Bounded Solutions of a Time-Varying Benjamin-Bona-Mahony Equation

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Abstract

In this note we study the existence, the stability and the smoothness of bounded solutions of the following time-varying BBM equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t - a\Delta u_t - b\Delta u = f(t,u), & t \geq 0, \quad x \in \Omega, \\ u(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases}$$

where a and b are positive numbers, $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous and globally Lipschitz function with a Lipschitz constant $l > 0$. Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$. This equation is a a form of Generalized Benjamin-Bona-Mahony equation. Under these conditions we shall prove the following statement: If λ_1 is the first eigenvalue of $-\Delta$ and $b\lambda_1 > l$, then the equation admits only one bounded solution on $\mathbb{R}$ which is exponentially stable. Also, we prove for a large class of functions f that this bounded solution is almost periodic.

Key words. bounded solutions, exponential stability, smoothness.

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1 Introduction

In this note we shall study the existence, the stability and the smoothness of a bounded solution for the following BBM equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_t - a\Delta u_t - b\Delta u = f(t,u), \qquad t \geq 0, \quad x \in \Omega, \\ u(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases} \tag{1.1}$$

where a and b are positive numbers, $f : I\!R \times I\!R \to I\!R$ is a continuous and globally Lipschitz function with a Lipschitz constant $l > 0$. i.e.,

$$|f(t,u) - f(t,v)| \leq l|u - v|, \quad t, u, v \in I\!R. \tag{1.2}$$

Ω is a bounded domain in $I\!R^N (N \geq 1)$. We shall assume the following hypothesis: there exists $l_f > 0$ such that

$$\|f(t,0)\| \leq l_f, \quad \forall t \in I\!R. \tag{1.3}$$

Under this assumption, we prove the following statements: If $b\lambda_1 > l$ (where λ_1 is the first eigenvalue of $-\Delta$), then the equation admits only one bounded solution which is exponentially stable. Also, we prove for a large class of functions f that this bounded solution is almost periodic. Some notation for this work can be found in [?], [?], [?] and [?].

The original Benjamin-Bona-Mahony equation was proposed in [?] for the case $n = 1$ as a model for the propagation of long waves. This equation and related types of Pseudo-Parabolic equations have been studied by many authors. Results about existence and uniqueness of solutions can be found in [?] and [?]. The long time behavior of solutions and the existence of attractors were studied by many authors to mention [?], [?] and [?].

2 Abstract Formulation of the Problem

In this section we choose the space in which this problem will be set as an abstract ordinary differential equation.

Let $X = L^2(\Omega) = L^2(\Omega, I\!R)$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, I\!R) \cap H_0^1(\Omega, I\!R). \tag{2.1}$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$

each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Therefore,

a) there exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of A.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n x, \tag{2.2}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_n x = \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k}. \tag{2.3}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in X and

$$x = \sum_{n=1}^{\infty} E_n x, \quad x \in X.$$

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \tag{2.4}$$

Hence, the equation (**??**) can be written as an abstract ordinary differential equation in X as follows

$$u' + aAu' + bAu = f^e(t,u), \quad t \geq 0, \tag{2.5}$$

where $f^e : I\!R \times X \to X$ is given by:

$$f^e(t,u)(x) = f(t,u(x)), \quad x \in \Omega, \quad u \in X.$$

The function f^e possess the following properties:

$$\|f^e(t,u) - f^e(t,v)\| \leq L\|u-v\|, \quad t \in I\!R, \quad u,v \in X, \tag{2.6}$$

$$\|f^e(t,0)\| \leq L_f, \quad t \in I\!R, \tag{2.7}$$

where

$$L = \frac{l}{1+a\lambda_1} \quad \text{and} \quad L_f = \mu(\Omega)^{1/2} l_f,$$

and $\mu(\Omega)$ is the Lebesgue measure of Ω.

Since $(I + aA) = a(A - (-\frac{1}{a})I)$ and $-\frac{1}{a} \in \rho(A)$, then the operator $I + aA : D(A) \to X$ is invertible with bounded inverse $(I+aA)^{-1} : X \to D(A)$. Therefore, the equation (**??**) also can be written as follows

$$u' + b(I+aA)^{-1}Au = (I+aA)^{-1}f^e(t,u), \quad t \geq 0. \tag{2.8}$$

Moreover, $(I + aA)$ and $(I + aA)^{-1}$ can be written in terms of the eigenvalues of A:

$$I + aAx = \sum_{n=1}^{\infty}(1+\lambda_n)E_n x \tag{2.9}$$

$$(I+aA)^{-1}x = \sum_{n=1}^{\infty} \frac{1}{1+a\lambda_n} E_n x. \tag{2.10}$$

Therefore, if we put $B = (I + aA)^{-1}$, the equation (**??**) can be written as follows

$$u' + bBAu = Bf^e(t, u), \quad t \geq 0, \tag{2.11}$$

Remark 2.1 *The operator $bAB : X \to D(A) \subset X$ is bounded and for $x \in D(A)$ we have that $ABx = BAx$. Therefore, we shall study the existence of a bounded solution $u_b(t)$ of the equation*

$$u' + bABu = Bf^e(t, u), \quad t \geq 0, \tag{2.12}$$

and prove that $u_b(t) \in D(A)$.

Now, we formulate a simple proposition.

Proposition 2.1 *The operators bAB and $T(t) = e^{-bABt}$ are given by the following expression*

$$bABx = \sum_{n=1}^{\infty} \frac{b\lambda_n}{1 + a\lambda_n} E_n x \tag{2.13}$$

$$T(t)x = e^{-bABt}x = \sum_{n=1}^{\infty} e^{\frac{-b\lambda_n}{1+a\lambda_n}t} E_n x, \tag{2.14}$$

and

$$\|T(t)\| \leq e^{-\beta t}, \quad t \geq 0, \tag{2.15}$$

where

$$\beta = \inf_{n \geq 1} \left\{ \frac{b\lambda_n}{1 + a\lambda_n} \right\} = \frac{b\lambda_1}{1 + a\lambda_1}.$$

3 Existence of the Bounded Solution

In this section we shall prove the existence and stability of unique bounded solutions of the equation (**??**). Since bBA is a bounded operator, $u(t)$ is a solution of the equation (**??**) if and only if

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} T(t-s)Bf^e(s, u(s))ds \tag{3.1}$$

We shall consider $X_b = C_b(\mathbb{R}, X)$ the space of bounded and continuous functions defined in $\mathbb{R}$ taking values in X. X_b is a Banach space with supremum norm

$$\|u\|_b = \sup\{\|u(t)\| : t \in \mathbb{R}\}, \quad u \in X_b.$$

A ball of radius $\rho > 0$ and center zero in this space is given by

$$B_\rho^b = \{u \in X_b : \|u(t)\|_b \leq \rho, \quad t \in \mathbb{R}\}.$$

Lemma 3.1 *Let u be in $X_b = C_b(R, X)$. Then u is a solution of (**??**) if and only if u is a solution of the following integral equation*

$$u(t) = \int_{-\infty}^{t} T(t-s)Bf^e(s, u(s))ds, \quad t \in R. \tag{3.2}$$

Proof Suppose that u is a solution of (**??**) . Then , from variation of constants formula (**??**) and uniqueness of the solution of (**??**) we obtain that

$$u(t) = T(t - t_0)u(t_0) + \int_{t_0}^{t} T(t-s)Bf(s, u(s))ds, \quad t \in R. \tag{3.3}$$

On the other hand, from (**??**) we obtain that

$$\|T(t - t_0)u(t_0)\| \leq e^{-\beta(t-t_0)}\|u(t_0)\|, \quad t \geq t_0,$$

and since $\|u(t)\| \leq m, \quad t \in R$, we get the following estimate

$$\|T(t - t_0)u(t_0)\| \leq me^{-\beta(t-t_0)}, \quad t \geq t_0.$$

Now, we shall prove that the improper integral $\int_{-\infty}^{t} T(t-s)Bf^e(s,u(s))ds$ exists. In fact,

$$\begin{aligned}\|\int_{-\infty}^{t} T(t-s)Bf^e(s,u(s))ds\| &\leq \int_{-\infty}^{t} e^{-\beta(t-s)}\|B\|\{l\|u(s)\|+L_f\}ds\\ &\leq \frac{L\|u\|_b+\|B\|L_f}{\beta}.\end{aligned}$$

Then, passing to the limit in (**??**) as t_0 goes to $-\infty$ we get

$$\lim_{t\to-\infty}\|T(t-t_0)u(t_0)\|=0.$$

and

$$u(t)=\int_{-\infty}^{t} T(t-s)Bf^e(s,u(s))ds, \quad t\in R.$$

Suppose that u is a solution of integral equation (**??**). Then , for all $t_0 \in R$ we get that

$$u(t)=\int_{-\infty}^{t_o} T(t-t_0)Bf^e(s,u(s))ds+\int_{t_0}^{t} T(t-t_0)Bf^e(s,u(s))ds.$$

Hence, for $t \geq t_0$ we get

$$\begin{aligned}u(t) &= T(t-t_0)\int_{-\infty}^{t_0} T(t_0-s)Bf^e(s,u(s))ds\\ &+ \int_{t_0}^{t} T(t-s)Bf^e(s,u(s))ds\\ &= T(t-t_0)u(t_0)+\int_{t_0}^{t} T(t-s)Bf^e(s,u(s))ds,\end{aligned}$$

where

$$u(t_0)=\int_{-\infty}^{t_0} T(t_0-s)Bf^e(s,u(s))ds.$$

Therefore, $u(t)$ is a solution of (**??**) on R. □

Theorem 3.1 *If $b\lambda_1 > l$, then equation (**??**) has an unique bounded solution which is exponentially stable.*

Proof Since $b\lambda_1 > l$, then $\frac{b\lambda_1}{1+a\lambda_1} > \frac{l}{1+a\lambda_1}$ and $\beta > L$. Then, for ρ big enough we have the following inequality

$$\rho(\beta - L) > \|B\|L_f. \tag{3.4}$$

For the existence of such a solution , we shall prove that the mapping

$$(Tu)(t) = \int_{-\infty}^{t} T(t-s)Bf^e(s,u(s))ds$$

is a contraction mapping from B_ρ into itself. In fact, for $u \in B_\rho$ we have

$$\|Tu(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)}(L\|u(s)\| + \|B\|L_f)ds \leq \frac{L\rho + \|B\|L_f}{\beta}.$$

The condition (**??**) implies that

$$L\rho + \|B\|L_f < \beta\rho \iff \frac{L\rho + \|B\|L_f}{\beta} < \rho.$$

Therefore, $Tu \in B_\rho^b$ for all $u \in B_\rho^b$.

Now, we shall see that T is a contraction mapping. In fact, for all $u_1, u_2 \in B_\rho^b$ we have that

$$\|Tu_1(t) - Tu_2(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)}L\|u_1(s) - u_2(s)\|ds \leq \frac{L}{\beta}\|u_1 - u_2\|_b, \quad t \in I\!R.$$

Hence,

$$\|Tu_1 - Tu_2\|_b \leq \frac{L}{\beta}\|u_1 - u_2\|_b, \quad u_1, u_2 \in B_\rho^b.$$

The condition (**??**) implies that

$$0 < \beta - L \iff L < \beta \iff \frac{L}{\beta} < 1.$$

Therefore, T has a unique fixed point u_b in B_ρ^b given by

$$u_b(t) = (Tu_b)(t) = \int_{-\infty}^{t} T(t-s)Bf(s,u_b(s))ds, \quad t \in R.$$

From Lemma **??**, u_b is a bounded solution of the equation (**??**). Since condition (**??**) holds for any $\rho > 0$ big enough independent of $L < \beta$, then u_b is the unique bounded solution of the equation (**??**).

To prove that $u_b(t)$ is exponentially stable in the large, we shall consider any other solution $u(t)$ of (**??**) and consider the following estimate for $t_0 \geq 0$

$$\begin{aligned} \|u(t) - u_b(t)\| &\leq \|T(t-t_0)(u(t_0) - u_b(t_0))\| \\ &+ \|\int_{t_0}^{t} T(t-s)\left\{Bf^e(s,u(s)) - Bf^e(s,u_b(s))\right\} ds\| \\ &\leq e^{-\beta(t-t_0)}\|(u(t_0) - u_b(t_0))\| + \int_{t_0}^{t} e^{-\beta(t-s)} L\|u(s) - u_b(s)\|ds. \end{aligned}$$

Then,

$$e^{\beta(t-t_0)}\|u(t) - u_b(t)\| \leq \|(u(t_0) - u_b(t_0))\| + \int_{t_0}^{t} e^{-\beta(t_0-s)} L\|u(s) - u_b(s)\|ds.$$

Hence, applying the Gronwall's inequality we obtain

$$\|u(t) - u_b(t)\| \leq e^{(L-\beta)(t-t_0)}\|(u(t_0) - u_b(t_0))\|, \quad t \geq t_0.$$

From (**??**) we get that $L - \beta < 0$ and therefore $u_b(t)$ is exponentially stable in the large. □

Proposition 3.1 *If $f(t,u) = f(t+\tau,u)$, then the unique bounded solution given by Theorem **??** is also periodic.*

Proof Let u_b be the bounded solution of the equation (**??**) given by Theorem **??**. Then, for $u(t) = u_b(t+\tau)$ we have that

$$u''(t) + bABu(t) + Bf(t+\tau,u(t)) = 0.$$

Since $Bf(t+\tau, u(t)) = Bf(t, u(t))$, then $u(t)$ is a solution of (**??**) with $\|u(t)\| \leq \rho$ for all $t \in I\!R$. Therefore, by the uniqueness of the fix point of T in B_ρ we get that $u(t) = u_b(t) = u_b(t+\tau)$, for all $t \in I\!R$. $\square$

Proposition 3.2 *If $f(t,u) = g(u)+p(t)$ with $g(u)$ a Lipschitz function and $p \in C_b(R)$ is almost periodic, then the bounded solution $u_b(t)$ is also almost periodic.*

Proof The proof follows using the same idea from [**?**] $\square$

4 Smoothness of the Bounded Solution

In this section we shall prove that the bounded solution of the equation (**??**) given by Theorem **??** is also solution of the original equation (**??**). That is to say, this bounded solution is a classic solution of the equation (**??**). To this end, we will use the following Theorem from [**?**].

Theorem 4.1 *Let A on $D(A)$ be a closed operator in the Banach space X and $x \in C([a,b);X)$ with $b \leq \infty$. Suppose that $x(t) \in D(A)$, $Ax(t)$ is continuous on $[a,b)$ and that the improper integrals*

$$\int_a^b x(s)ds \quad and \quad \int_a^b Ax(s)ds$$

exist. Then

$$\int_a^b x(s)ds \in D(A) \quad and \quad A\int_a^b x(s)ds = \int_a^b Ax(s)ds.$$

Theorem 4.2 *The bounded solution of the equation (**??**) given by Theorem **??** is also solution of the equation (**??**).*

Proof From remark (**??**) it is enough to prove that $u_b(t) \in D(A)$ for all $t \in \mathbb{R}$. From Lemma **??** we know that

$$\begin{aligned} u_b(t) &= \int_{-\infty}^{t} T(t-s)Bf^e(s,u_b(s))ds = \int_{-\infty}^{t} T(t-s)Bg(s)ds \\ &= \int_{0}^{\infty} T(s)Bg(t-s)ds = \int_{0}^{\infty} x(s)ds \end{aligned}$$

where $g(s) = f^e(s, u_b(s))$ and $x(s) = T(s)Bg(t-s)$. From condition (**??**) we obtain that

$$\|g(s)\| \leq l\|u_b(s)\| + L_f \leq l\|u_b\|_b + L_f, \quad s \in \mathbb{R}.$$

In order to apply Theorem **??** we need to prove the following claim:

Claim :

(a) $x \in C([0,\infty); X)$,

(b) $x(s) \in D(A), \quad s \in [0,\infty)$

(c) $Ax \in C([0,\infty); X)$,

(d) the improper integral

$$\int_0^\infty Ax(s)ds,$$

exists, where $A = -\Delta$.

Proof of part (a): Since bAB is bounded, then the semigroup $\{T(t)\}_{t\geq 0}$ is uniformly continuous. i.e.,

$$\lim_{s\to s_0} \|T(s) - T(s_0)\| = 0, \quad s, s_0 \in [0,\infty).$$

Then

$$\begin{aligned} \lim_{s\to s_0} \|x(s) - x(s_0)\| &\leq \lim_{s\to s_0} e^{-\beta s}\|B\|\|g(t-s) - g(t-s_0)\| \\ &+ \lim_{s\to s_0} \|B\|\|g\|_b\|T(s) - T(s_0)\| = 0. \end{aligned}$$

Proof of part (b): Since $B : X \to D(A)$ and $A(bAB)x = (bAB)Ax$ for $x \in D(A)$, then $T(s)Ax = AT(s)x$ for $x \in D(A)$ and $x(s) = T(s)Bg(t-s) \in D(A)$.

Proof of part (c): Since $Ax(s) = AT(s)Bg(t-s) = T(s)ABg(t-s)$, AB is bounded and $\{T(t)\}_{t\geq 0}$ is uniformly continuous, the proof follows as in part (a).

Proof of part (d): It follows from the commutativity of $T(s)$ and A, the boundedness of AB and the following estimate

$$\begin{aligned} \left\| \int_0^\infty Ax(s)ds \right\| &\leq \int_0^\infty \|AT(s)Bg(t-s)\|ds = \int_0^\infty \|T(s)ABg(t-s)\|ds \\ &\leq \|AB\| \int_0^\infty e^{-\beta s}\|g(t-s)\|ds \leq \frac{\|AB\|\|g\|_b}{\beta}. \end{aligned}$$

Then, applying Theorem **??**, we obtain that

$$\int_0^\infty x(s)ds \in D(A) \quad \text{and} \quad A\int_0^\infty x(s)ds = \int_0^\infty Ax(s)ds.$$

Hence,

$$u_b(t) = \int_{-\infty}^t T(t-s)Bf^e(s, u_b(s))ds \in D(A), \quad t \in I\!R,$$

and

$$u_b(t)' + bBAu_b(t) = Bf^e(t, u_b(t)), \quad t \in I\!R.$$

Then, applying the inverse of B in both sides of this equation we get

$$u_b(t)' + aAu_b(t)' + bAu_b(t) = f^e(t, u_b(t)), \quad t \in I\!R.$$

i.e.,

$$\begin{cases} (u_b)_t - a\Delta(u_b)_t - b\Delta u_b = f(t, u_b), & t \geq 0, \quad x \in \Omega, \\ u(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases}$$

$\square$

References

[1] H. LEIVA "Stability of a Periodic Solution for a System of Parabolic Equations" J. Applicable Analysis, Vol. 60, pp. 277-300(1996).

[2] H. LEIVA, "Existence of Bounded Solutions of a Second Order System with Dissipation" J. Math. Analysis and Appl. **237**, 288-302(1999).

[3] L. GARCIA and H. LEIVA, "Center Manifold and Exponentially Bounded Solutions of a Forced Newtonian System with Dissipation"E. Journal Differential Equations. conf. 05, 2000, pp. 69-77.

[4] H. LEIVA, "Existence of Bounded Solutions of a Second Order Evolution Equation and Applications"Journal Math. Physics. Vol. 41, N0 11, 2000.

[5] G.E. LADAS AND V. LAKSHMIKANTHAM, "Differential Equations in Abstract Sapces", Academic Press, Math. In science and Eng. Vol 85(1972).

[6] J.M. ALONSO, J. MAWHIN AND R. ORTEGA, "Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation", J.Math. Pures Appl., 78, 49-63 (1999).

[7] J. AVRIN AND J.A. GOLDTAEIN, "Global Existence for the Benjamin- Bona-Mahony Equation in Arbitrary Dimensions" Nonlinear Anal. **9** (1995), 861-865.

[8] T.B. BENJAMIN , J.L. BONA AND J.J. MAHONY, "Model Equations for Long Waves in Nonlinear Dispersive Systems", Philos. Trans. Roy. Soc. London Ser. A **272** (1972), 47-78.

[9] P. Biler, "Long Time Behavior of Solutions of the Generalized Benjamin-Bona-Mahony Equation in Two Space Dimensions", Differential Integral Equations **5**(1992), 891-901.

[10] A.O. CELEBI, V.K. KALANTAROV AND M. POLAT, "Attractors for the Generalized Benjamin-Bona-Mahony Equation", J. Differential Equations **157** (1999), 439-451.

[11] L.A. MEDEIROS AND G.P. MENZALA, "Existence and Uniqueness for periodic Solutions of the Benjamin-Bona-Mahony Equation", SIAM J. Math. Anal. **8** (1977), 792-799.

[12] I. CHUESHOV, M. POLAT AND S. SIEGMUND, "Gevrey Regularity of Global Attractor for Generalized Benjamin-Bona-Mahony Equation", (Submitted to J.D.E 2002)