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INVESTIGATION OF
A CURVED FREE SHEAR LAYER DEVELOPMENT

A THESIS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

NOMENCLATURE

English Notation

a, C, C_i	= constants, appearing in equations; i = 1, 2, 3, 4
C_f	= friction factor, $2\tau_w/\rho u_o^2$
d	= jet nozzle width
$g(\eta)$	= dimensionless velocity function, $g(\eta) = \bar{u}^2/\bar{u}_c^2$
$g_o(\eta)$	= dimensionless velocity function for free jet case
$g_1(\eta)$	= first order perturbation dimensionless velocity function
$G(\xi)$	= transformed dimensionless velocity function, $G(\xi) = \bar{u}^2/\bar{u}_c^2$
$G_o(\xi)$	= transformed dimensionless velocity function for free jet case
$G_1(\xi)$	= transformed first order perturbation dimensionless velocity function
$h(\eta)$	= dimensionless momentum function, $h(\eta) = \overline{uv}/\bar{u}_c^2$
$h_o(\eta)$	= dimensionless momentum function for free jet case
$h_1(\eta)$	= first order perturbation dimensionless momentum function
$H(\xi)$	= transformed dimensionless momentum function, $H(\xi) = \overline{uv}/\bar{u}_c^2$
$H_o(\xi)$	= transformed dimensionless momentum function for free jet case
$H_1(\xi)$	= transformed first order perturbation dimensionless momentum function
k	= constant
k_o, k_1	= constants, $k = k_o + \epsilon k_1$
K	= constant, $K/2 = 0.693$
L_o	= characteristic width of the jet
N_R	= slot Reynolds number, $u_o d/\nu$

NOMENCLATURE (Continued)

English Notation		
p	=	total level of static pressure, $p = \bar{p} + p'$
$\bar{p}$	=	time-averaged part of p
p'	=	fluctuating part of p
$\bar{p}_c$	=	time-averaged pressure at the jet centerline
$\bar{p}_s$	=	time-averaged pressure on the surface of the wall
p_∞	=	static pressure of the quiescent ambient
P	=	stagnation pressure
R	=	radius of curvature of the jet centerline
R_o	=	radius of curvature of the wall
u	=	total velocity component in the x-direction, $u = \bar{u} + u'$
$\bar{u}$	=	time-averaged part of u
u'	=	fluctuating part of u
$\bar{u}_c$	=	time-averaged jet centerline velocity component in the x-direction
$\bar{u}_m$	=	time-averaged maximum velocity component in the x-direction
u_o	=	nozzle exit velocity
v	=	total velocity component in the y-direction, $v = \bar{v} + v'$
$\bar{v}$	=	time-averaged part of v
v'	=	fluctuating part of v
x	=	curvilinear orthogonal coordinate along the jet centerline in the jet direction
y	=	curvilinear orthogonal coordinate normal to the jet centerline
$y_{c/2}$	=	that y location at which $\bar{u} = \frac{1}{2}\bar{u}_c$

NOMENCLATURE (Continued)

English or Greek <u>Notation</u>		
$y'_{c/2}$	=	that y location at which $\bar{u} = \frac{1}{2}\bar{u}_c$ in free jet case
$y_{m/2}$	=	that y location at which $\bar{u} = \frac{1}{2}\bar{u}_m$
δ	=	width of the outer free shear layer
δ_m	=	thickness of the boundary layer
ϵ	=	dimensionless perturbation parameter, $\epsilon = L_o/R$
ζ	=	dummy variable of integration
η	=	dimensionless y-coordinate, $\eta = y/L_o$
$\eta_{c/2}$	=	dimensionless y-coordinate, $\eta_{c/2} = y_{c/2}/L_o$
θ	=	angular distance along the cylindrical surface, measured from the slot
Λ	=	momentum transfer length in Reichardt's hypothesis
ν	=	kinematic viscosity
ν_T	=	eddy kinematic viscosity
ξ	=	dimensionless y-coordinate, $1.1774\xi = \eta$
ρ	=	fluid mass density
τ	=	turbulent shear stress
τ_w	=	shear stress at wall

SUMMARY

This thesis constitutes an investigation of the feasibility of developing an analytical model to study the growth of the outer free shear layer of a two-dimensional incompressible turbulent wall jet blowing over a curved wall. Suitable equations of motion are developed for the shear layer in a curvilinear orthogonal coordinate system. Simplification has been achieved by the application of Reichardt's concept of momentum transfer length which has been modified for the curved jet problem. A successful search has been made to obtain similarity transformations to render the equations to solvable forms. Physically pertinent parameters to the problem have been derived and their relationships with the solutions of the equations have been examined.

CHAPTER I

INTRODUCTION

The Wall Jet Problem

Turbulent free shear layer flow in the presence of a solid boundary has attracted significant attention of researchers and practicing engineers in recent years. Commonly termed a wall jet, such a flow field is generated when a two-dimensional jet blows over an adjacent curved or straight surface. Among its several applications to engineering systems are the problems of boundary layer control of aircraft wings and the operation of fluidic devices.

A widely accepted analytical model of the wall jet flow consists basically of dividing the flow field into an inner or wall layer and an outer or free shear layer. The inner layer adjacent to the wall is assumed to behave as a non-similar turbulent boundary layer, whereas the outer layer is normally treated as a self-similar free shear layer. The two solutions are then matched with suitable continuity conditions to yield continuous velocity profiles. A more detailed approach consists of the application of integral momentum and integral energy analyses to several subdivisions of the two basic regions, with matching boundary conditions for adjacent subdivisions, capable of yielding non-similar profiles for several streamwise stations.

One common feature of these analyses is the inclusion of only turbulent or eddy shear stresses in the equations of fluid motion in the free shear layer, thus neglecting the molecular viscous stresses. Subsequent observations have generally validated such an assumption. More important to the basic problem, however, is the use of various classical turbulent dissipation hypotheses, Prandtl's mixing length concept being the most often quoted mechanism, to formulate expressions for the shear stresses. What is somewhat disturbing in these investigations is an implied use of a significant amount of empirical information to formulate matching boundary conditions to patch adjacent subdivisions of the wall jet flow field.

A search for another hypothesis to formulate expressions for turbulent shear stresses in free shear layers seems most appropriate, both from the point of view of practical applications and a need to understand the basic mechanisms. One such hypothesis was proposed by Reichardt,[1] who applied it successfully to plane and axisymmetric free jet flows in quiescent medium in the absence of solid boundaries in the flow field. Two immediate problems which pose themselves concern the application of Reichardt's hypothesis in the free shear layer of a wall jet, and the modification of Reichardt's formulations to study curved turbulent free shear layers. Any known reference to resolve either of the two problems is not available in open literature at present. It seems quite reasonable to attempt to answer these problems by specifically considering the turbulent free shear layer of a wall jet blowing over a curved boundary, as shown in Figure 1.

The object of the present investigation is to examine the feasibility of using a modified Reichardt's hypothesis to develop an analytical model for the study of growth rate of outer free turbulent shear layer of a wall jet blowing over a curved surface. Analytical solutions thus

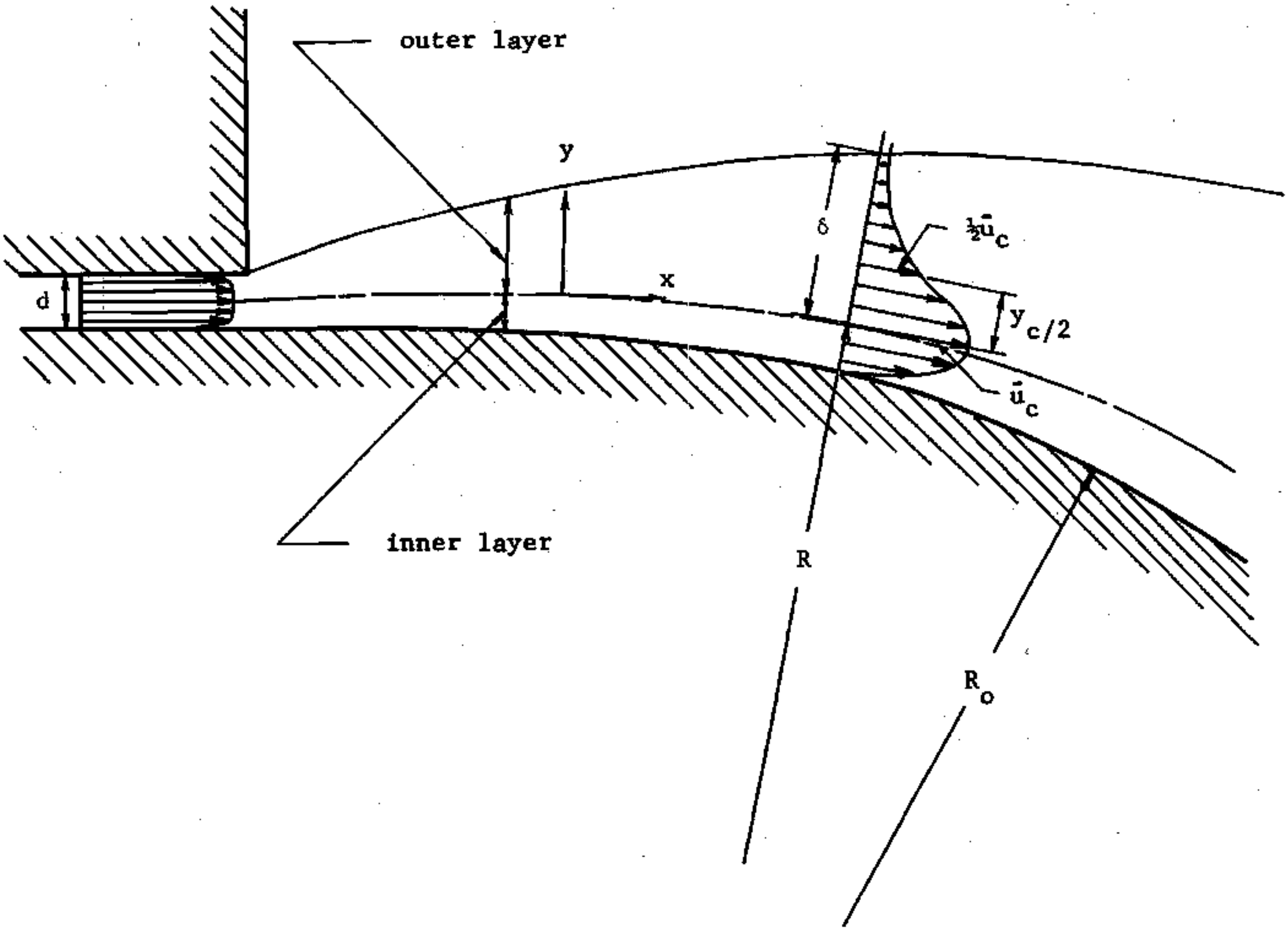

Figure 1. Wall Jet Blowing Over a Convex Surface

obtained should allow for a ready check against presently available experimental data in the literature on this subject.

A Brief Survey of Pertinent Literature

Analytical results for a wall jet blowing over a flat plate were first given by Glauert.(2) He divided the flow field at any streamwise station into two overlapping regions, as shown in Figure 2. Although his primary attention was focused on the laminar wall jet flow, he also obtained empirical formulas for similar turbulent profiles in the outer layer and for the streamwise growth rate of the inner layer.

In the region near the wall the shear stress was assumed to obey the Blasius formula and the outer region was characterized by a constant eddy viscosity. Such assumptions led to useful results in unrestricted shear flow.(3) Solutions were thus obtained for each region separately and then were matched at the transverse location of the maximum velocity, or zero shear stress.

Glauert(2) obtained formulas for the axial velocity decay and for the rate of spread of the wall jet, which were similar in form to the corresponding free jet formulas. He showed that

$$\bar{u}_m/u_o \propto x^{-0.583} \quad , \tag{1-1}$$

and

$$\delta \propto x \quad , \tag{1-2}$$

where $\bar{u}_m$ is the maximum velocity of the profile, u_o is the initial velocity where maximum velocity begins to decay, δ is the outer layer

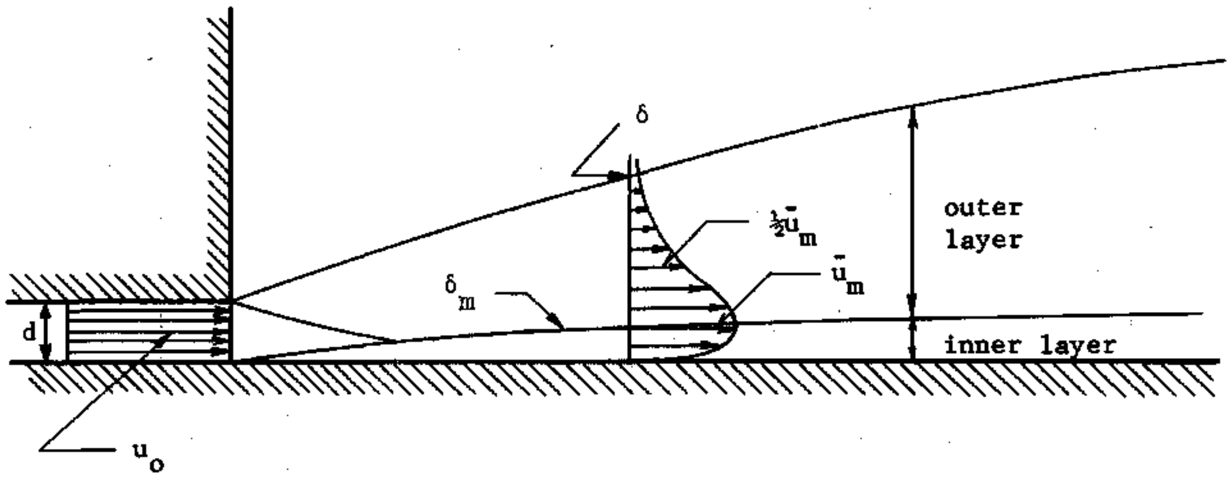

Figure 2. Wall Jet Blowing Over a Flat Plate

thickness, and x is the coordinate along the wall in the jet direction.

Sigalla[4] argued that most of the experimental data of the variation of maximum velocity along the wall could be well represented by the formula

$$\bar{u}_m/u_o = 3.45(d/x)^{1/2} \quad , \qquad (1\text{-}3)$$

where d is the width of the jet nozzle. He also obtained an empirical formula for the growth of the inner layer, given as

$$\delta_m = 0.182\ x/(\bar{u}_m x/\nu)^{1/5} \quad , \qquad (1\text{-}4)$$

where δ_m is the thickness of the boundary layer up to the maximum velocity point. The above formula is valid for $(x/d) > 25$ and is based on measurements taken in a range of nozzle Reynold's numbers, based on nozzle efflux velocity and nozzle width, from 2×10^4 to 5.2×10^4.

Myers, Schauer and Eustis[5] predicted the maximum velocity decay, jet thickness, and the shear stress analytically by using integral-momentum methods. Experimental data concerning velocity profiles, velocity decay and jet thickness agree well with previous investigations. Asymptotic solutions were found for large values of x, where it was shown that

$$u_m \propto x^{-0.5} \quad , \qquad (1\text{-}5)$$

$$\delta_m \propto x^{9/10} \quad , \text{ and} \qquad (1\text{-}6)$$

$$\delta - \delta_m \propto x \quad . \qquad (1\text{-}7)$$

They also obtained information regarding wall jet friction factor, given by

$$C_f N_R^{1/12}(x/d) = 0.1976 \tag{1-8}$$

where C_f, the friction factor, is given by $2\tau w/\rho u_o^2$, τw being the shear stress at the wall.

Schwarz and Cosart[6] showed by analysis that δ must vary as x, and u_m must vary as x^a over a range of self-preserving flows. The exponent "a" was empirically determined to be -0.555. Their Reynolds number was based on the maximum velocity and the thickness of the boundary layer, and varied from 22,000 to 106,000.

Experiments and analyses for the problem of a wall jet blowing over a curved surface, as shown in Figure 1, are also available in the literature. By considering the flow in the hodograph plane, potential theories have been obtained by Lighthill[7] and Woods[8] for two-dimensional incompressible jets blowing over a cylindrical surface with the surrounding fluid at rest. As in the work of Newman,[9] the surface pressure distribution and the maximum velocity decay for a turbulent jet around a circular cylinder were predicted by using an assumed empirical growth law, i.e.,

$$\frac{y_{m/2}}{R_o\theta} = 0.11(1 + 1.5\,\frac{y_{m/2}}{R_o}) \;, \tag{1-9}$$

where $y_{m/2}$ is the width of jet where $u = \frac{1}{2}u_m$, θ is the angular distance measured around the circular cylinder from the slot, and R_o is the radius of circular cylinder. The pressure distribution on the surface of the cylinder was then obtained as,

$$\frac{(p_\infty - \bar{p}_s)R_o}{(P - p_\infty)d} = \frac{8 \ln(1 + \frac{4}{3}\frac{y_{m/2}}{R_o})}{\left[(\frac{4}{3}\ \frac{y_{m/2}}{R_o} + 1)^2 - 1 + 2 \ln(1 + \frac{4}{3}\frac{y_{m/2}}{R_o})\right]} \tag{1-10}$$

where p_∞ is the static pressure of the surrounding fluid at rest, $\bar{p}_s$ is the static pressure on the surface of the cylinder, and P is the stagnation pressure of the fluid supplying the jet. The angle of separation was also obtained by Newman as

$$\theta_{sep} = 245 - 391 \frac{d}{R_o} \Big/ (1 + \frac{9}{8}\frac{d}{R_o}) \tag{1-11}$$

Newman's analysis assumed zero wall friction. This assumption was based on a survey of experimental work on plane wall jets which revealed that the behaviour of the jets depended largely on the outer layer.

Spalding[10] has set up a unified theory which is particularly successful in predicting the local drag force and his results are in good agreement with the experimental results of Sigalla,[4] and Bradshaw and Gee.[11]

Guitton[12] solved the boundary layer equations for a wall jet along curved surfaces for which (i) the radius is proportional to the jet thickness, and (ii) the radius is constant. Solutions were obtained in the form of a power series in terms of the ratio between shear layer thickness, δ, and the surface radius of curvature R_o, truncated after the first term. The experimental portion of Guitton's work was concerned with the flow of the jet on a concave surface of constant radius.

CHAPTER II

ANALYTICAL PRELIMINARIES

Some Remarks on Turbulent Jet Problems

The basic geometry of the flow for the development of an analytical model to predict the nature of self-similar velocity distributions in the outer free shear layer of an incompressible two-dimensional turbulent wall jet blowing over a convex solid surface is shown in Figure 1. A curvilinear orthogonal system of coordinates is introduced whose x-axis is defined along the maximum velocity line, the y-axis being perpendicular to it. The outer free shear layer of this flow is highly turbulent giving rise to eddy turbulent friction. Although the inner shear layer is governed by the influence of the wall with resultant non-similar boundary layer, the flow in the outer shear layer develops for the most part in a self-similar fashion.

Townsend(3) pointed out that in a fully isotropic turbulent flow, there exists a region, including almost all of the flow, over which the direct action of viscosity on the mean flow is negligible, i.e., the Reynolds stresses are large compared with the mean viscous stresses. Within this region, the mean motion of turbulence is determined by the boundary conditions of the flow alone, and is independent of the fluid viscosity. This is the so-called principle of Reynolds number similarity. In free turbulence, there is no region of the flow that is excluded from Reynolds number similarity. Therefore, in dealing with the present problem, if one starts from the Navier-Stokes equations, it seems reasonable to neglect the molecular viscous terms because the Reynolds stresses of turbulent flow predominate.

Methods for the calculation of turbulent flows, generally, are based on empirical hypotheses regarding turbulence shear stress or the turbulent diffusion coefficient, from which, with the aid of the equations of motion, and of continuity, and with assumed similarity conditions, velocity distributions are deduced. In solving the problems such as free turbulent flows, numerous examples have shown that the real phenomena can be described in an excellent way by the use of different semi-empirical hypotheses for turbulent stresses such as Prandtl's mixing length theory. However, such a deductive theory of turbulence suffers from the deficiency that it is impossible to determine which of the alternative a priori assumption comes closest to physical reality.

A significantly different approach to the study of free turbulence was taken by Reichardt,[1] who introduced the concept of a momentum transfer length unlike other authors who utilized Prandtl's mixing length hypothesis. Reichardt[1] and Hinze[13] discussed Reichardt's inductive theory, which is purely phenomenological, in more detail for a simple two-dimensional case.

After a critical examination of the voluminous experimental data on free turbulent flows, H. Reichardt discovered that the velocity profiles under consideration could be approximated very successfully by Gauss'-function, or by its integral, the error function. Starting with this premise, Reichardt attempted to cover all cases of free turbulent flow with the aid of a simple set of formulae instead of endeavouring to solve the differential equations of fluid dynamics. Before attempting to apply a modified version of Reichardt's inductive theory of turbulence to the present problem, it is proper to briefly summarize his original work.

Reichardt's Inductive Theory of Turbulent Diffusion

Reichardt considered the flow field of a two-dimensional incompressible jet in steady flow without adjacent boundary. For the free jet flow he neglected the streamwise pressure gradient term and terms containing the molecular viscosity. The resulting equation of motion for the instantaneous velocity u is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0 \qquad (2\text{-}1)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

or

$$u\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial y} = 0 \qquad (2\text{-}2)$$

If Equation (2-2) is added to Equation (2-1), one obtains,

$$\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y}(uv) = 0 \qquad (2\text{-}3)$$

In describing a turbulent flow in mathematical form, it is convenient to separate it into a mean motion and into a fluctuating motion, where the time-averages of all quantities describing the fluctuations are equal to zero. Thus

$$u = \bar{u} + u', \qquad v = \bar{v} + v', \qquad (2\text{-}4)$$

and

$$\overline{u'} = 0, \qquad \overline{v'} = 0, \tag{2-5}$$

where for any quantity A, $\bar{A} = \frac{1}{T}\int_{to}^{to+T} A dt$, with T being large compared with the time scale of the turbulent motions. Substitution of these expressions into Equation (2-3) and averaging with respect to time yields the required equation for turbulent flow,

$$\frac{\partial \overline{u^2}}{\partial x} + \frac{\partial}{\partial y} \overline{uv} = 0 \quad . \tag{2-6}$$

This is the equation for the conservation of the momentum component in the x-direction.

Reichardt argued that if $\overline{u^2}$ were to follow the Gaussian error function, it must satisfy the differential equation

$$\frac{\partial \overline{u^2}}{\partial x} = \Lambda \frac{\partial^2 \overline{u^2}}{\partial y^2} \quad . \tag{2-7}$$

In this equation, Λ has the dimension of length and may still be a function of x and y. In applying his theory to free turbulent flows, Reichardt assumed that Λ was a function of x alone, determined by the width of the mixing zone. In order to transform Equation (2-6) into Equation (2-7), he simply postulated that

$$\overline{uv} = - \Lambda(x)\frac{\partial \overline{u^2}}{\partial y} \quad . \tag{2-8}$$

This relation, which Reichardt called the momentum-transfer law, may be interpreted as, "the rate of transfer of u-momentum with the velocity v in the lateral y-direction is proportional to the gradient of the momentum flux $\overline{u^2}$ in that lateral direction."

As mentioned before, Reichardt's theory is purely phenomenological and is in good agreement with experimental results. On the other hand, Equation (2-7) has the advantage, of being linear in $\overline{u^2}$, so that the law of superposition of elementary solutions holds. The total momentum flux $\overline{u^2}$ contains the contributions of both the mean and the turbulent motions, namely,

$$\overline{u^2} = \bar{u}^2 + \overline{(u')^2} \ . \tag{2-9}$$

In view of the linearity of the differential Equation (2-7), the substitution of Equation (2-9) into Equation (2-7) immediately results in the set of equations,

$$\frac{\partial \bar{u}^2}{\partial x} = \Lambda(x)\frac{\partial^2 \bar{u}^2}{\partial y^2} \quad , \tag{2-10}$$

and

$$\frac{\partial \overline{(u')^2}}{\partial x} = \Lambda(x)\frac{\partial^2 \overline{(u')^2}}{\partial y^2} \quad . \tag{2-11}$$

Again, if Equation (2-9) is substituted into Equation (2-6), one gets,

$$\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \overline{(u')^2}}{\partial x} + \frac{\partial}{\partial y}\ \overline{uv} = 0$$

or

$$\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial \overline{(u')^2}}{\partial x} + \frac{\partial}{\partial y}(\bar{u}\bar{v}) + \frac{\partial}{\partial y}\ (\overline{u'v'}) = 0 \quad .$$

Following Abromovich[14], the term $\frac{\partial \overline{(u')^2}}{\partial x}$ can be neglected, because the velocities and fluctuations of velocity change much more slowly along the flow than they do in the transverse direction, and, the magnitude of u' and v' are of the same order. Thus,

$$\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial}{\partial y}\ (\bar{u}\bar{v}) + \frac{\partial}{\partial y}(\overline{u'v'}) = 0 \quad ,$$

or

$$\frac{\partial \bar{u}^2}{\partial x} + \frac{\partial}{\partial y}\overline{uv} = 0 \quad . \tag{2-12}$$

Again, in order to transform Equation (2-12) into Equation (2-10), it is proper to use

$$\overline{uv} = -\Lambda(x)\frac{\partial \bar{u}^2}{\partial y} \tag{2-13}$$

The relation in Equation (2-13) will be referred to later.

The merit of this theory is a practical one, namely, that solutions for distributions of transferable quantity in free turbulence can be constructed easily and that these solutions do agree very satisfactorily with experimental data for those cases which have been solved.

Equations of Motion

The full equations of motion and continuity for steady, incompressible, two-dimensional free shear layer providing radius of curvature R(x) of its centerline, where as before the viscous terms are considered negligible in comparison with the turbulent terms, are

$$\frac{R}{R+y}\,u\,\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{uv}{R+y} = -\frac{R}{R+y}\,\frac{1}{\rho}\,\frac{\partial p}{\partial x} \quad , \tag{2-14}$$

$$\frac{R}{R+y}\,u\,\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} - \frac{u^2}{R+y} = -\frac{1}{\rho}\,\frac{\partial p}{\partial y} \quad , \tag{2-15}$$

and

$$\frac{R}{R+y}\,\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{R+y} = 0 \tag{2-16}$$

If δ is the width of the free shear layer, under the boundary layer approximation, a resort to the order of magnitude analysis yields $\frac{\partial u}{\partial y} \sim O(\delta^{-1})$, $u \sim O(1)$, $\frac{\partial u}{\partial x} \sim O(1)$, $\frac{\partial v}{\partial y} \sim O(1)$, $v \sim O(\delta)$ and $\frac{\partial v}{\partial x} \sim O(\delta)$. Following Goldstein,[15] terms of the order δ and higher order may be dropped; also it may be assumed that y/R is small. If one carries out such a process of approximation for the present problem, the equations of motion and continuity can be reduced to

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \tag{2-17}$$

$$\frac{u^2}{R} = \frac{1}{\rho}\frac{\partial p}{\partial y} \quad , \quad \text{and} \tag{2-18}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad . \tag{2-19}$$

If Equation (2-19) is multiplied by u and then added to Equation (2-17), one obtains,

$$\frac{\partial u^2}{\partial x} + \frac{\partial}{\partial y} uv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad . \tag{2-20}$$

Quantities such as u, v and p in above equations are instantaneous values. Hence after substituting $u = \bar{u} + u'$, $v = \bar{v} + v'$ and $p = \bar{p} + p'$ into Equations (2-18) and (2-20), and carrying out the usual time-averaging analysis of the resulting equation term by term, one obtains the proper equations to be solved. Equation (2-18) can thus be written as

$$\frac{\overline{u^2}}{R} = \frac{1}{\rho}\frac{\partial \bar{p}}{\partial y} \quad .$$

Since $\overline{u^2} = \bar{u}^2 + \overline{(u')^2}$ and $\bar{u}^2/R$ is much larger than $\overline{(u')^2}/R$ if the radius of curvature of the jet centerline is assumed large enough, it seems reasonable to neglect the term $\overline{(u')^2}/R$ in comparison with $\bar{u}^2/R$. Such an argument results in the simplified equation for the y-momentum,

$$\frac{\bar{u}^2}{R} = \frac{1}{\rho}\frac{\partial\bar{p}}{\partial y} \quad . \tag{2-21}$$

If a similar analysis is performed on Equation (2-20), it gives,

$$\frac{\partial\overline{u^2}}{\partial x} + \frac{\partial}{\partial y}\overline{uv} = -\frac{1}{\rho}\frac{\partial\bar{p}}{\partial x} \quad ,$$

where if the term $\frac{\partial\overline{(u')^2}}{\partial x}$ is again neglected, the above equation can then be rewritten as

$$\frac{\partial}{\partial x}\left(\frac{\bar{p}}{\rho}\right) + \frac{\partial\bar{u}^2}{\partial x} + \frac{\partial}{\partial y}\overline{uv} = 0 \quad . \tag{2-22}$$

Equations (2-21) and (2-22) are the two equations of interest in the present problem. It is easy to see that the combination of these two equations can eliminate the pressure term in Equation (2-22). Once Reichardt's theory is introduced in Equation (2-22), one can solve for $\bar{u}^2$ from this equation under the assumption of self-similarity of the flow. However, it is necessary to modify Reichardt's original work because his inductive theory was derived in a form applicable to a two-dimensional cartesian coordinate frame.

CHAPTER III

DEVELOPMENT OF THE ANALYTICAL MODEL

A Modification of Reichardt's Inductive Theory

As mentioned in Chapter II, Reichardt assumed that the lateral transport of momentum in a turbulent free jet is proportional to the transverse gradient of the axial component of the momentum, viz.,

$$\overline{uv} = -\Lambda(x)\frac{\partial \overline{u^2}}{\partial y} \quad , \tag{2-8}$$

which is derived under the assumption that $\overline{u^2}$ is to follow the Gaussian error law

$$\frac{\partial \overline{u^2}}{\partial x} = \Lambda(x)\frac{\partial^2 \overline{u^2}}{\partial y^2} \quad . \tag{2-7}$$

Due to the linear nature of Equation (2-7), Equation (2-8) may be expressed as

$$\overline{uv} = -\Lambda(x)\frac{\partial \bar{u}^2}{\partial y} \tag{2-13}$$

where the streamwise gradient of the fluctuation in velocity may be neglected in the free shear layer. If Reichardt's theory is to be applied to the present problem, a modification of Equation (2-13) is necessary to include the effect of wall curvature. This influence has been incorporated in the transformed equations by Goldstein.[15] A kinematically and dynamically compatible form of the modified Reichardt's hypothesis proposed here is expressed as

$$\overline{uv} = -\Lambda(x)\left[\frac{\partial\bar{u}^2}{\partial y} + \frac{\bar{u}^2}{R+y}\right] \quad . \tag{3-1}$$

This equation reduces to

$$\overline{uv} = -\Lambda(x)\left[\frac{\partial\bar{u}^2}{\partial y} + \frac{\bar{u}^2}{R}\right] \tag{3-2}$$

if the effect of y/R is neglected.

Substitution of Equation (3-2) into Equation (2-22) yields,

$$\frac{\partial}{\partial x}\left(\frac{\bar{p}}{\rho}\right) + \frac{\partial\bar{u}^2}{\partial x} - \Lambda\frac{\partial^2\bar{u}^2}{\partial y^2} - \frac{\Lambda}{R}\frac{\partial\bar{u}^2}{\partial y} = 0 \quad . \tag{3-3}$$

It is possible to eliminate $\bar{p}$ from Equation (3-3) by using Equation (2-21); an integration of Equation (2-21) from y = y to y = δ, where δ is the thickness of the outer free shear layer, gives,

$$\left.\frac{\bar{p}(x,\zeta)}{\rho}\right|_{\zeta=y}^{\zeta=\delta} = \int_{\zeta=y}^{\zeta=\delta}\frac{\bar{u}^2(x,\zeta)}{R}\,d\zeta$$

The difference of pressures at y = δ and y = ∞ is negligible, so that one may write,

$$\frac{p_\infty - \bar{p}}{\rho} = \frac{1}{R}\int_{\zeta=y}^{\zeta=\infty}\bar{u}^2(x,\zeta)\,d\zeta \tag{3-4}$$

where p_∞ is the atmospheric pressure. The derivative of Equation (3-4) with respect to x is

$$\frac{\partial}{\partial x}\left(\frac{\bar{p}}{\rho}\right) = - \frac{\partial}{\partial x} \frac{1}{R} \int_{\zeta=y}^{\zeta=\infty} \bar{u}^2(x,\zeta)d\zeta \quad . \tag{3-5}$$

The term $\frac{\partial}{\partial x}(\frac{\bar{p}}{\rho})$ in Equation (3-3) may be eliminated with the final result

$$\frac{\partial \bar{u}^2}{\partial x} + \Lambda \frac{\partial^2 \bar{u}^2}{\partial y^2} + \frac{\Lambda}{R} \frac{\partial \bar{u}^2}{\partial y} - \frac{\partial}{\partial x} \frac{1}{R} \int_{\zeta=y}^{\zeta=\infty} \bar{u}^2(x,\zeta)d\zeta = 0 \quad . \tag{3-6}$$

The boundary conditions for Equation (3-6) are

$$\bar{u}\Big|_{y=0} = \bar{u}_c \tag{3-7a}$$

where $\bar{u}_c$ is the velocity at the jet centerline, and

$$\frac{\partial \bar{u}}{\partial y}\Big|_{y=0} = 0 \quad , \tag{3-7b}$$

since the turbulent shear stress may be assumed to vanish at the jet centerline. Furthermore, the following order requirements are imposed

$$\lim_{y\to\infty} \bar{u}(x,y) = 0, \quad \text{and} \tag{3-8a}$$

$$\lim_{y\to\infty} \frac{\partial \bar{u}(x,y)}{\partial y} = 0 \quad . \tag{3-8b}$$

Hypothesis of Self-Similarity

The additional assumption of self-similarity of the flow implies,

$$\bar{u}^2 = \bar{u}_c^2 g(\eta) \quad , \quad \text{and} \tag{3-9}$$

$$\overline{uv} = \bar{u}_c^2 \, h(\eta) \quad , \quad \eta = y/L_o \quad , \tag{3-10}$$

where $\bar{u}_c$ is the velocity of the jet along the centerline and L_o is the characteristic thickness of the outer shear layer. Both $\bar{u}_o$ and L_o are functions of x only. Substitution of Equation (3-9) into Equation (3-5) gives,

$$-\frac{d}{dx}\left(\frac{\bar{u}_c^2 L_o}{R}\right)\int_{\zeta=\eta}^{\zeta=\infty} g(\zeta)d\zeta - \frac{\bar{u}_c^2}{R}\frac{dL_o}{dx}\eta g + \frac{d\bar{u}_c^2}{dx} g - \bar{u}_c^2 \frac{1}{L_o}\frac{dL_o}{dx}\eta g'$$
$$- \Lambda \frac{\bar{u}_c^2}{L_o^2} g'' - \frac{\Lambda}{R}\frac{\bar{u}_c^2}{L_o} g' = 0 \quad .$$

The multiplication of the above equation by $(-R^2/\Lambda\,\bar{u}_c^2)$ and a rearrangement results in,

$$\left[\left(\frac{R^2}{\Lambda\bar{u}_c^2}\right)\frac{d}{dx}\left(\frac{\bar{u}_c^2 L_o}{R}\right)\right]\int_{\zeta=\eta}^{\zeta=\infty} g(\zeta)d\zeta + \left[\left(\frac{R}{L_o}\right)\left(\frac{L_o}{\Lambda}\frac{dL_o}{dx}\right)\right]\eta g - \left[\left(\frac{R^2}{\Lambda}\right)\left(\frac{1}{\bar{u}_c^2}\frac{du_c^2}{dx}\right)\right] g$$

$$+ \left[\left(\frac{R}{L_o}\right)^2\left(\frac{L_o}{\Lambda}\frac{dL_o}{dx}\right)\right]\eta g' + \left[\frac{R}{L_o}\right]^2 g'' + \left[\frac{R}{L_o}\right] g' = 0 \quad . \tag{3-11}$$

The modified Reichardt's inductive theory, in conjunction with the assumption of self-similarity of the flow, results in the additional requirement,

$$h + \left[\frac{\Lambda}{L_o}\right]\left\{g' + \left[\frac{L_o}{R}\right] g\right\} = 0 \quad . \tag{3-12}$$

Since R, Λ, L_o and $\bar{u}_c$ are functions of x only, it is observed that self-similarity is possible only if each of the square-bracket terms in Equation (3-11) and (3-12) is independent of x. When this criterion is applied to the coefficients on the left hand sides of both Equations (3-11) and (3-12), one obtains,

$$\left(\frac{R^2}{\Lambda \bar{u}_c^2}\right) \frac{d}{dx}\left(\frac{\bar{u}_c^2 L_o}{R}\right) = C_1 \quad , \tag{3-13}$$

$$\left(\frac{R^2}{\Lambda}\right)\left(\frac{1}{\bar{u}_c^2} \frac{d\bar{u}_c^2}{dx}\right) = C_2 \quad , \tag{3-14}$$

$$\frac{L_o}{\Lambda} \frac{dL_o}{dx} = C_3 \quad , \tag{3-15}$$

$$\frac{\Lambda}{L_o} = C_4 \quad , \text{ and} \tag{3-16}$$

$$\frac{L_o}{R} = \epsilon \quad , \tag{3-17}$$

where C_1, C_2, C_3, C_4 and ϵ are constants. Equations (3-15) and (3-16) together imply

$$\frac{dL_o}{dx} = k \tag{3-18}$$

where k is a constant. If Equation (3-18) is integrated with respect to x, one gets

$$L_o = kx \tag{3-19}$$

where the constant of integration vanishes when an infinitesimally small slit is assumed. Substitution of Equation (3-19) into Equation (3-17) gives

$$R = \frac{k}{\epsilon} x \quad . \tag{3-20}$$

From Equation (3-15),

$$\Lambda = \frac{L_o}{C_3} \frac{dL_o}{dx} \quad ,$$

where the constant, C_3, characterizes the particular form of momentum transfer length, Λ , basic to the flow geometry. For the present problem, following Abromovich,[14] the constant is incorporated into the characteristic jet thickness by assuming that

$$\Lambda = \frac{L_o}{2} \frac{dL_o}{dx} \quad . \tag{3-21}$$

L_o may be eliminated from Equations (3-21) and (3-19), i.e.,

$$\Lambda = \frac{k^2}{2} x \quad . \tag{3-22}$$

Substitution of Equations (3-20), (3-21) and (3-22) into Equations (3-13) and (3-14), yields,

$$\left(\frac{R^2}{\Lambda \bar{u}_c^2}\right) \frac{d}{dx} \left(\frac{\bar{u}_c^2 L_o}{R}\right) = \frac{2}{\epsilon} \frac{x}{\bar{u}_c^2} \frac{d\bar{u}_c^2}{dx} = C_1 \text{ , and} \tag{3-23}$$

$$\left(\frac{R^2}{\Lambda}\right)\left(\frac{1}{\bar{u}_c^2} \frac{d\bar{u}_c^2}{dx}\right) = \frac{2}{\epsilon^2} \frac{x}{\bar{u}_c^2} \frac{d\bar{u}_c^2}{dx} = C_2 \text{ ,} \tag{3-24}$$

respectively. From both Equations (3-23) and (3-24), it is seen that

$$\bar{u}_c^2 \propto x^a \quad , \tag{3-25}$$

where a is a constant. In view of Equation (3-20), if it is expressed in polar coordinates, the trajectory of the jet centerline observes a logarithmic spiral curve. Giles et al.[16] have demonstrated from analysis and experiments that for turbulent wall jets whose centerlines are logarithmic spirals, the velocity at the centerline may be expressed as

$$\bar{u}_c^2 \propto x^{-1} \quad , \tag{3-26}$$

a result which shall be adopted here.

If one substitutes Equations (3-19), (3-20), (3-22), and (3-26) into Equation (3-16), one obtains a complete dimensionless equation in terms of similar variables, viz.,

$$g'' + 2\eta g' + 2g + \epsilon\left\{2\eta g + g' - 2\int_{\zeta=\eta}^{\zeta=\infty} g(\zeta)d\zeta\right\} = 0 \quad . \tag{3-27}$$

In view of the boundary conditions for $\bar{u}$ in Equations (3-7a) and (3-7b), and the definition for $g(\eta)$ in Equation (3-9), the boundary conditions for $g(\eta)$ are obtained as

$$g(0) = 1 \quad , \qquad \text{and} \tag{3-28a}$$

$$g'(0) = 0 \quad , \tag{3-28b}$$

with order requirements,

$$\lim_{\eta\to\infty} g(\eta) = 0 \quad , \quad \text{and} \tag{3-29a}$$

$$\lim_{\eta\to\infty} g'(\eta) = 0 \quad . \tag{3-29b}$$

Equation (3-27), together with the boundary conditions (3-28a) and (3-28b), represent the complete analytical formulation of the problem at hand. For $\epsilon = 0$, Equation (3-27) reduces to

$$g'' + 2\eta g' + 2g = 0 \quad , \tag{3-30}$$

which is the governing equation for the free jet case.

In order to obtain $g(\eta)$, a perturbation method of solution is used with the plane free jet solution as the zeroth order approximation.

CHAPTER IV

ANALYTICAL SOLUTIONS

Assumptions and Boundary Conditions

Equation (3-27) derived in Chapter III is to be solved by using perturbation techniques. Equation (3-27) may be rewritten as

$$g'' + 2\eta g' + 2g + \epsilon\left\{2\eta g + g' - \int_{\zeta=\eta}^{\zeta=\infty} g(\zeta)d\zeta\right\} = 0 \quad , \tag{3-27}$$

subject to

$$g(0) = 1 \quad , \text{ and} \tag{3-28a}$$

$$g'(0) = 0 \quad , \tag{3-28b}$$

with order requirements,

$$\lim_{\eta\to\infty} g(\eta) = 0 \quad , \text{ and} \tag{3-29a}$$

$$\lim_{\eta\to\infty} g'(\eta) = 0 \quad . \tag{3-29b}$$

With the assumption that ϵ is a small quantity the solution of the present problem may be expressed in terms of a perturbation series of the form

$$g = g_o + \epsilon g_1 + \epsilon^2 g_2 + \ldots \quad . \tag{4-1}$$

In the present solution only terms up to order ϵ shall be retained, i.e.,

$$g = g_o + \epsilon g_1 \quad . \tag{4-2}$$

A substitution of Equation (4-2) into Equation (3-27) yields,

$$g_o'' + 2\eta g_o' + 2g_o + \epsilon\left\{g_1'' + 2\eta g_1' + 2g_1 + 2\eta g_o + g_o' - 2\int_{\zeta=\eta}^{\zeta=\infty} g_o(\zeta)d\zeta\right\} = 0. \tag{4-3}$$

If Equation (4-3) is to be satisfied identically in ϵ, then the coefficient of the powers of $0(\epsilon^{0})$ and $0(\epsilon)$ must separately vanish. One thus obtains,

$$g_o'' + 2\eta g_o' + 2g_o = 0 \quad , \text{ and} \tag{4-4}$$

$$g_1'' + 2\eta g_1' + 2g_1 + 2\eta g_o + g_o' - 2\int_{\zeta=\eta}^{\zeta=\infty} g_o(\zeta)d\zeta = 0 \quad . \tag{4-5}$$

After comparing Equation (4-4) to Equation (3-30), it is recognized that the solution for g_o in Equation (4-4) is the solution to the free jet case if ϵ is made to vanish. The boundary conditions for the free jet solution, g_o, are

$$g_o(0) = 1 \quad , \quad \text{and} \tag{4-6a}$$

$$g'_o(0) = 0 \quad , \tag{4-6b}$$

with order requirements

$$\lim_{\eta\to\infty} g_o(\eta) = 0 \quad , \text{ and} \tag{4-7a}$$

$$\lim_{\eta\to\infty} g_o'(\eta) = 0 \quad . \tag{4-7b}$$

The boundary conditions for g in Equations (3-28a) and (3-28b) and the definition for g in Equation (4-2) together imply that the boundary conditions for g_1 are

$$g_1(0) = 0 \quad , \text{ and} \tag{4-8a}$$

$$g'_1(0) = 0 \quad , \tag{4-8b}$$

with order requirements

$$\lim_{\eta\to\infty} g_1(\eta) = 0 \quad , \text{ and} \tag{4-9a}$$

$$\lim_{\eta\to\infty} g_1'(\eta) = 0 \quad . \tag{4-9b}$$

With the boundary conditions (4-6a) and (4-6b), (4-8a) and (4-8b), the solutions for g_o and g_1 may be obtained respectively. Once the solutions are obtained they are substituted into Equation (4-2), thus giving a complete solution for the present problem valid up to the first order of perturbation.

Solution for g_o—Two-Dimensional Free Jet Case

The differential equation governing g_o is

$$g_o'' + 2\eta g_o' + 2g_o = 0 \quad , \tag{4-4}$$

or

$$g_o'' + 2\,\frac{d}{d\eta}(\eta g_o) = 0 \quad ,$$

which is subject to the boundary conditions

$$g_o(0) = 1 \quad , \text{ and} \tag{4-6a}$$

$$g'_o(0) = 0 \quad , \tag{4-6b}$$

with order requirements

$$\lim_{\eta\to\infty} g_o(\eta) = 0 \quad , \text{ and} \tag{4-7a}$$

$$\lim_{\eta\to\infty} g_o'(\eta) = 0 \quad . \tag{4-7b}$$

If one first integrates Equation (4-4) twice with respect to η, and then applies the boundary conditions (4-6a) and (4-6b), one obtains

$$g_o = e^{-\eta^2} \quad . \tag{4-10}$$

It is interesting to note the asymptotic boundary conditions for a free jet case, namely the vanishing velocity and its gradient at $\eta \to \infty$ it is seen that Equation (4-10) satisfies the order requirements (4-7a) and (4-7b) automatically. As expected in this problem, the solution for g_o is unique. Results of computation of g_o are given in Table 1. A graphical representation of g_o is shown in Figure 3.

Solution for g_1

The substitution of the expression for g_o into Equation (4-5) gives the differential equation for g_1, viz.,

$$g_1'' + 2\eta g_1' + 2g_1 + 2\int_{\zeta=0}^{\zeta=\eta} e^{-\zeta^2}\, d\zeta - \sqrt{\pi} = 0 \quad , \tag{4-11}$$

with the boundary conditions

$$g_1(0) = 0 \quad , \tag{4-8a}$$

$$g'_1(0) = 0 \quad , \tag{4-8b}$$

and the order requirements

$$\lim_{\eta \to \infty} g_1(\eta) = 0 \quad , \tag{4-9a}$$

$$\lim_{\eta \to \infty} g_1'(\eta) = 0 \quad . \tag{4-9b}$$

Table 1. Solutions for g_o and g_1

η	g_o	g_1
.00000	1.000000000	.000000000
.05000	.997503117	.002212812
.10000	.990049824	.008486472
.15000	.977751233	.018605102
.20000	.960789435	.032135593
.25000	.939413056	.048645254
.30000	.913931184	.067670580
.35000	.884705909	.088728063
.40000	.852143787	.111325312
.45000	.816686481	.134971708
.50000	.778800778	.159188874
.55000	.738968484	.183519926
.60000	.697676331	.207538165
.65000	.655406259	.230853956
.70000	.612626396	.253120773
.75000	.569782823	.274039149
.80000	.527292423	.293359868
.85000	.485536896	.310884684
.90000	.444858074	.326466739
.95000	.405554507	.340008624
1.00000	.367879443	.351460226
1.05000	.332039956	.360814698
1.10000	.298197284	.368104674
1.15000	.266468301	.373396829
1.20000	.236927764	.376787238
1.25000	.209611395	.378395278
1.30000	.184519531	.378358975
1.35000	.161621194	.376829095
1.40000	.140858425	.373964965
1.45000	.122150674	.369929142
1.50000	.105399230	.364884354
1.55000	.090491446	.358989097
1.60000	.077304742	.352395758
1.65000	.065710275	.345247317
1.70000	.055576215	.337676592
1.75000	.046770625	.329804122
1.80000	.039163898	.321738366
1.85000	.032630759	.313574467
1.90000	.027051848	.305395316
1.95000	.022314916	.297270965
2.00000	.018315640	.289260246
2.05000	.014958136	.281410523

Table 1. (Continued)

η	g_o	g_1
2.10000	.012155180	.273759734
2.15000	.009828196	.266336273
2.20000	.007907055	.259161197
2.25000	.006329717	.252248157
2.30000	.005041761	.245605370
2.35000	.003995847	.239235770
2.40000	.003151112	.233138578
2.45000	.002472563	.227309475
2.50000	.001930454	.221742045
2.55000	.001499685	.216427429
2.60000	.001159229	.211356035
2.65000	.000891594	.206516758
2.70000	.000682328	.201898530
2.75000	.000519575	.197489597
2.80000	.000393669	.193278659
2.85000	.000296786	.189254194
2.90000	.000222630	.185405388
2.95000	.000166170	.181721603
3.00000	.000123410	.178193051
3.05000	.000091196	.174809979
3.10000	.000067055	.171563862
3.15000	.000049058	.168446077
3.20000	.000035713	.165449131
3.25000	.000025868	.162565656
3.30000	.000018644	.159789098
3.35000	.000013370	.157113178
3.40000	.000009540	.154532244
3.45000	.000006773	.152040901
3.50000	.000004785	.149634322

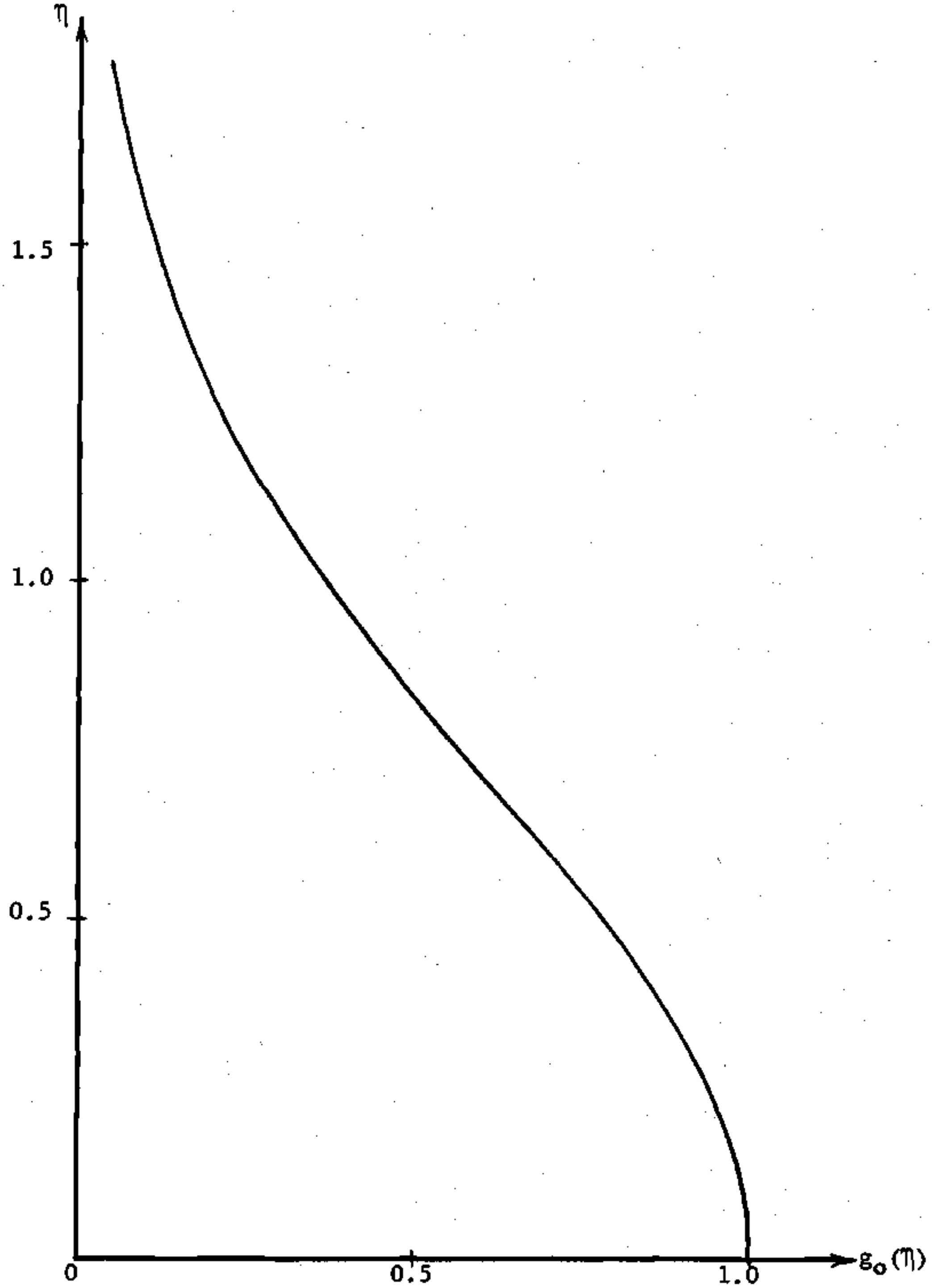

Figure 3. g_o, the Dimensionless Velocity Function for Free Jet Case

If one first integrates Equation (4-11) from $\eta = 0$ to $\eta = \eta$, and then applies the boundary conditions (4-8a) and (4-8b), one obtains,

$$g_1' + 2\eta g_1 + 2\eta\int_{\zeta=0}^{\zeta=\eta} e^{-\zeta^2}\, d\zeta + e^{-\eta^2} - \sqrt{\pi}\ \eta - 1 = 0 \quad . \tag{4-12}$$

After multiplication by the integrating factor, e^{η^2}, an integration of Equation (4-12) from $\eta = 0$ to $\eta = \eta$, with the boundary conditions (4-8a), gives,

$$g_1 = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\ e^{-\eta^2} - \int_{\zeta=0}^{\zeta=\eta} e^{-\zeta^2} d\zeta + e^{-\eta^2}\int_{\zeta=0}^{\zeta=\eta} e^{\zeta^2} d\zeta \quad . \tag{4-13}$$

Also it is interesting to note that g_1 vanishes as η approaches infinity in Equation (4-13); furthermore,

$$\lim_{\eta\to\infty} 2\eta g_1 = 1 \tag{4-14}$$

hence, from Equation (4-12), g_1' also vanishes as $\eta\to\infty$. In other words, both g_1 and g_1' satisfy the order requirements (4-9a) and (4-9b) automatically. Results of computation of g_1 are given in Table 1. A graphical representation of g_1 is shown in Figure 4.

Solution for g

Substitution of the solutions for g_o and g_1, which are expressed in Equations (4-10) and (4-13) respectively, into Equation (4-2), yields,

$$g(\eta) = e^{-\eta^2} + \epsilon\left\{\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\ e^{-\eta^2} - \int_{\zeta=0}^{\zeta=\eta} e^{-\zeta^2} d\zeta + e^{-\eta^2}\int_{\zeta=0}^{\zeta=\eta} e^{\zeta^2} d\zeta\right\} . \tag{4-15}$$

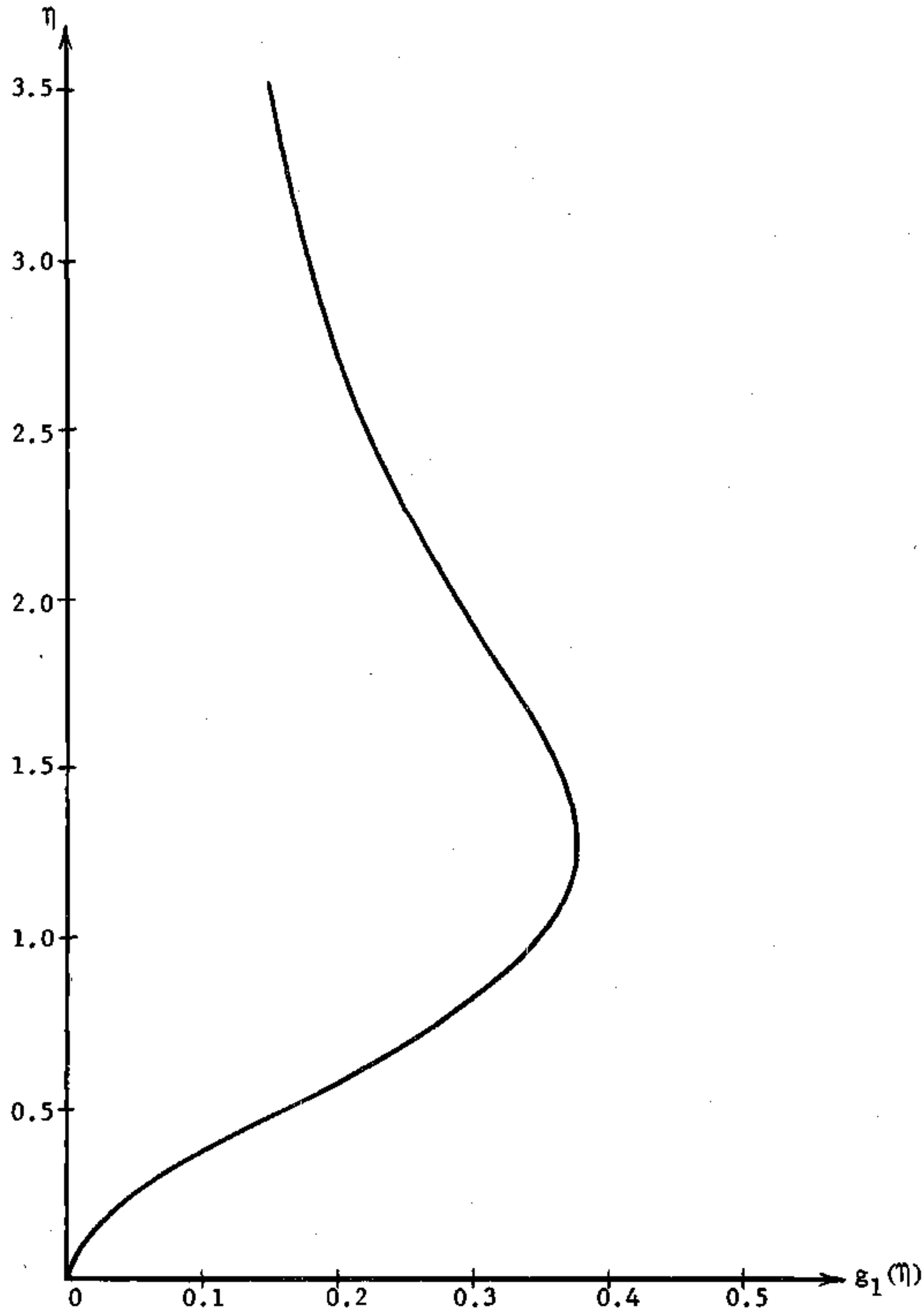

Figure 4. g_1, the First Order Perturbation Dimensionless Velocity Function

The Pressure Distribution Along the Jet Centerline

The pressure distribution along the jet centerline is obtained by setting $y = 0$ in Equation (3-4), viz.,

$$\frac{p_\infty - \bar{p}_c}{\rho} = \bar{u}_c^2 \ \epsilon\int_{\zeta=0}^{\zeta=\infty} g(\zeta)d\zeta \quad . \tag{4-16}$$

Substitution of Equations (4-2) and (3-16) into Equation (4-16), gives,

$$\frac{p_\infty - \bar{p}_c}{\rho} \propto \epsilon x^{-1}\int_{\zeta=0}^{\zeta=\infty}[g_o(\zeta) + \epsilon g_1(\zeta)]d\zeta \quad . \tag{4-17}$$

If terms of $O(\epsilon^2)$ and higher order are neglected, one obtains,

$$\frac{p_\infty - \bar{p}_c}{\rho} \propto \epsilon \frac{\sqrt{\pi}}{2} x^{-1} \quad . \tag{4-18}$$

The Growth of the Free Shear Layer

It is convenient to express the growth of the outer shear layer in terms of the trajectory of the half-maximum velocity point, i.e., at $\eta = \eta_{c/2}$ ($y = y_{c/2}$) where $\bar{u} = \frac{1}{2}\bar{u}_c$. From Equation (4-2),

$$\frac{1}{4} = g_o(\eta_{c/2}) + \epsilon g_1(\eta_{c/2}) \quad .$$

It is noted that $g_o(1.1774) = 1/4$ since $g_o = e^{-\eta^2}$, it follows that

$$g_o(1.1774) - g_o(\eta_{c/2}) = \epsilon g_1(\eta_{c/2}) \quad . \tag{4-19}$$

From the mean-value theorem,

$$g_o(\eta_{c/2}) - g_o(1.1774) = (\eta_{c/2} - 1.1774)g_o'(\zeta) \tag{4-20}$$

where $\eta_{c/2} < \zeta < 1.1774$. By approximation, it is possible to let

$$\zeta \approx 1.1774 \quad ,$$

Equation (4-20) then becomes,

$$g_o(\eta_{c/2}) - g_o(1.1774) = (\eta_{c/2} - 1.1774)g_o'(1.1774). \tag{4-21}$$

If one compares both Equations (4-19) and (4-21), one immediately obtains,

$$-(\eta_{c/2} - 1.1774)g_o'(1.1774) = \epsilon g_1(\eta_{c/2}) \quad ,$$

it is seen that $g_o'(1.1774) = -0.5887$ since $g_o = e^{-\eta^2}$, the above equation is therefore satisfied by

$$\eta_{c/2} - 1.1774 = \frac{\epsilon}{0.5887} g_1(\eta_{c/2}) \quad , \tag{4-22}$$

From Equation (4-19) if ϵ is small enough, it imples that the difference between $g_o(\eta_{c/2})$ and $g_o(1.1774)$ is small, and $\eta_{c/2}$ may be approximated by 1.1774. From Figure 4 or Equation (4-13), $g_1(1.1774) = 0.3754$, and the substitution of $g_1(1.1774)$ into Equation (4-22) along with $\eta_{c/2} = y_{c/2}/L_o$ results in

$$\frac{y_{c/2}}{L_o} = 1.1774 + \epsilon\left(\frac{0.3754}{0.5887}\right) \quad .$$

The substitution of Equation (3-19) into the above equation gives the expression for the growth of the outer shear layer, i.e.,

$$\frac{y_{c/2}}{x} = k\left[1.1774 + \epsilon(0.6376)\right] \quad . \tag{4-23}$$

If k is expanded to generalize the free jet case, it gives

$$k = k_o + \epsilon k_1 \quad , \tag{4-24}$$

where terms of $0(\epsilon^2)$ and higher have been neglected. Equation (4-23) then becomes

$$\frac{y_{c/2}}{x} = (1.1774)k_o + \epsilon\left[(0.6376)k_o + (1.1774)k_1\right] \quad , \tag{4-25}$$

again where term of order of $0(\epsilon^2)$ has been truncated. It appears that a certain amount of empirical information is needed in Equation (4-25) to evaluate both k_o and k_1.

Solution for h

It is recalled that,

$$h(\eta) = \frac{\overline{uv}}{\bar{u}_c^2} \quad ,$$

and from Equation (3-12)

$$h = -\frac{\Lambda}{L_o}\left\{g' + \epsilon g\right\} \quad ,$$

if the expressions for Λ and L_o, i.e., both Equations (3-22) and (3-19) respectively, are substituted into the above equation, one gets,

$$h = -\frac{k}{2}\left\{g' + \epsilon g\right\} \quad . \tag{4-26}$$

Subsitution of the expressions for g and k, i.e., Equations (4-15) and (4-24) respectively, into Equation (4-26), yields,

$$h = k_o \eta\, e^{-\eta^2} + \epsilon\left\{k_1 \eta\, e^{-\eta^2} + k_o\left[\eta\, e^{-\eta^2}\int_{\zeta=0}^{\zeta=\eta} e^{\zeta^2} d\zeta - \frac{\sqrt{\pi}}{2}\,\eta\, e^{-\eta^2} - \frac{1}{2}\right]\right\} \tag{4-27}$$

where terms of order of $O(\epsilon^2)$ and higher have been truncated.

A perturbation for $h(\eta)$, the transfer of x-momentum with the velocity v, may be expressed as

$$h = h_o + \epsilon h_1 \tag{4-28}$$

where h_o is the solution for free jet case and h is valid up to the first order of approximation. Substitution of Equation (4-28) into Equation (4-27), then a collection of the same orders for ϵ, yields,

$$h_o = k_o \eta\, e^{-\eta^2} \tag{4-29}$$

$$h_1 = k_1 \eta\, e^{-\eta^2} + k_o\left[\eta\, e^{-\eta^2}\int_{\zeta=0}^{\zeta=\eta} e^{\zeta^2} d\zeta - \frac{\sqrt{\pi}}{2}\,\eta\, e^{-\eta^2} - \frac{1}{2}\right] , \tag{4-30}$$

where k_o and k_1 have to be determined experimentally. A graphical representation for h_o/k_o, the free jet case, is shown in Figure 5.

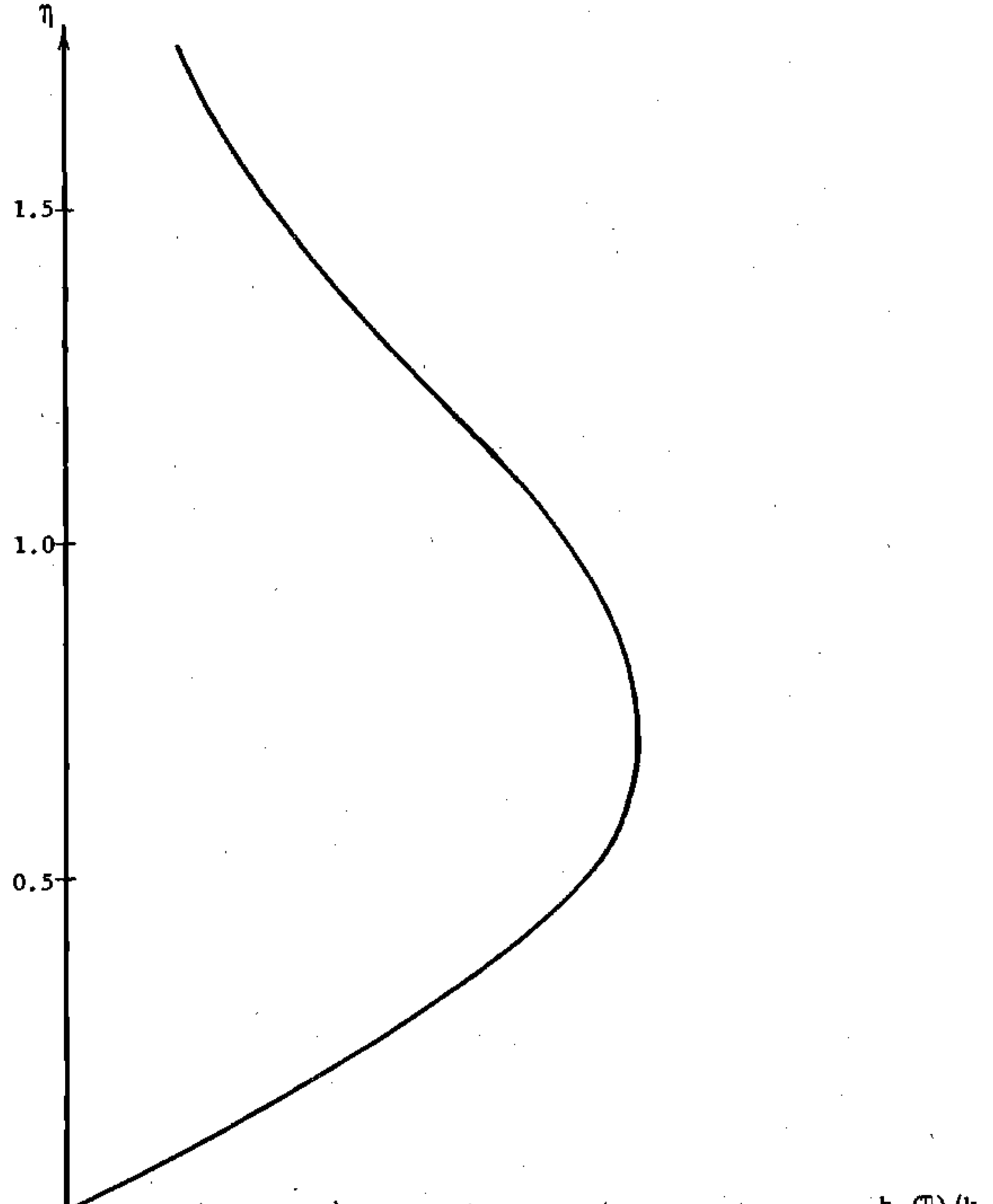

Figure 5. h_o/k_o, the Dimensionless Momentum Function for Free Jet Case

CHAPTER V

DISCUSSION OF RESULTS

Analytical work presented in this investigation concerns only the outer free shear layer of a curved wall jet; in order to present a total picture of the problem, it is necessary to study the development of a turbulent boundary layer over a curved wall. Solutions for the free shear layer and the boundary layer then need to be matched at the maximum velocity point, or the jet centerline. Meanwhile, the effect of separation of the flow has not been included in this problem because it is primarily a property of the turbulent boundary layer which is sandwiched between the free shear layer and the wall.

Experimental data available in the literature on plane wall jets[9] indicates that the behavior of the jet growth depends significantly on the development of the outer free shear layer. Hence if one assumes there is no wall friction, the radius of curvature of the jet centerline, R, obtained in Equation (3-20), may be qualitatively viewed as the locus of the wall, i.e.,

$$R_o = \frac{k}{\epsilon} x \qquad (5\text{-}1)$$

where R_o is the radius of curvature of the wall. This is the equation of a logarithmic spiral. In addition, from Equation (4-18),

$$\frac{p_\infty - \bar{p}_c}{\rho} \propto \epsilon \frac{\sqrt{\pi}}{2} \; x^{-1} \quad ,$$

where $\bar{p}_s$ is the pressure on the surface; in other words, the pressure distribution along the jet centerline can be identified with that along the surface of the wall if the wall friction is neglected. It is also seen from Equation (4-18) that, if ϵ is made to vanish, there is no difference in pressure between the jet and the ambient field, i.e., the pressure field is uniform. This is in agreement with the plane free jet case where the pressure gradient across and along the jet is assumed to vanish.

If one considers the solution for g which has been expressed as,

$$g = g_o + \epsilon g_1 \quad , \qquad (4\text{-}2)$$

where $g = \bar{u}^2/\bar{u}_c^2$,

$$g_o = e^{-\eta^2} \quad , \text{ and} \qquad (4\text{-}10)$$

$$g_1 = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2} e^{-\eta^2} - \int_{\zeta=0}^{\zeta=\eta} e^{-\zeta^2} d\zeta + e^{-\eta^2} \int_{\zeta=0}^{\zeta=\eta} e^{\zeta^2} d\zeta \quad , \qquad (4\text{-}13)$$

it can be seen that g_o exhibits a free jet profile. However, this solution may be rendered more useful by considering the influence of entrainment of fluid from the surrounding still medium. As pointed out by Newman[9], it becomes necessary to include the process of entrainment in any theoretical analysis of the jet flow. Newman[17] has suggested that the mean-velocity profile for a free jet issuing into the still air can be represented with fair accuracy by

$$\frac{\bar{u}}{\bar{u}_c} = e^{-\frac{K}{2}\xi^2} \qquad (5\text{-}2)$$

where $K/2 = 0.693$ and $\xi^2 = y/y'_{c/2}$, and $y'_{c/2}$ is the point where the velocity equals half of maximum jet velocity in free jet case. Equation

(5-2) has been observed to be consistent with experiments.[17] A comparison of Equation (4-10) to Equation (5-2) yields,

$$\sqrt{K}\,\xi = \eta \tag{5-3}$$

or

$$\eta = 1.1774\xi \quad . \tag{5-4}$$

The characteristic width of the jet, L_o, can then be redefined following Equation (5-4) as

$$L_o = \frac{y'_{c/2}}{1.1774} \quad . \tag{5-5}$$

On the other hand, if η is transformed according to Equation (5-4), the solution for the free jet case becomes

$$G_o(\xi) = g_o(1.1774\xi) = e^{-1.386\xi^2} \quad . \tag{5-6}$$

It follows that the solution for g_1 may then be expressed as

$$G_1(\xi) = g_1(1.1774\xi) = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\; e^{-1.386\xi^2} - 1.1774\int_{\zeta=0}^{\zeta=\xi} e^{-1.386\xi^2}\,d\zeta\; + $$
$$1.1774\, e^{-1.386\xi^2}\int_{\zeta=0}^{\zeta=\xi} e^{1.386\zeta^2}\,d\zeta \quad . \tag{5-7}$$

The solution for the velocity profile is therefore equivalent to,

$$G(\xi) = e^{-1.386\xi^2} + \epsilon\Big\{\frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\;{}^{-1.386\xi^2} - 1.1774\int_{\zeta=0}^{\zeta=\xi} e^{-1.386\zeta^2}\,d\zeta$$
$$+\; 1.1774\; e^{-1.386\xi^2}\int_{\zeta=0}^{\zeta=\xi} e^{1.386\zeta^2}\,d\zeta\Big\} \quad . \tag{5-8}$$

Results of computation of G_o and G_1 are given in Table 2. The graphical representations of G_o and G_1 are shown in Figures 6 and 7 respectively.

On the other hand, if both Equations (4-29) and (4-30), i.e., the transfer of x-momentum with the velocity v, are transformed according to Equation (5-4), one obtains,

$$H_o(\xi) = h_o(1.1774\xi) = 1.1774k_o\xi \; e^{-1.386\xi^2} , \tag{5-9}$$

$$H_1(\xi) = h_1(1.1774\xi) = 1.1774k_1\xi \; e^{-1.386\xi^2}$$
$$+ k_o\left[1.386 \; e^{-1.386\xi^2} \int_{\xi=0}^{\zeta=\xi} e^{1.386\zeta^2} d\zeta - \frac{\sqrt{\pi}}{2}(1.1774)\xi \; e^{-1.386\xi^2} - \frac{1}{2}\right]$$

where k_o and k_1 have to be determined experimentally. The final expression for the transfer of u-momentum with the velocity v is equivalent to

$$H(\xi) = k_o(1.1774)\xi \; e^{-1.386\xi^2} + \epsilon\left\{1.1774k_1\xi \; e^{-1.386\xi^2}\right. \tag{5-11}$$
$$\left. + k_o\left[1.386 \; e^{-1.386\xi^2} \int_{\zeta=0}^{\zeta=\xi} e^{1.386\zeta^2} d\zeta - \frac{\sqrt{\pi}}{2}(1.1774)\xi \; e^{-1.386\xi^2} - \frac{1}{2}\right]\right\}.$$

A graphical representation for h_o/k_o, the free jet case, is shown in Figure 8.

Table 2. Solutions for G_o and G_1

ξ	G_o	G_1
.00000	1.000000000	.000000000
.05000	.996540986	.003065488
.10000	.986235604	.011658403
.15000	.969296232	.025403712
.20000	.946068779	.043563809
.25000	.917020909	.065400321
.30000	.882726386	.090130411
.35000	.843846224	.116951630
.40000	.801107608	.145066343
.45000	.755281314	.173704516
.50000	.707158819	.202144783
.55000	.657529935	.229731496
.60000	.607161783	.255889453
.65000	.556780055	.280133188
.70000	.507052876	.302074112
.75000	.458577953	.321421526
.80000	.411873102	.337982379
.85000	.367370293	.351655126
.90000	.325413059	.362423748
.95000	.286257084	.370346457
1.00000	.250073608	.375546463
1.05000	.216955086	.378197983
1.10000	.186922731	.378516093
1.15000	.159935456	.376742747
1.20000	.135899460	.373137601
1.25000	.114678247	.367965721
1.30000	.096102504	.361491051
1.35000	.079979498	.353966638
1.40000	.066101757	.345631469
1.45000	.054254740	.336703364
1.50000	.044223458	.327378921
1.55000	.035797941	.317829281
1.60000	.028777544	.308202285
1.65000	.022974166	.298620261
1.70000	.018214453	.289184000
1.75000	.014341113	.279971637
1.80000	.011213467	.271043688
1.85000	.008707376	.262442335
1.90000	.006714675	.254196793
1.95000	.005142248	.246322764
2.00000	.003910853	.238827359

Table 2 (Continued)

ξ	G_o	G_1
2.05000	.002953794	.231708428
2.10000	.002215539	.224959174
2.15000	.001650322	.218566794
2.20000	.001220811	.212516889
2.25000	.000896847	.206791496
2.30000	.000654303	.201372955
2.35000	.000474056	.196241697
2.40000	.000341092	.191379772
2.45000	.000243726	.186768187
2.50000	.000172951	.182390304
2.55000	.000121881	.178228790
2.60000	.000085298	.174269026
2.65000	.000059283	.170495830
2.70000	.000040918	.166896749
2.75000	.000028047	.163458850
2.80000	.000019092	.160171632
2.85000	.000012906	.157024249
2.90000	.000008665	.154007975
2.95000	.000005777	.151113601
3.00000	.000003825	.148334073
3.05000	.000002515	.145661488
3.10000	.000001642	.143090103
3.15000	.000001065	.140613161
3.20000	.000000686	.138225922
3.25000	.000000439	.135922628
3.30000	.000000279	.133699317
3.35000	.000000176	.131551011
3.40000	.000000110	.129474431
3.45000	.000000068	.127465168
3.50000	.000000042	.125520544

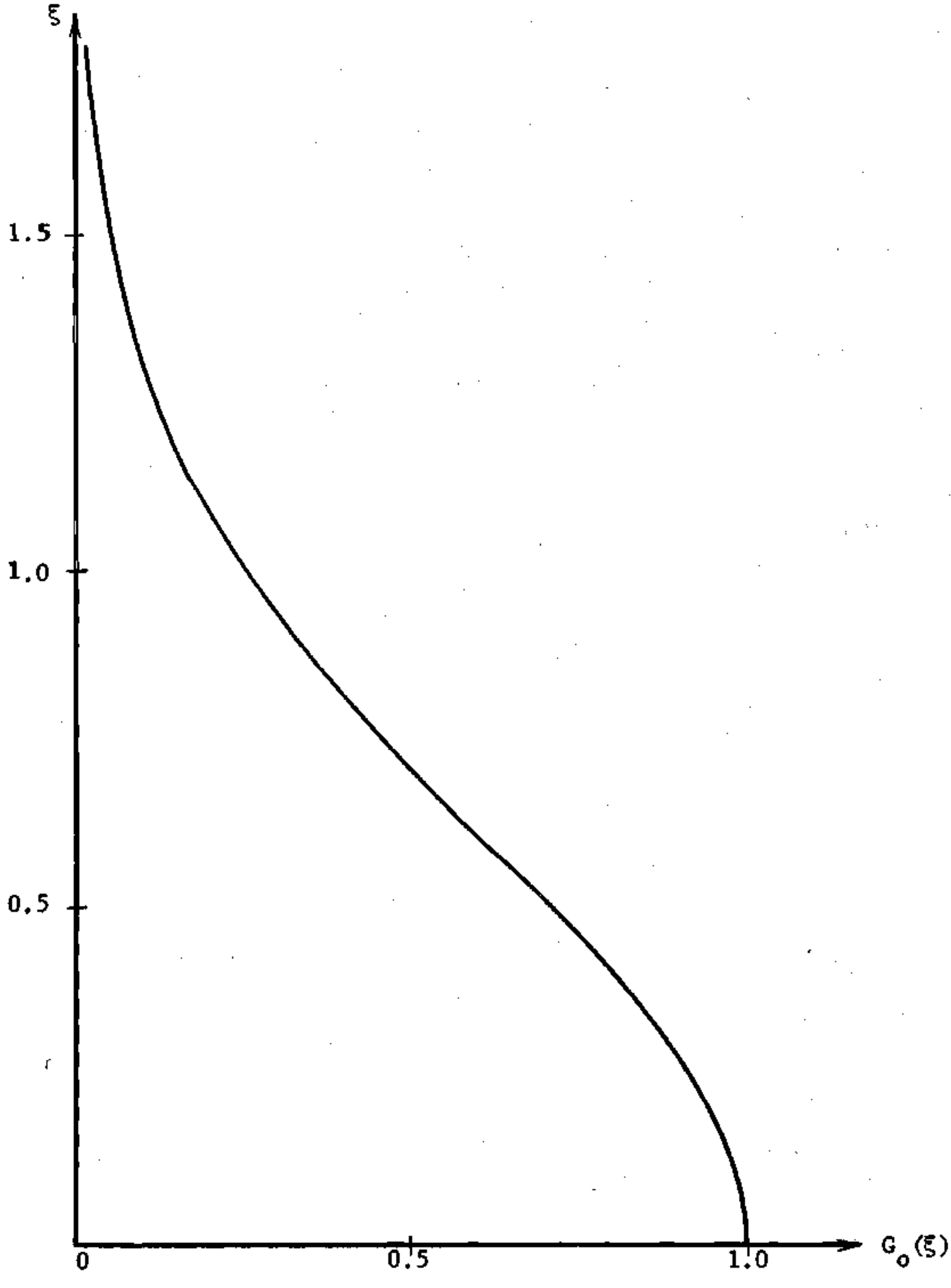

Figure 6. G_o, the Transformed Dimensionless Velocity Function for Free Jet Case

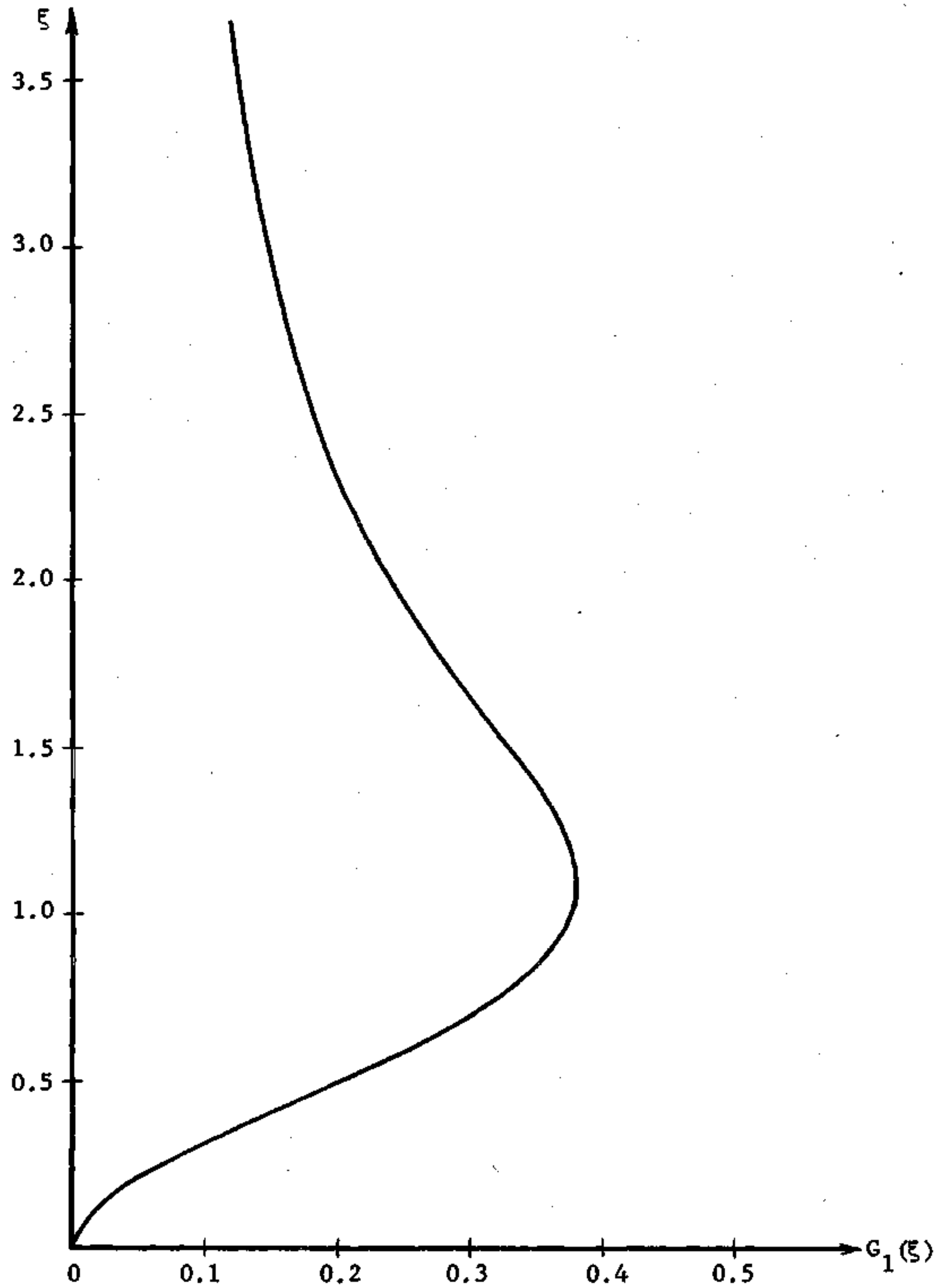

Figure 7. G_1, the Transformed First Order Perturbation Dimensionless Velocity Function

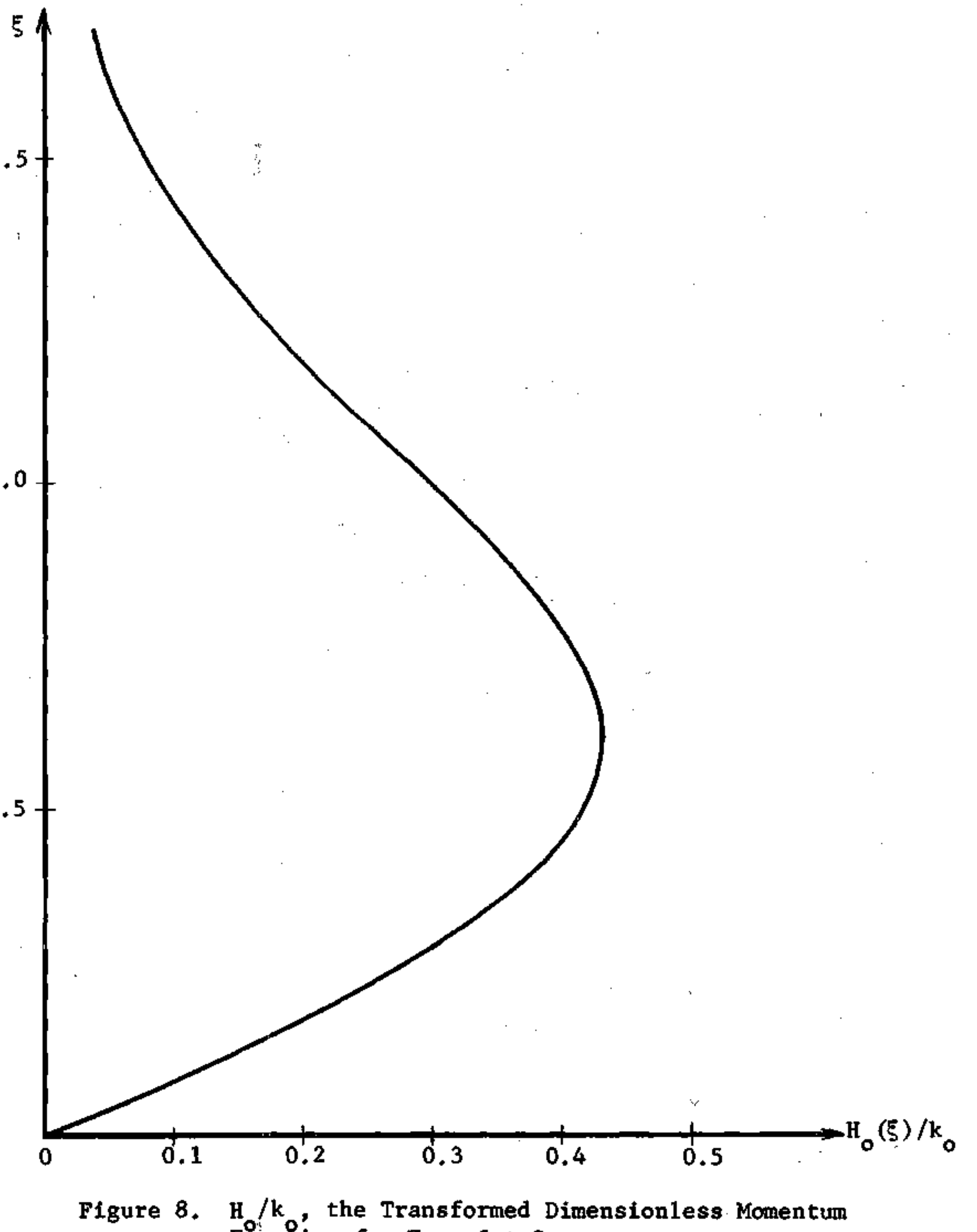

Figure 8. H_o/k_o, the Transformed Dimensionless Momentum Function for Free Jet Case

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

The fluid dynamics of the outer free shear layer of an incompressible two-dimensional turbulent jet blowing over a curved wall has been analyzed by utilizing the Reichardt's concept of turbulent momentum for self-similar velocity profiles. The results of the analysis indicate a successful use of the technique when applied to such flow fields.

The conclusions drawn from this investigation are:

1. The velocity profile for the curved free shear layer can be expressed as,

$$\bar{u}^2/\bar{u}_c^2 = g(\eta) = g_o(\eta) + \epsilon g_1(\eta) \; ,$$

where

$$g_o(\eta) = e^{-\eta^2} \tag{4-10}$$

$$g_1(\eta) = \frac{\sqrt{\pi}}{2} - \frac{\sqrt{\pi}}{2}\, e^{-\eta^2} - \int_{\zeta=0}^{\zeta=\eta} e^{-\zeta^2} d\zeta + e^{-\eta^2}\int_{\zeta=0}^{\zeta=\eta} e^{\eta^2} d\zeta \quad ; \tag{4-13}$$

and

$$\eta = y/L_o$$

$g_o(\xi)$ being the governing solution for plane free jet.

2. The radius of curvature of the jet centerline may be expressed by

$$R = \frac{k}{\epsilon}\; x \tag{3-20}$$

which implies that the jet centerline observes a logarithmic spiral for a self-similar flow.

3. The decay of maximum velocity can be expressed as

$$\bar{u}_c^2 \propto x^{-1} \quad . \tag{3-26}$$

4. The pressure distribution along the jet centerline is given by,

$$\frac{p_\infty - p_c}{\rho} \propto \epsilon \frac{\sqrt{\pi}}{2} x^{-1} \quad . \tag{4-18}$$

5. The growth of the outer shear layer is obtained as,

$$\frac{y_{c/2}}{x} = (1.1774)k_o + \epsilon[(0.6386)k_o + (1.1774)k_1] \quad . \tag{4-25}$$

where both k_o and k_1 must be determined experimentally.

6. The transfer of x-momentum in transverse direction can be expressed as,

$$\overline{uv}/\ \bar{u}_c^2 = h(\eta) = h_o(\eta) + \epsilon h_1(\eta)$$

where

$$h_o(\eta) = k_o \eta\, e^{-\eta^2} \quad , \tag{4-29}$$

$$h_1(\eta) = k_1 \eta\, e^{-\eta^2} + k_o[\eta\, e^{-\eta^2} \int_{\zeta=0}^{\zeta=\eta} e^{\zeta^2} d\zeta - \frac{\sqrt{\pi}}{2}\eta\, e^{-\eta^2} - \frac{1}{2}] \ . \tag{4-30}$$

$h_o(\eta)$ being the governing solution for the plane free jet.

Recommendations

Several suggestions for extending the present investigation can be summarized as follows:

1. Use of a more general formulation for shear stress, such as

$$\frac{\tau}{\rho} = \nu_T\left(\frac{\partial \bar{u}}{\partial y} - C\,\frac{\bar{u}}{R}\right)$$

where ν_T is the eddy kinematic viscosity, is recommended instead of the present formulation where C is assumed to vanish. It may be noted that such a modification will allow a nonvanishing shear stress at the jet centerline.

2. As mentioned before, the width of the slot at the nozzle exit is assumed infinitesimally small and therefore the present solutions are not valid near the exit. An improvement is necessary by which the down stream conditions can be matched to those at the slot, thus incorporating the width of the slot, d, in the solutions as a parameter.
3. A further look into the need for higher order corrections in terms of the parameter ϵ may prove fruitful.

APPENDIX

The computer program used to obtain the graphical representations of the functions g_o and g_1 is written in Fortran IV language and using the Burroughs B-5500 computer at the Rich Electronic Computer Center at the Georgia Institute of Technology. Following are the symbols and the program.

Symbols Used in this Work

C = $\sqrt{\pi}/2 = 0.88623$

Y = $\exp(-x^2)$

Z = $\exp(x^2)$

INTG1 = $\int_o^x \exp(-x^2)dx$

INTG2 = $\int_o^x \exp(x^2)dx$

Program

```
      DIMENSION Y(101),Z(101),INTG1(101),INTG2(101)
      REAL INTG1,INTG2
      READ(5.100)DX,N
      WRITE(6,150)
      C=0.88623
C     C=(PI/4)**1/2
      DO 5 I=1,4
      X=(I-1)*DX
      Y(I)=EXP(-X*X)
5     Z(I)=1./Y(I)
      INTG1(1)=0
      INTG1(2)=(Y(1)+Y(2))*DX/2.
      INTG1(3)=(Y(1)+4.*Y(2)+Y(3))*DX/3.
      INTG1(4)=(Y(1)+3.*Y(2)+3.*Y(3)+Y(4))*DX*3./8.
      INTG2(1)=0
      INTG2(2)=(Z(1)+Z(2))*DX/2.
      INTG2(3)=(Z(1)+4.*Z(2)+Z(3))*DX/3.
      INTG2(4)=(Z(1)+3.*Z(2)+3.*Z(3)+Z(4))*DX*3./8.
      DO 15 I=1,4
      X=(I-1)*DX
      A=-INTG1(I)
      B=Y(I)*INTG2(I)
      G1=C*(1-Y(I))+A+B
15    WRITE(6,200)X,Y(I),G1
```

Program (Continued)

```
      DO 20 I=5,101
      X=(I-1)*DX
      Y(I)=EXP(-X*X)
      Z(I)=1./Y(I)
      INTG1(I)=INTG1(I-2)+(Y(I-2)+4.*Y(I-1)+Y(I))*DX/3.
      INTG2(I)=INTG2(I-2)+(Z(I-2)+4.*Z(I-1)+Z(I))*DX/3.
      A=-INTG1(I)
      B=Y(I)*INTG2(I)
      G1=C*(1-Y(I))+A+B
20    WRITE(6,200)X,Y(I),G1
100   FORMAT(F10.5,I5)
150   FORMAT(15X,3HETA,12X,2HG0,15X,2HG1)
200   FORMAT(10X,F10.5,2F15.9)
      END
```

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