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Ezarchou "

RESONANCES IN TWO-LAYER COILS

A THESIS

Presented to

the Faculty of the Graduate Division

by

Nicholas John Exarchou

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Doctor of Philosophy

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February, 1959

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# RESONANCES IN TWO-LAYER COILS

Approved:

Date Approved by Chairman: March [illegible] 1954

## ACKNOWLEDGMENT

I am very thankful to Dr. B. J. Dasher for having been my thesis advisor and for having oriented my efforts in a certain area of work out of which the present subject emerged. His guidance, advice, and suggestions were indispensable for the satisfactory progress of this research.

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## ABSTRACT

During the course of research on transformer windings it was accidentally discovered that one of two particular types of two-layer coils possesses unusual resonant properties not shared by other coils; this coil has a multitude of resonances in the frequency spectrum. By contrast, the other two-layer type of coil has only one prominent resonance. The number of resonances associated with each one of these two different two-layer coils is not affected by opening or closing the connection between the layers; the latter operation changes the type of resonance existing at a certain frequency. A parallel resonance, for instance, is replaced by a series one that appears at or near the same frequency as before. A mathematical analysis of those resonance phenomena has been made and the results compared with experimental data.

The approximate mathematical analysis of the coil associated with the many resonances results in a wave equation; from it are obtained an expression for the lowest resonant frequency of the coil and other expressions justifying (1) the existence of overvoltages between the layers under certain circumstances, and (2) the pattern of the impedance-frequency curve obtained experimentally.

The analysis of the two-layer coil associated with the single resonance indicates that its behavior is equivalent to that of a series inductance-capacitance lumped-element circuit when the connection between the layers is open, and to that of a parallel inductance-capacitance lumped-element circuit when the connection between the layers is closed.

The analyses of the two-layer coils are restricted to the case where the difference between the diameters of the inner and outer layers is small.

Single-layer air-core coils have an impedance-frequency curve characterized by a portion in which the impedance increases gradually with frequency followed by another portion in which the impedance increases and decreases abruptly without following an orderly pattern.

A survey of resonances in transformer windings under no-load conditions indicates that, in general, although the spectrum of the low-voltage winding has two prominent parallel resonances, the spectrum of the high-voltage winding has only one.

# CHAPTER I

## INTRODUCTION

Coils intended for alternating-current operation are valued mainly for their inductive properties. In constructing a coil, however, it is impossible to impart only inductance to it; the end product will invariably have resistance and capacitance also. The latter property is in essence the overall effect of the infinitely many elementary capacitances that exist between any two points of its conductor material and is referred to as "distributed capacitance."

As expected from theoretical considerations based on the existence of the above properties, coils do display resonances when tested at sufficiently high frequencies. Figures 1, 2, and 4 to 11 show typical experimental impedance-frequency curves of a few different types of air-cored coils.

Single-layer air-core coils have, as in Figures 1 and 2, an impedance curve that increases gradually and linearly for a great portion of this phase, until a certain frequency, which depends on the particular coil, is reached. Beyond that frequency the curve adopts immediately an irregular and seemingly unpredictable pattern.

Four different types of two-layer coils are examined and analyzed. Figure 3 illustrates how each one is formed.

The word "coil" is used here with an extended definition that covers the cases of the two types, type M-open and type S-open, in which the connection between the two layers of turns is open.

A coil of type M has many resonances. If the connection between its layers is closed, resulting in a coil of type M-closed, its first resonance in the frequency spectrum is a parallel one (as in Figs. 4 and 6). If the connection between its layers is open, forming a coil of type M-open, its first resonance is a series one (as in Fig. 5). A coil of type S has a single resonance only. Figures 7 and 8 show the impedance-frequency curves of types S-closed and S-open respectively.

Figures 4 and 5, representing relations for the same two-layer type M coil (No. C-21), show how an impedance-frequency characteristic is altered when the interlayer connection is opened. Parallel resonances are replaced by series ones at almost the same frequencies; series resonances, however, are replaced by parallel resonances at different frequencies.

From an examination of Figures 4 and 6 which show curves, identical in pattern, of type M coils of rectangular and circular cross-sections respectively, it may be surmised that the existence of multiple resonances does not depend on the shape of the cross-section of a type M coil.

Figures 4 and 6 illustrate in turn the low- and high-frequency aspects of the impedance-frequency characteristics of type M-closed coils. In figure 4, at the low-frequency side of the spectrum, a practically linear variation of impedance values can be observed. Such a variation would be expected, at low frequencies, of the impedance of a coil having negligible resistance. Figure 6 shows the typical high-frequency leveling-off of the impedance curves of type M coils.

That the existence of the many resonances in two-layer coils does not depend on a critical placement of the two layers with respect to each other was ascertained with the following test: a special coil of type M-closed was constructed having two layers of turns of circular cross-sections, of the same length, but each one wound on a separate support; the inner one was of considerably smaller diameter than the outer one so as to be capable of occupying many different positions fully or partly inside the other one. The multiple-resonance pattern never failed to be observed as successive tests were performed for each one of the many different positions of the inner layer relatively to the other one. Three of the previous many positions are of interest; they are: the inner coil is entirely outside the outer one, their axes being approximately parallel; the inner layer is eccentrically located with respect to the outer one; and, the axis of the inner layer is at an angle with that of the other one.

Figures 9, 10, and 11 show the impedance-frequency curves of three-, four-, and six-layer air-core coils. Each one has only one prominent parallel resonance. It appears from this established trend that the single-resonance pattern is typical of all coils having three or more layers of turns.

Electrical engineering textbooks and handbooks describe how distributed capacitance arises in coils and they discuss the relative merits of a few ways of arranging turns for producing a minimum overall capacitance effect. They usually refer to only one resonance in coils and they present as an equivalent circuit for their representation an inductor and capacitor in parallel. No reference can be found in these works to the fact that some two-layer coils have many resonances.

In connection with methods of analyzing coils, the following two articles are noteworthy: the one by Blume and Boyajian (1) which offers a theory for single-layer coils on an iron-core, and the other by Rudenberg (2) which presents a theory for pancake-type coils.

The sequence of figures 1, 2, and 4 to 11 displays eloquently the general properties of layer-type coils. It may be pointed out, in conclusion, that the properties of two-layer type M coils are unique in resembling those of a two-parallel-wire transmission line.

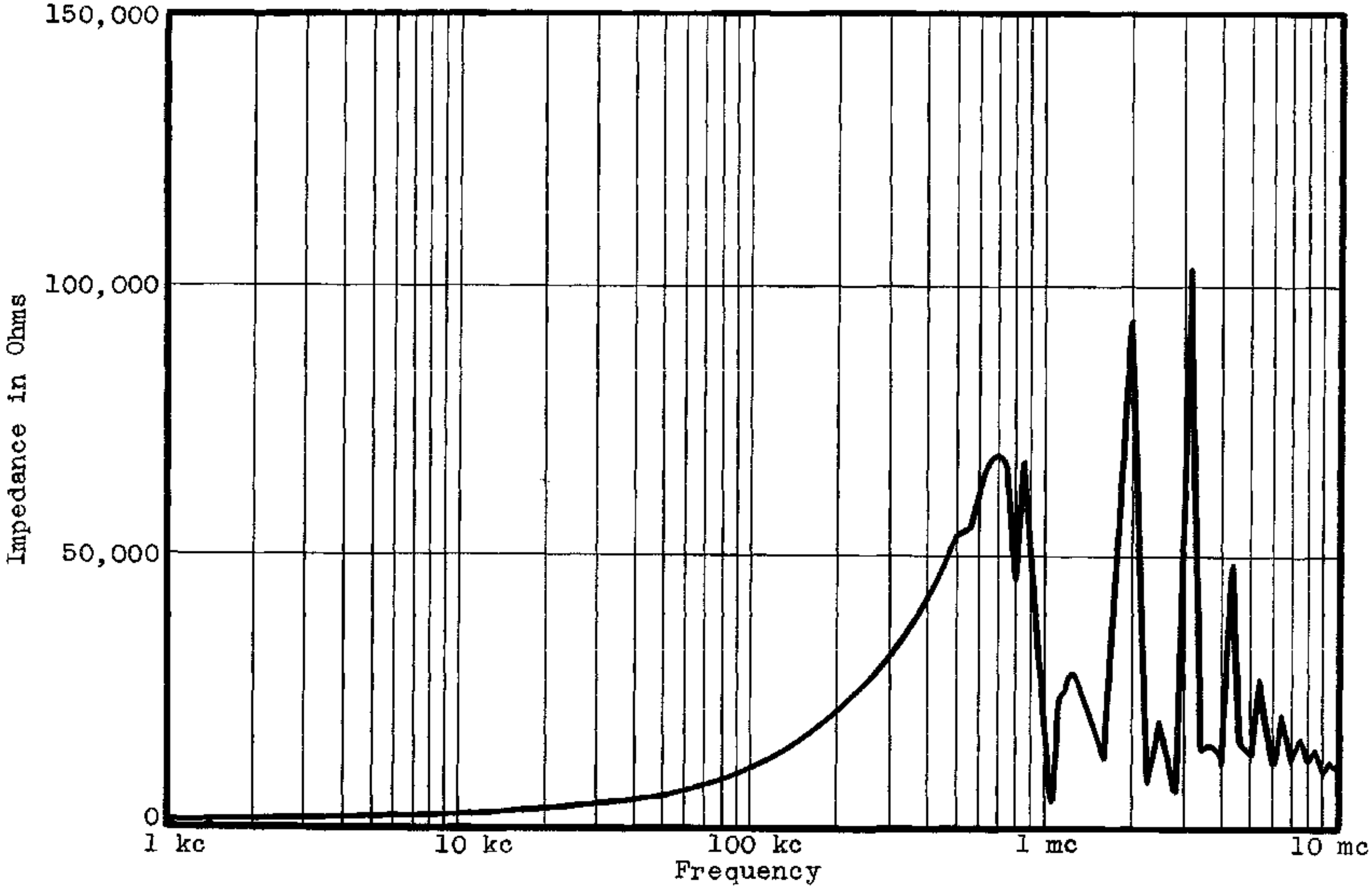



Figure 1. Impedance-frequency Characteristic of
Single-layer Air-core Coil No. C-1-25.

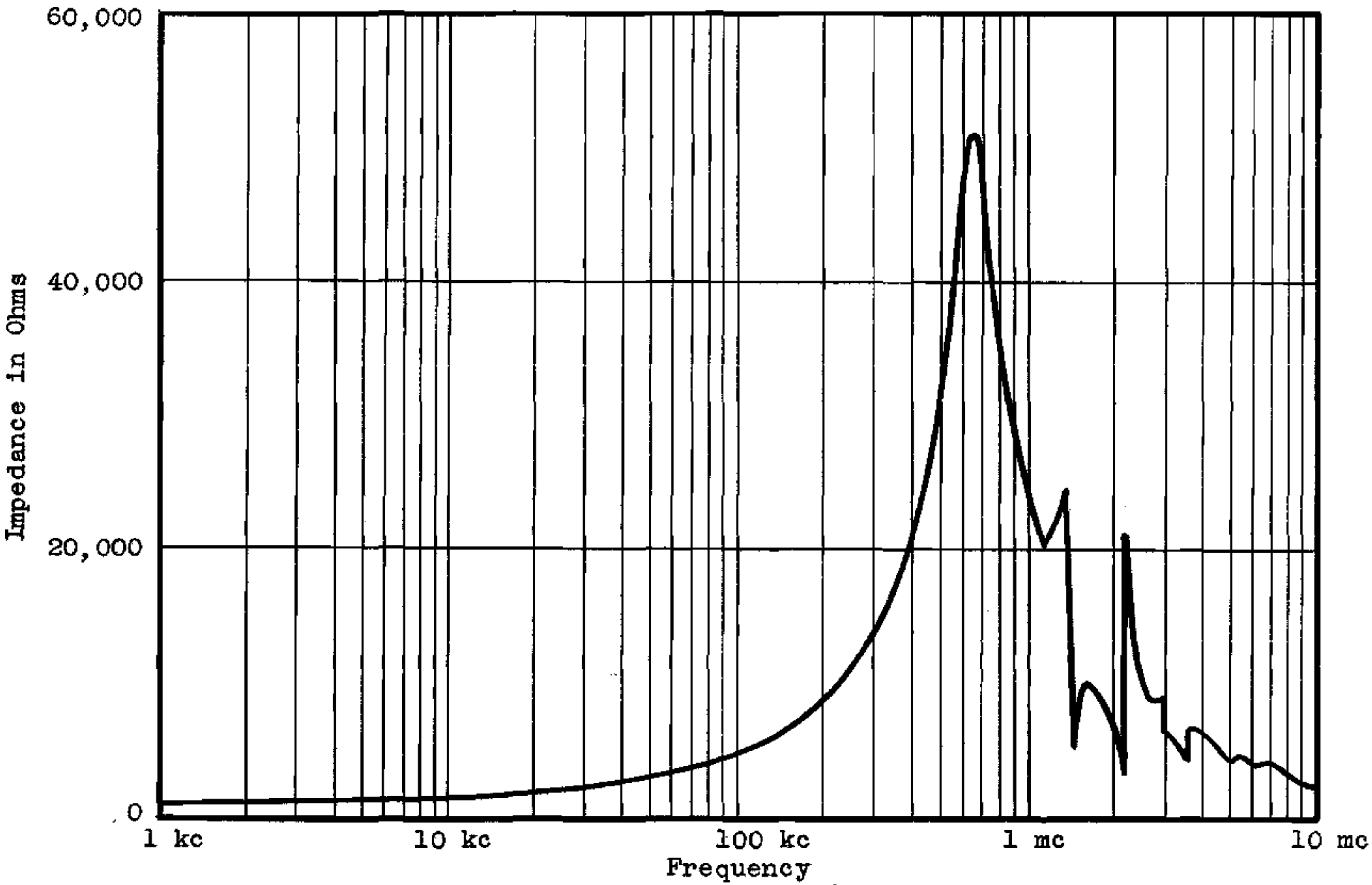
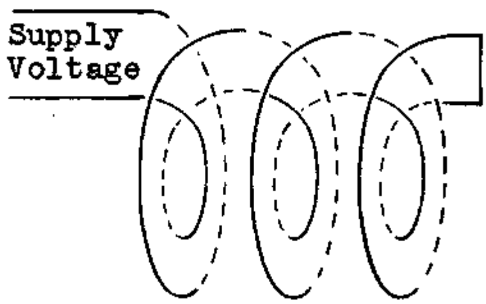
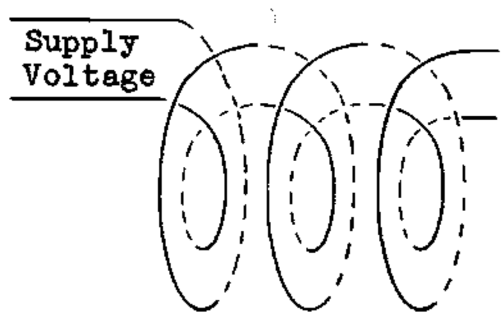


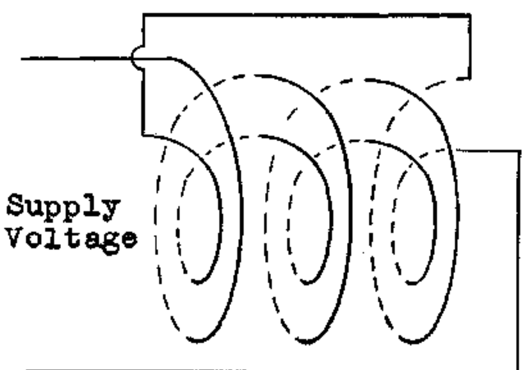

Figure 2. Impedance-frequency Characteristic of
Single-layer Air-core Coil No. C-1-120.



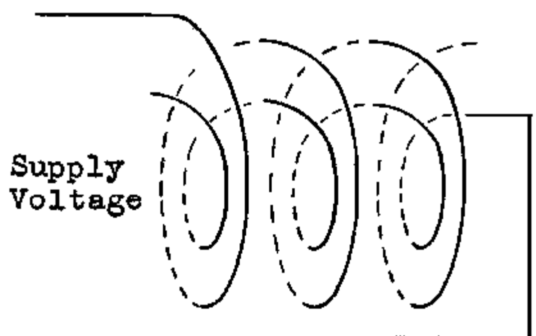

Type M-closed




Type M-open




Type S-closed




Type S-open

Figure 3. Different Types of Two-layer Coils.

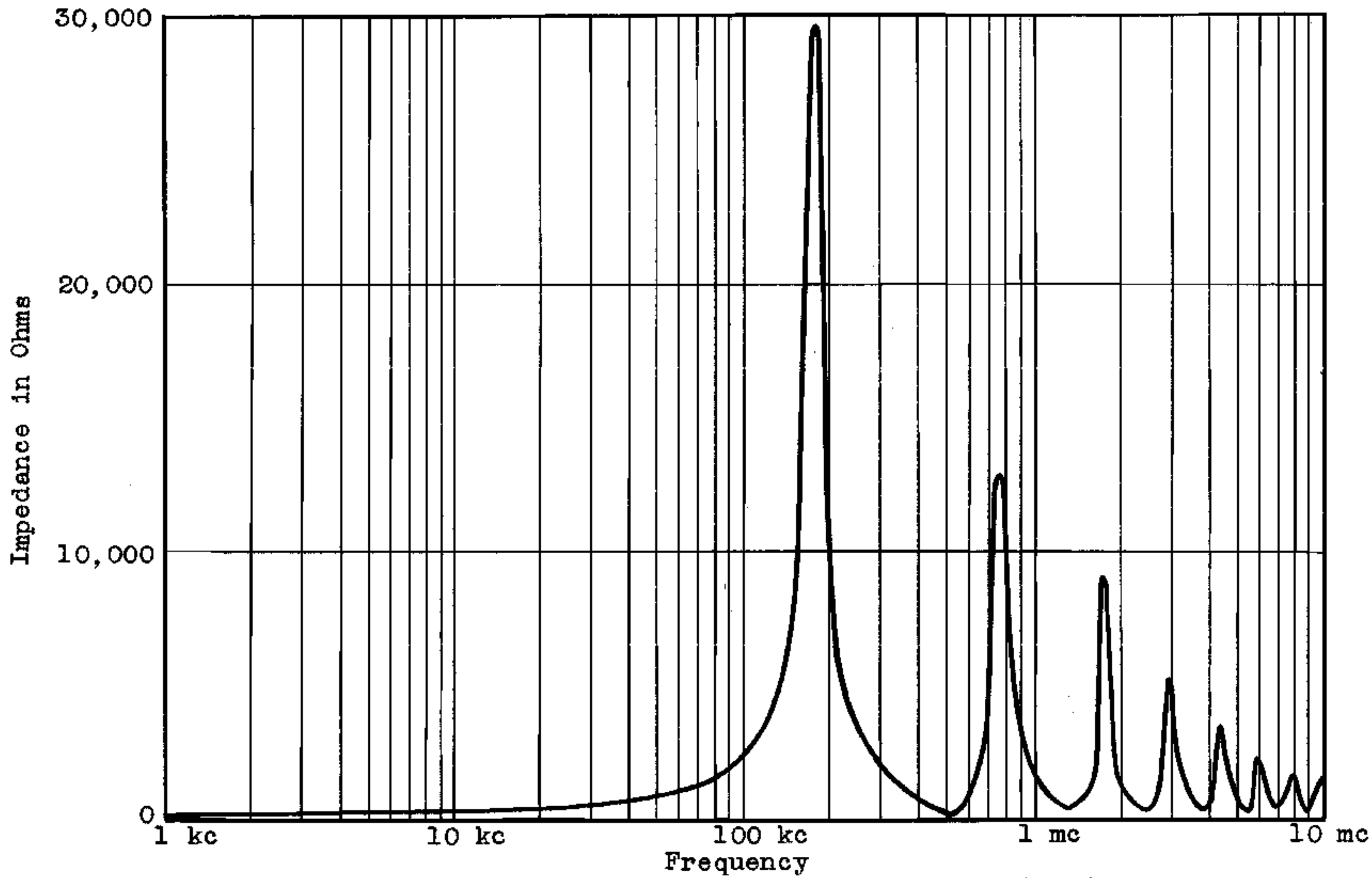



Figure 4. Impedance-frequency Characteristic of Two-layer Type M-closed Air-core Coil No. C-21.

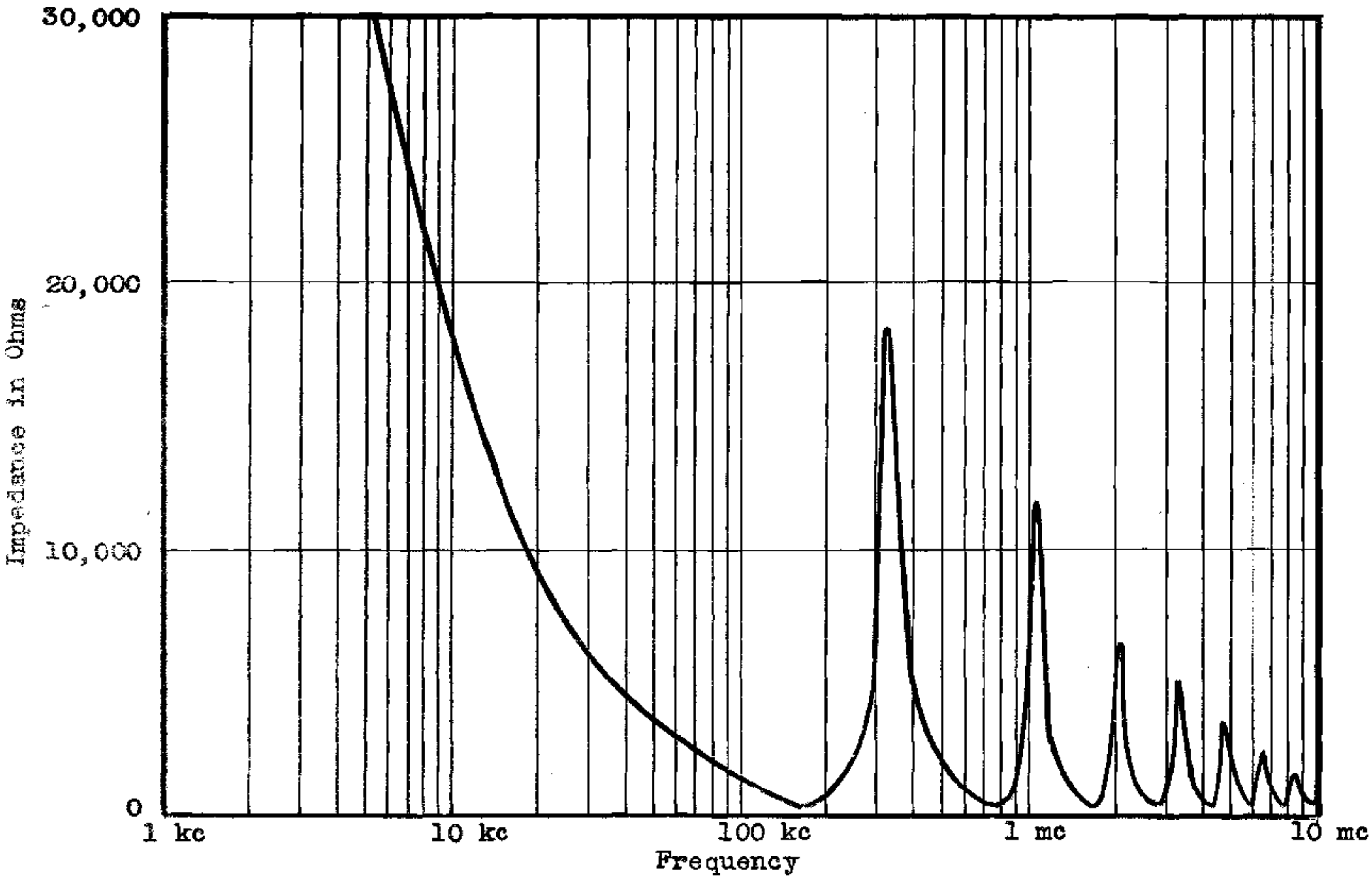



**Figure 5. Impedance-frequency Characteristic of Two-layer Type M-open Air-core Coil No. C-21.**

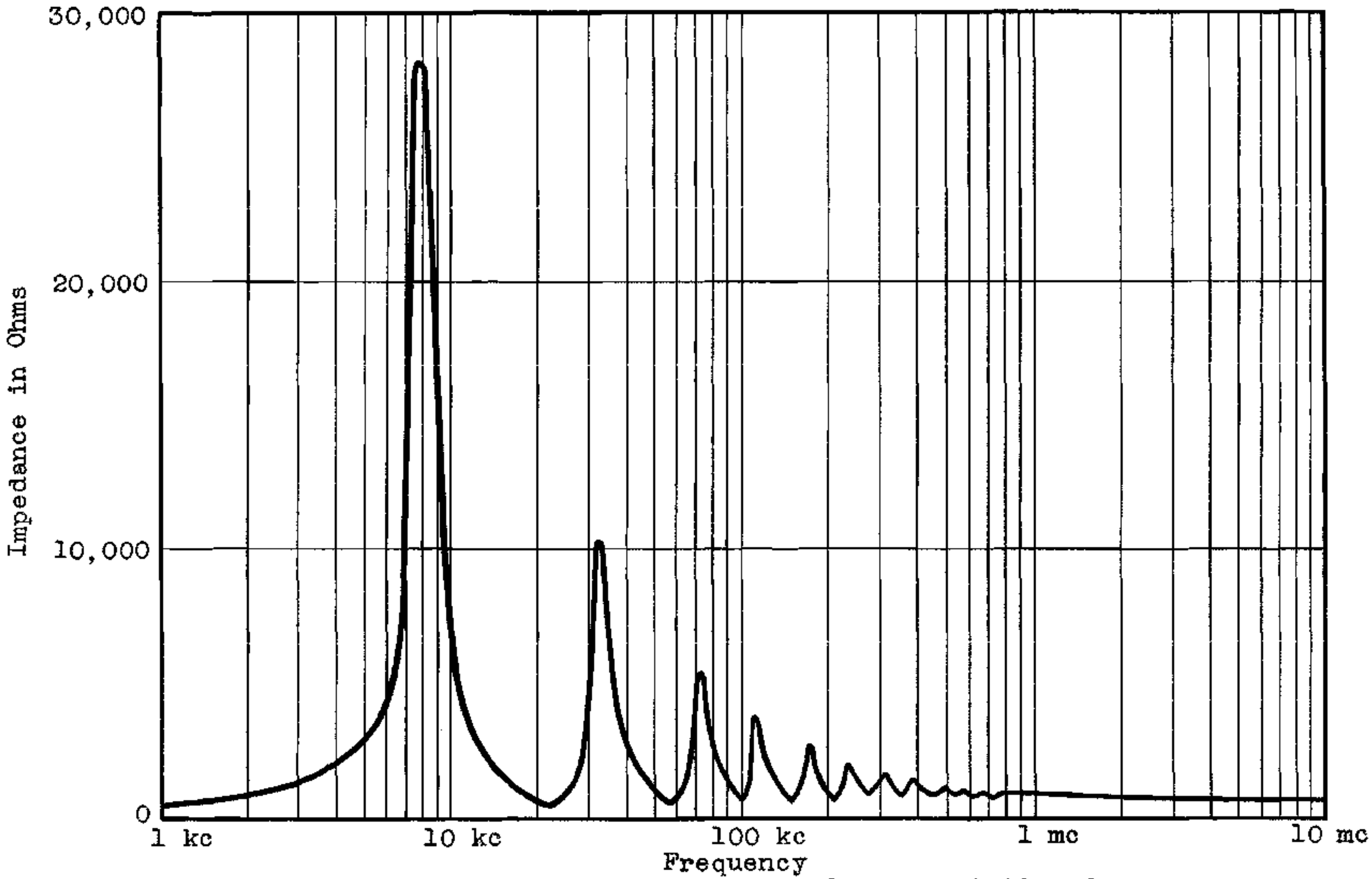



Figure 6. Impedance-frequency Characteristic of Two-layer Type M-closed Air-core Coil No. C-25.

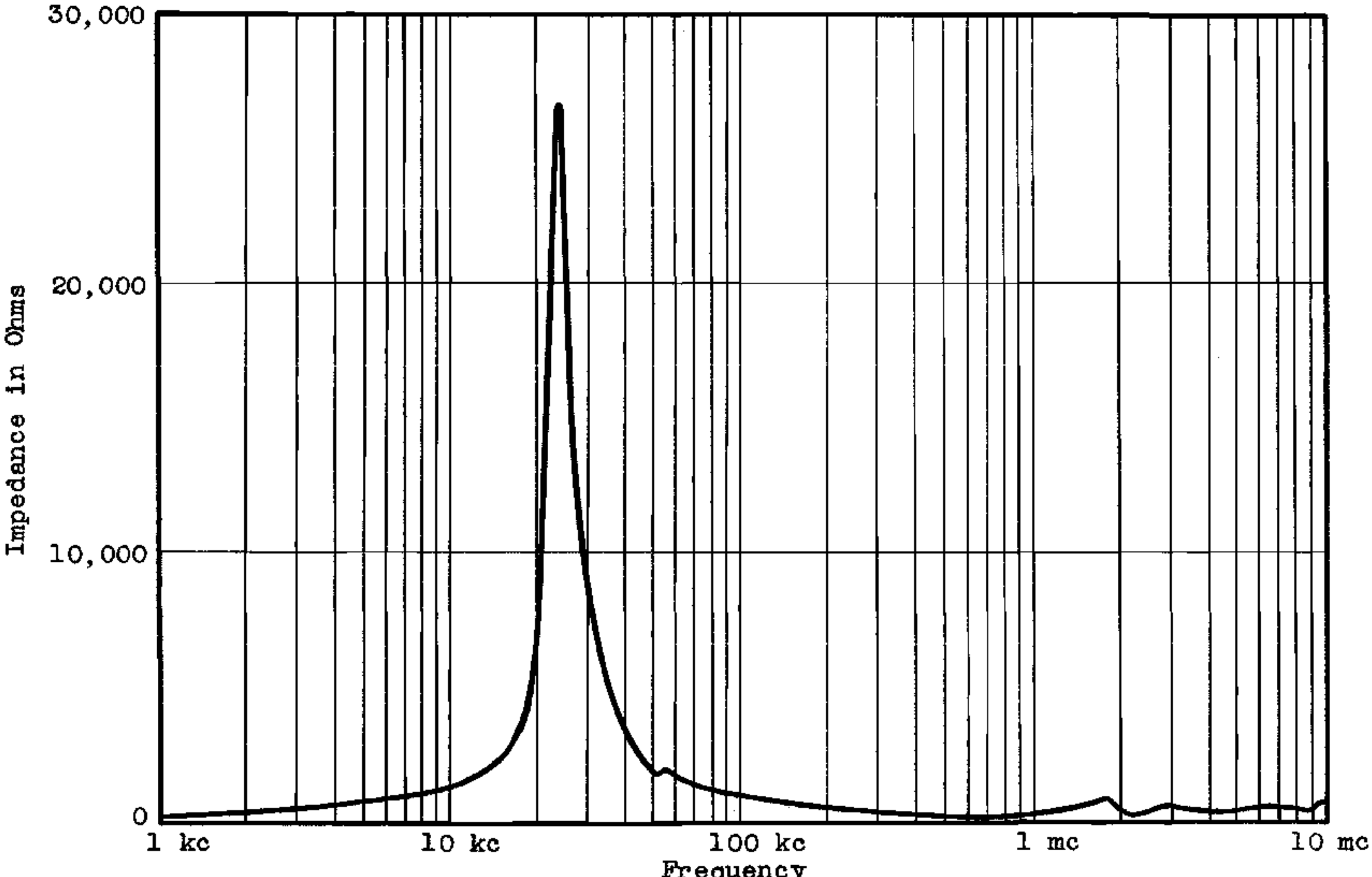



Figure 7. Impedance-frequency Characteristic of Two-layer Type S-closed Air-core Coil No. C-122.

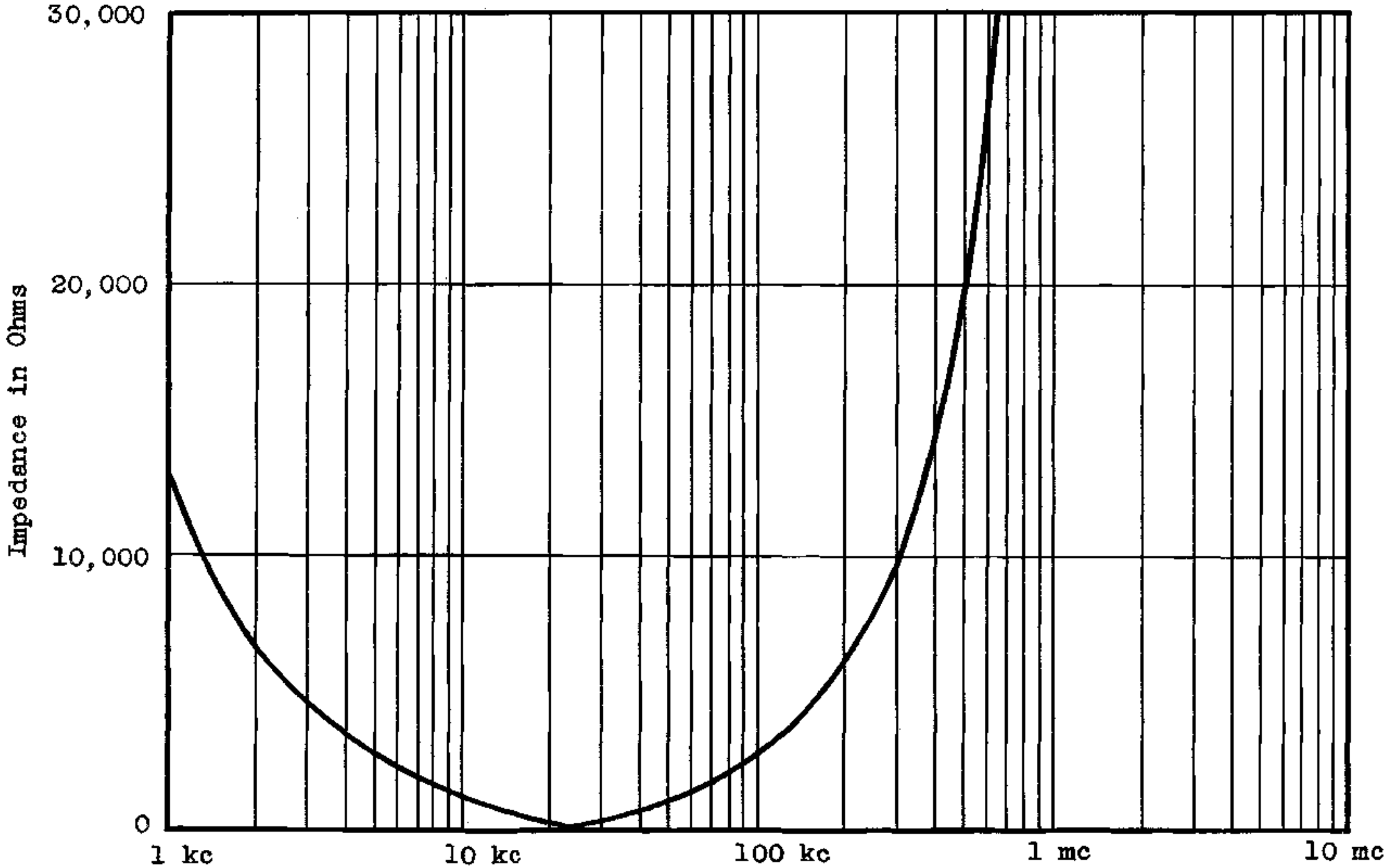


Figure 8. Impedance-frequency Characteristic of
Two-layer Type S-open Air-core Coil No. C-122.

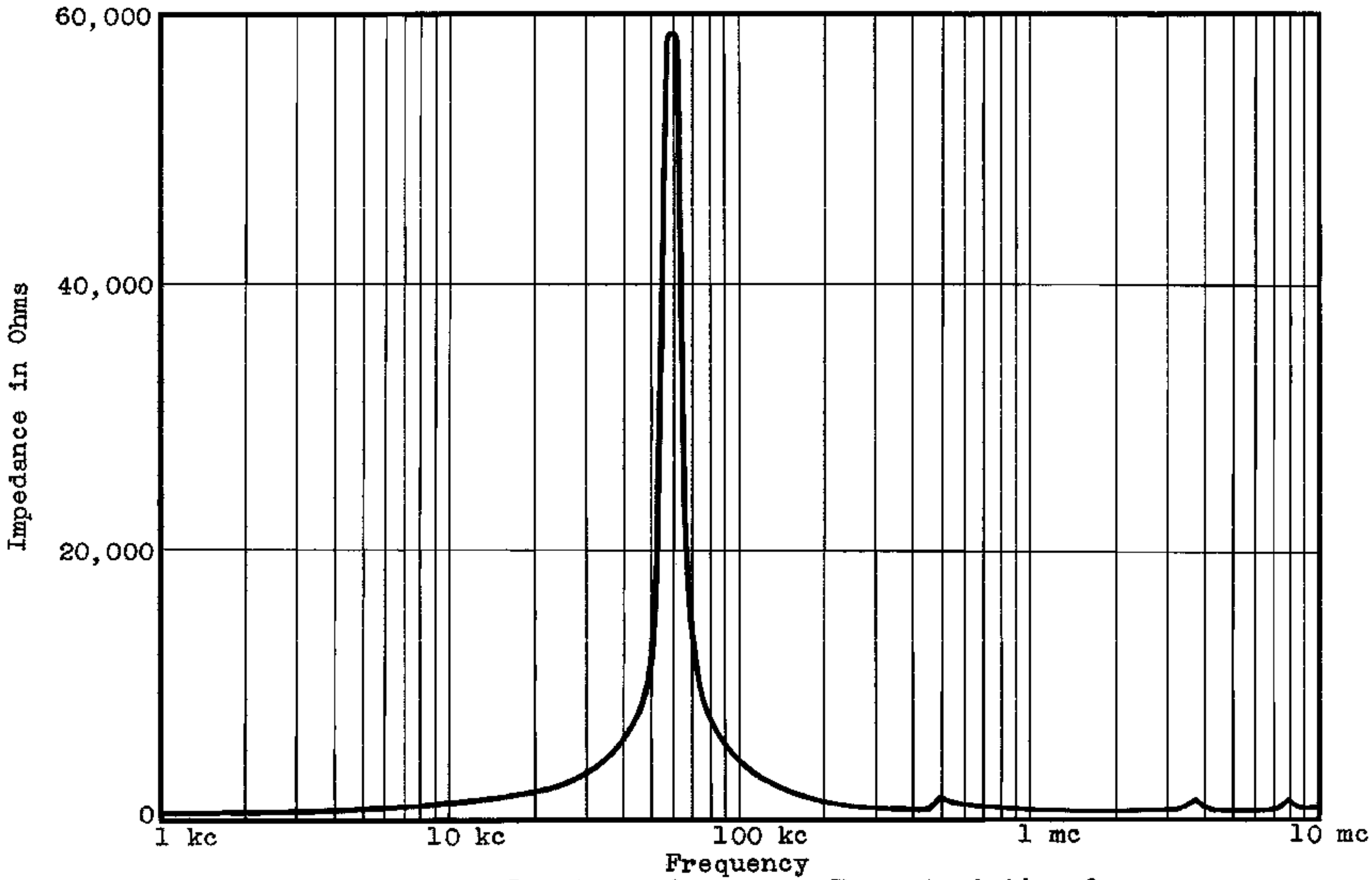



Figure 9. Impedance-frequency Characteristic of Three-layer Air-core Coil No. C-3-18.

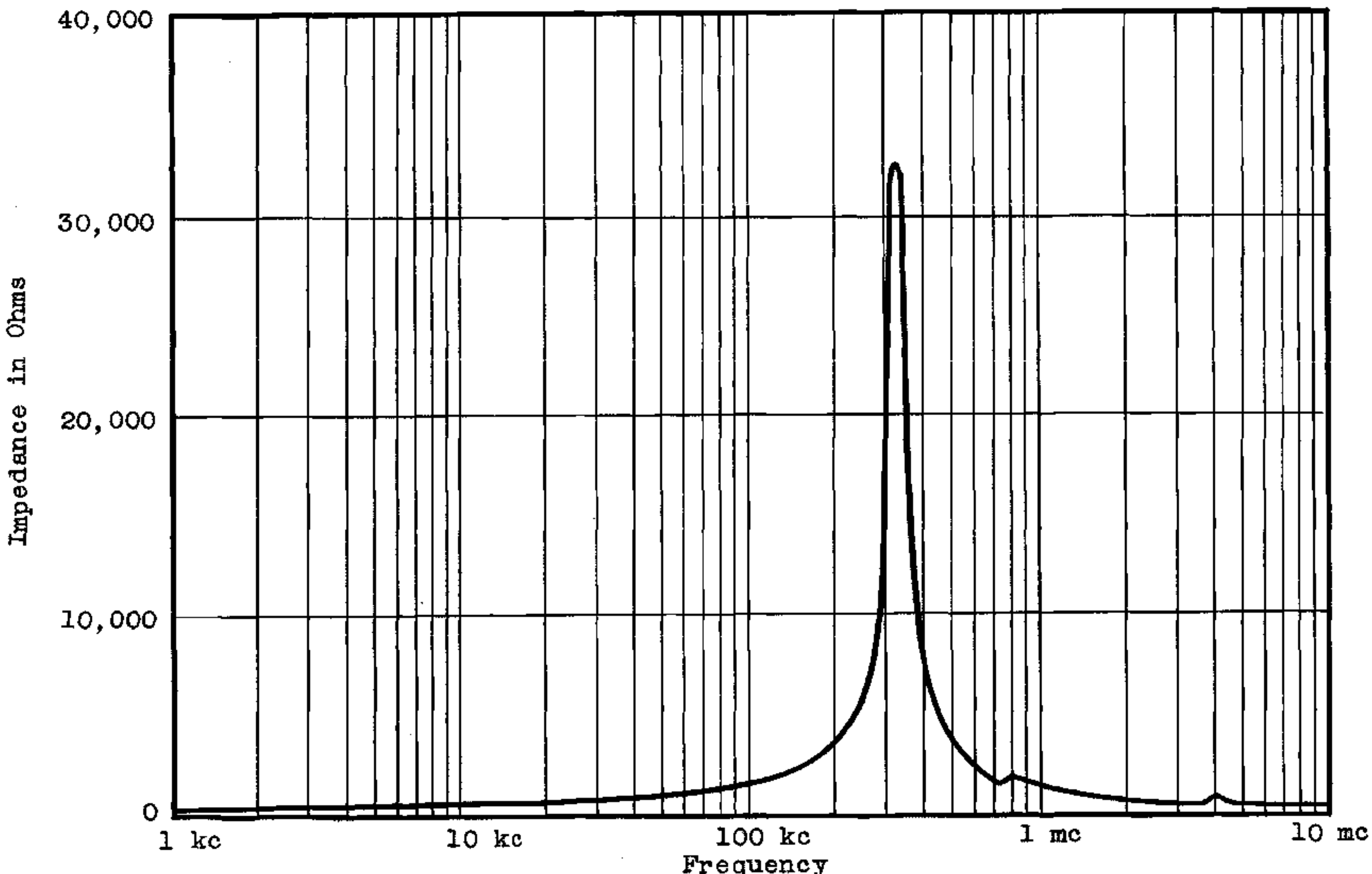



Figure 10. Impedance-frequency Characteristic of
Four-layer Air-core Coil No. C-4-17.

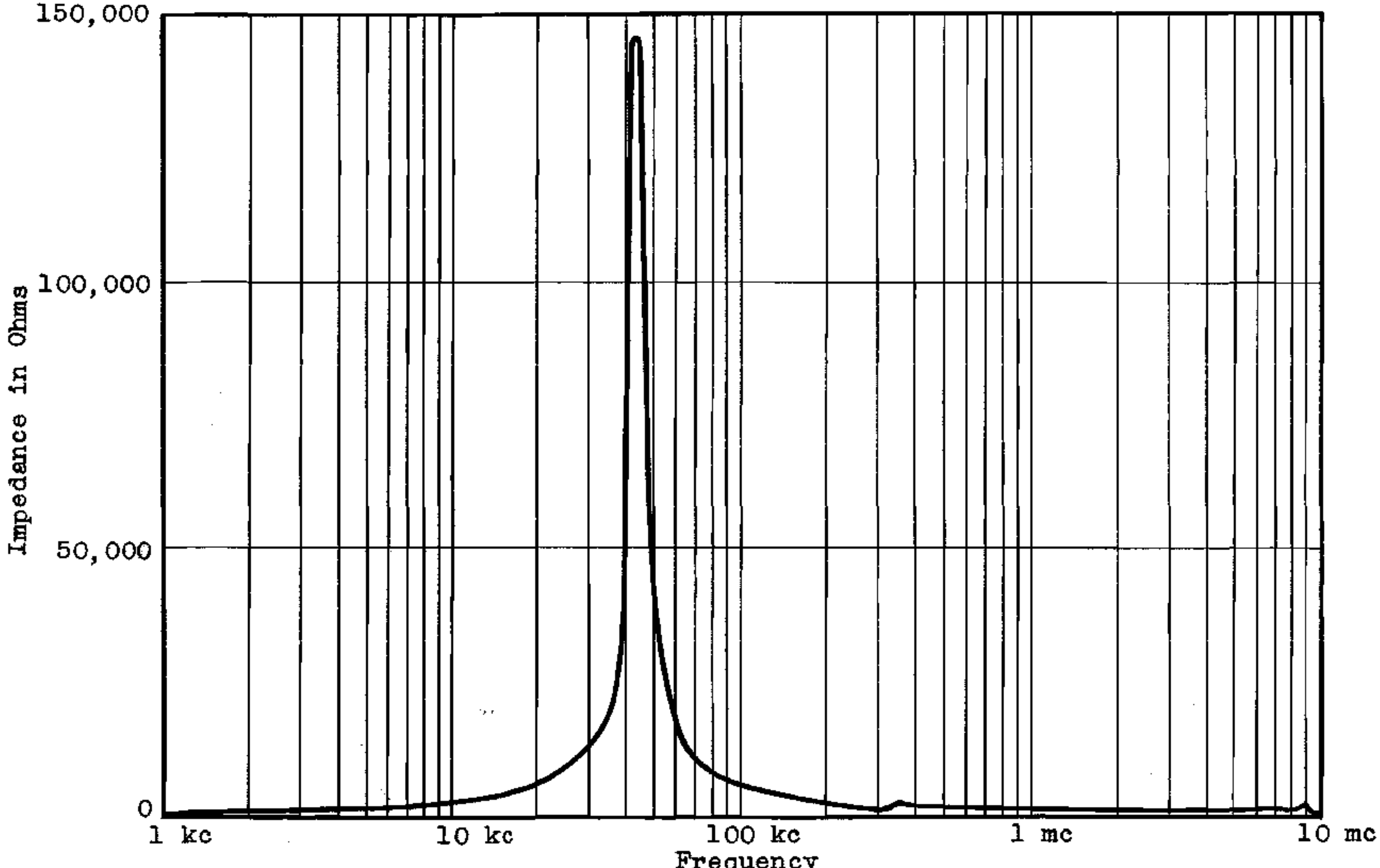



Figure 11. Impedance-frequency Characteristic of
Six-layer Air-core Coil No. C-6-11.

# CHAPTER II

## METHOD OF MEASURING IMPEDANCES

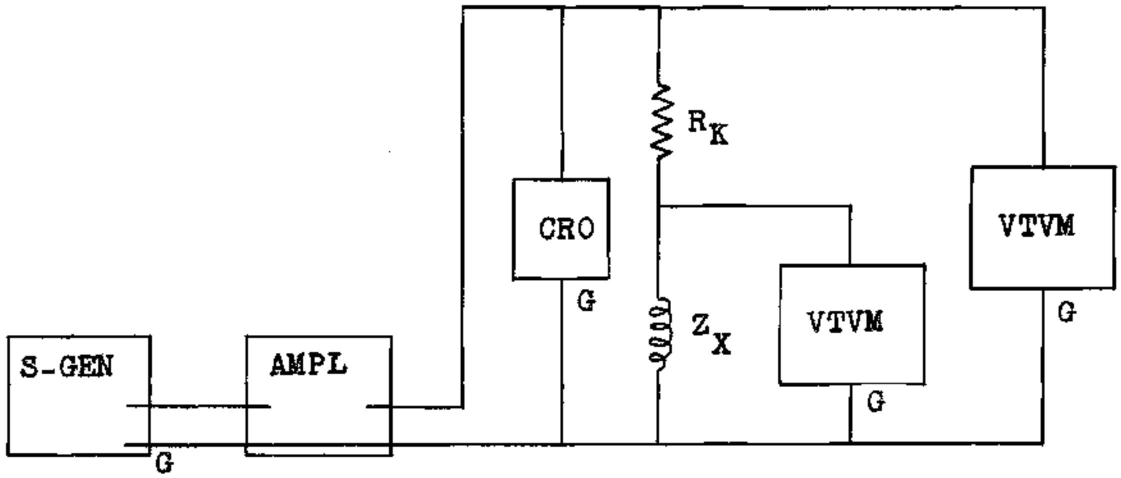
The impedances that were measured were those of coils and transformer windings. The measurements were extended to the frequency of 10 mc.

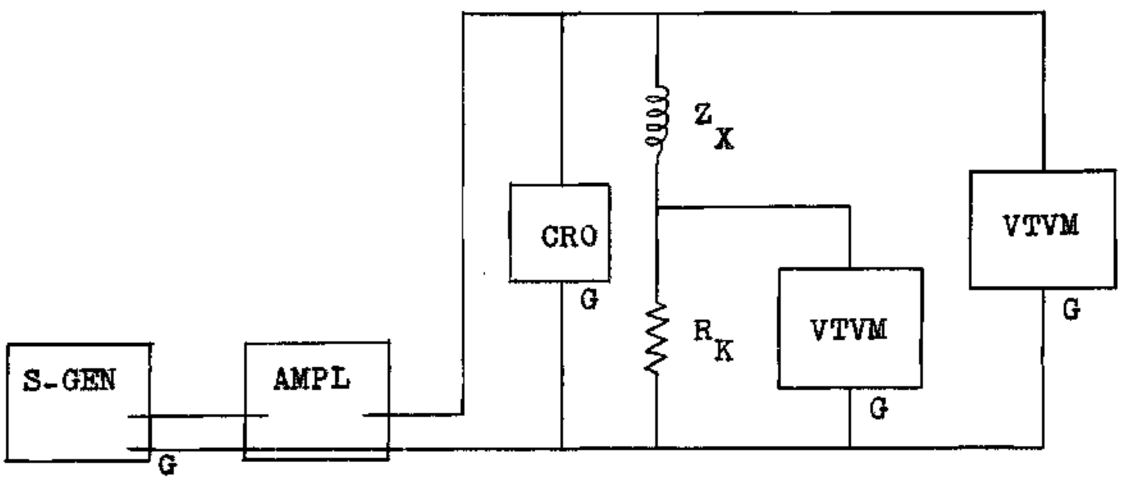
The measuring method employed is illustrated in Figure 12. The procedure was as follows: the unknown impedance was placed in series with a pure resistor of known value and this combination of elements was connected across the amplified output (see Fig. 14 for wiring diagram of wide band amplifier) of a signal generator. The voltage across the two-element circuit was kept at 15 volts rms for all frequencies below 5 mc and at not less than 7 volts rms for the remaining frequencies of the spectrum. An oscilloscope was connected across the output of the amplifier for ensuring through visual checks that the voltage remained at all times sinusoidal as desired. Two vacuum tube voltmeters were used: one was always connected across both the resistance and the unknown impedance; the other was connected across the element, resistance or unknown impedance, having one of its terminals connected to the common "ground" side of the circuit. When voltage readings had been taken for the entire frequency spectrum, the positions of the two elements were interchanged and a new sequence of readings was taken, throughout the spectrum, of the voltage across the other element. In this way the three "ground" terminals of the three voltmeters and of the signal generator were always connected to each other. The unknown impedance was determined from the relation

$$\text{impedance in ohms} = \left(\text{value of known resistor in ohms}\right) \cdot \frac{\text{voltage across impedance}}{\text{voltage across resistor}}$$

The necessity of using such an indirect method arises from the fact that when the three "ground" terminals of the three instruments are not connected to a common point, the distributed capacitances of their power transformers introduce into the circuit unwanted electromotive forces that alter the normal distribution of voltages between the two elements. This difficulty would not arise if the three instruments did not derive their power from the same supply. Figure 13 shows four different improper methods of connecting the voltmeters in the circuit; the gravity of the error of using them cannot be overemphasized.

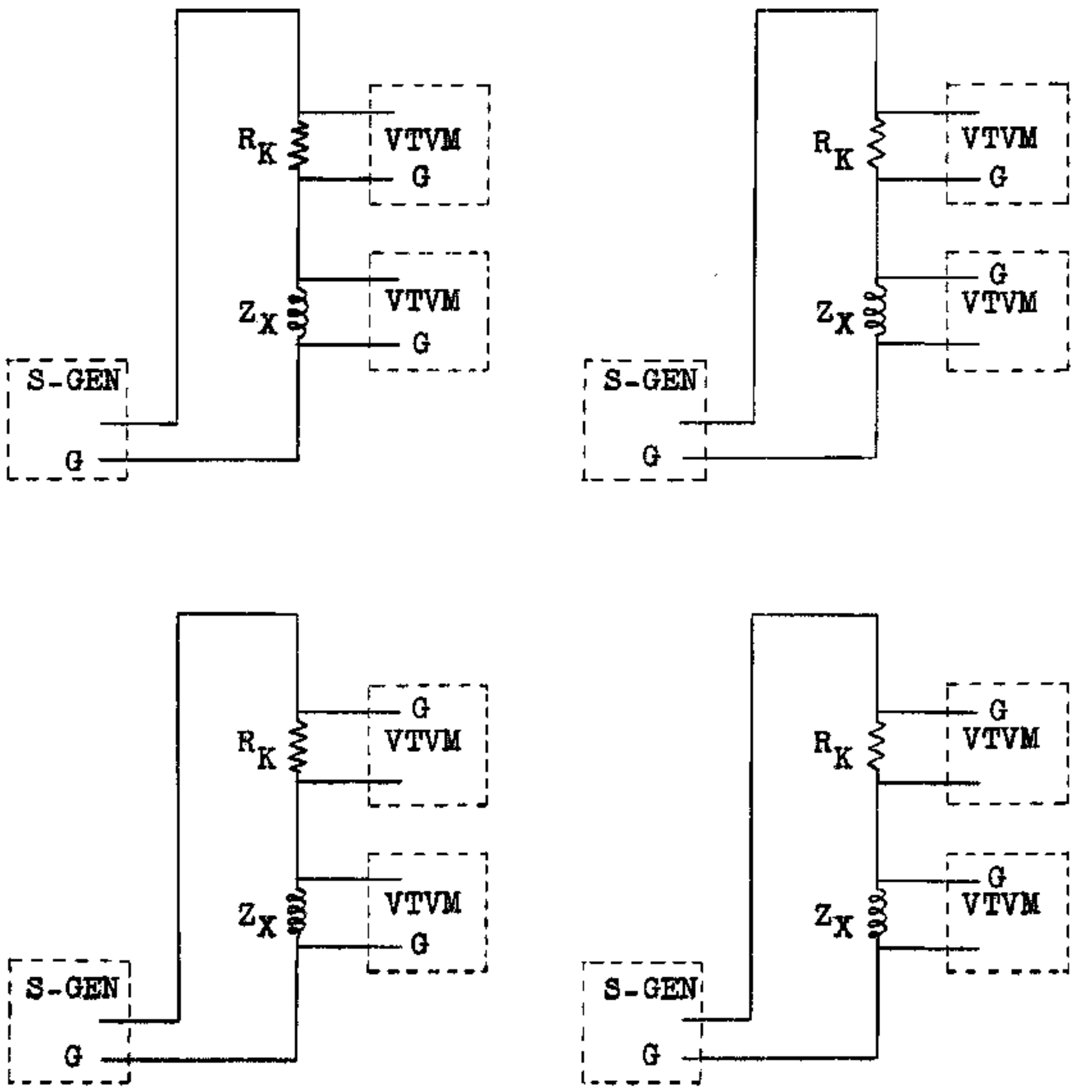
It may be added that since three voltage readings were obtained at each frequency, though with two voltmeters only and a transposition of elements, the result was equivalent to having used the "three-voltmeter method." From the data collected not only the absolute values of the impedances could be computed but also their phase angles.




| | | |
|---|---|---|
| S-GEN | : | Signal Generator |
| $Z_X$ | : | Unknown Impedance |
| $R_K$ | : | Resistor of Known Value |
| VTVM | : | Vacuum Tube Voltmeter |
| G | : | Ground Terminal of Electronic Instrument |
| AMPL | : | Amplifier |

Figure 12. Illustration of Method of Measuring Impedances.




S-GEN: Signal Generator

$Z_X$ : Unknown Impedance

$R_K$ : Resistor of Known Value

VTVM : Vacuum Tube Voltmeter

G : Ground Terminal of Electronic Instrument

Figure 13. Incorrect Methods of Using Vacuum Tube Voltmeters and Signal Generator.

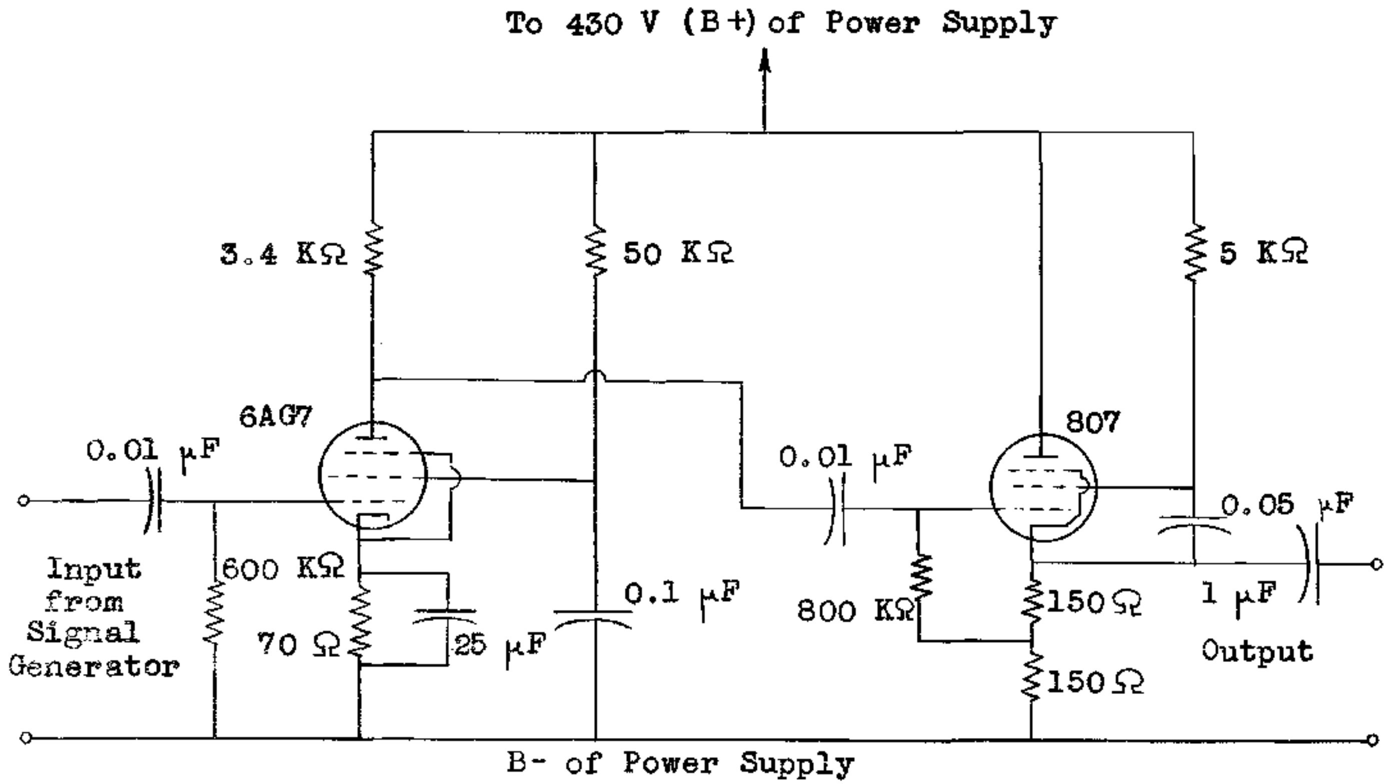



Figure 14. Amplifier Circuit.

# CHAPTER III

## ANALYSIS OF TWO-LAYER TYPE M COILS

The symbols that appear herein are defined in a special section of the Appendix entitled "List and Definitions of Symbols." Figure 15 is to be consulted for a visualization of the present situation.

Simplified analysis with resistance neglected.--The differential equation that describes approximately the behavior of two-layer type M coils is derived by two different methods. The turn-to-turn in a lengthwise direction capacitance is neglected because tests on the inner layer alone of a two-layer coil, before the addition of the second layer, indicate that the almost linear behavior, and therefore free of capacitive effects, of the impedance of this single layer is maintained throughout the span of frequencies in which the multiple resonances of the completed coil are manifested. The results appearing in Figures 1 and 6 are one example of the validity of the above assumption.

The first method for the derivation of the differential equation may be called "basic method" because it uses as starting points for the calculations nonspecialized expressions of electric laws. If Kirchhoff's law is used around a loop from one end of the supply, along the outer

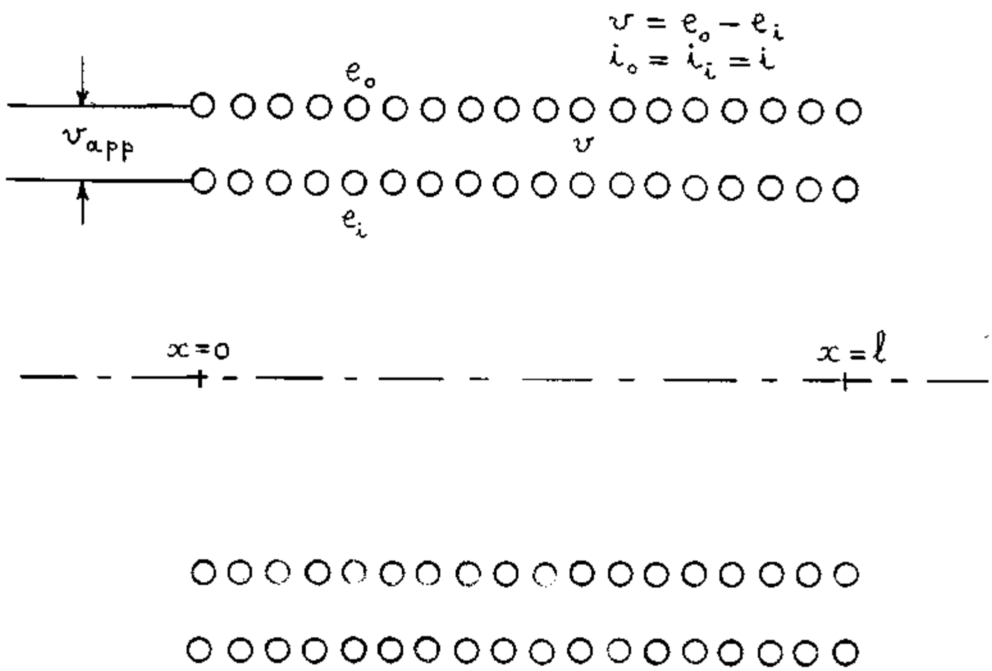



Figure 15. Section of a Two-layer Type M Coil.

layer of the coil, through the interlayer space, along the inner layer, and back to the other end of the supply, the following can be written:

$$v_{app} = \int_x^0 \frac{\partial e_o}{\partial x}\,dx + v + \int_0^x \frac{\partial e_i}{\partial x}\,dx$$

$$v_{app} = \int_x^0 \frac{\partial e_o}{\partial x}\,dx + v - \int_x^0 \frac{\partial e_i}{\partial x}\,dx$$

Since the voltage drops along two adjacent portions of the layers are equal and opposite in phase, it is written that

$$\int_x^0 \frac{\partial e_o}{\partial x}\,dx = -\int_x^0 \frac{\partial e_i}{\partial x}\,dx$$

and

$$v_{app} = 2\int_x^0 \frac{\partial e_o}{\partial x}\,dx + v \tag{1}$$

Faraday's law of induction must be used now. In preparation, the flux-current relation needs to be specified:

$$\phi = \frac{\text{magnetomotive force}}{\text{reluctance}} = \frac{i\,2n\,dx}{\frac{dx}{\mu A}} = 2n\mu A i \tag{2}$$

No distinction between the currents in the two layers was made because

$$i_i = i_o = i$$

It was assumed in equation (2) that the reluctance of each elementary coil partition $dx$ is concentrated in the space $dx$

in the inside of the coil.

If equation (2) at first is differentiated with respect to time and then used into Faraday's law, the following sequence of results ensues:

$$\frac{\partial \phi}{\partial t} = 2n\mu A \frac{\partial i}{\partial t}$$

$$\frac{\partial e_o}{\partial x} dx = -n\,dx \frac{\partial \phi}{\partial t} = -2n^2\mu A \frac{\partial i}{\partial t} dx$$

$$\frac{\partial e_o}{\partial x} = -2n^2\mu A \frac{\partial i}{\partial t} \tag{3}$$

If equation (1) is differentiated with respect to distance, there is obtained

$$\frac{d(v_{app})}{dx} = -2\frac{\partial e_o}{\partial x} + \frac{\partial v}{\partial x} = 0$$

The negative sign is due to the fact that $x$ is the lower limit of integration. The zero occurs because the applied voltage is not a function of the position $x$. From the last expression it follows that

$$\frac{\partial v}{\partial x} = 2\frac{\partial e_o}{\partial x} \tag{4}$$

If equation (3) is substituted into equation (4) it is found that

$$\frac{\partial v}{\partial x} = -4n^2\mu A \frac{\partial i}{\partial t} \tag{5}$$

A relation involving the displacement current between the layers is

$$\frac{\partial i}{\partial x}\,dx = -C\,dx\,\frac{\partial v}{\partial t}$$

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \tag{6}$$

A manipulation of equations (5) and (6) gives

$$\frac{\partial^2 v}{\partial x^2} = 4n^2\mu A C\frac{\partial^2 v}{\partial t^2} \tag{7}$$

which is the one-dimensional form of the wave equation.

In the second method advantage is taken of the similarity that exists between a two-wire transmission line and a two-layer coil. This method may be called "transmission-line method" and it yields the desired result, equation (7), quickly.

The inductance to be used here is obtained from an approximate relation for the inductance of a coil which is

$$L_T = \frac{N^2\mu A}{\ell} = \frac{(2n\ell)^2\mu A}{\ell} = 4n^2\ell\mu A$$

By differentiation of $L_T$ there is obtained:

$$\frac{dL_T}{d\ell} = 4n^2\mu A$$

$\frac{dL_T}{d\ell}$ can be written as $\frac{dL_T}{dx}$; thus:

$$\frac{dL_T}{dx} = 4n^2\mu A \tag{8}$$

The change in the voltage between the layers for two positions spaced by $dx$ is

$$-\frac{\partial v}{\partial x}dx = \frac{\partial i}{\partial t}dL_T = \frac{\partial i}{\partial t}4n^2\mu A dx$$

and finally

$$\frac{\partial v}{\partial x} = -4n^2\mu A\frac{\partial i}{\partial t} \tag{5}$$

which is the same result that was found previously. Equation (6) is applicable again and, when used in conjunction with equation (5), the wave equation (7) is obtained once more:

$$\frac{\partial^2 v}{\partial x^2} = 4n^2\mu AC\frac{\partial^2 v}{\partial t^2} \tag{7}$$

In the first method the need for deciding what inductance to use was by-passed by writing equation (2). In essence, however, the same assumption, concerning the choice of reluctance path for the flux, was made both times. The approach followed in the first method can be applied to the cases of the coils of types S-open and S-closed.

Solution of equation (7).--If a harmonic variation with respect to time is assumed for the voltage, then

$$v = V_m \sin\omega t$$

and

$$\frac{\partial^2 v}{\partial t^2} = -\omega^2 V_m \sin\omega t \tag{9}$$

If equation (9) is substituted into equation (7), there is obtained:

$$\frac{\partial^2(V_m \sin\omega t)}{\partial x^2} = \sqrt{2}\sin\omega t\,\frac{d^2V}{dx^2} = 4n^2\mu A(-\omega^2)\sqrt{2}\,V\sin\omega t$$

$$\frac{d^2V}{dx^2} + 4\omega^2 n^2 \mu A C V = 0 \qquad (10)$$

For convenience a new constant $k$ is defined as follows:

$$k^2 = 4\omega^2 n^2 \mu A C \qquad (11)$$

The general solution is

$$V = B_1 \cos kx + B_2 \sin kx \qquad (12)$$

The boundary conditions for the case of the type M-closed coil are

at $x = 0$, $V = V_s$ and
at $x = \ell$, $V = 0$

Therefore

$$B_1 = V_s \qquad B_2 = -\frac{V_s}{\tan k\ell} \qquad (13)$$

and

$$V = V_s \cos kx - \frac{V_s}{\tan k\ell}\sin kx \qquad (14)$$

According to the above equation the voltage between layers at different points along the length of the coil is either in phase or out of phase by 180°; this solution has been arrived at by neglecting R; if R had been taken into account the phase of the voltage would have changed gradually along the length of the coil.

It has been observed during this research that at the first parallel resonance the voltage distribution along the coil is approximately a cosine curve.

The frequency of the first parallel resonance can be obtained by setting

$$k\ell = \frac{\pi}{2}$$

$$2\omega n\sqrt{\mu AC}\,\ell = \frac{\pi}{2}$$

Thus

$$\omega = \frac{\pi}{4n\ell\sqrt{\mu AC}}$$

and

$$f = \frac{1}{8n\ell\sqrt{\mu AC}} \qquad (15)$$

Since $$L_T = 4n^2\ell\mu A \qquad (16)$$

and $$C_T = C\ell \qquad (17)$$

$$f = \frac{1}{4\sqrt{L_T C_T}} \qquad (18)$$

Equation (15) indicates that the frequency varies inversely as the coil length; in view of the simplifying assumptions that were made in this analysis, the previous statement is true only for coils having a large ratio of coil length to coil diameter. Figure 16 shows a relation that was obtained experimentally between "first resonant frequency" and "coil length."

If a coil of type M-open is considered, the same equation (18) will result for the first resonant frequency which, in this case, will be for a series resonance.

At low frequencies the voltage between the layers is known to vary linearly with length; the linear variation can be obtained as a special case of the general solution

$$V(x) = V_s \cos kx - \frac{V_s}{\tan kl} \sin kx \tag{14}$$

When the frequency is small the following approximations hold true:

$$\cos kx \cong 1$$
$$\sin kx \cong kx$$
$$\tan kl \cong kl$$

Then $V(x)$ becomes

$$V(x) = V_s \cdot 1 - \frac{V_s}{kl} kx$$

$$V(x) = V_s \left(1 - \frac{x}{l}\right) \tag{19}$$

which is a straight-line relation.

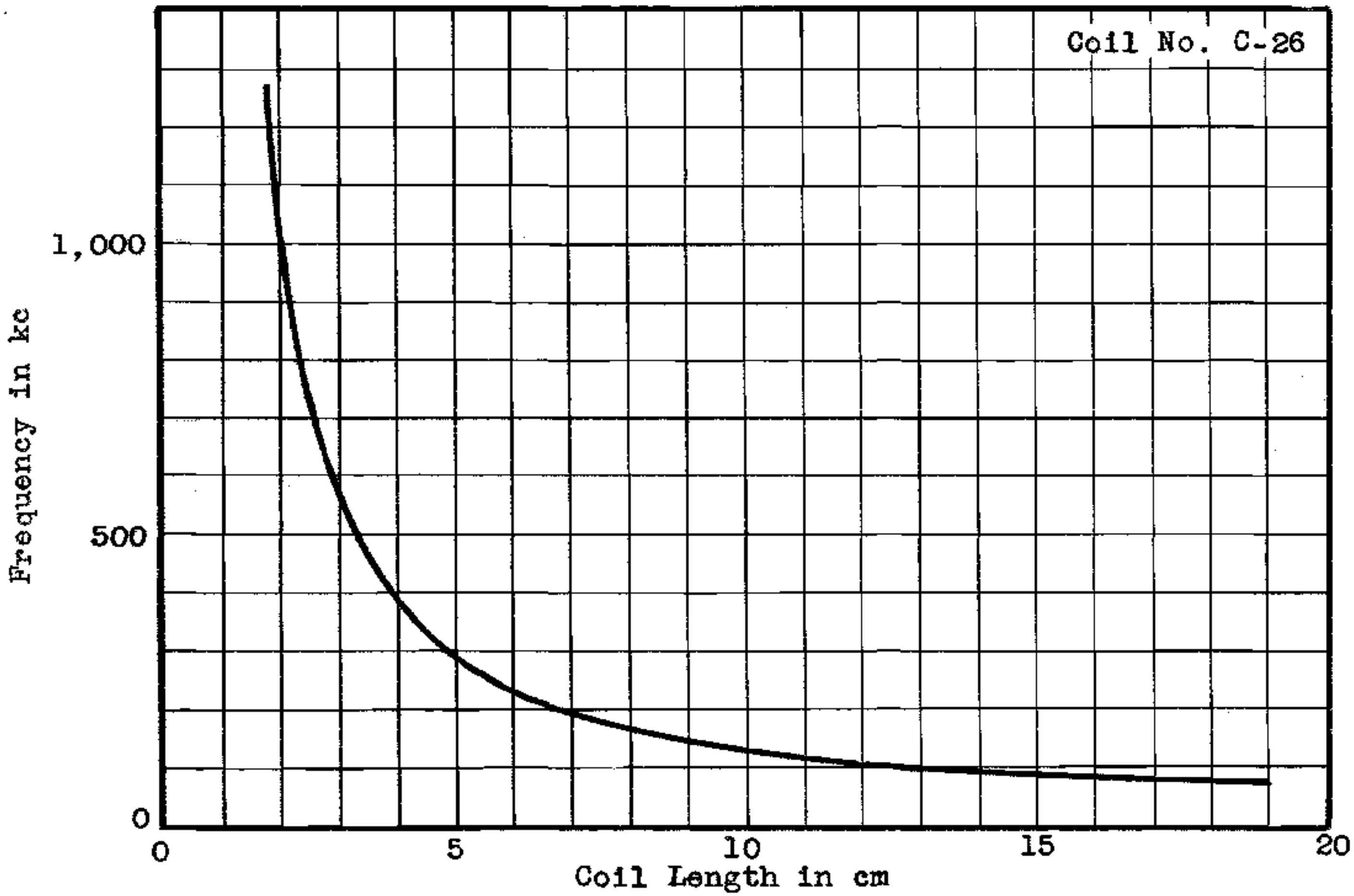



Figure 16. Typical Variation of First Resonant Frequency with Coil Length for a Type M Coil.

<u>Analysis with the resistance taken into account</u>.--The following relations can be written:

$$-\frac{\partial v}{\partial x} = 4n^2\mu A\frac{\partial i}{\partial t} + iR \tag{20}$$

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} \tag{6}$$

The time parameter can be eliminated with the assumption of a sinusoidal variation for the voltage; it is written then:

$$-\frac{d^2V}{dx^2} = \left(4n^2\mu AC\omega^2 - RCj\omega\right)V$$

$$\frac{d^2V}{dx^2} + \left(4n^2\mu AC\omega^2 - jRC\omega\right)V = 0 \tag{21}$$

$$\frac{d^2V}{dx^2} + \left(a - jb\right)V = 0$$

The voltage equation is

$$V = B_3\cos px + B_4\sin px$$

The following applies to the case of the M-closed type of coil:

$$V(x) = V_s\cos px - \frac{V_s}{\tan pl}\sin px \tag{22}$$

where

$$a = 4n^2\mu AC\omega^2 \tag{23}$$

$$b = RC\omega \tag{24}$$

$$p^2 = a - jb \tag{25}$$

The following is a modification of equation (20):

$$I(x) = -\frac{1}{R+j\omega 4n^2\mu A}\frac{dV}{dx} \tag{26}$$

Equation (26) in combination with equation (22) gives

$$I(x) = -\frac{j\omega C}{p}V_s\left(\sin px + \frac{1}{\tan pl}\cos px\right) \tag{27}$$

For $x=0$ equation (27) becomes

$$I = I_s = -\frac{j\omega C}{p}V_s\frac{1}{\tan pl}$$

Since

$$Z_s = \frac{V_s}{I_s}$$

it follows that

$$Z_s = +j\frac{p\tan pl}{\omega C} \tag{28}$$

Calculation of $|Z_s|$.--The curve of Figure 6 depicts the variation with frequency of the impedance of a type M-closed coil. This impedance is theoretically the absolute value of the expression of equation (28).

The following sequence of calculations has as end result the derivation of a formula giving the absolute value of $Z_S$ in terms of non-complex quantities.

It follows from equation (28) that

$$\left|Z_s\right| = \frac{|p|}{\omega C} \left|\tan pl\right| \tag{29}$$

Since $p=\sqrt{a-jb}$ , it is advantageous to define two new real constants $\alpha$ and $\beta$ such that

$$\beta - j\alpha = p \tag{30}$$

The constants $a$ and $b$ were defined previously by the equations (23) and (24).

If $\beta$ and $\alpha$ are substituted into equation (29) there results

$$\left|Z_s\right| = \frac{\sqrt{\alpha^2+\beta^2}}{\omega C} \left|\tan(\beta l - j\alpha l)\right|$$

The following identities are to be used:

$$\tan(\beta l - j\alpha l) = \frac{1}{j} \tanh j(\beta l - j\alpha l) = \frac{1}{j} \frac{\sinh(\alpha l + j\beta l)}{\cosh(\alpha l + j\beta l)}$$

$$\sinh(\alpha l + j\beta l) = \sinh\alpha l \cosh j\beta l + \cosh\alpha l \sinh j\beta l \tag{31}$$

$$\cosh(\alpha l + j\beta l) = \cosh\alpha l \cosh j\beta l + \sinh\alpha l \sinh j\beta l \tag{32}$$

The calculations for $\left|\tan(\beta l - j\alpha l)\right|$ are:

$$\left|\tan(\beta l - j\alpha l)\right| = \frac{\left|\sinh\alpha l \cos\beta l + \cosh\alpha l\, j\sin\beta l\right|}{\left|\cosh\alpha l \cos\beta l + j\sinh\alpha l \sin\beta l\right|}$$

$$= \frac{\sqrt{\sinh^2\alpha l \cos^2\beta l + \cosh^2\alpha l \sin^2\beta l}}{\sqrt{\cosh^2\alpha l \cos^2\beta l + \sinh^2\alpha l \sin^2\beta l}}$$

$$= \frac{\sqrt{(\sinh^2\alpha l)(1-\sin^2\beta l) + (\sin^2\beta l)(1+\sinh^2\alpha l)}}{\sqrt{(1+\sinh^2\alpha l)\cos^2\beta l + (\sinh^2\alpha l)(1-\cos^2\beta l)}}$$

$$\left|\tan(\beta l - j\alpha l)\right| = \frac{\sqrt{\sinh^2\alpha l - \cancel{\sinh^2\alpha l \sin^2\beta l} + \sin^2\beta l + \cancel{\sinh^2\alpha l \sin^2\beta l}}}{\sqrt{\cos^2\beta l + \cancel{\sinh^2\alpha l \cos^2\beta l} + \sinh^2\alpha l - \cancel{\sinh^2\alpha l \cos^2\beta l}}}$$

$$= \sqrt{\frac{\sinh^2\alpha l + \sin^2\beta l}{\sinh^2\alpha l + \cos^2\beta l}}$$

The expression for $\left|Z_s\right|$ finally is

$$\left|Z_s\right| = \frac{\sqrt{\alpha^2+\beta^2}}{\omega C}\sqrt{\frac{\sinh^2\alpha l + \sin^2\beta l}{\sinh^2\alpha l + \cos^2\beta l}} \qquad (33)$$

Equation (33) has a variation that is similar to that represented by the curve of Figure 6. At low frequencies $\sinh\alpha l$ is insignificant when compared with $\sin\beta l$ and $\cos\beta l$ and thus the trigonometric terms govern the variation of the function $\left|Z_s\right|$; at high frequencies $\sinh\alpha l$ becomes much greater than one and then the term

$$\sqrt{\frac{\sinh^2\alpha l + \sin^2\beta l}{\sinh^2\alpha l + \cos^2\beta l}}$$

approaches the value of one.

Equation (33) may hide the fact that a coil with negligible resistance acts at low frequencies as an inductor. The following simplifications reveal that the previous expression is reducible at low frequencies to $\omega L_T$, if $R_T$ is neglected.

If $R_T = 0$, then $b = 0$ because

$$b = \frac{R_T C_T}{\ell^2}\omega$$

Since $p=\sqrt{a-jb}=\beta-j\alpha$, then $\alpha=0$ , and

$$\beta = \sqrt{a} = \frac{\sqrt{L_T C_T}}{\ell}\omega$$

$$|Z_s|_{f\to 0} = \frac{\beta}{\omega C}\sqrt{\frac{0+\sin^2\beta\ell}{0+\cos^2\beta\ell}} = \frac{\beta}{\omega C}\tan\beta\ell$$

$$|Z_s| = \frac{\beta}{\omega C}\beta\ell = \frac{\beta^2\ell}{\omega C} = \frac{L_T C_T}{\ell^2}\omega^2\frac{\ell}{\omega\frac{C_T}{\ell}}$$

and finally

$$|Z_s| = \omega L_T$$

General case: an impedance of other than zero or infinite value is connected between the coil layers.--The circuit arrangement to be analyzed here is shown in Figure 17. It resembles the case of a transmission line terminated with a load impedance. In imitation of the terminology used in transmission lines, the end of the coil connected to the power supply will be called interchangeably "sending-end" or "supply-end," and the other end will be called "receiving-end."

The previously derived differential equation for a two-layer coil of type M is

$$\frac{d^2V}{dx^2} + (4n^2\mu AC\omega^2 - jRC\omega)V = 0 \tag{21}$$

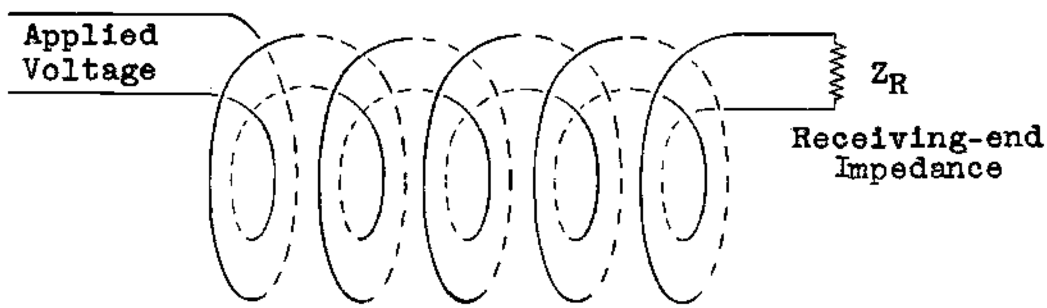



Figure 17. Two-layer Type M Coil Terminated in an Impedance of Value $Z_R$.

A solution for the voltage is

$$V = B_3 \cos px + B_4 \sin px$$

and a relation between the current and the voltage is

$$I = -\frac{1}{R + j4n^2\mu A\omega}\frac{dV}{dx} \tag{26}$$

The boundary conditions are

$$V = V_S \qquad I = I_S \qquad \text{at } x = 0 \quad \text{and}$$
$$V = V_R \qquad I = I_R \qquad \text{at } x = \ell$$

The following relations are also applicable:

$$V_S = I_S Z_S \quad \text{and} \quad V_R = I_R Z_R \tag{34}$$

From a proper combination of some of the previous relations, the expression for the coil impedance as considered from the supply-end, to be called henceforth "sending-end impedance," is found to be

$$Z_S = \frac{Z_R + \dfrac{R + j4n^2\mu A\omega}{p}\tan p\ell}{1 - Z_R \dfrac{p}{R + j4n^2\mu A\omega}\tan p\ell} \tag{35}$$

The expression for the voltage between two adjacent points of the two layers is

$$V = V_S \cos px - I_S \frac{R + j4n^2\mu A\omega}{p}\sin px \tag{36}$$

and the expression for the current in either layer at a distance $x$ from the sending-end is

$$I = I_s \cos px + \frac{p}{R + j4n^2\mu A\omega} V_s \sin px \tag{37}$$

<u>Selection of new constants to make the coil equations identical with those of a transmission line</u>.--Judicious groupings of the coil parameters into new constants will make the previous expressions for its voltage, current and sending-end impedance become identical with the corresponding ones for a transmission line. That this can be achieved must be expected from the fact that the voltage and current relations of both the coil and the transmission line stem from the same wave differential equation.

The new constants, $Z_o$ and $q$, to be introduced are defined as follows:

$$Z_o = j\,\frac{R + j4n^2\mu A\omega}{p} \tag{38}$$

$$q = -jp \tag{39}$$

They correspond to those called "characteristic impedance" and "propagation constant" in a transmission line. The constant $p$ was defined previously in equation (25).

If equations (38) and (39) are used in conjunction with equations (35), (36), and (37) the following relations ensue:

$$Z_s = Z_0 \frac{Z_R + Z_0 \tanh ql}{Z_0 + Z_R \tanh ql} \tag{40}$$

$$V = V_s \cosh qx - I_s Z_0 \sinh qx \tag{41}$$

$$I = I_s \cosh qx - \frac{V_s}{Z_0} \sinh qx \tag{42}$$

The constants $Z_0$ and $q$ can also be given in terms of $L_T$, $C_T$, and $R_T$ as follows:

$$Z_0 = \sqrt{\frac{R_T + j\omega L_T}{j\omega C_T}} \tag{43}$$

$$q = \frac{\sqrt{(R_T + j\omega L_T) j\omega C_T}}{l} \tag{44}$$

If the last five equations are compared with the corresponding ones for a two-wire transmission line, the similarity between the properties of the latter and those of a two-layer type M coil will become evident.

<u>Calculation of voltage between layers at mid-point of coil of type M-closed at first series resonance</u>.--The relation for the voltage between layers for a coil of type M-closed is

$$V(x) = V_s \cos px - \frac{V_s}{\tan pl} \sin px \tag{22}$$

where $p = \sqrt{a - jb} = \beta - j\alpha$

At the first series resonance

$$\beta l = \pi \tag{45}$$

and at the mid-point of the coil

$$x = \frac{l}{2} \quad \text{and} \quad \beta x = \beta \frac{l}{2} = \frac{\pi}{2} \tag{46}$$

A substitution of relations (45) and (46) into equation (22) gives

$$V(x) = V_s \cos(\beta - j\alpha)x - \frac{V_s}{\tan pl} \sin(\beta - j\alpha)x$$

$$V(x) = V_s \cos\beta x \cos j\alpha x + V_s \sin\beta x \sin j\alpha x - \\ - \frac{V_s}{\tan pl} \sin\beta x \cos j\alpha x + \frac{V_s}{\tan pl} \cos\beta x \sin j\alpha x$$

At this point it is convenient to calculate $\tan pl$ separately:

$$\tan pl = \tan(\beta l - j\alpha l) = \frac{\sin(\beta l - j\alpha l)}{\cos(\beta l - j\alpha l)} =$$

$$\tan pl = \frac{\sin\beta l \cos j\alpha l - \cos\beta l \sin j\alpha l}{\cos\beta l \cos j\alpha l + \sin\beta l \sin j\alpha l}$$

The use of the relations $\beta l = \pi$ and $\beta x = \frac{\pi}{2}$ into this last equation gives

$$\tan pl = \frac{-\cos\pi \sin j\alpha l}{\cos\pi \cos j\alpha l} = -\tan j\alpha l = -j\tanh\alpha l$$

The calculation for the voltage at the mid-point of the coil can be resumed now:

$$V\left(x = \frac{l}{2}\right) = V_s \cos\frac{\pi}{2} \cos j\alpha x + V_s \sin\frac{\pi}{2} \sin j\alpha x \\ - \frac{V_s}{-j\tanh\alpha l} \sin\frac{\pi}{2} \cos j\alpha x + \frac{V_s}{-j\tanh\alpha l} \cos\frac{\pi}{2} \sin j\alpha x$$

$$V\left(x=\frac{l}{2}\right) = V_s \sin j\alpha x + \frac{V_s}{j \tanh \alpha l} \cos j\alpha x$$

$$\frac{V\left(\frac{l}{2}\right)}{V_s} = \sin j\alpha x - j\frac{\cos j\alpha x}{\tanh \alpha l} = j \sinh \alpha x - j\frac{\cosh \alpha x}{\tanh \alpha l}$$

The imaginary unit $j$ is omitted now so that the absolute value of the voltage ratio may be found:

$$\left|\frac{V\left(\frac{l}{2}\right)}{V_s}\right| = \sinh \alpha x - \frac{\cosh \alpha x}{\tanh \alpha l}$$

At the mid-point of the coil $\alpha x = \alpha\frac{l}{2}$ and the use of this relation gives

$$\left|\frac{V\left(\frac{l}{2}\right)}{V_s}\right| = \sinh \frac{\alpha l}{2} - \frac{\cosh \frac{\alpha l}{2}}{\tanh \alpha l}$$

$$\left|\frac{V\left(\frac{l}{2}\right)}{V_s}\right| = \sinh \frac{\alpha l}{2} - \frac{\cosh \frac{\alpha l}{2}}{\frac{\sinh \alpha l}{\cosh \alpha l}} = \sinh \frac{\alpha l}{2} - \frac{\cosh \frac{\alpha l}{2}}{2 \sinh \frac{\alpha l}{2} \cosh \frac{\alpha l}{2}} \cosh \alpha l$$

$$\left|\frac{V\left(\frac{l}{2}\right)}{V_s}\right| = \frac{2 \sinh^2 \frac{\alpha l}{2} - \cosh \alpha l}{2 \sinh \frac{\alpha l}{2}} = \frac{2\,\frac{1}{2}\left(\cosh \alpha l - 1\right) - \cosh \alpha l}{2 \sinh \frac{\alpha l}{2}}$$

$$\left|\frac{V\left(\frac{l}{2}\right)}{V_s}\right| = \frac{\cosh \alpha l - 1 - \cosh \alpha l}{2 \sinh \frac{\alpha l}{2}}$$

$$\left|\frac{V\left(\frac{l}{2}\right)}{V_s}\right| = \frac{1}{2 \sinh \frac{\alpha l}{2}} \tag{47}$$

When $\frac{\alpha l}{2}$ is small enough for the relation

$$\sinh \frac{\alpha l}{2} \cong \frac{\alpha l}{2}$$

to hold approximately true, the voltage ratio becomes

$$\left|\frac{V(\frac{\ell}{2})}{V_s}\right| \cong \frac{1}{\alpha \ell} \tag{48}$$

It can be seen from either equation (47) or equation (48) that, under certain circumstances, a considerable overvoltage can be developed between the layers of a coil of type M-closed, at its mid-point, at the first series resonance. Laboratory experiments confirmed the existence of overvoltages under conditions as described previously. Figure 19, for instance, which gives the "voltage between layers vs. coil length" relation for a certain type M-closed coil at its first series resonance, indicates such an overvoltage of 5.8 to 1.

Voltage between layers distributions along coil length of coils of types M-closed and M-open at different resonant frequencies.--Figures 18 through 23 are in connection with type M-closed coils and Figures 24 through 27 in connection with type M-open coils. It can be noticed that the voltage distributions appearing in these figures are similar to those that are known to exist, under identical circumstances, along transmission lines.

Vector diagram of a type M-closed coil.--Figure 28 gives the vector diagram of a type M-closed coil and also defines the symbols used in it. This diagram indicates that there is symmetry, from an electrical point of view, between the

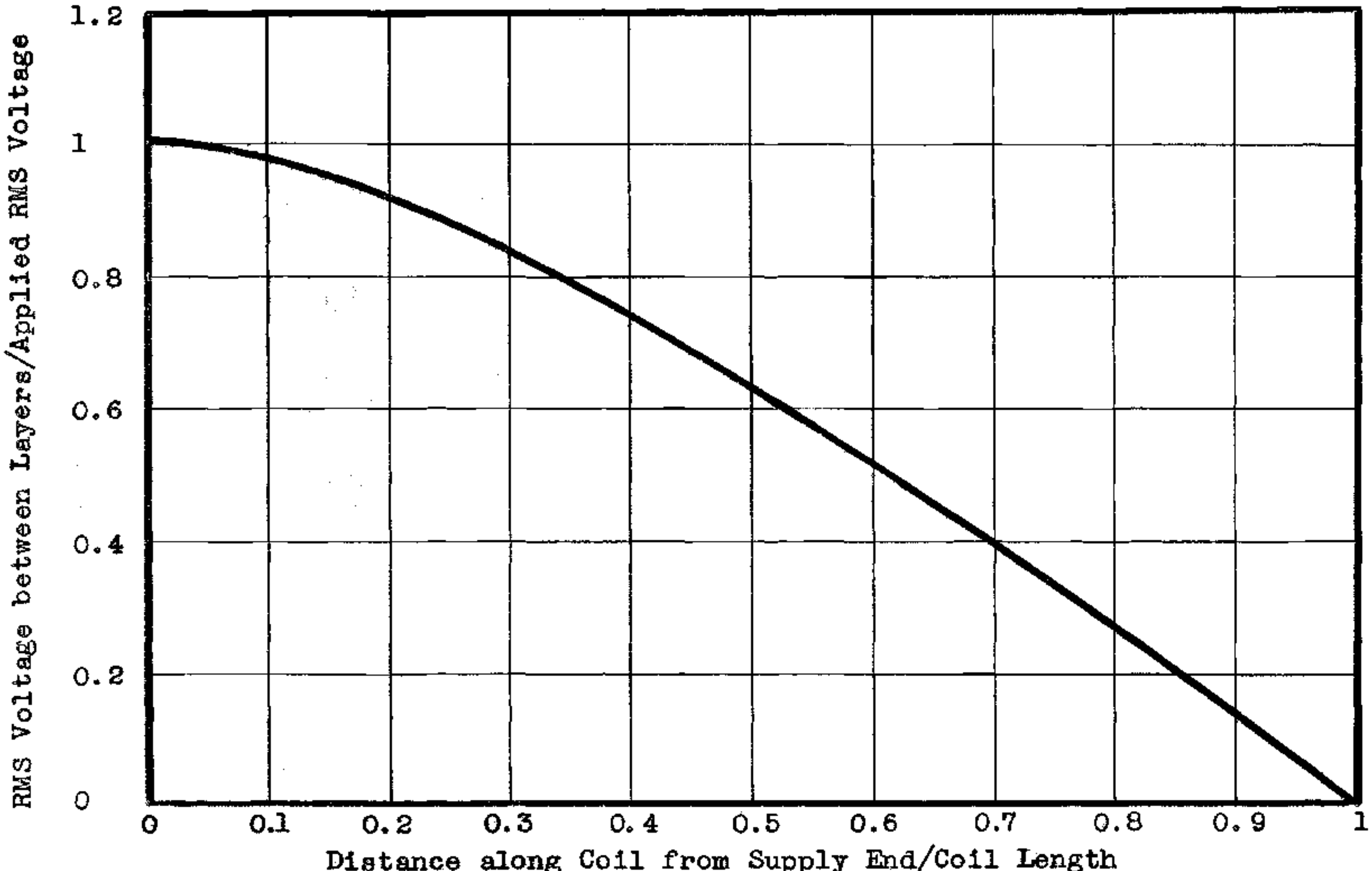



Figure 18. Distribution of RMS Voltage between Layers along Length of Type M-closed Air-core Coil No. C-25 at the Frequency of Its First Parallel Resonance.

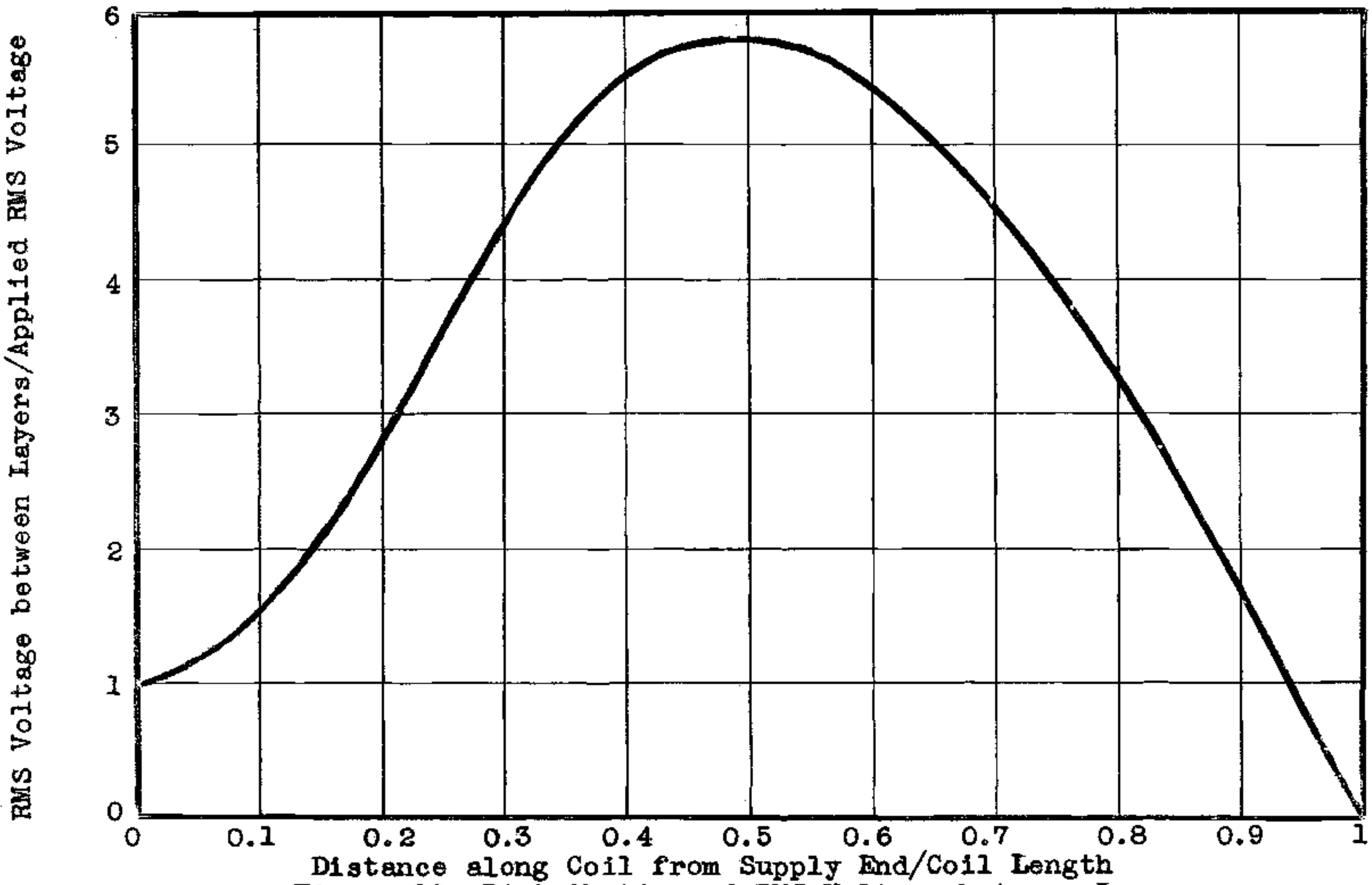



Figure 19. Distribution of RMS Voltage between Layers along Length of Type M-closed Air-core Coil No. C-25 at the Frequency of Its First Series Resonance.

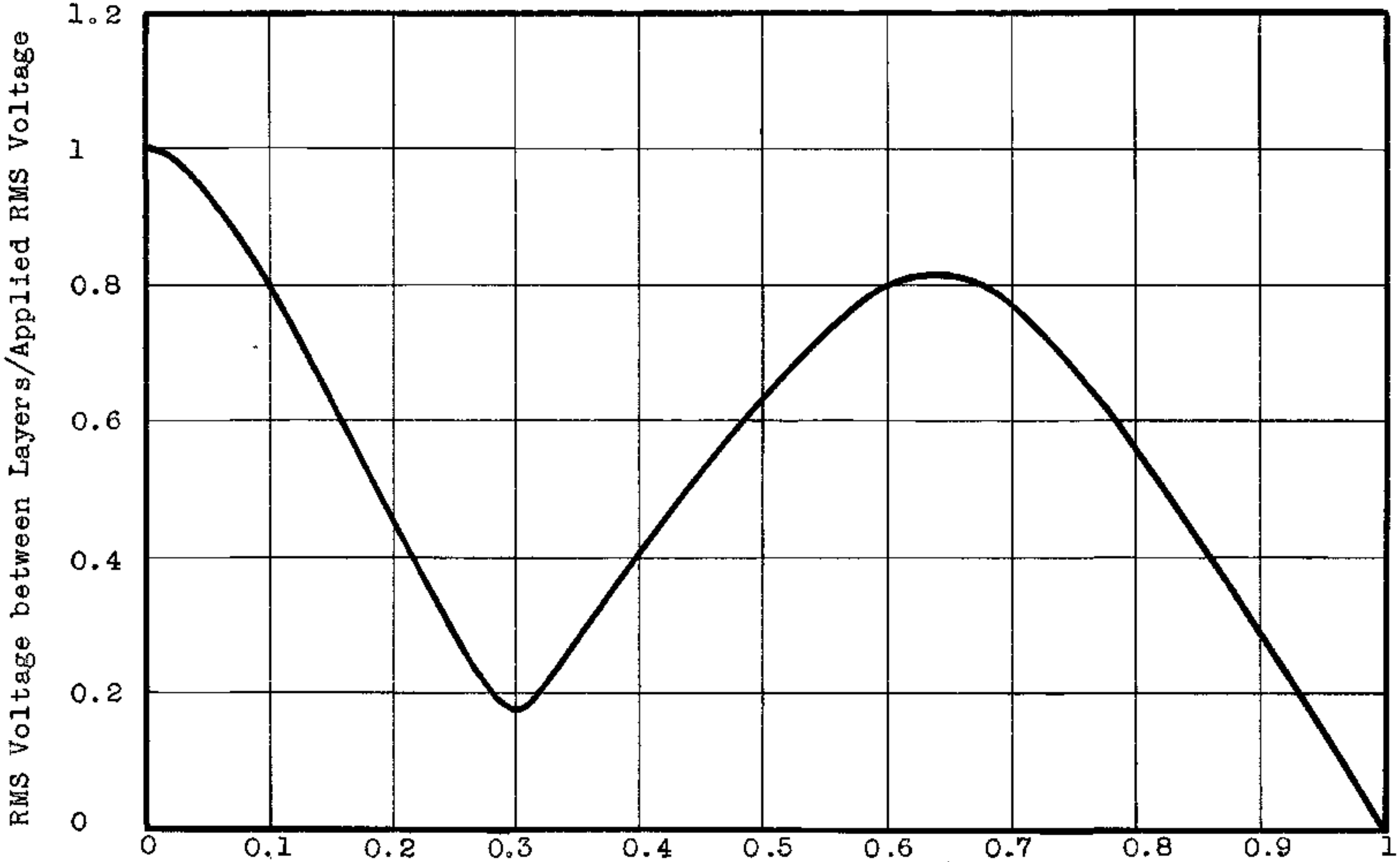



Figure 20. Distribution of RMS Voltage between Layers along Length of Type M-closed Air-core Coil No. C-25 at the Frequency of Its Second Parallel Resonance.

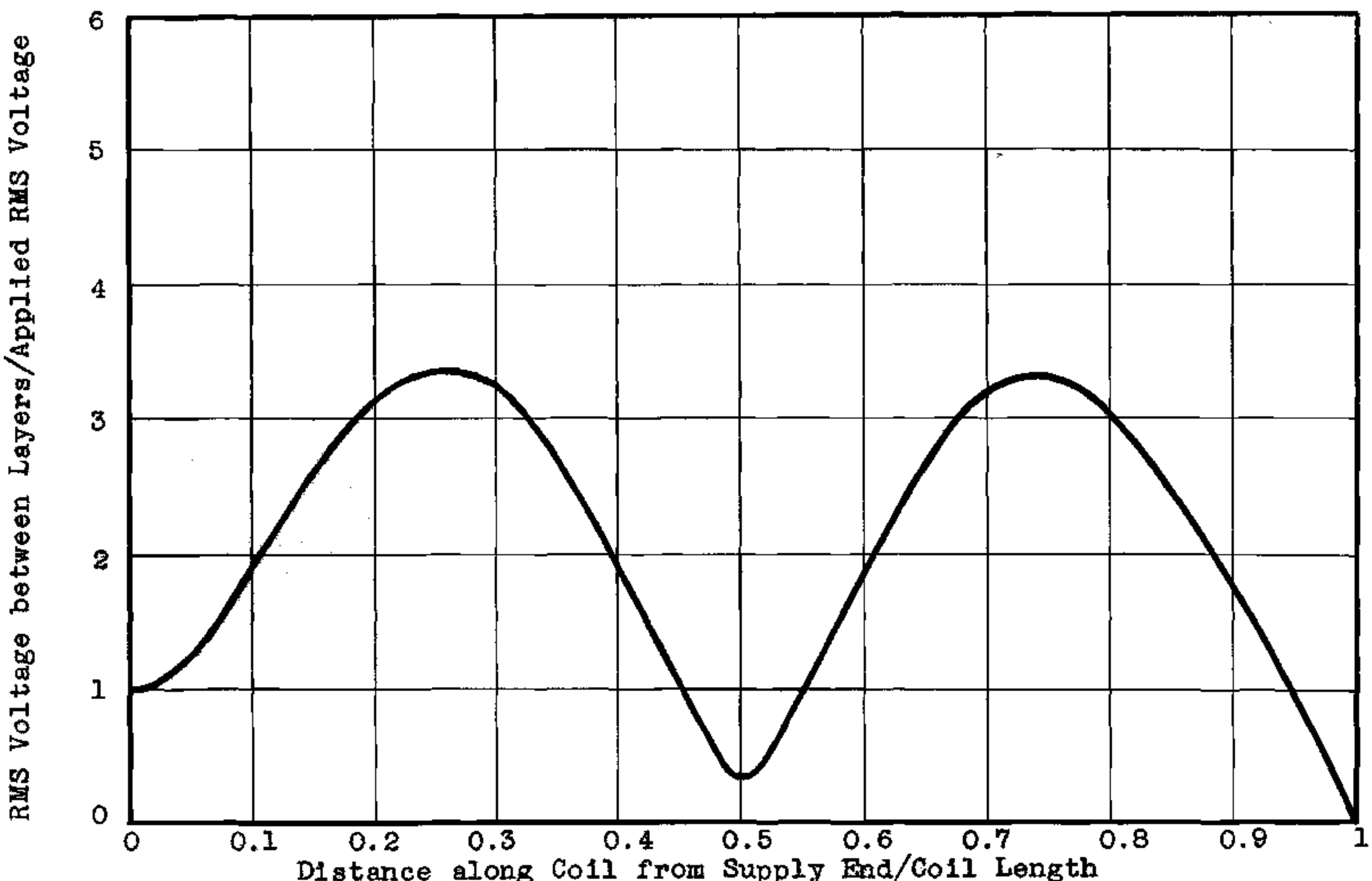



Figure 21. Distribution of RMS Voltage between Layers along Length of Type M-closed Air-core Coil No. C-25 at the Frequency of Its Second Series Resonance.

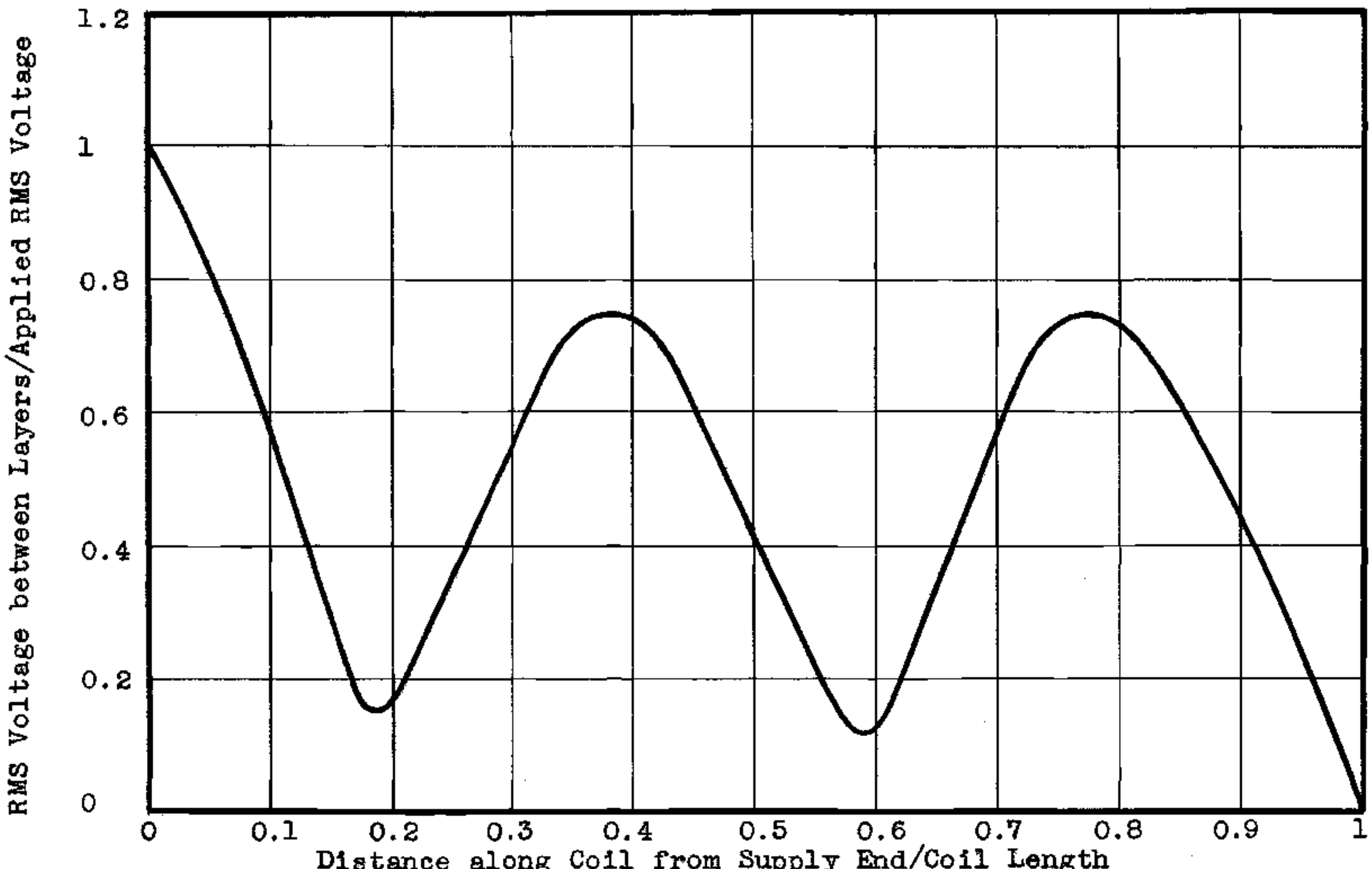



Figure 22. Distribution of RMS Voltage between Layers along Length of Type M-closed Air-core Coil No. C-25 at the Frequency of Its Third Parallel Resonance.

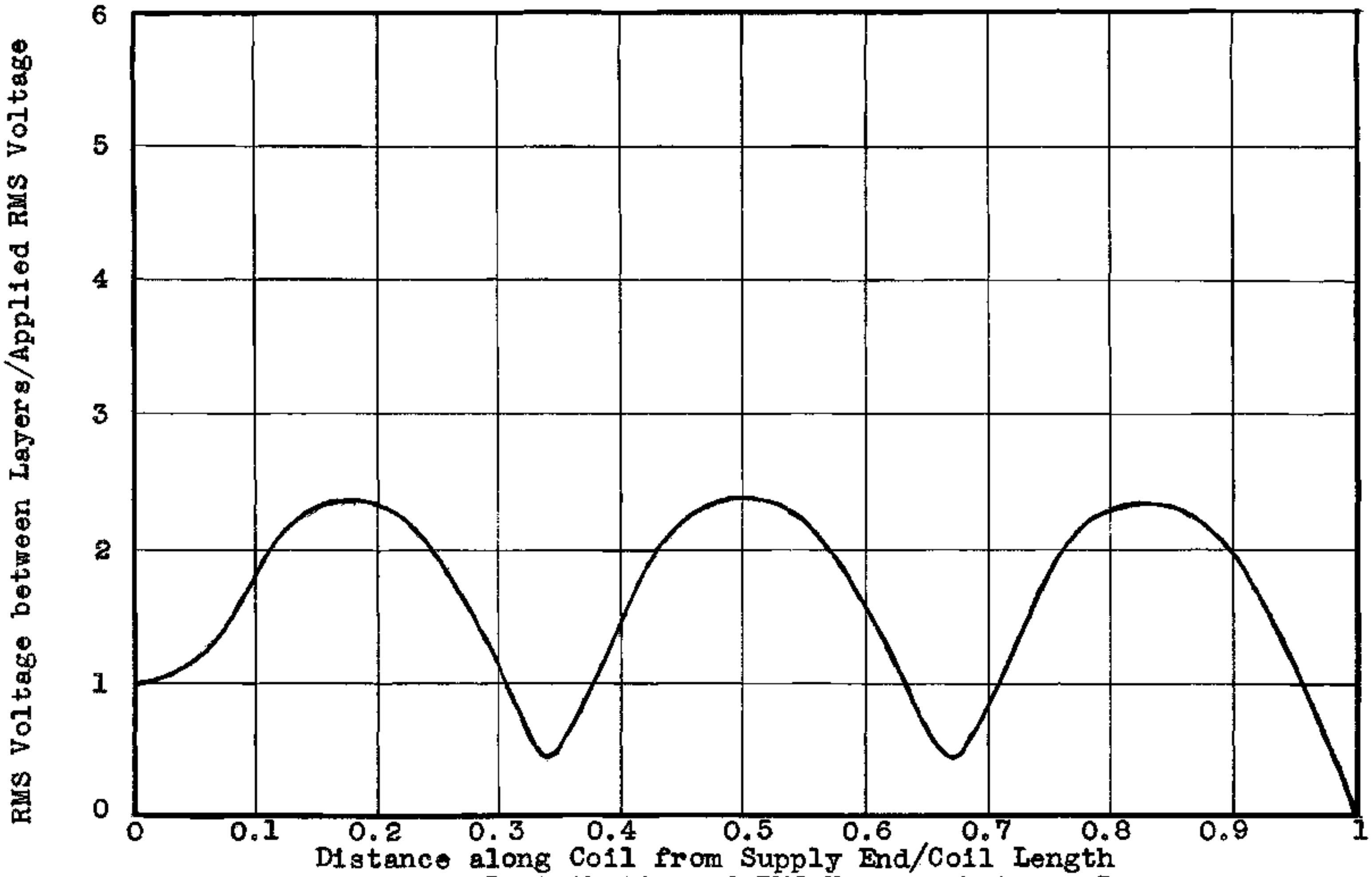


Figure 23. Distribution of RMS Voltage between Layers along Length of Type M-closed Air-core Coil No. C-25 at the Frequency of Its Third Series Resonance.

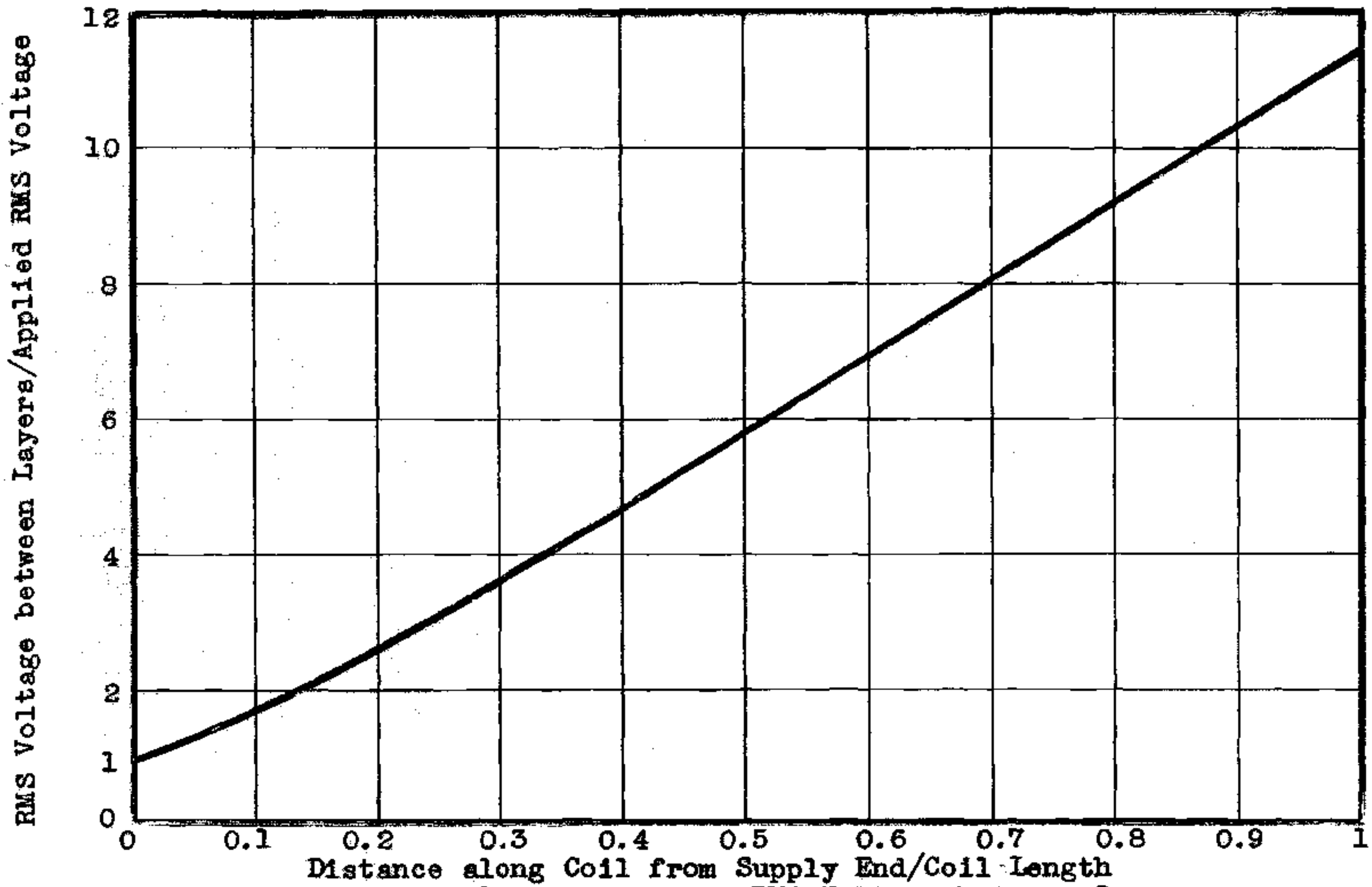



Figure 24. Distribution of RMS Voltage between Layers along Length of Type M-open Air-core Coil No. C-25 at the Frequency of Its First Series Resonance.

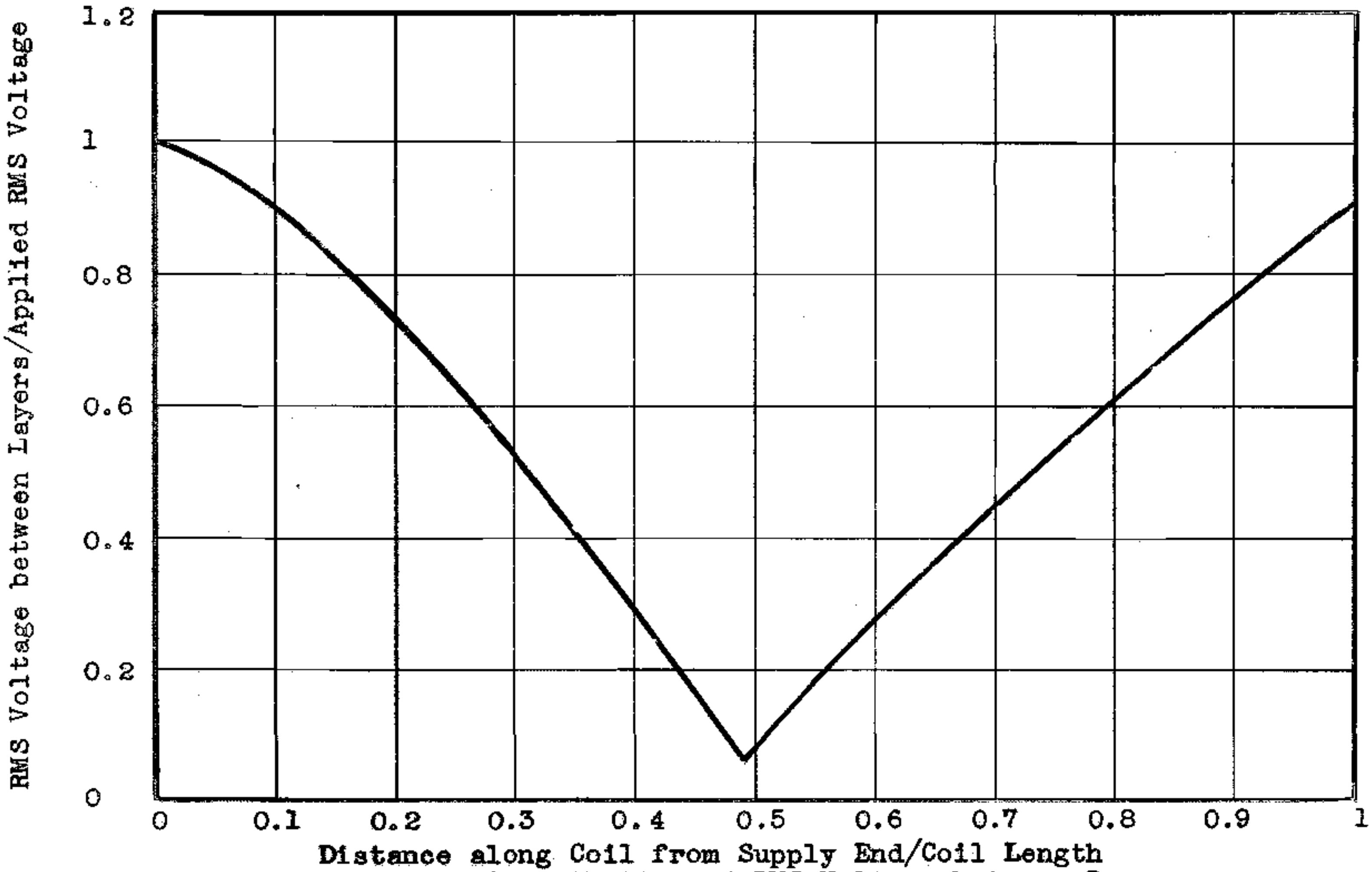



**Figure 25. Distribution of RMS Voltage between Layers along Length of Type M-open Air-core Coil No. C-25 at the Frequency of Its First Parallel Resonance.**

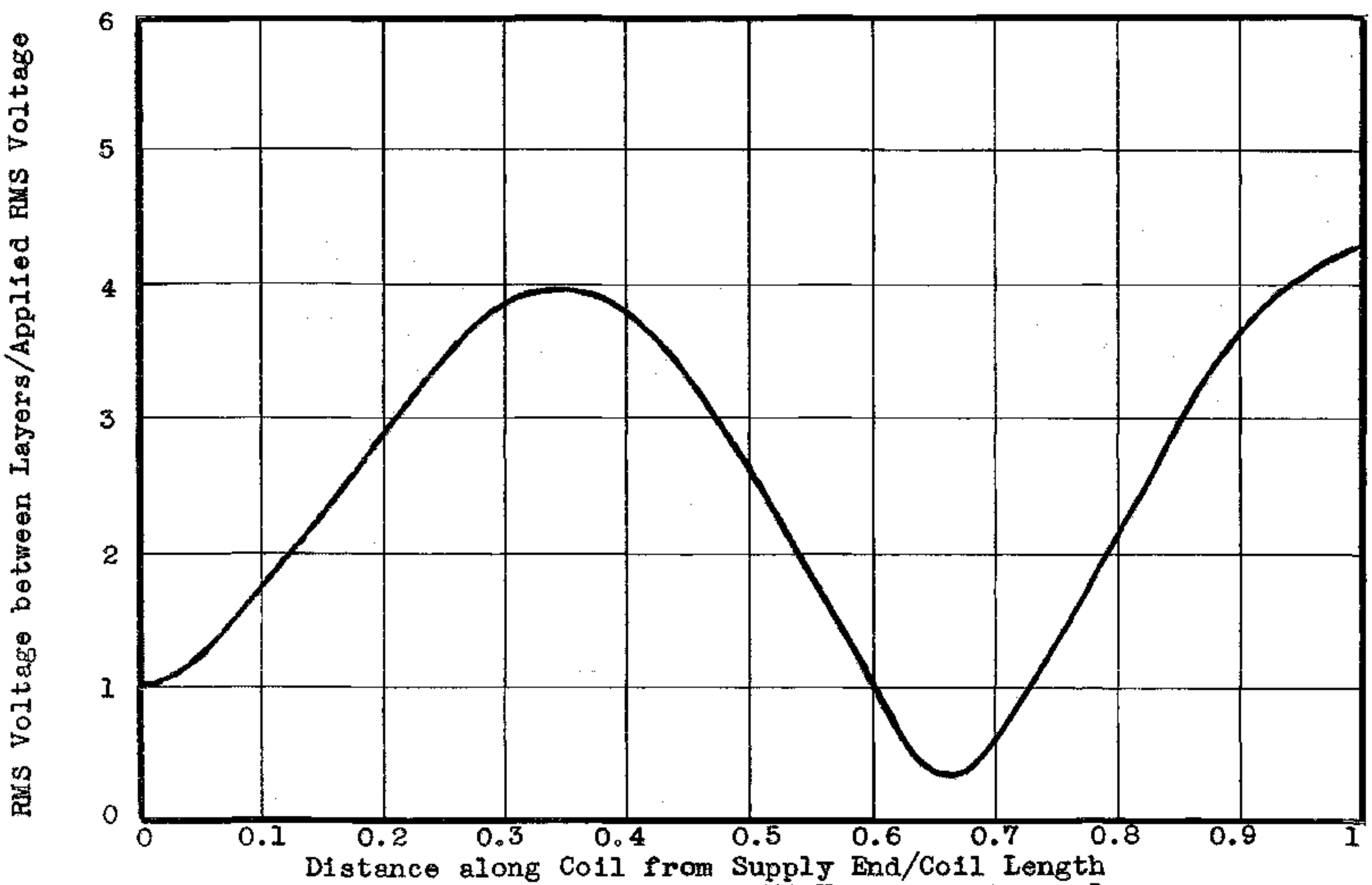


Figure 26. Distribution of RMS Voltage between Layers along Length of Type M-open Air-core Coil No. C-25 at the Frequency of Its Second Series Resonance.

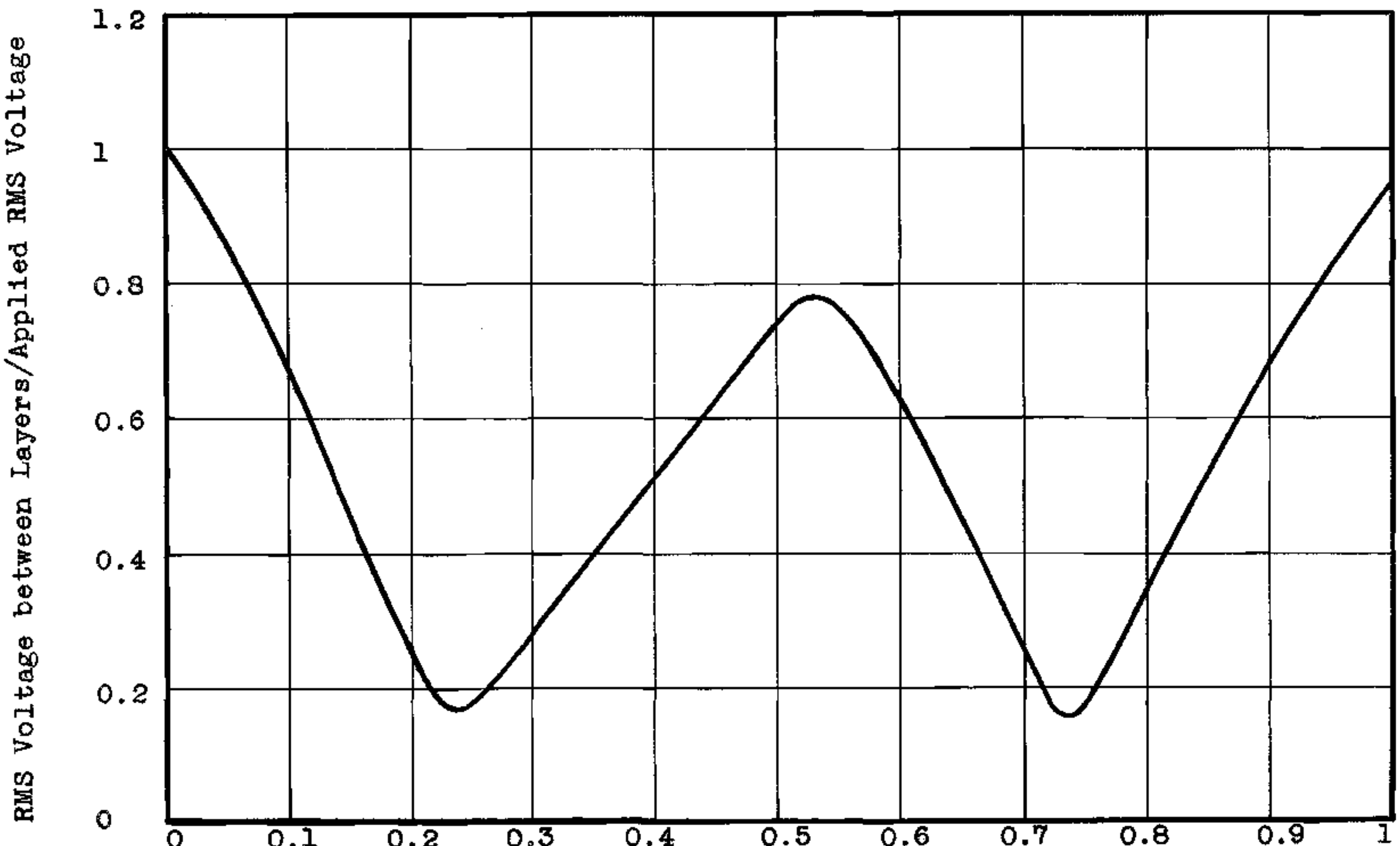



Figure 27. Distribution of RMS Voltage between Layers along Length of Type M-open Air-core Coil No. C-25 at the Frequency of Its Second Parallel Resonance.

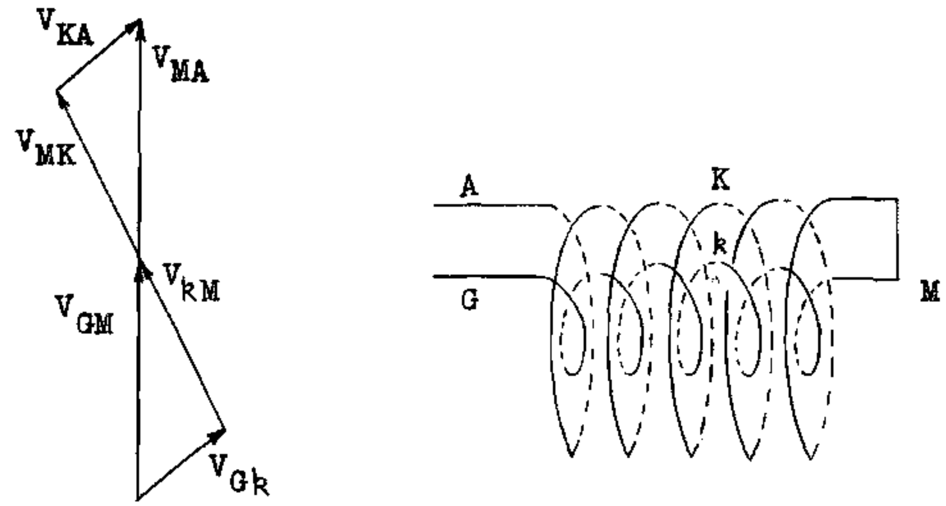



Figure 28. Vector Diagram of a Two-layer Type M-closed Coil.

outside and the inside layers. The following relations in connection with rms voltage values between different points of the coil can be written:

$$V_{AK} = V_{GK} \tag{49}$$

$$V_{MK} = V_{kM} \tag{50}$$

$$V_{kK} = 2V_{KM} \tag{51}$$

Equation (51) states that the voltage between adjacent points of the two layers is equal to twice the voltage from the end point of the coil to the point on the outside layer of the two adjacent ones considered. This last relation has a certain practical value: it allows an easy determination of the voltage between the layers, when the inner layer is inaccessible, with the measurement of a voltage on the outside layer.

Higher resonances.--The frequencies of the higher resonances of an air-cored coil of type M are not always simple multiples of the frequency of its first resonance, as is unrealistically indicated by the solution of the wave differential equation, but are so only when the ratio of the coil length to the coil diameter is, as found experimentally, larger than about thirty. This divergence from the theoretical expectations is partly due to the fact that the end effects of the coil, both for the inductance and the capacitance, have not been taken into consideration. A linear relation, $L_T = 4n^2\mu \ell A$, between inductance and length was used for the coil. This relation, which is theoretically valid only for coils of infinite length, can yield sufficiently accurate results for relatively short coils if amended with a correction multiplier $K$ so that $L_T'' = 4n^2\mu \ell A K$. This multiplier is a function of the coil length to diameter ratio and is defined graphically in Figure 29 (3). An examination of this figure will reveal that if the length of a certain coil is reduced, for instance, by one half, while its number of turns per unit length $n$ and its cross-sectional area $A$ are left unchanged, its inductance is reduced by more than one half. Without the use of $K$, however, a computation for the inductance of the previously halved in length coil will yield a value half as large as before and not less than that although it is less than that in reality. Resonant frequencies computed from formulas (18) and (97) which are

$$f = \frac{1}{4\sqrt{L_T C_T}} \tag{18}$$

and

$$f = \frac{1}{\pi\sqrt{L_T C_T}} \tag{97}$$

do not suffer in accuracy from the simplifications, the omission of the dependence of the coil inductance on the ratio of coil length to coil diameter and the assumption of a constant value of capacitance $C$ between layers per unit length for any position along the coil, utilized in the coil analysis if correct values for $L_T$ and $C_T$, such as values obtained from bridge measurements, are used in these formulas. In connection with the higher resonances, the use of the correction factor $K$ in conjunction with equation (18) results in a relation indicating that the ratio of the $m^{th}$ resonant frequency to the first one is larger than $m$ for coils of ordinary dimensions having a relatively small ratio, less than three, of length to diameter. If it is assumed that at the $m^{th}$ resonant frequency the directions of the currents and the magnetic fields divide a coil effectively into $m$ smaller coils each $1/m$ as long as the original one, then for each component coil the total inductance is $\frac{L_T K_m}{m}$ and the total capacitance is $\frac{C_T}{m}$. $K_m$ is the correction factor for the smaller coil with length to diameter ratio of $\frac{\ell}{mD}$ and $K_1$ is the correction factor for the entire undivided coil with length to diameter ratio of $\frac{\ell}{D}$. The expressions for the $m^{th}$ and the first resonant frequencies can be written as

$$f_m = \frac{1}{4\sqrt{\frac{L_T K_m}{m}\frac{C_T}{m}}} \tag{52}$$

and

$$f_1 = \frac{1}{4\sqrt{L_T K_1 C_T}} \tag{53}$$

The ratio of equation (52) to equation (53) is

$$\frac{f_m}{f_1} = m\sqrt{\frac{K_1}{K_m}} \tag{54}$$

Since $K_1$ is always larger than $K_m$, except for a coil of infinite length in which case they are both equal to one, the ratio $f_m/f_1$ is larger than $m$. If a sequence of values of the ratio $f_m/f_1$ is calculated for a coil becoming progressively shorter, it will be seen that this ratio becomes progressively higher. The calculated ratio is always smaller than the corresponding measured ratio and the difference between the two becomes larger as the coil under consideration becomes smaller. Therefore, although equation (54) is helpful in partly explaining the divergence between theoretical and experimental results in connection with the higher resonant frequencies of coils of ordinary lengths, it is not reliable for the determination of these frequencies. A comparison between calculated and measured frequency ratios can be found in Table 1.

Figure 30 shows a family of curves giving the ratios, as determined experimentally, of a few higher resonant frequencies to the first one for coils of length to diameter ratios from 3 to 1 to 0.2 to 1.

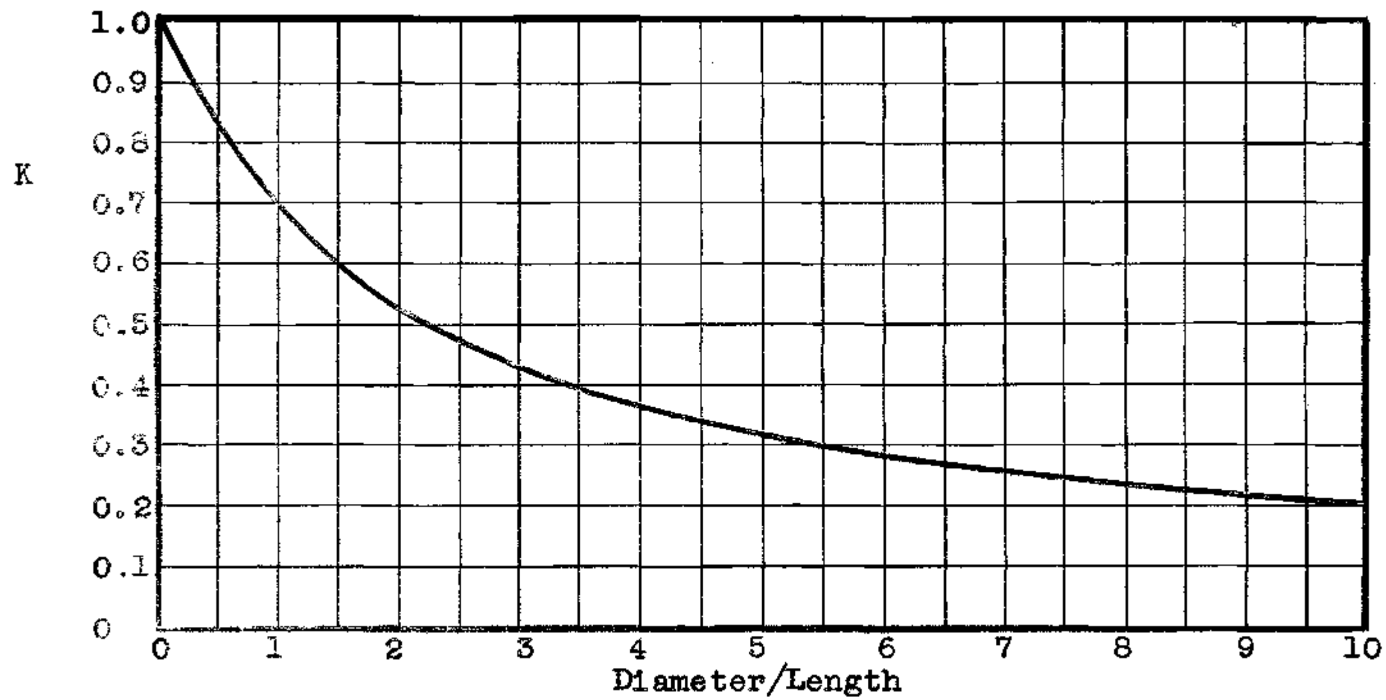



Figure 29. Value of Correction Factor K as a Function of the Ratio of Coil Diameter to Length.

Table 1. Comparison between Calculated and Measured $f_m/f_1$ Ratios for Different $\ell/D$ Ratios

| $\frac{\ell}{D}$ | Calc. $f_2/f_1$ | Meas. $f_2/f_1$ | Per Cent Diff. | Calc. $f_3/f_1$ | Meas. $f_3/f_1$ | Per Cent Diff. | Calc. $f_4/f_1$ | Meas. $f_4/f_1$ | Per Cent Diff. |
|---|---|---|---|---|---|---|---|---|---|
| $\infty$ | 2 | | | 3 | | | 4 | | |
| 4 | 2.10 | | | 3.28 | | | 4.57 | | |
| 3.5 | 2.10 | | | 3.31 | | | 4.60 | | |
| 3 | 2.12 | 2.25 | - 5.8 | 3.34 | 3.40 | - 1.8 | 4.71 | 5.00 | - 5.8 |
| 2.5 | 2.12 | 2.35 | - 9.8 | 3.38 | 3.50 | - 3.4 | 4.75 | 5.20 | - 8.6 |
| 2 | 2.18 | 2.40 | - 9.2 | 3.50 | 3.52 | - 0.6 | 5.00 | 5.50 | - 9.1 |
| 1.5 | 2.22 | 2.42 | - 8.3 | 3.64 | 3.68 | - 1.1 | 5.23 | 5.90 | -11 |
| 1 | 2.29 | 2.60 | -12 | 3.80 | 3.95 | - 3.8 | 5.45 | 6.56 | -17 |
| 0.8 | 2.32 | 2.75 | -16 | 3.90 | 4.00 | - 2.5 | 5.65 | 6.92 | -18 |
| 0.6 | 2.37 | 2.90 | -18 | 4.01 | 4.41 | - 9.1 | 5.84 | 7.48 | -22 |
| 0.5 | 2.38 | 3.05 | -22 | 4.12 | 4.50 | - 8.5 | 6.04 | 7.8 | -23 |
| 0.4 | 2.43 | 3.20 | -24 | 4.10 | 4.90 | -16 | 6.16 | 8.2 | -25 |
| 0.3 | 2.46 | 3.30 | -25 | 4.30 | 5.00 | -14 | | | |

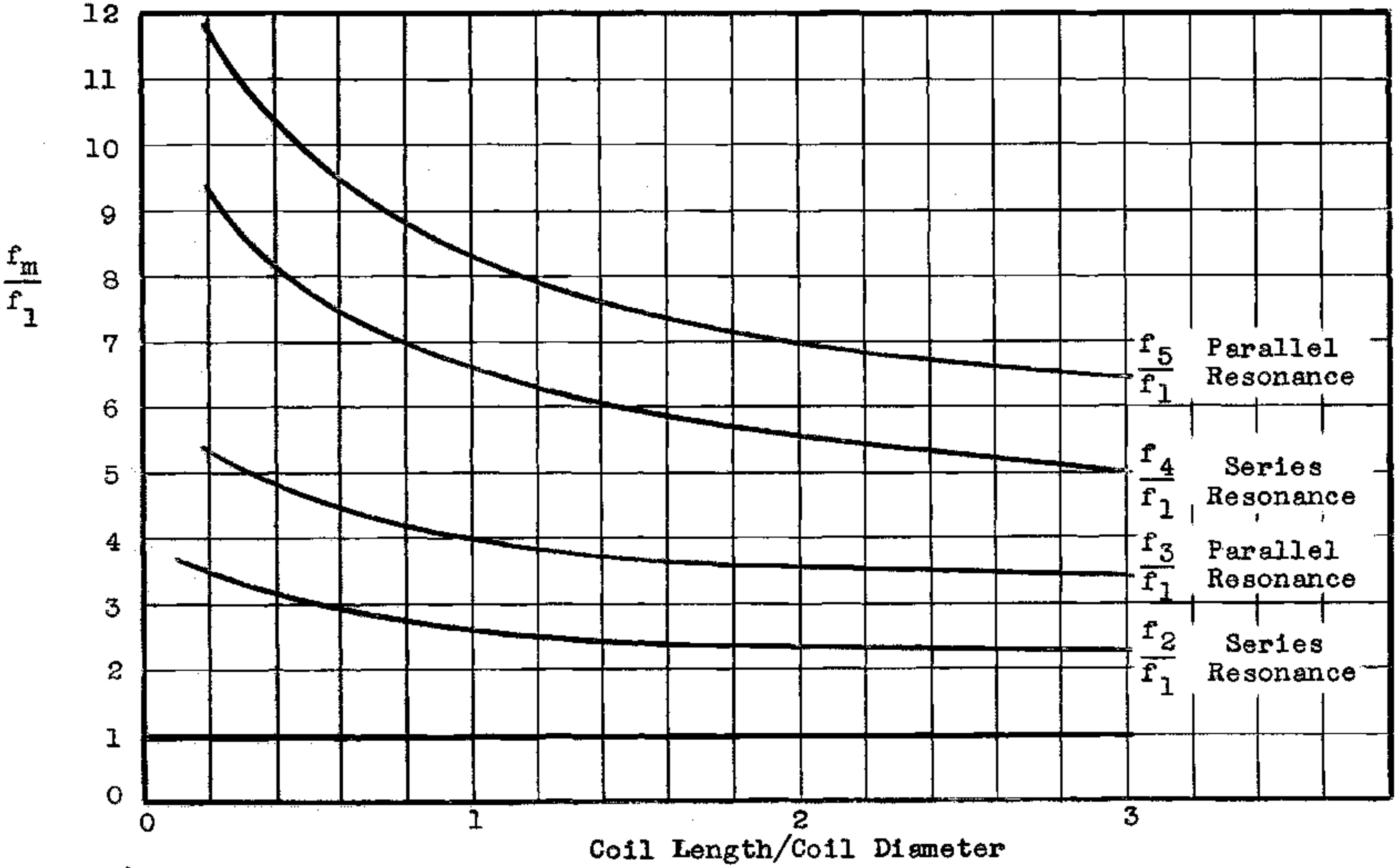



Figure 30. Variation of the Frequency Ratios $f_2/f_1$, $f_3/f_1$, $f_4/f_1$, and $f_5/f_1$ with the Ratio of Coil Length to Diameter for a Two-layer Type M-closed Coil.

# CHAPTER IV

## DETERMINATION OF RESONANT FREQUENCY OF TYPE S-OPEN COIL

A type S-open coil has a clearly defined series resonance at which its impedance is equal to its effective resistance. This resonance is due to the interaction of the distributed capacitance between the two layers with the distributed inductance of the turns and the determination of the frequency at which it occurs is the object of the calculations of this chapter. Other resonances, parallel and series, probably due to the additional effect of the distributed capacitance between turns in a lengthwise direction which is neglected in this analysis and to irregularities in the spacing betwwen successive turns, can also be observed at higher frequencies. Figure 8 presents the portion of the impedance-frequency curve of a type S-open coil which shows to advantage its series resonance. The voltage between two adjacent points of the two layers is constant, at a certain frequency, along the length of the coil; it is not the same, however, at all frequencies. At very low frequencies this voltage is approximately equal to the voltage applied to the coil. As the frequency is increased and that of the series resonance is approached,

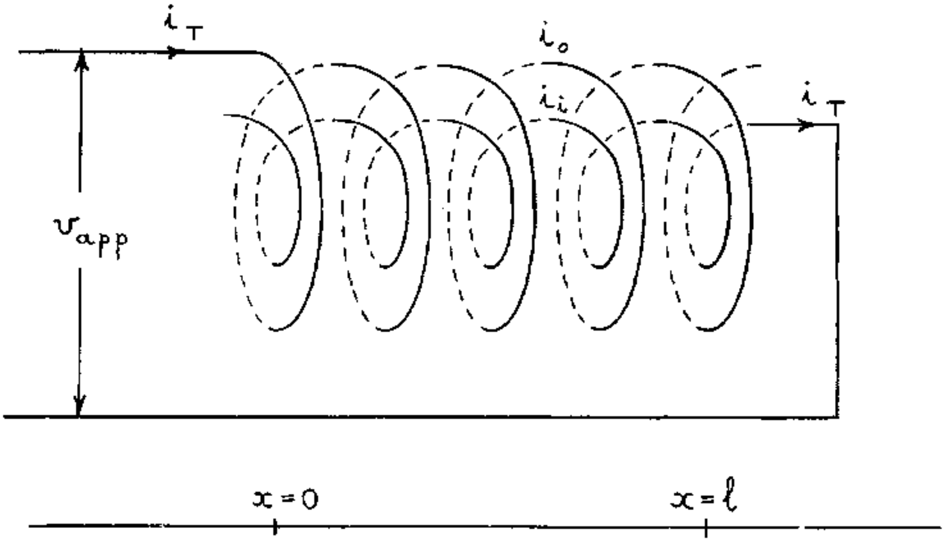



Figure 31. Type S-open Coil.

the voltage between the layers rises above the one applied to the coil terminals. At the frequency of the series resonance the ratio of the voltage between the layers to the applied voltage is at its maximum and that can be many times greater than one.

In the following calculations for the determination of the frequency of the series resonance of the coil the resistance parameter is neglected. Figure 31 is with reference to the present situation.

Here again, as in the case of the type M coil, the calculation starts with an equation based on one of Kirchhoff's laws; this equation expresses that the voltage applied to the coil is equal to the sum of the voltage drops along any path through the coil starting from one of its terminals and ending at the other one. The relation is

$$v_{app}(t) = \int_x^0 \frac{\partial e_o}{\partial x} dx + v + \int_\ell^x \frac{\partial e_i}{\partial x} dx \tag{55}$$

A voltage induced in a few turns of <u>one</u> layer is due to changing flux created by appropriate magnetomotive forces of <u>both</u> layers.

Equation (2), when adapted to the present case, becomes

$$\phi(t) = n\mu A\left[i_i(x,t) + i_o(x,t)\right]$$

or simply

$$\phi = n\mu A\,(i_i + i_o) \tag{56}$$

In accordance with Faraday's law and the previous explanations, it can be written that

$$\frac{\partial e_o}{\partial x}\,dx = -n\,dx\,\frac{d\phi}{dt} = -n\,dx\,n\mu A\left(\frac{\partial i_i}{\partial t} + \frac{\partial i_o}{\partial t}\right)$$

or

$$\frac{\partial e_o}{\partial x} = -n^2\mu A\left(\frac{\partial i_i}{\partial t} + \frac{\partial i_o}{\partial t}\right) \tag{57}$$

$$\frac{\partial e_i}{\partial x} = -n^2\mu A\left(\frac{\partial i_i}{\partial t} + \frac{\partial i_o}{\partial t}\right) \tag{58}$$

Since

$$i_i(x,t) + i_o(x,t) = i_T(t) \tag{59}$$

it follows that

$$\frac{\partial e_o}{\partial x} = \frac{\partial e_i}{\partial x} = -n^2\mu A\,\frac{di_T}{dt} \tag{60}$$

The members of these two equations are constant with respect to distance.

Equation (55) becomes, after integration is performed as indicated,

$$v_{app} = \frac{\partial e_o}{\partial x}\Big[x\Big]_x^0 + v + \frac{\partial e_i}{\partial x}\Big[x\Big]_\ell^x$$

If equation (60) is used

$$v_{app} = -n^2 \mu A \frac{di_T}{dt}(-x) + v - n^2 \mu A (x - \ell) \frac{di_T}{dt}$$

and

$$v_{app} = v + n^2 \mu A \frac{di_T}{dt} \ell \tag{61}$$

Since $v_{app}$ and $i_T$ do not depend on $x$, it follows that $v$ also does not depend on $x$.

Equation (6) applies to this case and, with two slight modifications, it becomes

$$\frac{\partial i_o}{\partial x} = -C \frac{dv}{dt} \tag{62}$$

Since $v$ is independent of $x$ it follows that the rate of change of the current $i_o$ with respect to $x$ is constant and this means that the expression for $i_o$ is a linear function of $x$. Therefore,

$$i_o(x,t) = \frac{\partial i_o}{\partial x} x + i_T \tag{63}$$

The last term, $i_T$, of equation (63) is in agreement with the boundary condition that

at $x = 0$ $\qquad i_o = i_T \qquad$ and $i_i = 0$

For the determination of the slope of the straight-line relation the following condition is to be considered:

$$\text{at} \quad x = \ell \qquad\qquad i_o = 0$$

$$0 = \frac{\partial i_o}{\partial x}\ell + i_T$$

and

$$\frac{\partial i_o}{\partial x} = -\frac{i_T}{\ell} \tag{64}$$

Therefore

$$i_o = -\frac{i_T}{\ell}x + i_T \tag{65}$$

The combination of equations (62) and (64) gives

$$\frac{dv}{dt} = \frac{i_T}{C\ell} \tag{66}$$

which can be written as

$$j\omega v = \frac{i_T}{C\ell}$$

from which

$$v = -j\frac{i_T}{\omega C\ell} \tag{67}$$

A substitution of equation (67) into equation (61) gives

$$v_{app} = -j\frac{i_T}{\omega C\ell} + j\omega n^2\mu A\ell i_T$$

$$v_{app} = j\left(\omega n^2\mu A\ell - \frac{1}{\omega C\ell}\right)i_T \tag{68}$$

The ratio

$$\frac{v_{app}}{i_T}$$

is the coil impedance and therefore

$$Z = j\left(\omega n^2 \mu A \ell - \frac{1}{\omega C \ell}\right) \tag{69}$$

If the relations

$$n^2 \mu A \ell = \frac{L_T}{4}$$

$$C\ell = C_T$$

are used, equation (69) becomes

$$Z = j\left(\omega \frac{L_T}{4} - \frac{1}{\omega C_T}\right) \tag{70}$$

It can be said on the basis of equation (70) that the equivalent circuit of a type S-open coil is a series inductance-capacitance circuit of lumped values $L_T/4$ and $C_T$.

The result expressed by equation (70) indicates that a coil of type S-open behaves at high frequencies almost as a pure inductance, if its resistance can be neglected, of value

$$L = \frac{L_T}{4} \tag{71}$$

where $L_T$ is the low-frequency inductance of the ensemble of its two layers when they are connected to each other as in a type S-closed coil. Expression (70), however, is valid only for a limited range of frequencies above the one of the series resonance of the coil. Factors that are not important for the determination of the frequency of the series resonance become predominant at higher frequencies and invalidate expression (70).

The frequency of the series resonance of a type S-open air-core coil, as derived from equation (70), is

$$f = \frac{1}{\pi\sqrt{L_T C_T}} \tag{72}$$

This frequency is slightly higher than the frequency of the first resonance, a series one, of a type M-open coil.

# CHAPTER V

## DETERMINATION OF RESONANT FREQUENCY OF TYPE S-CLOSED COIL

A typical impedance-frequency curve of a type S-closed air-core coil appears in Figure 7. The only prominent resonance that the coil has is a parallel one. The voltage between any two adjacent points of the two layers is at all frequencies equal to one half of the applied voltage.

Figure 32 may be consulted in connection with the calculations that follow.

A relation enuntiated in the previous chapter is

$$v_{app} = \int_x^0 \frac{\partial e_o}{\partial x} dx + v + \int_\ell^x \frac{\partial e_i}{\partial x} dx \qquad (55)$$

which can also be written as

$$v_{app} = \int_\ell^0 \frac{\partial e_o}{\partial x} dx + \int_x^\ell \frac{\partial e_o}{\partial x} dx + v + \int_\ell^x \frac{\partial e_i}{\partial x} dx \qquad (73)$$

If equations (57) and (58) are reexamined it can be seen that

$$\frac{\partial e_o}{\partial x} = \frac{\partial e_i}{\partial x} \qquad (74)$$

If equation (74) is substituted into equation (73) it

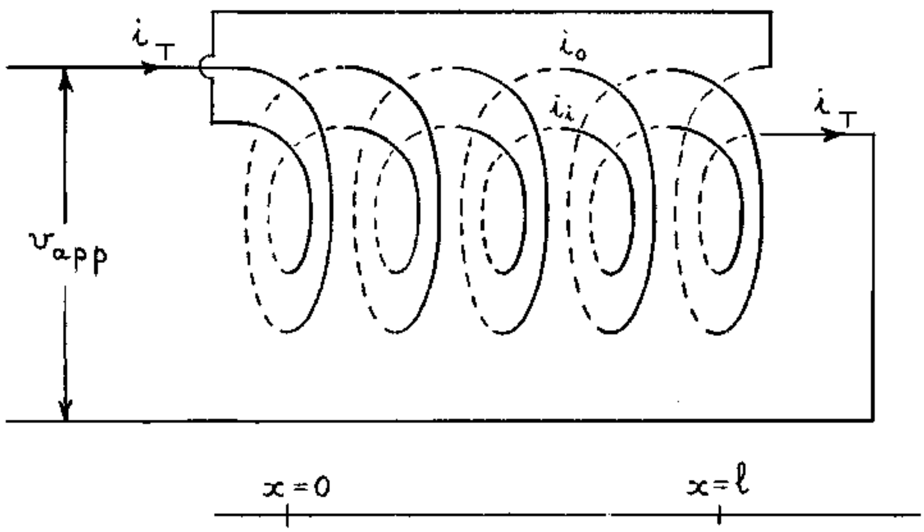



Figure 32. Type S-closed Coil.

is obtained that

$$v_{app} = \int_{\ell}^{0} \frac{\partial e_o}{\partial x}\, dx + v \tag{75}$$

Since

$$\int_{\ell}^{0} \frac{\partial e_o}{\partial x}\, dx = \frac{v_{app}}{2}$$

then

$$v_{app} = \frac{v_{app}}{2} + v$$

and

$$\frac{v_{app}}{2} = v \tag{76}$$

which shows that $v$ is constant along the length of the coil.

If Kirchhoff's law is used along the entire length of the coil it follows that

$$v_{app} = \int_{\ell}^{0} -n^2 \mu A \left(\frac{\partial i_o}{\partial t} + \frac{\partial i_i}{\partial t}\right) dx + \int_{\ell}^{0} -n^2 \mu A \left(\frac{\partial i_o}{\partial t} + \frac{\partial i_i}{\partial t}\right) dx$$

$$v_{app} = 2n^2 \mu A \int_{0}^{\ell} \frac{\partial i_o}{\partial t}\, dx + 2n^2 \mu A \int_{0}^{\ell} \frac{\partial i_i}{\partial t}\, dx$$

$$v_{app} = 2n^2 \mu A \int_{0}^{\ell} \left(\frac{\partial i_o}{\partial t} + \frac{\partial i_i}{\partial t}\right) dx \tag{77}$$

Equation (62) is

$$\frac{\partial i_o}{\partial t} = -C \frac{dv}{dt} = -j\omega C v \tag{62}$$

A similar relation can be written for the current in the inside layer:

$$\frac{\partial i_i}{\partial x} = C\frac{dv}{dt} = +j\omega C v \tag{78}$$

The difference between equations (62) and (78) is due to the fact that as $x$ is increased $i_o$ decreases and $i_i$ increases. It follows from equations (62) and (78) that $i_o$ and $i_i$ can be expressed by linear functions of $x$. Thus

$$i_o = i_T + \frac{\partial i_o}{\partial x}x \tag{79}$$

where $i_T$, a constant, satisfies the boundary condition that at $x=0$, $i_o = i_T$. Similarly

$$i_i = B_5 + \frac{\partial i_i}{\partial x}x$$

where $B_5$ is a constant that must be determined.

At $x = \ell$

$$i_i(x=\ell) = B_5 + \frac{\partial i_i}{\partial x}\ell = i_T$$

and

$$B_5 = i_T - \frac{\partial i_i}{\partial x}\ell$$

With the use of equations (62) and (78) the expressions for $i_o$ and $i_i$ become:

$$i_o = i_T - j\omega C v x \tag{80}$$
$$i_i = i_T + j\omega C v(x - \ell) \tag{81}$$

Equations (80) and (81) are in agreement with the known fact that $i_o$ at $x = \ell$ is equal to $i_i$ at $x = 0$. The current in the connection between the layers is

$$i_o(x=\ell) = i_i(x=0) = i_T - j\omega C v \ell$$

The sum of the currents in the two layers is

$$i_o + i_i = 2 i_T - j\omega C v \ell \tag{82}$$

If equation (82) is differentiated with respect to time there is obtained:

$$\frac{\partial i_o}{\partial t} + \frac{\partial i_i}{\partial t} = 2\frac{d i_T}{dt} - j\omega C \frac{dv}{dt}\ell$$
$$\frac{\partial i_o}{\partial t} + \frac{\partial i_i}{\partial t} = j\omega(2 i_T - j\omega C v \ell) \tag{83}$$

A substitution of equation (83) into equation (77) gives

$$v_{app} = j\omega 2 n^2 \mu A \int_0^{\ell} (2 i_T - j\omega C v \ell)\, dx$$

and since none of the terms under the integral sign is a function of $x$

$$v_{app} = j\omega 2 n^2 \mu A (2 i_T - j\omega C v \ell)\ell \tag{84}$$

With the use of equation (76), equation (84) becomes

$$v_{app} - \omega^2 n^2 \mu A \ell^2 C v_{app} = j\omega 4 n^2 \mu A \ell i_T \tag{85}$$

The substitution of relations

$$C\ell = C_T$$
$$4n^2\mu A\ell = L_T$$

into equation (85) gives

$$v_{app}\left(1 - \omega^2 \frac{L_T C_T}{4}\right) = j\omega L_T i_T$$

and thus

$$Z = \frac{j\omega L_T}{1 - \omega^2 \frac{L_T C_T}{4}} \qquad (86)$$

which can also be written as

$$Z = \frac{j\omega L_T (-j) \frac{1}{\omega \frac{C_T}{4}}}{j\left(\omega L_T - \frac{1}{\omega \frac{C_T}{4}}\right)} \qquad (87)$$

An examination of equation (87) reveals that the type S-closed air-core coil is equivalent to an inductance-capacitance parallel circuit of values $L_T$ and $C_T/4$.

From equation (86) it is found that the resonant frequency of a type S-closed air-core two-layer coil is

$$f = \frac{1}{\pi\sqrt{L_T C_T}} \qquad (88)$$

It is noted that both type S-open and type S-closed coils have their resonance at the same frequency if, with the exception of the interlayer connection, they are constructed identically.

# CHAPTER VI

## RESONANCES AND OVERVOLTAGES IN TRANSFORMER WINDINGS UNDER NO-LOAD CONDITIONS

Since transformer windings are basically iron-cored coils, their impedance-frequency characteristics are modified versions of the characteristics of the coils of which they have been constructed. When a coil becomes part of a transformer, its impedance-frequency curve is altered because of the influences of the iron-core and the other windings. How an iron-core alone affects the frequency curve of a coil can be learned from an examination and comparison of the curves of Figures 33 and 34. Figure 33 shows the impedance-frequency characteristic of a two-layer air-core coil of type M-closed. Figure 34 shows the characteristic of the same coil when an iron-core is added to it. It can be seen that the total number of resonances is the same in both cases: neither new resonances have been created nor any have been made to disappear. The effect of the addition of the iron-core is the lowering of the frequencies and the Q's of some of the existing resonances. This effect, which is quite pronounced on resonances that occur in the lower side of the spectrum, becomes progressively smaller at higher frequencies. It appears that

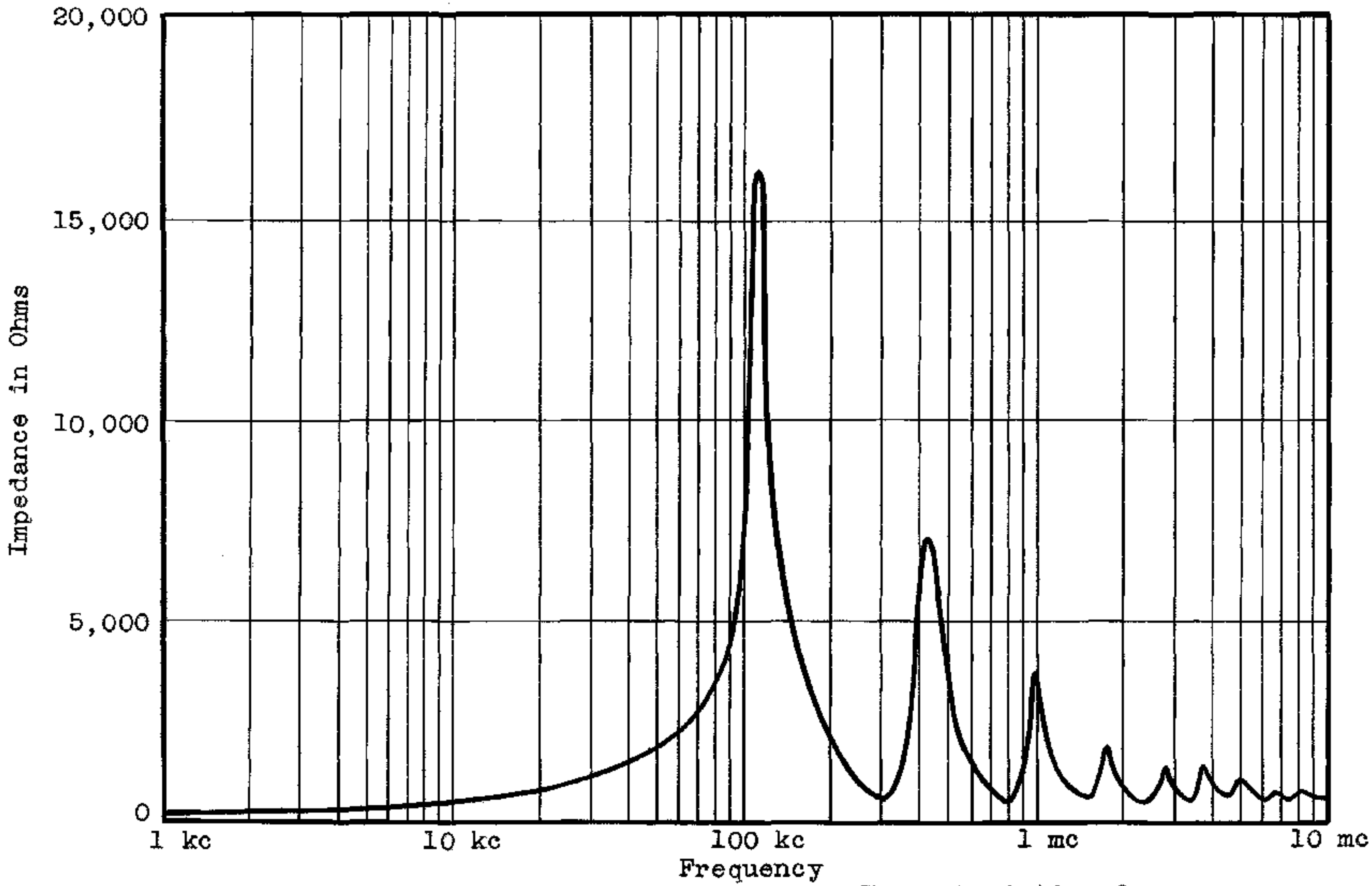



Figure 33. Impedance-frequency Characteristic of Two-layer Type M-closed Air-core Coil No. C-14.

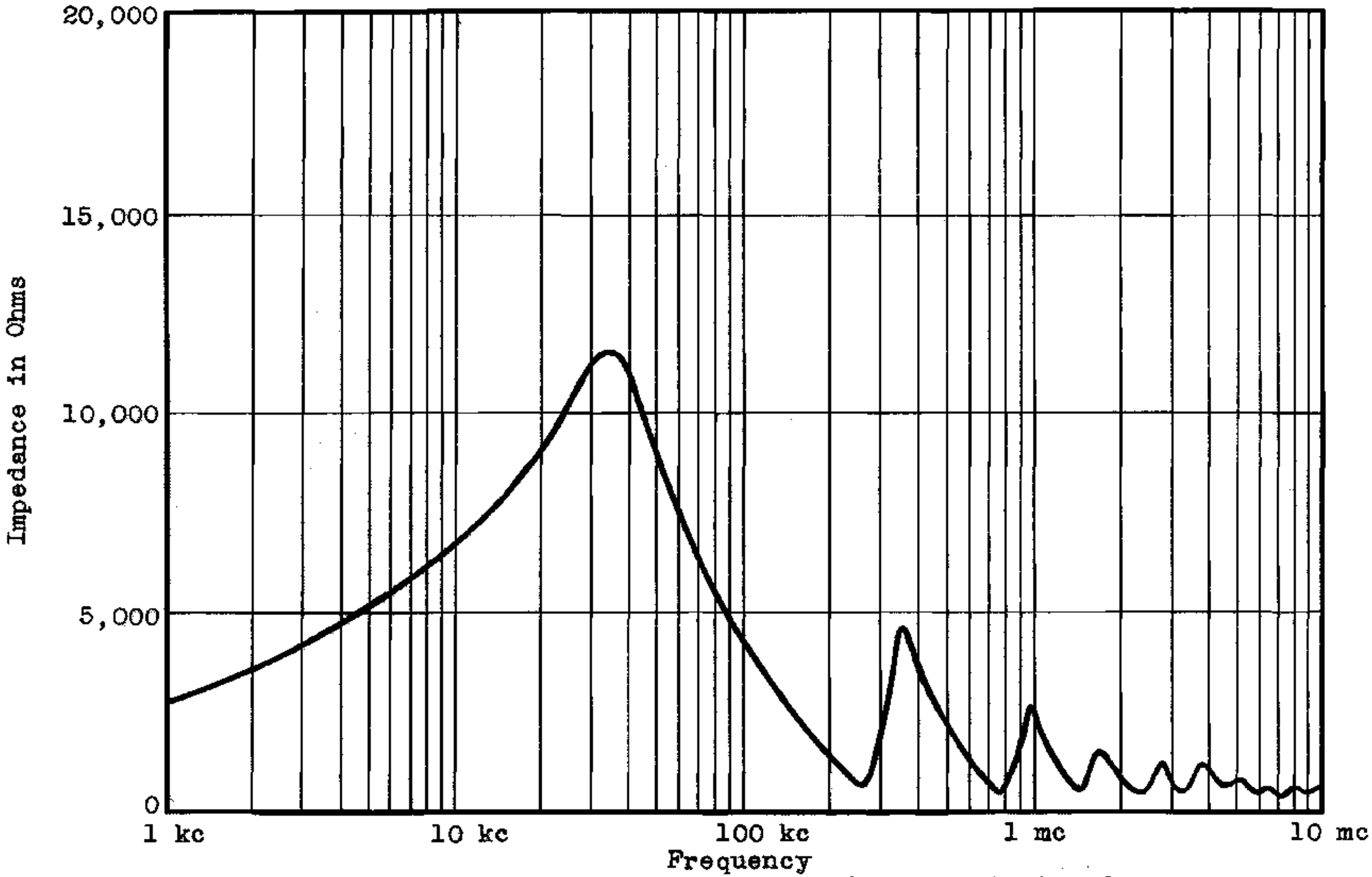


Figure 34. Impedance-frequency Characteristic of
Two-layer Type M-closed Iron-core Coil No. C-14.

resonant frequencies and Q's of impedance peaks cease to be affected by the addition of an iron-core above the frequencies of 200 kc and 1 mc respectively.

Since transformer windings are mutually coupled through the iron-core, resonances that are a property of one of them appear also in the spectra of the others. The effect of the iron-core in linking magnetically the windings becomes smaller at higher frequencies and, for practical purposes, it ceases to exist above 100 kc in transformers designed for operation at 60 cps. Therefore only the resonances of windings that occur below 100 kc are reflected into other windings. Figures 35 through 38 show the impedance-frequency curves of the low- and high-voltage sides of two transformers. In these cases the low-voltage sides display two prominent resonances whereas the high-voltage sides only one. Each one of these windings, however, would have only one resonance if taken out of its transformer and tested, if it is assumed that it is made of more than two layers of turns. Of the two windings of each one of these transformers, the low-voltage one, having the smaller number of turns and layers, will have the higher frequency resonance. The difference in the number of observable resonances between the two sides is due to the fact that a low-frequency resonance, created by a high-voltage winding and occurring in the frequency region of below 100 kc, appears twice, one time in its own side and another time in the other side by reflection, whereas a high-frequency resonance,

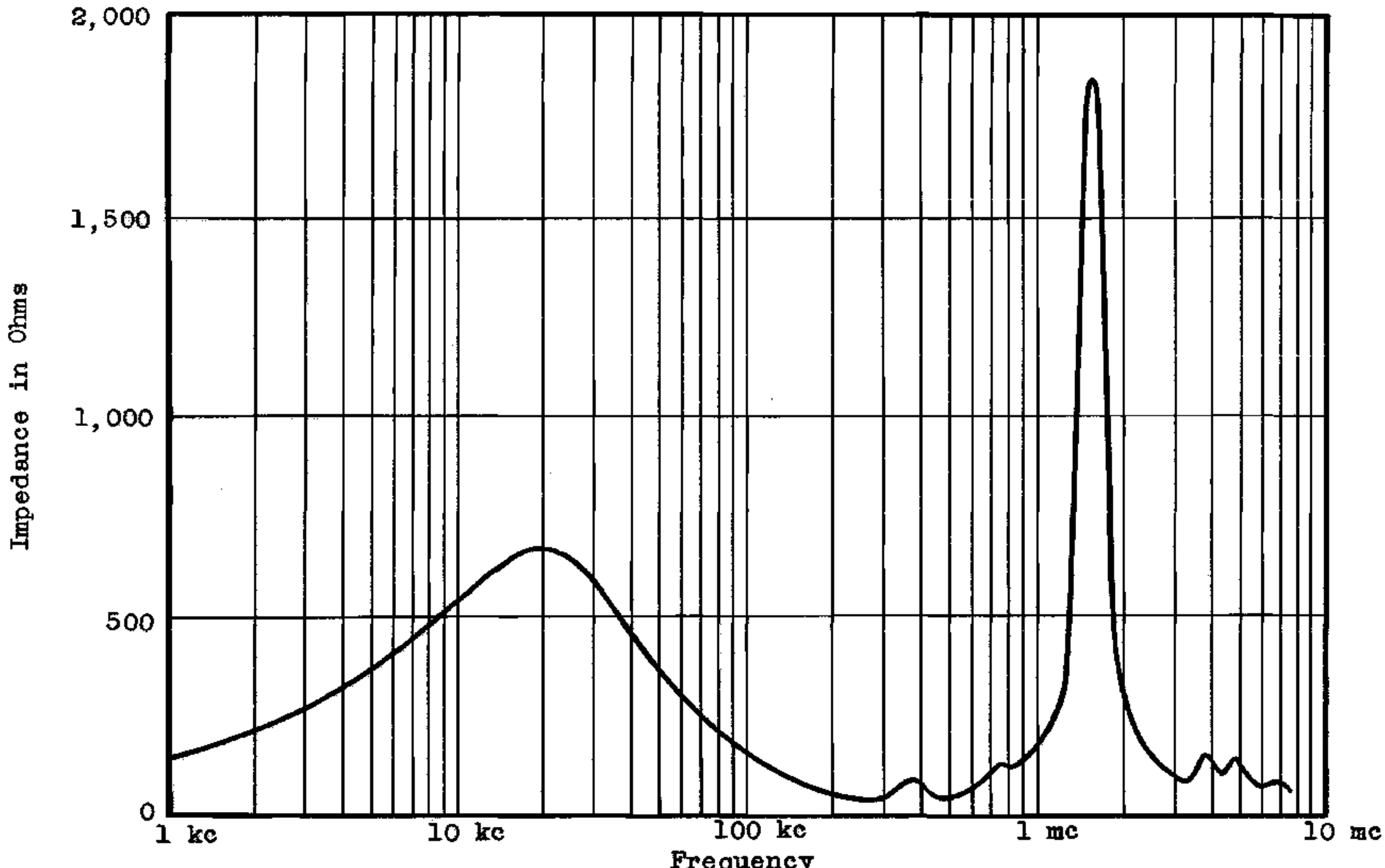



Figure 35. Impedance-frequency Characteristic of the Low-voltage Winding of Transformer No. T-14.

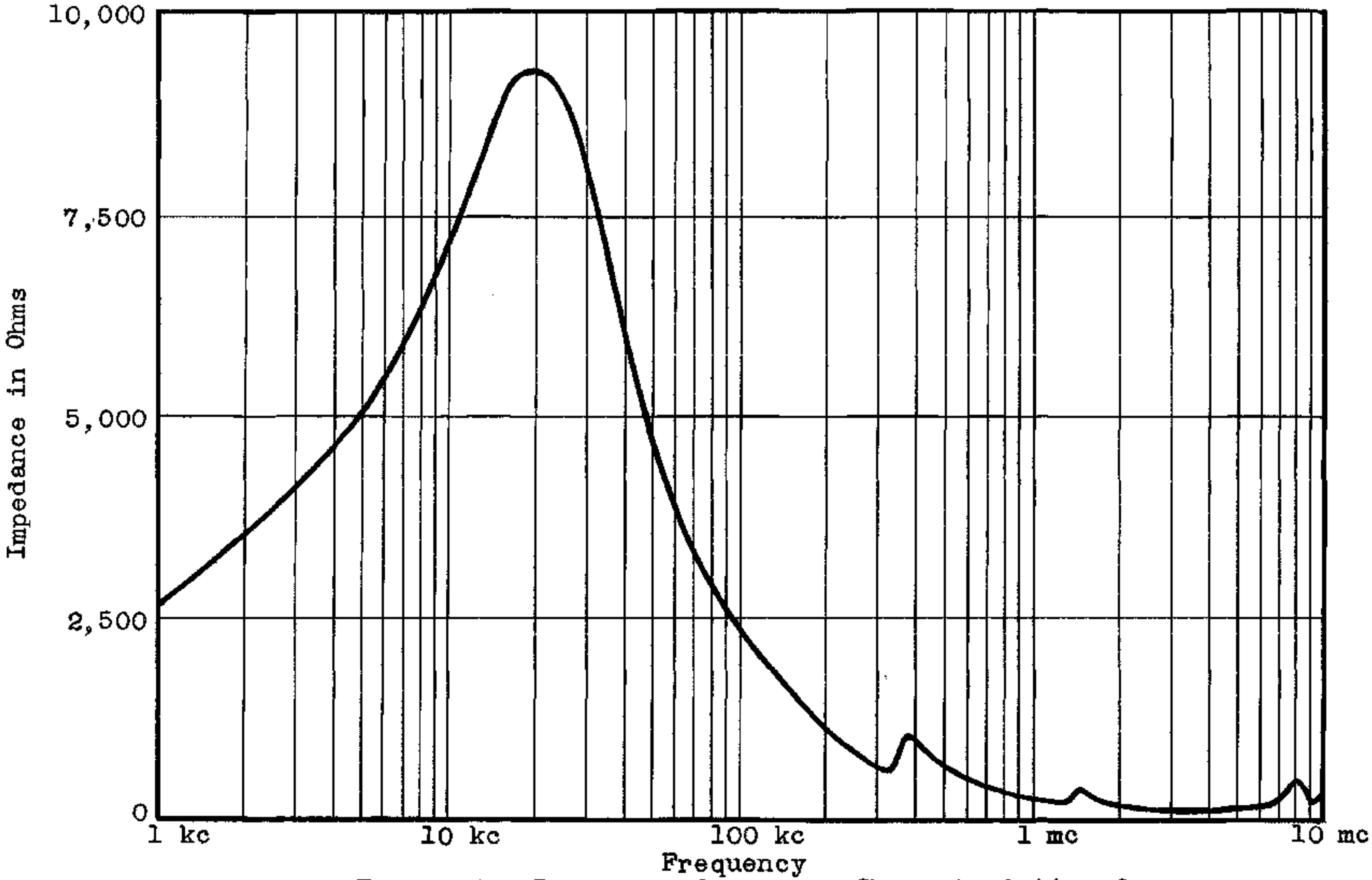



Figure 36. Impedance-frequency Characteristic of
the High-voltage Winding of Transformer No. T-14.

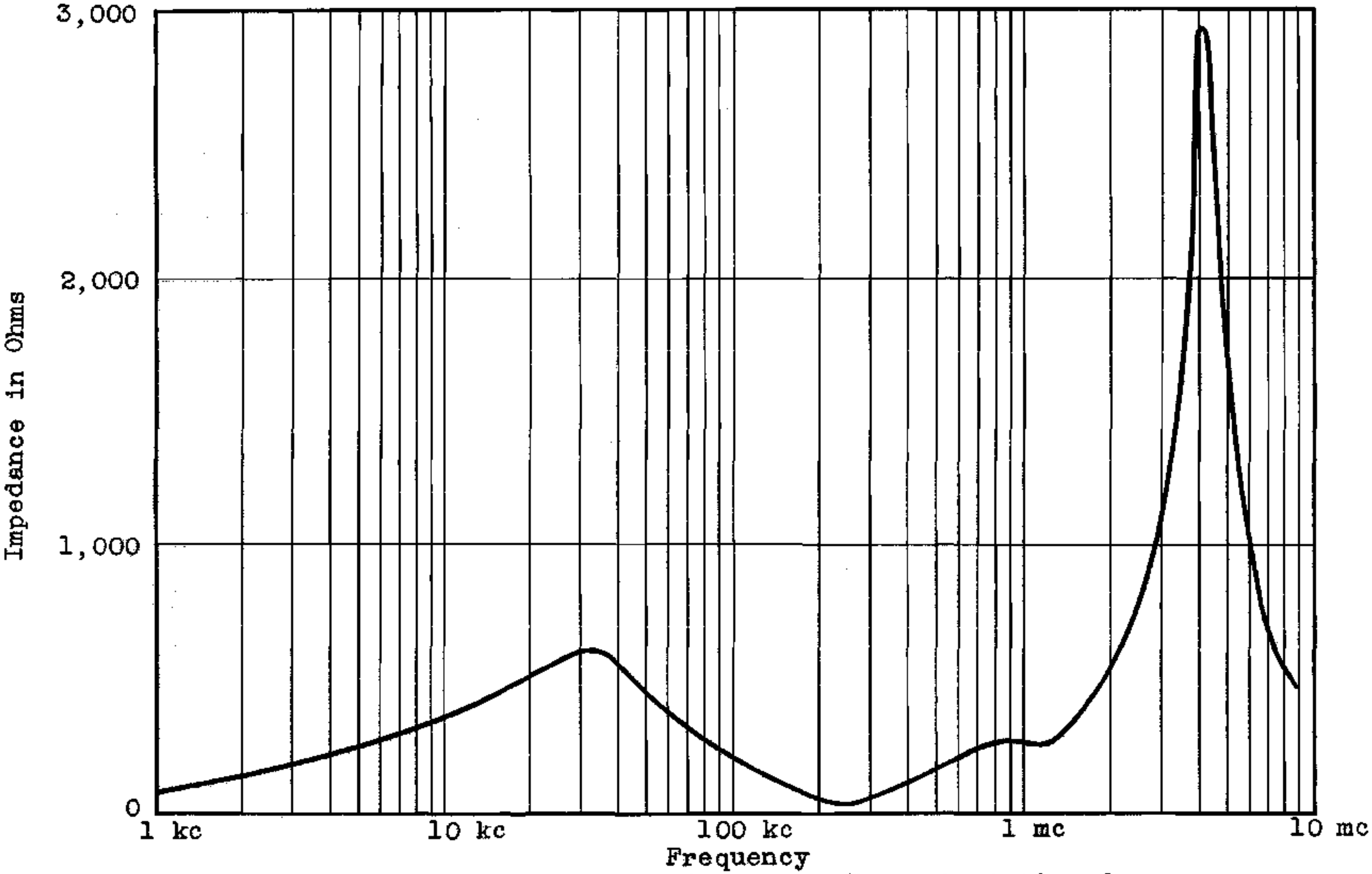



Figure 37. Impedance-frequency Characteristic of
the Low-voltage Winding of Transformer No. T-80.

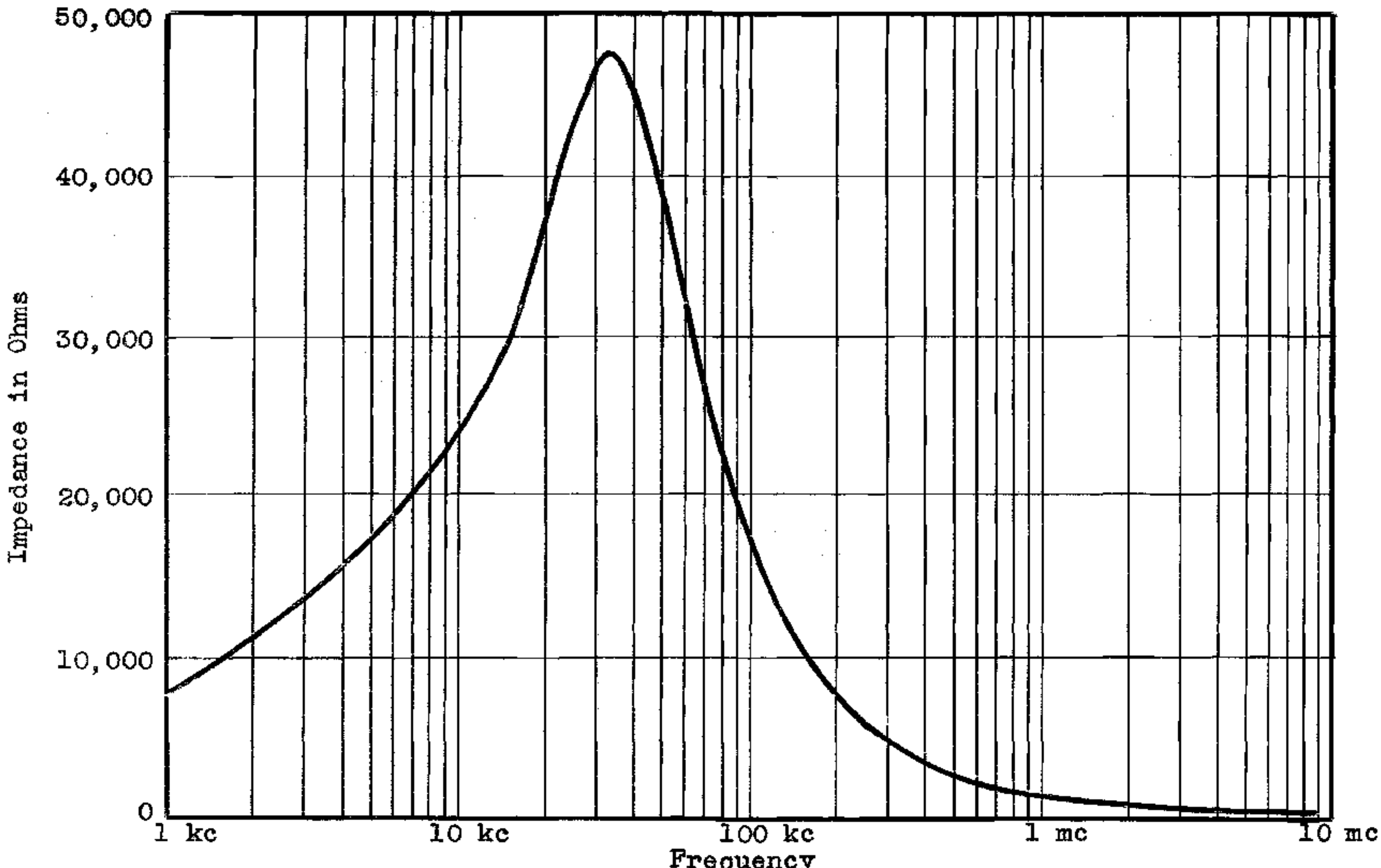


Figure 38. Impedance-frequency Characteristic of the High-voltage Winding of Transformer No. T-80.

occurring much above 100 kc, cannot be reflected to another side and thus appears only once in its own side.

Some of the resonances of transformer windings are responsible, under appropriate circumstances, for the creation of overvoltages. The first situation to be described in which an overvoltage can be developed is in connection with the series resonance of the low-voltage winding. The high- to low-voltage side voltage ratio of a transformer is not constant at all frequencies. Whenever it increases above its normal value, an overvoltage exists across the high-voltage side. Figures 39 and 40 show two typical variations with frequency of the normalized voltage ratios (high- to low-voltage sides) of two transformers. The two highest overvoltages are in each case 3.92 to 1 and 3.40 to 1 respectively. In these cases the low-voltage sides were connected to a variable frequency supply and the high-voltage sides were unloaded. It can be observed that the normalized voltage ratios stay constant and equal to one for a large portion of the spectrum. Eventually, however, they start increasing and after they go through one or more maxima in their variations, they decrease to an insignificant value. A transformer loses its usefulness when this low value of voltage ratio is reached because, naturally, no significant amount of power can then be transferred from one of its sides to the other. If Figures 39 and 40 are compared with Figures 35 and 37, which give the impedance-frequency curves of the low-voltage sides of the two transformers in question, it will be noticed that the maxima of the voltage ratios occur in the neighborhood of frequencies at which the low-voltage side (supply-side) impedances are minimum. These impedances go through a minimum, and are then composed effectively of resistance only, at frequencies where the overall capacitive and inductive effects cancel each other. At such a frequency the potential difference across the low-voltage side is equal to the product of its effective resistance by the current flowing; the potential difference across the other side is equal to the number of its turns multiplied by the rate of change of the mutual flux created by the current in the low-voltage side. The voltage ratio of the latter to the former is a number greater than the ratio of turns of the high- to the low-voltage side. It is helpful to remember at this point that the well-known relation that the ratio of the secondary voltage to the primary voltage is equal to the ratio of the secondary turns to the primary turns, which is valid for a considerable portion of the spectrum, was derived for conditions where the resistive and capacitive effects are negligible when compared with the inductive effect; at the point of maximum voltage ratio the previous approximation cannot be made and hence this relation is no longer valid; it is the resistance parameter alone, as it was stated previously, that is of importance at such an isolated point of the frequency spectrum.

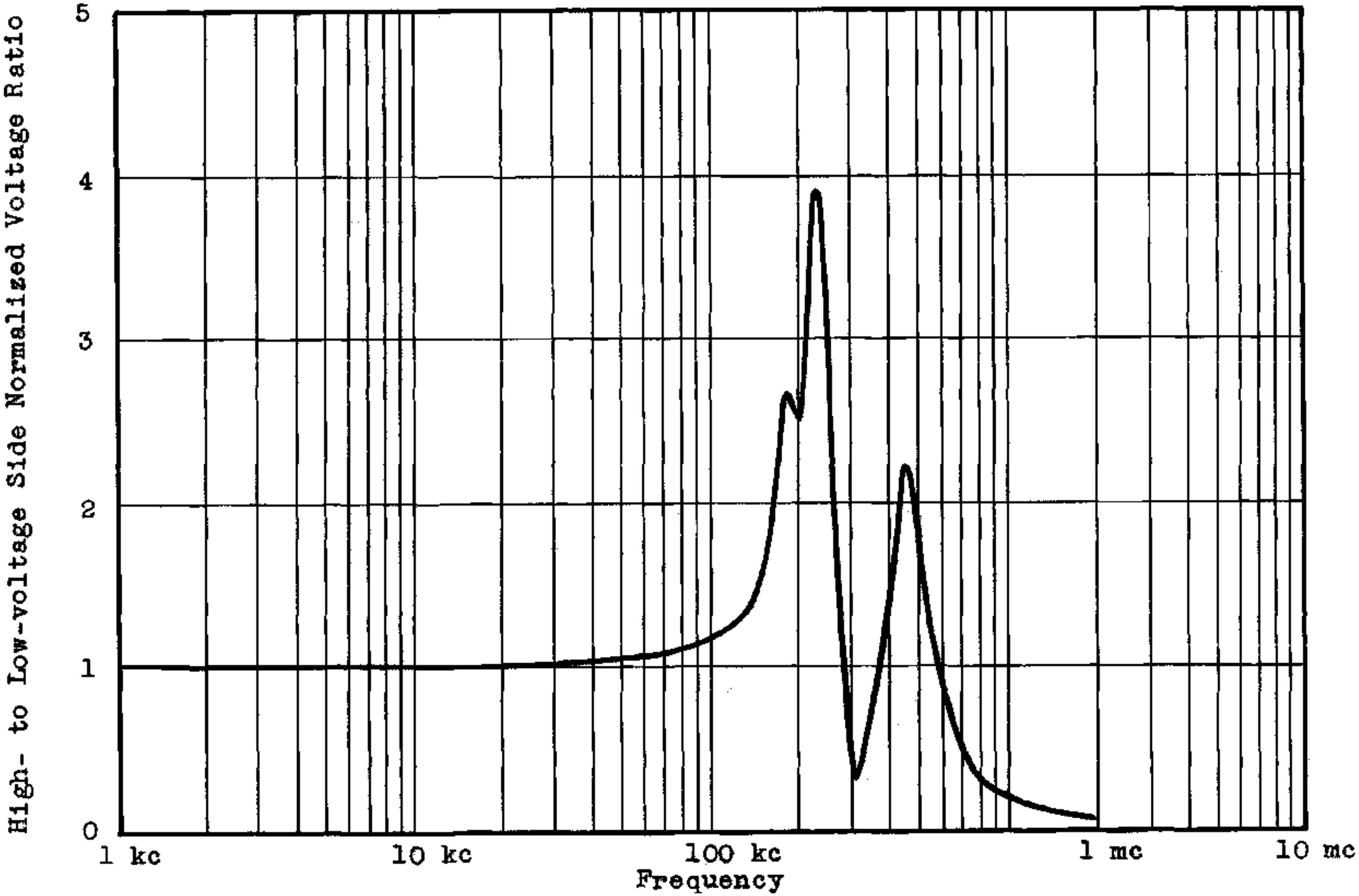


Figure 39. Variation of Normalized Voltage Ratio with Frequency of Transformer No. T-14.

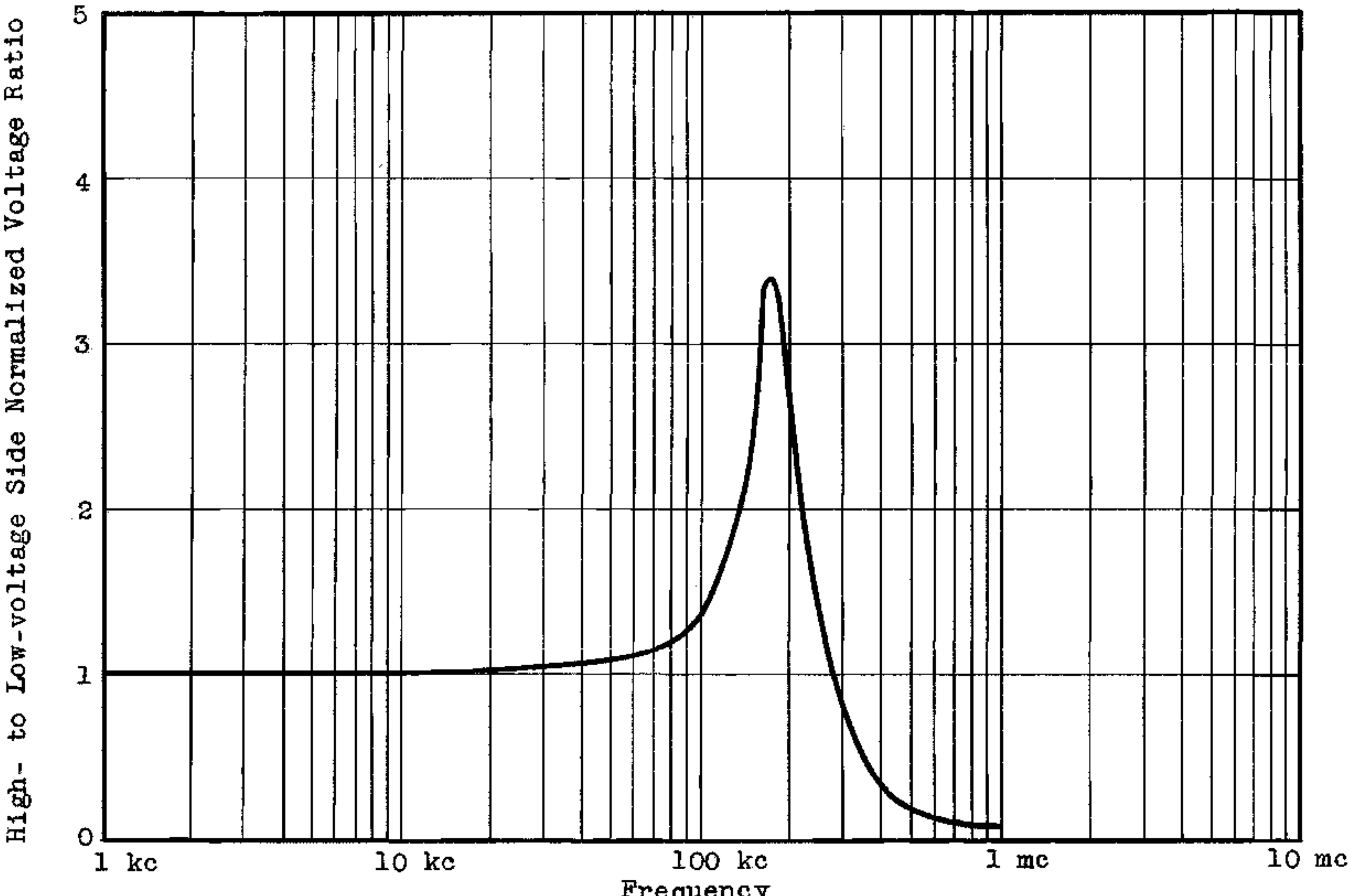



Figure 40. Variation of Normalized Voltage Ratio with Frequency of Transformer No. T-80.

The overvoltage of the high-voltage side at a point of minimum impedance of the low-voltage side is considered to be a curiosity rather than a nuisance because of the unusual frequencies at which it occurs.

A second situation in which an overvoltage can be made to appear in a transformer is associated with the second prominent parallel resonance of its low-voltage winding. This resonance, as explained previously, is manifested through the properties of the winding itself and is not made to appear in this winding through reflection from another winding. Figures 35, 37, and 41 (resonance at 4.4 mc) show such resonances. Figure 42 presents the frequency curve of the impedance, to be designated by $Z_{AN}$, between the iron-core and one end of the low-voltage winding of the transformer that is associated with the results of Figure 41. The distributed capacitance between iron-core and low-voltage winding forms the seemingly nonexistent connection between the two terminal points of this impedance. Figure 43 shows the circuit arrangement that was used for the detection of the overvoltages; in this figure points A and N can be identified as the terminal points of the impedance $Z_{AN}$. The frequency curve of $Z_{AN}$ in Figure 42 has two minima which, with an examination of Figure 41, can be seen to exist at frequencies a little before and a little after the frequency of the prominent parallel resonance of the low-voltage side. Overvoltages appear across the low-voltage winding at frequencies in the neighborhood of each one of the two minima of impedance. By "overvoltage" it is meant that the ratio of the voltage across the low-voltage winding, with terminals A and B, to the supply voltage across terminals A and N, is larger than one. The two overvoltages in the case of the transformer associated with Figures 41 and 42 are: 8.50 to 1 at 2.32 mc and 2.83 to 1 at 6.60 mc.

The magnitudes of the overvoltages that may be detected in transformers under circumstances as described previously depend, of course, on the particular constants of these transformers. The above overvoltage of 8.50 to 1 is considered unusually high since it is the largest one found from tests on five transformers. The overvoltage of the latter case, as the one of the former, is also regarded as an oddity disassociated with any responsibility for transformer failures.

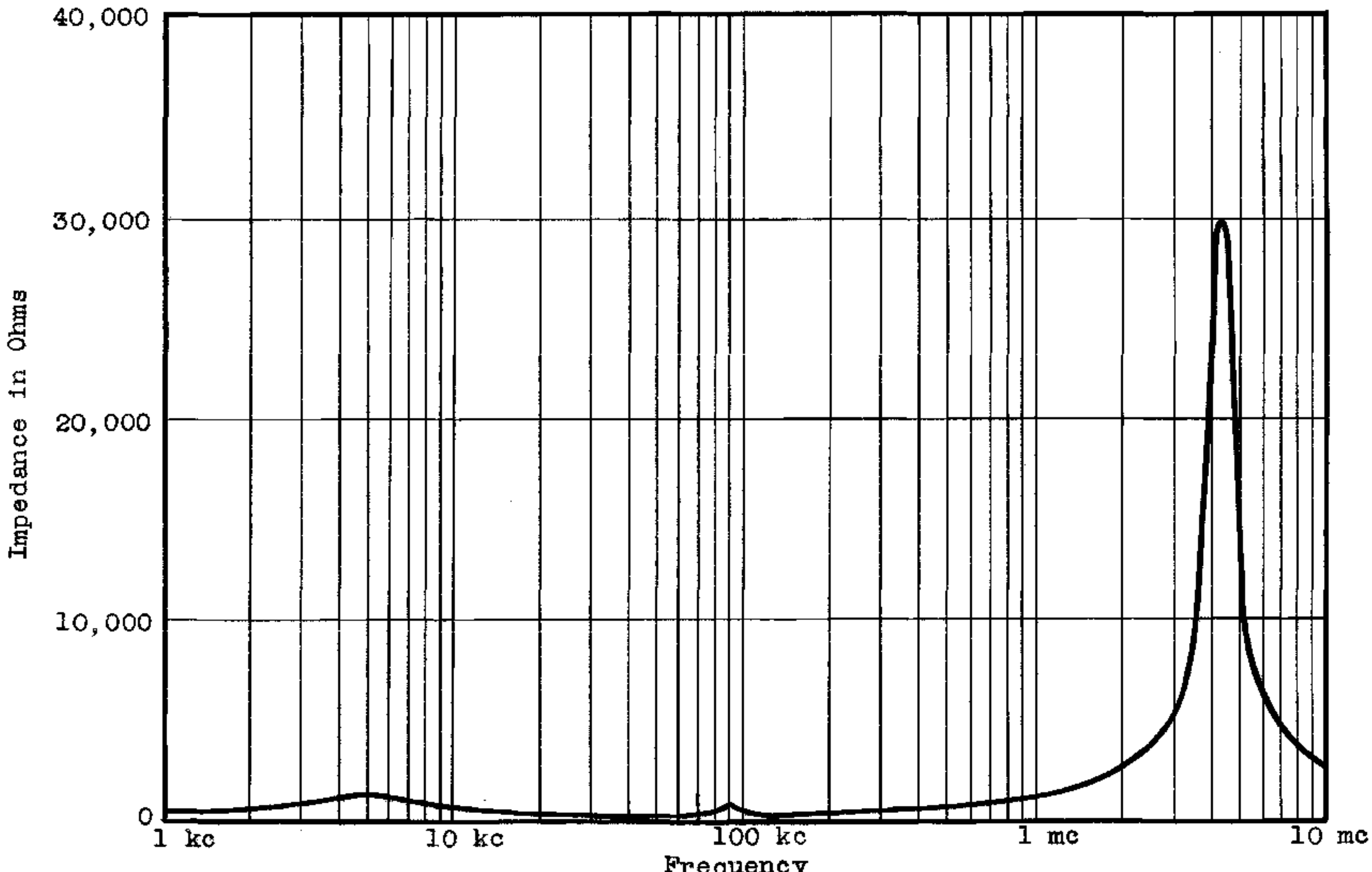



Figure 41. Impedance-frequency Characteristic of the Low-voltage Winding of Transformer No. T-90.

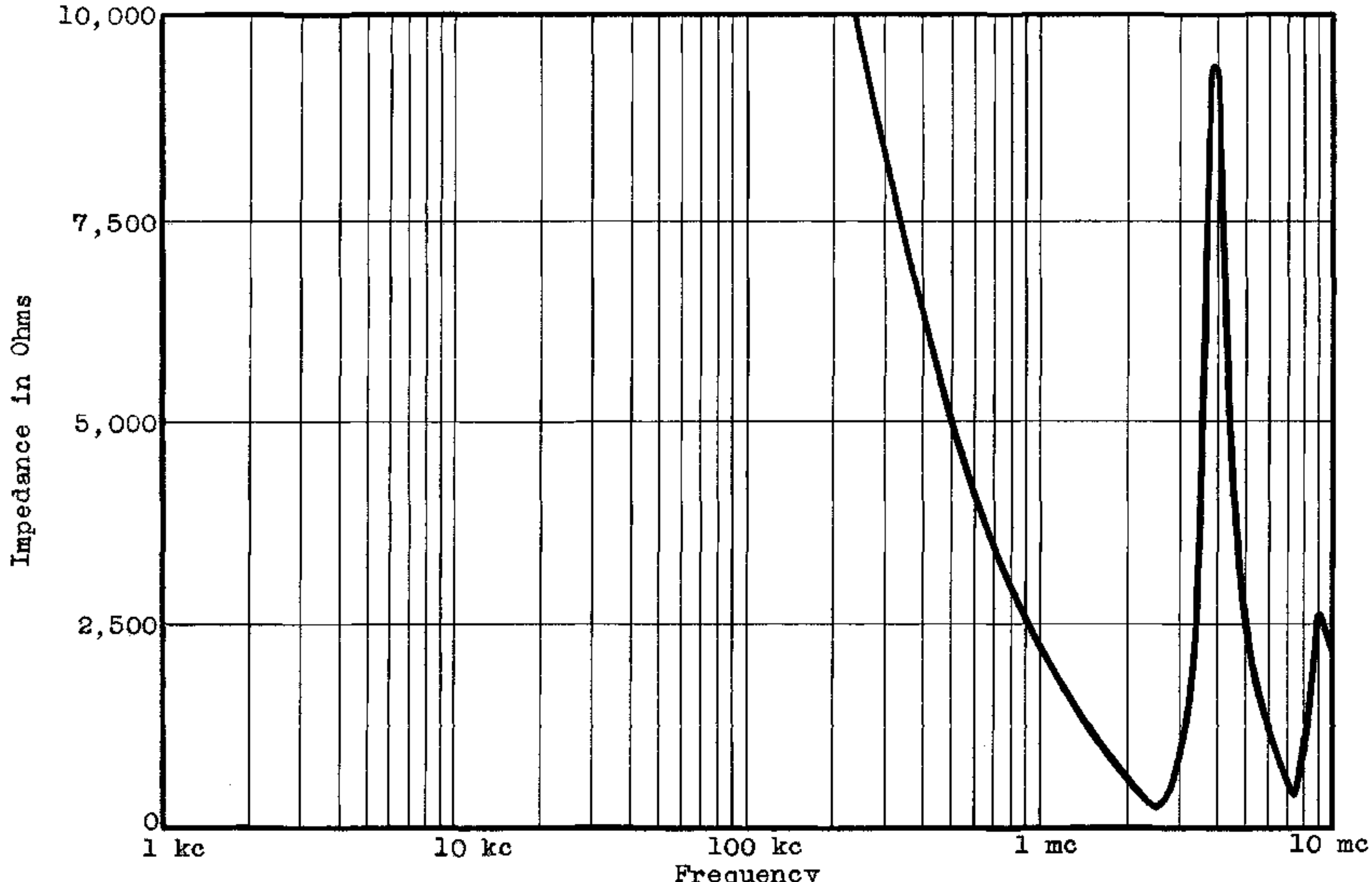
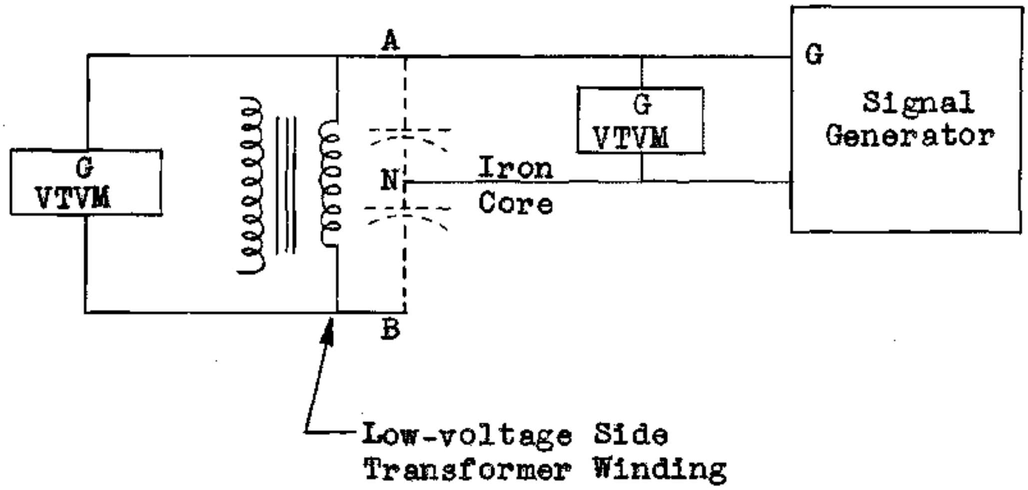



Figure 42. Variation with Frequency of the Impedance between the Iron Core and One Terminal of the Low-voltage Winding of Transformer No. T-90.




VTVM: Vacuum Tube Voltmeter

G : Ground Terminal of Electronic Instrument

Figure 43. Circuit Arrangement for Detecting the Overvoltages Associated with the Parallel Resonance of the Low-voltage Winding of a Transformer.

# CHAPTER VII

## CONCLUSION

General.--The preceding treatment describes mathematically and graphically the properties of two-layer coils and also focuses attention on some high-frequency aspects of power transformers. In connection with two-layer coils the mathematical treatment specifically accomplishes the following: it confirms that there is a similarity between the properties of a type M coil and a two-parallel-wire transmission line since both are described by the same differential equation; it justifies the results obtained experimentally and thus eliminates doubts about their correctness; in some cases it shows how the properties of these coils are affected by variations in the coil parameters; it indicates indisputably that the basic difference between a type M coil and a type S coil is that the former has a wave-equation behavior but the latter has a lumped-element behavior; it provides formulas for the calculation of some resonant frequencies of the two types of coils which in turn allows estimates to be made of the upper limits of the frequency ranges in which both coils behave as inductors.

Limitations.--Because of the many simplifications that are made in the analysis, many of the results serve only to justify the existence of the observed phenomena and not to calculate actual values of electrical quantities. Simplifications made are the omission of the turn-to-turn capacitance in a lengthwise direction, the omission in some cases of the resistance, the assumption of an inductance per unit length that is constant at all points along the length of the coil and at all frequencies, the assumption of a constant capacitance per unit length which amounts to the neglect of the end effect, and the omission of the skin effect which affects the high-frequency resistance.

The high-frequency resistance of an isolated conductor varies approximately as the square root of the frequency. No expression, however, is available for the high-frequency resistance of the wire of a coil where so many conductors are near each other. It is necessary, nevertheless, to assume a reasonable increase with frequency in the effective resistance of the coil wire in order to be able to deduce, from an examination of the coil expressions, realistic trends of the coil properties with increasing frequencies.

Validity of assumed relation between flux and current in a coil.--The agreement of some results calculated from theoretical expressions with their corresponding measured values indicates that the initially assumed relation between flux and current in a coil is valid, since the entire theory is based on it. This relation, which must not be considered to be exact, is

$$\phi(x,t) = n\mu A \sum_{s=1}^{s} i_s(x,t)$$

for a coil having s layers of turns, a cross-sectional area A, and n turns per unit length in each layer. The restriction that succeeding turns must be as close to each other as possible must be added. This relation may prove useful in the analysis of other types of coils.

Evaluation of expressions for the first resonant frequency of a type M coil.--The two expressions found for the first resonant frequency of a type M coil are

$$f = \frac{1}{8n\ell\sqrt{\mu AC}} \tag{15}$$

and

$$f = \frac{1}{4\sqrt{L_T C_T}} \tag{18}$$

The first expression shows to advantage how the first resonant frequency depends on the coil parameters. It can be seen, for instance, that if the coil length is doubled while all the other factors are left unchanged, the frequency is halved. This expression makes use of an approximate formula for the inductance of a coil: the formula gives an inductance value that is ten per cent high for a coil of length to diameter ratio of four but it improves in accuracy for coils of larger length to diameter ratios.

The second expression is more suitable than the first for the calculation of a resonant frequency since it does not force the use of an approximate inductance formula. Table 2 gives a comparison between calculated and measured resonant frequencies of type M and type S coils. The values used for $L_T$ and $C_T$ were obtained by bridge measurements. For type M coils the per cent differences between the two corresponding frequencies range between -7.4 per cent and -24 per cent. The per cent differences that are over 20 per cent, however, are for coils made of cotton-enamel insulated wire that keeps succeeding turns, because of its thick insulation, further apart than does the plain enameled wire of the remaining coils. The coils made of enameled wire gave per cent differences ranging between -7.4 per cent and -14 per cent. The consistency of the direction of the divergence between the two results suggests that if the calculated frequency is multiplied by a certain correction factor, for instance by 1.1 or 1.2 according to whether the wire is enamel- or cotton-enameled-insulated or by some other correction factor appropriate for another condition, a frequency value closer to the true one will be obtained.

Another expression, differing from the others by including the resistance parameter, for the first resonant frequency of a type M coil can be obtained by setting $\beta \ell = \frac{\pi}{2}$. Since

$$\beta = \frac{1}{\ell}\sqrt{\frac{1}{2}\left[\omega C_T \sqrt{R_T^2 + \omega^2 L_T^2} + \omega^2 L_T C_T\right]}$$

Table 2. Comparison between Calculated and Measured Resonant Frequencies of Two-layer Coils

| Coil Type | Coil Number | Calculated Frequency in kc | Measured Frequency in kc | Per Cent Difference |
|---|---|---|---|---|
| M | C- 14 | 98.7 | 110.0 | -10 |
| M | C- 17 | 492.0 | 541.0 | - 9.1 |
| M | C- 21 | 132.1 | 170.0 | -22 |
| M | C- 25 | 6.7 | 7.8 | -14 |
| M | C- 26 | 18.2 | 24.0 | -24 |
| M | C- 28 | 43.8 | 55.8 | -22 |
| M | C-120 | 6.1 | 6.6 | - 7.4 |
| M | C-128 | 1.9 | 2.1 | -10 |
| S | C-122 | 24.6 | 24.0 | + 2.5 |
| S | C-124 | 8.0 | 8.2 | - 2.4 |

In the case of a type M coil, the resonant frequency given is the lowest one of all its resonant frequencies.

the expression for the frequency is

$$f = \frac{1}{4\sqrt{L_T C_T \left(1 + \frac{C_T R_T^2}{L_T \pi^2}\right)}}$$

It can be seen that the inclusion of the coil resistance in the calculation results in a frequency value lower than the one calculated with expression (18). This widens the difference between calculated and measured results. It must be remarked, however, that the contribution of the term $\frac{C_T R_T^2}{L_T \pi^2}$ is negligible. For example, for the coil No. C-128 which has the highest $C_T$ and $R_T$ of all the coils tested, but also the highest $L_T$, the term $\frac{C_T R_T^2}{L_T \pi^2}$ is 0.00146 which is negligible in comparison with one. It is fortunate that the effective resistance of a coil, which cannot be known exactly at high frequencies because of the lack of an expression properly taking the skin effect into account, plays such an insignificant role in the determination of the resonant frequency.

Expression (18) is identical with the expression for the first resonant frequency of a lossless transmission line. It is well known that a transmission line is in resonance when its length is one quarter of the wavelength long:

$$\ell' = \frac{\lambda}{4} \tag{89}$$

A relation between the wavelength, the frequency, and the phase velocity is

$$\lambda f = v \tag{90}$$

The phase velocity of a lossless transmission line is

$$v = \frac{1}{\sqrt{L'C'}} \tag{91}$$

The substitution of relations (90) and (91) into relation (89) gives

$$\ell' = \frac{1}{4}\,\frac{v}{f}$$

$$\ell' = \frac{1}{4}\,\frac{1}{f\sqrt{L'C'}}$$

$$f = \frac{1}{4\ell'\sqrt{L'C'}} = \frac{1}{4\sqrt{(L'\ell')(C'\ell')}}$$

$$f = \frac{1}{4\sqrt{L'_T C'_T}}$$

which is identical with expression (18).

The two-layer type M coil is, within the limits of accuracy of this theory, a scaled down version from the frequency point of view of a transmission line. Coils, when compared with transmission lines, have an extraordinarily large concentration of inductance and capacitance per unit length. This is the reason why the phenomena that occur in transmission lines at high frequencies occur in coils at so much lower frequencies.

<u>Evaluation of the expression for the absolute value of the impedance of a type M-closed coil.</u>--The expression for the absolute value of the impedance of a type M-closed coil is

$$|Z_s| = \frac{\sqrt{\alpha^2+\beta^2}}{\omega C}\sqrt{\frac{\sinh^2\alpha l+\sin^2\beta l}{\sinh^2\alpha l+\cos^2\beta l}} \tag{33}$$

which can also be written as

$$|Z_s| = \sqrt[4]{\frac{R_T^2+\omega^2 L_T^2}{\omega^2 C_T^2}}\sqrt{\frac{\sinh^2\alpha l+\sin^2\beta l}{\sinh^2\alpha l+\cos^2\beta l}} \tag{92}$$

or

$$|Z_s| = |Z_o|\sqrt{\frac{\sinh^2\alpha l+\sin^2\beta l}{\sinh^2\alpha l+\cos^2\beta l}} \tag{93}$$

The high-frequency limit of $Z_o$ is

$$Z_{o,h\text{-}f} = \sqrt{\frac{L_T}{C_T}}$$

which is a constant and real number. For the coils tested this limit is reached in the low audio-frequency range before even the occurrence of the first resonance. For example, the $|Z_o|$ at 1 kc of coil No. C-28 is 544 ohms; its $Z_{o,h\text{-}f}$ is 539 ohms. The impedance expression can, therefore, be written as

$$|Z_s| = \sqrt{\frac{L_T}{C_T}}\sqrt{\frac{\sinh^2\alpha l+\sin^2\beta l}{\sinh^2\alpha l+\cos^2\beta l}} \tag{94}$$

Expression (94) indicates that the high-frequency value of $|Z_s|$ is constant and equal to $Z_{o,h\text{-}f}$. This, however, does not agree with experimental results: it can be observed, for instance, from Figure 6 that after 700 kc the impedance-frequency curve of coil No. C-25 starts decreasing slowly. The $Z_{o,h-f}$ of coil No. C-25 is 1,700 ohms; the almost level part of its curve passes through 650 ohms at 1 mc and through 336 ohms at 10 mc. Coil No. C-128 has a $Z_{o,h-f}$ of 2,370 ohms; at 250 kc, the beginning of the level portion of its curve, its impedance is 1,300 ohms; at 1 mc it is 800 ohms. But by contrast coil No. C-26 which has a $Z_{o,h-f}$ of 783 ohms has an impedance of 775 ohms at 2 mc and of 720 ohms at 4.5 mc.

Expression (94) cannot be used for the calculation of the impedance of a coil at a certain frequency. There are two reasons for this: (1) there is no expression available for the effective resistance of a coil as a function of frequency and (2) the resonances of a coil do not occur at the frequencies predicted by the theory.

The expression for $\alpha$ for a type M coil is

$$\alpha = \frac{1}{\ell}\sqrt{\frac{1}{2}\left[\omega C_T\sqrt{R_T^2+\omega^2L_T^2} - \omega^2 L_T C_T\right]} \tag{95}$$

The $\alpha$ of coils similar to those tested in the laboratory approaches its high-frequency limit, which is

$$\alpha_{h-f} = \frac{R_T}{2\ell Z_{o,h-f}} \tag{96}$$

in the low audio-frequency range. For instance, the $\alpha_{h-f}$ of coil No. C-28 is 0.00382; its $\alpha$ at an angular frequency $\omega$

of 2,000 radians per second is 0.00366 and this represents a difference of 4.2 per cent. Without the consideration of the skin effect, $\alpha$ reaches a limit at a frequency too low to be of use in justifying the coil phenomena through the equations obtained. If the skin effect is considered, however, $\alpha$ keeps increasing through the increase of $R_T$ with frequency. It is imperative that $\sinh \alpha l$ of expression (93) increase with frequency until it reaches a value substantially above one, at least around ten, in order to justify the typical "up and down" impedance variation, as in Figure 4. Since a relation between $R_T$ and $f$ is not available, this situation suggests a method for computing $R_T$ by using the measured value of $Z_5$. At parallel and series resonances expression (94) becomes

$$Z_{max} = Z_{o,h-f} \frac{1}{\tanh \alpha l} \tag{97}$$

and

$$Z_{min} = Z_{o,h-f} \tanh \alpha l \tag{98}$$

The multitude of resonances gives a series of $Z_{max}$ and $Z_{min}$ values which in turn allow $\alpha$ to be determined at many frequencies. A curve of $\alpha$ vs. frequency permits the $\alpha$ of a certain coil to be found by intrapolation at frequencies at which the coil has no resonances. With $\alpha$ known, the effective resistance of the coil can be calculated with expression (96). If the previous effective resistance is divided by the length of the wire in the coil, an average value and not a true value of resistance per unit length will be obtained because the current is not the same in all the parts of the coil.

The $\alpha$'s of three coils were computed from expressions (97) and (98) and their frequency variations were plotted. For each coil the curve formed by the $\alpha$'s calculated from the impedances at the parallel resonances was different from the curve formed by the $\alpha$'s calculated from the impedances at the series resonances. Figure 44 shows two typical $\alpha$ vs. frequency curves for coil No. C-28.

Evaluation of expression (47).--Expression (47) gives the voltage between the layers at the mid-point of a type M-closed coil at the first series resonance. When $\alpha\ell$ is less than 0.1 this expression assumes the following simplified form:

$$\left|\frac{V\left(\frac{\ell}{2}\right)}{V_s}\right| \cong \frac{1}{\alpha\ell} \qquad (48)$$

The $\alpha_{\ell - f}$ computed by using the low-frequency resistance cannot be used here. At a frequency at which a resonance occurs the $\alpha$ has already increased above its limit as determined by the direct-current resistance of the coil. Therefore, since the $\alpha$ corresponding to a certain frequency cannot be found with the information available in the literature, this expression has no other practical value aside from

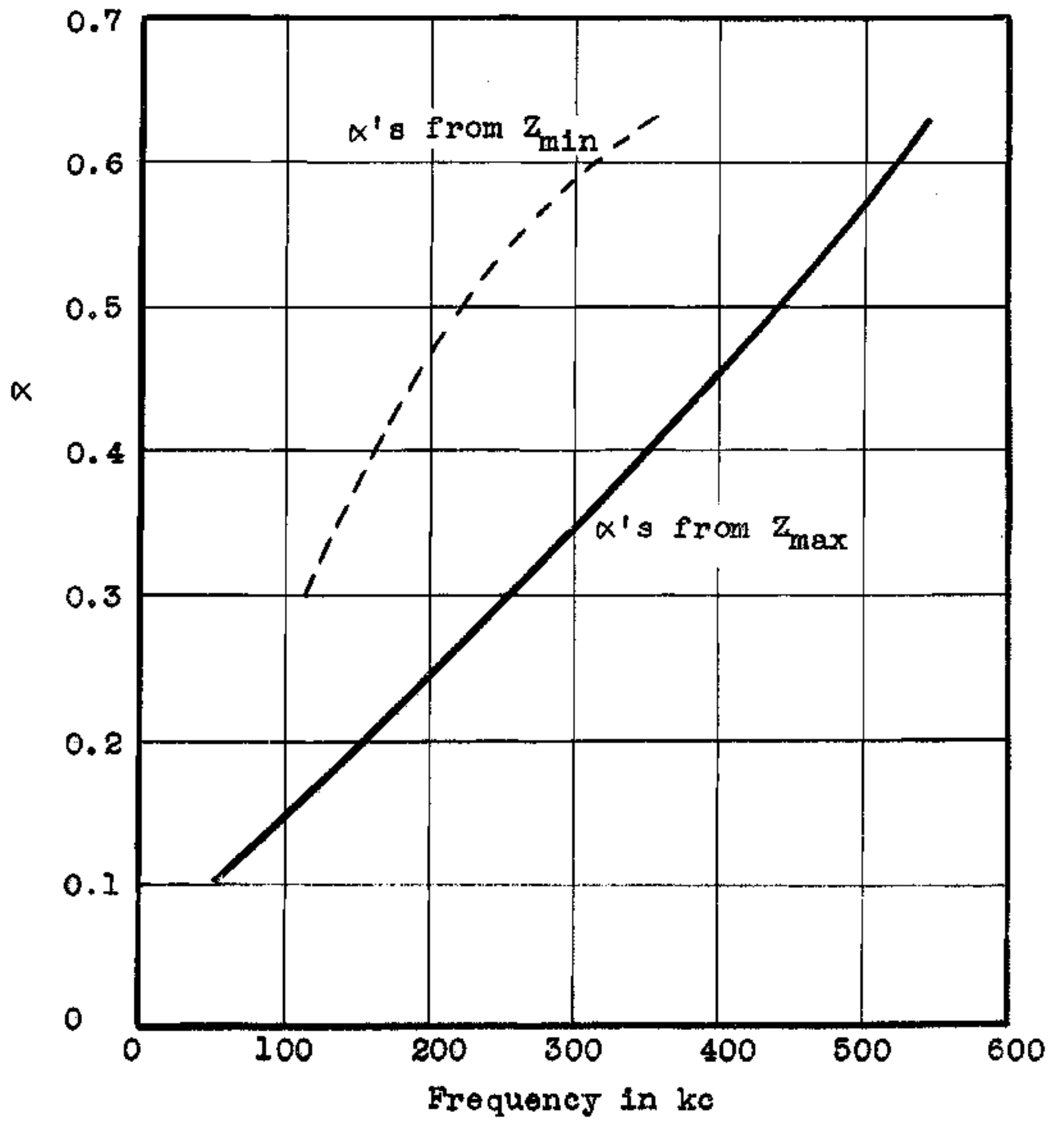



Figure 44. Variation with Frequency of the α's Calculated from $Z_{max}$ and $Z_{min}$ of Type M-closed Coil No. C-28.

confirming the existence of an overvoltage.

The $\alpha$'s of some coils used in this research have been determined with the previously suggested method and a comparison between measured overvoltages and overvoltages calculated with expression (48) is given: for coil No. C-25 the measured overvoltage is 5.8 and the calculated overvoltage is 8.4; for coil No. C-128 the measured overvoltage is 9.2 and the calculated overvoltage is 10.

Limitations of theory in connection with higher resonances of type M coils; evaluation of expression (54).--The higher resonances of a type M coil are not simple multiples of its first resonance as the theory predicts. For example, for the short coil No. C-25 of type M-closed of length to diameter ratio of 0.77 the successive frequency ratios are

1, 2.64, 4.14, 6.98, 9.00, 12.50, 14.70, 18.50, 21.60, 25.60

instead of being, according to the theory,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10

It has been noticed that the frequency ratios of longer coils give better agreement with the theory. An extremely long coil was built, coil No. C-28 of type M-closed of length to diameter ratio of twenty five, in an effort to find frequency ratios agreeing with the theoretical ratios. The measured frequency ratios for this coil are

1, 2.09, 3.05, 4.19, 5.19, 6.40, 7.41, 8.61, 9.80, 10.80

instead of being 1, 2, 3, ..., 10, and this indicates that the theory fits better the case of large rather than small ratio of length to diameter.

Expression (54) is

$$\frac{f_m}{f_1} = m\sqrt{\frac{K_1}{K_m}} \tag{54}$$

and it gives the ratio of the $m^{th}$ resonant frequency to the first of a type M coil. The correction factors $K_1$ and $K_m$ can be obtained from the graph of Figure 29. This expression gives results of better accuracy for coils of larger rather than shorter ratios of length to diameter. It can be seen from Table 1, which gives a comparison between frequency ratios calculated with expression (54) and measured frequency ratios, that the per cent error of the calculated $f_2/f_1$ is less than 12 per cent when $l/D$ is greater than 1, that the per cent error of the calculated $f_3/f_1$ is less than 10 per cent when $l/D$ is greater than 0.5, and that the per cent error of the calculated $f_4/f_1$ is less than 11 per cent when $l/D$ is greater than 1.5. For the extremely long coil No. C-28 of type M-closed, the calculated frequency ratios almost perfectly agree with the theoretical ratios given previously; they are

1, 2.02, 3.08, 4.12, 5.20, 6.26, 7.39, 8.54, 9.66, 10.89

When the connection between the layers is opened, changing thus the type of a coil from M-closed to M-open, the frequency ratios are slightly changed. For instance, for the coil No. C-21 the ratios corresponding to the M-closed case are 1, 3.00, 4.48, 7.63, 9.83,..., while those corresponding to the M-open case are 1, 1.96, 4.48, 6.33, 9.70,.... It can be noticed that it is the ratios corresponding to resonances of even order that differ. As the $l/D$ ratio increases, the two series of frequency ratios tend to better agree with each other.

<u>Termination of a type M coil in a resistance equal to its</u> $Z_{o,h-f}$.--According to the theory a type M coil terminated in a resistance equal to its $Z_{o,h-f}$ should have an impedance that is constant and equal to $Z_{o,h-f}$ at all high frequencies. In reality such a termination produces only a considerable, but not complete, reduction in the prominence of the resonances that appear in the impedance-frequency curves of this coil when its "receiving-end" is either open or closed. This effect can be studied from a comparison of the curve of Figure 45 with those of Figures 4 and 5. Figure 45 gives the impedance-frequency curve of coil No. C-21 terminated in a resistance of 2,000 ohms; the other two figures give its curves when its "receiving-end" is open- or short-circuited. The resistance of 2,000 ohms produces a slightly better smoothing of the resonance peaks than a resistance equal to 1,519 ohms, the value of its $Z_{o,h-f}$.

A test on the extremely long coil No. C-28 terminated in a resistance equal to its $Z_{o,h-f}$ failed to give a smoother impedance-frequency curve than the one shown in Figure 45 for the short by comparison coil No. C-21.

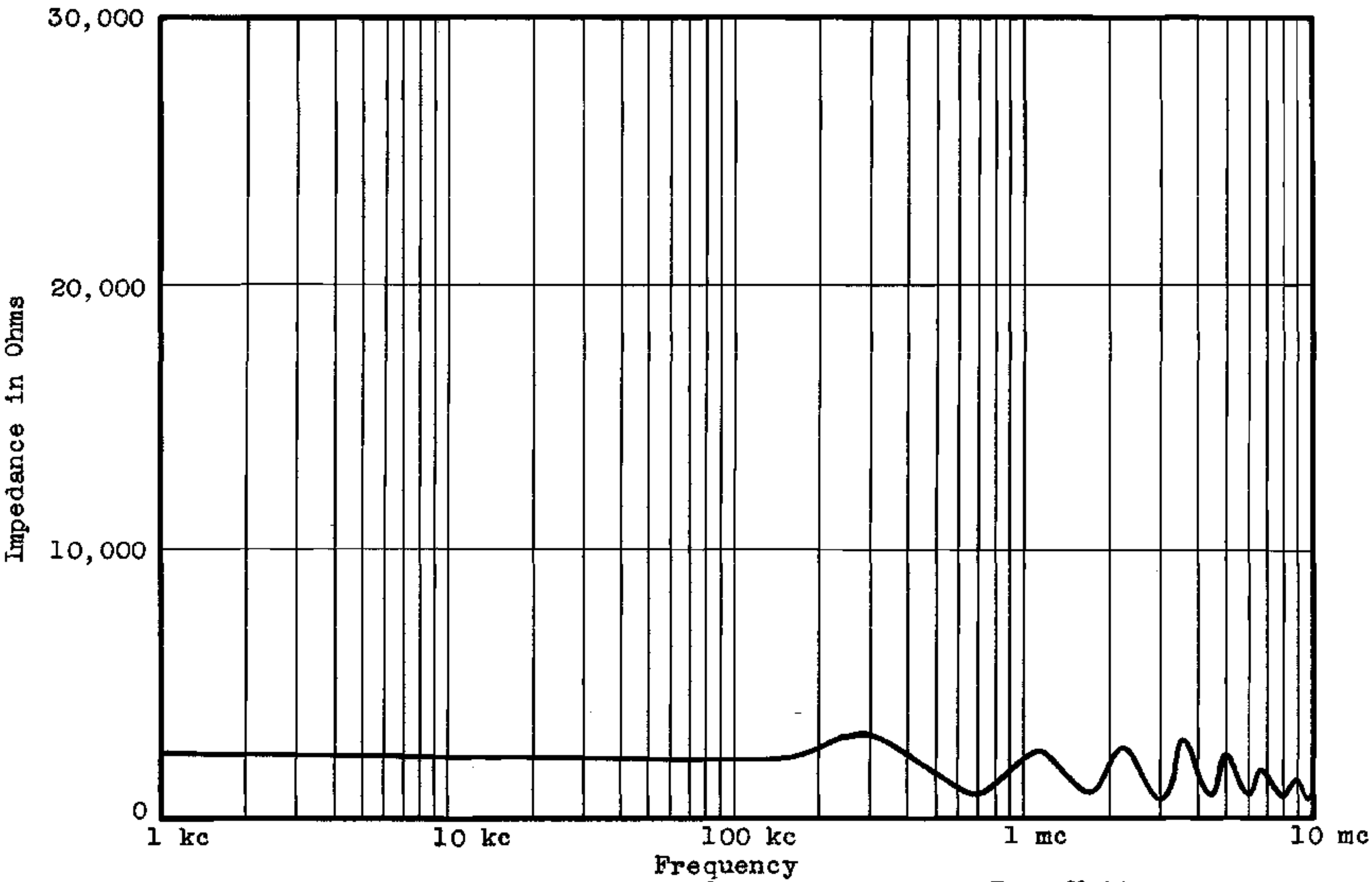



Figure 45. Impedance-frequency Characteristic of Type M Air-core
Coil No. C-21 Terminated in a Resistance of 2,000 Ohms.

Evaluation of the expression for the resonant frequency of a type S coil.--The expression for the resonant frequency of a type S-open or S-closed coil is

$$f = \frac{1}{\pi\sqrt{L_T C_T}}$$

Two type S coils were built and tested. These coils were made of enameled wire allowing neighboring turns to be very close to each other. The per cent differences obtained between calculated and measured resonant frequencies are +2.5 per cent and -2.4 per cent. In view of this close correspondence the expression for the resonant frequency is considered reliable.

Evaluation of the expression for the impedance of a type S-open coil.--The expression for the impedance of a type S-open coil, which is

$$Z = j\left(\frac{\omega L_T}{4} - \frac{1}{\omega C_T}\right) \tag{70}$$

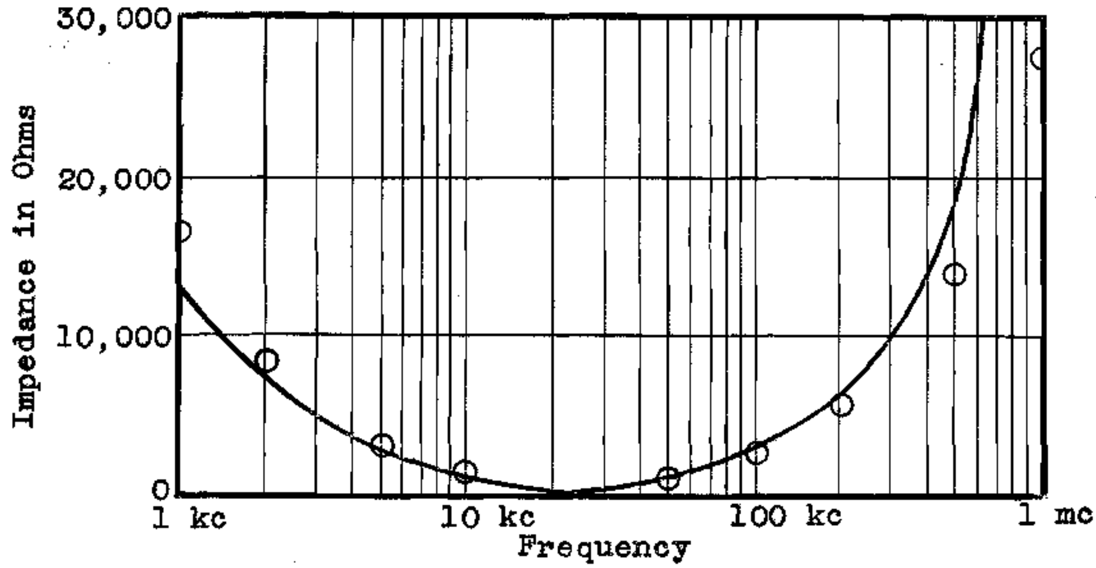
was derived without taking the resistance of the coil into consideration. The omission of the resistance becomes noticeable, principally at the frequency of resonance of the coil and, to a lesser degree, at frequencies near it. At the frequency of resonance, for example, the expression indicates that the impedance of the coil is zero, although in reality the impedance is equal to the omitted effective resistance. At other frequencies the reactance term of the impedance expression overshadows in importance the neglected resistance term.

Figure 46 gives a comparison between the experimentally obtained impedance-frequency curve of coil No. C-122 of type S-open and the corresponding calculated curve from expression (70). The agreement between the two curves is good only in a certain frequency region extending from both sides of the frequency of resonance of the coil. At frequencies far distant from the one of the resonance, the divergence between the two curves is substantial.

The impedance of a type S-open coil does not increase indefinitely as the frequency is increased as expression (70) and the curve of Figure 46 indicate. Factors that were omitted in the derivation of expression (70), as the turn-to-turn capacitance in a lengthwise direction, become important at high frequencies and curb the upward trend of the impedance curve. At such high frequencies the impedance curve fluctuates irregularly at high levels of impedance.

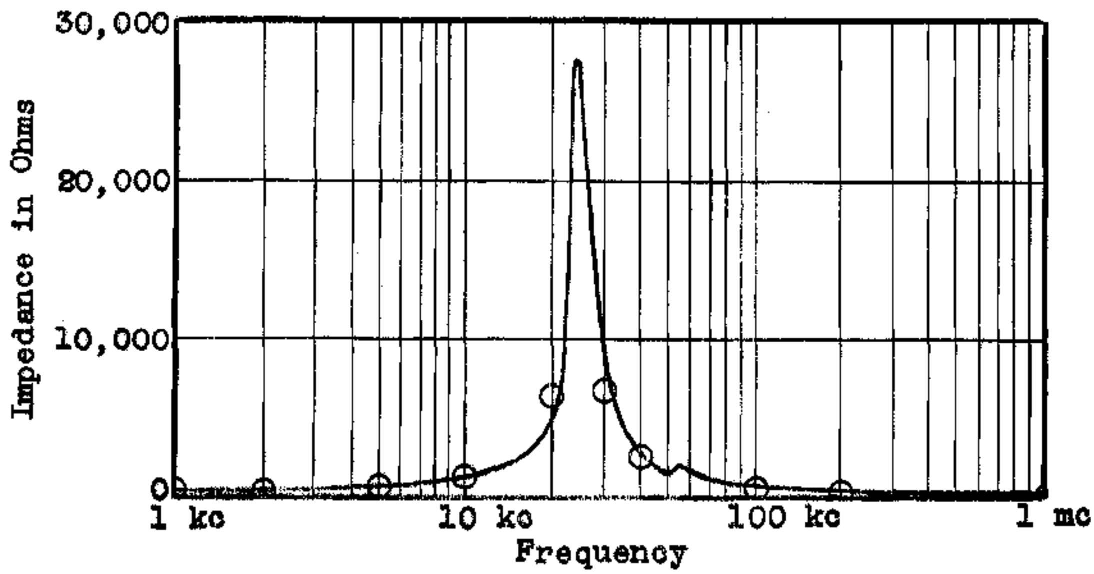
Evaluation of the expression for the impedance of a type S-closed coil.--Figure 47 giving results for coil No. C-122 of type S-closed indicates that, with the exception of a small frequency region in the neighborhood of the frequency of resonance, values calculated from the expression

$$Z = \frac{j\omega L_T}{1 - \omega^2 \frac{L_T C_T}{4}} \qquad (86)$$




Measured Values
Calculated Values

Figure 46. Comparison between Calculated and Measured Impedance Values for Type S-open Coil No. C-122.




Measured Values
Calculated Values

Figure 47. Comparison between Calculated and Measured Impedance Values for Type S-closed Coil No. C-122.

are in good agreement with corresponding values obtained experimentally. Expression (86), having been derived without taking the resistance of the coil into account, is unable to predict impedance values at in the vicinity of the resonant frequency of the coil.

Application.--Type M coils can find immediate application in the educational field for laboratory instruction: the properties of distributed parameters can be more easily studied on these coils than on transmission lines. For the observation of resonant effects on a transmission line it is necessary to have either an inconveniently long line with a supply voltage of a not too low frequency or a shorter line with a supply voltage of an exceptionally high frequency, both frequencies being in or above the radio-frequency range. The equipment needed for such an experiment not only is cumbersome but also is not likely to be found in large quantities in schools to allow individual experimentation by students. These same resonant effects can be observed on type M coils, which are of manageable dimensions and which can be built in large numbers at an insignificant cost, at low audio frequencies which are produced by relatively inexpensive and widely available signal generators. A coil having a self-resonant frequency of less than 1 kc meets the following specifications: length, 80 cm; wire, number 28 American Wire Gage with enamel insulation; turns laid orderly and as closely to each other as possible; cellophane tape placed on top of inner layer to facilitate laying of outer layer. The supporting frame should have small holes at half an inch intervals as points of access to the inner layer for measuring purposes.

# APPENDIX I

## LIST AND DEFINITIONS OF SYMBOLS

| | |
|---|---|
| $v_{app}$ | voltage applied across terminals of coil; function of time only-volts |
| $v$ | voltage between adjacent layers; in the general case, function of time and position-volts |
| $e_o$ | voltage between a certain point of the outer layer and the ground point of the supply; function of time and position-volts |
| $e_i$ | voltage between a certain point of the inner layer and the ground point of the supply; function of time and position-volts |
| $i_o$ | current flowing in the outer layer; function of time and position-amperes |
| $i_i$ | current flowing in the inner layer; function of time and position-amperes |
| $i_T$ | current entering or leaving coil from supply points; function of time only-amperes |
| $i = i_o = i_i$ | whenever this relation applies |
| $\phi$ | flux inside coil; in the general case, function of time and position-webers |
| $n$ | number of turns per unit length for one layer only-turns/meter |
| $N$ | total number of turns of both layers |
| $C$ | capacitance between layers per unit length-farads/meter |
| $C_T$ | total capacitance between layers-farads - $C_T = C\ell$ |
| $\ell$ | length of coil-meters |
| $\mu$ | permeability of air-$\mu \cong \mu_o = \frac{4\pi}{10^7}$ henry/meter |

| | |
|---|---|
| $A$ | cross-sectional area of coil-square meters |
| $f$ | frequency-cycles per second |
| $\omega$ | angular frequency-radians per second |
| $L_T$ | d-c inductance of coil for both layers-henrys |
| $V$ | rms value of voltage between layers; function of position only-volts |
| $V_s$ | rms value of applied voltage-volts |
| $B_1, B_2, \ldots, B_5$ | constants |
| $r$ | resistance of one turn-ohms/turn |
| $R_T$ | total resistance of both layers-ohms – $R_T = R\ell = 2nr\ell$ |
| $D$ | coil diameter-meters |
| $R$ | resistance per unit length for two layers-ohms |
| $\ell'$ | length of a transmission line-meters |
| $L'$ | inductance per unit length of a transmission line-henrys/meter |
| $C'$ | capacitance per unit length of a transmission line-farads/meter |
| $L'_T$ | total inductance of a transmission line-henrys |
| $C'_T$ | total capacitance of a transmission line-farads |

$$a = \frac{L_T C_T}{\ell^2}\omega^2$$

$$b = \frac{R_T C_T}{\ell}\omega$$

$$p = \sqrt{a - jb} = \beta - j\alpha$$

$$\alpha = \frac{1}{\ell}\sqrt{\frac{1}{2}\left[\omega C_T\sqrt{R_T^2 + \omega^2 L_T^2} - \omega^2 L_T C_T\right]}$$

$$\beta = \frac{1}{\ell}\sqrt{\frac{1}{2}\left[\omega C_T\sqrt{R_T^2 + \omega^2 L_T^2} + \omega^2 L_T C_T\right]}$$

# APPENDIX II

Table 3. Specifications of Two-layer Coils Tested

| Coil Number | Coil Type | Conductor Number AWG | Metal of Conductor | Conductor Insulation |
|---|---|---|---|---|
| C- 14 | M | 26 | copper | enamel |
| C- 17 | M | 18 | aluminum | thermoplastic |
| C- 21 | M | 24 | copper | cotton-enamel |
| C- 25 | M | 26 | copper | enamel |
| C- 26 | M | 24 | copper | cotton-enamel |
| C- 28 | M | 18 | copper | cotton-enamel |
| C-120 | M | 20 | copper | enamel |
| C-128 | M | 28 | copper | enamel |
| C-122 | S | 22 | copper | enamel |
| C-124 | S | 24 | copper | enamel |

Table 3 (Continued). Specifications of Two-layer Coils Tested

| Coil Number | Total Inductance (for Both Layers) in Millihenrys | Capacitance between Layers in Micromicro-farads | Total Resistance (for Both Layers) in Ohms |
|---|---|---|---|
| C- 14 | 4.62 | 1,388 | 9.60 |
| C- 17 | 0.37 | 700 | 0.76 |
| C- 21 | 2.88 | 1,240 | 5.31 |
| C- 25 | 63.10 | 22,200 | 51.00 |
| C- 26 | 10.70 | 17,650 | 19.34 |
| C- 28 | 3.08 | 10,600 | 3.86 |
| C-120 | 52.12 | 32,130 | 18.26 |
| C-128 | 321.00 | 57,000 | 285.40 |
| C-122 | 17.57 | 9,613 | 10.56 |
| C-124 | 74.26 | 21,400 | 40.00 |

Table 3 (Continued). Specifications of Two-layer Coils Tested

| Coil Number | Axial Length in Centimeters | Diameter in Centimeters | Total Number of Turns (for Both Layers) |
|---|---|---|---|
| C- 14 | 5.00 | 8.8X7.5 * | 220 |
| C- 17 | 9.40 | 6.60 | 100 |
| C- 21 | 5.90 | 12.3X9.5 * | 154 |
| C- 25 | 14.45 | 18.80 | 640 |
| C- 26 | 46.80 | 5.47 | 1334 |
| C- 28 | 93.80 | 3.55 | 1492 |
| C-120 | 48.55 | 15.24 | 1088 |
| C-128 | 48.15 | 15.24 | 2600 |
| C-122 | 14.20 | 15.24 | 400 |
| C-124 | 20.00 | 20.32 | 694 |

* Dimensions in centimeters of rectangular cross-section of coil.

The diameters of the dimensions of the rectangular cross-sections given are those of the outer surfaces of the frames supporting the turns of the coils.

The two layers of turns are separated from each other with one layer of cellophane tape.

# APPENDIX III

Table 4. Specifications of Nontwo-layer Coils Tested

| Coil Number | Number of Layers | Conductor Number AWG | Metal of Conductor | Conductor Insulation |
|---|---|---|---|---|
| C-1- 25 | 1 | 26 | copper | enamel |
| C-1-120 | 1 | 20 | copper | enamel |
| C-3- 18 | 3 | 26 | copper | enamel |
| C-4- 17 | 4 | 18 | aluminum | thermoplastic |
| C-6- 11 | 6 | 26 | copper | enamel |

Table 4 (Continued).
Specifications of Nontwo-layer Coils Tested

| Coil Number | Total Inductance in Millihenrys | Total Resistance in Ohms | Axial Length in Centimeters | Diameter in Centimeters | Total Number of Turns |
|---|---|---|---|---|---|
| C-1- 25 | 15.89 | 25.50 | 14.45 | 18.80 | 320 |
| C-1-120 | 6.11 | 4.79 | 26.80 | 15.24 | 300 |
| C-3- 18 | 11.72 | 15.93 | 5.37 | 8.9×7.6 * | 360 |
| C-4- 17 | 1.64 | 1.52 | 9.40 | 6.60 | 200 |
| C-6- 11 | 36.63 | 25.90 | 5.22 | 8.6×5.2 * | 700 |

* Dimensions in centimeters of rectangular cross-section of coil.

The diameters or the dimensions of the rectangular cross-sections given are those of the outer surfaces of the frames supporting the turns of the coils.

Successive layers of turns are separated from each other with one layer of cellophane tape.

# APPENDIX IV

## Ratings of Transformers Cited in the Text

<u>Transformer No. T-14</u>

Rating: 15 kva
Voltage rating: 117/450 volts
Manufacturer: Westinghouse
Serial number of nameplate: 3606616

<u>Transformer No. T-80</u>

Rating: 10 volt-amperes
Voltage ratio: 10 to 1
Manufacturer: Jefferson Electric
Serial number of nameplate: 467-747

<u>Transformer No. T-90</u>

Turns of high-voltage winding: 700
Turns of low-voltage winding: 70
Both windings were constructed of number 26 AWG enameled wire
This transformer was assembled in the laboratory

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