TWO-PHASE FRICTIONAL PRESSURE DROP
IN LAMINAR, STRATIFIED FLOWS

A THESIS

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TWO-PHASE FRICTIONAL PRESSURE DROP

IN LAMINAR, STRATIFIED FLOWS

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NOMENCLATURE

A_1	area occupied by phase 1
A_2	area occupied by phase 2
A_c	cross-sectional area
$\overline{\alpha}$	area-averaged void fraction, $\overline{\alpha} \equiv \frac{A_2}{A_c}$
b	length of the parallel plates
β	"flowing" volumetric concentration, $\beta \equiv \frac{Q_2}{Q_1+Q_2}$
C_1	constant defined by equation (15), Chapter I
C_2	constant defined by equation (16), Chapter I
D	diameter of circular pipe
D_H	hydraulic diameter, $D_H \equiv \frac{4A_c}{P_\omega}$
f_M	two-phase Moody friction factor
G	total mass flux, $G = \rho_1 j_1 + \rho_2 j_2$
h	distance between parallel plates
j	velocity for center of volume, $j = j_1 + j_2$
j_1	superficial velocity for phase 1, $j_1 = \frac{Q_1}{A_c}$
j_2	superficial velocity for phase 2, $j_2 = \frac{Q_2}{A_c}$
k	parameter defined by equation (16), Chapter III
ℓ	characteristic length
$(\frac{\Delta p}{L})$	pressure drop per unit length
ρ_1	density for phase 1
ρ_2	density for phase 2
ρ_m	mixture density

Q_1 volumetric flow rate for phase 1

Q_2 volumetric flow rate for phase 2

Φ_L square root of the two-phase frictional pressure drop to the frictional pressure drop that would result if the liquid phase occupied the entire pipe alone

Φ_G square root of the two-phase frictional pressure drop to the frictional pressure drop that would result if the gas phase occupied the entire pipe alone

Re_H Reynolds number based on hydraulic diameter

S "slip" ratio, $S \equiv \dfrac{\bar{v}_2}{\bar{v}_1}$

v_{fg} difference in specific volumes of saturated liquid and vapor

v_f specific volume of liquid

$\bar{v}_1$ average axial velocity for phase 1

$\bar{v}_2$ average axial velocity for phase 2

v_r relative velocity, $v_r \equiv \bar{v}_2 - \bar{v}_1$

v_m velocity for the center of mass

ν_m mixture kinematic viscosity

μ_1 absolute viscosity for phase 1

μ_2 absolute viscosity for phase 2

μ_m mixture absolute viscosity

X_{vv} Martinelli parameter for viscous-viscous flow, $X_{vv}^2 \equiv \dfrac{Q_1/\mu_1}{Q_2/\mu_2}$

(x,y,z) rectangular Cartesian coordinates

χ mass quality, $\chi \equiv \dfrac{\rho_2 Q_2}{\rho_1 Q_1 + \rho_2 Q_2}$

NOMENCLATURE

Appendix A

A_c	cross-sectional area of rectangular duct
$A_1'(n)$	coefficient defined by equation (26)
$A_2'(n)$	coefficient defined by equation (27)
2a	width of the rectangular duct
b_1	depth of the less viscous phase 1
b_2	depth of the more viscous phase 2
2b	height of the rectangular duct
$B_1'(n)$	coefficient defined by equation (28)
$B_2'(n)$	coefficient defined by equation (29)
ch	hyperbolic cosine function
D_H	hydraulic diameter
f_M	Moody friction factor
g_c	dimensional conversion factor = 32.174 lb_m-ft/lb_f-sec^2
h	location of the interface with respect to the centerline
k_1, k_2	functions defined by equation (9)
m	absolute viscosity ratio, (μ_2/μ_1)
n	variable defined by equation (20)
p	static pressure
Q_1	volumetric flow rate for phase 1
Q_2	volumetric flow rate for phase 2
ρ_1	density for phase 1

ρ_2	density for phase 2
sh	hyperbolic sine function
v_1	local velocity for phase 1 in z-direction
v_2	local velocity for phase 2 in z-direction
μ_1	absolute viscosity for phase 1
μ_2	absolute viscosity for phase 2
(x,y,z)	rectangular Cartesian coordinates
Y_1, Y_2	variables defined by equation (23)

NOMENCLATURE

Appendix B

$A_1(m)$	coefficient defined by equation (36)
$A_2(m)$	coefficient defined by equation (37)
$2a$	length of the interface
$\bar{\alpha}$	area-averaged void fraction
$B_1(m)$	coefficient defined by equation (22)
$B_2(m)$	coefficient defined by equation (23)
ch	hyperbolic cosine function
ctg	cotangent function
dA_c	elementary cross-sectional area
(ε,θ)	bi-polar coordinates in Figure 8
g	gravitational constant = 32.174 ft/sec^2
γ	pipe inclination angle
k_1	parameter defined by equation (3)
k_2	parameter defined by equation (4)
m	dummy integration variable
p	static pressure
Q_1	volumetric flow rate for phase 1
Q_2	volumetric flow rate for phase 2
ρ_1	density for phase 1
ρ_2	density for phase 2
r_1, r_2	radial lengths defined in Figure 8

R	pipe radius
sh	hyperbolic sine function
Θ_1, Θ_2	angular measures defined in Figure 8
v_1	axial velocity for phase 1
v_2	axial velocity for phase 2
μ_1	absolute viscosity for phase 1
μ_2	absolute viscosity for phase 2
(x,y)	rectangular Cartesian coordinates
$\xi_1(m)$	function defined by equation (14)
$\xi_2(m)$	function defined by equation (15)

SUMMARY

The purpose of this thesis is threefold. First, the proper definition for the mixture kinematic viscosity, ν_m, will be developed for stratified, horizontal flow systems. Secondly, using this definition for ν_m, an appropriate expression for the two-phase Reynolds number will be derived which is used as the similarity parameter for modeling the friction factor in two-phase, separated flow. And thirdly, a correlation is presented for predicting the volumetric concentration in separated, two-phase flow.

The results of this analysis will show that the two-phase friction factor reduces to the well-known pressure drop correlation applicable to single-phase flow in terms of the Moody friction factor and the Reynolds number defined in this analysis. In addition, the author's void fraction correlation will show that Hewitt's "triangular" relationship for the volume flow rates, overall pressure drop, and the void fraction, does not hold for laminar, horizontal, stratified flows.

The three separated flow systems analyzed are: (1) flow between wide, horizontal parallel plates; (2) flow through horizontal, rectangular ducts; and (3) flow through horizontal, circular pipes. In all three cases, the drift or diffusion flow model is used to establish the correct expression for the mixture viscosity. Experimental data are used to test the validity of this analysis for predicting the frictional pressure drop and void fraction. Excellent agreement is shown between experimental and predicted results.

CHAPTER I

INTRODUCTION

1.1 Significance of the Problem

The frequent occurrence of two-phase, single- and/or two-component flow in pipelines is characteristic of many modern petroleum, chemical, and nuclear systems. Two current problem areas are the ability to accurately predict the pressure losses and the volumetric concentration in these pipelines. One of the flow regimes frequently observed is stratified flow. This type of two-phase flow has been demonstrated experimentally by several investigators [4,5,28, 33] and it is the subject of this work.

1.2 Review of the Literature

A literature search with respect to horizontal, two-phase flow in conduits revealed that a logical and general method for accurately predicting the frictional pressure drop and volumetric concentration does not exist. Numerous good references [3,10,14,19,29,31] containing discussions of horizontal flow correlations point out that most of these correlations are empirical; therefore, they are subject to the limitations of their own data.

Many investigators seem to rely on the work of Lockhart and Martinelli [21] and Martinelli et al. [23] as the classical approach to this problem. The correlation of the latter was established by performing numerous experiments in horizontal, circular pipes covering a wide range of flow rates at atmospheric pressures. The experimental data were separated into four basic groups depending on whether each phase was flowing in the laminar or turbulent state. The parameters Φ_L and Φ_G, defined in the Nomenclature, were determined and plotted against the Martinelli parameter, X, which is a function of input system quantities and fluid properties only. The Martinelli parameter for laminar-laminar flow, X_{VV}, is also defined in the Nomenclature. Their correlations for predicting the frictional pressure drop and the void fraction are shown in Figure 1. Although the Lockhart-Martinelli correlation gives good agreement in a number of cases, it has been shown to be quite inaccurate in the case of stratified fluid flow [2,5,8,11].

Another and more commonly used approach is to treat the two fluids as if they were a homogeneous mixture with appropriately defined mixture properties (i.e. mixture density and mixture viscosity). Herein lies the major problem. How does one define these mixture properties?

Many investigators [1,2,7,9,12,16,18,22,24,25,26] have tried to establish a frictional pressure drop correlation similar to that obtained in single-phase flow (in terms of the Moody friction factor and the fluid Reynolds number) by choosing arbitrary and artificial definitions

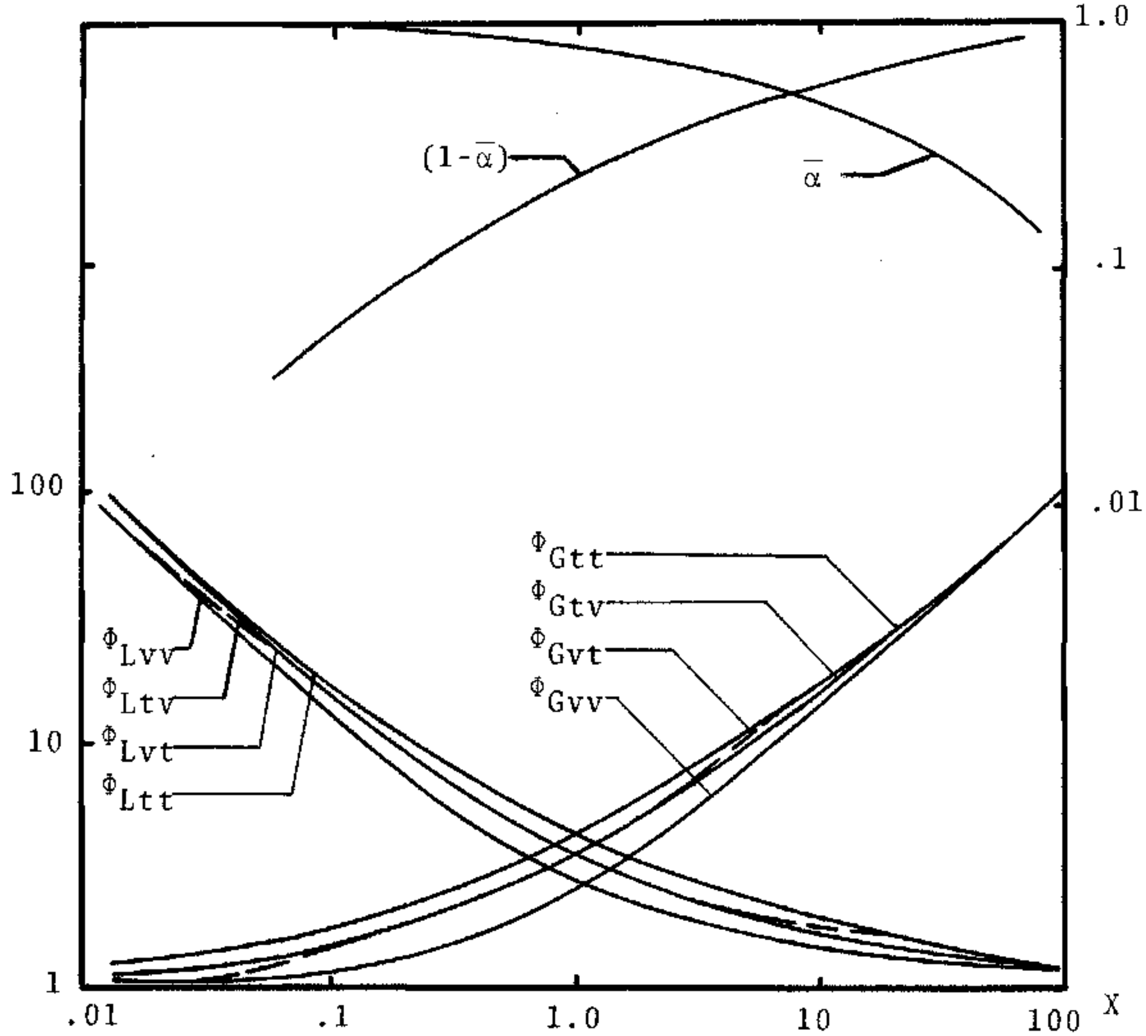

Figure 1. Lockhart-Martinelli Correlation

for the mixture viscosity to use in the definition of the Reynolds number. Numerous definitions for this two-phase viscosity are found in the literature and the researchers all claim that their expressions are the proper definition to use. Seven of the most publicized definitions plus a Russian definition are presented below to illustrate the wide variety from which one has had to choose. They are:

(1) $\mu_m = \mu_1$, (Owens [26], 1962)

(2) $\frac{1}{\mu_m} = \frac{1-\chi}{\mu_1} + \frac{\chi}{\mu_2}$, (Isbin et al. [18], 1957 and McAddams et al. [24], 1962)

(3) $\mu_m = (1-\chi)\mu_1 + \chi\mu_2$, (Cicchitti et al. [7], 1960)

(4) $\mu_m = \mu_1^{1-\chi}\mu_2^{\chi}$, (Hagendorn [16], 1965)

(5) $\mu_m = (1-\overline{\alpha})\mu_1 + \overline{\alpha}\mu_2$, (Bankoff [1], 1960)

(6) $\mu_m = (1-\beta)\mu_1 + \beta\mu_2 C_2$, (Dukler et al. [12], 1954 and Ngyuen and Spedding [25], 1973)

(7) $\frac{1}{\nu_m} = \frac{1-\beta}{\nu_1} + \frac{\beta}{\nu_2}$, (Mamaev et al. [22], 1969)

(8) $\mu_m = \mu_1[1+\chi\frac{v_{fg}}{v_f}]$, (Davidson [9], 1948)

where the mass quality, χ, the volumetric flux concentration, β, and the void fraction, $\overline{\alpha}$, are defined in the Nomenclature. The parameter C_2 is defined in equation (16).

Notice that in every expression but (8), the mixture viscosity reduces to the correct result at the extremes; that is, when

$$\left.\begin{array}{l}\chi \to o\\ \beta \to o\\ \overline{\alpha} \to o\end{array}\right. \quad \equiv \quad \begin{array}{c}\mu_m = \mu_1\\ (\nu_m = \nu_1)\end{array} \tag{9}$$

and when

$$\left.\begin{array}{l}\chi \to 1\\ \beta \to 1\\ \overline{\alpha} \to 1\end{array}\right. \quad \equiv \quad \begin{array}{c}\mu_m = \mu_2\\ (\nu_m = \nu_2)\end{array} \quad . \tag{10}$$

Three of the previous expressions for the mixture viscosity are plotted versus the quality in Figure 2 to point out the large differences one could encounter during any analysis involving a definition for the viscosity depending on his choice of equations (1) through (8). It is easily seen in Figure 2 that the values can differ by a factor of 5 in the worst case!

An entirely different and novel approach to the problem of predicting the frictional pressure drop was undertaken by Dukler, Wicks, and Cleveland [12] in 1964 and rederived by Ngyuen and Spedding [25] in 1973. Dukler et al. introduced and developed a correlation through similarity analysis. Their resulting expressions for the Euler number and the mixture Reynolds number are

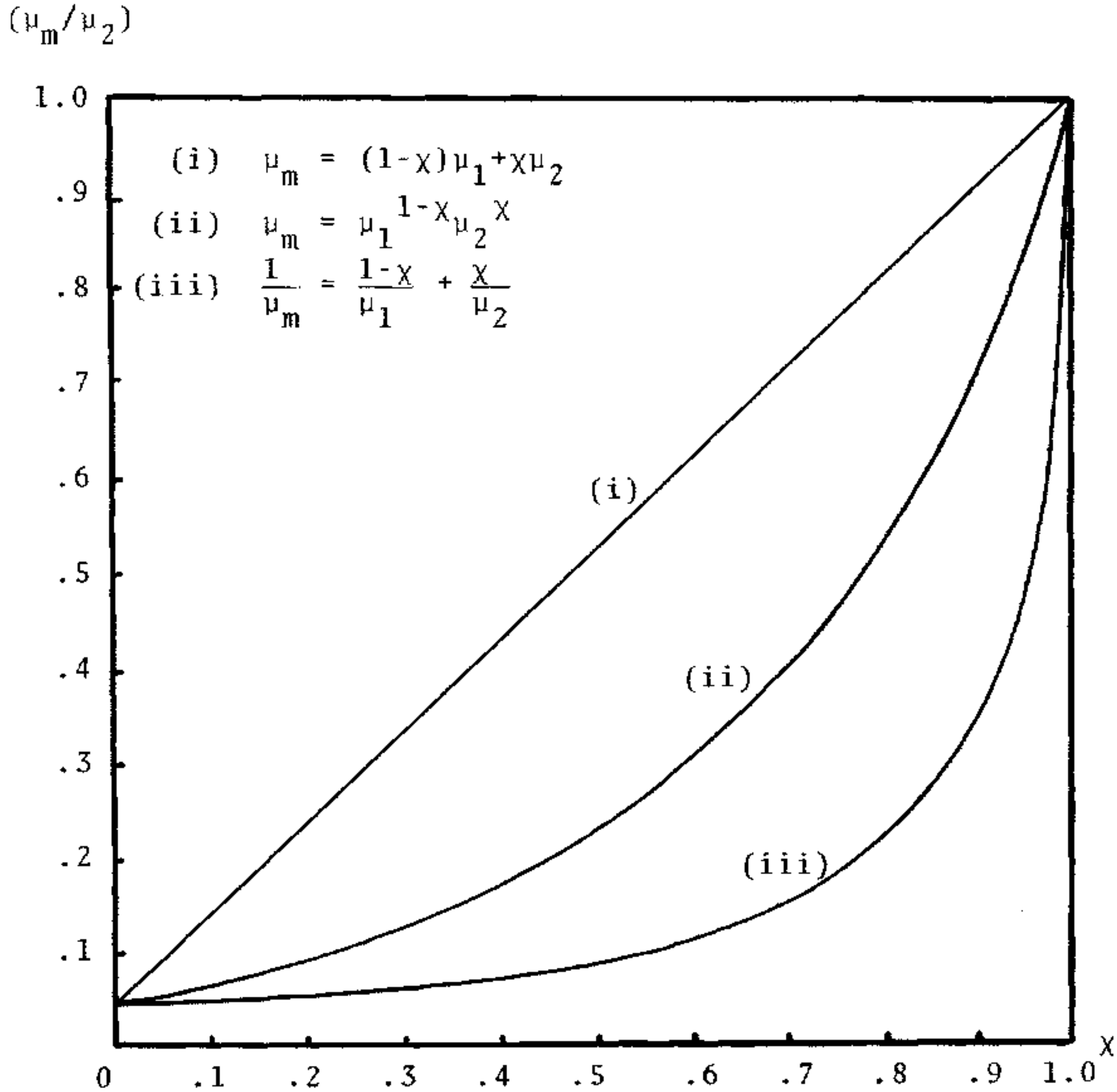

Figure 2. Non-Dimensional Viscosity Ratio versus Quality for $(\mu_2/\mu_1) = 20.1$

$$N_{Eu_{TP}} \equiv 2f_M = \{\frac{(\frac{dp}{dz})}{\frac{j^2}{g_c \ell}}\}[\frac{1}{(1-\beta)\rho_1+\beta\rho_2 C_1}] \tag{11}$$

and

$$N_{Re_{TP}} \equiv \ell j\{\frac{(1-\beta)\rho_1+\beta\rho_2 C_1}{(1-\beta)\mu_1+\beta\mu_2 C_2}\} \quad . \tag{12}$$

As a consequence of this approach, the mixture density, ρ_{TP}, was defined as

$$\rho_{TP} = (1-\beta)\rho_1+\beta\rho_2 C_1 \tag{13}$$

and the mixture viscosity, μ_{TP}, as

$$\mu_{TP} = (1-\beta)\mu_1+\beta\mu_2 C_2 \quad , \tag{14}$$

where the constants C_1 and C_2 are given by

$$C_1 \equiv (\frac{\overline{v}_L}{\overline{v}_G})^2 (\frac{\frac{dv_G}{dz}}{\frac{dv_L}{dz}}) (\frac{v_G}{\overline{v}_G}) [\frac{R_G \overline{R}_L}{\overline{R}_G R_L}] \tag{15}$$

and

$$C_2 \equiv \left(\frac{\overline{v}_L}{\overline{v}_G}\right)\left(\frac{\frac{d^2 v_G}{dn^2}}{\frac{d^2 v_L}{dn^2}}\right)\left[\frac{R_G \overline{R}_L}{\overline{R}_G R_L}\right] \quad . \tag{16}$$

Dukler et al. considered four special cases and made various assumptions in each case to evaluate C_1 and C_2. The two more important cases of interest are the case of flow without "slip" and the case of flow with "slip". In both cases, C_1 and C_2 were assumed to be equal to one.

In Chapter II a brief discussion on the separated flow models (i.e. the two-fluid model and the diffusion or drift model) will show why Dukler's et al. method, as well as the methods of many other investigators who used the homogeneous model approach, are not consistent with these models.

In 1967, Yu [33] did analytical and experimental research on the two-phase frictional pressure drop in laminar, stratified flow in horizontal conduits of varying cross section. In his analysis he introduces the concept of an apparent mixture viscosity which is very similar to the author's expression for flow in rectangular ducts in Chapter IV. However, he did not use this definition to propose a correlation for predicting the frictional pressure drop, nor did he propose a method for predicting the void fraction.

1.3 Purpose

The purpose of this thesis is threefold: (1) to develop the correct definition for the mixture kinematic viscosity, ν_m, for stratified flow systems; (2) to derive an appropriate expression for the mixture Reynolds number which can be used as a similarity parameter for modeling the friction factor in two-phase flow; and (3) to present a correlation for predicting the void fraction in horizontal, separated flow.

The results of this analysis will show that when the Reynolds number similarity group defined in this analysis is used, the two-phase friction factor reduces to the well-known Moody friction factor applicable to single-phase flow systems. Thus, the Moody friction factor in both two-phase and single-phase flow can be correlated on the same diagram. This confirms that the Reynolds number, as defined in this analysis, is the correct similarity group to be used in frictional pressure drop models.

Furthermore, the void fraction correlation as a result of this analysis will show that the "triangular" relationship claimed by Hewitt [17] does not hold for separated, two-phase flow. Hewitt bases his "triangular" relationship on three parameters: the individual volume flow rates, the overall pressure drop, and the void fraction. He claims that in order to calculate the void fraction, a knowledge of both the volume flow rates and the overall pressure drop must be known. In other words, according to Hewitt, to predict any one of the three previously mentioned parameters, one must know the other two parameters.

Experimental data are used to check the validity of both the pressure drop and void fraction correlations.

CHAPTER II

FLOW MODEL FORMULATION

2.1 General

The numerous analyses based on the area-averaged separated flow model can be divided into two basic groups. One is the two-fluid model which is formulated by considering each phase separately, whereas the second is the diffusion or drift model which is formulated by considering the entire mixture. It is this latter flow model which will be the basis of the analyses in Chapters III, IV, and V to develop pressure drop and void fraction correlations for horizontal, separated flow.

Before discussing the drift model, a brief discussion of the two-fluid model is included to point out the major differences in these two models.

2.2 Two-Fluid Model

The two-fluid model is formulated by considering each phase separately. Therefore, this formulation is expressed in terms of six field equations: two continuity equations, two momentum equations, and two energy equations.

This model will yield satisfactory results whenever the two mixture components are weakly coupled, that is when equalization of velocities does not occur [20]. This can be expected whenever there is a large difference between the densities and the velocities of the two components.

Thus, the model will be applicable to problems concerned with the dynamics of the interface and other interactions between the two phases. Since any formulation based on this model is represented in terms of six field equations, a mathematical analysis may be quite difficult; thus, it is not an effective model for system dynamics analyses or for determining mixture properties.

2.3 Diffusion or Drift Model

In contrast to the two-fluid model, the diffusion or drift model is formulated by considering the entire mixture. Therefore, the resulting formulation is expressed in terms of four field equations: three for the mixture plus the void propagation equation for one of the phases [34]. As pointed out by [20] the drift model follows the similar well-established approach used to analyze the dynamic behavior of chemically reacting binary mixtures. It is therefore applicable whenever the two mixture components are closely coupled, that is, whenever they interact so that their differences between the velocities and the other properties are small. Hence, attention is focused on the relative motion rather than the motion of the individual phases. And the field equations must be based on the baricenter, or center of mass, of the mixture.

This last requirement that the conservation equations be expressed in terms of the baricenter is where so many investigators' formulations have been wrong. Although the many traditional formulations were based on the three conservation equations, they did not express these equations in terms of the baricenter. Thus, one important consequence from this is that the mixture properties were not properly defined! In fact, various authors were forced to introduce no less than four definitions for the mixture density [20]. And note that there are many more expressions for the mixture viscosity similar to the eight expressions in Chapter I.

Therefore, in the following chapters, the author will show the correct and consistent approach to use in developing a pressure drop correlation based on this drift model.

CHAPTER III

STRATIFIED, LAMINAR FLOW BETWEEN WIDE, HORIZONTAL PARALLEL PLATES

3.1 Governing Equations

The two-phase frictional pressure drop will be analyzed analytically for the case of stratified, laminar flow between wide, horizontal parallel plates. The flow model is depicted in Figure 3. The basic differential equation governing the laminar, horizontal, fully-developed flow of an incompressible Newtonian fluid is

$$0 = -\frac{dp}{dz} + \mu\left(\frac{d^2v}{dy^2}\right) \tag{1}$$

Thus, for phase 1 and phase 2, respectively, one has

$$0 = -\frac{dp}{dz} + \mu_1\left(\frac{d^2v_1}{dy^2}\right) \tag{2}$$

and

$$0 = -\frac{dp}{dz} + \mu_2\left(\frac{d^2v_2}{dy^2}\right) \tag{3}$$

or rearranging equations (2) and (3), one obtains

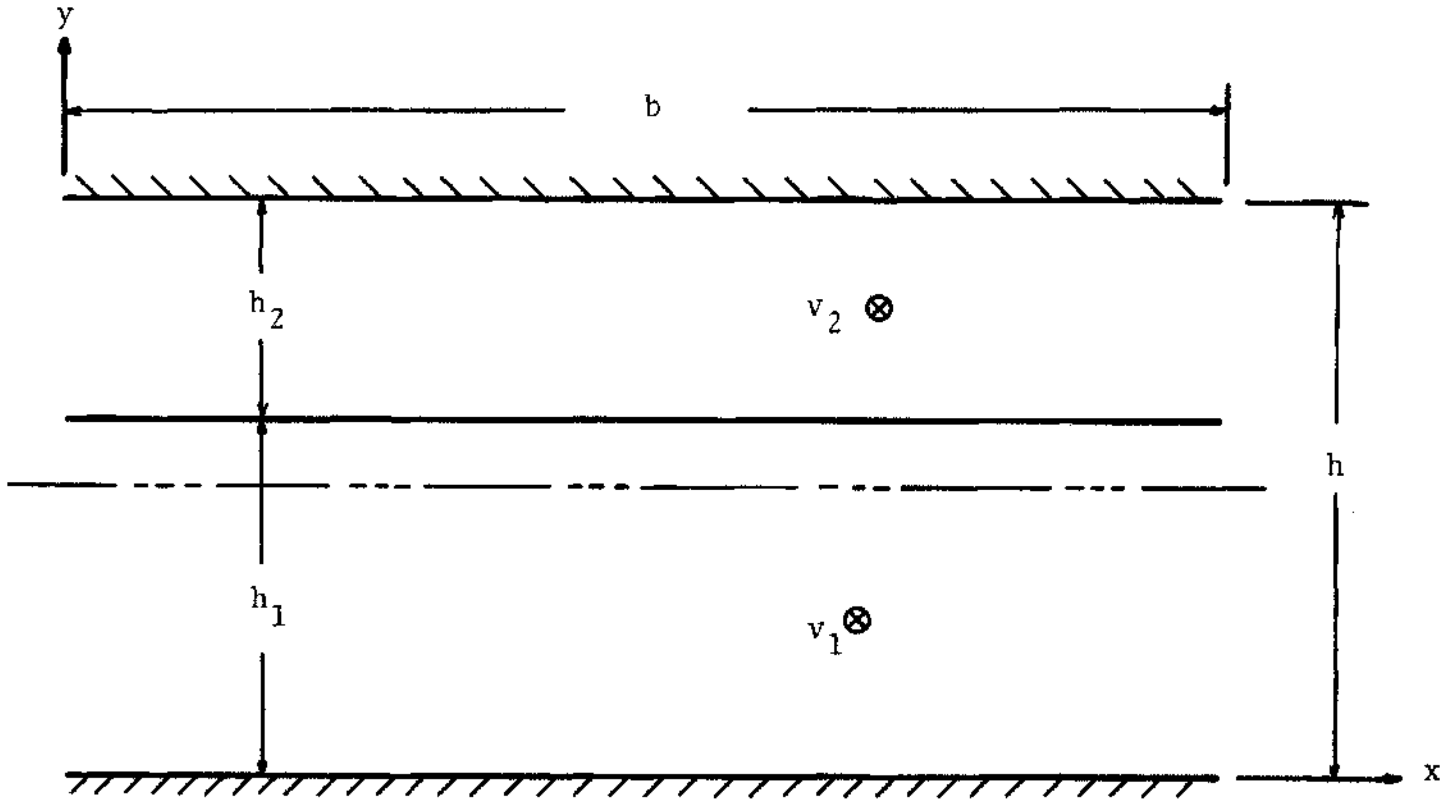

Figure 3. Separated Flow Model for Wide, Horizontal Parallel Plates

$$\frac{d^2v_1}{dy^2} = \frac{2}{\mu_1} k \tag{4}$$

and

$$\frac{d^2v_2}{dy^2} = \frac{2}{\mu_2} k \tag{5}$$

where

$$k \equiv \frac{1}{2}\left(\frac{dp}{dz}\right) \quad . \tag{6}$$

The solutions to equations (4) and (5) yield the velocity distribution for each phase, respectively [2,22,32]. Note that the velocity distribution for phase 1 in Whitaker [32] contains an error. The area-averaged velocity for each phase is obtained by integrating the solutions to equations (4) and (5) over their respective depths, h_1 and h_2. Thus, the average velocities for phase 1 and phase 2, respectively, become [22]:

$$\overline{v}_1 = -\frac{h^2}{2}k(1-\overline{\alpha})\left[\frac{(1-\overline{\alpha})}{3\mu_1} + \frac{\overline{\alpha}}{(1-\overline{\alpha})\mu_2+\overline{\alpha}\mu_1}\right] \tag{7}$$

and

$$\overline{v}_2 = -\frac{h^2}{2}k\overline{\alpha}\left[\frac{\overline{\alpha}}{3\mu_2} + \frac{(1-\overline{\alpha})}{(1-\overline{\alpha})\mu_2+\overline{\alpha}\mu_1}\right] \tag{8}$$

where $\overline{\alpha}$ is defined to be the area-averaged void fraction per unit depth; i.e.

$$\begin{aligned} \overline{\alpha} &\equiv \frac{h_2}{h}, \\ 1-\overline{\alpha} &\equiv \frac{h_1}{h}. \end{aligned} \tag{9}$$

3.2 Velocity for the Center of Volume, j

The superficial velocity for phase 1 and phase 2, respectively, is defined as

$$j_1 \equiv (1-\overline{\alpha})\overline{v}_1,\ j_2 \equiv \overline{\alpha}\ \overline{v}_2 . \tag{10}$$

Substitution of equation (9) into equation (10) gives

$$j_1 = -\frac{h^2(1-\overline{\alpha})^2k}{2}\left[\frac{(1-\overline{\alpha})}{3\mu_1} + \frac{\overline{\alpha}}{(1-\overline{\alpha})\mu_2+\alpha\mu_1}\right] \tag{11}$$

and

$$j_2 = -\frac{h^2\overline{\alpha}^2k}{2}\left[\frac{\overline{\alpha}}{3\mu_2} + \frac{(1-\overline{\alpha})}{(1-\overline{\alpha})\mu_2+\overline{\alpha}\mu_1}\right] . \tag{12}$$

The velocity for the center of volume, j, is obtained by adding the superficial velocity for phase 1 and the superficial velocity for phase 2; thus, j becomes

$$j \equiv j_1 + j_2 = -\frac{h^2 k}{6}\left[\frac{(1-\bar{\alpha})^3}{\mu_1} + \frac{\bar{\alpha}^3}{\mu_2} + \frac{3\bar{\alpha}(1-\bar{\alpha})}{(1-\bar{\alpha})\mu_2 + \bar{\alpha}\mu_1}\right] \tag{13}$$

or substituting for k defined in equation (6), one gets

$$j = -\frac{h^2}{12}\left(\frac{dp}{dz}\right)\left[\frac{(1-\bar{\alpha})^3}{\mu_1} + \frac{\bar{\alpha}^3}{\mu_2} + \frac{3\bar{\alpha}(1-\bar{\alpha})}{(1-\bar{\alpha})\mu_2 + \bar{\alpha}\mu_1}\right] . \tag{14}$$

3.3 Relative Velocity, v_r

The relative velocity, v_r, is defined as the difference between the average velocity for phase 2 and the average velocity for phase 1; i.e.

$$v_r \equiv \bar{v}_2 - \bar{v}_1 . \tag{15}$$

Substitution for $\bar{v}_2$ and $\bar{v}_1$ defined in equations (7) and (8) into equation (15) gives

$$v_r = -\frac{h^2}{6}k\left[\frac{\bar{\alpha}^2}{\mu_2} - \frac{(1-\bar{\alpha})^2}{\mu_1}\right] \tag{16}$$

or substituting for k from equation (6), v_r becomes

$$v_r = -\frac{h^2}{12}\left(\frac{dp}{dz}\right)\left[\frac{\overline{\alpha}^2}{\mu_2} - \frac{(1-\overline{\alpha})^2}{\mu_1}\right] \quad . \tag{17}$$

3.4 Velocity for the Center of Mass, v_m

Zuber and Dougherty [34] have shown that the velocity for the baricenter or the center of mass can be computed by

$$v_m \equiv j - \overline{\alpha}(1-\overline{\alpha})\left[\frac{\rho_1 - \rho_2}{\rho_m}\right] v_r \tag{18}$$

Therefore, the defining equation for v_m becomes

$$\rho_m v_m \equiv -\frac{h^2}{12}\left(\frac{dp}{dz}\right)\left[\frac{(1-\overline{\alpha})^3}{\nu_1} + \frac{\overline{\alpha}^3}{\nu_2} + \frac{3\overline{\alpha}(1-\overline{\alpha})\rho_m}{(1-\overline{\alpha})\mu_2 + \overline{\alpha}\mu_1}\right] \tag{19}$$

after substituting equations (14) and (17) into equation (18).

Notice that in equation (19) the expression inside the brackets must have the units of a kinematic viscosity. Hence, one can define the mixture kinematic viscosity based on the diffusion model, ν_m, as

$$\frac{1}{\nu_m} \equiv \frac{(1-\overline{\alpha})^3}{\nu_1} + \frac{\overline{\alpha}^3}{\nu_2} + \frac{3\overline{\alpha}(1-\overline{\alpha})\rho_m}{(1-\overline{\alpha})\mu_2 + \overline{\alpha}\mu_1} \quad . \tag{20}$$

Equation (20) can be expressed as

$$\frac{1}{\nu_m} = \frac{(1-\overline{\alpha})}{\nu_1} + \frac{\overline{\alpha}}{\nu_2} + I\{\overline{\alpha}, \text{ properties}\} \tag{21}$$

where the function, I, which accounts for the interaction between the two phases, becomes

$$I \equiv -\bar{\alpha}(1-\bar{\alpha})\left[\frac{\mu_2-\mu_1}{(1-\bar{\alpha})\mu_2+\bar{\alpha}\mu_1}\right]\left\{\frac{(1-\bar{\alpha})(2-\bar{\alpha})}{\nu_1}-\frac{\bar{\alpha}(1+\bar{\alpha})}{\nu_2}\right\} - \frac{\bar{\alpha}(1-\bar{\alpha})(2\bar{\alpha}-1)(\rho_1-\rho_2)}{(1-\bar{\alpha})\mu_2+\bar{\alpha}\mu_1} \quad . \tag{22}$$

The absolute mixture viscosity, μ_m, can then be computed from

$$\mu_m \equiv \rho_m \nu_m \tag{23}$$

where the mixture density, ρ_m, is defined as

$$\rho_m \equiv (1-\bar{\alpha})\rho_1+\bar{\alpha}\rho_2 \quad . \tag{24}$$

From an electrical analog, the kinematic mixture viscosity defined in equations (21) and (22) can be thought of as the sum of two "resistances" acting in parallel (i.e. $\frac{(1-\bar{\alpha})}{\nu_1}$ and $\frac{\bar{\alpha}}{\nu_2}$) plus an interaction term, I, due to both "resistances." It is important to note that in all other expressions for the mixture viscosity [1,2,7,9,12,16,18,22,24, 25,26] this interaction term has <u>not</u> been included. It is readily seen in equation (22) that the interaction expression is negligible whenever the void fraction, $\bar{\alpha}$, is very close to zero or one, and whenever the difference between the absolute viscosities of the two fluids is small. All other cases must be determined from experiment.

3.5 Moody Friction Factor, f_M

By considering a force balance on the flow system, one can always write

$$A_c \Delta p \equiv \tau_\omega P_\omega L \tag{25}$$

or

$$\tau_\omega \equiv \left(\frac{\Delta p}{L}\right)\left(\frac{A_c}{P_\omega}\right) \tag{26}$$

where P_ω is the wetted perimeter. By definition, the wall shear stress, τ_ω, can be expressed in terms of the Moody friction factor, f_M, as

$$\tau_\omega \equiv \frac{f_M}{8}\rho_m v_m{}^2 \quad . \tag{27}$$

Substitution of equation (27) into equation (26) gives

$$f_M = \frac{8}{\rho_m v_m{}^2}\left(\frac{\Delta p}{L}\right)\left(\frac{A_c}{P_\omega}\right) \quad . \tag{28}$$

Rewriting equation (19) as

$$\rho_m v_m \nu_m = \frac{h^2}{12}\left(-\frac{dp}{dz}\right) \equiv \frac{h^2}{12}\left(\frac{\Delta p}{L}\right) \tag{29}$$

and substituting equation (29) into equation (28), one gets

$$f_M = \frac{96}{h^2}\left(\frac{A_c}{P_\omega}\right)\frac{\nu_m}{v_m} \quad . \tag{30}$$

Multiplication of the RHS of equation (30) by one (i.e. D_H/D_H, where D_H is the hydraulic diameter defined to be equal four times the cross-sectional area divided by the wetted perimeter) results in the following expression:

$$f_M = \frac{384}{h^2}\left(\frac{A_c}{P_\omega}\right)^2\frac{1}{Re_H} \quad . \tag{31}$$

From Figure 3, one can write

$$A_c = bh, \; P_\omega = 2b \tag{32}$$

and substituting equation (32) into equation (31) gives the following expression for the two-phase Moody friction factor, f_M,:

$$f_M = 96/Re_H \tag{33}$$

where the mixture Reynolds number based on the hydraulic diameter, Re_H, is defined as

$$Re_H \equiv \frac{v_m D_H}{\nu_m} \; . \tag{34}$$

Notice that equation (33) is identical to the equation used for the case of single-phase flow through wide, horizontal parallel plates when correlating the single-phase friction factor with the fluid Reynolds number. Thus, by appropriately defining the two-phase mixture kinematic viscosity as in equation (16), the two-phase flow problem reduces to a "pseudo-homogeneous" flow with constant properties.

3.6 Void Fraction Correlation

Recalling the definition of j_1 and j_2 in equations (11) and (12), respectively, one can form the ratio, j_2/j_1; i.e.

$$\frac{j_2}{j_1} = \frac{Q_2/A_c}{Q_1/A_c} = \frac{Q_2}{Q_1} = \frac{\bar{\alpha}^2\left[\frac{\bar{\alpha}}{3\mu_2}+\frac{(1-\bar{\alpha})}{(1-\bar{\alpha})\mu_2+\bar{\alpha}\mu_1}\right]}{(1-\bar{\alpha})^2\left[\frac{(1-\alpha)}{3\mu_1}+\frac{\alpha}{(1-\bar{\alpha})\mu_2+\bar{\alpha}\mu_1}\right]} \tag{35}$$

or, in terms of the "slip", $\bar{v}_2/\bar{v}_1$, one obtains

$$S \equiv \frac{\overline{v}_2}{\overline{v}_1} = \frac{j_2/\overline{\alpha}}{j_1/(1-\overline{\alpha})} = \frac{\overline{\alpha}\left[\frac{\overline{\alpha}}{3\mu_2} + \frac{(1-\overline{\alpha})}{(1-\overline{\alpha})\mu_2+\overline{\alpha}\mu_1}\right]}{(1-\overline{\alpha})\left[\frac{(1-\overline{\alpha})}{3\mu_1} + \frac{\overline{\alpha}}{(1-\overline{\alpha})\mu_2+\overline{\alpha}\mu_1}\right]} \tag{36}$$

or, in terms of the "flowing" volumetric concentration, β, one gets the following expression:

$$\beta \equiv \frac{Q_2}{Q_1+Q_2} = \frac{j_2}{j_1+j_2} = \frac{\overline{\alpha}^2\left[\frac{\overline{\alpha}}{3\mu_2} + \frac{(1-\overline{\alpha})}{(1-\overline{\alpha})\mu_2+\overline{\alpha}\mu_1}\right]}{\frac{(1-\overline{\alpha})^3}{3\mu_1} + \frac{\overline{\alpha}^3}{3\mu_2} + \frac{\overline{\alpha}(1-\overline{\alpha})}{(1-\overline{\alpha})\mu_2+\overline{\alpha}\mu_1}} \quad . \tag{37}$$

After algebraically manipulating equations (35), (36), and (37), an expression of the following form can be obtained:

$$A_0+A_1\overline{\alpha}+A_2\overline{\alpha}^2+A_3\overline{\alpha}^3+A_4\overline{\alpha}^4 \equiv 0 \tag{38}$$

where the coefficients, A_i's, are given in equations (39), (40), and (41) corresponding to the previous three different ratios, j_2/j_1, $\overline{v}_2/\overline{v}_1$; and β. They are:

$$\begin{aligned} A_0 &= \left(\frac{\mu_2}{\mu_1}\right)^2\left(\frac{Q_2}{Q_1}\right) \\ A_1 &= -\,4\left(\frac{\mu_2}{\mu_1}\right)\left[\frac{\mu_2}{\mu_1}-1\right]\left(\frac{Q_2}{Q_1}\right) \end{aligned} \tag{39}$$

$$A_2 = \{3(\frac{\mu_2}{\mu_1})[2(\frac{\mu_2}{\mu_1})(\frac{Q_2}{Q_1})-1]-9(\frac{\mu_2}{\mu_1})(\frac{Q_2}{Q_1})\}$$

$$A_3 = -\{2(\frac{\mu_2}{\mu_1})[2(\frac{\mu_2}{\mu_1})(\frac{Q_2}{Q_1})-1]-6(\frac{\mu_2}{\mu_1})(\frac{Q_2}{Q_1})\} \qquad (39)$$

$$A_4 = (\frac{\mu_2}{\mu_1}-1)\{(\frac{\mu_2}{\mu_1})(\frac{Q_2}{Q_1})+1\}$$

$$A_0 = (\frac{\mu_2}{\mu_1})S$$

$$A_1 = \{S[4-3(\frac{\mu_2}{\mu_1})]+3\}$$

$$A_2 = \{S[3(\frac{\mu_2}{\mu_1})-5]+2\} \qquad (40)$$

$$A_3 = \{1+S[1-(\frac{\mu_2}{\mu_1})]-\frac{\mu_1}{\mu_2}\}$$

$$A_4 = 0$$

$$A_0 = \frac{1}{3}\beta(\frac{\mu_2}{\mu_1})$$

$$A_1 = -\{\frac{2}{3}\beta[(\frac{\mu_2}{\mu_1})-2]\}$$

$$(41)$$

$$A_2 = (2\beta-1)$$

$$A_3 = -\{2\beta(\frac{\mu_2}{\mu_1})-\frac{1}{3}(\beta+8)\}$$

$$A_4 = \{\beta(\frac{\mu_2}{\mu_1})+(\beta-1)(\frac{\mu_1}{\mu_2})+\frac{1}{3}(2-\beta)$$

It is clearly seen from equations (39), (40), and (41) that the void fraction, $\bar{\alpha}$, can be computed solely from the input volumetric flow rates for each phase and their respective properties. The void fraction does not depend on a knowledge of the pressure drop. Therefore, the "triangular" relationship as claimed by Hewitt [17] does not exist!

CHAPTER IV

STRATIFIED, LAMINAR FLOW THROUGH HORIZONTAL, RECTANGULAR DUCTS

4.1 General

The two-phase frictional pressure drop for stratified, laminar flow through horizontal, rectangular ducts will be analyzed analytically and then tested with the available experimental data. The flow model is depicted in Figure 4. The exact analytical solution for the velocity distribution for each phase and their subsequent volume flow rates have been derived by Charles and Lilleleht [5] and Yu [33]. It should be noted that the Fourier coefficients A^*_{in} and B^*_{in} in Yu's analysis are incorrectly printed.

4.2 Velocity for the Center of Mass, v_m

The theoretical volume flow rates for phase 1 and phase 2, respectively, have been derived by [5] and are presented in Appendix A, equations (35) and (36). They are:

$$Q_1 = \frac{128a^4}{\pi^5} \sum_{i=o}^{\infty} \left(\frac{1}{2i+1}\right)^5 [A_1'(n)\{1-ch(nb_1)\}+B_1'(n)sh(nb_1)] - \frac{2}{3}k_1 a^3 b_1 \tag{1}$$

and

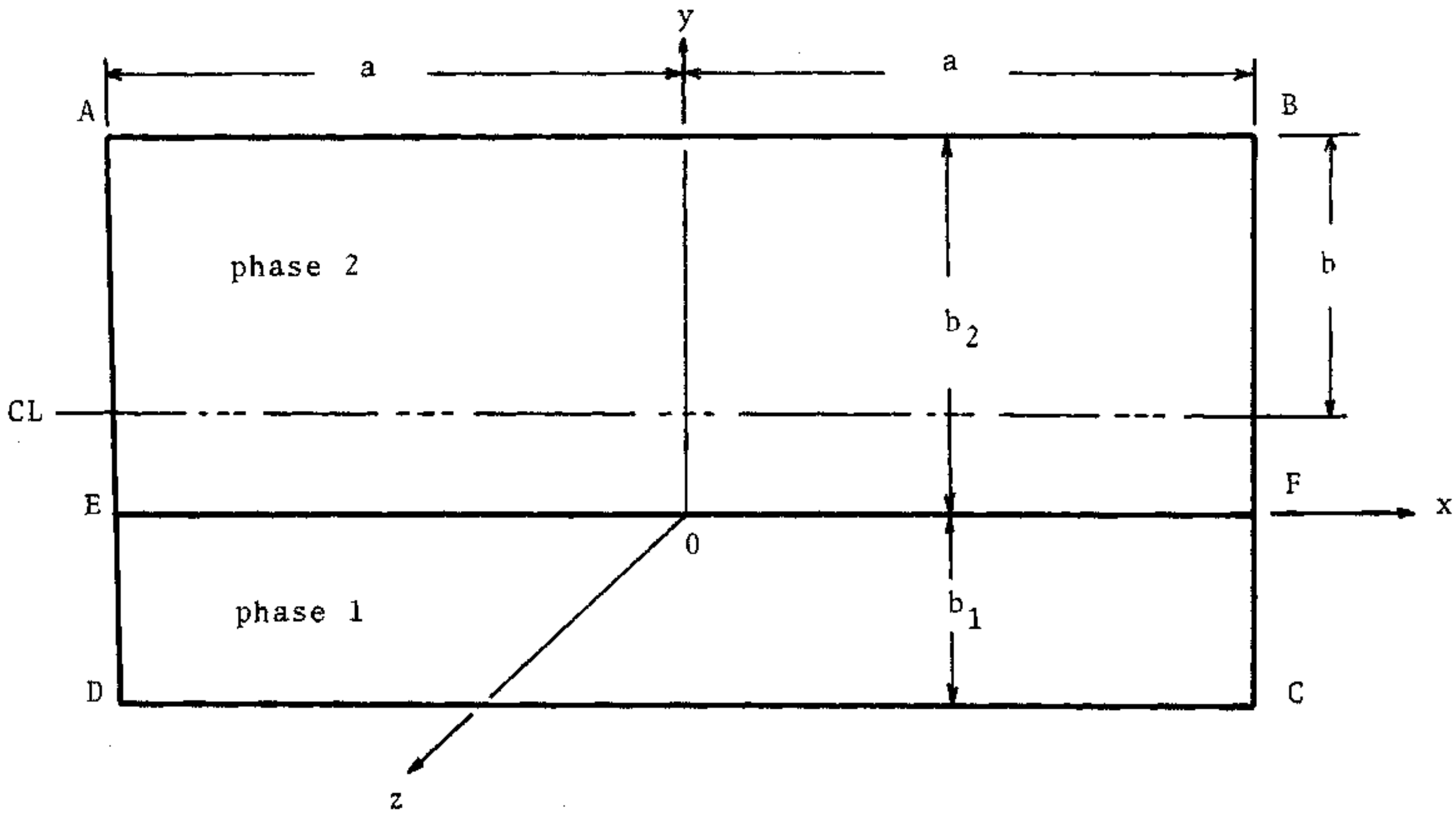

Figure 4. Separated Flow Model for Horizontal, Rectangular Ducts

$$Q_2 = \frac{128a^4}{\pi^5} \sum_{i=o}^{\infty} \left(\frac{1}{2i+1}\right)^5 [A_2'(n)\{ch(nb_2)-1\}+B_2'(n)sh(nb_2)]$$

$$- \frac{2}{3}k_2 a^3 b_2 \ . \tag{2}$$

Equations (1) and (2) can be rewritten as

$$Q_1 = - \frac{2}{3}k_1 ab^3\{(\frac{a}{b})^2 2(1-\bar{\alpha}) - \frac{6}{a^2b^3} \sum_{n}^{\infty} [\frac{1}{n^5} \frac{f_1(n)}{k_1}]\} \tag{3}$$

and

$$Q_2 = - \frac{2}{3}k_2 ab^3\{(\frac{a}{b})^2 2\bar{\alpha} - \frac{6}{a^2b^3} \sum_{n}^{\infty} [\frac{1}{n^5} \frac{f_2(n)}{k_2}]\} \tag{4}$$

where the functions $f_1(n)$ and $f_2(n)$ are defined to be

$$f_1(n) \equiv A_1'(n)\{1-ch(nb_1)\}+B_1'(n)sh(nb_1) \tag{5}$$

and

$$f_2(n) \equiv A_2'(n)[ch(nb_2)-1]+B_2'(n)sh(nb_2) \ . \tag{6}$$

The functions $A_1'(n)$, $A_2'(n)$, $B_1'(n)$, and $B_2'(n)$ are defined in Appendix A, equations (26), (27), (28), and (29), respectively.

From Figure 4, one can write

$$b_1 = 2b(1-\bar{\alpha}), \; b_2 = 2b\bar{\alpha} \tag{7}$$

and dividing the expressions defined in equations (1) and (2) by the cross-sectional area, A_c, which is equal to 4ab, one gets the superficial velocities for phase 1 and phase 2, respectively, as

$$j_1 \equiv \frac{Q_1}{A_c} = -\frac{1}{3}k_1 b^2\{(\frac{a}{b})^2(1-\bar{\alpha})-\frac{3}{a^2b^3}\sum_n^\infty \frac{1}{n^5}f_1(n)\} \tag{8}$$

and

$$j_2 \equiv \frac{Q_2}{A_c} = -\frac{1}{3}k_2 b^2\{(\frac{a}{b})^2\bar{\alpha}-\frac{3}{a^2b^3}\sum_n^\infty \frac{1}{n^5}f_2(n)\} \quad . \tag{9}$$

The defining equation for the velocity for the center of mass can be obtained from

$$G \equiv \rho_m v_m = G_1+G_2 = \rho_1 j_1+\rho_2 j_2 \quad . \tag{10}$$

Thus, after substitution of equations (8) and (9) into equation (10), one obtains

$$\rho_m v_m = -\frac{1}{3}g_c(\frac{dp}{dz})b^2[\frac{F_1'}{\nu_1}+\frac{F_2'}{\nu_2}] \tag{11}$$

where the functions F_1' and F_2' are defined to be

$$F_1' \equiv (\frac{a}{b})^2(1-\overline{\alpha}) - \frac{3}{a^2 b^3} \sum_{n}^{\infty} \frac{1}{n^5} f_1(n) \tag{12}$$

and

$$F_2' \equiv (\frac{a}{b})^2 \overline{\alpha} - \frac{3}{a^2 b^3} \sum_{n}^{\infty} \frac{1}{n^5} f_2(n) \quad . \tag{13}$$

From equations (12) and (13) it is seen that the functions F_1' and F_2' depend on both the void fraction, $\overline{\alpha}$, and the aspect ratio, a/b. The author chooses to express the functions defined in equations (12) and (13) as

$$F_1' = F\ F_1 \tag{14}$$

and

$$F_2' = F\ F_2 \tag{15}$$

where the function F (see Appendix A, equation (31)) is defined to be

$$F \equiv 1 - \frac{192b}{\pi^5 a} \sum_{n=1,3,5\ldots}^{\infty} [\frac{1}{n^5} \tanh(\frac{n\pi a}{2b})] \quad . \tag{16}$$

Now, rewriting equation (11) as

$$\rho_m v_m = -\frac{1}{3} g_c \left(\frac{dp}{dz}\right) F b^2 \left[\frac{F_1}{\nu_1}+\frac{F_2}{\nu_2}\right] \tag{17}$$

one can choose to define the mixture kinematic viscosity for the two-phase flow through horizontal, rectangular ducts as

$$\frac{1}{\nu_m} \equiv \frac{F_1}{\nu_1}+\frac{F_2}{\nu_2} \quad . \tag{18}$$

Equation (18) is in a deceivingly compact form, and it appears as though no interaction term results similar to that obtained in Chapter III for the case of flow through horizontal parallel plates. However, this is not the case. An expression for the mixture kinematic viscosity, ν_m, can be obtained after some lengthy algebraic manipulations in a form similar to the one for parallel plates. The result is

$$\frac{1}{\nu_m} = \frac{1-\bar{\alpha}}{\nu_1}+\frac{\bar{\alpha}}{\nu_2}-\frac{3}{a^4 b}\left(\frac{1}{\rho_m}\right) \sum_n^{\infty} \left(\frac{1}{n^5}\left[\frac{f_1}{\nu_1}+\frac{f_2}{\nu_2}\right]\right) \quad . \tag{19}$$

As before, the mixture kinematic viscosity can be thought of as the sum of two "resistances" acting in parallel plus an interaction term.

4.3 Moody Friction Factor, f_M

Recall from Chapter III, equation (28) that the Moody friction factor is defined as

$$f_M \equiv \frac{8}{\rho_m v_m{}^2}\left(\frac{\Delta p}{L}\right)\left(\frac{A_c}{P_\omega}\right) \quad . \tag{20}$$

Rewriting equation (17) as

$$\left(-g_c\frac{dp}{dz}\right) = \frac{3\rho_m v_m \nu_m}{F\ b^2} \equiv \frac{\Delta p}{L} \tag{21}$$

and substituting equation (21) into equation (20) results in

$$f_M = \frac{24}{F}\left(\frac{1}{b^2}\right)\left(\frac{A_c}{p_\omega}\right)\frac{\nu_m}{v_m} \quad . \tag{22}$$

Similarly, by multiplying the RHS of equation (22) by one (i.e. D_H/D_H) and simplifying, one obtains f_M as

$$f_M = \frac{96}{F}\left(\frac{A_c}{P_\omega}\right)^2\left(\frac{1}{b^2}\right)\frac{1}{Re_H} \tag{23}$$

where the mixture Reynolds number based on the hydraulic diameter, Re_H, is

$$Re_H \equiv \frac{v_m D_H}{\nu_m} \quad . \tag{24}$$

Now, from Figure 4, one can write

$$A_c = 4ab,\ P_\omega = 4(a+b) \tag{25}$$

and substitution of equation (25) into equation (23) gives

$$f_M = \frac{96}{F}\left[\frac{(\frac{a}{b})}{1+(\frac{a}{b})}\right]^2 \frac{1}{Re_H} \tag{26}$$

where a/b is the aspect ratio. Equation (26) will be checked with experimental data in Section 4.5.

4.4 Void Fraction Correlation

Due to the nature of the expressions defined in equations (1) and (2), one cannot explicitly solve for the void fraction, $\bar{\alpha}$, by forming any of the ratios (j_2/j_1), $(\bar{v}_2/\bar{v}_1)$, or β. Although the problem is complicated by the fact that the volume flow rates for each phase are dependent on the system geometry and the void fraction, the infinite series in equations (1) and (2) are in no way a deterrent to the practical use of these theoretical expressions. These series converge very rapidly; hence, only 4-10 terms were needed to obtain an answer accurate to within 1/10,000th absolute error. All machine computations were done on the UNIVAC 1108 at the Rich Computer Center on the Georgia Tech campus.

One simple but effective method which was used by the author for predicting the void fraction is described here.

The functions F_1 and F_2 are computed as a function of the void fraction, $\overline{\alpha}$, for a particular aspect ratio, a/b, and for constant fluid properties. The void fraction is incremented in steps of 0.05 covering the range from zero to one. Then, the ratio β, which is the "flowing" volumetric concentration defined to be equal to $Q_2/(Q_1+Q_2)$, is formed for each void fraction from zero to one. Thus, β becomes

$$\beta(\alpha) \equiv \frac{Q_2}{Q_1+Q_2} = \frac{F_2/\mu_2}{F_1/\mu_1+F_2/\mu_2} \tag{27}$$

or rearranging, one gets

$$\beta(\alpha) = \frac{F_2(\alpha)}{(\frac{\mu_2}{\mu_1})F_1(\alpha)+F_2(\alpha)} \quad . \tag{28}$$

Therefore, for any selected aspect ratio, a/b, and constant viscosity ratio, (μ_2/μ_1), the void fraction, $\overline{\alpha}$, can be determined from equation (28) by knowing the input volume flow rates for each phase and then computing the "flowing" volumetric concentration, β. It does not require a knowledge of the pressure drop; therefore, as before, the "triangular" relationship claimed by Hewitt does not exist!

The author has chosen to correlate $\overline{\alpha}$ versus β because upon inspection of the ratio F_2/F_1 (or F_1/F_2), one can easily see that its value ranges from zero to infinity (or infinity to zero). This is not a physically appealing nor a convenient range over which to interpolate any function. The increment on the void fraction of 0.05 is small enough so that interpolation using equation (28) is good to four decimal places.

4.5 Experimental Verification

A search through the two-phase literature for complete sets of experimental data subject to the earlier assumptions of Section 4.1 yielded only the one data set of Hao-Sheng Yu [33] with which to test the validity of this analysis.

Yu performed experiments with a light paraffin oil and water in a 25 foot long rectangular duct of a/b = 2.0. Measurements were made on the local pressure, interface shape, flow rates, and fluid temperatures. With this data, the Moody friction factor defined in equation (20) was plotted versus the two-phase mixture Reynolds number defined in equation (24). The result is shown in Figure 5. The solid line in this figure is the theoretical result predicted by equation (23); that is,

$$f_M \cong 62.19213 \, Re_H^{-1} \tag{29}$$

where the function F is equal to 0.68605 for an a/b = 2.0.

As before, notice that equation (29) is identical to

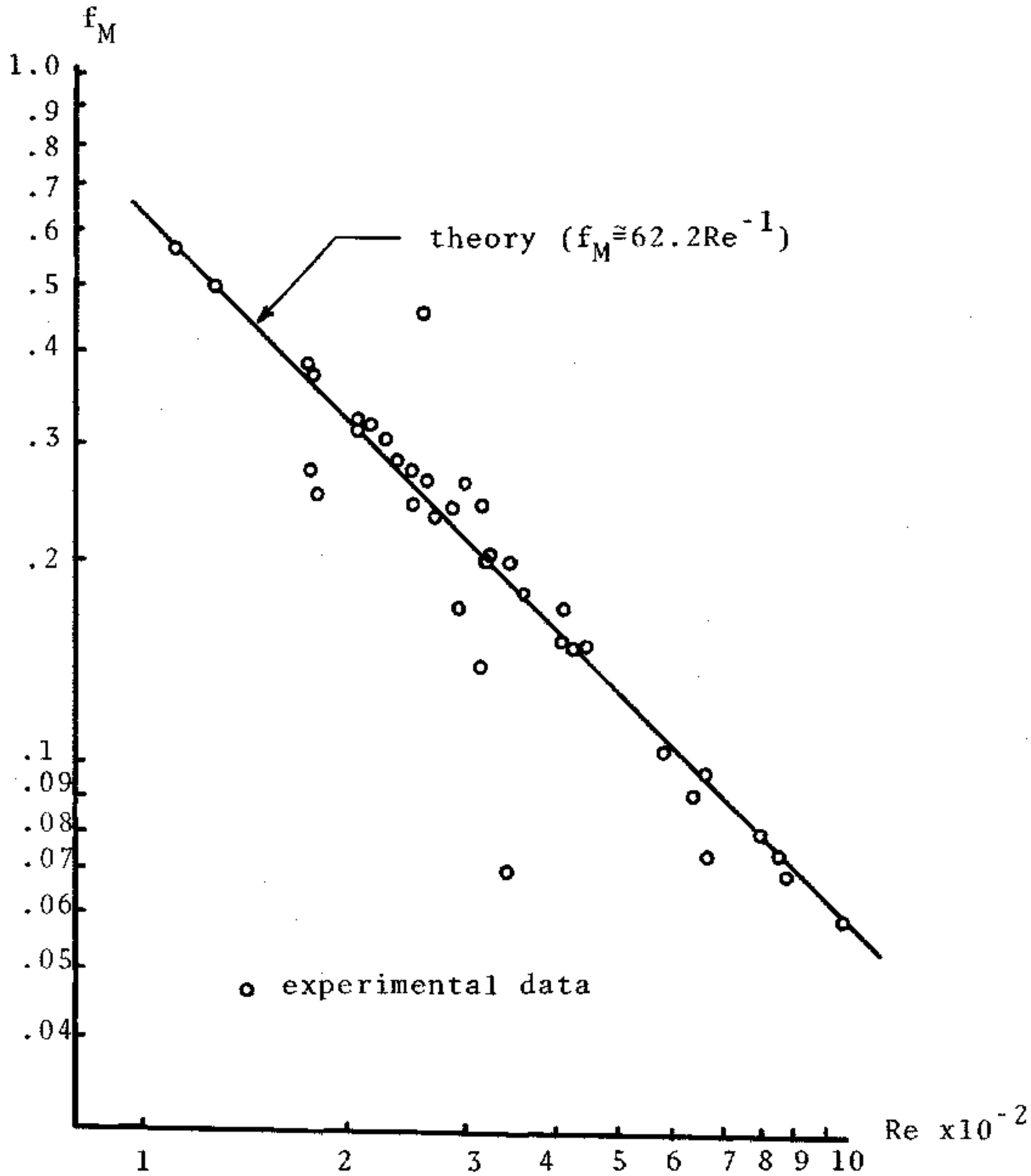

Figure 5. Two-Phase Moody Friction Factor versus Mixture Reynolds Number for Horizontal, Rectangular Ducts of a/b = 2.0

the single-phase expression for predicting the friction factor as a function of the Reynolds number. Thus, by appropriately defining the two-phase mixture kinematic viscosity in equation (18), the two-phase flow problem reduces to a "pseudo-homogeneous" single-phase flow one.

Figure 6 is a plot of the predicted void fraction, $\overline{\alpha}$, versus the "flowing" volumetric concentration, β, defined in equation (28) for an $a/b = 2.0$ by the author's theory (solid line) and by the Lockhart-Martinelli correlation (dotted line). Notice that the experimental data are in excellent agreement with the author's theory.

Figure 7 is a plot of the two-phase Moody friction factor, f_M, versus the mixture Reynolds number using the author's theoretical value for $\overline{\alpha}$ from Figure 6 to evaluate the kinematic mixture viscosity, ν_m, instead of Yu's experimentally determined value. Notice that the resulting pressure drop correlation in Figure 7 is better than the one in Figure 6.

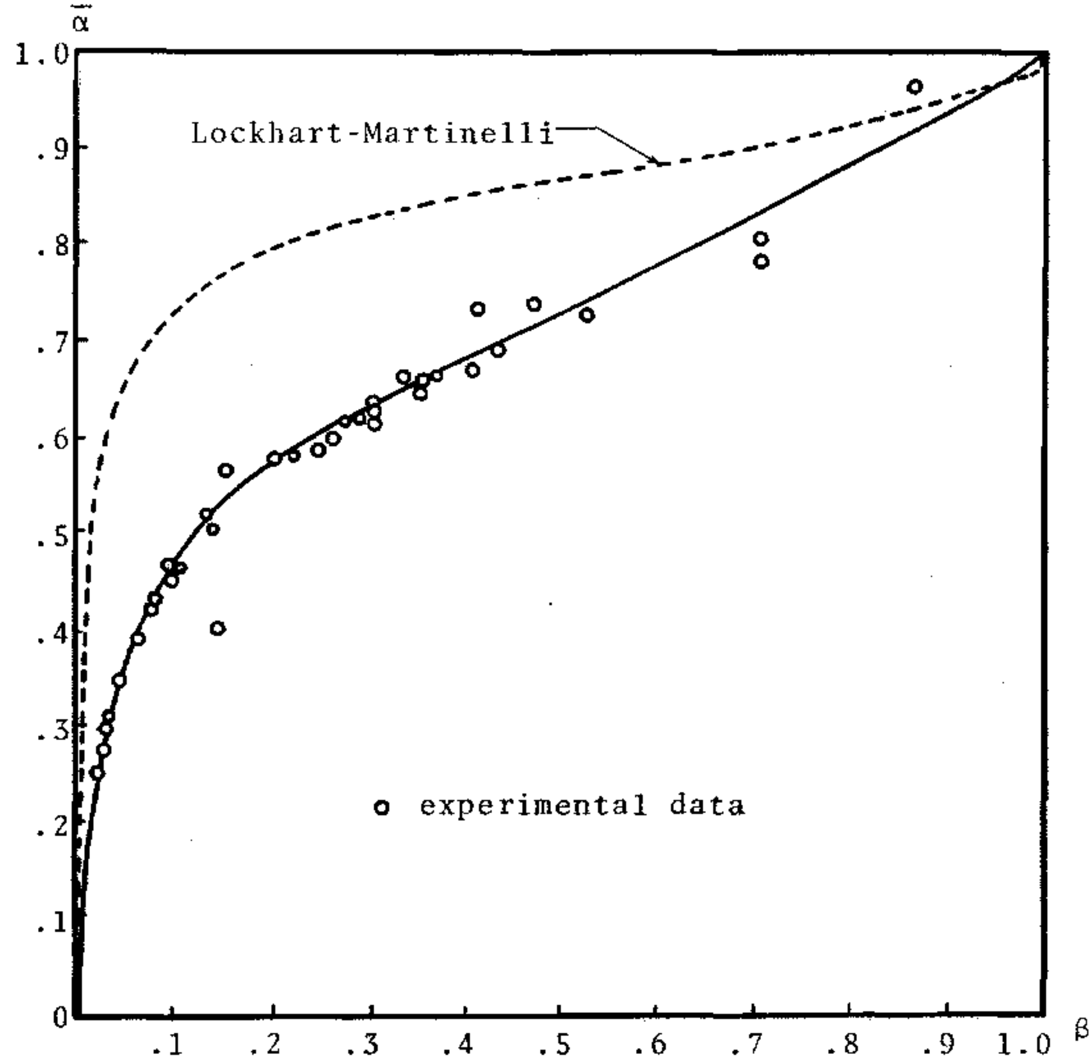

Figure 6. Void Fraction versus "Flowing" Volumetric Concentration for Horizontal, Rectangular Ducts of a/b = 2.0 and $(\mu_2/\mu_1) = 28.8$

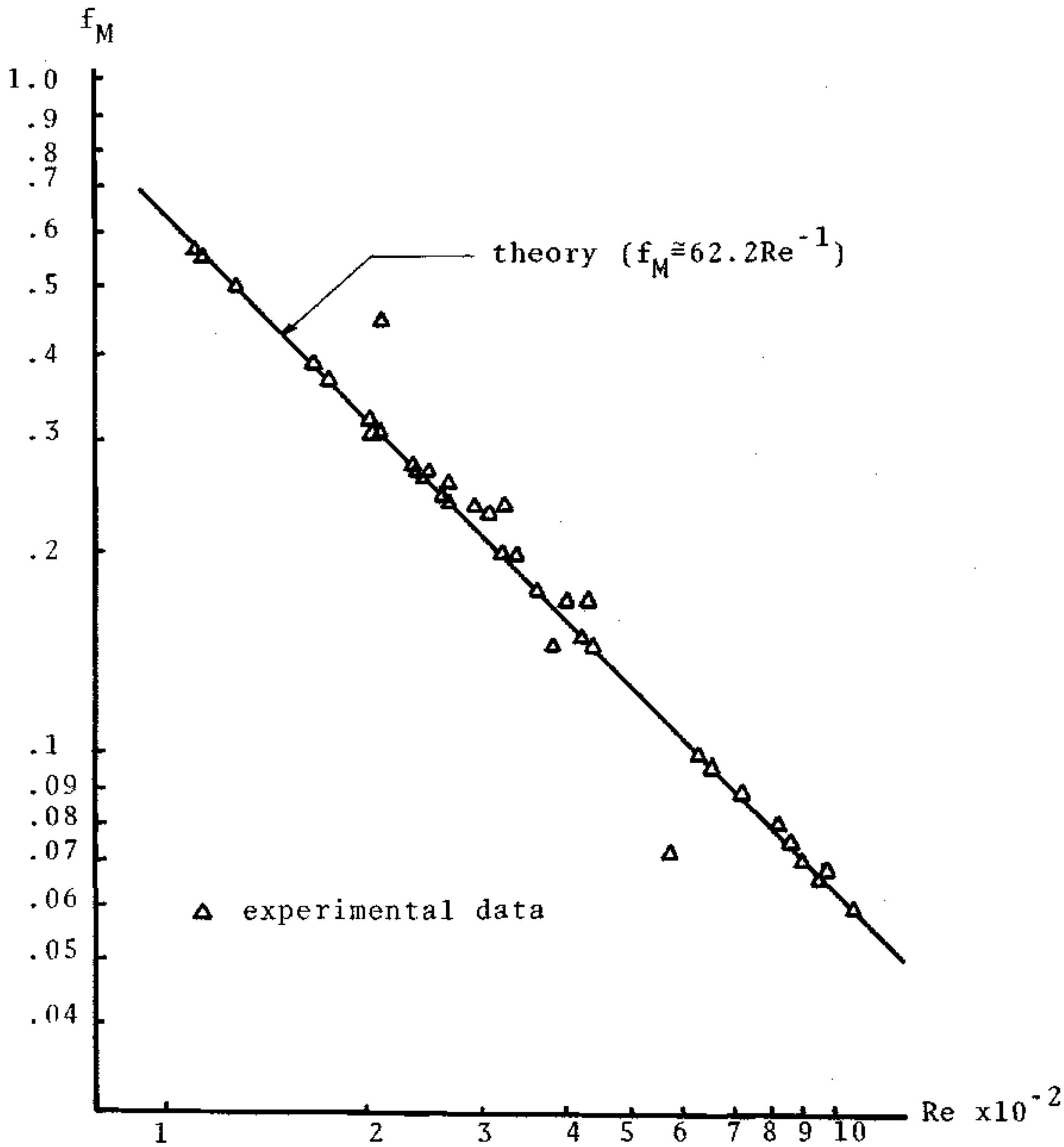

Figure 7. Two-Phase Moody Friction Factor versus Mixture Reynolds Number for Horizontal, Rectangular Ducts of a/b = 2.0 with a Theoretical Value for $\overline{\alpha}$

CHAPTER V

STRATIFIED, LAMINAR FLOW THROUGH HORIZONTAL, CIRCULAR PIPES

5.1 General

The two-phase frictional pressure drop for stratified, laminar flow through horizontal, circular pipes will be analyzed analytically and then tested with the available experimental data. The flow model is shown in Figure 8. The exact analytical solution for the velocity distribution for the two phases and their corresponding volume flow rates has been derived by Mamaev et al. [22]. Their analysis is presented in Appendix B for reference.

5.2 Velocity for the Center of Volume, j

The theoretical volume flow rates for the two phases have been derived by [22] and are presented in Appendix B, equations (54) and (55). They are

$$Q_1 = - \frac{\pi R^4}{8\mu_1}\left(\frac{dp}{dz}\right)F_1 \qquad (1)$$

and

$$Q_2 = - \frac{\pi R^4}{8\mu_2}\left(\frac{dp}{dz}\right)F_2 \qquad (2)$$

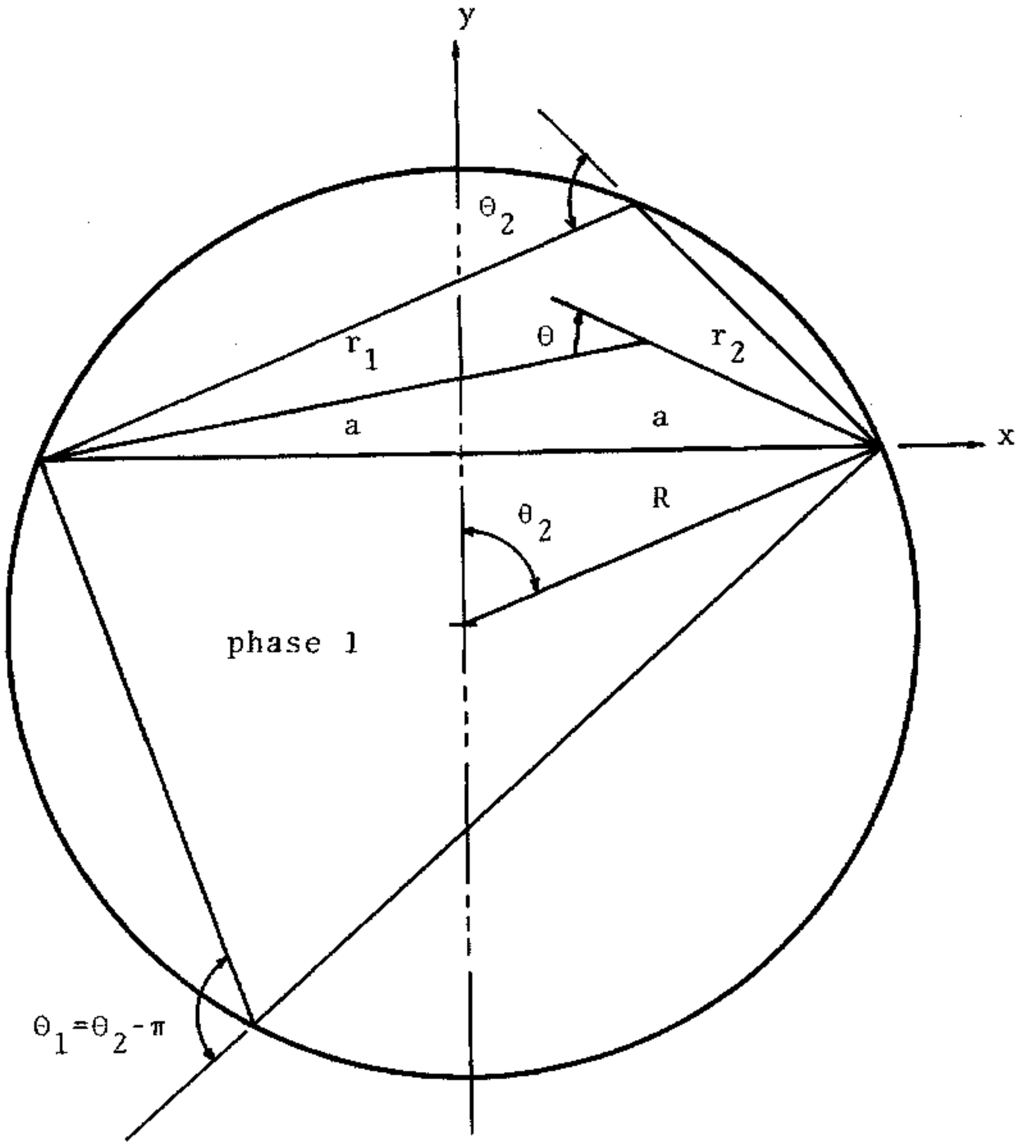

Figure 8. Separated Flow Model for Horizontal, Circular Pipes

where the functions F_1 and F_2, defined in Appendix B, equations (56) and (57), are

$$F_1 \equiv - \frac{1}{\pi}\{\Theta_1 - \frac{1}{6}(3+2\sin^2\Theta_2)\sin 2\Theta_2\} - 2\sin^4\Theta_2 \times$$
$$\times \int_o^\infty [\mathrm{ctg}\Theta_1 \mathrm{sh}(m\Theta_1) - m\,\mathrm{ch}(m\Theta_1)]\frac{mA_1(m)}{\mathrm{sh}(m\pi)}dm \tag{3}$$

and

$$F_2 \equiv \frac{1}{\pi}\{\Theta_2 - \frac{1}{6}(3+2\sin^2\Theta_2)\sin 2\Theta_2\} + 2\sin^4\Theta_2 \times$$
$$\times \int_o^\infty [\mathrm{ctg}\Theta_2 \mathrm{ch}(m\Theta_2) - m\,\mathrm{sh}(m\Theta_2)]\frac{mA_2(m)}{\mathrm{sh}(m\pi)}dm \quad . \tag{4}$$

The functions $A_1(m)$ and $A_2(m)$, also defined in Appendix B, equations (36) and (37), are

$$A_1(m) = \frac{8}{\mathrm{sh}(m\pi)}[\frac{m\,\mathrm{ch}(m\Theta_2)(\dot{\mu}_2 - \dot{k}_2) - \mathrm{ctg}\Theta_2 \mathrm{sh}(m\Theta_2)(1-\dot{k}_2)}{(1-\dot{\mu}_2)\mathrm{sh}[m(\pi - 2\Theta_2)] - (1+\dot{\mu}_2)\mathrm{sh}(m\pi)}] \tag{5}$$

and

$$A_2(m) = \frac{8}{\mathrm{sh}(m\pi)}[\frac{m\,\mathrm{ch}(m\Theta_1)(\dot{k}_1 - \dot{\mu}_1) - \mathrm{ctg}\Theta_2 \mathrm{sh}(m\Theta_1)(\dot{k}_1 - 1)}{(\dot{\mu}_1 - 1)\mathrm{sh}(m(\pi - 2\Theta_2)] - (1+\dot{\mu}_1)\mathrm{sh}(m\pi)}] \quad . \tag{6}$$

Adding equations (1) and (2) and then dividing by the cross-sectional area, A_c, gives an expression for the velocity for the center of volume, j, as

$$j = j_1 + j_2 = \frac{Q_1+Q_2}{A_c} = -\frac{R^2(\frac{dp}{dz})}{8}\{\frac{1-\bar{\alpha}}{\mu_1} + \frac{\bar{\alpha}}{\mu_2} + \frac{\sin^2\Theta_2 \sin 2\Theta_2}{3\pi}(\frac{1}{\mu_1} - \frac{1}{\mu_2}) + (\frac{I_1}{\mu_1} + \frac{I_2}{\mu_2})\} \tag{7}$$

where the integral terms, I_1 and I_2, are defined to be

$$I_1 \equiv -2\sin^4\Theta_2 \int_0^\infty [\mathrm{ctg}\Theta_1 \mathrm{sh}(m\Theta_1) - m\,\mathrm{ch}(m\Theta_1)]\frac{mA_1(m)}{\mathrm{sh}(m\pi)}dm \tag{8}$$

and

$$I_2 \equiv 2\sin^4\Theta_2 \int_0^\infty [\mathrm{ctg}\Theta_2 \mathrm{ch}(m\Theta_2) - m\,\mathrm{sh}(m\Theta_2)]\frac{mA_2(m)}{\mathrm{sh}(m\pi)}dm \ . \tag{9}$$

Rewriting j in terms of the functions F_1 and F_2, one obtains

$$j = -\frac{R^2(\frac{dp}{dz})}{8}[\frac{F_1}{\mu_1} + \frac{F_2}{\mu_2}] \ . \tag{10}$$

5.3 Relative Velocity, v_r

The average velocity for each phase can be computed as

$$\bar{v}_1 \equiv \frac{Q_1}{A_1} = -\frac{R^2(\frac{dp}{dz})}{8\mu_1}\left[1+\frac{\sin^2\Theta_2 \sin 2\Theta_2}{3\pi(1-\bar{\alpha})}+\frac{I_1}{(1-\bar{\alpha})}\right] \tag{11}$$

and

$$\bar{v}_2 \equiv \frac{Q_2}{A_2} = -\frac{R^2(\frac{dp}{dz})}{8\mu_2}\left[1-\frac{\sin^2\Theta_2 \sin 2\Theta_2}{3\pi\bar{\alpha}}+\frac{I_2}{\bar{\alpha}}\right] \tag{12}$$

or expressing the average velocities in terms of F_1 and F_2, one can obtain

$$\bar{v}_1 = -\frac{R^2(\frac{dp}{dz})}{8\mu_1}\frac{F_1}{(1-\bar{\alpha})} \tag{13}$$

and

$$\bar{v}_2 = -\frac{R^2(\frac{dp}{dz})}{8\mu_2}\frac{F_2}{\bar{\alpha}} \quad . \tag{14}$$

Thus, the relative velocity, v_r, is obtained by subtracting equation (12) from equation (11); i.e.

$$v_r \equiv \bar{v}_2-\bar{v}_1 = -\frac{R^2(\frac{dp}{dz})}{8}\left[\frac{1}{\mu_2}-\frac{1}{\mu_1}-\frac{\sin^2\Theta_2 \sin 2\Theta_2}{3\pi}\left(\frac{1}{(1-\bar{\alpha})\mu_1}+\frac{1}{\bar{\alpha}\mu_2}\right) + \frac{I_2}{\bar{\alpha}\mu_2}-\frac{I_1}{(1-\bar{\alpha})\mu_1}\right. \tag{15}$$

or using the definitions in equations (13) and (14), one gets

$$\bar{v}_r = - \frac{R^2(\frac{dp}{dz})}{8}[\frac{F_2}{\bar{\alpha}\mu_2} - \frac{F_1}{(1-\bar{\alpha})\mu_1}] \quad . \tag{16}$$

5.4 Velocity for the Center of Mass, v_m

As in Chapter III, the defining equation for the velocity for the center of mass, v_m, is

$$v_m \equiv j - \bar{\alpha}(1-\bar{\alpha})(\frac{\rho_1 - \rho_2}{\rho_m})v_r \quad . \tag{17}$$

Substitution of equations (7) and (15) into equation (17) results in

$$v_m = - \frac{R^2(\frac{dp}{dz})}{8}(\frac{1}{\rho_m})[\frac{1-\bar{\alpha}}{\nu_1} + \frac{\bar{\alpha}}{\nu_2} + \frac{\sin^2\Theta_2 \sin 2\Theta_2}{3\pi}(\frac{1}{\nu_1} - \frac{1}{\nu_2}) + + (\frac{I_1}{\nu_1} + \frac{I_2}{\nu_2})] \tag{18}$$

or

$$v_m = - \frac{R^2(\frac{dp}{dz})}{8}(\frac{1}{\rho_m})[\frac{F_1}{\nu_1} + \frac{F_2}{\nu_2}] \tag{19}$$

after substituting equations (10) and (16) into equation (17).

It is readily seen from equations (18) and (19) that the expression inside the brackets has the units of a kinematic viscosity. Therefore, one can define the two-phase mixture kinematic viscosity, ν_m, as

$$\frac{1}{\nu_m} \equiv \frac{1-\bar{\alpha}}{\nu_1}+\frac{\bar{\alpha}}{\nu_2}+\frac{\sin^2\theta_2 \sin 2\theta_2}{3\pi}\left(\frac{1}{\nu_1}-\frac{1}{\nu_2}\right)+\left(\frac{I_1}{\nu_1}+\frac{I_2}{\nu_2}\right) \tag{20}$$

or, in terms of the functions F_1 and F_2, as

$$\frac{1}{\nu_m} \equiv \frac{F_1}{\nu_1}+\frac{F_2}{\nu_2} \quad . \tag{21}$$

As with the other two stratified flow cases, the kinematic mixture viscosity can be thought of as the sum of two "resistances" acting in parallel plus an interaction term, I, where this interaction term for horizontal, circular pipes becomes

$$I \equiv \frac{\sin^2\Theta_2 \sin 2\Theta_2}{3\pi}\left(\frac{1}{\nu_1}-\frac{1}{\nu_2}\right)+\left(\frac{I_1}{\nu_1}+\frac{I_2}{\nu_2}\right) \quad . \tag{22}$$

5.5 Moody Friction Factor, f_M

Recall that the Moody friction factor can be defined as

$$f_M = \frac{8}{\rho_m v_m^2}\left(\frac{\Delta p}{L}\right)\left(\frac{A_c}{P_\omega}\right) \tag{23}$$

or (expressing it in terms of the hydraulic diameter) as

$$f_M = \frac{2}{\rho_m v_m^2}\left(\frac{\Delta p}{L}\right)D_H \quad . \tag{24}$$

Rewriting equation (19) as

$$-\frac{dp}{dz} = \frac{8\rho_m v_m \nu_m}{R^2} \equiv \frac{\Delta p}{L} \tag{25}$$

and substituting equation (24) into equation (23) yields the following expression for f_M:

$$f_M = \frac{16\nu_m D_H}{v_m R^2} = \frac{16\nu_m D_H}{v_m\left(\frac{D^2}{4}\right)} \quad . \tag{26}$$

But, the hydraulic diameter for a circular pipe is identical to the pipe diameter; thus, equation (26) becomes

$$f_M = \frac{64\nu_m}{v_m D} = 64\mathrm{Re}_D^{-1} \quad . \tag{27}$$

This resulting equation is seen to be identical to the equation used in single-phase flow. Therefore, by appropriately defining the two-phase mixture kinematic viscosity, ν_m, once again, the two-phase flow problem reduces to a "pseudo-homogeneous" single-phase problem. The validity of this analysis and equation (27) is checked with experimental data in Section 5.7.

5.6 Void Fraction Correlation

The procedure for predicting the void fraction, $\overline{\alpha}$, in stratified flow through horizontal, circular pipes is the same that was used in Chapter IV. It is not repeated here. This method for predicting the void fraction is tested with experimental data in Section 5.7.

5.7 Experimental Verification

The only available set of experimental data subject to the author's previous assumptions is that of Russell, Hodson, and Govier [28]. Russell et al. performed experiments with a fairly viscous oil and water in a 0.806 inch I.D. horizontal, circular pipe. The Moody friction factor was plotted versus the two-phase mixture Reynolds number. The result is shown in Figure 9. The solid line in this figure is the theoretical result predicted by equation (27). The limited experimental data are in fair agreement with the author's theory.

Figure 10 is a plot of the predicted void fraction

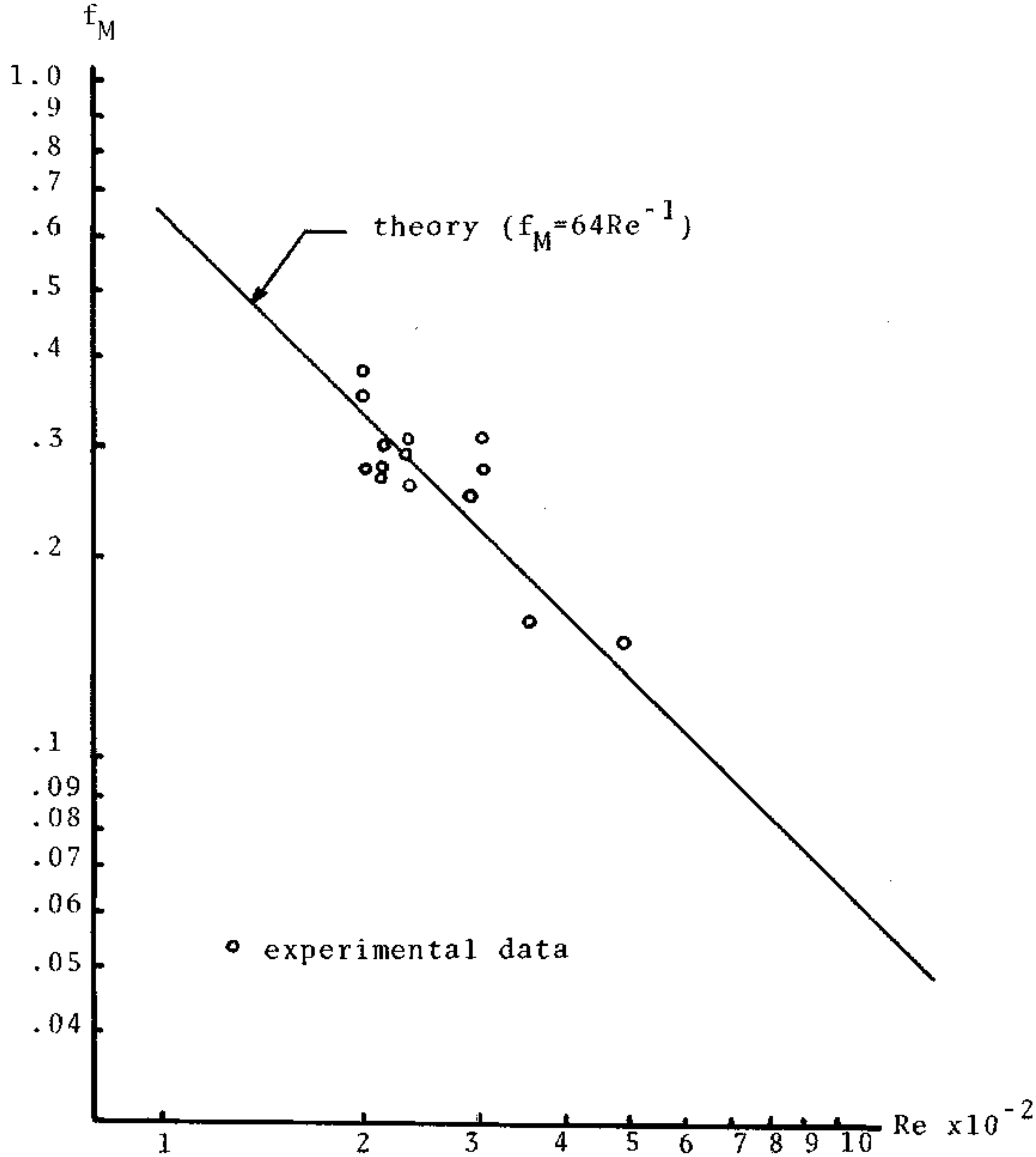

Figure 9. Two-Phase Moody Friction Factor versus Mixture Reynolds Number for Horizontal, Circular Pipes

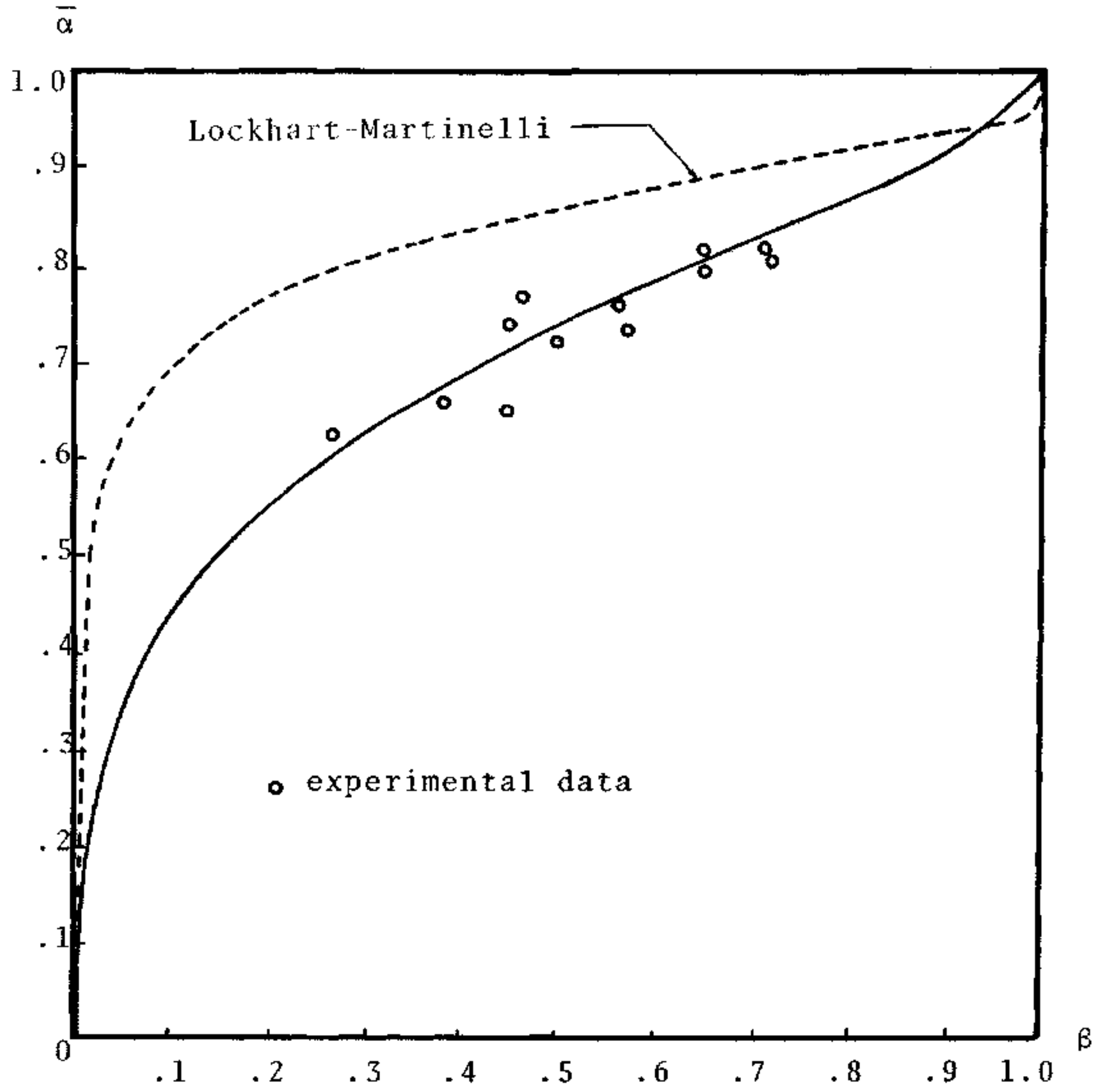

Figure 10. Void Fraction versus "Flowing" Volumetric Concentration for Horizontal, Circular Pipes with $(\mu_2/\mu_1) = 20.1$

versus the "flowing" volumetric concentration by the author's theory (solid line) and by the Lockhart-Martinelli correlation (dotted line). Notice again that the experimental are in better agreement with the author's correlation. Using the author's correlation for $\bar{\alpha}$ in the expression for the mixture kinematic viscosity defined in equation (21), a plot of f_M versus the Re was obtained. The result is shown in Figure 11.

Figure 12 shows the effect of this interaction term, I, in the expression for ν_m. Notice the significant differences between the curves for the second and third definitions in this figure. In the worst case, they can differ by as much as a factor of 5!

Figure 13 is a plot of the average void fraction, $\bar{\alpha}$, versus β for various viscosity ratios (μ_2/μ_1) ranging from 0.001 to 1000. This plot clearly shows that $\bar{\alpha}$ can be determined <u>solely</u> from a knowledge of input system quantities and fluid properties.

Figure 14 is a plot of the functions used in computing ν_m, $F_1(\alpha)$ and $F_2(\alpha)$ versus the void fraction for selected constant viscosity ratios, $\dot{\mu}_2$. These functions are defined in equations (3) and (4) of Section 5.2. The experimental data used correspond to a $\dot{\mu}_2 \simeq 20.1$.

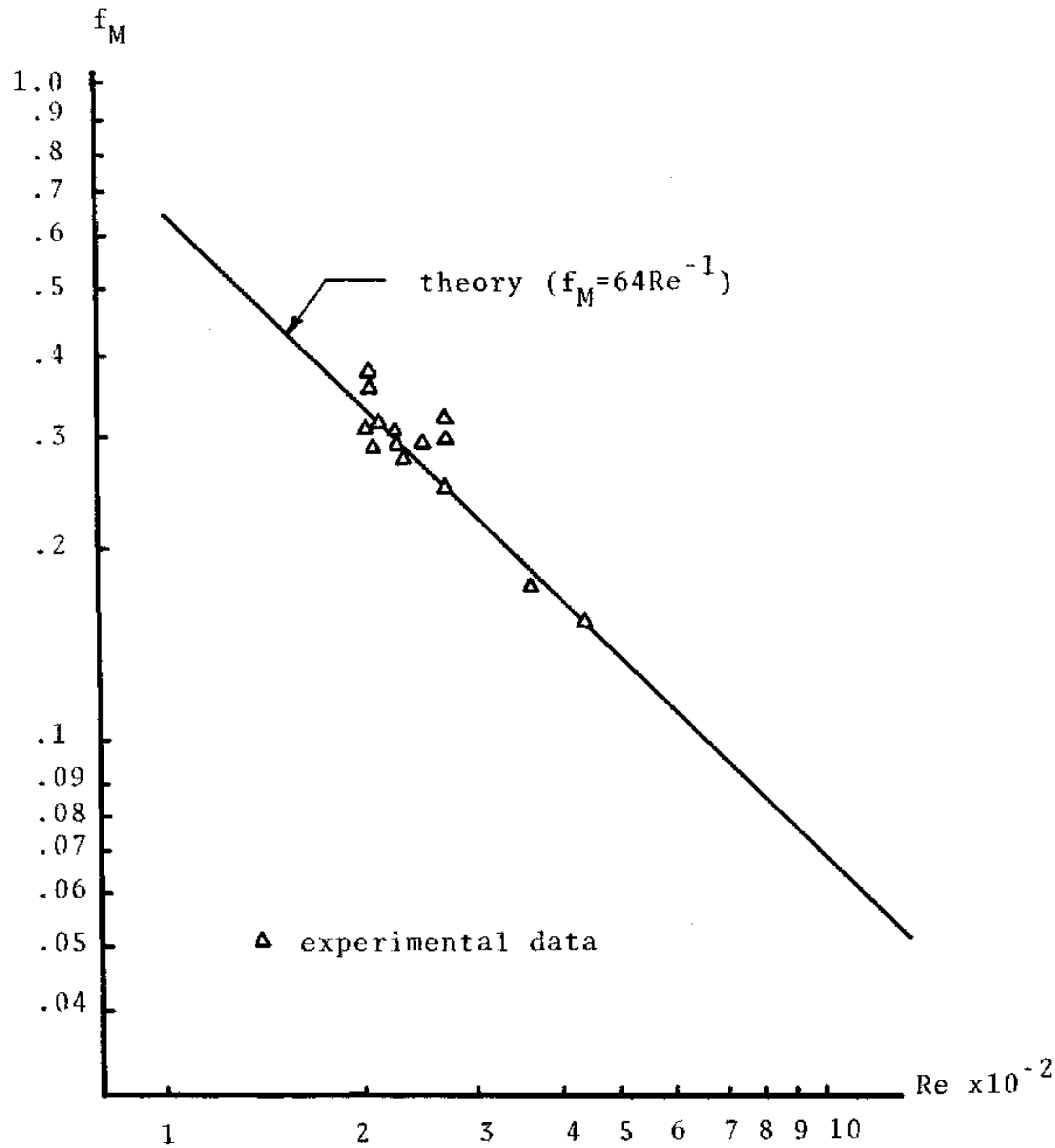

Figure 11. Two-Phase Moody Friction Factor versus Mixture Reynolds Number for Horizontal, Circular Pipes with Theoretical Value for α

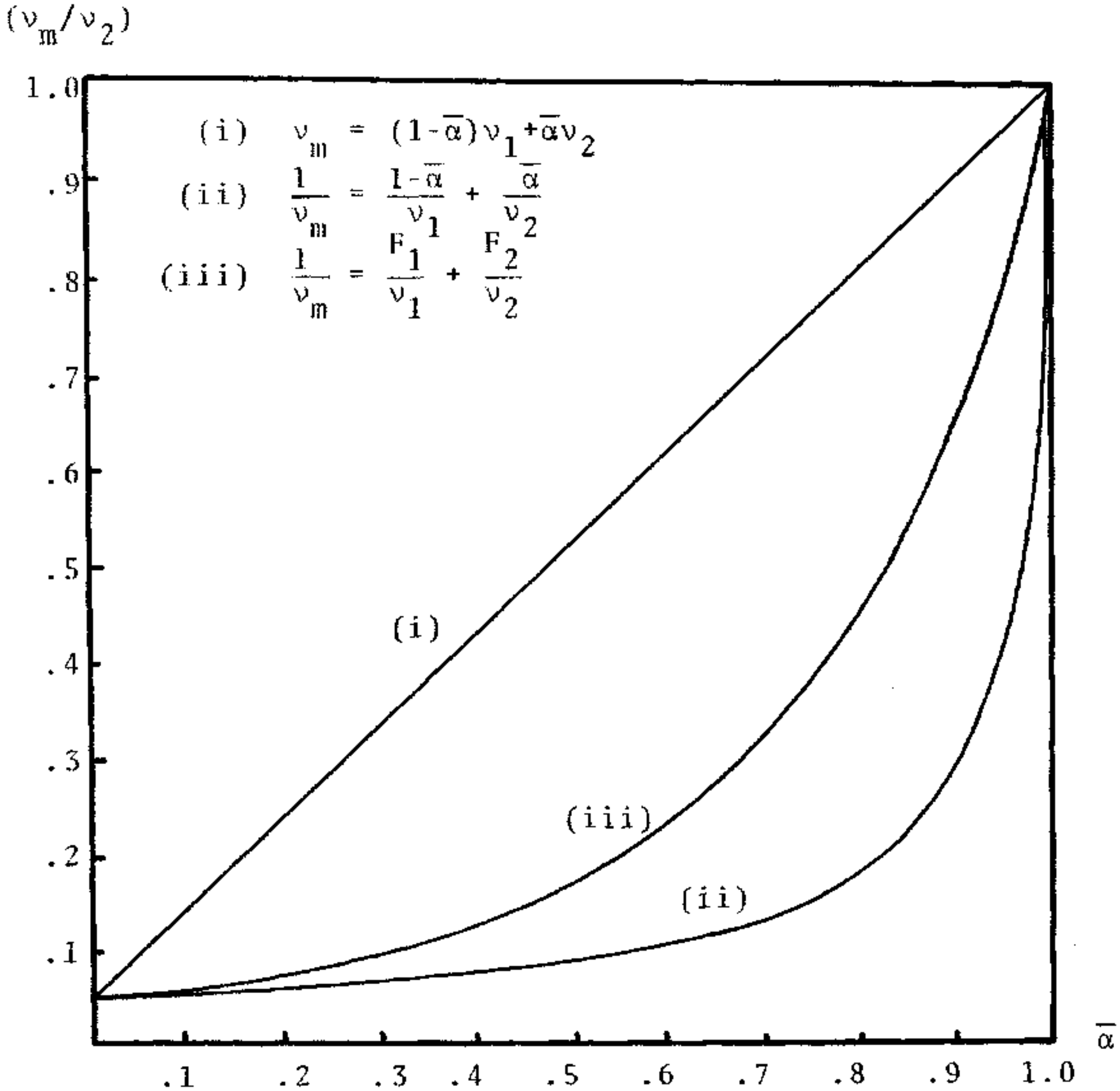

Figure 12. Non-Dimensional Kinematic Mixture Viscosity versus Void Fraction for Horizontal, Circular Pipes with $(\mu_2/\mu_1) = 20.1$ and $(\rho_2/\rho_1) = .834$

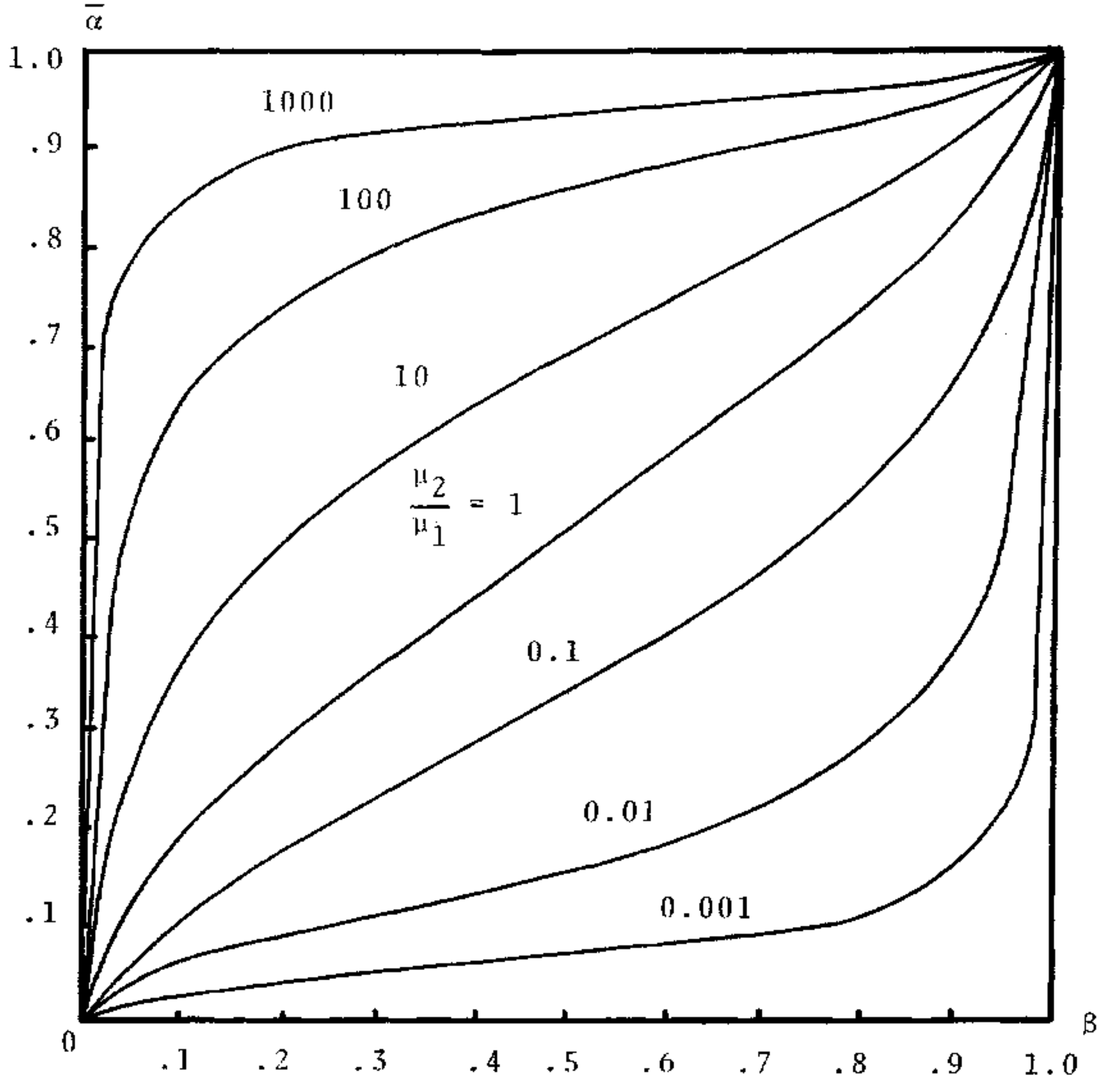

Figure 13. Average Void Fraction versus "Flowing" Volumetric Concentration for Horizontal, Circular Pipes for Various Viscosity Ratios, (μ_2/μ_1) = constant

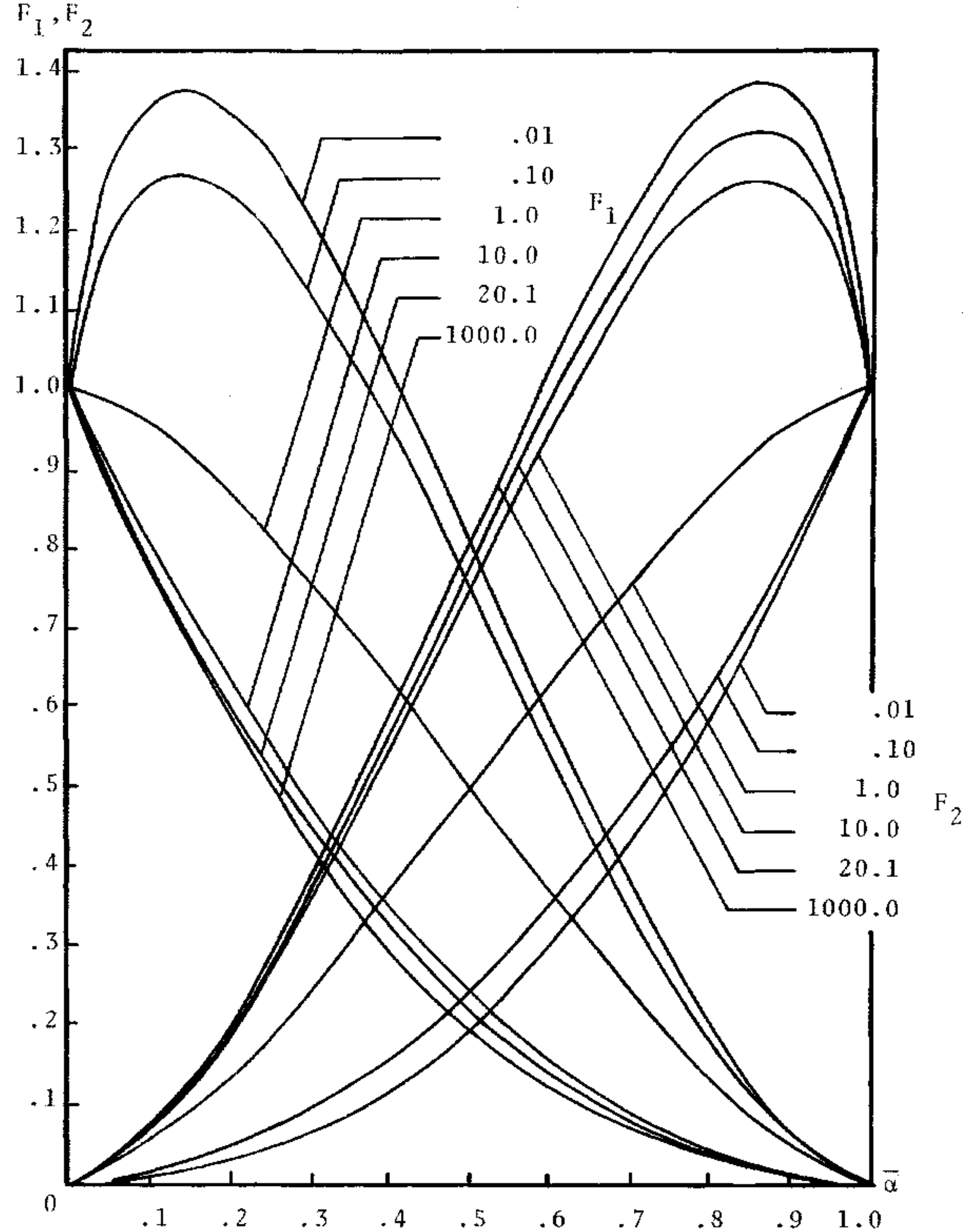

Figure 14. Functions F_1 and F_2 versus the Void Fraction for Selected Constant Viscosity Ratios, $\hat{\mu}_2$

CHAPTER VI

CONCLUSIONS

The following conclusions are made by the author as a result of the previous analyses:

(1) The diffusion or drift flow model is the proper model to use for predicting the two-phase, frictional pressure drop for the horizontal, stratified flow of two Newtonian fluids with a flat interface.

(2) As a direct consequence from using the drift model, the velocity for the two-phase mixture <u>must</u> be defined in terms of the baricenter instead of the center of volume; this results in the proper definition for the mixture kinematic viscosity.

(3) The mixture kinematic viscosity, ν_m, can be thought of as the sum of two "resistances" acting in parallel <u>plus</u> an interaction term which can be neglected under special conditions.

(4) The mixture kinematic viscosity together with the baricenter velocity and the hydraulic diameter define the Reynolds number for separated, two-phase flow.

(5) Using this Reynolds number as a similarity parameter, the frictional pressure drop for separated, two-phase flow can be correlated in terms of the standard Moody friction factor.

(6) Thus, the results of this analysis show that the correlation for frictional pressure drop in single- and two-phase flow are identical if the two-phase Reynolds number, defined in this analysis, is used as the similarity parameter.

(7) In addition to the frictional pressure drop correlation, the analysis developed in this thesis yields an expression for the void fraction in terms of known input parameters, that is, flow rates, duct geometry, and fluid properties.

(8) And finally, as a result of the author's analysis, it can be concluded that the "triangular" relationship claimed by Hewitt [17] does not hold since both the frictional pressure drop and the void fraction can be computed <u>simultaneously</u> given the input parameters (i.e., flow rates, duct geometry, and fluid properties of the individual phases or components.

APPENDICES

APPENDIX A

SEPARATED FLOW IN A RECTANGULAR CONDUIT [5]

Charles and Lilleleht [5] have derived expressions for both the velocity distribution and the volumetric flow rates for the co-current laminar, stratified flow of two immiscible, Newtonian fluids in horizontal, rectangular conduits. Their analysis is presented in this Appendix.

The flow model is depicted in Figure 4. The flow is assumed to be fully developed and the interface between the fluid layers is assumed to be smooth and horizontal. The conduit is represented by ABCD and the interface by EF. Fluid 2 flows above fluid 1, the depth being b_1 and b_2, respectively. The width of the duct is 2a. The (x,y,z) coordinate system has the origin at 0 with the x-axis coincident with the interface and the flow in the z-direction.

The basic differential equation governing the flow of both phases is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{g_c}{\mu}\left(\frac{\partial p}{\partial z}\right) \tag{1}$$

which may be written for each phase as

$$\frac{\partial^2 v_j}{\partial x^2} + \frac{\partial^2 v_j}{\partial y^2} = \frac{g_c}{\mu_j}\left(\frac{\partial p}{\partial z}\right), \ j=1,2 \tag{2}$$

where v is the local velocity which is a function of both x and y. The boundary conditions, which are similar to those used in the previous studies [6,8] of stratified flow in a circular pipe, are:

$$v_1=0 \text{ when } x=\pm a, \ -b_1 \leq y \leq 0 \tag{3}$$

$$v_1=0 \text{ when } -a \leq x \leq a, \ y=-b_1 \tag{4}$$

$$v_2=0 \text{ when } x=\pm a, \ 0 \leq y \leq b_2 \tag{5}$$

$$v_2=0 \text{ when } -a \leq x \leq a, \ y=b_2 \tag{6}$$

$$v_1=v_2 \qquad \text{when } -a \leq x \leq a \tag{7}$$

$$\mu_1\left(\frac{\partial v_1}{\partial y}\right)=\mu_2\left(\frac{\partial v_2}{\partial y}\right) \qquad y=0 \tag{8}$$

Let
$$k_j \equiv \frac{g_c\left(\frac{\partial p}{\partial z}\right)}{\mu_j} \tag{9}$$

and
$$m \equiv \frac{\mu_2}{\mu_1} \tag{10}$$

If a variable V_j is defined by the relationship

$$v_j \equiv V_j + \frac{k_j}{2}(x^2-a^2) \tag{11}$$

then substitution for v_j into the differential equation (2) gives

$$\frac{\partial^2 V_j}{\partial x^2} + \frac{\partial^2 V_j}{\partial y^2} \equiv 0 \tag{12}$$

The boundary conditions defined in equations (3) through (8) now become

$$V_1=0 \text{ when } x=\pm a,\ -b_1\leq y\leq 0 \tag{13}$$

$$V_1=-\frac{k_1}{2}(x^2-a^2) \text{ when } -a\leq x\leq a,\ y=-b_1 \tag{14}$$

$$V_2=0 \text{ when } x=\pm a,\ 0\leq y\leq b_2 \tag{15}$$

$$V_2=-\frac{k_2}{2}(x^2-a^2) \text{ when } -a\leq x\leq a,\ y=b_2 \tag{16}$$

$$V_2-V_1 = \frac{1}{2}(k_1-k_2)(x^2-a^2) \qquad -a\leq x\leq a \tag{17}$$

$$\frac{\partial V_1}{\partial y}-m\frac{\partial V_2}{\partial y} = 0 \qquad y=0 \tag{18}$$

Boundary conditions (13) and (15) give $V_j = 0$ when $x = \pm a$ and are satisfied by terms of the form

$$Y_j \cos\left[\frac{(2i+1)\pi}{2a}\right] \tag{19}$$

where Y_j is a function of y only and i is an integer. Let

$$n \equiv \frac{(2i+1)\pi}{2a} \tag{20}$$

and substitute

$$V_j = Y_j \cos(nx) \tag{21}$$

into equation (12); thus one gets

$$\frac{\partial^2 Y_j}{\partial y^2} - n^2 Y_j = 0 \tag{22}$$

from which it follows that

$$Y_j = A_j(n)\mathrm{sh}(ny)+B_j(n)\mathrm{ch}(ny) \tag{23}$$

in which $A_j(n)$ and $B_j(n)$ are functions of n and, therefore, of i but not of x and y. The solution may therefore be written in the form

$$V_j = \sum_{i=o}^{\infty} [A_j(n)\mathrm{sh}(ny)+B_j(n)\mathrm{ch}(ny)]\cos(nx) \tag{24}$$

which may be considered as a cosine Fourier series, the two coefficients being functions of y. Two of the six boundary conditions have been used. Four remain, therefore, for the evaluation of $A_j(n)$ and $B_j(n)$, $j = 1,2$.

Hence, by using the boundary conditions defined in equations (14), (16), (17) and (18) in conjunction with equation (24), the velocity distribution is obtained as

$$v_j = \frac{16a^2}{\pi^3} \sum_{i=o}^{\infty} \frac{(-1)^i}{(2i+1)^3} [A_j'(n)sh(ny)+B_j'(n)ch(ny)]cos(nx) + $$

$$+ \frac{k_j}{2}(x^2-a^2) \tag{25}$$

where the coefficients $A_j'(n)$ and $B_j'(n)$ for $j = 1,2$ become

$$A_1'(n) = mA_2'(n) \tag{26}$$

$$A_2'(n) = \frac{k_2ch(nb_1)-k_1ch(nb_2)+(k_1-k_2)ch(nb_1)ch(nb_2)}{ch(nb_1)sh(nb_2)+msh(nb_1)ch(nb_2)} \tag{27}$$

$$B_1'(n) = \frac{k_1sh(nb_2)+mk_2sh(nb_1)+m(k_1-k_2)sh(nb_1)ch(nb_2)}{ch(nb_1)sh(nb_2)+msh(nb_1)ch(nb_2)} \tag{28}$$

and

$$B_2'(n) = \frac{mk_2sh(nb_1)+k_1sh(nb_2)-(k_1-k_2)sh(nb_2)ch(nb_1)}{ch(nb_1)sh(nb_2)+msh(nb_1)ch(nb_2)} \tag{29}$$

A limiting case is of interest: if $m = 1$ and, therefore, $k_1 = k_2$, then the fluids 1 and 2 are identical as far as the problem is concerned. If, in addition, $b_1 = b_2 = b$, then the x-axis lies equidistant between the top and bottom of the conduit. With these substitutions, equation (25) reduces to the expression derived by Cornish (8) for the velocity distribution in single-phase flow given below as

$$v = - \frac{16b^2 g_c}{\pi^3 \mu}\left(\frac{\partial p}{\partial z}\right) \sum_{n=1,3...}^{\infty} \left[\frac{(-1)^{\frac{n-1}{2}}}{n^3}\left(\frac{ch(\frac{n\pi x}{2b})}{ch(\frac{n\pi a}{2b})}\right)\right]\cos\left(\frac{n\pi x}{2b}\right) - \frac{g_c}{2\mu}\left(\frac{\partial p}{\partial z}\right)(b^2 - y^2) \tag{30}$$

and for the volumetric flux, Q, as

$$Q = - \frac{4}{3}\,\frac{ab^3 g_c}{\mu}\left(\frac{\partial p}{\partial z}\right)\left\{1 - \frac{192b}{\pi^5 a} \sum_{n=1,3...}^{\infty} \left[\frac{1}{n^5}\tanh\left(\frac{n\pi a}{2b}\right)\right]\right\} \tag{31}$$

or

$$Q = - \frac{4}{3}\,\frac{a^3 b\ g_c}{\mu}\left(\frac{\partial p}{\partial z}\right)F\ . \tag{32}$$

Cornish considered a conduit of width 2a and depth 2b and he located the origin of his coordinates at the axis of the conduit, with the x-axis horizontal, the y-axis vertical, and the flow in the z-direction.

The volumetric flow rates, Q_1, and Q_2, are obtained by evaluating the integrals

$$Q_1 \equiv \int_{-b_1}^{o} \int_{-a}^{+a} v_1 dxdy \tag{33}$$

and

$$Q_2 \equiv \int_{o}^{b_2} \int_{-a}^{+a} v_2 dxdy \tag{34}$$

Substitution for v_1 and v_2 from equation (25) into equations (33) and (34), respectively, gives

$$Q_1 = \frac{128a^4}{\pi^5} \sum_{i=o}^{\infty} \left(\frac{1}{2i+1}\right)^5 [A_1'(n)(1-ch(nb_1))+B_1'(n)sh(nb_1)] - \\ - \frac{2}{3}k_1 a^3 b_1 \tag{35}$$

and

$$Q_2 = \frac{128a^4}{\pi^5} \sum_{i=o}^{\infty} \left(\frac{1}{2i+1}\right)^5 [A_2'(n)(ch(nb_2)-1)+B_2'(n)sh(nb_2)] - \\ - \frac{2}{3}k_2 a^3 b_2 \tag{36}$$

with $A_1'(n)$, $A_2'(n)$, $B_1'(n)$, and $B_2'(n)$ given by equations (26) through (29).

Expressions have been derived in open form for the velocity distribution and the volumetric flow rates for the stratified, laminar flow of two immiscible fluids in a closed rectangular conduit. It is seen from the analysis that the velocity distribution and flow rates can, therefore, be calculated from a knowledge of the physical dimensions of the conduit, the fluid properties, the pressure gradient, and the position of the interface.

APPENDIX B

SEPARATED FLOW IN THE LAMINAR REGIME THROUGH A HORIZONTAL, CIRCULAR PIPE* [22]

The system for the separated flow of two immiscible fluids with a flat interface is shown in Figure 8. Fluid 1 forms the upper layer and has a density less than that of fluid 2. The flow is also assumed to be isothermal, fully developed, with incompressible, Newtonian fluids of constant viscosities flowing in a constant cross-sectional pipe. The basic differential equation governing this type of flow has the same form in both phases; thus, for phase 1 and phase 2, respectively, in rectangular Cartesian coordinates, it becomes

$$\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} = \frac{2k_1}{\mu_1} \tag{1}$$

and

$$\frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} = \frac{2k_2}{\mu_2} . \tag{2}$$

---

*It was discovered while examining the original work that many typographical errors were present in the equations. The author, together with his advisor, Dr. Novak Zuber, translated and verified their analysis, and corrected the expressions containing the errors. Their analysis with corrected equations is presented in this Appendix for reference.

The quantities k_1 and k_2 are defined as

$$k_1 \equiv \frac{1}{2}(\frac{\partial p}{\partial x} - g\rho_1 \cos\gamma) \tag{3}$$

and

$$k_2 \equiv \frac{1}{2}(\frac{\partial p}{\partial x} - g\rho_2 \cos\gamma) \tag{4}$$

where the angle γ is the pipe inclination angle as measured from the vertical axis. Hence, for horizontal flow, $(\gamma = \frac{\pi}{2})$, the effects due to gravity forces on the flow are zero.

Since the rectangular coordinate system is not a very convenient system to use with this type of flow system geometry, equations (1) and (2) are transformed to the bi-polar coordinate system, (ε,Θ). Referring to Figure 8, the bi-polar coordinate ε is defined to be $\varepsilon = \ln(r_1/r_2)$, and thus, (x,y) can be expressed in terms of (ε,Θ) as

$$x = \frac{a\ \mathrm{sh}(\varepsilon)}{\mathrm{ch}(\varepsilon)+\cos\Theta},\ y = \frac{a\ \sin\Theta}{\mathrm{ch}(\varepsilon)+\cos\Theta}\ . \tag{5}$$

Using these relationships in equation (5), for example:

$$\frac{\partial v}{\partial x} = \frac{(\frac{\partial y}{\partial \Theta})(\frac{\partial v}{\partial \varepsilon}) - (\frac{\partial y}{\partial \varepsilon})(\frac{\partial v}{\partial \Theta})}{(\frac{\partial x}{\partial \varepsilon})(\frac{\partial y}{\partial \Theta}) - (\frac{\partial x}{\partial \Theta})(\frac{\partial y}{\partial \varepsilon})}, \tag{6}$$

$$\frac{\partial v}{\partial y} = \frac{(\frac{\partial x}{\partial \varepsilon})(\frac{\partial v}{\partial \Theta}) - (\frac{\partial x}{\partial \Theta})(\frac{\partial v}{\partial \varepsilon})}{(\frac{\partial x}{\partial \varepsilon})(\frac{\partial y}{\partial \Theta}) - (\frac{\partial x}{\partial \Theta})(\frac{\partial y}{\partial \varepsilon})} , \tag{7}$$

equations (1) and (2) can be expressed in terms of (ε,Θ) as

$$\frac{\partial^2 v_1}{\partial \varepsilon^2} + \frac{\partial^2 v_1}{\partial \Theta^2} = \frac{2k_1 a^2}{\mu_1[ch(\varepsilon)+\cos\Theta]^2} \tag{8}$$

and

$$\frac{\partial^2 v_2}{\partial \varepsilon^2} + \frac{\partial^2 v_2}{\partial \Theta^2} = \frac{2k_2 a^2}{\mu_2[ch(\varepsilon)+\cos\Theta]^2} \tag{9}$$

The solutions to equations (8) and (9) yield the local velocity distribution for each phase, respectively. Thus, one obtains

$$v_1(\varepsilon,\Theta) = -\frac{k_1 a^2}{2\mu_1}[\xi_1(\varepsilon,\Theta) + \frac{2\cos\Theta}{ch(\varepsilon)+\cos\Theta}] \tag{10}$$

and

$$v_2(\varepsilon,\Theta) = -\frac{k_2 a^2}{2\mu_2}[\xi_2(\varepsilon_1\Theta) + \frac{2\cos\Theta}{ch(\varepsilon)+\cos\Theta}] \tag{11}$$

where the terms $\frac{-k_1a^2\cos\Theta}{\mu_1[ch(\varepsilon)+\cos\Theta]}$ and $\frac{-k_2a^2\cos\Theta}{\mu_2[ch(\varepsilon)+\cos\Theta]}$ are the particular solutions to equations (8) and (9), respectively, and the functions $\xi_1(\varepsilon,\Theta)$ and $\xi_2(\varepsilon,\Theta)$ are the complementary solutions to the following set of homogeneous equations:

$$\frac{\partial^2\xi_1}{\partial\varepsilon^2} + \frac{\partial^2\xi_1}{\partial\Theta^2} \equiv 0 \tag{12}$$

and

$$\frac{\partial^2\xi_2}{\partial\varepsilon^2} + \frac{\partial^2\xi_2}{\partial\Theta^2} \equiv 0 \ . \tag{13}$$

The functions ξ_1 and ξ_2 can be expressed in terms of Fourier Integrals as

$$\xi_1(\varepsilon,\Theta) = \int_0^\infty \{A_1(m)sh[m(\Theta-\Theta_1)]+B_1(m)sh(m\Theta)\}\cos(m\varepsilon)dm \tag{14}$$

and

$$\xi_2(\varepsilon_1\Theta) = \int_0^\infty \{A_2(m)sh[m(\Theta-\Theta_2)]+B_2(m)sh(m\Theta)\}\cos(m\varepsilon)dm \tag{15}$$

where A_1, B_1, A_2, and B_2 are functions of m and are evaluated from the following four boundary conditions:

(i) on the boundary wetted by phase 1, the velocity of phase 1 must be zero; $v_1(\varepsilon,\Theta) = 0$;

(ii) on the boundary ($\Theta = \Theta_2$ = constant) wetted by phase 2, the velocity of phase 2 must be zero; $v_2(\varepsilon,\Theta) = 0$;

(iii) on the interface ($\Theta = 0$), there must be equality of velocities; thus, $v_1(\varepsilon,0) = v_2(\varepsilon,0)$;

(iv) on the interface, there must also be equality of shear stress; thus, $\mu_1(\frac{\partial v_1}{\partial y})_{\Theta=0} = \mu_2(\frac{\partial v_2}{\partial y})_{\Theta=0}$.

Applying the first two boundary conditions to equations (10) and (11) and (14) and (15), one can obtain

$$\int_0^\infty B_1(m)\,\mathrm{sh}(m\Theta_1)\cos(m\varepsilon)\,dm = -\frac{2\cos\Theta_1}{\mathrm{ch}(\varepsilon)+\cos\Theta_1} \qquad (16)$$

and

$$\int_0^\infty B_2(m)\,\mathrm{sh}(m\Theta_2)\cos(m\varepsilon)\,dm = -\frac{2\cos\Theta_2}{\mathrm{ch}(\varepsilon)+\cos\Theta_2} \qquad (17)$$

where the right hand side (RHS) of equations (16) and (17), respectively, can be expressed as

$$-\frac{2\cos\Theta_1}{\mathrm{ch}(\varepsilon)+\cos\Theta_1} = -4\,\mathrm{ctg}\Theta_1\int_0^\infty \frac{\mathrm{sh}(m\Theta_1)\cos(m\varepsilon)\,dm}{\mathrm{sh}(m\pi)} \qquad (18)$$

and

$$-\frac{2\cos\Theta_2}{\mathrm{ch}(\varepsilon)+\cos\Theta_2} = -4\,\mathrm{ctg}\Theta_2\int_0^\infty \frac{\mathrm{sh}(m\Theta_2)\cos(m\varepsilon)\,dm}{\mathrm{sh}(m\pi)}\,. \qquad (19)$$

Substituting equations (18) and (19) into equations (16) and (17), respectively, one obtains

$$\int_0^\infty B_1(m)\,sh(m\Theta_1)\cos(m\varepsilon)\,dm = -\ 4ctg\Theta_1 \int_0^\infty \frac{sh(m\Theta_1)\cos(m\varepsilon)\,dm}{sh(m\pi)} \tag{20}$$

and

$$\int_0^\infty B_2(m)\,sh(m\Theta_2)\cos(m\varepsilon)\,dm = -\ 4ctg\Theta_2 \int_0^\infty \frac{sh(m\Theta_2)\cos(m\varepsilon)\,dm}{sh(m\pi)} \tag{21}$$

and from equations (20) and (21), the functions $B_1(m)$ and $B_2(m)$ are determined to be

$$B_1(m) = -\ \frac{4ctg\Theta_1}{sh(m\pi)} \tag{22}$$

and

$$B_2(m) = -\ \frac{4ctg\Theta_2}{sh(m\pi)}\ . \tag{23}$$

From the third boundary condition requiring equality of velocities on the interface, one can obtain

$$\frac{k_1}{\mu_1}\left[\frac{2}{1+ch(\varepsilon)} - \int_0^\infty A_1(m)\,sh(m\Theta_1)\cos(m\varepsilon)\,dm\right] =$$

$$\frac{k_2}{\mu_2}\left[\frac{2}{1+ch(\varepsilon)} - \int_0^\infty A_2(m)\,sh(m\Theta_2)\cos(m\varepsilon)\,dm\right]\ . \tag{24}$$

Rewriting the first term inside the brackets on the LHS of equation (24) as

$$\frac{2}{1+ch(\varepsilon)} = 4 \int_{o}^{\infty} \frac{m \cos(m\varepsilon)}{sh(m\pi)} dm \tag{25}$$

and substituting the RHS of equation (25) back into equation (24), one can obtain the following:

$$\frac{k_1}{\mu_1}\{\int_{o}^{\infty} \cos(m\varepsilon)[\frac{4m}{sh(m\pi)} - A_1(m)sh(m\Theta_1)]dm =$$

$$\frac{k_2}{\mu_2}\{\int_{o}^{\infty} \cos(m\varepsilon)[\frac{4m}{sh(m\pi)} - A_2(m)sh(m\Theta_2)]dm \; . \tag{26}$$

Thus, from equation (26), the following relationship between $A_1(m)$ and $A_2(m)$ becomes

$$\frac{k_1}{\mu_1}[\frac{4m}{sh(m\pi)} - A_1(m)sh(m\Theta_1)] = \frac{k_2}{\mu_2}[\frac{4m}{sh(m\pi)} - A_2(m)sh(m\Theta_2)] \; . \tag{27}$$

To explicitly solve for $A_1(m)$ and $A_2(m)$, one must apply the remaining boundary condition requiring equality of shear stress on the interface; thus, transforming this boundary condition from rectangular Cartesian coordinates to the bi-polar coordinates results in

$$\mu_1(\frac{\partial v_1}{\partial \Theta})_{\Theta=0} = \mu_2(\frac{\partial v_2}{\partial \Theta})_{\Theta} = 0 \tag{28}$$

where from equation (5) it is seen that

$$(\frac{\partial x}{\partial \Theta})_{\Theta=0} = 0, \ (\frac{\partial y}{\partial \Theta})_{\Theta=0} = 1 . \tag{29}$$

Performing the partial differentiation indicated in equation (28) on equations (10) and (11) and evaluating at the interface, one finds that

$$\mu_1(\frac{\partial v_1}{\partial \Theta})_{\Theta=0} = - \frac{a^2 k_1}{2}(\frac{\partial \xi_1}{\partial \Theta})_{\Theta=0} \tag{30}$$

and

$$\mu_2(\frac{\partial v_2}{\partial \Theta})_{\Theta=0} = - \frac{a^2 k_2}{2}(\frac{\partial \xi_2}{\partial \Theta})_{\Theta=0} . \tag{31}$$

Now, substituting equations (30) and (31) into equation (28) yields

$$k_1(\frac{\partial \xi_1}{\partial \Theta})_{\Theta=0} = k_2(\frac{\partial \xi_2}{\partial \Theta})_{\Theta=0} . \tag{32}$$

Similarly, performing the partial differentiation indicated in equation (32) on the functions ξ_1 and ξ_2 defined in equations (14) and (15), respectively, and then evaluating at the interface, one can obtain

$$\left(\frac{\partial \xi_1}{\partial \Theta}\right)_{\Theta=0} = \int_0^\infty [A_1(m)\mathrm{ch}(m\Theta_1)+B_1(m)]m\cos(m\varepsilon)dm \tag{33}$$

and

$$\left(\frac{\partial \xi_2}{\partial \Theta}\right)_{\Theta=0} = \int_0^\infty [A_2(m)\mathrm{ch}(m\Theta_2)+B_2(m)]m\cos(m\varepsilon)dm \ . \tag{34}$$

Substituting for the integrals involving $B_1(m)$ and $B_2(m)$ from equations (20) and (21), respectively, into equations (33) and (34), and then substituting this result into equation (32), one gets

$$k_1[A_1(m)\mathrm{ch}(m\Theta_1) - \frac{4\mathrm{ctg}\Theta_1}{\mathrm{sh}(m\pi)}] = k_2[A_2(m)\mathrm{ch}(m\Theta_2) - \frac{4\mathrm{ctg}\Theta_2}{\mathrm{sh}(m\pi)}] \ . \tag{35}$$

By solving equations (27) and (35) simultaneously, the following expressions for the coefficients $A_1(m)$ and $A_2(m)$ are obtained:

$$A_1(m) = \frac{8}{\mathrm{sh}(m\pi)}\left[\frac{m\mathrm{ch}(m\Theta_2)(\dot{\mu}_2-\dot{k}_2)-\mathrm{ctg}\Theta_2\mathrm{sh}(m\Theta_2)(1-\dot{k}_2)}{(1-\dot{\mu}_2)\mathrm{sh}[m(\pi-2\Theta_2)]-(1+\dot{\mu}_2)\mathrm{sh}(m\pi)}\right] \tag{36}$$

and

$$A_2(m) = \frac{8}{\mathrm{sh}(m\pi)}\left[\frac{m\mathrm{ch}(m\Theta_1)(\dot{k}_1-\dot{\mu}_1)-\mathrm{ctg}\Theta_2\mathrm{sh}(m\Theta_1)(\dot{k}_1-1)}{(\dot{\mu}_1-1)\mathrm{sh}[m(\pi-2\Theta_2)]-(1+\dot{\mu}_1)\mathrm{sh}(m\pi)}\right] \ . \tag{37}$$

where, for convenience, one can define

$$\dot{\mu}_1 \equiv \frac{\mu_1}{\mu_2},\ \dot{\mu}_2 \equiv \frac{\mu_2}{\mu_1} \tag{38}$$

and

$$\dot{k}_1 \equiv \frac{k_1}{k_2},\ \dot{k}_2 \equiv \frac{k_2}{k_1}. \tag{39}$$

Notice for horizontal flow that $\dot{k}_1 = \dot{k}_2 = 1.0$.

Substituting for $A_1(m)$, $A_2(m)$, $B_1(m)$, and $B_2(m)$ from equations (36), (37), (20), and (21), respectively, the local velocity distribution for phase 1 and phase 2 can be expressed as

$$v_1(\varepsilon,\Theta) = -\frac{a^2k_1}{\mu_1}\left\{\frac{\sin(\Theta_2-\Theta)}{[ch(\varepsilon)+\cos\Theta]\sin\Theta_2} + \right. \tag{40}$$

$$\left. 4\int_0^\infty \frac{[m\,ch(m\Theta_2)(\dot{\mu}_2-\dot{k}_2)-ctg\Theta_2\,sh(m\Theta_2)(1-\dot{k}_2)]\cos(m\varepsilon)\,sh[m(\Theta-\Theta_1)]\,dm}{[(1-\dot{\mu}_2)sh[m(\pi-2\Theta_2)]-(1+\dot{\mu}_2)sh(m\pi)]sh(m\pi)}\right\}$$

and

$$v_2(\varepsilon,\Theta) = -\frac{a^2k_2}{\mu_2}\left\{\frac{\sin(\Theta_2-\Theta)}{[ch(\varepsilon)+\cos\Theta]\sin\Theta_2} + \right. \tag{41}$$

$$\left. 4\int_0^\infty \frac{[m\,ch(m\Theta_1)(\dot{k}_1-\dot{\mu}_1)-ctg\Theta_1\,sh(m\Theta_1)(\dot{k}_1-1)]\cos(m\varepsilon)\,sh[m(\Theta-\Theta_2)]\,dm}{[(\dot{\mu}_1-1)sh[m(\pi-2\Theta_2)]-(1+\dot{\mu}_1)sh(m\pi)]sh(m\pi)}\right\}$$

Equations (40) and (41) are valid for all $|\Theta|<\pi$.

From the local velocity distributions defined in equations (40) and (41), one can derive the volumetric fluxes for each phase by integrating over an elementary area. The elementary area in bi-polar coordinates is defined as

$$dA_c \equiv (d\ell)_\Theta \, (d\ell)_\varepsilon \tag{42}$$

where $(d\ell)_\Theta$ and $(d\ell)_\varepsilon$ can be written in terms of (x,y) and (ε,Θ) as

$$(d\ell)_\Theta = [(\frac{\partial x}{\partial \varepsilon})^2 + (\frac{\partial y}{\partial \varepsilon})^2]^{1/2} d\varepsilon \tag{43}$$

and

$$(d\ell)_\varepsilon = [(\frac{\partial x}{\partial \Theta})^2 + (\frac{\partial y}{\partial \Theta})^2]^{1/2} d\Theta \ . \tag{44}$$

Thus, substituting for the elementary, dA_c, from equation (42) and performing the indicated differentiation in equations (43) and (44), one obtains Q_1 and Q_2 as

$$Q_1 = a^2 \int_{\Theta_1}^{\circ} d\Theta \int_{-\infty}^{\infty} \frac{v_1(\Theta,\varepsilon)\,d\varepsilon}{[ch(\varepsilon)+\cos\Theta]^2} \tag{45}$$

$$Q_2 = a^2 \int_{o}^{\Theta_2} d\Theta \int_{-\infty}^{\infty} \frac{v_2(\Theta,\varepsilon)d\varepsilon}{[ch(\varepsilon)+\cos\Theta]^2} . \tag{46}$$

After substituting for the local velocity distributions defined in equations (40) and (41), one obtains the following expressions:

$$Q_1 = \frac{a^4 k_1}{\mu_1}\{ \int_{o}^{\Theta_1} \cos\Theta d\Theta \int_{-\infty}^{\infty} \frac{d\varepsilon}{[ch(\varepsilon)+\cos\Theta]^2} -$$

$$- ctg\Theta_1 \int_{o}^{\infty} \sin\Theta d\Theta \int_{-\infty}^{\infty} \frac{d\varepsilon}{[ch(\varepsilon)+\cos\Theta]^3} + \tag{47}$$

$$+ \frac{1}{2} \int_{o}^{\infty} A_1(m)dm \int_{o}^{\infty} sh[m(\Theta-\Theta_1)]d\Theta \int_{-\infty}^{\infty} \frac{\cos(m\varepsilon)d\varepsilon}{[ch(\varepsilon)+\cos\Theta]^2}$$

and

$$Q_2 = - \frac{a^4 k_2}{\mu_2}\{ \int_{o}^{\Theta_2} \cos\Theta d\Theta \int_{-\infty}^{\infty} \frac{d\varepsilon}{[ch(\varepsilon)+\cos\Theta]^2} -$$

$$- ctg\Theta_2 \int_{o}^{\Theta_2} \sin\Theta d\Theta \int_{-\infty}^{\infty} \frac{d\varepsilon}{[ch(\varepsilon)+\cos\Theta]^3} + \tag{48}$$

$$+ \frac{1}{2} \int_{o}^{\infty} A_2(m)dm \int_{o}^{\infty} sh[m(\Theta-\Theta_2)]d\Theta \int_{-\infty}^{\infty} \frac{\cos(m\varepsilon)d\varepsilon}{[ch(\varepsilon)+\cos\Theta]}\} .$$

Now, integrating equations (47) and (48) with respect to the variable ε gives

$$Q_1 = \frac{a^4 k_1}{\mu_1}\{ \int_\circ^{\Theta_1} [\Theta(3-2\sin^2\Theta)-3\sin\Theta\cos\Theta]\frac{\cos\Theta}{\sin^5\Theta}d\Theta -$$

$$- \operatorname{ctg}\Theta_1 \int_\circ^{\Theta_1} [\Theta(3-2\sin^2\Theta)-3\sin\Theta\cos\Theta]\frac{d\Theta}{\sin^4\Theta} + \tag{49}$$

$$+ \pi \int_\circ^\infty \frac{A_1(m)}{\operatorname{sh}(m\pi)}dm \int_\circ^{\Theta_1} \frac{\operatorname{sh}[m(\Theta-\Theta_1)]}{\sin^3\Theta}[m\operatorname{ch}(m\Theta)\sin\Theta-\operatorname{sh}(m\Theta)\cos\Theta]d\Theta\}$$

and

$$Q_2 = - \frac{a^4 k_2}{\mu_2}\{ \int_\circ^{\Theta_2} [\Theta(3-2\sin^2\Theta)-3\sin\Theta\cos\Theta]\frac{\cos\Theta}{\sin^5\Theta}d\Theta -$$

$$- \operatorname{ctg}\Theta_2 \int_\circ^{\Theta_2} [\Theta(3-2\sin^2\Theta-3\sin\Theta\cos\Theta]\frac{d\Theta}{\sin^4\Theta} + \tag{50}$$

$$+ \pi \int_\circ^\infty \frac{A_2(m)dm}{\operatorname{sh}(m\pi)} \int_\circ^{\Theta_2} \frac{\operatorname{sh}[m(\Theta-\Theta_2)]}{\sin^3\Theta}[m\operatorname{ch}(m\Theta)\sin\Theta-\operatorname{sh}(m\Theta)\cos\Theta]d\Theta\}$$

Integrating these equations with respect to the variable Θ yields the following expressions for the volumetric fluxes, Q_1 and Q_2, respectively, as

$$Q_1 = \frac{a^4 k_1}{\mu_1} \{\frac{1}{4}[\frac{\Theta_1(1-3ctg^2\Theta_1)}{\sin^2\Theta_1} + ctg\Theta_1(1+3ctg^2\Theta_1)] -$$

$$- \frac{ctg\Theta_1}{\sin^2\Theta_1}(1-\Theta_1 ctg\Theta_1) + \frac{1}{3}ctg\Theta_1 + \tag{51}$$

$$+ \frac{\pi}{2}\int_0^\infty \frac{A_1(m)}{sh(m\pi)}[msh(m\Theta_1)ctg\Theta_1 - m^2 ch(m\Theta_1)]dm\}$$

and

$$Q_2 = - \frac{a^4 k_2}{\mu_2} \{\frac{1}{4}[\frac{\Theta_2(1-3ctg^2\Theta_2)}{\sin^2\Theta_2} + ctg\Theta_2(1+3ctg^2\Theta_2)] -$$

$$- \frac{ctg\Theta_2}{\sin^2\Theta_2}(1-\Theta_2 ctg\Theta_2) + \frac{1}{3}ctg\Theta_2 + \tag{52}$$

$$+ \frac{\pi}{2}\int_0^\infty \frac{A_2(m)}{sh(m\pi)}[msh(m\Theta_2)ctg\Theta_2 - m^2 ch(m\Theta_2)]dm\} .$$

From Figure 8, one can write

$$a = R\sin\Theta_2 \tag{53}$$

and substituting for k_1 and k_2 from equations (3) and (4) into equations (51) and (52), respectively, one can express the volumetric fluxes, Q_1 and Q_2, as

$$Q_1 = -\frac{\pi R^4}{8\mu_1}\left\{\frac{dp}{dx} - g\rho_1\cos\gamma\right\}F_1 \tag{54}$$

and

$$Q_2 = -\frac{\pi R^4}{8\mu_2}\left\{\frac{dp}{dx} - g\rho_2\cos\gamma\right\}F_2 \tag{55}$$

where the functions F_1 and F_2 are expressed as follows:

$$F_1 \equiv -\frac{1}{\pi}[\Theta_1-\frac{1}{6}\{3+2\sin^2\Theta_2\}\sin 2\Theta_2]-2\sin^4\Theta_2 \; x$$
$$x \int_0^\infty [\mathrm{ctg}\Theta_1 \mathrm{sh}(m\Theta_1)-m\,\mathrm{ch}(m\Theta_1)]\frac{mA_1(m)}{\mathrm{sh}(m\pi)}dm \tag{56}$$

and

$$F_2 \equiv \frac{1}{\pi}[\Theta_2-\frac{1}{6}\{3+2\sin^2\Theta_2\}\sin 2\Theta_2+2\sin^4\Theta_2 \; x$$
$$x \int_0^\infty [\mathrm{ctg}\Theta_2 \mathrm{sh}(m\Theta_2)-m\,\mathrm{ch}(m\Theta_2)]\frac{mA_2(m)}{\mathrm{sh}(m\pi)}dm \; . \tag{57}$$

From the system geometry, the ratio of the area occupied by phase 1 to the total area of the pipe and the ratio of the area occupied by phase 2 to the total area of the pipe can be written as

$$\frac{A_1}{A_T} \equiv (1-\overline{\alpha}) = -\frac{1}{2\pi}[2\Theta_1 - \sin 2\Theta_1] \tag{58}$$

and

$$\frac{A_2}{A_T} \equiv \overline{\alpha} = \frac{1}{2\pi}[2\Theta_2 - \sin 2\Theta_2] \tag{59}$$

Consider the following special cases:

(i) When $\overline{\alpha} = 0$, $\Theta_1 = -\pi$, $\Theta_2 = 0$; then

$$F_1 = 1.0,\ F_2 = 0 \tag{60}$$

(ii) When $\overline{\alpha} = 1$, $\Theta_1 = 0$, $\Theta_2 = \pi$; then

$$F_1 = 0,\ F_2 = 1.0 \tag{61}$$

(iii) When $\overline{\alpha} = 0.5$, $\Theta_1 = \Theta_2 = -\pi/2$; then

$$(F_1)_{\overline{\alpha}=.5} = \frac{1}{2}[1 + \frac{32(\dot{k}_2 - \dot{\mu}_2)}{1+\dot{\mu}_2} \int_0^\infty \frac{m^3 ch^2(\frac{m\pi}{2})\, dm}{sh^3(m\pi)}] \tag{62}$$

and

$$(F_2)_{\overline{\alpha}=.5} = \frac{1}{2}[1 - \frac{32(\dot{k}_2 - \dot{\mu}_2)}{\dot{k}_2(1+\dot{\mu}_2)} \int_0^\infty \frac{m^3 ch^2(\frac{m\pi}{2})\, dm}{sh^3(m\pi)}] \ . \tag{63}$$

Evaluating the integral terms in equations (62) and (63), one obtains

$$(F_1)_{\bar{\alpha}=.5} = \frac{1}{2}[1 + \frac{(\dot{k}_2 - \dot{\mu}_2)}{1+\dot{\mu}_2}(\frac{16-\pi^2}{\pi^2})] \tag{64}$$

and

$$(F_2)_{\bar{\alpha}=.5} = \frac{1}{2}[1 - \frac{(\dot{k}_2 - \dot{\mu}_2)}{\dot{k}_2(1+\dot{\mu}_2)}(\frac{16-\pi^2}{\pi^2})] \; . \tag{65}$$

Therefore, when the flow is horizontal ($\dot{k}_1 = \dot{k}_2 = 1.0$), one gets the following result which is valid for all viscosity ratios, $\dot{\mu}_2$:

$$(F_1)_{\bar{\alpha}=.5} + (F_2)_{\bar{\alpha}=.5} = 1 \tag{66}$$

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