated in [25] for the sharp fractional Hardy inequality with remainder on the halfspace.

THEOREM 2.4.10. *Let $n \geq 1$, $2 \leq p < \infty$, and $0 < \alpha < p$ with $\alpha \neq 1$. Then, letting $g(x) = x_n^{(1-\alpha)/p} f(x)$, for all $f \in \dot{W}_0^{\alpha/2,p}(\mathbb{H}^n)$,*

$$\int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\, \mathrm{d}x\, \mathrm{d}y - D_{n,p,\alpha} \int_{\mathbb{H}^n} \frac{|f(x)|^p}{x_n^{\alpha}}\, \mathrm{d}x \geq m_p \int_{\mathbb{H}^n}\int_{\mathbb{H}^n} \frac{|g(x)-g(y)|^p}{|x-y|^{n+\alpha}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}}$$

where $0 < m_p \leq 1$ is given by (9), and the optimal constant in the Hardy inequality is

$$D_{n,p,\alpha} = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \int_0^1 \frac{|1-r^{(\alpha-1)/p}|^p}{(1-r)^{1+\alpha}}\, \mathrm{d}r.$$

If $p = 2$, then this is an equality with $m_p = 1$.

It was computed in [10] that, for $p = 2$,

$$D_{n,2,\alpha} = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \frac{\beta\left(\frac{1+\alpha}{2}, 1-\frac{\alpha}{2}\right) - 2^{\alpha}}{\alpha 2^{\alpha}},$$

and it will be shown later that this is also the sharp constant for the fractional Hardy inequality over convex domains. Also important later is the relationship

$$D_{n,p,\alpha} = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} D_{1,p,\alpha}. \tag{10}$$

At times hereinafter, for notational convenience, we may write

$$I_{\alpha,p}^{\Omega}(f) = \int_{\Omega\times\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\, \mathrm{d}x\, \mathrm{d}y,$$

and

$$J_{\alpha,p}^{\Omega}(f) = \int_{\Omega\times\Omega} \frac{|x_n^{(1-\alpha)/p} f(x) - y_n^{(1-\alpha)/p} f(y)|^p}{|x-y|^{n+\alpha}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}}.$$

We particularly are interested in $J_{\alpha,p}^{\mathbb{H}^n}(f)$ because x_n is the distance to the boundary of $\mathbb{H}^n$. The notation $J_{\alpha,p}^{\Omega}(f)$ is the restriction of that integral to $\Omega \times \Omega \subseteq \mathbb{H}^n \times \mathbb{H}^n$.

Since we will be often focusing on the case $p = 2$, we further denote

$$I_\alpha^\Omega = I_{\alpha,2}^\Omega, \quad J_\alpha^\Omega = J_{\alpha,2}^\Omega.$$

Thus, using this notation, Theorem 2.4.10 can be written as

$$I_{\alpha,p}^{\mathbb{H}^n}(f) - D_{n,p,\alpha} \int\limits_{\mathbb{H}^n} |f(x)|^p x_n^{-\alpha} \, \mathrm{d}x \geq m_p J_{\alpha,p}^{\mathbb{H}^n}(f).$$

Now, if we consider $p = 2$, then it was also shown in [10] that

$$(f, (-\Delta)^{\alpha/2} f) = a_{n,\alpha} J_\alpha^{\mathbb{H}^n}(f) + \frac{1}{\pi}\Gamma\left(\frac{1+\alpha}{2}\right)^2 \int\limits_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha} \, \mathrm{d}x, \tag{11}$$

or, alternatively,

$$J_\alpha^{\mathbb{H}^n}(f) = I_\alpha^{\mathbb{R}^n}(f) - b_{n,\alpha} \int\limits_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha} \, \mathrm{d}x, \tag{12}$$

where

$$b_{n,\alpha} = \frac{1}{\pi a_{n,\alpha}} \Gamma\left(\frac{1+\alpha}{2}\right)^2.$$

2.4.4 Fractional Sobolev inequality

Finally, from [2], Theorems 7.34 and 7.47, we have the Sobolev inequality for the fractional integral.

THEOREM 2.4.11. *Let $n \geq 2$, $p \geq 2$, and $1 < \alpha < \min\{p, n\}$. Then, for all $f \in \dot{W}_0^{\alpha/2,p}(\mathbb{R}^n)$, there exists $S_{n,p,\alpha} > 0$ so that*

$$\int\limits_{\mathbb{R}^n} \int\limits_{\mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \geq S_{n,p,\alpha} \|f\|_{p^*}^p,$$

where $p^* = pn/(n-\alpha)$ is the critical Sobolev exponent for fractional integrals.

The sharp constant in this inequality is not generally known, see [2], [11], [33], and [24]. However, certain estimates and bounds were obtained in [11] and [33]. Notwithstanding, due to Lieb, when $p = 2$ the sharp constant and all optimizers are known. See, e.g., [29].

The following restatement, using the operator in (6), of Theorem 2.4.11 for $p = 2$ is known, but we state it for completeness. Its proof follows nearly word for word from [29].

THEOREM 2.4.12. *Let $n \geq 2$, $0 < \alpha < 2$, and let $2^* = \frac{2n}{n-\alpha}$ be the critical Sobolev exponent. Then, for all $f \in \dot{W}_0^{\alpha/2,2}(\mathbb{R}^n)$,*

$$(f, (-\Delta)^{\alpha/2} f) \geq S'_{n,\alpha} \, \|f\|_{2^*}^2 \, ,$$

where

$$S'_{n,\alpha} = \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})} \, |\mathbb{S}^n|^{\alpha/n} \, .$$

This inequality is an equality if and only if f is of the form

$$A \left(\gamma^2 + |x-a|^2\right)^{-(n-\alpha)/2} \, ,$$

where $A, \gamma \in \mathbb{R}$ are nonzero, and $a \in \mathbb{R}^n$.

It should be noted that, for $p = 2$, then we have the following relationship between the two aforementioned fractional Sobolev constants

$$S_{n,2,\alpha} = \frac{S'_{n,\alpha}}{a_{n,\alpha}}.$$

Further, we have the following relationship with the classical Sobolev constant

$$\lim_{p\to 2} S_{n,p} = \lim_{\alpha \to 2} S'_{n,\alpha},$$

where $S_{n,p}$ is the sharp constant for the classical Sobolev inequality in Theorem 2.3.4. Note that, effectively, $\alpha = 2$ in the classical case.

As we have seen, there exist analogues for fractional integrals to all the classical results presented above, with the exception of the Hardy-Sobolev-Maz'ya inequality in Theorem 2.3.6. In this paper, we will prove such a fractional Hardy-Sobolev-Maz'ya inequality in the case $p = 2$, along with other related results.

CHAPTER III

TRANSFORMATIONS AND REARRANGEMENTS

3.1 Conformal Transformations

3.1.1 Conformal Transformations of the Upper Halfspace

For $n \geq 3$, the set of conformal mappings is limited to four classes of transformations and their compositions.

THEOREM 3.1.1 (Liouville's Theorem on Conformal Mappings)**.** *Let $n \geq 3$, then any smooth conformal map on a domain of $\mathbb{R}^n$ can be expressed as a composition of translations, similarities, orthogonal transformations and inversions.*

In general, a similarity is uniform scaling, which can be accomplished by scaling about the origin and then translation to the original location, if necessary. The set of all orthogonal transformations make up the orthogonal group, which is generated by reflections. Hence, Liouville's theorem can be restated to say that any smooth conformal map on $\Omega \subseteq \mathbb{R}^n$ is a composition of translations, scaling about the origin, reflections, and inversions.

Further, every conformal transformation in $\mathbb{R}^n$ must be conformal on the boundary of the upper halfspace $\mathbb{H}^n$, and every conformal transformation on $\mathbb{H}^n$ must be conformal in $\mathbb{R}^n$. Thus, scaling about the origin and inversions are conformal in the upper halfspace. Translations $(x', x_n) \mapsto (x' + a', x_n), a' \in \mathbb{R}^{n-1}$ are conformal in $\mathbb{H}^n$, as well as reflections about the x_n-axis. By restriction to the boundary of $\mathbb{H}^n$, we see that, for all $n \geq 4$, all conformal transformations of $\mathbb{H}^n$ are compositions of translations parallel to the boundary, scaling about the origin, reflections about the x_n-axis, and inversions. It is beyond the scope of this paper to show this is true as well for $n = 3$, since the boundary of the 3-dimensional upper halfspace is $\mathbb{R}^2$.

An interesting and very useful result is the conformal invariance of the operators $(f,(-\Delta)^{\alpha/2}f)$ and $J_\alpha^{\mathbb{H}^n}(f)$.

THEOREM 3.1.2. *Let $n \geq 1$ and $0 < \alpha < 2$. Let $f \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$, and*

$$F(w) = |J_S(w)|^{1/2^*} f(Sw),$$

where S is a transformation that is either a translation, scaling about the origin, an inversion, or an orthogonal transformation, and J_S is the Jacobian of S. Then,

$$(f,(-\Delta)^{\alpha/2}f) = (F,(-\Delta)^{\alpha/2}F),$$

and, therefore, $F \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$.

Proof. Note that if $f \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$ and $(f,(-\Delta)^{\alpha/2}f) = (F,(-\Delta)^{\alpha/2}F)$, this implies that $F \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$ by definition of $\dot{W}^{\alpha/2,2}(\mathbb{R}^n)$. Thus, by showing invariance, we show inclusion.

We use the identity, from Definition 2.4.1 and Lemma 2.4.2, that

$$(f,(-\Delta)^{\alpha/2}f) = a_{n,\alpha} I_\alpha^{\mathbb{R}^n}(f).$$

First, if S is a translation, then $F(w) = f(w-h), h \in \mathbb{R}^n$. Using the substition $x = u - h$, we obtain

$$I_\alpha^{\mathbb{R}^n}(f) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(u-h)-f(v-h)|^2}{|u-v|^{n+\alpha}}\,\mathrm{d}u\,\mathrm{d}v = I_\alpha^{\mathbb{R}^n}(F).$$

Second, if S is scaling about the origin, then $F(w) = \lambda^{(\alpha-n)/2} f(w/\lambda)$, $\lambda > 0$, and using the substition $x = u/\lambda$,

$$\begin{aligned} I_\alpha^{\mathbb{R}^n}(f) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(u/\lambda)-f(v/\lambda)|^2}{|u/\lambda - v/\lambda|^{n+\alpha}}\,\lambda^{-2n}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\lambda^{(\alpha-n)/2}f(u/\lambda)-\lambda^{(\alpha-n)/2}f(v/\lambda)|^2}{|u-v|^{n+\alpha}}\,\mathrm{d}u\,\mathrm{d}v \qquad = I_\alpha^{\mathbb{R}^n}(F). \end{aligned}$$

Third, if S is an orthogonal transformation, $F(w) = f(Qw)$, where Q is an orthogonal matrix. Using the substitution $x = Qu$, we compute

$$I_\alpha^{\mathbb{R}^n}(f) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|f(Qu) - f(Qv)|^2}{|Qu - Qv|^{n+\alpha}} |Q| \,\mathrm{d}u\,\mathrm{d}v = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \frac{|f(Qu) - f(Qv)|^2}{|Q|^{n+\alpha}|u - v|^{n+\alpha}} \,\mathrm{d}u\,\mathrm{d}v = I_\alpha^{\mathbb{R}^n}(F).$$

Finally, if S is an inversion, then $F(w) = |w|^{\alpha-n} f(w/|w|^2)$. This result is more complicated computationally than the above results, so we prove it separately in the Lemma to follow. $\square$

LEMMA 3.1.3. *Let $n \geq 1$ and $0 < \alpha < 2$. Let $F(w) = |w|^{\alpha-n} f(w/|w|^2)$, for all $f \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$. Then*

$$(f, (-\Delta)^{\alpha/2} f) = (F, (-\Delta)^{\alpha/2} F). \tag{13}$$

Proof. As above in Theorem 3.1.2, we prove (13) using the representation $I_\alpha^{\mathbb{R}^n}(f)$. For fixed ϵ, consider the regions

$$R_\epsilon^1 := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|x|}{|y|} > 1 + \epsilon \right\},$$

and

$$R_\epsilon^2 := \left\{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \frac{|y|}{|x|} > 1 + \epsilon \right\}.$$

Changing variables $x = v/|v|^2$, which leaves $R_\epsilon^1 \cup R_\epsilon^2$ invariant in $\mathbb{R}^n \times \mathbb{R}^n$, and noting

$$\left| \frac{v}{|v|^2} - \frac{w}{|w|^2} \right|^2 = \frac{1}{|v|^2} - \frac{2v \cdot w}{|v|^2|w|^2} + \frac{1}{|w|^2} = \frac{|v - w|^2}{|v|^2|w|^2},$$

we obtain

$$\begin{aligned}
&\int\limits_{R_\epsilon^1 \cup R_\epsilon^2} \frac{|f(x) - f(y)|^2}{|x - y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int\limits_{R_\epsilon^1 \cup R_\epsilon^2} \left| f\left(\frac{v}{|v|^2}\right) - f\left(\frac{w}{|w|^2}\right) \right|^2 \left| \frac{v}{|v|^2} - \frac{w}{|w|^2} \right|^{-n-\alpha} \frac{\mathrm{d}v}{|v|^{2n}} \frac{\mathrm{d}w}{|w|^{2n}} \\
&= \int\limits_{R_\epsilon^1 \cup R_\epsilon^2} \frac{\left| |v|^{n-\alpha} F(v) - |w|^{n-\alpha} F(w) \right|^2}{|v|^{n-\alpha}|w|^{n-\alpha}|v - w|^{n+\alpha}} \,\mathrm{d}v\,\mathrm{d}w
\end{aligned}$$

$$= \int\limits_{R_\epsilon^1 \cup R_\epsilon^2} \frac{|F(v) - F(w)^2}{|v-w|^{n+\alpha}} \,\mathrm{d}v\,\mathrm{d}w + 2 \int\limits_{R_\epsilon^1 \cup R_\epsilon^2} \frac{|w|^{\alpha-n} - |v|^{\alpha-n}}{|v-w|^{n+\alpha}} |v|^{n-\alpha} |F(v)|^2 \,\mathrm{d}v\,\mathrm{d}w, \quad (14)$$

as a result of Fubini's theorem. The second integral in (14) can be written as

$$2 \int\limits_{\mathbb{R}^n} \mathrm{d}v |v|^{n-\alpha} |F(v)|^2 \int\limits_{\{w:(v,w)\in R_\epsilon^1 \cup R_\epsilon^2\}} \mathrm{d}w \frac{|w|^{\alpha-n} - |v|^{\alpha-n}}{|v-w|^{n+\alpha}},$$

and we can evaluate its inside integral, using the substitutions $r = |v|t$ and $s = \cos\phi$,

$$\int\limits_{\{w:(v,w)\in R_\epsilon^1 \cup R_\epsilon^2\}} \mathrm{d}w \frac{|w|^{\alpha-n} - |v|^{\alpha-n}}{|v-w|^{n+\alpha}}$$

$$= |\mathbb{S}^{n-2}| \int\limits_{\{r>(1+\epsilon)|v|\}\cup\{r<\frac{|v|}{1+\epsilon}\}} \mathrm{d}r\; r^{n-1}(r^{\alpha-n} - |v|^{\alpha-n}) \int\limits_0^\pi \mathrm{d}\phi \; \frac{\sin^{n-2}\phi}{[r^2 + |v|^2 - 2r|v|\cos\phi]^{\frac{n+\alpha}{2}}}$$

$$= |\mathbb{S}^{n-2}| |v|^{-n} \int\limits_{\{t>1+\epsilon\}\cup\{t<\frac{1}{1+\epsilon}\}} \mathrm{d}t \left(t^{\alpha-1} - t^{n-1}\right) \int\limits_{-1}^{1} \mathrm{d}s \frac{(1-s^2)^{(n-3)/2}}{(t^2+1-2st)^{\frac{n+\alpha}{2}}},$$

Thus, the second integral in (14) is

$$2|\mathbb{S}^{n-2}| \left(\int\limits_{\mathbb{R}^n} |F(v)|^2 |v|^{-\alpha} \,\mathrm{d}v \right) \left(\int\limits_{\{t>1+\epsilon\}\cup\{t<\frac{1}{1+\epsilon}\}} \mathrm{d}t \left(t^{\alpha-1} - t^{n-1}\right) \int\limits_{-1}^{1} \mathrm{d}s \frac{(1-s^2)^{(n-3)/2}}{(t^2+1-2st)^{\frac{n+\alpha}{2}}} \right).$$

We'll show the right integral is zero, the left integral finite, so the product is zero.

First, note that, in the right integral, t is never 1, so there is no singularity. Since the part of the integral where $t < \frac{1}{1+\epsilon}$ is finite, then the integral is finite if the sum is. We compute

$$\int\limits_{1+\epsilon}^{\infty} \mathrm{d}t \left(t^{\alpha-1} - t^{n-1}\right) \int\limits_{-1}^{1} \mathrm{d}s \frac{(1-s^2)^{(n-3)/2}}{(t^2+1-2st)^{(n+\alpha)/2}}$$

$$= \int\limits_0^{\frac{1}{1+\epsilon}} \frac{\mathrm{d}t}{t^2} \left(t^{1-\alpha} - t^{1-n}\right) \int\limits_{-1}^{1} \mathrm{d}s \frac{(1-s^2)^{(n-3)/2}}{(1/t^2+1-2s/t)^{(n+\alpha)/2}}$$

$$= -\int\limits_0^{\frac{1}{1+\epsilon}} \mathrm{d}t \left(t^{\alpha-1} - t^{n-1}\right) \int\limits_{-1}^{1} \mathrm{d}s \frac{(1-s^2)^{(n-3)/2}}{(1+t^2-2st)^{(n+\alpha)/2}},$$

so the sum is zero.

Next, by a simple change of variables,

$$\int_{\mathbb{R}^n} |F(v)|^2 |v|^{-\alpha} \,\mathrm{d}v = \int_{\mathbb{R}^n} \left| \left| \frac{x}{|x|^2} \right| f(x) \right|^2 \left| \frac{x}{|x|^2} \right| \frac{\mathrm{d}x}{|x|^{2n}} = \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} \,\mathrm{d}x.$$

Then, from the fractional Hardy inequality, Theorem 2.4.9 above, there exists $c > 0$ so that

$$\int_{\mathbb{R}^n} |F(v)|^2 |v|^{-\alpha} \,\mathrm{d}v = \int_{\mathbb{R}^n} |f(x)|^2 |x|^{-\alpha} \,\mathrm{d}x \leq c I_\alpha^{\mathbb{R}^n}(f) < \infty,$$

as $f \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$.

Therefore, the second integral in (14) is zero, and

$$\int_{R_\epsilon^1 \cup R_\epsilon^2} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} \,\mathrm{d}x \,\mathrm{d}y = \int_{R_\epsilon^1 \cup R_\epsilon^2} \frac{|F(v) - F(w)^2}{|v-w|^{n+\alpha}} \,\mathrm{d}v \,\mathrm{d}w.$$

Taking the limit as $\epsilon \to 0$, we obtain (13). □

We show invariance under $J_\alpha^{\mathbb{H}^n}$ in the following Corollary to Theorem 3.1.2.

COROLLARY 3.1.4. *Let $n \geq 1$ and $1 < \alpha < 2$. Let $f \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$, and*

$$F(w) = |J_S(w)|^{1/2^*} f(Sw),$$

where S is a transformation that is either a translation parallel to the boundary, scaling about the origin, a reflection about the x_n-axis, or an inversion. Then, $J_\alpha^{\mathbb{H}^n}(f) = J_\alpha^{\mathbb{H}^n}(F)$.

Proof. Note that we prove for a more limited set of transformations in this Corollary, than in Theorem 3.1.2. As was stated in (12),

$$J_\alpha^{\mathbb{H}^n}(f) = I_\alpha^{\mathbb{R}^n}(f) - b_{n,\alpha} \int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha} \,\mathrm{d}x.$$

It was shown in Theorem 3.1.2 that $I_\alpha^{\mathbb{R}^n}(f) = I_\alpha^{\mathbb{R}^n}(F)$, so it is only necessary to prove

$$\int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha} \,\mathrm{d}x = \int_{\mathbb{H}^n} |F(w)|^2 w_n^{-\alpha} \,\mathrm{d}w.$$

It should be clear that this Hardy inequality is invariant under translation parallel to the boundary and reflections about the x_n-axis.

First, let $F(w) = \lambda^{(\alpha-n)/2} f(w/\lambda)$, $\lambda > 0$, so F is a scaling of f, then

$$\int_{\mathbb{H}^n} |F(w)|^2 w_n^{-\alpha}\, \mathrm{d}w = \lambda^{\alpha-n} \int_{\mathbb{H}^n} |f(w/\lambda)|^2 w_n^{-\alpha}\, \mathrm{d}w = \int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha}\, \mathrm{d}x.$$

Next, consider inversion, letting $F(w) = |w|^{\alpha-n} f(w/|w|^2)$, then

$$\begin{aligned}\int_{\mathbb{H}^n} |F(w)|^2 w_n^{-\alpha}\, \mathrm{d}w &= \int_{\mathbb{H}^n} \left||w|^{\alpha-n} f(w/|w|^2)\right|^2 w_n^{-\alpha}\, \mathrm{d}w \\ &= \int_{\mathbb{H}^n} \left||x|^{n-\alpha} f(x)\right|^2 \left(\frac{x_n}{|x|^2}\right)^{-\alpha} \frac{\mathrm{d}x}{|x|^{2n}} \\ &= \int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha}\, \mathrm{d}x,\end{aligned}$$

as desired. □

3.1.2 Conformal Transformation from $\mathbb{H}^n$ to the Unit Ball

Of considerable importance throughout this paper is the conformal transformation between $B = B_1(\mathbf{0})$, the unit ball centered at the origin in $\mathbb{R}^n$, and the upper halfspace $\mathbb{H}^n$. In particular, this transformation is a composition of several transformations.

- Translation 1 unit up: $(w', w_n) \mapsto (w', w_n + 1)$;
- Scaling by 1/2: $(w', w_n + 1) \mapsto \frac{1}{2}(w', w_n + 1)$;
- Inversion: $\frac{1}{2}(w', w_n + 1) \mapsto \frac{\frac{1}{2}(w', w_n+1)}{\frac{1}{4}(|w'|^2 + (w_n+1)^2)}$;
- Translation 1 unit down: $\frac{2w', 2w_n+2)}{|w'|^2+(w_n+1)^2} \mapsto \frac{2w', 2w_n+2-|w'|^2-(w_n+1)^2)}{|w'|^2+(w_n+1)^2}$,

where $w = (w', w_n) \in \mathbb{R}^n$, with $w' \in \mathbb{R}^{n-1}$, $w_n \in \mathbb{R}$. Indeed, then $T : B \to \mathbb{H}^n$ is given by

$$Tw = \left(\frac{2w', 1 - |w|^2}{|w'|^2 + (w_n + 1)^2}\right) = \eta(w) \left(w', \frac{1 - |w|^2}{2}\right), \tag{15}$$

where

$$\eta(w) = \frac{2}{|w'|^2 + (w_n + 1)^2} = \frac{2}{|w|^2 + 2w_n + 1}.$$

It is a straightforward calculation to verify that T is an involution, and it is also known that its Jacobian is $\eta(w)^n$. See, *e.g.*, Appendix, [15]. Note that if $n = 1$, then

$$Tx = \frac{1 - x}{1 + x}, \quad \eta(x) = \frac{2}{(1 + x)^2}.$$

The following computational lemma will be useful.

LEMMA 3.1.5. *Let $x = (x', x_n) \in \mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$, and $x_n \in \mathbb{R}$. Then*

$$|T(x)|^2 = \frac{|x'|^2 + (x_n - 1)^2}{|x'|^2 + (x_n + 1)^2},$$

where T is the transformation defined in (15).

Proof. Let T, η be as above. Then,

$$\begin{aligned} |T(x)|^2 &= \frac{\eta(x)^2}{4}\left[4|x'|^2 + \left(1 - |x'|^2 - x_n^2\right)^2\right] \\ &= \frac{\eta(x)^2}{4}\left[4|x'|^2 + 1 + |x'|^4 + x_n^4 - 2|x'|^2 - 2x_n^2 + 2x_n^2|x'|^2\right] \\ &= \frac{\eta(x)^2}{4}\left[|x'|^4 + (x_n + 1)^2(x_n - 1)^2 + |x'|^2(x_n - 1)^2 + |x'|^2(x_n + 1)^2\right] \\ &= \frac{\eta(x)^2}{4}\left(|x'|^2 + (x_n + 1)^2\right)\left(|x'|^2 + (x_n - 1)^2\right), \end{aligned}$$

and the result follows. □

For every function f with support in $\mathbb{H}^n$, we can use the transformation T to associate with f a new function with support in B. As with our treatment of conformal transformations above, for any given α and p, we define

$$\widetilde{f}(w) := |J_T(w)|^{1/p^*} f(Tw) = \eta(w)^{n/p^*} f\left(Tw\right),$$

where $p^* = np/(n - \alpha)$, for $0 < \alpha < 2$, is the critical Sobolev exponent for fractional integrals. If we write $w = Tx$, then it is a straightforward computation to show

$$\eta(Tx) = \frac{1}{\eta(x)}$$

and thus

$$f(x) = \eta(x)^{n/p^*}\widetilde{f}(Tx).$$

From this, we see

$$\int\limits_{\mathbb{H}^n} |f(x)|^{p^*}\,\mathrm{d}x = \int\limits_{B} |f(Tw)|^{p^*}\eta(w)^n\,\mathrm{d}w = \int\limits_{B} |\widetilde{f}(w)|^{p^*}\,\mathrm{d}w,$$

so the Sobolev norms of the two associated functions are equal.

We also need to consider the balls $B_R(\mathbf{0})$, $0 < R < 1$, and their images under the conformal transformation T. We denote

$$B^R = \{Tx \in \mathbb{R}^n : x \in B_R(\mathbf{0})\}. \tag{16}$$

Thus, if supp $\widetilde{f} \subseteq B_R(\mathbf{0})$, then supp $f \subseteq B^R$.

It turns out that B^R is also a ball, contained in the upper halfspace. If we consider two different sized balls, say $0 < R_1 < R_2 < 1$, then $B^{R_1} \subset B^{R^2}$; however, the spheres that make up the boundaries of these balls are not concentric.

LEMMA 3.1.6. *Let* $0 < R < 1$*. If* B^R *is the domain defined in* (16)*, then*

$$B^R := \left\{(x', x_n) \in \mathbb{H}^n : |x'|^2 + \left(x_n - \frac{1+R^2}{1-R^2}\right)^2 < \left(\frac{2R}{1-R^2}\right)^2\right\}. \tag{17}$$

Proof. Let T be as defined in (15), so for any $x \in \mathbb{H}^n$, then $Tx \in B$, as T is an involution. We prove the Lemma by mapping the boundary of $B_R(\mathbf{0})$ to the upper halfspace using the transformation T. Since every point on the boundary has length R, suppose $x \in \mathbb{H}^n$ such that $|Tx| = R$. We then consider the preimage $x \in \mathbb{H}^n$. Noting that $x_n > 0$, we compute

$$\begin{aligned} & R^2 = \frac{|x'|^2 + (x_n - 1)^2}{|x'|^2 + (x_n + 1)^2} \\ \Rightarrow\quad & (1 - R^2)|x'|^2 + (1 - R^2)x_n^2 - 2(1 + R^2)x_n = R^2 - 1 \\ \Rightarrow\quad & |x'|^2 + \left(x_n - \frac{1+R^2}{1-R^2}\right)^2 = \left(\frac{2R}{1-R^2}\right)^2, \end{aligned}$$

using completing the square and proving (17). $\square$

A Corollary to Theorem 3.1.2 is the invariance of the integral $I_\alpha^{\mathbb{R}^n}$ under the operation $f \mapsto \widetilde{f}$.

THEOREM 3.1.7. *Let* $n \geq 1$ *and* $0 < \alpha < 2$. *Then,*

$$I_\alpha^{\mathbb{R}^n}(f) = I_\alpha^{\mathbb{R}^n}(\widetilde{f}),$$

for all $f \in \dot{W}^{\alpha/2,2}(\mathbb{R}^n)$.

Proof. The result follows from the definition of $\widetilde{f}$ as a composition of conformal transformations for which $I_\alpha^{\mathbb{R}^n}$ is invariant under Theorem 3.1.2. □

3.2 Rearrangements

3.2.1 Spherically symmetric rearrangement

One common tool in optimization problems are rearrangements of a function. Here we define one particular rearrangement common in the study of Hardy and Sobolev inequalities. We first define the notion of rearrangement of a set.

DEFINITION 3.2.1. Let $\Omega \subset \mathbb{R}^n$ be a Borel set of fintite Lebesgue measure. We define the spherically symmetric rearrangement of Ω to be the open ball centered at the origin whose measure is the same as Ω. This set shall be denoted Ω^*.

One set of considerable interest is the set

$$\{x \in \mathbb{R}^n : f(x) > a\},$$

called the level set of f at height a. This is often referred to simply as $\{f > a\}$. Note that if $a \geq b$, then

$$\{f > a\} \subseteq \{f > b\},$$

and

$$|\{f > a\}| \leq |\{f > b\}|,$$

where $|\{f > a\}|$ represents the measure of the level set.

Next, we define the rearrangement of a function f, and this should be done in terms of the rearrangements of the level sets of $|f|$. To make the definition the most broad, we need the following concept.

DEFINITION 3.2.2. Let f be a Borel measurable function. We say f vanishes at infinity if the measure of its level set $\{|f| > t\}$ is finite for all $t > 0$.

The following definition for the rearrangement of a function comes from [13].

DEFINITION 3.2.3. Let f be a Borel measurable function that vanishes at infinity. Then, we define

$$f^*(x) = \sup \{a > 0 : |\{f > a\}| \geq |B_1(x)||x|^n\},$$

called the spherically symmetric decreasing rearrangement of f.

Alternatively, the following equivalent definition comes from [29].

DEFINITION 3.2.4. Let f be a Borel measurable function that vanishes at infinity. Then, we define

$$f^*(x) = \int_0^\infty 1^*_{\{|f(x)|>t\}} \, \mathrm{d}t,$$

called the spherically symmetric decreasing rearrangement of f.

We recall in the above definition that

$$1_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega \\ 0, & \text{if } x \notin \Omega \end{cases}.$$

is the indicator function.

It should be obvious that spherically symmetric decreasing rearrangement of the indicator function on the set Ω is

$$1^*_\Omega = 1_{\Omega^*}.$$

Note that the second definition for the rearrangement of a function is most notable when compared with the layer cake representation

$$|f(x)| = \int_0^\infty 1_{\{|f(x)|>t\}} \,\mathrm{d}t.$$

See, e.g., [29].

The spherically symmetric decreasing rearrangement has the following notable and apparent properties.

1. By its definition, f^* is nonnegative, radially symmetric, and nonincreasing as a function of $|x|$;

2. As desired, $\{f^* > a\} = \{|f| > a\}^*$, for all $a > 0$. That is, the level sets of f^* are the rearrangements of the level sets of $|f|$;

3. The functions f^* and $|f|$ are equimeasurable; that is, their level sets have the same measure. Hence, if $f \in L^p(\mathbb{R}^n)$, then $\|f\|_p = \|f^*\|_p$, for all $1 \leq p \leq \infty$.

4. The rearrangement is order preserving. Let f, g vanish at infinity and let $f(x) \geq g(x) \geq 0$ for all $x \in \mathbb{R}^n$. Then, $f^*(x) \geq g^*(x)$ as well.

The following results are well known, but the reader may refer to [27] and [29] for proofs and details.

THEOREM 3.2.5 (Hardy-Littlewood rearrangement inequality)**.** *Let $p > 1$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^{p'}(\mathbb{R}^n)$ be nonnegative. Then,*

$$\int_{\mathbb{R}^n} fg \leq \int_{\mathbb{R}^n} f^* g^*.$$

Similarly, we have

THEOREM 3.2.6 (Riesz-Sobolev rearrangement inequality)**.** *Let $r, s, t > 1$ with $\frac{1}{r} + \frac{1}{s} + \frac{1}{t}$. Let $f \in L^r(\mathbb{R}^n)$, $g \in L^s(\mathbb{R}^n)$, and $h \in L^t(\mathbb{R}^n)$ be nonnegative. Then,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)g(x-y)h(y)\,\mathrm{d}x\,\mathrm{d}y \leq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^*(x)g^*(x-y)h^*(y)\,\mathrm{d}x\,\mathrm{d}y.$$

As a result of the following, we say that the spherically symmetric decreasing rearrangement is nonexpansive.

THEOREM 3.2.7. *Let $J : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function such that $J(0) = 0$. Let f, g be nonnegative and vanishing at infinity. Then,*

$$\int_{\mathbb{R}^n} J(f^* - g^*) \leq \int_{\mathbb{R}^n} J(f - g).$$

A particular example of this would be that if $f, g \in L^p(\mathbb{R}^n)$, for some $p \geq 1$, then

$$\|f^* - g^*\|_p \leq \|f - g\|_p.$$

A similar result involving a double integral and a kernel was proven in [24].

THEOREM 3.2.8. *Let $J : \mathbb{R} \to \mathbb{R}$ be a nonnegative convex function such that $J(0) = 0$, and let $k \in L^1(\mathbb{R}^n)$ be a symmetric decreasing function. Let f be nonnegative and vanishing at infinity. If we denote*

$$E[f] = \int_{\mathbb{R}^n \times \mathbb{R}^n} J\left(f(x) - f(y)\right) k(x-y)\, \mathrm{d}x\, \mathrm{d}y,$$

then

$$E[f] \geq E[f^*],$$

where it is understood that $E[f] = \infty$ if $E[f^*] = \infty$.

The following is then a special case of this Theorem.

THEOREM 3.2.9. *Let $f \in W_0^{\alpha/2,2}(\mathbb{R}^n)$ be nonnegative, $0 < \alpha < 2$. Then,*

$$I_{\alpha,p}^{\mathbb{R}^n}(f) \geq I_{\alpha,p}^{\mathbb{R}^n}(f^*).$$

Similarly, we have this well-known classical result.

THEOREM 3.2.10 (Polyá-Szegő inequality)**.** *Let $f \in W_0^{1,p}(\Omega)$ with $p \geq 1$. Then,*

$$\int_{\Omega} |\nabla f|^p \geq \int_{\Omega} |\nabla f^*|^p.$$

Many of the preceding results are of particular use in the Hardy and Sobolev optimization problems discussed in Chapter 2. Many of the historical results can be, or have been proven, using the spherically symmetric decreasing rearrangement and these Theorems.

3.2.2 Rearrangement on the Upper Halfspace

We have demonstrated so far a certain interrelationship between those functions with support on the halfspace and those with support on the ball through the use of the transformation T, given in (15). This relationship will be used in this section to obtain a rearrangement for functions on the halfspace. As discussed, the spherically symmetric decreasing rearrangement often assists in minimizing certain optimization problems on $\mathbb{R}^n$. However, such a tool is not useful on the halfspace, so we must devise another.

3.2.2.1 Properties

Let $n \geq 2$ and $f \in L^{p^*}(\mathbb{H}^n)$. We consider two operations on nonnegative f. First, let Vf be the $(n-1)$-dimensional spherically symmetric decreasing rearrangement of f in hyperplanes parallel to the boundary of $\mathbb{H}^n$. That is, for each $a > 0$, we define $f_a(x') = f(x', a)$, where $x' \in \mathbb{R}^{n-1}$. Then, $Vf(x', a) = f_a^*(x')$, where the rearrangment is the spherically symmetric decreasing rearrangement in $\mathbb{R}^{n-1}$.

Next, let Uf be the transformation of f obtained by a certain fixed rotation of $\widetilde{f}$. In particular, using the rotation

$$\mathbf{R} : (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_n, -x_{n-1}), \quad x_i \in \mathbb{R}, i = 1, \ldots, n,$$

then U maps

$$f(x) \mapsto \widetilde{f}(x) \mapsto \widetilde{f}(\mathbf{R}x) \mapsto \eta(x)^{n/p^*} \widetilde{f}(\mathbf{R}Tx).$$

Note how the last transformation mimics the map $\widetilde{f} \mapsto f$.

THEOREM 3.2.11. *Let $n \geq 2$, $f \in L^{p^*}(\mathbb{H}^n)$, and let U, V be defined as above. If we let $f_k = (VU)^k f$, then, there exists $f^{\#} \in L^{p^*}(\mathbb{H}^n)$ with the following properties*

1. *$f^{\#}$ is nonnegative and spherically symmetric decreasing in hyperplanes parallel to the boundary of $\mathbb{H}^n$;*
2. *$\widetilde{f^{\#}}$ is radially symmetric;*
3. *$\|f\|_{p^*} = \left\|f^{\#}\right\|_{p^*}$; and*
4. *$\lim_{k\to\infty} f_k = f^{\#}$ in $L^{p^*}(\mathbb{H}^n)$.*

Proof. The result follows from Theorem 2.4 in [15]. □

By passing to a subsequence, we can assume, without loss of generality, that $f_k \to f^{\#}$ pointwise almost everywhere. We will refer to $f^{\#}$ as the pseudosymmetric halfspace rearrangment of f.

3.2.2.2 Results

As a result of the above, we can explicitly write $\widetilde{f^{\#}}$ as the product of two radial functions in the ball picture, a specific, known spherically symmetric increasing function and a spherically symmetrically decreasing function.

THEOREM 3.2.12. *Let $n \geq 2$, $f \in L^{p^*}(\mathbb{H}^n)$, where $f = f^{\#}$. Then, there exists a decreasing function $h : [0,1] \to [0,\infty]$, where $h(1) = 0$, so that*

$$\widetilde{f}(w) = \left(\frac{2}{1-|w|^2}\right)^{n/p^*} h(|w|).$$

Proof. Let $x \in \mathbb{H}^n$ such that $x_n = 1$, and, recalling that T is an involution, let $w = Tx$. Thus, if we restrict T to the hyperplane

$$H = \{(x', x_n) \in \mathbb{H}^n : x' \in \mathbb{R}^{n-1}, x_n = 1\},$$

then the image, or stereographic projection, of H under T is the sphere

$$S = \left\{ w = (w', w_n) : w' \in \mathbb{R}^{n-1}, w_n \in \mathbb{R}, |w'|^2 + \left(w_n + \frac{1}{2}\right)^2 = \frac{1}{4} \right\}$$

whose north and south poles pass through the origin and the point $(0, \ldots, 0, -1)$, respectively. Indeed, starting from the sphere S and moving back onto the halfspace via $w = Tx$, then

$$\begin{aligned}
& |w'|^2 + \left(w_n + \frac{1}{2}\right)^2 = \frac{1}{4} \\
\Rightarrow \quad & |w|^2 + w_n = 0 \\
\Rightarrow \quad & \frac{|x'|^2 + (x_n - 1)^2}{|x'|^2 + (x_n + 1)^2} + \frac{1 - |x|^2}{|x'|^2 + (x_n + 1)^2} = 0 \\
\Rightarrow \quad & |x|^2 - 2x_n + 1 + 1 - |x|^2 = 0 \\
\Rightarrow \quad & x_n = 1,
\end{aligned}$$

so the preimage of S is H, as desired. Thus, for all $w \in S$, we have that

$$w_n = -|w|^2, \quad \eta(w) = \frac{2}{1 - |w|^2}.$$

Further,

$$\frac{2}{1 - |w|^2} = \eta(w) = \eta(Tx) = \frac{1}{\eta(x)} = \frac{|x'|^2 + 4}{2},$$

whenever $x \in H$. Hence,

$$|x'|^2 = \frac{4|w|^2}{1 - |w|^2}, \tag{18}$$

for all $x \in H$, and, since f is radial with respect to the $(n-1)$-dimensional hyperplane H, then, for all $w \in S$, we have

$$f(Tw) = f(x', 1) = f\left(|x'|, 1\right) = f\left(|x'| = \frac{2|w|}{\sqrt{1 - |w|^2}}, 1\right).$$

Also, as

$$\widetilde{f}(w) = \eta(w)^{n/p^*} f(Tw),$$

then

$$\widetilde{f}(w) = \left(\frac{2}{1-|w|^2}\right)^{n/p^*} f\left(|x'| = \frac{2|w|}{\sqrt{1-|w|^2}}, 1\right) = \left(\frac{2}{1-|w|^2}\right)^{n/p^*} h(|w|),$$

where

$$h(r) = f\left(|x'| = \frac{2r}{\sqrt{1-r^2}}, 1\right).$$

Note that f is radially symmetric decreasing on H, $f \in L^{p^*}(\mathbb{H}^n)$, and

$$\lim_{r \uparrow 1} \frac{2r}{\sqrt{1-r^2}} = \infty.$$

In particular, $2r/\sqrt{1-r^2}$ is monotone increasing. Thus, $h(r)$ must be a decreasing function, defined only on the interval $[0,1]$, such that $h(1) = 0$.

Further note that for each particular radius R in the unit ball, the corresponding sphere $\partial B_R(\mathbf{0})$ intersects S. This radius then corresponds to a particular radius in H as given by (18). But, on the ball, $\widetilde{f}$ is a radial function, so if w is any point in the unit ball, there exists some rotation R_w and $w_S \in S$ so that $w = R_w w_S$. Therefore,

$$\widetilde{f}(w) = \widetilde{f}(w_S) = \left(\frac{2}{1-|w_S|^2}\right)^{n/p^*} h(|w_S|) = \left(\frac{2}{1-|w|^2}\right)^{n/p^*} h(|w|),$$

for all $w \in B$. □

From this, we can also compute the representation for the rearranged function $f^{\#}(x)$ as it lies in the upper halfspace.

COROLLARY 3.2.13. *Let $n \geq 2$ and $f \in L^{p^*}(\mathbb{H}^n)$. Then the pseudosymmetric halfspace rearrangement can be written as*

$$f^{\#}(x) = x_n^{-n/p^*} h(|Tx|),$$

where $h : [0,1] \to [0,\infty]$ is a decreasing function with $h(1) = 0$.

Proof. The proof is a simple computation using the conformal transformation T.

$$f^{\#}(x) = \eta(x)^{n/p^*} \widetilde{f^{\#}}(Tx)$$

$$
\begin{aligned}
&= \eta(x)^{n/p^*} \left(\frac{2}{1 - |Tx|^2} \right)^{n/p^*} h(|Tx|) \\
&= \eta(x)^{n/p^*} \left(\frac{|x'|^2 + (x_n + 1)^2}{2x_n} \right)^{n/p^*} h(|Tx|) \\
&= x_n^{-n/p^*} h(|Tx|).
\end{aligned}
$$
$\square$

Although not symmetric, there is nonetheless a certain symmetry to $f^{\#}$, as the level sets of $h(|Tx|)$ are the balls B^R, $0 < R < 1$, as in (17). These two representations in Theorems 3.2.12 and Corollary 3.2.13 will be key to the later proofs of the existence and minimization results for the Hardy-Sobolev-Maz'ya inequality on the halfspace. Both will be used repeatedly throughout the paper.

CHAPTER IV

HARDY INEQUALITIES FOR FRACTIONAL INTEGRALS ON CONVEX AND GENERAL DOMAINS

The sharp fractional Hardy inequality on the upper halfspace, stated above as Theorem 2.4.10, says that for $n \geq 1$, $2 < p \leq \infty$, and $0 < \alpha < p$ with $\alpha \neq 1$,

$$\int\limits_{\mathbb{H}^n}\int\limits_{\mathbb{H}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \geq D_{n,p,\alpha}\int\limits_{\mathbb{H}^n} \frac{|f(x)|^p}{x_n^\alpha}\,\mathrm{d}x, \tag{19}$$

for all $f \in \dot{W}_0^{\alpha/2,p}(\mathbb{H}^n)$, where $D_{n,p,\alpha}$ is the best possible constant. Further, there is a more general fractional Hardy inequality, Theorem 2.4.8 above, that states if $n \geq 1$, $\alpha > 1$, and $0 < p < \infty$, then there exists $D^\Omega_{n,p,\alpha} > 0$ such that

$$\int\limits_{\Omega}\int\limits_{\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \geq D^\Omega_{n,p,\alpha}\int\limits_{\Omega} \frac{|f(x)|^p}{d(x)^\alpha}\,\mathrm{d}x, \tag{20}$$

for all $f \in \dot{W}_0^{\alpha/2,p}(\Omega)$, where Ω is a bounded Lipschitz domain.

In this Chapter, we prove a sharp Hardy inequality for fractional integrals for functions that are supported in a general domain. When that domain is convex, we also prove that the Hardy term is stronger than that weighted by the usual distance function raised to a power. In the latter case, we also show that the best possible constant is $D_{n,p,\alpha}$, the same sharp constant as in (19) above.

To accomplish this, we shall first prove a one dimensional improved Hardy inequality for an interval. We will then reduce the higher dimensional problem to a one dimensional problem so that we may apply the one dimensional inequality previously obtained. The final result ensues after some discussion of geometric details.

4.1 One Dimensional Hardy Inequalities

4.1.1 Case $p = 2$

We first consider the case where $p = 2$. Recall that if $f \in C_c^\infty(0,\infty)$, then

$$\widetilde{f}(w) = \left(\frac{2}{(1+w)^2}\right)^{\frac{1-\alpha}{2}} f\left(\frac{1-w}{1+w}\right) \in C_c^\infty\big((-1,1)\big),$$

and, as proven above in Theorem 3.1.7, in one-dimension,

$$\int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y = \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \frac{|\widetilde{f}(v)-\widetilde{f}(w)|^2}{|v-w|^{1+\alpha}}\,\mathrm{d}v\,\mathrm{d}w.$$

The idea of the following proof is to use Theorem 3.1.7 to transform the problem on the interval to a problem on the half-line. A similar approach, using inversion symmetry, was used in [15] to obtain sharp functional inequalities therein.

THEOREM 4.1.1. *Let $1 < \alpha < 2$, then*

$$\int\limits_{(a,b)}\int\limits_{(a,b)} \frac{|f(x)-f(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \geq D_{1,2,\alpha}\int\limits_a^b |f(x)|^2\left(\frac{1}{x-a}+\frac{1}{b-x}\right)^\alpha \mathrm{d}x,$$

for all $f \in C_c^\infty(a,b)$.

Proof. By translation and scaling it suffices to prove the result for the interval $(-1,1)$. This is the ball in one dimension, so we choose to use the notation $\widetilde{g}$ instead of f. Let $\widetilde{g} \in C_c^\infty((-1,1))$, then using Theorem 3.1.7,

$$\begin{aligned}
&\int\limits_{-1}^{1}\int\limits_{-1}^{1} \frac{|\widetilde{g}(v)-\widetilde{g}(w)|^2}{|v-w|^{1+\alpha}}\,\mathrm{d}v\,\mathrm{d}w \\
&= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \frac{|\widetilde{g}(v)-\widetilde{g}(w)|^2}{|v-w|^{1+\alpha}}\,\mathrm{d}v\,\mathrm{d}w - 2\int\limits_{-1}^{1}\mathrm{d}v|\widetilde{g}(v)|^2 \int\limits_{\mathbb{R}\setminus(-1,1)} \frac{\mathrm{d}w}{|v-w|^{1+\alpha}} \\
&= \int\limits_{\mathbb{R}}\int\limits_{\mathbb{R}} \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y - 2\int\limits_{-1}^{1}\mathrm{d}v|\widetilde{g}(v)|^2 \int\limits_{\mathbb{R}\setminus(-1,1)} \frac{\mathrm{d}w}{|v-w|^{1+\alpha}} \\
&= \int\limits_{0}^{\infty}\int\limits_{0}^{\infty} \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y + 2\int\limits_{0}^{\infty}\mathrm{d}x|g(x)|^2 \int\limits_{-\infty}^{0} \frac{\mathrm{d}y}{|x-y|^{1+\alpha}}
\end{aligned}$$

$$- 2\int_{-1}^{1} \mathrm{d}v|\widetilde{g}(v)|^2 \int_{\mathbb{R}\setminus(-1,1)} \frac{\mathrm{d}w}{|v-w|^{1+\alpha}}.$$

If $x > 0$, we calculate

$$\int_{-\infty}^{0} \frac{\mathrm{d}y}{|x-y|^{1+\alpha}} = \frac{1}{\alpha}x^{-\alpha},$$

and, for $-1 < x < 1$, then

$$\begin{aligned}\int_{\mathbb{R}\setminus(-1,1)} \frac{\mathrm{d}w}{|v-w|^{1+\alpha}} &= \int_{1}^{\infty} (w-v)^{-1-\alpha}\,\mathrm{d}w + \int_{-\infty}^{-1} (v-w)^{-1-\alpha}\,\mathrm{d}w \\ &= \frac{1}{\alpha}(1-v)^{-\alpha} + \frac{1}{\alpha}(v+1)^{-\alpha},\end{aligned}$$

Thus, we have

$$\begin{aligned}&\int_{-1}^{1}\int_{-1}^{1} \frac{|\widetilde{g}(v)-\widetilde{g}(w)|^2}{|v-w|^{1+\alpha}}\,\mathrm{d}v\,\mathrm{d}w \\ &= \int_{0}^{\infty}\int_{0}^{\infty} \frac{|g(x)-g(y)|^2}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y + \frac{2}{\alpha}\int_{0}^{\infty} \frac{|g(x)|^2}{x^\alpha}\,\mathrm{d}x \\ &\qquad - \frac{2}{\alpha}\int_{-1}^{1} |\widetilde{g}(v)|^2 \left((1-v)^{-\alpha} + (1+v)^{-\alpha}\right)\,\mathrm{d}v \\ &\geq D_{1,2,\alpha}\int_{0}^{\infty} \frac{|g(x)|^2}{x^\alpha}\,\mathrm{d}x + \frac{2}{\alpha}\int_{0}^{\infty} \frac{|g(x)|^2}{x^\alpha}\,\mathrm{d}x - \frac{2}{\alpha}\int_{-1}^{1} |\widetilde{g}(v)|^2 \left((1-v)^{-\alpha} + (1+v)^{-\alpha}\right)\,\mathrm{d}v,\end{aligned}$$

where the inequality uses the sharp fractional Hardy inequality, Theorem 2.4.10 above. We seek to transform back from g to $\widetilde{g}$, so that we may express the inequality in terms of a single function. Thus, we obtain

$$\begin{aligned}\int_{0}^{\infty} \frac{|g(x)|^2}{x^\alpha}\,\mathrm{d}x &= \int_{-1}^{1} g^2(Tv)\left(\frac{1-v}{1+v}\right)^{-\alpha} \frac{2}{(1+v)^2}\,\mathrm{d}v \\ &= \int_{-1}^{1} \widetilde{g}^2(v)\left(\frac{1+v}{1-v}\right)^{\alpha}\left(\frac{2}{(1+v)^2}\right)^{\alpha}\,\mathrm{d}v \\ &= \int_{-1}^{1} \widetilde{g}^2(v)\left(\frac{2}{1-v^2}\right)^{\alpha}\,\mathrm{d}v,\end{aligned}$$

since, for $n = 1$,

$$\widetilde{g}(v) = \left(\frac{2}{(1+v)^2}\right)^{\frac{1-\alpha}{2}} g(Tv).$$

Hence, we arrive at the inequality

$$\int_{-1}^{1}\int_{-1}^{1} \frac{|\widetilde{g}(v) - \widetilde{g}(w)|^2}{|v-w|^{1+\alpha}} \, \mathrm{d}v \, \mathrm{d}w \geq D_{1,2,\alpha} \int_{-1}^{1} |\widetilde{g}(v)|^2 \left(\frac{1}{1-v} + \frac{1}{1+v}\right)^{\alpha} \mathrm{d}v$$
$$+ \frac{2}{\alpha} \int_{-1}^{1} |\widetilde{g}(v)|^2 \left(\frac{2^\alpha - (1+v)^\alpha - (1-v)^\alpha}{(1+v)^\alpha (1-v)^\alpha}\right) \mathrm{d}v.$$

Finally, we note that for $1 < \alpha < 2$, then

$$2^\alpha - (1+x)^\alpha - (1-x)^\alpha \geq 0,$$

for all $x \in [-1, 1]$, so we are done. $\square$

Theorem 4.1.1 generalizes easily to open sets on the real line.

COROLLARY 4.1.2. *Let $J \subset \mathbb{R}$ be any open set, and let $1 < \alpha < 2$. Then,*

$$\int_J \int_J \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \geq D_{1,2,\alpha} \int_J |f(x)|^2 \left(\frac{1}{d_J(x)} + \frac{1}{\delta_J(x)}\right)^{\alpha} \mathrm{d}x,$$

for any $f \in C_c^\infty(J)$, where $\delta_J(x) = \sup\{|t| : x + t \in J\}$.

Proof. Since any open set $J \subset \mathbb{R}$ is a countable union of disjoint intervals I_k we find, using Theorem 4.1.1, that

$$\int_J \int_J \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \geq \sum_{k=1}^{\infty} \int_{I_k} \int_{I_k} \frac{|f(x) - f(y)|^2}{|x-y|^{1+\alpha}} \, \mathrm{d}x \, \mathrm{d}y$$
$$\geq \sum_{k=1}^{\infty} D_{1,2,\alpha} \int_{I_k} |f(x)|^2 \left(\frac{1}{d_{I_k}(x)} + \frac{1}{\delta_{I_k}(x)}\right)^{\alpha} \mathrm{d}x$$
$$\geq D_{1,2,\alpha} \int_J |f(x)|^2 \left(\frac{1}{d_J(x)} + \frac{1}{\delta_J(x)}\right)^{\alpha} \mathrm{d}x.$$

$\square$

4.1.2 Case $p > 1$

For general $p > 1$, the following result of Rupert Frank and Robert Seiringer was presented in [29] with their permission.

THEOREM 4.1.3. *Let $1 < p < \infty$ and $1 < \alpha < p$. Then*

$$\int_0^1 \int_0^1 \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \geq D_{1,p,\alpha} \int_0^1 \frac{|f(x)|^p}{x^\alpha} \,\mathrm{d}x, \tag{21}$$

for all smooth functions f with $f(0) = 0$

On any interval, we then have

THEOREM 4.1.4. *Let $f \in C_c^\infty(a, b)$. Then for all $1 < p < \infty$ and $1 < \alpha < p$,*

$$\int_a^b \int_a^b \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \geq D_{1,p,\alpha} \int_a^b \frac{|f(x)|^p}{\min\{(x-a), (b-x)\}^\alpha} \,\mathrm{d}x,$$

with $-\infty < a < b < \infty$.

Proof. This proof is similar to Proposition 3 in [4]. Note that by the scale invariance of (21), for any $c > 0$,

$$\int_0^c \int_0^c \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \geq D_{1,p,\alpha} \int_0^c \frac{|f(x)|^p}{x^\alpha} \,\mathrm{d}x,$$

for all smooth functions f with $f(0) = 0$. Now, let $f \in C_c^\infty(a, b)$. Then,

$$\begin{aligned}
&\int_a^b \int_a^b \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\
&\geq \int_a^{\frac{b+a}{2}} \int_a^{\frac{b+a}{2}} \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y + \int_{\frac{b+a}{2}}^b \int_{\frac{b+a}{2}}^b \frac{|f(x) - f(y)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^{\frac{b-a}{2}} \int_0^{\frac{b-a}{2}} \frac{|f(x+a) - f(y+a)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y + \int_0^{\frac{b-a}{2}} \int_0^{\frac{b-a}{2}} \frac{|f(b-x) - f(b-y)|^p}{|x - y|^{1+\alpha}} \,\mathrm{d}x\,\mathrm{d}y
\end{aligned}$$

$$\geq D_{1,p,\alpha}\left(\int_0^{\frac{b-a}{2}} \frac{|f(x+a)|^p}{x^\alpha}\,\mathrm{d}x + \int_0^{\frac{b-a}{2}} \frac{|f(b-x)|^p}{x^\alpha}\,\mathrm{d}x\right)$$

$$= D_{1,p,\alpha}\left(\int_a^{\frac{b+a}{2}} \frac{|f(x)|^p}{(x-a)^\alpha}\,\mathrm{d}x + \int_{\frac{b+a}{2}}^{b} \frac{|f(b-x)|^p}{(b-x)^\alpha}\,\mathrm{d}x\right)$$

$$= D_{1,p,\alpha}\int_a^b \frac{|f(x)|^p}{\min\{(x-a),(b-x)\}^\alpha}\,\mathrm{d}x. \qquad \square$$

Once again, we generalize to any open set on the real line, and the proof is nearly word for word as the one in Corollary 4.1.2.

COROLLARY 4.1.5. *Let $J \subset \mathbb{R}$ be open, and let $1 < p < \infty$ and $1 < \alpha < p$. Then,*

$$\int_J \int_J \frac{|f(x)-f(y)|^p}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \geq D_{1,p,\alpha}\int_J \frac{|f(x)|^p}{d_J(x)^\alpha}\,\mathrm{d}x,$$

for any $f \in C_c^\infty(J)$.

4.1.3 Extension of general p

The following is an extension of the above result for $p > 1$, which will allow us to add an extra term for the convex Hardy inequality, where $1 < \alpha < 2$. Using [30], a similar result was found in the case $p = 2$ in [20].

THEOREM 4.1.6. *Let $f \in C_c^\infty(0,1)$, $1 < p < \infty$, and $1 < \alpha < 2$. Then,*

$$\int_0^1 \int_0^1 \frac{|f(x)-f(y)|^p}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y - D_{1,p,\alpha}\int_0^1 |f(x)|^p\, x^{-\alpha}\,\mathrm{d}x$$

$$\geq (2-\alpha)D_{1,p,\alpha}\int_0^1 \frac{|f(x)|^p}{x^{\alpha-1}}\,\mathrm{d}x + m_p\int_0^1 \frac{|g(x)-g(y)|^p}{|x-y|^{1+\alpha}}\,[xy(1-x)(1-y)]^{\frac{\alpha-1}{2}}\,\mathrm{d}x\,\mathrm{d}y,$$

where $g(x) = \left(\frac{1-x}{x}\right)^{\frac{\alpha-1}{p}} f(x)$ and m_p is as given in (9).

Note that m_p is the sharp constant for the remainder in the sharp fractional Hardy inequality on the upper halfspace, Theorem 2.4.10 above.

Proof. Note that

$$(1-x)^{\alpha-2} - 1 \geq (2-\alpha)x$$

for all $0 \leq x \leq 1$, since this is an equality at zero, and the derivative of the left-hand side is at least that of the right-hand side on the interval $[0,1]$. Further, consider the map from $(0,1) \mapsto (0,\infty)$ given by $s = \frac{1}{1-x} - 1$. Using these, we obtain

$$\begin{aligned}
\int_0^\infty \left| f\left(\frac{s}{s+1}\right)\right|^p s^{-\alpha}\,\mathrm{d}s &= \int_0^1 |f(x)|^p x^{-\alpha}(1-x)^{\alpha-2}\,\mathrm{d}x \\
&= \int_0^1 |f(x)|^p x^{-\alpha}\,\mathrm{d}x + \int_0^1 |f(x)|^p x^{-\alpha}((1-x)^{\alpha-2}-1)\,\mathrm{d}x \\
&\geq \int_0^1 |f(x)|^p x^{-\alpha}\,\mathrm{d}x + (2-\alpha)\int_0^1 |f(x)|^p x^{1-\alpha}\,\mathrm{d}x,
\end{aligned}$$

Hence, from Theorem 2.4.10,

$$\begin{aligned}
&\int_0^1\int_0^1 \frac{|f(x)-f(y)|^p}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^\infty\int_0^\infty \frac{\left|f\left(\frac{s}{s+1}\right) - f\left(\frac{t}{t+1}\right)\right|^p}{|s-t|^{1+\alpha}}(s+1)^{\alpha-1}(t+1)^{\alpha-1}\,\mathrm{d}s\,\mathrm{d}t \\
&\geq D_{1,p,\alpha}\int_0^\infty \left|f\left(\frac{s}{s+1}\right)\right|^p s^{-\alpha}\,\mathrm{d}s + m_p\int_0^\infty\int_0^\infty \frac{\left|s^{\frac{1-\alpha}{p}} f\left(\frac{s}{s+1}\right) - t^{\frac{1-\alpha}{p}} f\left(\frac{t}{t+1}\right)\right|}{|s-t|^{1+\alpha}}(st)^{\frac{\alpha-1}{2}}\,\mathrm{d}s\,\mathrm{d}t \\
&\geq D_{1,p,\alpha}\int_0^\infty \left|f\left(\frac{s}{s+1}\right)\right|^p s^{-\alpha}\,\mathrm{d}s + m_p\int_0^1 \frac{|g(x)-g(y)|^p}{|x-y|^{1+\alpha}}\left[xy(1-x)(1-y)\right]^{\frac{\alpha-1}{2}}\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}$$

and the result follows. □

We believe it is possible for this inequality to be further improved. This proof uses $(s+1)^{\alpha-1} \geq 1$ and $(t+1)^{\alpha-1} \geq 1$, which throws out a lot of information. It should be fairly straightforward to improve these and gain an additional remainder term, or improve the ones contained herein.

Similar to Theorem 4.1.4, we seek to extend this result to any interval. The proof of the following result is very similar to the proof of Theorem 4.1.4. Note, however, that the additional term is not scale invariant; hence, the factor $\frac{1}{b-a}$ preceding the integral.

THEOREM 4.1.7. *Let $1 < p < \infty$ and $1 < \alpha < 2$. Then,*

$$\int_a^b \int_a^b \frac{|f(x)-f(y)|^p}{|x-y|^{1+\alpha}}\,\mathrm{d}x\,\mathrm{d}y - D_{1,p,\alpha}\int_a^b \frac{|f(x)|^p}{\min\{(x-a),(b-x)\}^\alpha}\,\mathrm{d}x$$
$$\geq \frac{(2-\alpha)D_{1,p,\alpha}}{b-a}\int_a^b \frac{|f(x)|^p}{\min\{(x-a),(b-x)\}^{\alpha-1}}\,\mathrm{d}x,$$

for all $f \in C_c^\infty(a,b)$ with $-\infty < a < b < \infty$.

4.2 Hardy Inequalities on General Domains in $\mathbb{R}^n$

4.2.1 On general domains

Throughout this section, we assume $n \geq 2$. Before we can prove the main results of this section, we must first reduce the double integrals in such a way that we can apply the one dimensional Hardy inequalities proven in Theorem 4.1.1.

THEOREM 4.2.1. *Let Ω be any region in $\mathbb{R}^n$ and assume that $f \in C_c^\infty(\Omega)$. Then*

$$\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{1}{2}\int_{\mathbb{S}^{n-1}} \mathrm{d}w \int_{\{x:x\cdot w=0\}} \mathrm{d}\mathcal{L}_w(x) \int_{\{x+sw\in\Omega\}} \mathrm{d}s \int_{\{x+tw\in\Omega\}} \mathrm{d}t \frac{|f(x+sw)-f(x+tw)|^p}{|s-t|^{1+\alpha}} \quad (22)$$

where $\mathcal{L}_w$ denotes the $(n-1)$-dimensional Lebesgue measure on the plane $x \cdot w = 0$.

Proof. We write the expression

$$I_{\alpha,p}^\Omega(f) = \int_\Omega\int_\Omega \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y = \int_\Omega \mathrm{d}x \int_{\{x+z\in\Omega\}} \mathrm{d}z \frac{|f(x)-f(x+z)|^p}{|z|^{n+\alpha}},$$

and using polar coordinates $z = rw$ we arrive at the expression

$$I_{\alpha,p}^\Omega(f) = \int_\Omega \mathrm{d}x \int_{\mathbb{S}^{n-1}} \mathrm{d}w \int_{\left\{\substack{x+rw\in\Omega,\\ r>0}\right\}} \mathrm{d}r \frac{|f(x)-f(x+rw)|^p}{r^{1+\alpha}}$$

$$= \frac{1}{2} \int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w \int\limits_{\Omega} \mathrm{d}x \int\limits_{\{x+hw\in\Omega\}} \mathrm{d}h \frac{|f(x) - f(x+hw)|^p}{|h|^{1+\alpha}}.$$

Thus, the domain of integration in the innermost integral is the line $x+hw$ intersected with the domain Ω. Splitting the variable x into components perpendicular to w and parallel to w, i.e., replacing x by $x + sw$, where $x \cdot w = 0$, we arrive at

$$\frac{1}{2} \int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w \int\limits_{\{x:x\cdot w=0\}} \mathrm{d}\mathcal{L}_w(x) \int\limits_{\{x+sw\in\Omega\}} \mathrm{d}s \int\limits_{\{x+(s+h)w\in\Omega\}} \mathrm{d}h \frac{|f(x+sw) - f(x+(s+h)w)|^p}{|h|^{1+\alpha}}.$$

The variable change $t = s + h$ yields (22). $\square$

Since we prove a stronger result than (20), the general fractional Hardy inequality from Theorem 2.4.8 above, we need a few definitions before we can state the result. The following is motivated by [18]. Let Ω be any domain in $\mathbb{R}^n$ with non-empty boundary. Fix a direction $w \in \mathbb{S}^{n-1}$ and define

$$d_{w,\Omega}(x) = \min\{|t| \ : \ x + tw \notin \Omega\},$$

and

$$\delta_{w,\Omega}(x) = \sup\{|t| : x + tw \in \Omega\}.$$

That is, consider all the points of intersection of the line $x+tw$ with the boundary of Ω, then $d_{w,\Omega}(x)$ and $\delta_{w,\Omega}(x)$ represent the distance to the points that are closest and furthest from x, respectively. Now set

$$\frac{1}{M_\alpha(x)^\alpha} := \frac{\int_{\mathbb{S}^{n-1}} \left[\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)}\right]^\alpha \mathrm{d}w}{\int_{\mathbb{S}^{n-1}} |w_n|^\alpha \, \mathrm{d}w}. \tag{23}$$

Note the integral in the denominator can be easily computed

$$\int\limits_{\mathbb{S}^{n-1}} |w_n|^\alpha \, \mathrm{d}w = |\mathbb{S}^{n-2}| \int\limits_0^\pi |\cos^\alpha \phi| \sin^{n-2} \phi \, \mathrm{d}\phi$$

$$= 2|\mathbb{S}^{n-2}| \int\limits_0^{\pi/2} \cos^\alpha \phi \sin^{n-2} \phi \, \mathrm{d}\phi$$

$$= 2\left(\frac{2\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n-1}{2}\right)}\right)\beta\left(\frac{n-1}{2},\frac{1+\alpha}{2}\right)$$
$$= 4\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})},$$

where $\beta(\cdot,\cdot)$ is the Euler beta function. Note that this evaluation of the integral is a known identity. Hence, using the identity

$$D_{n,p,\alpha} = 2\pi^{\frac{n-1}{2}}\frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})}D_{1,p,\alpha}$$

from (10), we get

$$\int\limits_{\mathbb{S}^{n-1}} |w_n|^\alpha \,\mathrm{d}w = 2\frac{D_{n,p,\alpha}}{D_{1,p,\alpha}}. \tag{24}$$

With these concepts in mind, we can state the following.

THEOREM 4.2.2. *Let $1 < \alpha < 2$, and let Ω be any domain with non-empty boundary. Then,*

$$\int\limits_\Omega\int\limits_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \geq D_{n,2,\alpha}\int\limits_\Omega \frac{|f(x)|^2}{M_\alpha(x)^\alpha}\,\mathrm{d}x,$$

for any $f \in C_c^\infty(\Omega)$.

Proof. By Theorem 4.2.1 and Corollary 4.1.2 we find that

$$\begin{aligned}
&\int\limits_\Omega\int\limits_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2}\int\limits_{\mathbb{S}^{n-1}}\mathrm{d}w \int\limits_{\{x:x\cdot w=0\}} d\mathcal{L}_w(x)\int\limits_{x+sw\in\Omega}\mathrm{d}s\int\limits_{x+tw\in\Omega}\mathrm{d}t\frac{|f(x+sw)-f(x+tw)|^2}{|s-t|^{1+\alpha}} \\
&\geq \frac{D_{1,2,\alpha}}{2}\int\limits_{\mathbb{S}^{n-1}}\mathrm{d}w\int\limits_{\{x:x\cdot w=0\}} d\mathcal{L}_w(x)\int\limits_{x+sw\in\Omega}\mathrm{d}s|f(x+sw)|^2\left[\frac{1}{d_w(x+sw)}+\frac{1}{\delta_w(x+sw)}\right]^\alpha \\
&= \frac{D_{1,2,\alpha}}{2}\int\limits_{\mathbb{S}^{n-1}}\mathrm{d}w\int\limits_\Omega |f(x)|^2\left[\frac{1}{d_{w,\Omega}(x)}+\frac{1}{\delta_{w,\Omega}(x)}\right]^\alpha \mathrm{d}x \\
&= D_{n,2,\alpha}\int\limits_\Omega \frac{|f(x)|^2}{M_\alpha(x)^\alpha}\,\mathrm{d}x,
\end{aligned}$$

where we have used (23) and (24) in the last equation. Also, note the use of $d_w(x+sw)$ to refer to the distance from x to the boundary of Ω along the line $x + sw$, and $\delta_w(x + sw)$ is similarly used. $\square$

4.2.2 On convex domains

If the domain Ω is convex, the quantity $M_\alpha(x)$ can be bounded in terms of the distance to the boundary $d_\Omega(x)$ and in terms of the quantity $D_\Omega(x)$, the *'width of Ω with respect to x'*. For convex domains with smooth boundary, this quantity is given by the width of the smallest slab that contains Ω and consists of two parallel hyper-planes one of which is tangent to $\partial\Omega$ at the point closest to x.

For general convex sets we define $D_\Omega(x)$ as follows. Fix $x \in \Omega$ arbitrary and pick a point z on the boundary of Ω that is closest to x, so that

$$d_\Omega(x) = |x - z|.$$

In general, there may be more than one such point. Denote by P_z the set of supporting hyper-planes of Ω that pass through the point z and set

$$\mathcal{P}_x = \bigcup_{\substack{z \in \partial\Omega \\ d_\Omega(x) = |z - x|}} P_z.$$

For $P \in \mathcal{P}_x$, we denote by $S(P)$ the smallest slab that contains Ω and is bounded by P on one side and a hyper-plane parallel to it on the other. Such a slab might be a half space if Ω is unbounded. The width $D_{S(P)}$ of the slab $S(P)$ is, naturally, the distance between the two bounding hyper-planes. We set $D_{S(P)} = \infty$ if $S(P)$ is a half space. Now we define

$$D_\Omega(x) = \inf_{P \in \mathcal{P}_x} D_{S(P)}. \tag{25}$$

With these definitions, we can restate Theorem 4.2.2 for convex domains.

THEOREM 4.2.3. *Let Ω be a convex domain, and let $1 < \alpha < 2$. Then, for any $f \in C_c^\infty(\Omega)$,*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y \geq D_{n,2,\alpha} \int_\Omega |f(x)|^2 \left[\frac{1}{d_{\Omega(x)}} + \frac{1}{D_\Omega(x) - d_\Omega(x)} \right]^\alpha \mathrm{d}x, \quad (26)$$

where the constant $D_{n,2,\alpha}$ is the best possible.

Proof. Fix $x \in \Omega$, and let P be a supporting hyperplane to Ω through the closest point $z \in \Omega$ to x. Pick coordinates so that the standard vector e_n is normal to the plane P. Then,

$$d_{w,\Omega}(x) \leq d_{w,S(P)}(x), \quad \delta_{w,\Omega}(x) \leq \delta_{w,S(P)}(x).$$

Further, note that $d_{w,S(P)}(x) + \delta_{w,S(P)}(x)$ is the length of the segment given by intersecting the slab $S(P)$ with the line $x + tw$. Projecting this segment onto the line normal to the slab yields

$$d_{w,S(P)}(x)|w_n| = d_\Omega(x), \quad \delta_{w,S(P)}(x)|w_n| = D_{S(P)} - d_\Omega(x).$$

Note that there may exist directions w where the length of the latter segment is not finite, in which case we set $D_{S(P)} = \infty$. Thus,

$$\left[\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)} \right]^\alpha \geq |w_n|^\alpha \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_{S(P)} - d_\Omega(x)} \right]^\alpha$$

holds for all $P \in \mathcal{P}_x$. Taking the supremum over $\mathcal{P}_x$ and integrating with respect to w over the unit sphere yields

$$\int_{\mathbb{S}^{n-1}} dw \left[\frac{1}{d_{w,\Omega}(x)} + \frac{1}{\delta_{w,\Omega}(x)} \right]^\alpha \geq \int_{\mathbb{S}^{n-1}} dw |w_n|^\alpha \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_\Omega(x) - d_\Omega(x)} \right]^\alpha .$$

Then, using (23), we get the inequality

$$\frac{1}{M_\alpha(x)^\alpha} \geq \left[\frac{1}{d_\Omega(x)} + \frac{1}{D_\Omega(x) - d_\Omega(x)} \right]^\alpha ,$$

from which (26) follows.

It remains to show that the constant $D_{n,2,\alpha}$ in (26) is the best possible. The following ideas are derived from the proof of Theorem 5 in [31], in which the author proves a similar result for the classical Hardy inequality.

Pick a hyperplane P that is tangent to Ω at some point $z \in \partial\Omega$. Such hyperplanes exist since Ω is convex. See, e.g., [6]. We can assume, without loss of generality, that P is the hyperplane $\{x : x_n = 0\}$. It was shown in [10] and [25] that the constant for the halfspace problem, $D_{n,2,\alpha}$, is sharp by constructing a sequence of trial functions. Since the Hardy inequality in Theorem 2.4.10 is invariant under scaling and translation that is parallel to the boundary of $\mathbb{H}^n$, we can transplant these trial functions using scaling and lateral translation to Ω near the point z, thereby showing that $D_{n,2,\alpha}$ is also optimal for (26).

Indeed, let $\{f_k\}$ denote the series of trial functions as is used in either [10] or [25]. Denote the Rayleigh quotient

$$\chi^{\Omega}_{n,\alpha}(f) = \frac{\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} \,\mathrm{d}x \,\mathrm{d}y}{\int_\Omega |f(x)|^2 d_\Omega(x)^{-\alpha} \,\mathrm{d}x},$$

so that

$$\lim_{k\to\infty} \chi^{\mathbb{H}^n}_{n,\alpha}(f_k) = D_{n,2,\alpha},$$

since $d_{\mathbb{H}^n}(x) = x_n$. As Ω is contained in $\mathbb{H}^n$, it is clear that $d_\Omega \le x_n$ for all $x \in \Omega$. Hence, if $f \in C_c^\infty(\Omega)$,

$$\int\limits_\Omega |f(x)|^2 d_\Omega(x)^{-\alpha} \,\mathrm{d}x \ge \int\limits_\Omega |f(x)|^2 x_n^{-\alpha} \,\mathrm{d}x = \int\limits_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha} \,\mathrm{d}x,$$

because supp $f \subset \Omega$. Thus, if $f \in C_c^\infty(\Omega)$, then

$$\chi^{\Omega}_{n,\alpha}(f) \le \chi^{\mathbb{H}^n}_{n,\alpha}(f),$$

where we assume that $\Omega \subset \mathbb{H}^n$. Therefore, we end up with

$$\inf_{f\in C_c^\infty(\Omega)} \frac{\int_\Omega \int_\Omega \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} \,\mathrm{d}x \,\mathrm{d}y}{\int_\Omega |f(x)|^2 \left[\frac{1}{d_{\Omega(x)}} + \frac{1}{D_\Omega(x) - d_\Omega(x)}\right]^\alpha \mathrm{d}x} \le \inf_{f\in C_c^\infty(\Omega)} \chi^{\Omega}_{n,\alpha}(f) \le \chi^{\mathbb{H}^n}_{n,\alpha}(f_k),$$

and taking the limit as $k \to \infty$, the result follows. $\square$

4.2.3 For general p

Now, for $p > 1$, we can state the Hardy inequality for general domains.

THEOREM 4.2.4. *Let $1 < p < \infty$ and $1 < \alpha < p$. Then for any domain Ω with non-empty boundary and any $f \in C_c^\infty(\Omega)$,*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \geq D_{n,p,\alpha} \int_\Omega \frac{|f(x)|^p}{m_\alpha(x)^\alpha} \,\mathrm{d}x, \tag{27}$$

where

$$\frac{1}{m_\alpha(x)^\alpha} := \frac{\int_{\mathbb{S}^{n-1}} \frac{1}{d_{w,\Omega}(x)^\alpha} \,\mathrm{d}w}{\int_{\mathbb{S}^{n-1}} |w_n|^\alpha \,\mathrm{d}w}.$$

In particular, for Ω convex

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y \geq D_{n,p,\alpha} \int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^\alpha} \,\mathrm{d}x. \tag{28}$$

In (28), the constant $D_{n,p,\alpha}$ is the best possible.

The proof of Theorem 4.2.4 is a straightforward modification of the proofs given for Theorem 4.2.2 and Theorem 4.2.3 using Corollary 4.1.5 instead of Corollary 4.1.2.

In the convex case, where $1 < \alpha < 2$, we can improve on this result using Theorem 4.1.7. The proof is similar to the proofs of Theorems 4.2.2 and 4.2.3 above, but it also accomodates the additional term that Theorem 4.1.7 incorporates over Theorem 4.1.4. While a similar result was obtained in [20] in the $p = 2$ case, that result was proportional to the diameter of Ω and not the inradius d_Ω^0.

THEOREM 4.2.5. *Let $\Omega \subset \mathbb{R}^n$ be convex, and let $1 < p < \infty$ and $1 < \alpha < 2$. If $d_\Omega^0 < \infty$, then, for any $f \in C_c^\infty(\Omega)$,*

$$\int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y - D_{n,p,\alpha} \int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^\alpha} \,\mathrm{d}x \geq \frac{c}{d_\Omega^0} \int_\Omega \frac{|f(x)|^p}{d_\Omega(x)^{\alpha-1}} \,\mathrm{d}x, \tag{29}$$

where $c = \frac{2-\alpha}{2n} D_{n,p,\alpha}$.

Proof. Consider the line segment $x + sw$, where $w \in \mathbb{S}^{n-1}$ and $x \in \Omega$. We define the breadth of Ω through x in the direction of w as

$$b_{w,\Omega}(x) = d_{w,\Omega}(x) + \delta_{w,\Omega}(x).$$

Note that the breadth of the interval (a, b) on $\mathbb{R}$ is $b - a$.

Now, from [28] and from [6] [Theorems 2.2 and 2.4, Chapter 5], amongst all ellipsoids contained in Ω, there exists a unique ellipsoid E of maximum volume, and if E were centered at the origin, then $\Omega \subseteq nE$. Thus, for convenience, we will refer to nE as the ellipsoid containing Ω such that E and nE share the same center and nE is proportional to E by a factor of n. Let L refer to the shortest semi-principal axis of nE. Then, Ω is contained in a slab S whose boundaries are the parallel planes that are tangent and normal to the endpoints of L. Clearly, $2nd^0_\Omega$ is no less than the length of the shortest semi-principal axis of E, so $2nd^0_\Omega \geq |L|$.

Choose coordinates so the standard vector e_n is normal to S and parallel to L. Projecting $b_{w,\Omega}(x)$ onto L, then, as in the proof of Theorem 4.2.3, we have

$$|w_n| b_{w,\Omega}(x) \leq |L| \leq 2nd^0_\Omega.$$

As we showed earlier in the proof of Theorem 4.2.3, we can assume that projecting the line segment $x + sw$ onto L yields

$$d_{w,\Omega}(x) \leq d_{w,S}(x) = \frac{d_\Omega(x)}{|w_n|}.$$

Thus,

$$\int\limits_{\mathbb{S}^{n-1}} \frac{\mathrm{d}w}{b_{w,\Omega}(x) d_{w,\Omega}(x)^{\alpha-1}} \geq \frac{1}{2nd^0_\Omega} d_\Omega(x)^{1-\alpha} \int\limits_{\mathbb{S}^{n-1}} |w_n|^\alpha \,\mathrm{d}w.$$

Hence, using Theorem 4.1.7 and Theorem 4.2.1, we obtain

$$\int\limits_\Omega \int\limits_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+\alpha}} \,\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{1}{2} \int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w \int\limits_{\{x : x\cdot w = 0\}} d\mathcal{L}_w(x) \int\limits_{x+sw\in\Omega} \mathrm{d}s \int\limits_{x+tw\in\Omega} \mathrm{d}t \frac{|f(x + sw) - f(x + tw)|^p}{|s - t|^{1+\alpha}}$$

$$
\begin{aligned}
&\geq \frac{D_{1,p,\alpha}}{2} \int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w \int\limits_{\{x:x\cdot w=0\}} d\mathcal{L}_w(x) \int\limits_{x+sw\in\Omega} \mathrm{d}s \left[\frac{|f(x+sw)|^p}{d_w(x+sw)^\alpha}\right.\\
&\qquad \left. + (2-\alpha)\frac{|f(x+sw)|^p}{d_w(x+sw)^{\alpha-1}}\left(\frac{1}{d_w(x+sw)+\delta_w(x+sw)}\right)\right]\\
&= \frac{D_{1,p,\alpha}}{2}\left(\int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w \int\limits_{\Omega} \mathrm{d}x \frac{|f(x)|^p}{d_{w,\Omega}(x)^\alpha} + (2-\alpha)\int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w \int\limits_{\Omega} \mathrm{d}x \frac{|f(x)|^p}{d_{w,\Omega}(x)^{\alpha-1}} b_{w,\Omega}(x)^{-1}\right)\\
&\geq D_{n,p,\alpha}\int\limits_{\Omega} \frac{|f(x)|^p}{d_\Omega(x)^\alpha}\,\mathrm{d}x + \frac{c}{d_\Omega^0}\int\limits_{\Omega} \frac{|f(x)|^p}{d_\Omega(x)^{\alpha-1}}\,\mathrm{d}x,
\end{aligned}
$$

where $c = \frac{2-\alpha}{2n} D_{n,p,\alpha}$. □

CHAPTER V

A FRACTIONAL HARDY-SOBOLEV-MAZ'YA INEQUALITY ON THE UPPER HALFSPACE

In this Chapter, we prove the fractional analogue of Theorem 2.3.6, the Hardy-Sobolev-Maz'ya inequality for functions whose support lies in the upper halfspace. Our result is for the case where $p = 2$. That is, for $n \geq 2$ and $1 < \alpha < 2$, there exists $M_{n,\alpha} > 0$ so that

$$\int\limits_{\mathbb{H}^n \times \mathbb{H}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y - D_{n,2,\alpha} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha} \, \mathrm{d}x \geq M_{n,\alpha} \left(\int\limits_{\mathbb{H}^n} |f(x)|^{2^*} \, \mathrm{d}x \right)^{2/2^*}$$

for all $f \in C_c^\infty(\mathbb{H}^n)$.

As they will be used frequently in this Chapter, we recall the notation

$$I^\Omega_{\alpha,p}(f) = \int\limits_{\Omega \times \Omega} \frac{|f(x) - f(y)|^p}{|x-y|^{n+\alpha}} \, \mathrm{d}x \, \mathrm{d}y,$$

and

$$J^\Omega_{\alpha,p}(f) = \int\limits_{\Omega \times \Omega} \frac{|x_n^{(1-\alpha)/p} f(x) - y_n^{(1-\alpha)/p} f(y)|^p}{|x-y|^{n+\alpha}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}}.$$

Recall as well that we are interested primarily in $J^{\mathbb{H}^n}_{\alpha,p}(f)$, since x_n is the distance to the boundary of $\mathbb{H}^n$, and that the symbol $J^\Omega_{\alpha,p}(f)$ refers to the restriction of the integral $J^{\mathbb{H}^n}_{\alpha,p}(f)$ to $\Omega \times \Omega \subseteq \mathbb{H}^n \times \mathbb{H}^n$. Also, as we will usually refer to the case $p = 2$, we further denote $I^\Omega_\alpha = I^\Omega_{\alpha,2}$ and $J^\Omega_\alpha = J^\Omega_{\alpha,2}$.

In particular, our concern here is to minimize, or at least bound below, the Rayleigh quotient

$$\Psi_{n,\alpha}(f) := \frac{I^{\mathbb{H}^n}_\alpha(f) - D_{n,2,\alpha} \int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha} \, \mathrm{d}x}{\|f\|^2_{2^*}} = \frac{J^{\mathbb{H}^n}_\alpha(f)}{\|f\|^2_{2^*}}. \tag{30}$$

We'll show that $\Psi_{n,\alpha}$ decreases under the pseudosymmetric halfspace rearrangement $f \mapsto f^{\#}$. Then, we decompose the rearranged function by truncation so that we are left with the sum of two functions, one an "upper" function that has support in a fixed ball and the other a "lower" function that is uniformly bounded. We then use two inequalites, one a bounded Sobolev inequality, and the other a weighted Hardy-Sobolev-Maz'ya inequality, to bound the L^{2^*}-norms of these "upper" and "lower" functions and prove the Hardy-Sobolev-Maz'ya inequality for the upper halfspace.

5.1 Preliminary Results

In this section, we shall prove the two inequalities mentioned above and a Lemma establishing that the Rayleigh quotient $\Psi_{n,\alpha}$ decreases under the pseudosymmetric halfspace rearrangement. The first inequality we prove is for $I^{\Omega}_{\alpha,p}$ with respect to functions whose support is contained in convex sets or bounded Lipschitz domains.

THEOREM 5.1.1. *Let $p \geq 2$ and $1 < \alpha < \min\{n, p\}$. Let $\Omega \subseteq \mathbb{R}^n$ be convex or a bounded Lipschitz domain. Then, for all $f \in C^{\infty}_c(\Omega)$, there exists $S^{\Omega}_{n,p,\alpha} > 0$ such that*

$$I^{\Omega}_{\alpha,p}(f) \geq S^{\Omega}_{n,p,\alpha} \|f\|^p_{p^*}.$$

Proof. From the general fractional Hardy inequality in Theorem 2.4.8, and from the sharp Hardy inequality in Theorem 4.2.4, we know there exists $c > 0$ so that

$$I^{\Omega}_{\alpha,p}(f) \geq c \int\limits_{\Omega} |f(x)|^p d_{\Omega}(x)^{-\alpha} \,\mathrm{d}x,$$

for all $f \in C^{\infty}_c(\Omega)$. Further, by the fractional Sobolev inequality in Theorem 2.4.11, there exists $S_{n,p,\alpha} > 0$ such that

$$I^{\mathbb{R}^n}_{\alpha,p}(f) \geq S_{n,p,\alpha} \|f\|^p_{p^*},$$

for all $f \in C^{\infty}_c(\mathbb{R}^n)$. Now, if $x \in \Omega$, then $B_{d_{\Omega}(x)}(x) \subseteq \Omega$. Thus, since

$$\int\limits_{\Omega^c} |x-y|^{-n-\alpha} \,\mathrm{d}y \leq \int\limits_{\left(B_{d_{\Omega}(x)}(x)\right)^c} |x-y|^{-n-\alpha} \,\mathrm{d}y = \frac{1}{\alpha} |\mathbb{S}^{n-1}| d_{\Omega}(x)^{-\alpha},$$

we have

$$\begin{aligned} S_{n,p,\alpha}\|f\|_{p^*}^p &\le I_{\alpha,p}^{\mathbb{R}^n}(f) \\ &= I_{\alpha,p}^{\Omega}(f) + 2\int\limits_{\Omega} \mathrm{d}x|f(x)|^p \int\limits_{\Omega^c} \mathrm{d}y|x-y|^{-n-\alpha} \\ &\le I_{\alpha,p}^{\Omega}(f) + \frac{2}{\alpha}|\mathbb{S}^{n-1}| \int\limits_{\Omega} |f(x)|^p d_\Omega(x)^{-\alpha}\,\mathrm{d}x \\ &\le \left(1 + \frac{2|\mathbb{S}^{n-1}|}{c\alpha}\right) I_{\alpha,p}^{\Omega}(f), \end{aligned}$$

from which the result follows. □

The next inequality could be seen either as a weighted fractional Sobolev inequality for the term $J_{\alpha,p}^{\mathbb{H}^n}(f)$, or, since $J_{\alpha,p}^{\mathbb{H}^n}(f)$ is a lower bound to the remainder of the fractional Hardy inequality on the halfspace, see Theorem 2.4.10, as a weighted Hardy-Sobolev-Maz'ya inequality. When $p = 2$, it is clearly the latter, since $J_{\alpha}^{\mathbb{H}^n}(f)$ is precisely the remainder. Note, however, that the inequality is scale invariant, so the exponent of the function differs from the critical Sobolev exponent by the exponent of that weight.

The following proof of the weighted Hardy-Sobolev-Maz'ya inequality uses ideas from [14], as well as the idea from Theorem 4.3 in [29], to write the integral in terms of its layer cake representation. The layer cake representation is the identity

$$f(x) = \int\limits_0^\infty 1_{\{f(x)>t\}}\,\mathrm{d}t,$$

as proven in Theorem 1.13 of [29].

THEOREM 5.1.2. *Let $p \ge 2$ and $1 < \alpha < \min\{n, p\}$. Then, there exists $w_{n,p,\alpha} > 0$ such that*

$$J_{\alpha,p}^{\mathbb{H}^n}(f) \ge w_{n,p,\alpha} \left(\int\limits_{\mathbb{H}^n} |f(x)|^q x_n^{-n+nq/p^*}\,\mathrm{d}x \right)^{p/q},$$

where $q = q_(n,\alpha) = p\left(\frac{n+\frac{\alpha-1}{2}}{n-1}\right)$.

Proof. We can assume that $f \geq 0$, since, by virtue of the triangle inequality,

$$|x_n^{(1-\alpha)/p} f(x) - y_n^{(1-\alpha)/p} f(y)| \geq \big||x_n|^{(1-\alpha)/p}|f(x)| - |y_n|^{(1-\alpha)/p}|f(y)|\big| .$$

Thus, we see that $J_{\alpha,p}^{\mathbb{H}^n}(f) \geq J_{\alpha,p}^{\mathbb{H}^n}(|f|)$.

We need a few preliminary results. Recall that 1_Ω is the indicator function on the set Ω, then, for any $s \in \mathbb{R}$,

$$\int_0^\infty st^{-s-1} 1_{\{|x|<t\}}\, \mathrm{d}t = \int_{|x|}^\infty st^{-s-1}\, \mathrm{d}t = |x|^{-s}.$$

The following is motivated by the Appendix in [14]. Let $t \geq 0$, and define the function

$$t_+ = \begin{cases} t, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases},$$

known as the positive part of t. Then,

$$\int_0^\infty (t-a)_+ a^{p-2}\, \mathrm{d}a = \int_0^t (t-a) a^{p-2}\, \mathrm{d}a = \frac{1}{p(p-1)} t^p.$$

Further note that

$$(|t| - a)_+ = (t-a)_+ + (-t-a)_+,$$

so, letting $a \geq 0$, we obtain

$$\begin{aligned} |g(x) - g(y)|^p &= p(p-1) \int_0^\infty \left(|g(x) - g(y)| - a\right)_+ a^{p-2}\, \mathrm{d}a \\ &= p(p-1) \int_0^\infty \left[(g(x) - g(y) - a)_+ + (g(y) - g(x) - a)_+\right] a^{p-2}\, \mathrm{d}a \\ &= p(p-1) \int_0^\infty \mathrm{d}a\ a^{p-2} \int_0^\infty \mathrm{d}b \left(1_{\{g(x)-a>b\}} 1_{\{g(y)<b\}} + 1_{\{g(y)-a>b\}} 1_{\{g(x)<b\}}\right), \end{aligned}$$

where the last equality is an identity in [14] that is an exercise using the layer cake representation. Now, let us define

$$g(x) = x_n^{\frac{1-\alpha}{p}} f(x).$$

Then, using the results above,

$$\begin{aligned}
&\int\limits_{\mathbb{H}^n}\int\limits_{\mathbb{H}^n} \frac{|g(x)-g(y)|^p}{|x-y|^{n+\alpha}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}} \\
&= p(p-1)(n+\alpha) \int\limits_{\mathbb{H}^n\times\mathbb{H}^n} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}} \int\limits_0^\infty \frac{\mathrm{d}c}{c}\, c^{-n-\alpha} 1_{\{|x-y|<c\}} \int\limits_0^\infty \mathrm{d}a\, a^{p-2} \int\limits_0^\infty \mathrm{d}b \\
&\qquad \times \left[1_{\{g(x)>a+b\}} 1_{\{g(y)<b\}} + 1_{\{g(y)>a+b\}} 1_{\{g(x)<b\}}\right] \\
&= 2p(p-1)(n+\alpha) \int\limits_{\mathbb{H}^n\times\mathbb{H}^n} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}} \int\limits_0^\infty \frac{\mathrm{d}c}{c}\, c^{-n-\alpha} \int\limits_0^\infty \mathrm{d}a\, a^{p-2} \int\limits_0^\infty \mathrm{d}b\, 1_{\{|x-y|<c\}} \\
&\qquad \times \left(1 - 1_{\{g(y)\geq b\}}\right) 1_{\{g(x)>a+b\}}.
\end{aligned}$$

To simplify, let us make a few definitions. We write

$$\lambda(a) = \int\limits_{\mathbb{H}^n} 1_{\{g(x)>a\}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}},$$

and

$$u(a,c) = \int\limits_{\mathbb{H}^n\times\mathbb{H}^n} 1_{\{g(x)>a\}} 1_{\{|x-y|<c\}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}}.$$

From our discussion regarding level sets and rearrangements in Chapter 3, we know that $b \geq a$ implies $\{f > b\} \subseteq \{f > a\}$. As an extension of that, it must be true that

$$u(a,c) \geq u(b,c), \quad \lambda(a) \geq \lambda(b),$$

where $b \geq a$. Further, since $\alpha > 1$, we obtain

$$u(a,c) \geq \int\limits_{\mathbb{H}^n} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} 1_{\{g(x)>a\}} \int\limits_{\mathbb{H}^n} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}} 1_{\{|y|<c\}} = Dc^{n+\frac{\alpha-1}{2}} \lambda(a),$$

where

$$D = \int\limits_{\mathbb{H}^n} 1_{\{|y|<1\}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}}.$$

Using Fubini, and letting $A = 2p(p-1)(n+\alpha)$,

$$\int\limits_{\mathbb{H}^n}\int\limits_{\mathbb{H}^n} \frac{|g(x)-g(y)|^p}{|x-y|^{n+\alpha}} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}}$$

$$= A \int_0^\infty \mathrm{d}a\, a^{p-2} \int_0^\infty \mathrm{d}b \int_0^\infty \frac{\mathrm{d}c}{c}\, c^{-n-\alpha} \int_{\mathbb{H}^n \times \mathbb{H}^n} \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} \frac{\mathrm{d}y}{y_n^{\frac{1-\alpha}{2}}} \Big(1_{\{|x-y|<c\}} 1_{\{g(x)>a+b\}}$$

$$- 1_{\{|x-y|<c\}} 1_{\{g(x)>a+b\}} 1_{\{g(y)\geq b\}} \Big)$$

$$\geq A \int_0^\infty \mathrm{d}a\, a^{p-2} \int_0^\infty \mathrm{d}b \int_0^\infty \frac{\mathrm{d}c}{c}\, c^{-n-\alpha} \Big(u(a+b,c) - \min\{u(a+b,c), u(b,c), \lambda(a+b)\lambda(b)\} \Big)$$

$$\geq A \int_0^\infty \mathrm{d}a\, a^{p-2} \int_0^\infty \mathrm{d}b \int_0^\infty \frac{\mathrm{d}c}{c}\, c^{-n-\alpha} \Big(u(a+b,c) - \min\{u(a+b,c), \lambda(a+b)\lambda(b)\} \Big)$$

$$\geq A \int_0^\infty \mathrm{d}a\, a^{p-2} \int_0^\infty \mathrm{d}b \int_0^\infty \frac{\mathrm{d}c}{c}\, c^{-n-\alpha} \Big(u(a+b,c) - \lambda(a+b)\lambda(b) \Big)_+$$

$$\geq A \int_0^\infty \mathrm{d}a\, a^{p-2} \int_0^\infty \mathrm{d}b \int_0^\infty \frac{\mathrm{d}c}{c}\, c^{-n-\alpha} \lambda(a+b) \left(D c^{n+\frac{\alpha-1}{2}} - \lambda(b) \right)_+ .$$

From the layer cake representation, we can derive the formula

$$g^q(x) = \int_0^\infty q a^{q-1} 1_{\{g(x)>a\}}\, \mathrm{d}a.$$

Thus, since $g \geq 0$, we denote

$$\|g\|_{q(\nu)}^q = \int_{\mathbb{H}^n} |g(x)|^q \frac{\mathrm{d}x}{x_n^{\frac{1-\alpha}{2}}} = \int_0^\infty q a^{q-1} \lambda(a)\, \mathrm{d}a \geq \int_0^b q a^{q-1} \lambda(a)\, \mathrm{d}a = \lambda(b) b^q.$$

Using the substitution

$$c = \left(\frac{\lambda(b)}{D} \right)^{\frac{2}{2n+\alpha-1}} t,$$

and the identity

$$1 - \frac{p}{q} = \frac{\alpha+1}{2n+\alpha-1},$$

then

$$J_{\alpha,p}^{\mathbb{H}^n}(f) \geq A D^{-\frac{2n+2\alpha}{2n+\alpha-1}} \int_0^\infty \mathrm{d}a\, a^{p-2} \int_0^\infty \mathrm{d}b\, \lambda(a+b) \lambda(b)^{-\frac{\alpha+1}{2n+\alpha-1}} \int_1^\infty \mathrm{d}t\, t^{-n-\alpha-1} \left(t^{n+\frac{\alpha-1}{2}} - 1 \right)$$

$$\geq A \frac{2n+\alpha-1}{(\alpha+1)(n+\alpha)} D^{-\frac{2n+2\alpha}{2n+\alpha-1}} \|g\|_{q(\nu)}^{p-q} \int_0^\infty \mathrm{d}a\, a^{p-2} \int_0^a \mathrm{d}b\, \lambda(a+b) b^{q-p}$$

$$\geq \frac{A(2n+\alpha-1) D^{-\frac{2n+2\alpha}{2n+\alpha-1}}}{(\alpha+1)(n+\alpha)(q-p+1)} \|g\|_{q(\nu)}^{p-q} \int_0^\infty a^{q-1} \lambda(2a)\, \mathrm{d}a$$

$$= \frac{A(2n+\alpha-1)D^{-\frac{2n+2\alpha}{2n+\alpha-1}}2^{-q}}{(\alpha+1)(n+\alpha)(q-p+1)q}\|g\|_{q(\nu)}^{p}$$
$$= \frac{A(2n+\alpha-1)D^{-\frac{2n+2\alpha}{2n+\alpha-1}}2^{-q}}{(\alpha+1)(n+\alpha)(q-p+1)q}\left(\int\limits_{\mathbb{H}^n} |f(x)|^q x_n^{\frac{\alpha-1}{2}-q\frac{\alpha-1}{p}}\,\mathrm{d}x\right)^{p/q}.$$

Since

$$\frac{\alpha-1}{2} - q\frac{\alpha-1}{p} = -n + \frac{nq}{p^*},$$

we are done. $\square$

We want to note a certain relationship between the weighted Sobolev term in Theorem 5.1.2 and the usual Sobolev term. Indeed, if we let $f \in L^{p^*}$ so that $f = f^{\#}$, then, by Corollary 3.2.13,

$$f(x) = x_n^{-n/p^*} h(|Tx|),$$

where $h : [0,1] \to [0,\infty]$ is a decreasing function with $h(1) = 0$. Hence, we get

$$\left(\int\limits_{\mathbb{H}^n} |f(x)|^q x_n^{-n+nq/p^*}\,\mathrm{d}x\right)^{p/q} = \left(\int\limits_{\mathbb{H}^n} \left|x_n^{\frac{n}{p^*}} f(x)\right|^q \frac{\mathrm{d}x}{x_n^n}\right)^{p/q} = \left(\int\limits_{\mathbb{H}^n} |h(|Tx|)|^q\,\mathrm{d}\mu(x)\right)^{p/q},$$

where $\mathrm{d}\mu(x) = x_n^{-n}\,\mathrm{d}x$. Similarly,

$$\left(\int\limits_{\mathbb{H}^n} |f(x)|^{p^*}\,\mathrm{d}x\right)^{p/p^*} = \left(\int\limits_{\mathbb{H}^n} \left|x_n^{-\frac{n}{p^*}} h(|Tx|)\right|^{p^*} \frac{\mathrm{d}x}{x_n^n}\right)^{p/p^*} = \left(\int\limits_{\mathbb{H}^n} |h(|Tx|)|^{p^*}\,\mathrm{d}\mu(x)\right)^{p/p^*}.$$

We have already proven that $J_{\alpha,p}^{\mathbb{H}^n}(f)$ dominates the former, and we will later show that $J_{\alpha}^{\mathbb{H}^n}(f)$ dominates the latter. Thus, there is a clear relationship between the pseudosymmetric halfspace rearrangement, the weighted Hardy-Sobolev-Maz'ya inequality presented in Theorem 5.1.2 above, and the fractional Sobolev norm with the usual critical Sobolev exponent. Future exploration into this would be interesting.

Finally, we show that $\Psi_{n,\alpha}$ decreases under the transformation $f \mapsto f^{\#}$. Using this, we can approach our optimization problem with a restricted class of functions with explicit properties.

LEMMA 5.1.3. *Let $n \geq 2$, and let $f \in C_c^\infty(\mathbb{H}^n)$. Then,*

$$\Psi_{n,\alpha}(f) \geq \Psi_{n,\alpha}(f^\#).$$

Proof. Let us recall the two operations U, V on nonnegative f. First, let V be the spherically symmetric decreasing rearrangement in hyperplanes parallel to the boundary of $\mathbb{H}^n$. Then, let Uf be the transformation of f obtained by a certain fixed rotation of $\widetilde{f}$. In particular, using the rotation

$$\mathbf{R} : (x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_n, -x_{n-1}), \quad x_i \in \mathbb{R}, i = 1, \ldots, n,$$

then U maps

$$f(x) \mapsto \widetilde{f}(x) \mapsto \widetilde{f}(\mathbf{R}x) \mapsto \eta(x)^{n/2^*} \widetilde{f}(\mathbf{R}Tx).$$

Then, we define $F_k := (VU)^k f$ and recall that $F_k \to f^\#$ almost everywhere.

The operations U, V require that $f \geq 0$. To make this assumption, we need $\Psi_{n,\alpha}$ to be nonincreasing under the map $f \mapsto |f|$. Clearly, the Hardy term and the $L^{2^*}(\mathbb{H}^n)$-norm are invariant, and since

$$|f(x) - f(y)| \geq \Big| |f|(x) - |f|(y) \Big|$$

implies

$$I_\alpha^{\mathbb{H}^n}(f) \geq I_\alpha^{\mathbb{H}^n}(|f|),$$

then $\Psi_{n,\alpha}$ is nonincreasing under absolute value, as desired.

We claim $\Psi_{n,\alpha}(F_k)$ is decreasing as $k \to \infty$. However, it is enough to show that

$$\Psi_{n,\alpha}(f) \geq \Psi_{n,\alpha}(VUf),$$

since $F_0 = f$. In Chapter 2, we showed that we can write the remainder term as

$$J_\alpha^{\mathbb{H}^n}(f) = I_\alpha^{\mathbb{R}^n}(f) - b_{n,\alpha} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha} \, \mathrm{d}x,$$

where the constant $b_{n,\alpha}$ is defined there. See (12). Hence, we can write

$$\Psi_{n,\alpha}(f) = \frac{I_\alpha^{\mathbb{R}^n}(f) - b_{n,\alpha}\int_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha}\,\mathrm{d}x}{\|f\|_{2^*}^2}.$$

Applying the transformation U to f, we claim $\Psi_{n,\alpha}$ is invariant. Indeed, we showed in Theorem 3.1.7 that $I_\alpha^{\mathbb{R}^n}(f) = I_\alpha^{\mathbb{R}^n}(\widetilde{f})$, the latter of which is invariant under rotations. Hence, $I_\alpha^{\mathbb{R}^n}(f)$ is invariant under U. Further, since

$$\begin{aligned}\int\limits_{\mathbb{H}^n} f^2(x)x_n^{-\alpha}\,\mathrm{d}x &= \int\limits_B f^2(Tw)\left(\frac{1-|w|^2}{2}\eta(w)\right)^{-\alpha}\eta(w)^n\,\mathrm{d}w\\ &= \int\limits_B \left(\frac{2}{1-|w|^2}\right)^{\alpha}\widetilde{f}^2(w)\,\mathrm{d}w,\end{aligned}$$

then it is invariant under U as well. It is clear that the L^{2^*}-norm is also invariant under U. Hence, $\Psi_{n,\alpha}(f) = \Psi_{n,\alpha}(Uf)$.

Next, we show that $\Psi_{n,\alpha}(Uf) \geq \Psi_{n,\alpha}(VUf)$. Recall that Theorem 3.2.8, as proven in [24], states for $J : \mathbb{R} \to \mathbb{R}$ nonnegative, convex such that $J(0) = 0$, $k \in L^1(\mathbb{R}^n)$ a symmetric decreasing function, and f nonnegative, vanishing at infinity, then

$$\int\limits_{\mathbb{R}^n\times\mathbb{R}^n} J\left(f(x)-f(y)\right)k(x-y)\,\mathrm{d}x\,\mathrm{d}y \geq \int\limits_{\mathbb{R}^n\times\mathbb{R}^n} J\left(f^*(x)-f^*(y)\right)k(x-y)\,\mathrm{d}x\,\mathrm{d}y,$$

where it is understood that if the left is unbounded, so is the right.

Using this result, it can be shown that $I_\alpha^{\mathbb{R}^n}(f)$ decreases under the spherically symmetric decreasing rearrangement. Thus, we use this Theorem to show $I_\alpha^{\mathbb{R}^n}(f)$ decreases under the rearrangement V as well. Indeed, for any fixed $x_n \in \mathbb{R}$, we write

$$f_n(x') = f(x', x_n), \quad x' \in \mathbb{R}^{n-1},$$

and denote the kernel

$$k(x') = \left(|x'|^2 + |x_n - y_n|^2\right)^{\frac{n+\alpha}{2}},$$

which is symmetric decreasing and in $L^1(\mathbb{R}^{n-1})$, so long as $|x_n - y_n| > 0$. Then, letting $J(x) = x^2$, we can apply Theorem 3.2.8 to obtain

$$I_\alpha^{\mathbb{R}^n}(f) = \int\limits_0^\infty \mathrm{d}x_n \int\limits_0^\infty \mathrm{d}y_n \int\limits_{\mathbb{R}^{n-1}} \mathrm{d}x' \int\limits_{\mathbb{R}^{n-1}} \mathrm{d}y' \frac{|f_n(x') - f_n(y')|^2}{\left(|x'-y'|^2 + |x_n - y_n|^2\right)^{\frac{n+\alpha}{2}}}$$

$$= \lim_{\epsilon\to 0} \iint\limits_{\substack{|x_n-y_n|>\epsilon\\ x_n,y_n>0}} \mathrm{d}x_n\,\mathrm{d}y_n \left(\int\limits_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} J(f_n(x')-f_n(y'))k(x'-y')\,\mathrm{d}x'\,\mathrm{d}y' \right)$$

$$\geq \lim_{\epsilon\to 0} \iint\limits_{\substack{|x_n-y_n|>\epsilon\\ x_n,y_n>0}} \mathrm{d}x_n\,\mathrm{d}y_n \left(\int\limits_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} J(f_n^*(x')-f_n^*(y'))k(x'-y')\,\mathrm{d}x'\,\mathrm{d}y' \right)$$

$$= I_\alpha^{\mathbb{R}^n}(Vf),$$

where f_n^* is the $(n-1)$-dimensional spherically symmetric decreasing rearrangement of f_n.

Further, as the rearrangement under V is only along hyperplanes parallel to the boundary of $\mathbb{H}^n$ (*i.e.*, where x_n is fixed), the integral

$$\int\limits_{\mathbb{H}^n} f^2(x)x_n^{-\alpha}\,\mathrm{d}x$$

must be invariant under V. Again, it is clear that the L^{2^*}-norm is invariant under V.

Therefore, applying Fatou's lemma, and the properties of the pseudosymmetric halfspace rearrangement, we obtain

$$\Psi_{n,\alpha}(f) = \frac{J_\alpha^{\mathbb{H}^n}(f)}{\|f\|_{2^*}^2} = \frac{J_\alpha^{\mathbb{H}^n}(F_0)}{\|f^\#\|_{2^*}^2} \geq \lim_{k\to\infty} \frac{J_\alpha^{\mathbb{H}^n}(F_k)}{\|f^\#\|_{2^*}^2} \geq \frac{J_\alpha^{\mathbb{H}^n}(f^\#)}{\|f^\#\|_{2^*}^2} = \Psi_{n,\alpha}(f^\#),$$

as desired. □

5.2 Main Result

The main result of this Chapter is the following fractional Hardy-Sobolev-Maz'ya inequality on the upper halfspace, in the case where $p=2$.

THEOREM 5.2.1. *Let $n \geq 2$ and $1 < \alpha < 2$. Then, there exists $M_{n,\alpha} > 0$ so that*

$$\int\limits_{\mathbb{H}^n\times\mathbb{H}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\,\mathrm{d}x\,\mathrm{d}y - D_{n,2,\alpha}\int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha}\,\mathrm{d}x \geq M_{n,\alpha}\left(\int\limits_{\mathbb{H}^n} |f(x)|^{2^*}\,\mathrm{d}x\right)^{2/2^*} \quad (31)$$

for all $f \in C_c^\infty(\mathbb{H}^n)$. Alternatively, we write (31) *as $J_\alpha^{\mathbb{H}^n}(f) \geq M_{n,\alpha}\|f\|_{2^*}^2$.*

Proof. As a result of Lemma 5.1.3, we can assume that $f = f^{\#}$. Thus, recalling the relationship between a function f on the halfspace and $\widetilde{f}$ on the unit ball, then, as stated in Theorem 3.2.12 above,

$$\widetilde{f}(w) = \left(\frac{2}{1-|w|^2}\right)^{n/2^*} h(|w|),$$

where $h(r)$ is a decreasing function on $[0,1]$ and $h(1) = 0$. Further, from Corollary 3.2.13, we can write

$$f(x) = \eta(x)^{n/2^*}\widetilde{f}(Tx) = x_n^{-n/2^*}h(|Tx|). \tag{32}$$

Finally, from Lemma 5.1.3, we know that

$$I_\alpha^{\mathbb{R}^n}(f) < \infty, \quad \|f\|_{2^*}^2 < \infty,$$

since the original function was in $C_c^\infty(\mathbb{H}^n)$. However, we note that f is no longer necessarily in $C_c^\infty(\mathbb{H}^n)$.

We decompose $h = h_1 + h_0$ by truncation, fixing $0 < R < 1$ so that

$$h_0(r) = \min\{h(r), h(R)\}.$$

In this way, we "cut" the "upper" function h_1 off the top of h, so that the "lower" function h_0 remains. As such, $h_1(|Tx|)$ has support in the fixed ball B^R, for all $x \in \mathbb{H}^n$, and h_0 is uniformly bounded by $h(R)$. We further write

$$f = f_1 + f_0,$$

where the definitions of f_1, f_0 follow from (32).

We claim there exists $c, d > 0$, each dependent only on R, n and α, such that

$$J_\alpha^{\mathbb{H}^n}(f) \geq c\,\|f_1\|_{2^*}^2\,, \tag{33}$$

and

$$J_\alpha^{\mathbb{H}^n}(f) \geq d\,\|f_0\|_{2^*}^2\,. \tag{34}$$

Fixing $0 < \lambda < 1$, then by the triangle inequality for the L^p-norm and the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned}
J_\alpha^{\mathbb{H}^n}(f) &= \lambda J_\alpha^{\mathbb{H}^n}(f) + (1-\lambda) J_\alpha^{\mathbb{H}^n}(f) \\
&\geq \min\{\lambda c, (1-\lambda) d\} \left(\|f_1\|_{2^*}^2 + \|f_0\|_{2^*}^2 \right) \\
&\geq \frac{1}{2} \min\{\lambda c, (1-\lambda) d\} \left(\|f_1\|_{2^*} + \|f_0\|_{2^*} \right)^2 \\
&\geq \frac{1}{2} \min\{\lambda c, (1-\lambda) d\} \|f\|_{2^*}^2 .
\end{aligned}$$

Since $1 > \lambda, R > 0$, the constant is strictly greater than zero. So, taking the supremum over λ and R, the result will follow.

We need to prove (33) and (34). We start with the former. Since the support of h_1 is contained in the interval $[0, R]$; therefore, supp $f_1 \subseteq B^R$, where B^R is as defined in (16) above as

$$B^R = \{Tx \in \mathbb{R}^n : x \in B_R(\mathbf{0})\},$$

or, equivalently as

$$B^R = \left\{ (x', x_n) \in \mathbb{H}^n : |x'|^2 + \left(x_n - \frac{1+R^2}{1-R^2} \right)^2 < \left(\frac{2R}{1-R^2} \right)^2 \right\},$$

as proven by Lemma 3.1.6. Hence, for all $x, y \in B^R$,

$$\begin{aligned}
&\left| x_n^{\frac{1-\alpha}{2}} f_1(x) - y_n^{\frac{1-\alpha}{2}} f_1(y) \right|^2 \\
&\quad = \left| x_n^{\frac{1-n}{2}} (h(|Tx|) - h(R)) - y_n^{\frac{1-n}{2}} (h(|Ty|) - h(R)) \right|^2 \\
&\quad = \left| \left(x_n^{\frac{1-n}{2}} h(|Tx|) - y_n^{\frac{1-n}{2}} h(|Ty|) \right) + h(R) \left(y_n^{\frac{1-n}{2}} - x_n^{\frac{1-n}{2}} \right) \right|^2 \\
&\quad \leq 2 \left| x_n^{\frac{1-\alpha}{2}} f(x) - y_n^{\frac{1-\alpha}{2}} f(y) \right|^2 + 2h^2(R) \left| y_n^{\frac{1-n}{2}} - x_n^{\frac{1-n}{2}} \right|^2 .
\end{aligned}$$

We see that for any $0 < R < 1$,

$$J_\alpha^{B^R}(f) + A_1 h^2(R) \geq \frac{1}{2} J_\alpha^{B^R}(f_1). \tag{35}$$

where

$$A_1 = \int\limits_{B^R \times B^R} \frac{\left| y_n^{\frac{1-n}{2}} - x_n^{\frac{1-n}{2}} \right|^2}{|x-y|^{n+\alpha}} x_n^{\frac{\alpha-1}{2}} y_n^{\frac{\alpha-1}{2}} \, \mathrm{d}x \, \mathrm{d}y,$$

so that A_1 is dependent only on R, n and α.

We show here that A_1 is finite. Note that the ball B^R is symmetric about the x_n-axis. As such, we could say that the north and south poles of B^R lie at the points satisfying

$$\left(x_n - \frac{1+R^2}{1-R^2}\right)^2 = \left(\frac{2R}{1-R^2}\right)^2,$$

or

$$x_n = \frac{1-R}{1+R}, \frac{1+R}{1-R}.$$

As a result, if x is any point in B^R, then

$$\frac{1-R}{1+R} < x_n < \frac{1+R}{1-R}. \tag{36}$$

Noting that the radius of B^R is $\frac{2R}{1-R^2}$, we compute

$$\begin{aligned}
&\int\limits_{B^R \times B^R} \frac{\left|y_n^{\frac{1-n}{2}} - x_n^{\frac{1-n}{2}}\right|^2}{|x-y|^{n+\alpha}} x_n^{\frac{\alpha-1}{2}} y_n^{\frac{\alpha-1}{2}} \,\mathrm{d}x\,\mathrm{d}y \\
&\leq \left(\tfrac{1+R}{1-R}\right)^{\alpha-1} \int\limits_{B^R \times B^R} \frac{\left|y_n^{\frac{1-n}{2}} - x_n^{\frac{1-n}{2}}\right|^2}{|x-y|^{n+\alpha}} \\
&\leq \left(\tfrac{1+R}{1-R}\right)^{\alpha-1} \int\limits_{\frac{1-R}{1+R}}^{\frac{1+R}{1-R}} \mathrm{d}x_n \int\limits_{\frac{1-R}{1+R}}^{\frac{1+R}{1-R}} \mathrm{d}y_n \left|y_n^{\frac{1-n}{2}} - x_n^{\frac{1-n}{2}}\right|^2 \int\limits_{|x'|<\frac{2R}{1-R^2}} \mathrm{d}x' \int\limits_{\mathbb{R}^{n-1}} \frac{\mathrm{d}y'}{(|x'-y'|^2 + |x_n - y_n|^2)^{\frac{n+\alpha}{2}}} \\
&= \left(\tfrac{2R}{1-R^2}\right)^{n-1} \frac{\beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right)}{2(n-1)} \left|\mathbb{S}^{n-2}\right|^2 \left(\tfrac{1+R}{1-R}\right)^{\alpha-1} \int\limits_{\frac{1-R}{1+R}}^{\frac{1+R}{1-R}} \mathrm{d}x_n \int\limits_{\frac{1-R}{1+R}}^{\frac{1+R}{1-R}} \mathrm{d}y_n \frac{\left|y_n^{\frac{1-n}{2}} - x_n^{\frac{1-n}{2}}\right|^2}{|x_n - y_n|^{1+\alpha}},
\end{aligned}$$

which is finite because $x_n^{\frac{1-n}{2}}$ is Lipschitz continuous on any closed interval that does not include zero.

We claim now, and prove later, that we can approximate $x_n^{\frac{1-\alpha}{2}} f_1(x)$ by functions in $C_c^\infty(B^R)$ under the $\dot{W}^{\alpha/2,2}(B^R)$-norm, and, therefore, we can apply Theorem 5.1.1 so that

$$I_\alpha^\Omega\left(x_n^{\frac{1-\alpha}{2}} f_1(x)\right) \geq S_{n,2,\alpha}^\Omega \left\|x_n^{\frac{1-\alpha}{2}} f_1(x)\right\|_{2^*}^2.$$

Using this and (36),

$$
\begin{aligned}
J_\alpha^{B^R}(f_1) &\geq \left(\frac{1-R}{1+R}\right)^{\alpha-1} I_\alpha^{B^R}\left(x_n^{\frac{1-\alpha}{2}} f_1(x)\right) \\
&\geq S^{\Omega}_{n,2,\alpha}\left(\frac{1-R}{1+R}\right)^{\alpha-1}\left(\int_{B^R} x_n^{\left(\frac{1-\alpha}{2}\right)2^*} \left|f_1(x)\right|^{2^*}\,\mathrm{d}x\right)^{2/2^*} \\
&\geq S^{\Omega}_{n,2,\alpha}\left(\frac{1-R}{1+R}\right)^{2\alpha-2} \|f_1\|^2_{2^*}.
\end{aligned}
$$

Recall that $q = q_(n,\alpha) = p\left(\frac{n+\frac{\alpha-1}{2}}{n-1}\right)$. Then, from Theorem 5.1.2,

$$
\begin{aligned}
J_\alpha^{\mathbb{H}^n}(f) &\geq w_{n,2,\alpha}\left(\int_{\mathbb{H}^n} \left|f(x)\right|^q x_n^{-n+nq/2^*}\,\mathrm{d}x\right)^{2/q} \\
&= w_{n,2,\alpha}\left(\int_{\mathbb{H}^n} \left|h(|Tx|)\right|^q x_n^{-n}\,\mathrm{d}x\right)^{2/q} \\
&\geq w_{n,2,\alpha} h^2(R)\left(\int_{B^R} x_n^{-n}\,\mathrm{d}x\right)^{2/q} \\
&= A_2 h^2(R),
\end{aligned}
$$

where

$$
A_2 = w_{n,2,\alpha}\left(\int_{B^R} x_n^{-n}\,\mathrm{d}x\right)^{2/q} < \infty,
$$

is also dependent only on R, n and α. Therefore, from (35),

$$
\left(1+\frac{A_1}{A_2}\right) J_\alpha^{\mathbb{H}^n}(f) \geq J_\alpha^{B^R}(f) + A_1 h^2(R) \geq \frac{1}{2} c_{n,\alpha}\left(\frac{1-R}{1+R}\right)^{2\alpha-2} \|f_1\|^2_{2^*},
$$

which proves (33).

In establishing (34), we use the inequality

$$
\int_0^S \frac{r^{n-1}}{(1-r^2)^n}\,\mathrm{d}r = \int_{1/S}^{\infty} \frac{r^{n-1}}{(r^2-1)^n}\,\mathrm{d}r \geq S^{2-n}\int_{1/S}^{\infty} \frac{r}{(r^2-1)^n}\,\mathrm{d}r = \frac{1}{2(n-1)}\frac{S^n}{(1-S^2)^{n-1}}, \quad (37)
$$

for all $0 < S < 1$. We note that h_0 is constant on $[0, R]$, and it is decreasing to zero on $[R, 1]$. Thus, the following establishes how fast h_0 vanishes at 1. From Theorem

5.1.2, we can compute

$$
\begin{aligned}
J_\alpha^{\mathbb{H}^n}(f) &\geq w_{n,2,\alpha}\left(\int\limits_{\mathbb{H}^n} \left|f(x)\right|^q x_n^{-n+nq/2^*}\,\mathrm{d}x\right)^{2/q}\\
&= w_{n,2,\alpha}\left(\int\limits_B \left|\widetilde{f}(w)\right|^q \left(\frac{2}{1-|w|^2}\right)^{n-nq/2^*}\,\mathrm{d}w\right)^{2/q}\\
&= w_{n,2,\alpha}\left(\int\limits_B \left|h(|w|)\right|^q \left(\frac{2}{1-|w|^2}\right)^{n}\,\mathrm{d}w\right)^{2/q}\\
&\geq w_{n,2,\alpha}\left(2^n|\mathbb{S}^{n-1}|\int\limits_0^1 \frac{r^{n-1}}{(1-r^2)^n}h_0(r)^q\,\mathrm{d}r\right)^{2/q}\\
&\geq w_{n,2,\alpha}h_0(S)^2\left(\frac{2^{n-1}}{n-1}|\mathbb{S}^{n-1}|\right)^{2/q}\left(\frac{S^n}{(1-S^2)^{n-1}}\right)^{2/q},
\end{aligned}
$$

where $0<S<1$. Thus,

$$
h_0(r)^{2^*} \leq w_{n,2,\alpha}^{-2^*/2}\left(\frac{2^{n-1}}{n-1}|\mathbb{S}^{n-1}|\right)^{-2^*/q}\left(\frac{(1-r^2)^{n-1}}{r^n}\right)^{2^*/q} J_\alpha^{\mathbb{H}^n}(f)^{2^*/2}.
$$

Then, we calculate

$$
\begin{aligned}
\|f_0\|_{2^*}^{2^*} &= \int\limits_B |\widetilde{f}_0(w)|^{2^*}\,\mathrm{d}w\\
&= 2^n|\mathbb{S}^{n-1}|\int\limits_0^1 \frac{r^{n-1}}{(1-r^2)^n}h_0(r)^{2^*}\,\mathrm{d}r\\
&= 2^n|\mathbb{S}^{n-1}|\left(h_0(R)^{2^*}\int\limits_0^R \frac{r^{n-1}}{(1-r^2)^n}\,\mathrm{d}r + \int\limits_R^1 \frac{r^{n-1}}{(1-r^2)^n}h_0(r)^{2^*}\,\mathrm{d}r\right)\\
&\leq 2^n|\mathbb{S}^{n-1}|w_{n,2,\alpha}^{-2^*/2}\left(\frac{2^{n-1}}{n-1}|\mathbb{S}^{n-1}|\right)^{-2^*/q}\left[\left(\frac{(1-R^2)^{n-1}}{R^n}\right)^{2^*/q}\int\limits_0^R \frac{r^{n-1}}{(1-r^2)^n}\,\mathrm{d}r\right.\\
&\qquad\left.+\int\limits_R^1 \frac{r^{n-1}}{(1-r^2)^n}\left(\frac{(1-r^2)^{n-1}}{r^n}\right)^{2^*/q}\,\mathrm{d}r\right] J_\alpha^{\mathbb{H}^n}(f)^{2^*/2}.
\end{aligned}
$$

As $2^*>q$, the claim follows.

Finally, we show we can approximate $x_n^{\frac{1-\alpha}{2}}f_1(x)$ by functions in $C_c^\infty(B^R)$ under

$\dot{W}^{\alpha/2,2}(B^R)$. First, we define

$$g(x) = x_n^{\frac{1-\alpha}{2}} f_1(x),$$

and

$$g_c(x) = \min\left[\max\left(g(x) - c, 0\right), 1/c\right]. \tag{38}$$

Then, by monotone convergence,

$$I_\alpha^{B^R}(g_c) \to I_\alpha^{B^R}(g),$$

as $c \to 0$. Since

$$|g_c(x) - g_c(y)| \leq |g(x) - g(y)|, \tag{39}$$

then, using (35)

$$I_\alpha^{B^R}(g_c) \leq I_\alpha^{B^R}(x_n^{\frac{1-\alpha}{2}} f_1(x)) \leq \left(\frac{1+R}{1-R}\right)^{\alpha-1} J_\alpha^{B^R}(f_1) < \infty.$$

Now, based on the symmetry of h and the definition of h_1, it is clear that supp g_c is a proper subset of the open set B^R; that is, there exists $\epsilon > 0$ such that $d_{B^R}(x) > \epsilon$ for all $x \in$ supp g_c. From (38), g_c is uniformly bounded as well, so $g_c \in L^2(B^R)$, and, similar to the proof of Theorem 5.1.1,

$$I_\alpha^{\mathbb{R}^n}(g_c) \leq I_\alpha^{B^R}(g_c) + \frac{2}{\alpha}|\mathbb{S}^{n-1}| \int\limits_{B^R} |g_c(x)|^2 d_{B^R}(x)^{-\alpha}\, \mathrm{d}x < \infty.$$

Thus, $g_c \in W^{\alpha/2,2}(\mathbb{R}^n)$. Now, Theorem 2.4.7, as given in [2], states

$$W_0^{\alpha/2,2}(\mathbb{R}^n) = W^{\alpha/2,2}(\mathbb{R}^n).$$

Since supp g_c is a proper subset of the open set B^R, we know there exists a sequence $\{g_c^j\} \subset C_c^\infty(B^R)$ such that

$$\|g_c - g_c^j\|_{W^{\alpha/2,2}(\mathbb{R}^n)} \to 0$$

as $j \to \infty$. Therefore,

$$I_\alpha^{B^R}(g_c^j) \to I_\alpha^{B^R}(g_c),$$

also as $j \to \infty$. From this, it follows easily that $g \in \dot{W}_0^{\alpha/2,2}(B^R)$. $\square$

CHAPTER VI

ON THE EXISTENCE OF MINIMIZERS

This Chapter builds on the results in Chapter 5. We discuss some results we've obtained towards proving the existence of a minimizer for the fractional Hardy-Sobolev-Maz'ya inequality on the upper halfspace, Theorem 5.2.1 above, as well as the existence of a minimizer in a limited case. In this discussion, we use the Rayleigh quotient

$$\Psi_{n,\alpha}(f) := \frac{I_\alpha^{\mathbb{H}^n}(f) - D_{n,2,\alpha}\int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha}\, \mathrm{d}x}{\|f\|_{2^*}^2} = \frac{J_\alpha^{\mathbb{H}^n}(f)}{\|f\|_{2^*}^2}$$

from (30) above.

6.1 Preliminary Results

We start with a definition of the space in which the minimizer will exist.

DEFINITION 6.1.1. Let $1 < \alpha < 2$, and define the space $\dot{S}_0^\alpha(\mathbb{H}^n)$ as the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|f\|_{\dot{S}_0^\alpha(\mathbb{H}^n)} = \left(\int\limits_{\mathbb{H}^n\times\mathbb{H}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\, \mathrm{d}x\, \mathrm{d}y - D_{n,2,\alpha} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha}\, \mathrm{d}x \right)^{1/2}.$$

Alternatively, we write $\|f\|_{\dot{S}_0^\alpha(\mathbb{H}^n)} = \sqrt{J_\alpha^{\mathbb{H}^n}(f)}$.

Let $f \in \dot{S}_0^\alpha(\mathbb{H}^n)$, and let $f^\#$ be its pseudosymmetric halfspace rearrangement, as defined in Corollary 3.2.13. Using a similar argument as we used at the end of the proof of Theorem 5.2.1, we can show that $f^\# \in \dot{S}_0^\alpha(\mathbb{H}^n)$ as well. Thus, since $\Psi_{n,\alpha}$ decreases under the pseudosymmetric halfspace rearrangement, in searching for a minimizer, we will be able to assume that all functions have the properties of that rearrangement.

LEMMA 6.1.2. *Let $1 < \alpha < 2$, and let $f \in \dot{S}_0^{\alpha}(\mathbb{H}^n)$. Then, $f^{\#} \in \dot{S}_0^{\alpha}(\mathbb{H}^n)$.*

Proof. First, note that

$$\int_{-\infty}^{0} \mathrm{d}y_n \int_{\mathbb{R}^{n-1}} \mathrm{d}y' \left(|x'-y'|^2 + |x_n - y_n|^2\right)^{-\frac{n+\alpha}{2}} = \frac{|\mathbb{S}^{n-2}|}{2\alpha}\beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right) x_n^{-\alpha}.$$

Thus, using the fractional Hardy inequality for the halfspace, Theorem 2.4.10,

$$\begin{aligned} I_\alpha^{\mathbb{R}^n}(f) &= I_\alpha^{\mathbb{H}^n}(f) + 2\left(\frac{|\mathbb{S}^{n-2}|}{2\alpha}\beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right)\right) \int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha}\,\mathrm{d}x \\ &\le \left(1 + \frac{|\mathbb{S}^{n-2}|\beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right)}{\alpha D_{n,2,\alpha}}\right) I_\alpha^{\mathbb{H}^n}(f), \end{aligned}$$

Now, since $f \in \dot{S}_0^{\alpha}(\mathbb{H}^n)$, then $I_\alpha^{\mathbb{H}^n}(f)$ must be finite in order for $\|f\|_{\dot{S}_0^{\alpha}(\mathbb{H}^n)}$ to be defined and finite. Hence, if we define $f_c^{\#} = \left(f^{\#}\right)_c$, as is defined in (38) above, then from (39) and the proof of Lemma 5.1.3, we have

$$I_\alpha^{\mathbb{R}^n}(f_c^{\#}) \le I_\alpha^{\mathbb{R}^n}(f^{\#}) \le I_\alpha^{\mathbb{R}^n}(f) < \infty.$$

Further, since $f_c^{\#}$ is uniformly bounded with compact support away from the boundary of $\mathbb{H}^n$, then $f_c^{\#} \in L^2(\mathbb{R}^n)$, so

$$f_c^{\#} \in W^{\alpha/2,2}(\mathbb{R}^n) = W_0^{\alpha/2,2}(\mathbb{R}^n).$$

Since supp $f_c^{\#}$ is a proper subset of the open set $\mathbb{H}^n$, we know there exists a sequence $\{f_c^j\} \subset C_c^\infty(\mathbb{H}^n)$ so that

$$\|f_c^{\#} - f_c^j\|_{W^{\alpha/2,2}(\mathbb{R}^n)} \to 0$$

as $j \to \infty$. By monotone convergence,

$$I_\alpha^{\mathbb{H}^n}(f_c^{\#}) \to I_\alpha^{\mathbb{H}^n}(f^{\#})$$

and

$$\int_{\mathbb{H}^n} |f_c^{\#}|^2 x_n^{-\alpha}\,\mathrm{d}x \to \int_{\mathbb{H}^n} |f^{\#}|^2 x_n^{-\alpha}\,\mathrm{d}x$$

as $j \to \infty$. Thus, it follows that $f^{\#} \in \dot{S}_0^{\alpha}(\mathbb{H}^n)$. $\square$

We now show that for certain sequences $\{f_k\}$ in $\dot{S}_0^\alpha(\mathbb{H}^n)$ that converge pointwise, but not strongly, to zero, then $\Psi_{n,\alpha}(f_k)$ is bounded below by $S_{n,2,\alpha}$, the sharp constant for the fractional Sobolev inequality in Theorem 2.4.11. This computational Lemma provides a very important estimate in proving our result regarding the existence of minimizers for the fractional Hardy-Sobolev-Maz'ya inequality on the halfspace.

LEMMA 6.1.3. *Let $n \geq 3$ and $1 < \alpha < 2$. Let $\{f_k\}_{k=1}^\infty \subset \dot{S}_0^\alpha(\mathbb{H}^n)$ such that $f_k = f_k^\#$, $\forall k$, $f_k \to 0$ pointwise, $\|f_k\|_{2^*}$ is uniformly bounded, $\forall k$, and $\lim_{k\to\infty} \|f_k\|_{2^*} > 0$. Then,*

$$\lim_{k\to\infty} \Psi_{n,\alpha}(f_k) \geq S_{n,2,\alpha}. \tag{40}$$

Proof. Let $\{f_k\}$ be a sequence for $\Psi_{n,\alpha}$ in $\dot{S}_0^\alpha(\mathbb{H}^n)$ satisfying the requirements above. Using truncation of the sequence, if necessary, we can assume that $\Psi_{n,\alpha}(f_k)$ is uniformly bounded, since otherwise (40) would already be true. As in Theorem 5.2.1, we use the halfspace representation from Corollary 3.2.13, so, for all k,

$$f_k(x) = x_n^{-n/2^*} h_k(|Tx|),$$

where $h_k : [0,1] \to [0,\infty]$ is a decreasing function with $h_k(1) = 0$. Since $f_k \to 0$ pointwise, then h_k converges pointwise to zero as well.

We want to define cutoff functions so that we can isolate the center of the function, where there might be a singularity, from the boundary. We will do this on the half-line, and then pull it back first to the unit ball B, and then to the upper halfspace $\mathbb{H}^n$. Now, there exists $\psi \in C^\infty(0,\infty)$ so that

$$\psi(t) = \begin{cases} 0, & \text{if } 0 < t \leq 1 \\ 1, & \text{if } 2 \leq t < \infty \end{cases}.$$

Now we define the following pairs of functions

$$\phi(t) = \sin\left(\frac{\pi}{2}\psi(t)\right), \qquad \overline{\phi}(t) = \cos\left(\frac{\pi}{2}\psi(t)\right),$$

and

$$\phi_R(t) = \phi(t/R), \qquad \overline{\phi}_R(t) = \overline{\phi}(t/R),$$

for all $R > 0$. Finally, consider the map

$$x \mapsto -\ln|Tx|, \quad x \in \mathbb{R}^n. \tag{41}$$

Note that (41) sends $\mathbb{H}^n \mapsto (0, \infty)$. Also, recalling that, from Lemma 3.1.6,

$$B^R = \left\{ (x', x_n) \in \mathbb{H}^n : |x'|^2 + \left(x_n - \frac{1+R^2}{1-R^2} \right)^2 < \left(\frac{2R}{1-R^2} \right)^2 \right\},$$

then (41) maps $B^{e^{-s}} \mapsto (s, \infty)$. Hence, we also define

$$\varphi_R(x) = \phi_R(-\ln|Tx|), \qquad \overline{\varphi}_R(x) = \overline{\phi}_R(-\ln|Tx|).$$

It should be observed that $\varphi_R(x) = 0$ whenever $x \notin B^{e^{-R}}$ and that $\overline{\varphi}_R(x) = 0$ whenever $x \in B^{e^{-2R}}$. Note that, like the other pairs of functions, $\varphi_R^2(t) + \overline{\varphi}_R^2(t) = 1$.

We can write

$$\begin{aligned}
&|\varphi_R(x) f_k(x) - \varphi_R(y) f_k(y)|^2 + |\overline{\varphi}_R(x) f_k(x) - \overline{\varphi}_R(y) f_k(y)|^2 \\
&\quad = f_k^2(x) + f_k^2(y) - 2 f_k(x) f_k(y) (\varphi_R(x)\varphi_R(y) + \overline{\varphi}_R(x)\overline{\varphi}_R(y)) \\
&\quad = |f_k(x) - f_k(y)|^2 + f_k(x) f_k(y) (2 - 2\varphi_R(x)\varphi_R(y) - 2\overline{\varphi}_R(x)\overline{\varphi}_R(y)) \\
&\quad = |f_k(x) - f_k(y)|^2 + \left(|\varphi_R(x) - \varphi_R(y)|^2 + |\overline{\varphi}_R(x) - \overline{\varphi}_R(y)|^2 \right) f_k(x) f_k(y).
\end{aligned}$$

Let

$$V_{k,R} = \int\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|\varphi_R(x) - \varphi_R(y)|^2 + |\overline{\varphi}_R(x) - \overline{\varphi}_R(y)|^2}{|x-y|^{n+\alpha}} f_k(x) f_k(y) \, \mathrm{d}x \, \mathrm{d}y.$$

We recall once more that, from (12) in Chapter 2, we can write

$$J_\alpha^{\mathbb{H}^n}(f) = I_\alpha^{\mathbb{R}^n}(f) - b_{n,\alpha} \int\limits_{\mathbb{H}^n} \frac{|f(x)|^2}{x_n^\alpha} \, \mathrm{d}x.$$

Hence,

$$\begin{aligned}
J_\alpha^{\mathbb{H}^n}(f_k) = I_\alpha^{\mathbb{R}^n}(\varphi_R f_k) + I_\alpha^{\mathbb{R}^n}(\overline{\varphi}_R f_k) - V_{k,R} - b_{n,\alpha} \int\limits_{\mathbb{H}^n} |\varphi_R(x) f_k(x)|^2 x_n^{-\alpha} \, \mathrm{d}x \\
- b_{n,\alpha} \int\limits_{\mathbb{H}^n} |\overline{\varphi}_R(x) f_k(x)|^2 x_n^{-\alpha} \, \mathrm{d}x
\end{aligned}$$

$$= I_\alpha^{\mathbb{R}^n}(\varphi_R f_k) + J_\alpha^{\mathbb{R}^n}(\overline{\varphi}_R f_k) - V_{k,R} - b_{n,\alpha} \int\limits_{\mathbb{H}^n} |\varphi_R(x) f_k(x)|^2 x_n^{-\alpha} \, \mathrm{d}x$$

$$\geq S_{n,2,\alpha} \|\varphi_R f_k\|_{2^*}^2 - V_{k,R} - b_{n,\alpha} \int\limits_{\mathbb{H}^n} |\varphi_R(x) f_k(x)|^2 x_n^{-\alpha} \, \mathrm{d}x. \tag{42}$$

We can assume that $J_\alpha^{\mathbb{H}^n}(f_k)$ is uniformly bounded since

$$J_\alpha^{\mathbb{H}^n}(f_k) = \Psi_{n,\alpha}(f_k) \|f_k\|_{2^*}^2,$$

and the right side of this equation is uniformly bounded. From this, one can prove that h is uniformly pointwise bounded. Indeed, using Theorem 5.2.1, Corollary 3.2.13, and (37), then the following establishes how fast h vanishes at 1:

$$J_\alpha^{\mathbb{H}^n}(f_k) \geq M_{n,\alpha} \left(\int\limits_{\mathbb{H}^n} \left| h_k(|Tx|) \right|^{2^*} x_n^{-n} \, \mathrm{d}x \right)^{2/2^*}$$

$$= M_{n,\alpha} \left(2^n |\mathbb{S}^{n-1}| \int\limits_0^1 \frac{r^{n-1}}{(1-r^2)^n} h_k(r)^{2^*} \, \mathrm{d}r \right)^{2/2^*}$$

$$\geq M_{n,\alpha} h_k(S)^2 \left(\frac{2^{n-1}}{n-1} |\mathbb{S}^{n-1}| \right)^{2/2^*} \left(\frac{S^n}{(1-S^2)^{n-1}} \right)^{2/2^*},$$

where $0 < S < 1$. From Theorem 5.1.2,

$$J_\alpha^{\mathbb{H}^n}(f) \geq w_{n,2,\alpha} \left(\int\limits_{\mathbb{H}^n} \left| h(|Tx|) \right|^q x_n^{-n} \, \mathrm{d}x \right)^{2/q},$$

where $q = \frac{2n+\alpha-1}{n-1}$, since $p = 2$ in Theorem 5.2.1. Hence, replacing 2^* with q above, recalling that $J_\alpha^{\mathbb{H}^n}(f_k)$ is uniformly bounded, and letting $C > 0$ be a fixed constant, then we have the uniform pointwise bound

$$h_k(r) \leq C \min \left\{ \left(\frac{(1-r^2)^{n-1}}{r^n} \right)^{1/2^*}, \left(\frac{(1-r^2)^{n-1}}{r^n} \right)^{1/q} \right\}. \tag{43}$$

Now, fix $S > 0$ and consider

$$\int\limits_{\mathbb{H}^n \setminus B^S} |f_k|^{2^*} = \int\limits_{B \setminus B_S(\mathbf{0})} \left| \widetilde{f}_k(w) \right|^{2^*} \, \mathrm{d}w$$

$$= |\mathbb{S}^{n-1}| \int\limits_S^1 |h_k(r)|^{2^*} \left(\frac{2r}{1-r^2}\right)^n \frac{\mathrm{d}r}{r}$$

$$\leq C^{2^*} |\mathbb{S}^{n-1}| \int\limits_S^1 \left(\frac{(1-r^2)^{n-1}}{r^n}\right)^{2^*/q} \left(\frac{2r}{1-r^2}\right)^n \frac{\mathrm{d}r}{r},$$

which is integrable by the easily verified fact that $2^* > q$. By dominated convergence,

$$\left\| 1_{\mathbb{H}^n \setminus B^S} f_k \right\|_{2^*} \to 0, \tag{44}$$

for all $S > 0$. Then,

$$\int\limits_{\mathbb{H}^n} |f_k|^{2^*} = \int\limits_{\mathbb{H}^n} |\varphi_R f_k|^{2^*} + \int\limits_{\mathbb{H}^n \setminus B^{e^{-2R}}} |f_k|^{2^*} - \int\limits_{B^{e^{-R}} \setminus B^{e^{-2R}}} |\varphi_R f_k|^{2^*}.$$

It should be clear that the latter two integrals converge to zero because of (44). Thus,

$$\lim_{k\to\infty} \|f_k\|_{2^*}^2 = \lim_{k\to\infty} \|\varphi_R f_k\|_{2^*}^2. \tag{45}$$

Now, we define the following function, having support on the halfline $(0, \infty)$,

$$g_k(t) = \frac{h_k(e^{-t})}{(\sinh t)^{n/2^*}},$$

and we seek to show that

$$V_{k,R} + b_{n,\alpha} \int\limits_{\mathbb{H}^n} |\varphi_R(x) f_k(x)|^2 x_n^{-\alpha} \, \mathrm{d}x \leq \frac{c}{R^\alpha} \int\limits_0^\infty g_k^2(t) \, \mathrm{d}t, \tag{46}$$

where $c > 0$ is a fixed constant. Using that $\sinh t \geq t$,

$$\begin{aligned} \int\limits_{\mathbb{H}^n} |\varphi_R(x) f_k(x)|^2 x_n^{-\alpha} \, \mathrm{d}x &= |\mathbb{S}^{n-1}| \int\limits_0^1 \phi_R^2(-\ln r) \left(\frac{2r}{1-r^2}\right)^n h_k^2(r) \frac{\mathrm{d}r}{r} \\ &= |\mathbb{S}^{n-1}| \int\limits_0^\infty \phi_R^2(t) h_k^2(e^{-t}) \sinh^{-n} t \, \mathrm{d}t \\ &\leq |\mathbb{S}^{n-1}| \int\limits_R^\infty h_k^2(e^{-t}) \sinh^{-n} t \, \mathrm{d}t \\ &\leq \frac{|\mathbb{S}^{n-1}|}{R^\alpha} \int\limits_0^\infty g_k^2(t) \, \mathrm{d}t. \end{aligned}$$

From its Taylor series, $\cosh t \geq 1 + t^2/2$. Using that $n \geq 3$, we consider the function

$$\begin{aligned}
Q(t) &:= \int\limits_{\mathbb{S}^{n-1}} \int\limits_{\mathbb{S}^{n-1}} (\cosh t - w \cdot w')^{-\frac{n+\alpha}{2}} \, \mathrm{d}w \, \mathrm{d}w' \\
&= |\mathbb{S}^{n-2}||\mathbb{S}^{n-1}| \int\limits_0^\pi (\sin^{n-2}\theta)(\cosh t - \cos\theta)^{-\frac{n+\alpha}{2}} \, \mathrm{d}\theta \\
&= |\mathbb{S}^{n-2}||\mathbb{S}^{n-1}| \int\limits_{-1}^1 (1-u^2)^{\frac{n-3}{2}} (\cosh t - u)^{-\frac{n+\alpha}{2}} \, \mathrm{d}u \\
&\leq 2^{\frac{n-3}{2}} |\mathbb{S}^{n-2}||\mathbb{S}^{n-1}| \int\limits_{-1}^1 (1-u)^{\frac{n-3}{2}} (\cosh t - u)^{-\frac{n+\alpha}{2}} \, \mathrm{d}u \\
&\leq 2^{\frac{n-3}{2}} |\mathbb{S}^{n-2}||\mathbb{S}^{n-1}| \int\limits_{-1}^1 (\cosh t - u)^{-\frac{3+\alpha}{2}} \, \mathrm{d}u \\
&\leq \frac{2^{\frac{n-1}{2}}}{1+\alpha} |\mathbb{S}^{n-2}||\mathbb{S}^{n-1}| \, (\cosh t - 1)^{-\frac{1+\alpha}{2}} \\
&\leq \frac{2^{\frac{n+\alpha}{2}}}{1+\alpha} |\mathbb{S}^{n-2}||\mathbb{S}^{n-1}||t|^{-1-\alpha}.
\end{aligned}$$

Let L_ϕ denote the Lipschitz constant for the cutoff function ϕ on the halfline. Transforming first from the halfspace to the unit ball under T, and then from the unit ball to the half-line via $r = e^{-s}$, we get

$$\begin{aligned}
&\int\limits_{\mathbb{H}^n \times \mathbb{H}^n} \frac{f_k(x) f_k(y)}{|x-y|^{n+\alpha}} \Big| \varphi_R(x) - \varphi_R(y) \Big|^2 \mathrm{d}x \, \mathrm{d}y \\
&= \int\limits_{\mathbb{H}^n \times \mathbb{H}^n} \frac{f_k(Tx) f_k(Ty)}{\eta(x)^{\frac{n+\alpha}{2}} |x-y|^{n+\alpha} \eta(y)^{\frac{n+\alpha}{2}}} \Big| \varphi_R(Tx) - \varphi_R(Ty) \Big|^2 \eta(x)^n \eta(y)^n \, \mathrm{d}x \, \mathrm{d}y \\
&= \int\limits_{B \times B} \frac{\widetilde{f_k}(x) \widetilde{f_k}(y)}{|x-y|^{n+\alpha}} \Big| \phi_R(-\ln|x|) - \phi_R(-\ln|y|) \Big|^2 \mathrm{d}x \, \mathrm{d}y \\
&= \int\limits_0^1 \frac{\mathrm{d}r}{r} \int\limits_0^1 \frac{\mathrm{d}\rho}{\rho} \left(\frac{2}{1-r^2} \frac{2}{1-\rho^2} \right)^{\frac{n-\alpha}{2}} h_k(r) h_k(\rho) \Big| \phi_R(-\ln r) - \phi_R(-\ln\rho) \Big|^2 \int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w \int\limits_{\mathbb{S}^{n-1}} \mathrm{d}w' \\
&\qquad \times \frac{(r\rho)^n}{(r^2 + \rho^2 - 2r\rho w \cdot w')^{\frac{n+\alpha}{2}}} \\
&= 2^{-\frac{n+\alpha}{2}} \int\limits_0^1 \frac{\mathrm{d}r}{r} \int\limits_0^1 \frac{\mathrm{d}\rho}{\rho} \left(\frac{2r}{1-r^2} \frac{2\rho}{1-\rho^2} \right)^{\frac{n-\alpha}{2}} h_k(r) h_k(\rho) \Big| \phi_R(-\ln r) - \phi_R(-\ln\rho) \Big|^2 Q\left(\ln \frac{\rho}{r} \right)
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-\frac{n+\alpha}{2}} \int_0^\infty \int_0^\infty g_k(s) g_k(t) \Big| \phi_R(s) - \phi_R(t) \Big|^2 Q(s-t) \, \mathrm{d}s \, \mathrm{d}t \\
&\leq \frac{|\mathbb{S}^{n-2}||\mathbb{S}^{n-1}|}{1+\alpha} \int_0^\infty \mathrm{d}s \; g_k^2(s) \int_{\mathbb{R}} \mathrm{d}t \frac{\Big|\phi_R(s+t) - \phi_R(t)\Big|^2}{|t|^{1+\alpha}} \\
&\leq \frac{|\mathbb{S}^{n-2}||\mathbb{S}^{n-1}|}{1+\alpha} \int_0^\infty \mathrm{d}s \; g_k^2(s) \left(\frac{L_\phi}{R^2} \int_{|t|<R} \frac{\mathrm{d}t}{|t|^{\alpha-1}} + \int_{|t|\geq R} \frac{\mathrm{d}t}{|t|^{1+\alpha}} \right) \\
&\leq \frac{|\mathbb{S}^{n-2}||\mathbb{S}^{n-1}|}{(1+\alpha)(2-\alpha)} R^{-\alpha} \int_0^\infty g_k^2(t) \, \mathrm{d}t.
\end{aligned}
$$

Repeat the above computations replacing φ_R with $\overline{\varphi}_R$, and we establish (46).

Next, we show $\|g_k\|_2^2$ is uniformly bounded. Noting that $J_\alpha^{\mathbb{H}^n}(f_k)$ is uniformly bounded, we do this by proving the following sequence of inequalities.

1. $\|g_k\|_2^2 \leq c_1 I_\alpha^{\mathbb{R}^n}(\varphi_S f_k) + c_2$,

2. $I_\alpha^{\mathbb{R}^n}(\varphi_S f_k) \leq c_3 J_\alpha^{B^{e^{-S}}}(\varphi_S f_k) + c_4 \|\varphi_S f_k\|_2^2$,

3. $J_\alpha^{B^{e^{-S}}}(\varphi_S f_k) \leq c_5 J_\alpha^{\mathbb{H}^n}(f_k) + c_6 \|1_{B^{e^{-S}}} f_k\|_2^2$, and

4. $\|\varphi_S f_k\|_2^2 \leq \|1_{B^{e^{-S}}} f_k\|_2^2 \leq c_7$,

where $c_i > 0$ is a fixed constant, for all $i = 1, 2, \ldots, 7$.

From Theorem 2.4.9, we recall the fractional Hardy inequality which states

$$
I_\alpha^{\mathbb{R}^n}(f) \geq C_{n,2,\alpha} \int_{\mathbb{R}^n} f^2(x) |x|^{-\alpha} \, \mathrm{d}x.
$$

Fix $S > 0$, then using (43),

$$
\begin{aligned}
&\int_0^\infty g_k^2(t) \, \mathrm{d}t \\
&= \int_0^1 \left(\frac{2r}{1-r^2} \right)^{n-\alpha} h_k^2(r) \frac{\mathrm{d}r}{r} \\
&= \int_0^1 \left(\frac{2r}{1-r^2} \right)^{n-\alpha} \phi_S^2(-\ln r) h_k^2(r) \frac{\mathrm{d}r}{r} + \int_0^1 \left(\frac{2r}{1-r^2} \right)^{n-\alpha} \overline{\phi}_S^2(-\ln r) h_k^2(r) \frac{\mathrm{d}r}{r}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{B} \varphi_S^2(Tw) \widetilde{f}_k^2(w) |w|^{-\alpha} \, \mathrm{d}w + C^2 \int\limits_{-\ln 2S}^{1} \left(\frac{2r}{1-r^2} \right)^{n-\alpha} \left(\frac{(1-r^2)^{n-1}}{r^n} \right)^{2/q} \frac{\mathrm{d}r}{r} \\
&\leq \frac{C_{n,2,\alpha}^{-1}}{|\mathbb{S}^{n-1}|} I_\alpha^{\mathbb{R}^n} \left(\varphi_S(Tw) \widetilde{f}_k(w) \right) + c_S \\
&= \frac{C_{n,2,\alpha}^{-1}}{|\mathbb{S}^{n-1}|} I_\alpha^{\mathbb{R}^n} (\varphi_S f_k) + c_S,
\end{aligned}
$$

where

$$
c_S = C^2 \int\limits_{-\ln 2S}^{1} \left(\frac{2r}{1-r^2} \right)^{n-\alpha} \left(\frac{(1-r^2)^{n-1}}{r^n} \right)^{2/q} \frac{\mathrm{d}r}{r}.
$$

Note that $c_S < \infty$ because $0 < S < 1$ and

$$
\frac{2}{q}(n-1) - (n-\alpha) > -1.
$$

This last inequality holds as $q = \frac{2n+\alpha-1}{n-1}$.

Now, from the proof of Theorem 5.1.1, there exists $c_\Omega > 0$ so that

$$
c_\Omega I_\alpha^\Omega(f) \geq I_\alpha^{\mathbb{R}^n}(u).
$$

Further, we estimate

$$
\begin{aligned}
&\left| \varphi_S f_k(x) - \varphi_S f_k(y) \right|^2 \\
&= \left| x_n^{\frac{\alpha-1}{2}} \left(x_n^{\frac{1-\alpha}{2}} \varphi_S f_k(x) - y_n^{\frac{1-\alpha}{2}} \varphi_S f_k(y) \right) + y_n^{\frac{1-\alpha}{2}} \varphi_S f_k(y) \left(x_n^{\frac{\alpha-1}{2}} - y_n^{\frac{\alpha-1}{2}} \right) \right|^2 \\
&\leq 2 x_n^{\alpha-1} \left| x_n^{\frac{1-\alpha}{2}} \varphi_S f_k(x) - y_n^{\frac{1-\alpha}{2}} \varphi_S f_k(y) \right|^2 + 2 y_n^{1-\alpha} (\varphi_S f_k)^2(y) \left| x_n^{\frac{\alpha-1}{2}} - y_n^{\frac{\alpha-1}{2}} \right|^2.
\end{aligned}
$$

Thus, there exists $A > 0$ so that

$$
I_\alpha^{\mathbb{R}^n}(\varphi_S f_k) \leq c_\Omega I_\alpha^{B^{e^{-s}}}(\varphi_S f_k) \leq c_\Omega \left(J_\alpha^{B^{e^{-s}}}(\varphi_S f_k) + A \|\varphi_S f_k\|_2^2 \right).
$$

Indeed, the norm on the right is established as follows

$$
2 \int\limits_{B^{e^{-s}}} \int\limits_{B^{e^{-s}}} \frac{\left| x_n^{\frac{\alpha-1}{2}} - y_n^{\frac{\alpha-1}{2}} \right|^2}{|x-y|^{n+\alpha}} y_n^{1-\alpha} (\varphi_S f_k)^2(y) \, \mathrm{d}x \, \mathrm{d}y
$$

$$\leq 2\left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} \int\limits_{B^{e^{-s}}} \mathrm{d}y(\varphi_S f_k)^2(y) \int\limits_{\frac{1-e^{-s}}{1+e^{-s}}}^{\frac{1+e^{-s}}{1-e^{-s}}} \mathrm{d}x_n \int\limits_{\mathbb{R}^{n-1}} \mathrm{d}x' \frac{\left|x_n^{\frac{\alpha-1}{2}} - y_n^{\frac{\alpha-1}{2}}\right|^2}{(|x'-y'|^2+|x_n-y_n|^2)^{\frac{n+\alpha}{2}}}$$

$$= \left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} |\mathbb{S}^{n-2}|\beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right) \int\limits_{B^{e^{-s}}} \mathrm{d}y(\varphi_S f_k)^2(y) \int\limits_{\frac{1-e^{-s}}{1+e^{-s}}}^{\frac{1+e^{-s}}{1-e^{-s}}} \mathrm{d}x_n \frac{\left|x_n^{\frac{\alpha-1}{2}} - y_n^{\frac{\alpha-1}{2}}\right|^2}{|x_n-y_n|^{1+\alpha}}$$

$$\leq \left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} |\mathbb{S}^{n-2}|\beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right) L^2 \int\limits_{B^{e^{-s}}} \mathrm{d}y(\varphi_S f_k)^2(y) \int\limits_{\frac{1-e^{-s}}{1+e^{-s}}}^{\frac{1+e^{-s}}{1-e^{-s}}} \mathrm{d}x_n |x_n-y_n|^{1-\alpha}$$

$$\leq \left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} \beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right) \frac{4L^2|\mathbb{S}^{n-2}|}{(2-\alpha)\sinh S} \|\varphi_S f_k\|_2^2,$$

where L is the Lipschitz constant for $x_n^{\frac{\alpha-1}{2}}$ on the interval $\left(\tfrac{1-e^{-s}}{1+e^{-s}}, \tfrac{1+e^{-s}}{1-e^{-s}}\right)$.

A similar estimate as above gives

$$\begin{aligned}
&\left|x_n^{\frac{1-\alpha}{2}}\varphi_S f_k(x) - y_n^{\frac{1-\alpha}{2}}\varphi_S f_k(y)\right|^2 \\
&= \left|\varphi_S(x)\left(x_n^{\frac{1-\alpha}{2}} f_k(x) - y_n^{\frac{1-\alpha}{2}} f_k(y)\right) + y_n^{\frac{1-\alpha}{2}} f_k(y)\left(\varphi_S(x)-\varphi_S(y)\right)\right|^2 \\
&\leq 2\varphi_S^2(x)|x_n^{\frac{1-\alpha}{2}} f_k(x) - y_n^{\frac{1-\alpha}{2}} f_k(y)|^2 + 2y_n^{1-\alpha} f_k^2(y)|\varphi_S(x)-\varphi_S(y)|^2.
\end{aligned}$$

Thus, noting that φ is Lipschitz continuous on $\mathbb{H}^n$, with Lipschitz constant L_φ,

$$\begin{aligned}
&J_\alpha^{B^{e^{-s}}}(\varphi_S f_k) \\
&\leq 2J_\alpha^{B^{e^{-s}}}(f_k) + 2\left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} \int\limits_{B^{e^{-s}}} \mathrm{d}y f_k^2(y) \int\limits_{B^{e^{-s}}} \mathrm{d}x \frac{|\varphi_S(x)-\varphi_S(y)|^2}{|x-y|^{n+\alpha}} \\
&\leq 2J_\alpha^{\mathbb{H}^n}(f_k) + 2\left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} L^2 \int\limits_{B^{e^{-s}}} \mathrm{d}y f_k^2(y) \int\limits_{\frac{1-e^{-s}}{1+e^{-s}}}^{\frac{1+e^{-s}}{1-e^{-s}}} \mathrm{d}x_n \int\limits_{\mathbb{R}^{n-1}} \frac{\mathrm{d}x'}{(|x'-y'|^2+|x_n-y_n|^2)^{\frac{n+\alpha}{2}-1}} \\
&= 2J_\alpha^{\mathbb{H}^n}(f_k) + \left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} |\mathbb{S}^{n-2}|\beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right) L^2 \int\limits_{B^{e^{-s}}} \mathrm{d}y f_k^2(y) \int\limits_{\frac{1-e^{-s}}{1+e^{-s}}}^{\frac{1+e^{-s}}{1-e^{-s}}} \mathrm{d}x_n |x_n-y_n|^{1-\alpha} \\
&\leq 2J_\alpha^{\mathbb{H}^n}(f_k) + \left(\tfrac{1+e^{-s}}{1-e^{-s}}\right)^{\alpha-1} \beta\left(\frac{n-1}{2}, \frac{1+\alpha}{2}\right) \frac{4L^2|\mathbb{S}^{n-2}|}{(2-\alpha)\sinh S} \|1_{B^{e^{-s}}} f_k\|_2^2,
\end{aligned}$$

from similar calculations as above.

Finally, using (43),

$$\begin{aligned}
\|\varphi_S f_k\|_2^2 &\le \|1_{B^{e^{-S}}} f_k\|_2^2 \\
&= \int\limits_{B_{e^{-S}}(\mathbf{0})} \widetilde{f}_k^2(w)\eta(w)^\alpha \, \mathrm{d}w \\
&\le \left(\frac{2}{(1-e^{-S})^2}\right)^\alpha |\mathbb{S}^{n-1}| \int\limits_0^{e^{-S}} \left(\frac{2}{1-r^2}\right)^{n-\alpha} h_k^2(r) r^{n-1} \, \mathrm{d}r \\
&\le \frac{2^n C^2}{(1-e^{-S})^{2\alpha}} |\mathbb{S}^{n-1}| \int\limits_0^{e^{-S}} \frac{r^{n-1}}{(1-r^2)^{n-\alpha}} \left(\frac{(1-r^2)^{n-1}}{r^n}\right)^{2/q} \mathrm{d}r,
\end{aligned}$$

which is finite because $0 < S < 1$ and $2 < q$.

Hence, we have established our sequence of inequalities. Thus, since $J_\alpha^{\mathbb{H}^n}(f_k)$ is uniformly bounded, then so is $\int_0^\infty g_k^2(t)\, \mathrm{d}t$. Therefore, from (42), (45), and (46),

$$\begin{aligned}
\lim_{k\to\infty} J_\alpha^{\mathbb{H}^n}(f_k) &\ge \lim_{k\to\infty} \left(S_{n,2,\alpha} \|\varphi_R f_k\|_{2^*}^2 - V_{k,R} - b_{n,\alpha} \int\limits_{\mathbb{H}^n} |\varphi_R(x) f_k(x)|^2 x_n^{-\alpha} \, \mathrm{d}x \right) \\
&\ge S_{n,2,\alpha} \lim_{k\to\infty} \|f_k\|_{2^*}^2 - \frac{c}{R^\alpha},
\end{aligned}$$

and, letting $R \to \infty$, the result follows. □

6.2 Sufficient Condition for Existence

In what follows, we prove that if the sharp constant for the fractional Hardy-Sobolev-Maz'ya inequality is strictly less than the sharp constant for the fractional Sobolev inequality, then a minimizer exists. For the proof, we need the following extension of Fatou's lemma from Theorem 1.9 in [29].

THEOREM 6.2.1 (Missing term in Fatou's lemma). *Let $0 < p < \infty$ and let $\Omega \subseteq \mathbb{R}^n$. Let $f_k \in L^p(\Omega)$ for all k. Assume there exists $f \in L^p(\Omega)$ such that $f_k \to f$ almost everywhere and that $\|f_k\|_p$ is uniformly bounded. Then,*

$$\|f_k\|_p^p = \|f\|_p^p + \|f - f_k\|_p^p + o(1),$$

where $o(1)$ indicates a quantity that vanishes as $j \to \infty$.

The above Theorem is a special case of a broader result found in [12]. We use this result to prove the following, recalling that $M_{n,\alpha}$ is the sharp constant for the fractional Hardy-Sobolev-Maz'ya inequality.

THEOREM 6.2.2. *Let $n \geq 3$ and $1 < \alpha < 2$. If $S_{n,2,\alpha} > M_{n,\alpha}$, then $\Psi_{n,\alpha}$ has a minimizer in $\dot{S}_0^\alpha(\mathbb{H}^n)$.*

Proof. Let $\{f_k\}$ be a minimizing sequence for $\Psi_{n,\alpha}$ in $\dot{S}_0^\alpha(\mathbb{H}^n)$. In particular, we can assume that

$$f_k(x) = x_n^{-n/2^*} h_k(|Tx|),$$

from Lemma 5.1.3. By the scale invariance of $\Psi_{n,\alpha}$, we can also assume $\|f_k\|_{2^*} = 1$ for all k. It should also be noted that, since $\Psi_{n,\alpha}(f_k) \to M_{n,\alpha}$ as $k \to \infty$, then by truncation, if necessary, we may further assume that for any $\epsilon > 0$, then

$$\left| J_\alpha^{\mathbb{H}^n}(f_k) - M_{n,\alpha} \right| < \epsilon,$$

so $J_\alpha^{\mathbb{H}^n}(f_k)$ is uniformly bounded.

As above in (43), letting $C > 0$ be a fixed constant, we know that

$$h_k(r) \leq C \min \left\{ \left(\frac{(1-r^2)^{n-1}}{r^n} \right)^{1/2^*}, \left(\frac{(1-r^2)^{n-1}}{r^n} \right)^{1/q} \right\}, \tag{47}$$

for all k. Hence, h_k is monotonic and uniformly pointwise bounded, so by Helly's selection theorem, and passing to a subsequence if necessary, there exists h such that $h_k \to h$ pointwise.

Define $d_k = f_k - f$, so $d_k \to 0$ pointwise. Using the inequality

$$(a+b)^p < a^p + b^p, \quad a, b > 0, 0 < p < 1,$$

and applying Theorem 6.2.1, the missing term in Fatou's lemma, we obtain

$$\|f_k\|_{2^*}^2 = \left(\|f_k\|_{2^*}^{2^*}\right)^{2/2^*} = \left(\|f\|_{2^*}^{2^*} + \|d_k\|_{2^*}^{2^*} + o(1)\right)^{2/2^*} \leq \|f\|_{2^*}^2 + \|d_k\|_{2^*}^2 + o(1), \tag{48}$$

as $\|f_k\|_{2^*}^2$ is uniformly bounded, for all k, and $2^* > 2$. Also, since $J_\alpha^{\mathbb{H}^n}(f_k)$ is uniformly bounded, further application of Theorem 6.2.1 results in

$$J_\alpha^{\mathbb{H}^n}(f_k) = J_\alpha^{\mathbb{H}^n}(f) + J_\alpha^{\mathbb{H}^n}(d_k) + o(1). \tag{49}$$

We denote

$$\lim_{k\to\infty} \|d_k\|_{2^*}^2 = L,$$

then from Lemma 6.1.3, we get the critical estimate

$$\lim_{k\to\infty} J_\alpha^{\mathbb{H}^n}(d_k) \geq \lim_{k\to\infty} S_{n,2,\alpha}\|d_k\|_{2^*}^2 = S_{n,2,\alpha}L \tag{50}$$

Using (48),(49), and (50), we compute

$$\begin{aligned}
M_{n,\alpha} &= \lim_{k\to\infty} \Psi_{n,\alpha}(f_k) \\
&\geq \lim_{k\to\infty} \frac{J_\alpha^{\mathbb{H}^n}(f) + J_\alpha^{\mathbb{H}^n}(d_k) + o(1)}{\|f\|_{2^*}^2 + \|d_k\|_{2^*}^2 + o(1)} \\
&\geq \frac{M_{n,\alpha}\|f\|_{2^*}^2 + S_{n,2,\alpha}L}{\|f\|_{2^*}^2 + L} \\
&= M_{n,\alpha} + \frac{(S_{n,2,\alpha} - M_{n,\alpha})\, L}{\|f\|_{2^*}^2 + L}.
\end{aligned}$$

Hence, $L = 0$, since it is given that $S_{n,2,\alpha} > M_{n,\alpha}$. Therefore, $f_k \to f$ in $L^{2^*}(\mathbb{H}^n)$, and

$$M_{n,\alpha} = \lim_{k\to\infty} \frac{J_\alpha^{\mathbb{H}^n}(f_k)}{\|f_k\|_{2^*}^2} = \frac{J_\alpha^{\mathbb{H}^n}(f) + \lim_{k\to\infty} J_\alpha^{\mathbb{H}^n}(d_k)}{\|f\|_{2^*}^2} = \Psi_{n,\alpha}(f) + \lim_{k\to\infty} J_\alpha^{\mathbb{H}^n}(d_k) \geq M_{n,\alpha},$$

so $J_\alpha^{\mathbb{H}^n}(d_k) \to 0$ and f is a minimizer. □

6.3 *Case of Existence*

In this section, we show that, for each $n \geq 3$, there exists an interval $\Omega_n \subset (1,2)$ so that when $\alpha \in \Omega_n$, then $S_{n,2,\alpha} > M_{n,\alpha}$. In particular, we use a test function g to get an exact value for $\Psi_{n,\alpha}(g)$. We prove some general results about the test function, then, using estimation and numerical results, we show $\Psi_{n,\alpha}(g) < S_{n,2,\alpha}$ when $\alpha \in \Omega_n$. Hence, the sharp constant $M_{n,\alpha}$ must be less than $S_{n,2,\alpha}$ in Ω_n as well. As a result of Theorem 6.2.2, a minimizer then exists for each $n \geq 3$ such that $\alpha \in \Omega_n$.

Let $n \geq 3$ and let $1 < \alpha < 2$. Our test function is defined as

$$g(x) = 1_{\mathbb{H}^n}(x)(2x_n)^{\alpha/2}\eta(x)^{n/2},$$

so that, from the definition in (15),

$$Tw = \left(\frac{2w', 1-|w|^2}{|w'|^2 + (w_n+1)^2}\right) = \eta(w)\left(w', \frac{1-|w|^2}{2}\right),$$

where $w = (w', w_n) \in \mathbb{R}^n$, with $w' \in \mathbb{R}^{n-1}$, and $w_n \in \mathbb{R}$, we obtain

$$\widetilde{g}(w) = \eta(w)^{\frac{n-\alpha}{2}} g(Tw) = \eta(w)^{-\alpha/2} 1_{\mathbb{H}^n}(Tw)\left(\eta(w)(1-|w|^2)\right)^{\alpha/2} = 1_B(w)(1-|w|^2)^{\alpha/2},$$

where B is the unit ball centered at the origin in $\mathbb{R}^n$.

We recall here several identities and inequalities stated earlier. From Definition 2.4.1 and Lemma 2.4.2,

$$(f, (-\Delta)^{\alpha/2} f) = \int\limits_{\mathbb{R}^n} (2\pi|k|)^{\alpha} |\hat{f}(k)|^2 \, \mathrm{d}k = a_{n,\alpha} I_\alpha^{\mathbb{R}^n}(f),$$

where

$$a_{n,\alpha} = 2^{\alpha-1}\pi^{-n/2}\frac{\Gamma(\frac{n+\alpha}{2})}{\left|\Gamma(-\frac{\alpha}{2})\right|}.$$

Further, from (11),

$$J_\alpha^{\mathbb{H}^n}(f) = \frac{1}{a_{n,\alpha}}(f, (-\Delta)^{\alpha/2} f) - \frac{1}{\pi a_{n,\alpha}}\Gamma\left(\frac{1+\alpha}{2}\right)^2 \int\limits_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha} \, \mathrm{d}x.$$

From Theorems 2.4.11 and 2.4.12, we have the related Sobolev inequalities

$$I_\alpha^{\mathbb{R}^n}(f) \geq S_{n,2,\alpha}\|f\|_{2^*}^2, \text{ and} (f, (-\Delta)^{\alpha/2} f) \geq S'_{n,\alpha} \|f\|_{2^*}^2,$$

where $S_{n,2,\alpha}$ and $S'_{n,\alpha}$ are sharp constants, and

$$S'_{n,\alpha} = \frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})} |\mathbb{S}^n|^{\alpha/n}.$$

Note the relationship $S_{n,2,\alpha} = S'_{n,\alpha}/a_{n,\alpha}$.

With these in mind, and the function g fixed as above, we define

$$\phi(n,\alpha) := \frac{J_\alpha^{\mathbb{H}^n}(g)}{S_{n,2,\alpha}\|g\|_{2^*}^2}$$

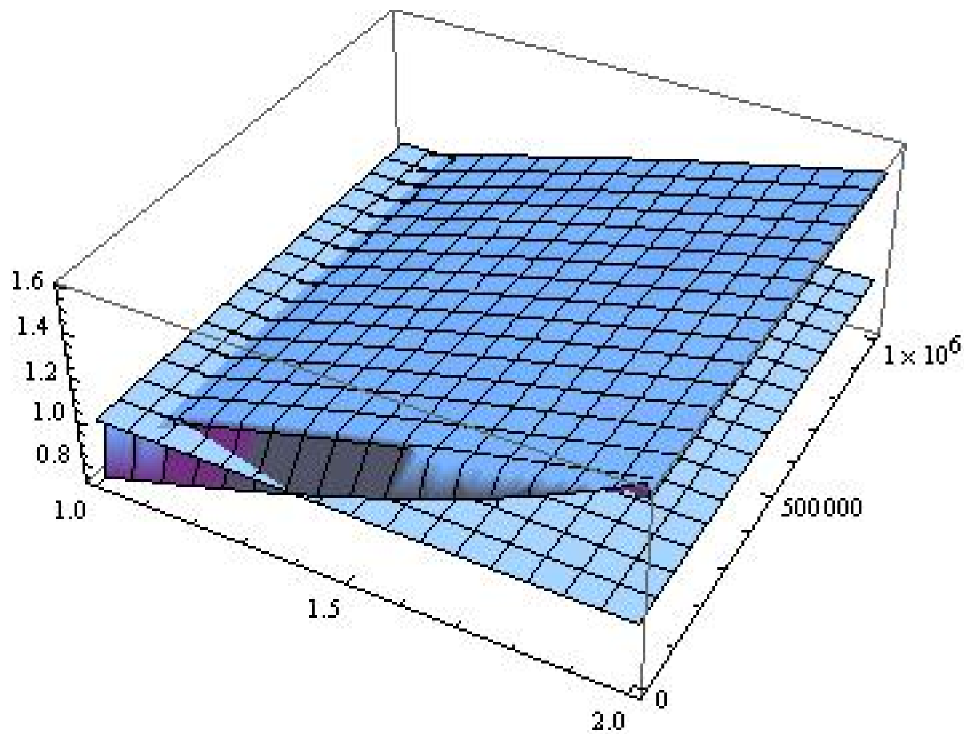

Figure 1: Graph of the surface $\phi(n, \alpha)$ and the plane $\{x \in \mathbb{R}^3 : x_3 = 1\}$

and seek to find n, α so that $\phi(n, \alpha) < 1$.

For the reader's visualization, we present Figures 1 and 2, two graphs produced in *Mathematica* showing the surface $\phi(n, \alpha)$ and the plane $\{x \in \mathbb{R}^3 : x_3 = 1\}$. From the graphs, we see that $\phi(n, \alpha)$ is increasing along both n and α.

Further, note that the intersection of the surface $\phi(n, \alpha)$ and the plane $\{x \in \mathbb{R}^3 : x_3 = 1\}$ in Figures 1 and 2 is defined by the implicit function $\phi(n, \alpha) = 1$. In Table 1, we make some rough approximate numerical calculations of that implicit function. These computations show that, after an increase from $n = 3$ to $n = 4$, there is a decline in α as $n \to \infty$. For reasons discussed below, the limiting value is approximately 1.099.

Table 1: Values of n, α along $\phi(n, \alpha) = 1$

n	3	4	5	10	25	50	100	250	500	1000	5000
α	1.39	1.41	1.40	1.32	1.22	1.17	1.14	1.12	1.11	1.11	1.10

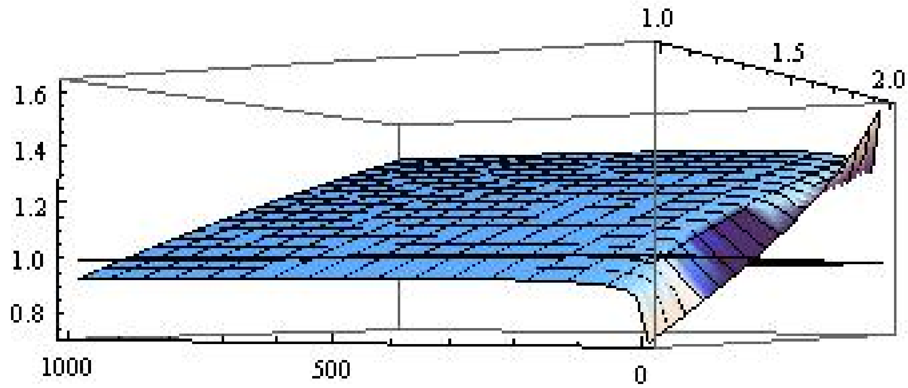

Figure 2: A different angle emphasizing the increase in $\phi(n, \alpha)$ with the variable n

While we have a good picture, from the graphs and numerical results, of the domain of values of (n, α) for which a minimizer does exist, we seek to prove this more conclusively.

6.3.1 Results on the Test Function

We now state a series of results regarding our test function g and $\phi(n, \alpha)$. The first establishes that $g \in \dot{S}_0^\alpha(\mathbb{H}^n)$. Then, we compute exactly the function $\phi(n, \alpha)$. Afterwards, we consider the limit of $\phi(n, \alpha)$ as $n \to \infty$.

LEMMA 6.3.1. *Let $n \geq 3$ and $1 < \alpha < 2$. Then,*

$$g(x) = 1_{\mathbb{H}^n}(x)(2x_n)^{\alpha/2}\eta(x)^{n/2} \in \dot{S}_0^\alpha(\mathbb{H}^n).$$

Proof. First, note that$\dot{S}_0^\alpha(\mathbb{H}^n)$ is the completion of $C_c^\infty(\mathbb{H}^n)$ with respect to the norm

$$\|f\|_{\dot{S}_0^\alpha(\mathbb{H}^n)} = \sqrt{J_\alpha^{\mathbb{H}^n}(f)},$$

and that

$$\sqrt{J_\alpha^{\mathbb{H}^n}(f)} = \left(\frac{1}{a_{n,\alpha}}(f, (-\Delta)^{\alpha/2} f) - \frac{1}{\pi a_{n,\alpha}}\Gamma\left(\frac{1+\alpha}{2}\right)^2 \int_{\mathbb{H}^n} |f(x)|^2 x_n^{-\alpha}\, \mathrm{d}x. \right)^{1/2}$$

$$= \left(\frac{1}{a_{n,\alpha}}(\widetilde{f}, (-\Delta)^{\alpha/2} \widetilde{f}) - \frac{1}{\pi a_{n,\alpha}}\Gamma\left(\frac{1+\alpha}{2}\right)^2 \int_B |\widetilde{f}(w)|^2 \left(\frac{2}{1-|w|^2}\right) \mathrm{d}w \right)^{1/2}. \quad (51)$$

Denote (51) by $\sqrt{\widetilde{J}_\alpha^B(f)}$, and consider the scaled function

$$\widetilde{g}_{1-1/k}(w) = 1_{\{|w|\leq 1-1/k\}}(1-1/k)^{(\alpha-n)/2}(1-|w/(1-1/k)|^2)^{\alpha/2},$$

and note that $\widetilde{g}_{1-1/k} \to \widetilde{g}$ as $k \to \infty$. Since $\widetilde{g}_{1-1/k}$ is uniformly bounded with compact support away from the boundary of B, then $\widetilde{g}_{1-1/k} \in L^2(\mathbb{R}^n)$, and, by dominated convergence, $\sqrt{\widetilde{J}_\alpha^B(\widetilde{g}_{1-1/k})} \to \sqrt{\widetilde{J}_\alpha^B(\widetilde{g})}$ as $k \to \infty$. Thus,

$$\widetilde{g}_{1-1/k} \in W^{\alpha/2,2}(\mathbb{R}^n) = W_0^{\alpha/2,2}(\mathbb{R}^n).$$

From this, we know that, as $\widetilde{g}_{1-1/k}$ is a proper subset of the open set B, there exists a sequence $\{\widetilde{g}^j_{1-1/k}\}_{j=1}^\infty \subset C_c^\infty(B)$ so that

$$\left\|\widetilde{g}^j_{1-1/k} - \widetilde{g}_{1-1/k}\right\|_{W^{\alpha/2,2}(\mathbb{R}^n)} \to 0.$$

as $j \to \infty$. Now, it is a simple exercise to show that $\widetilde{g}^k_{1-1/k} \in C_c^\infty(B)$ converges to $\widetilde{g}$ under the norm $\sqrt{\widetilde{J}_\alpha^B(\widetilde{g})}$. Hence, $g \in \dot{S}_0^\alpha(\mathbb{H}^n)$. $\square$

LEMMA 6.3.2. *Let $n \geq 3$ and $1 < \alpha < 2$. Then,*

$$\phi(n,\alpha) = \frac{2}{n}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\frac{1}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{\alpha}{n}}\left(\frac{\Gamma\left(\frac{\alpha n}{n-\alpha}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)}\right)^{1-\frac{\alpha}{n}}\gamma(n,\alpha),$$

where

$$\gamma(n,\alpha) = \frac{1}{1+\alpha/n}\Gamma\left(\frac{\alpha}{2}+1\right)^2 - \frac{1}{\pi}\Gamma\left(\frac{1+\alpha}{2}\right)^2.$$

Proof. Note that

$$\begin{aligned}\phi(n,\alpha) &= \frac{\frac{1}{a_{n,\alpha}}(g,(-\Delta)^{\alpha/2}g) - \frac{1}{\pi a_{n,\alpha}}\Gamma\left(\frac{1+\alpha}{2}\right)^2\int_{\mathbb{H}^n}|g(x)|^2 x_n^{-\alpha}\,\mathrm{d}x}{S_{n,2,\alpha}\|g\|_{2^*}^2} \\ &= \frac{(\widetilde{g},(-\Delta)^{\alpha/2}\widetilde{g}) - \frac{1}{\pi}\Gamma\left(\frac{1+\alpha}{2}\right)^2\int_B|\widetilde{g}(w)|^2\left(\frac{2}{1-|w|^2}\right)^\alpha\mathrm{d}w}{S'_{n,\alpha}\|\widetilde{g}\|_{2^*}^2}.\end{aligned}$$

The equivalence of the Hardy terms in the two latter expressions is a simple computation that was done in Lemma 5.1.3 above. Now, we need two results to make the computations. First, from Theorem 4.15, [37],

$$\widehat{\widetilde{g}}(k) = \frac{\Gamma\left(\frac{\alpha}{2}+1\right)}{\pi^{\alpha/2}}|k|^{-\frac{n}{2}-\frac{\alpha}{2}}J_{\frac{n}{2}+\frac{\alpha}{2}}(2\pi|k|),$$

where $J_{\frac{n}{2}+\frac{\alpha}{2}}$ is a Bessel function of the first kind. Further, from formula 11.4.6 in [1],

$$\int_0^\infty t^{-1} J^2_{\frac{n}{2}+\frac{\alpha}{2}}(t)\,\mathrm{d}t = \frac{1}{n+\alpha}.$$

Thus, using these results, we compute

$$\begin{aligned}
(\widetilde{g}, (-\Delta)^{\alpha/2}\,\widetilde{g}) &= \int_{\mathbb{R}^n} (2\pi|k|)^\alpha \left|\pi^{-\alpha/2}\Gamma\left(\frac{\alpha}{2}+1\right)|k|^{-\frac{n}{2}-\frac{\alpha}{2}} J_{\frac{n}{2}+\frac{\alpha}{2}}(2\pi|k|)\right|^2 \mathrm{d}k \\
&= 2^\alpha \Gamma\left(\frac{\alpha}{2}+1\right)^2 \int_{\mathbb{R}^n} |k|^{-n} J^2_{\frac{n}{2}+\frac{\alpha}{2}}(2\pi|k|)\,\mathrm{d}k \\
&= 2^\alpha \Gamma\left(\frac{\alpha}{2}+1\right)^2 |\mathbb{S}^{n-1}| \int_0^\infty r^{-1} J^2_{\frac{n}{2}+\frac{\alpha}{2}}(2\pi r)\,\mathrm{d}r \\
&= 2^\alpha \Gamma\left(\frac{\alpha}{2}+1\right)^2 |\mathbb{S}^{n-1}| \int_0^\infty s^{-1} J^2_{\frac{n}{2}+\frac{\alpha}{2}}(s)\,\mathrm{d}s \\
&= \frac{2^\alpha}{n+\alpha}\Gamma\left(\frac{\alpha}{2}+1\right)^2 |\mathbb{S}^{n-1}|.
\end{aligned}$$

Also,

$$\int_B |\widetilde{g}(w)|^2 \left(\frac{2}{1-|w|^2}\right)^\alpha \mathrm{d}w = 2^\alpha \left(\int_B \mathrm{d}w\right) = \frac{2^\alpha}{n}|\mathbb{S}^{n-1}|.$$

And lastly,

$$\begin{aligned}
\|\widetilde{g}\|^2_{2^*} &= \left(\int_B \left|1-|w|^2\right|^{\frac{\alpha}{2}\frac{2n}{n-\alpha}} \mathrm{d}x\right)^{2/2^*} \\
&= \left(\left|\mathbb{S}^{n-1}\right| \int_0^1 r^{n-1}(1-r^2)^{\frac{\alpha n}{n-\alpha}}\,\mathrm{d}r\right)^{1-\frac{\alpha}{n}} \\
&= \left(\frac{|\mathbb{S}^{n-1}|}{2} \int_0^1 s^{\frac{n}{2}-1}(1-s)^{\frac{\alpha n}{n-\alpha}}\,\mathrm{d}r\right)^{1-\frac{\alpha}{n}} \\
&= \left(\frac{|\mathbb{S}^{n-1}|}{2}\beta\left(\frac{n}{2}, \frac{\alpha n}{n-\alpha}+1\right)\right)^{1-\frac{\alpha}{n}}.
\end{aligned}$$

Hence,using the Gamma function identity

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z),$$

for $z \in \mathbb{C}$, we obtain

$$\begin{aligned}
\phi(n,\alpha) &= \frac{\frac{2^\alpha}{n+\alpha}|\mathbb{S}^{n-1}|\Gamma\left(\frac{\alpha}{2}+1\right)^2 - \frac{2^\alpha}{\pi n}|\mathbb{S}^{n-1}|\Gamma\left(\frac{1+\alpha}{2}\right)^2}{\frac{\Gamma\left(\frac{n+\alpha}{2}\right)}{\Gamma\left(\frac{n-\alpha}{2}\right)}|\mathbb{S}^n|^{\frac{\alpha}{n}}\left(\frac{|\mathbb{S}^{n-1}|}{2}\beta\left(\frac{n}{2},\frac{\alpha n}{n-\alpha}+1\right)\right)^{1-\frac{\alpha}{n}}} \\
&= \frac{2^{\alpha+1}}{n}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\left(\frac{\Gamma\left(\frac{\alpha n}{n-\alpha}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)}\right)^{1-\frac{\alpha}{n}}\left(\frac{|\mathbb{S}^{n-1}|}{2|\mathbb{S}^n|}\right)^{\frac{\alpha}{n}}\gamma(n,\alpha) \\
&= \frac{2^{\alpha+1}}{n}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\left(\frac{\Gamma\left(\frac{\alpha n}{n-\alpha}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)}\right)^{1-\frac{\alpha}{n}}\left(\frac{\pi^{n/2}\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{(n+1)/2}\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{\alpha}{n}}\gamma(n,\alpha) \quad (52) \\
&= \frac{2^{\alpha+1}}{n}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\left(\frac{\Gamma\left(\frac{\alpha n}{n-\alpha}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)}\right)^{1-\frac{\alpha}{n}}\left(\frac{\pi^{n/2}2^{1-n}\sqrt{\pi}\,\Gamma(n)}{2\pi^{(n+1)/2}\Gamma\left(\frac{n}{2}\right)^2}\right)^{\frac{\alpha}{n}}\gamma(n,\alpha) \\
&= \frac{2}{n}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\frac{1}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{\alpha}{n}}\left(\frac{\Gamma\left(\frac{\alpha n}{n-\alpha}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)}\right)^{1-\frac{\alpha}{n}}\gamma(n,\alpha) \qquad \square
\end{aligned}$$

Having stated $\phi(n,\alpha)$ in terms of Gamma functions, we can use their asymptotic relationships to compute the limit of $\phi(n,\alpha)$ as $n \to \infty$.

LEMMA 6.3.3. *Let $n \geq 3$ and $1 < \alpha < 2$. Then, $\phi(n,\alpha) \to 2^\alpha\bar{\gamma}(\alpha)/\Gamma(\alpha+1)$ as $n \to \infty$, where $\bar{\gamma}(\alpha) = \Gamma\left(\frac{\alpha}{2}+1\right)^2 - \frac{1}{\pi}\Gamma\left(\frac{1+\alpha}{2}\right)^2$.*

Proof. From (52) in Lemma 6.3.2 above, it follows that

$$\phi(n,\alpha) = \frac{2^{\alpha+1-\alpha/n}}{n\pi^{\alpha/2n}}\frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)}\frac{\Gamma\left(\frac{\alpha n}{n-\alpha}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{\alpha n}{n-\alpha}+1+\frac{n}{2}\right)}\right)^{\frac{\alpha}{n}}\frac{\gamma(n,\alpha)}{\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)^{1-\alpha/n}}.$$

It can be seen either from the asymptotic Stirling formula, as in formula 6.1.39 in [1], or from Wendel [40], the following asymptotic relation

$$\frac{\Gamma(x+b)}{\Gamma(x+a)} \sim x^{b-a},$$

for all $x > 0$. Thus,

$$\phi(n,\alpha) \sim \frac{2^{\alpha+1-\alpha/n}}{n\pi^{\alpha/2n}}\left(\frac{n}{2}\right)^{-\alpha}\left(\frac{n}{2}\right)^{\frac{\alpha n}{n-\alpha}+1}\left(\frac{n}{2}\right)^{-\frac{\alpha}{n}\left(\frac{\alpha n}{n-\alpha}+\frac{1}{2}\right)}\frac{\gamma(n,\alpha)}{\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)^{1-\alpha/n}}.$$

Using the identity

$$-\alpha + \frac{\alpha n}{n-\alpha} + 1 - \frac{\alpha}{n}\left(\frac{\alpha n}{n-\alpha}+\frac{1}{2}\right) = 1 - \frac{\alpha}{2n},$$

then

$$\lim_{n\to\infty} \phi(n,\alpha) = \lim_{n\to\infty} \frac{2^\alpha n^{-\alpha/2n}}{\pi^{\alpha/2n}} \frac{\gamma(n,\alpha)}{\Gamma\left(\frac{\alpha n}{n-\alpha}+1\right)^{1-\alpha/n}} = \frac{2^\alpha \bar{\gamma}(\alpha)}{\Gamma(\alpha+1)}. \qquad \square$$

As discussed above, we now establish, for the curve defined by the implicit function $\phi(n,\alpha) = 1$, an approximate limiting value of α as $n \to \infty$.

LEMMA 6.3.4. *There exists a solution $\bar{\alpha}$ to the equation*

$$\frac{2^\alpha}{\Gamma(\alpha+1)}\, \bar{\gamma}(\alpha) = 1. \tag{53}$$

in the interval $\left(\frac{12}{11}, \frac{10}{9}\right)$. *If there exist any other solutions to* (53) *in the interval* $(1,2)$, *then $\bar{\alpha}$ has the lowest value.*

Proof. Denote by $L(\alpha)$ the left-hand side of (53). We'll prove that $L(\alpha)$ is increasing, at least in the interval $(1, 1.47)$. It can be shown numerically that

$$L\left(\tfrac{12}{11}\right) < 1 < L\left(\tfrac{10}{9}\right).$$

Since $L(\alpha)$ is continuous on the entire interval $(1,2)$, $\bar{\alpha}$ is the unique solution in the interval $\left(\frac{12}{11}, \frac{10}{9}\right)$. Hence, it is the solution with the lowest value in the interval $(1,2)$. As stated above, by a rough numerical approximation, we compute $\bar{\alpha} \approx 1.099$.

To prove our claim, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\alpha}\bar{\gamma}(\alpha) &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\Gamma\left(\frac{\alpha}{2}+1\right)^2 - \frac{1}{\pi}\,\Gamma\left(\frac{1+\alpha}{2}\right)^2\right) \\
&= \Gamma\left(\frac{\alpha}{2}+1\right)\Gamma'\left(\frac{\alpha}{2}+1\right) - \frac{1}{\pi}\,\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma'\left(\frac{1+\alpha}{2}\right) \\
&= \Gamma\left(\frac{\alpha}{2}+1\right)^2 \psi\left(\frac{\alpha}{2}+1\right) - \frac{1}{\pi}\,\Gamma\left(\frac{1+\alpha}{2}\right)^2 \psi\left(\frac{1+\alpha}{2}\right),
\end{aligned}$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the Digamma function. Note that $\Gamma\left(\frac{\alpha}{2}+1\right)$ is minimized by $\alpha = 1$ and $\Gamma\left(\frac{1+\alpha}{2}\right)$ is maximized by that same value. Thus,

$$\frac{\Gamma\left(\frac{\alpha}{2}+1\right)^2}{\Gamma\left(\frac{1+\alpha}{2}\right)^2} \geq \frac{\Gamma\left(\frac{3}{2}\right)^2}{\Gamma(1)^2} = \frac{\pi}{4} > \frac{1}{\pi}\frac{\psi\left(\frac{1+\alpha}{2}\right)}{\psi\left(\frac{\alpha}{2}+1\right)}, \tag{54}$$

as $\psi(x)$ is an increasing function for $x > 0$. Cross-multiplying (54), we see that the derivative of

$$\Gamma\left(\frac{\alpha}{2}+1\right)^2 - \frac{1}{\pi}\,\Gamma\left(\frac{1+\alpha}{2}\right)^2 \tag{55}$$

is positive for all $\alpha \in (1, 2)$, so (55) is strictly increasing on that interval.

Next, we consider

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{2^\alpha}{\Gamma(\alpha+1)} = \frac{2^\alpha\Gamma(\alpha+1)\ln 2 - 2^\alpha\Gamma'(\alpha+1)}{\Gamma(\alpha+1)^2} = \frac{2^\alpha}{\Gamma(\alpha+1)}\left(\ln 2 - \psi(\alpha+1)\right). \tag{56}$$

The only zero of (56) is when $\psi(\alpha+1) = \ln 2$, and it can be determined numerically that occurs at approximately $\alpha = 1.48$. Since $\psi(x)$ is an increasing function for $x > 0$, then (56) is positive for all $\alpha \in (1, 1.47)$, and $2^\alpha/\Gamma(\alpha+1)$ is increasing on the interval $(1, 1.47)$. If two functions are increasing on an interval, then so is their product. Hence, $2^\alpha\bar{\gamma}(\alpha)/\Gamma(\alpha+1)$ is increasing on the interval $(1, 1.47)$, proving the claim. $\square$

6.3.2 Existence of a Minimizer

We now state and prove that there exists an interval for each $n \geq 3$ so that a minimizer exists for $\Psi_{n,\alpha}$ in that interval, deferring the necessary computational Lemmas until afterwards. While this Theorem does not provide exact values for the right endpoint of these intervals, the reader may refer back to the numerical results as shown in Figures 1 and 2, as well as Table 1, for a better understanding.

THEOREM 6.3.5. *For all $n \geq 3$, there exists α_n such that $\phi(n, \alpha_n) = 1$, and $\phi(n, \alpha) < 1$ for all $\alpha < \alpha_n$. Therefore, for all $n \geq 3$ and all $\alpha \in (1, \alpha_n)$, there exists a minimizer for $\Psi_{n,\alpha}$ in $\dot{S}_0^\alpha(\mathbb{H}^n)$.*

Proof. We show in Lemmas 6.3.6 and 6.3.7 below that

$$\phi(n, 1) < 1 < \phi(n, 2)$$

for all $n \geq 3$. Since $\phi(n, \alpha)$ is continuous, there must exist a solution α_n of the equation $\phi(n, \alpha) = 1$ such that $\phi(n, \alpha) < 1$ for all $\alpha < \alpha_n$.

From Lemma 6.3.1, we know that $g \in \dot{S}_0^\alpha(\mathbb{H}^n)$, so it follows from Theorem 6.2.2 that, for all $n \geq 3$ and all $\alpha \in (1, \bar{\alpha})$, there exists a minimizer for $\Psi_{n,\alpha}$ in $\dot{S}_0^\alpha(\mathbb{H}^n)$. $\square$

We now prove Lemmas 6.3.6 and 6.3.7, as cited in Theorem 6.3.5. In both of the following Lemmas, we use the Gamma function identity

$$\Gamma(x)\Gamma\left(x + \tfrac{1}{2}\right) = 2^{1-2x}\sqrt{\pi}\,\Gamma(2x),$$

as well as the following inequality from [16]

$$\frac{b^{b-1}}{a^{a-1}}\,e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-1/2}}{a^{a-1/2}}\,e^{a-b},$$

for all $b > a > 0$.

LEMMA 6.3.6. *Let $n \geq 3$, then $\phi(n, 1) < 1$.*

Proof. First note that

$$\gamma(n,1) = \frac{1}{1+1/n}\Gamma\left(\tfrac{3}{2}\right)^2 - \frac{1}{\pi}\Gamma\left(1\right)^2 = \frac{n}{n+1}\left(\frac{\pi}{4}\right) - \frac{1}{\pi} \leq \frac{\pi^2-4}{4\pi}.$$

From Lemma 6.3.2, we compute

$$\begin{aligned}
\phi(n,1) &= \frac{2}{n}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}\frac{1}{\Gamma\left(\frac{n}{2}\right)}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}\left(\frac{\Gamma\left(\frac{n}{n-1}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{n-1}+1\right)}\right)^{1-\frac{1}{n}}\gamma(n,1) \\
&= \frac{2}{n}\frac{2}{n-1}\frac{\pi^2-4}{4\pi}\left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}\left(\frac{\Gamma\left(\frac{n}{n-1}+\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{n-1}+1\right)}\right)^{1-\frac{1}{n}} \\
&= \frac{\pi^2-4}{\pi n(n-1)}\left(\frac{2^{n-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}\left(\frac{\frac{n}{n-1}+\frac{n}{2}}{\frac{n}{n-1}}\frac{\Gamma\left(\frac{n}{n-1}+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{n-1}\right)}\right)^{1-\frac{1}{n}} \\
&= \frac{\pi^2-4}{\pi n(n-1)}\frac{2^{1-1/n}}{\pi^{1/2n}}\left(\frac{n+1}{2}\right)^{1-\frac{1}{n}}\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}\left(\frac{\Gamma\left(\frac{n}{n-1}+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{n-1}\right)\Gamma\left(\frac{n}{2}\right)}\right)^{1-\frac{1}{n}}
\end{aligned}$$

Simplifying and bounding the expressions in the parentheses,

$$\left(\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{1}{n}}\left(\frac{\Gamma\left(\frac{n}{n-1}+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{n-1}\right)\Gamma\left(\frac{n}{2}\right)}\right)^{1-\frac{1}{n}}$$

$$< \left(\frac{\left(\frac{n+1}{2}\right)^{\frac{n}{2}}}{\left(\frac{n}{2}\right)^{\frac{n-1}{2}}} e^{-\frac{1}{2}} \right)^{\frac{1}{n}} \left(\frac{\left(\frac{n}{n-1}+\frac{n}{2}\right)^{\frac{n}{n-1}+\frac{n-1}{2}}}{\Gamma\left(\frac{n}{n-1}\right)\left(\frac{n}{2}\right)^{\frac{n-1}{2}}} e^{-\frac{n}{n-1}} \right)^{1-\frac{1}{n}}$$

$$= \left(\left(1+\frac{1}{n}\right)^{\frac{n}{2}} \left(\frac{n}{2}\right)^{\frac{1}{2}} e^{-\frac{1}{2}} \right)^{\frac{1}{n}} \left(\left(1+\frac{2}{n-1}\right)^{\frac{n-1}{2}} \left(\frac{n}{n-1}+\frac{n}{2}\right)^{\frac{n}{n-1}} \frac{e^{-\frac{n}{n-1}}}{\Gamma\left(\frac{n}{n-1}\right)} \right)^{\frac{n-1}{n}}$$

$$< \left(\frac{n}{2}\right)^{\frac{1}{2n}} \left(\frac{n}{n-1}+\frac{n}{2}\right) \Gamma\left(\tfrac{n}{n-1}\right)^{-1}$$

Note we used that for $n \geq 3$, then $\Gamma\left(\frac{n}{n-1}\right) < 1$. Hence,

$$\begin{aligned}
\phi(n,1) &< \frac{\pi^2-4}{\pi n(n-1)} \frac{2^{1-1/n}}{\pi^{1/2n}} \left(\frac{n+1}{2}\right)^{1-\frac{1}{n}} \left(\frac{n}{2}\right)^{\frac{1}{2n}} \left(\frac{n}{n-1}+\frac{n}{2}\right) \Gamma\left(\tfrac{n}{n-1}\right)^{-1} \\
&< \frac{\pi^2-4}{\pi} \frac{n+1}{n(n-1)} \frac{n(n+1)}{2(n-1)} \Gamma\left(\tfrac{n}{n-1}\right)^{-1} n^{\frac{1}{2n}} \\
&= \frac{\pi^2-4}{2\pi} \left(\frac{n+1}{n-1}\right)^2 \Gamma\left(\tfrac{n}{n-1}\right)^{-1} n^{\frac{1}{2n}}. \qquad (57)
\end{aligned}$$

Denote the equation in (57) by $\phi_1(n)$, and observe that $\phi_1(n)$ is a decreasing function whose limit is $(\pi^2-4)/2\pi < 1$ as $n \to \infty$. We choose $N > 0$ large enough so that for all $n \geq N$, then $\phi_1(n) < 1$. By direct computation, $\phi_1(100) \approx 0.990826$. Therefore, for all $n \geq 100$, $\phi(n,1) < \phi_1(n) < 1$, and for all $3 \leq n < 100$, we can directly compute that $\phi(n,1) < 1$ as well. These numerical results are not difficult to verify. □

LEMMA 6.3.7. *Let $n \geq 3$, then $\phi(n,2) > 1$.*

Proof. First note that

$$\gamma(n,2) = \frac{1}{1+2/n}\Gamma(2)^2 - \frac{1}{\pi}\Gamma\left(\tfrac{3}{2}\right)^2 = \frac{n}{n+2} - \frac{1}{4} = \frac{3n-2}{4(n+2)}.$$

From Lemma 6.3.2, we compute

$$\begin{aligned}
\phi(n,2) &= \frac{2}{n}\frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n}{2}+1\right)} \frac{1}{\Gamma\left(\frac{n}{2}\right)} \left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{2}{n}} \left(\frac{\Gamma\left(\frac{2n}{n-2}+1+\frac{n}{2}\right)}{\Gamma\left(\frac{2n}{n-2}+1\right)}\right)^{1-\frac{2}{n}} \gamma(n,2) \\
&= \frac{2}{n}\frac{3n-2}{4(n+2)}\frac{4}{n(n-2)} \left(\frac{2^{n-1}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{2n}{n-2}+1\right)}{\Gamma\left(\frac{2n}{n-2}+\frac{n}{2}+1\right)}\right)^{\frac{2}{n}} \frac{\Gamma\left(\frac{2n}{n-2}+\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n}{n-2}+1\right)} \\
&= \frac{2(3n-2)}{n^2(n^2-4)}\frac{n+2}{4} \left(\frac{2^{n+1}}{\sqrt{\pi}(n+2)} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{2n}{n-2}\right)}{\Gamma\left(\frac{2n}{n-2}+\frac{n}{2}\right)}\right)^{\frac{2}{n}} \frac{\Gamma\left(\frac{2n}{n-2}+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n}{n-2}\right)}
\end{aligned}$$

If we require that $n \geq 10$, then

$$\frac{2n}{n-2} + \frac{n}{2} \leq \frac{n+1}{2} + 2,$$

so that

$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{2n}{n-2} + \frac{n}{2}\right)} \geq \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2} + 2\right)} = \frac{4}{(n+1)(n+3)}.$$

As $\Gamma\left(\frac{2n}{n-2}\right) > 1$ for $n \geq 3$, then

$$\left(\frac{2^{n+1}}{\sqrt{\pi}(n+2)} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{2n}{n-2}\right)}{\Gamma\left(\frac{2n}{n-2} + \frac{n}{2}\right)}\right)^{\frac{2}{n}} \geq \left(\frac{2^{n+3}}{\sqrt{\pi}(n+1)(n+2)(n+3)}\right)^{\frac{2}{n}} > \left(\frac{2^{n+3}}{\sqrt{\pi}(n+3)^3}\right)^{\frac{2}{n}}.$$

Furthermore,

$$\begin{aligned}
\frac{\Gamma\left(\frac{2n}{n-2} + \frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{2n}{n-2}\right)} &> \frac{\left(\frac{2n}{n-2} + \frac{n}{2}\right)^{\frac{2n}{n-2} + \frac{n}{2} - 1}}{\left(\frac{n}{2}\right)^{\frac{n}{2} - 1}} e^{-\frac{2n}{n-2}} \, \Gamma\left(\frac{2n}{n-2}\right)^{-1} \\
&= \left(\frac{n}{2}\right)^{\frac{2n}{n-2}} \left(1 + \frac{4}{n-2}\right)^{\frac{2n}{n-2} + \frac{n}{2} - 1} e^{-\frac{2n}{n-2}} \, \Gamma\left(\frac{2n}{n-2}\right)^{-1} \\
&> \frac{n^2}{4} \left(1 + \frac{4}{n-2}\right)^{\frac{n-2}{2}} e^{-\frac{2n}{n-2}} \, \Gamma\left(\frac{2n}{n-2}\right)^{-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\phi(n,2) &> \frac{3n-2}{2n^2(n-2)} \left(\frac{2^{n+3}}{\sqrt{\pi}(n+3)^3}\right)^{\frac{2}{n}} \left(\frac{n^2}{4}\right) \left(1 + \frac{4}{n-2}\right)^{\frac{n-2}{2}} e^{-\frac{2n}{n-2}} \, \Gamma\left(\frac{2n}{n-2}\right)^{-1} \\
&> \frac{3}{2} \left(\frac{2}{n+3}\right)^{\frac{6}{n}} \left(1 + \frac{4}{n-2}\right)^{\frac{n-2}{2}} \Gamma\left(\frac{2n}{n-2}\right)^{-1} e^{-\frac{2n}{n-2}} \, \pi^{-1/n}. \qquad (58)
\end{aligned}$$

Denote the equation in (58) by $\phi_2(n)$, and observe that $\phi_2(n)$ is an increasing function whose limit is $\frac{3}{2} > 1$ as $n \to \infty$. As a result of our requirement above, we must choose $N \geq 10$ and large enough so that for all $n \geq N$, then $\phi_2(n) > 1$. By direct computation, $\phi_2(100) \approx 1.06096$. Therefore, for all $n \geq 100$, $\phi(n,2) > \phi_2(n) > 1$, and for all $3 \leq n < 100$, we can directly compute that $\phi(n,2) > 1$ as well. These numerical results are not difficult to verify. □

CHAPTER VII

CONCLUSION

In this paper, we have presented two new significant results, as well as numerous supporting results that we believe are interesting and important in their own right. In particular, we proved a sharp fractional Hardy inequality over general domains, and an improved sharp fractional Hardy inequality when those domains are convex, as well as showing that the sharp constant for the convex inequality was the same as that for the halfspace. Further, we established the existence of a fractional Hardy-Sobolev-Maz'ya inequality on the halfspace, and the existence of a minimizer in a limited case. These findings have application to, among others, stochastic processes and mathematical physics.

The study of fractional integral inequalities is a fairly new area of study, with most results published in the last 10 years. The field has been very active in that time, as fractional analogues for many classical results have been derived. Nevertheless, this is still much work that can be done. For instance, the exact value of the sharp constant for the Hardy-Sobolev-Maz'ya inequality on the upper halfspace is unknown, and, therefore, the existence or nonexistence of a minimizer in all cases is yet to be established. Also, it isn't even known whether there exists a Hardy-Sobolev-Maz'ya inequality on convex domains. If there is, what is its sharp constant? Is it the same as that for the halfspace, like the fractional Hardy inequality on convex domains in Theorem 4.2.3? These problems are very challenging, and it remains to be seen if and when the answers are to be discovered.

APPENDIX A

TABLES OF CONSTANTS

For the reader's convenience, we list here the sharp constants mentioned in this paper, and their exact value, if known, as well as certain other constants mentioned herein. Some of the constants may be applicable as well to other inequalities not listed.

HARDY INEQUALITY CONSTANTS

Constant	Equation	Value
D_p^Ω	$\int_\Omega \lvert\nabla f\rvert^p \geq D_p^\Omega \int_\Omega \lvert f\rvert^p d_\Omega^{-p}$, Ω Lipschitz.	If Ω is convex then $D_p^\Omega = \left(\frac{p-1}{p}\right)^p$, else unknown.
$D_{n,p,\alpha}^\Omega$	$I_{\alpha,p}^\Omega(f) \geq D_{n,p,\alpha}^\Omega \int_\Omega \lvert f\rvert^p d_\Omega^{-\alpha}$, Ω bounded Lipschitz.	Unknown
$D_{n,p,\alpha}$	$I_{\alpha,p}^\Omega(f) \geq D_{n,p,\alpha} \int_\Omega \lvert f\rvert^p d_\Omega^{-\alpha}$, Ω convex.	$2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \int_0^1 \frac{\lvert 1-r^{(\alpha-1)/p}\rvert^p}{(1-r)^{1+\alpha}}\,\mathrm{d}r$ $2\pi^{\frac{n-1}{2}} \frac{\Gamma(\frac{1+\alpha}{2})}{\Gamma(\frac{n+\alpha}{2})} \frac{\beta\left(\frac{1+\alpha}{2},1-\frac{\alpha}{2}\right)-2^\alpha}{\alpha 2^\alpha}$ $(p=2)$
$C_{n,p,\alpha}$	$I_{\alpha,p}^{\mathbb{R}^n}(f) \geq C_{n,p,\alpha} \int_{\mathbb{R}^n} \lvert f(x)\rvert^p \lvert x\rvert^{-\alpha}\,\mathrm{d}x$	$2\int_0^1 r^{\alpha-1}\lvert 1-r^{(n-\alpha)/p}\rvert^p \Phi_{n,\alpha}\,\mathrm{d}r$, $\Phi_{n,\alpha} = \lvert\mathbb{S}^{n-2}\rvert \int_{-1}^1 \frac{(1-t^2)^{(n-3)/2}}{(1-2tr+r^2)^{(n+\alpha)/2}}\,\mathrm{d}t$.

SOBOLEV INEQUALITY CONSTANTS

Constant	Equation	Value
$S_{n,p}$	$\lVert\nabla f\rVert_p^p \geq S_{n,p}\lVert f\rVert_{\frac{pn}{n-p}}^p$	$\pi n^{2/p}\left(\frac{n-p}{p-1}\right)^{2-\frac{2}{p}}\left(\frac{\Gamma(n/p)\Gamma(1+n-n/p)}{\Gamma(1+n/2)\Gamma(n)}\right)^{2/n}$
$S'_{n,\alpha}$	$(f,(-\Delta)^{\alpha/2}f) \geq S'_{n,\alpha}\lVert f\rVert_{2^*}^2$	$\frac{\Gamma(\frac{n+\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}\lvert\mathbb{S}^n\rvert^{\alpha/n}$
$S_{n,p,\alpha}$	$I_{\alpha,p}^{\mathbb{R}^n}(f) \geq S_{n,p,\alpha}\lVert f\rVert_{p^*}^p$	$S_{n,2,\alpha} = \frac{S'_{n,\alpha}}{a_{n,\alpha}}$, else unknown.
$S_{n,p,\alpha}^\Omega$	$I_{\alpha,p}^\Omega(f) \geq S_{n,p,\alpha}^\Omega\lVert f\rVert_{p^*}^p$, Ω convex or bounded Lipschitz.	Unknown
$w_{n,p,\alpha}$	$J_{\alpha,p}^{\mathbb{H}^n}(f) \geq$ $w_{n,p,\alpha}\left(\int_{\mathbb{H}^n}\lvert f(x)\rvert^q x_n^{-n+nq/p^*}\,\mathrm{d}x\right)^{p/q}$	Unknown

MAZ'YA INEQUALITY CONSTANTS

Constant	Equation	Value
M_n	$\int_{\mathbb{H}^n} \lvert\nabla f\rvert^2 - \frac{1}{4}\int_{\mathbb{H}^n} \lvert f(x)\rvert^2 x_n^{-2}\,\mathrm{d}x \geq M_n \lVert f\rVert^2_{\frac{2n}{n-2}}$	$M_3 = S_{3,2}$, otherwise unknown.
$M_{n,\alpha}$	$I^{\mathbb{H}^n}_{\alpha,p}(f) - D_{n,p,\alpha}\int_{\mathbb{H}^n} \lvert f(x)\rvert^p x_n^{-\alpha}\,\mathrm{d}x \geq M_{n,\alpha}\lVert f\rVert^2_{2^*}$	Unknown
m_p	$I^{\mathbb{H}^n}_{\alpha,p}(f) - D_{n,p,\alpha}\int_{\mathbb{H}^n} \lvert f(x)\rvert^p x_n^{-\alpha}\,\mathrm{d}x \geq m_p J^{\mathbb{H}^n}_{\alpha,p}(f)$	$\min_{0<u<1/2}\left((1-u)^p - u^p + pu^{p-1}\right)$

OTHER CONSTANTS

Constant	Equation	Value
$a_{n,\alpha}$	$(f, (-\Delta)^{\alpha/2} f) = a_{n,\alpha} I^{\mathbb{R}^n}_{\alpha}(f)$	$2^{\alpha-1}\pi^{-n/2}\frac{\Gamma(\frac{n+\alpha}{2})}{\lvert\Gamma(-\frac{\alpha}{2})\rvert}$
$b_{n,\alpha}$	$I^{\mathbb{R}^n}_{\alpha}(f) = J^{\mathbb{H}^n}_{\alpha}(f) + b_{n,\alpha}\int_{\mathbb{H}^n} \lvert f(x)\rvert^2 x_n^{-\alpha}\,\mathrm{d}x$	$\frac{1}{\pi a_{n,\alpha}}\Gamma\left(\frac{1+\alpha}{2}\right)^2$
p^*	$I^{\mathbb{R}^n}_{\alpha,p}(f) \geq S_{n,p,\alpha}\lVert f\rVert^p_{p^*}$	$\frac{pn}{n-\alpha}$
q	$J^{\mathbb{H}^n}_{\alpha,p}(f) \geq w_{n,p,\alpha}\left(\int_{\mathbb{H}^n} \lvert f(x)\rvert^q x_n^{-n+nq/p^*}\,\mathrm{d}x\right)^{p/q}$	$p\left(\frac{n+\frac{\alpha-1}{2}}{n-1}\right)$

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