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A STUDY OF LINEAR TRANSFORMATION METHODS OF THREE-PHASE SYSTEMS

A THESIS

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OF THREE-PHASE SYSTEMS

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ABSTRACT

The increasing dependence of the electric power utilities upon the network calculator for system analysis shows the need for better and clearer methods of power system representation and analysis. Symmetrical components and modifications of symmetrical components have supplied a suitable method for the analysis of balanced systems, but there is need for a more satisfactory analysis for unbalanced systems.

This thesis investigates the application of linear matrix transformations to the simplification of the solution of unbalanced three-phase systems. The existing components are shown to be but special types of matrix transformations. The advantages and disadvantages of these transformations are discussed. The special characteristics of the transformations of tensor analysis as applied to electrical systems are discussed.

The application of the congruent matrix transformation to diagonalize a symmetric matrix is discussed in detail as a means of determining new and advantageous components for use in solving unbalanced three-wire and four-wire three-phase systems. An example of a flat-spaced three-wire series three-phase system is solved by the congruent transformation method.

Complex-similarity matrix transformations are discussed. This linear transformation is shown to be a general transformation. It is shown how symmetrical components can be derived by this linear transformation as a special case of the complex-similarity transformation. It is pointed out that alpha, beta, zero components may be derived by this general transformation. It is noted that the digital computer is needed for stages of the development of these transformation matrices.

A linear matrix transformation has been found that will diagonalize any symmetrical impedance matrix. This transformation will make it possible to represent a three-phase network as three independent single-phase systems.

LIST OF ILLUSTRATIONS

CHAPTER I

INTRODUCTION

The solution of linear three-phase electrical networks is an important problem to those concerned with the generation, distribution, and utilization of electrical power. In a power system, the known quantities are most frequently the generated voltages and the system impedances. The unknown quantities are the currents. Unfortunately, the arithmetic labor in the solution of linear three-phase electrical networks renders a solution prohibitive for a great number of practical problems.

A great stride toward the reduction of the labor involved in the solving of three-phase electrical networks was made by C. L. Fortescue (1), in 1913, with the introduction of "symmetrical components." Various other types of components followed: "modified symmetrical components" by W. W. Lewis (2), in 1917, and "alpha, beta, zero components" by Edith Clarke (3), in 1938. Other types of components have been utilized for the solutions of further specific problems. The two reaction method for solutions involving salient pole three-phase machines and the double revolving field theory of single-phase motors may be cited as examples.

Each of these methods was proposed to solve a particular problem, and each of them evolved from the special characteristics of the physical system under consideration. This approach resulted in the loss of the fundamental nature of the methods: the use of each of these "components" involved substitution of a new set of variables. These new variables resulted from the familiarity of individuals with the solutions of specific networks and did not result from a systematic mathematical investigation of the solutions of networks by substitution of variables.

The purpose of this thesis is to investigate analytically the application of linear transformations to three-phase electrical systems and to present information concerning the nature of the transformations which yield the simplest solution in terms of the new variables.

CHAPTER II

EXISTING COMPONENTS AS LINEAR TRANSFORMATIONS

Linear transformations are to be applied to the general equations for a series three-phase network as shown in Fig. 1. The general equations in terms of the original variables are:

$$\begin{aligned} V_a &= Z_{aa} \cdot I_a + Z_{ab} \cdot I_b + Z_{ac} \cdot I_c \\ V_b &= Z_{ba} \cdot I_a + Z_{bb} \cdot I_b + Z_{bc} \cdot I_c \\ V_c &= Z_{ca} \cdot I_a + Z_{cb} \cdot I_b + Z_{cc} \cdot I_c \end{aligned} \tag{1}$$

The analysis utilized in this work can be described best in matrix notation; therefore, Eq. 1 will be expressed as

$$\begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} = \begin{bmatrix} Z_{aa} & Z_{ab} & Z_{ac} \\ Z_{ba} & Z_{bb} & Z_{bc} \\ Z_{ca} & Z_{cb} & Z_{cc} \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} \tag{2}$$

or

$$V = Z \cdot I \tag{3}$$

The matrix Z is symmetrical by the reciprocity theorem; i. e., $Z_{ij} = Z_{ji}$ (i, j = a, b, c for all values).

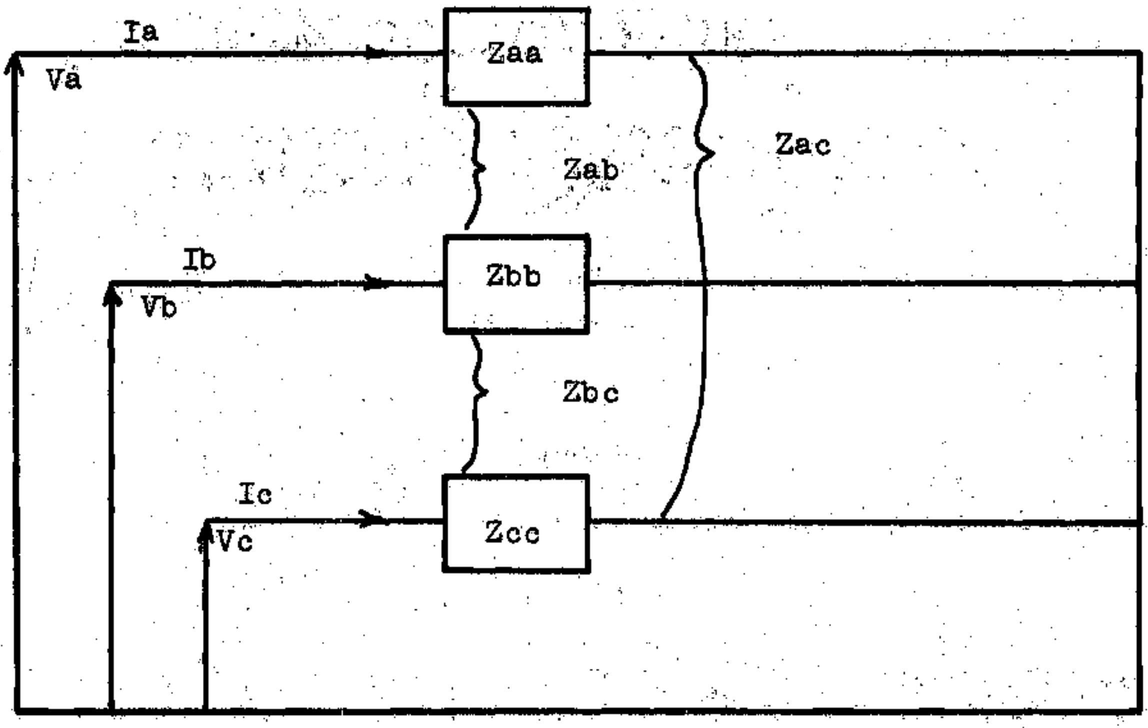

Fig. 1

A series three-phase system

The previously utilized "components" transform the currents, voltages and impedances to new variables as shown below. A transformation matrix C is utilized to transform both the currents and voltages. For instance:

$$V' = C \cdot V \tag{4}$$

or

$$\begin{bmatrix} V_{ao} \\ V_{a1} \\ V_{a2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \cdot \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} \tag{5}$$

and

$$I' = C \cdot I \tag{6}$$

or

$$\begin{bmatrix} I_{ao} \\ I_{a1} \\ I_{a2} \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix}$$

where: 1, a, and a^2 are the first, second, and third cube roots of unity, respectively.

Premultiplying both sides of Eq. 4 and Eq. 6 by C^{-1} (the inverse of C) and substituting the results into Eq. 3, the following equation results:

$$C^{-1} \cdot V' = Z \cdot C^{-1} \cdot I' \tag{8}$$

Premultiplying both sides of Eq. 8 by C yields

$$V' = (C \cdot Z \cdot C^{-1}) \cdot I' \tag{9}$$

and by letting

$$Z' = C \cdot Z \cdot C^{-1} \tag{10}$$

the equation in terms of the new variables becomes:

$$V' = Z' \cdot I' \tag{11}$$

The use of these components is of advantage if the resulting matrix Z' is a diagonal matrix, as the three-phase system can then be replaced by three independent single-phase systems. This feature is of great advantage in system solutions with a network calculator. The reduction of the three-phase network into three independent single-phase systems no longer necessitates the solving of the network by a system of simultaneous equations. In balanced three-phase systems (a system in which the phases are electrically identical) all the before mentioned components transform the Z matrix into a diagonal matrix. On the other hand, for unbalanced three-phase systems, the resulting Z will not be of the diagonal form; therefore, the use of these components is of little or no advantage and will frequently add to the labor of the solutions of unbalanced networks.

The methods presented in this thesis will utilize transformations to produce a diagonal matrix Z'. The procedure of this thesis may be compared to that of Gabriel Kron (4). In Kron's work, all the transformations are tensor transformations in which he makes no attempt to use transformations which yield diagonal impedance tensors. This investigation will use linear transformations which will transform the matrix Z of both balanced and unbalanced systems directly into a diagonal matrix. No other restrictions will be placed upon the new variables.

CHAPTER III

LINEAR TRANSFORMATIONS

A linear transformation of the impedance matrix Z into a diagonal matrix may be accomplished by two methods: (1) directly, by a "congruent transformation (5)" and (2) directly, by a "complex-similarity transformation (6)." The congruent transformation will be discussed in this chapter and illustrated in Chapters IV and V. The second method will be discussed in Chapter VI.

The matrix Z is a symmetric matrix with elements in the field of complex numbers; i. e., its elements are complex. This type of matrix, even though it is common in electrical engineering, has not been of much interest to the mathematician. The advantage in transforming Z into a diagonal matrix is made evident by the ease of operating with this type matrix, in particular, if the diagonal matrix operates on a column matrix. The elements of a matrix resulting from the product of a diagonal matrix with a column matrix are independent. This independence is from the result that none of the elements of the product matrix contain a common factor from the two operating matrices; i. e., the element of the $\underline{i}$th row and the $\underline{i}$th column of the diagonal matrix forms a product with only the ith element in the column matrix. If, then, the impedance matrix is diagonalized, a three-phase system can be analyzed in terms of the new variables as three independent single-phase systems.

The congruent transformation is the most direct method of diagonalizing a symmetric matrix. A congruent transformation is defined as one in which the transformation is accomplished by premultiplying with a transformation matrix P and postmultiplying with the transpose of P. The transformation matrix P must have the following characteristics: its determinant must be nonzero and it must be obtainable by a succession of elementary row operations of the type which will be later denoted as $O_{ij}(k)$. The consequence of performing each row operation of this type is that a zero will be introduced below the main diagonal of the resulting matrix. If this operation is repeated three times, all the elements of a three-by-three matrix below the main diagonal will be made zero. Thus in the transformation

$$Z' = P \cdot Z \cdot P_t \tag{12}$$

the matrix P is developed with this succession of row operations, and the product P·Z will have only zeros below the main diagonal.

The diagonalization of Z will then be completed by a series of column operations which will be denoted by $C_{ij}(k)$ and will be discussed later. This series of column operations will yield zeros above the main diagonal. It will be shown that this series of column operations will yield the transpose of P so that Eq. 12 is satisfied. It will also be shown that the result of the three row operations and the three column operations will indeed diagonalize the matrix. These operations may be performed in either order as indicated in Eq. 12; i. e., P may be premultiplied into Z and the result postmultiplied by P_t, or Z may be postmultiplied by P_t and that result premultiplied by P.

Thus, for any three-phase circuit of the type shown in Fig. 1, the matrix P which reduces the impedance matrix to diagonal form may be found. Chapter IV will show how this may be accomplished.

CHAPTER IV

CONGRUENT TRANSFORMATIONS
APPLIED TO THREE-PHASE SYSTEMS

The application of congruent transformations to diagonalize the impedance matrix of a three-wire three-phase system follows. This type of transformation requires

$$Z' = P \cdot Z \cdot P_t \tag{13}$$

where: Z is the symmetrical impedance matrix,

Z' is the new diagonal impedance matrix,

P is the transformation matrix to be developed, and

P_t is the transpose of P.

In order to simplify the writing of equations, let the impedance of Fig. 1 be denoted as follows:

$$\begin{aligned} Z_{aa} &= a \\ Z_{bb} &= d \\ Z_{cc} &= e \\ Z_{ab} = Z_{ba} &= b \\ Z_{ac} = Z_{ca} &= c \\ Z_{cb} = Z_{bc} &= f \end{aligned} \tag{14}$$

Thus

$$Z = \begin{bmatrix} a & b & c \\ b & d & f \\ c & f & e \end{bmatrix} \tag{15}$$

The elements of P are obtained by first operating on the unit matrix U with the row operation and premultiplying the result into Z. This is done in succession, with the result of the row operation on the unit matrix being premultiplied into the preceding result. This row operation is denoted as $O_{ij}(k)$ and acts on the matrix which follows it. $O_{ij}(k)$ multiplies the $\underline{j}$th row of the matrix by $\underline{k}$ and adds the result to the $\underline{i}$th row. The result of this will be a matrix denoted as $O_{(p)}$ with p the number in the sequence of operations.

The second series of steps utilizes elementary column operations which act, in this work, on the unit matrix U and the result postmultiplied into the result of the preceding step. The column operation will be denoted as $C_{ij}(k)$ and acts on the matrix which precedes it. $C_{ij}(k)$ multiplies the $\underline{j}$th column by $\underline{k}$ and adds the result to the $\underline{i}$th column. The result of this operation on U will be denoted by $C_{(p)}$ with p the number in the sequence. These are illustrated below:

$$O_{12}(k)\cdot U = O_{12}(k)\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&k&0\\0&1&0\\0&0&1\end{bmatrix} \tag{16}$$

$$O_{23}(k)\cdot U = O_{23}(k)\cdot\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&k\\0&0&1\end{bmatrix}$$

Similarly:

$$C_{12}(k)\cdot U = C_{12}(k)\cdot \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\k&1&0\\0&0&1\end{bmatrix} \tag{17}$$

$$C_{23}(k)\cdot U = C_{23}(k)\cdot \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&k&1\end{bmatrix}$$

These steps will be performed on the unit matrix in the following sequence, and each operation will act on Z by introducing a zero below the main diagonal of Z. The use of the first elementary row operation will introduce a zero as the element in the first column and the second row, as can be seen below.

$$O_{21}(-b/a)\cdot U = \begin{bmatrix}1&0&0\\-b/a&1&0\\0&0&1\end{bmatrix} \tag{18}$$

Let this operation be denoted as $O_{(1)}$, so that

$$O_{(1)}\cdot Z = \begin{bmatrix}a&b&c\\0&-\frac{b^2}{a}+d&-\frac{bc}{a}+f\\c&f&e\end{bmatrix} \tag{19}$$

The result of the next row operation $O_{(2)}$ will be premultiplied into the result of Eq. 19 and will introduce a zero as the element in the third row and first column.

$$O_{(2)} = O_{31}(-c/a)\cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -c/a & 0 & 1 \end{bmatrix} \tag{20}$$

and

$$O_{(2)}\cdot O_{(1)}\cdot Z = \begin{bmatrix} a & b & c \\ 0 & -\frac{b^2}{a}+d & -\frac{bc}{a}+f \\ 0 & -\frac{bc}{a}+f & -\frac{c^2}{a}+e \end{bmatrix} \tag{21}$$

The triangularization of the matrix will be completed by the operation

$$O_{(3)} = O_{32}\left(-\frac{-\frac{bc}{a}+f}{-\frac{b^2}{a}+d}\right)\cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{-\frac{bc}{a}+f}{-\frac{bb}{a}+d} & 1 \end{bmatrix} \tag{22}$$

Then

$$O_{(3)}\cdot O_{(2)}\cdot O_{(1)}\cdot Z = \begin{bmatrix} a & b & c \\ 0 & -\frac{b^2}{a}+d & -\frac{bc}{a}+f \\ 0 & 0 & -\frac{\left(-\frac{bc}{a}+f\right)^2}{-\frac{bb}{a}+d} - \frac{c^2}{a} + e \end{bmatrix} \tag{23}$$

Since all the elements below the main diagonal are zeros in Eq. 23, it will be noted that P has been developed. From Eq. 23,

$$P \cdot Z = O_{(3)} \cdot O_{(2)} \cdot O_{(1)} \cdot Z \tag{24}$$

therefore,

$$P = O_{(3)} \cdot O_{(2)} \cdot O_{(1)} \tag{25}$$

Substituting Eqs. 18, 20, and 22 into Eq. 25, P is defined.

$$P = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{b}{a} & 1 & 0 \\ \left(\frac{b}{a}\right)\left(\frac{-\frac{bc}{a}+f}{-\frac{bb}{a}+d}\right)-\frac{c}{a} & -\frac{-\frac{bc}{a}+f}{-\frac{bb}{a}+d} & 1 \end{bmatrix} \tag{26}$$

Completion of the diagonalization of the right side of Eq. 23 will be accomplished by the development of P_t. P_t may be derived from Eq. 26 by a simple transposition of the rows for the columns, or it may be developed as a series of elementary column operations $C_{ij}(k)$. It should be noted that $C_{21}(k) \cdot U$ is the transpose of $O_{21}(k) \cdot U$.

The steps of the development of P_t, by the latter method, are the following:

$$C_{(1)} = C_{21}(-b/a) . U = \begin{bmatrix} 1 & -\frac{b}{a} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

$$C_{(2)} = C_{31}(-c/a) \cdot U = \begin{bmatrix} 1 & 0 & -c/a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{28}$$

and (29)

$$C_{(3)} = C_{32}\left(-\frac{\frac{-bc}{a}+f}{\frac{-bb}{a}+d}\right)\cdot U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\frac{-bc}{a}+f}{\frac{-bb}{a}+d} \\ 0 & 0 & 0 \end{bmatrix}$$

so (30)

$$P\cdot Z\cdot C_{(1)}\cdot C_{(2)}\cdot C_{(3)} = \begin{bmatrix} a & 0 & 0 \\ 0 & \frac{-b^2}{a}+d & 0 \\ 0 & 0 & -\frac{\left(\frac{-bc}{a}+f\right)^2}{\frac{-bb}{a}+d} \quad \frac{-c^2}{a}+e \end{bmatrix}$$

From Eq. 30,

$$P_t = C_{(1)}\cdot C_{(2)}\cdot C_{(3)} \tag{31}$$

and substituting Eqs. 27, 28, and 29 into Eq. 31 defines the elements of P_t:

$$P_t = \begin{bmatrix} 1 & -\frac{b}{a} & -\frac{c}{a}+\left(\frac{b}{a}\right)\left(\frac{\frac{-bc}{a}+f}{\frac{-bb}{a}+d}\right) \\ 0 & 1 & -\frac{\frac{-bc}{a}+f}{\frac{-bb}{a}+d} \\ 0 & 0 & 1 \end{bmatrix} \tag{32}$$

It may be noted that P_t is the transpose of P by comparing Eq. 26 with Eq. 32. Further it may be noted that $P\cdot Z\cdot P_t$ is the diagonal matrix of Eq. 30 and is the Z' desired in Eqs. 12 and 13.

To determine how the voltages and currents are transformed, both sides of Eq. 12 are premultiplied by P^{-1} and postmultiplied by P_t^{-1}. The result is

$$Z = P^{-1} \cdot Z' \cdot P_t^{-1} \tag{33}$$

Eq. 33 substituted into Eq. 3 yields

$$V = P^{-1} \cdot Z' \cdot P_t^{-1} \cdot I \tag{34}$$

Both sides of Eq. 34 are premultiplied by P, and

$$P \cdot V = Z' \cdot (P_t^{-1} \cdot I) \tag{35}$$

Let

$$V' = P \cdot V \tag{36}$$

and

$$I' = P_t^{-1} \cdot I \tag{37}$$

and in terms of the new variables, the desired form of Eq. 11 is obtained. It can be shown that the instantaneous power W is invariant under congruent transformation. Instantaneous power is defined

$$W = V_t \cdot I \tag{38}$$

Premultiplying Eq. 36 by P^{-1},

$$V = P^{-1} \cdot V' \tag{39}$$

and transposing V defines V_t in terms of the new variables:

$$V_t = V_t' \cdot P_t^{-1} \tag{40}$$

Premultiplying both sides of Eq. 37 by P_t defines I in terms of the new variables:

$$I = P_t \cdot I' \tag{41}$$

Substituting Eqs. 40 and 41 into Eq. 38 defines the power in terms of the new variables:

$$W = V_t' \cdot P_t^{-1} \cdot P_t \cdot I' \tag{42}$$

or

$$W = V_t' \cdot I' \tag{43}$$

Thus, the instantaneous power is invariant under congruent transformation.

To determine the transforming matrix for the currents, the inverse of P_t must be obtained. There are three steps in obtaining the inverse of a matrix P_t: (1) evaluate the determinant of P_t, (2) form the adjoint of P_t, Adj P_t , and (3) divide the Adj P_t by the determinant of P_t.

Inspection of Eq. 32 shows that the determinant of P_t is unity.

The Adj P_t is formed by transposing P_t, replacing each element by its cofactor, and multiplying each cofactor by $(-1)^{i+j}$, where $\underline{i}$ and $\underline{j}$ are the number of the row and column, respectively. The Adj P_t is

$$\begin{bmatrix} (-1)^2(1) & (-1)^3(-b/a) & (-1)^4(c/a) \\ 0 & (-1)^4(1) & (-1)^5\left(\dfrac{\frac{-bc+f}{a}}{-\frac{bb+d}{a}}\right) \\ 0 & 0 & (-1)^6(1) \end{bmatrix} \tag{44}$$

Thus,

$$P_t^{-1} = \begin{bmatrix} 1 & (b/a) & (c/a) \\ 0 & 1 & \dfrac{\frac{-bc+f}{a}}{-\frac{bb+d}{a}} \\ 0 & 0 & 1 \end{bmatrix} \tag{45}$$

Substituting Eq. 26 into Eq. 36,

$$V' = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{b}{a} & 1 & 0 \\ -\frac{c}{a} + \left(\frac{b}{a}\right)\left(\dfrac{\frac{-bc+f}{a}}{-\frac{b^2+d}{a}}\right) - \dfrac{\frac{-bc+f}{a}}{-\frac{bb+d}{a}} & & 1 \end{bmatrix} \cdot \begin{bmatrix} V_a \\ V_b \\ V_c \end{bmatrix} \tag{46}$$

Substituting Eq. 45 into Eq. 37,

$$I' = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} 1 & (b/a) & (c/a) \\ 0 & 1 & \dfrac{\dfrac{-bc}{a}+f}{\dfrac{-bb}{a}+d} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} I_a \\ I_b \\ I_c \end{bmatrix} \tag{47}$$

Substituting Eqs. 46, 47, and 30 into Eq. 10, the results are

$$\begin{aligned} V_1 &= aI_1 \\ V_2 &= \left(-\frac{b}{a}^2+d\right)(I_2) \\ V_3 &= \left(e - \frac{c}{a}^2 - \frac{\left(\frac{-bc}{a}+f\right)^2}{\frac{-bb}{a}+d}\right)\left(I_3\right) \end{aligned} \tag{48}$$

As an illustration, the circuit shown in Fig. 2 will be solved. The circuit represents a three-wire transmission line one mile long, composed of three 2/0 solid copper conductors. The conductors lie in a horizontal plane with conductor a in the center 30 inches from c and 45 inches from b. The conductors are 50 feet above the ground. The values for the impedances were determined using Carson's Equations for the self and mutual impedances of transmission lines in the presence of ground.

The system impedance values for Eq. 48 are:

$$\begin{aligned} a = d = e &= 0.523+j1.5 \text{ ohms} \\ b &= 0.092+j0.804 \text{ ohms} \\ c &= 0.092+j0.854 \text{ ohms} \\ f &= 0.092+j0.734 \text{ ohms} \end{aligned} \tag{49}$$

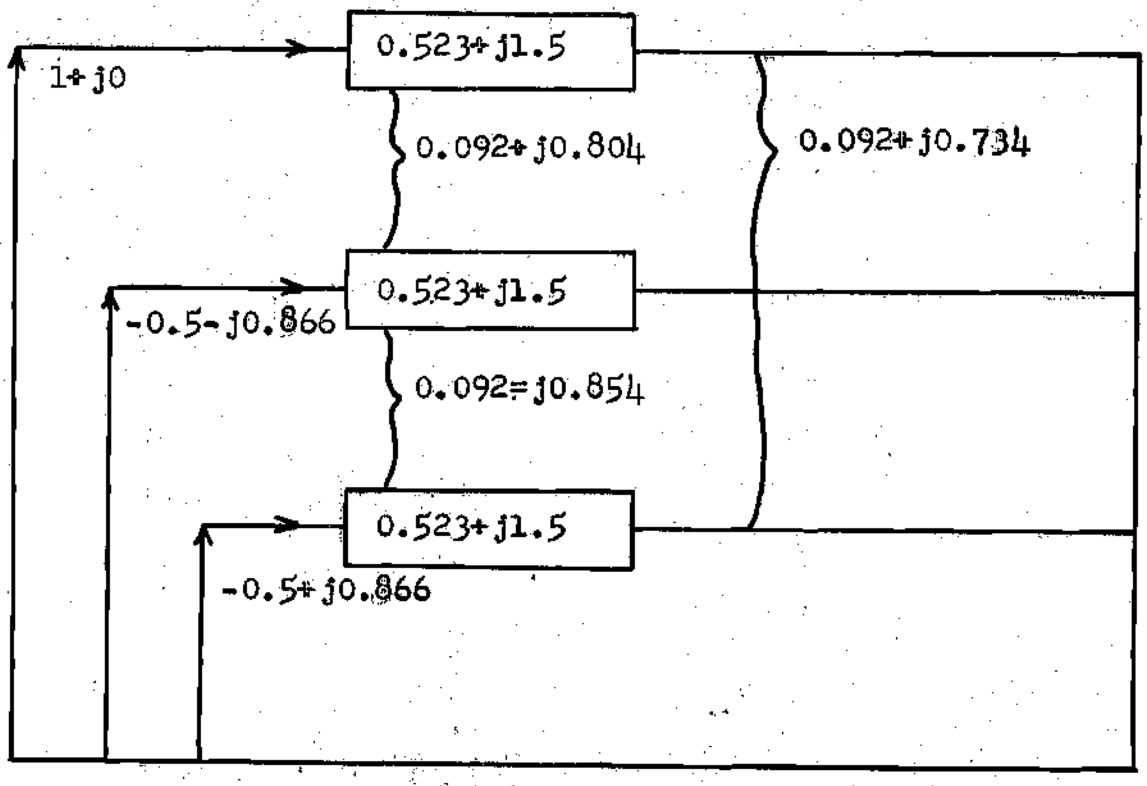

Fig. 2

Series three-wire three-phase system

The applied voltages are balanced, so

$$\begin{aligned} V_a &= 1 + j0 \text{ volts} \\ V_b &= -0.5 - j0.866 \text{ volts} \\ V_c &= -0.5 + j0.866 \text{ volts} \end{aligned} \tag{50}$$

Subsituting Eq. 50 into Eq. 46

$$\begin{aligned} V_1 &= 1 + j0 \text{ volts} \\ V_2 &= -0.997 - j0.978 \text{ volts} \\ V_3 &= -0.768 + j1.293 \text{ volts} \end{aligned} \tag{51}$$

Then substituting Eqs. 51 and 49 into Eq. 48

$$\begin{aligned} I_1 &= 0.208 - j0.596 \text{ amperes} \\ I_2 &= -1.066 + j0.325 \text{ amperes} \\ I_3 &= 0.678 + j1.188 \text{ amperes} \end{aligned} \tag{52}$$

and finally substituting the values of Eq. 52 into Eq. 47

$$\begin{aligned} I_a &= 0.611 - j1.167 \text{ amperes} \\ I_b &= -1.237 - j0.004 \text{ amperes} \\ I_c &= 0.678 + j1.188 \text{ amperes} \end{aligned} \tag{53}$$

The values of Eqs. 49, 50 and 53 were substituted into Eq. 1, and the results check very accurately.

CHAPTER V

CONGRUENT TRANSFORMATIONS APPLIED TO FOUR-WIRE SERIES THREE-PHASE SYSTEMS

In the presentation of the application of the congruent transformation to four-wire systems, let the impedance matrix of Fig. 3 be denoted as

$$Z = \begin{bmatrix} a & b & c & g \\ b & d & f & h \\ c & f & e & k \\ g & h & k & n \end{bmatrix} \tag{54}$$

The impedance matrix will be transformed with the transformation matrix P. The same type of elementary row operation which was explained in the last chapter will be used in the development of this P. It will be shown that the transformation matrix P for a four-wire system contains as a submatrix the transformation matrix P for a three-wire system. Thus, the four-wire system P can be developed from the P of the three-wire system.

Let the first operation be

$$O_{(1)} = O_{21}(-b/a)\cdot U = O_{21}(k_1)\cdot U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -b/a & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{55}$$

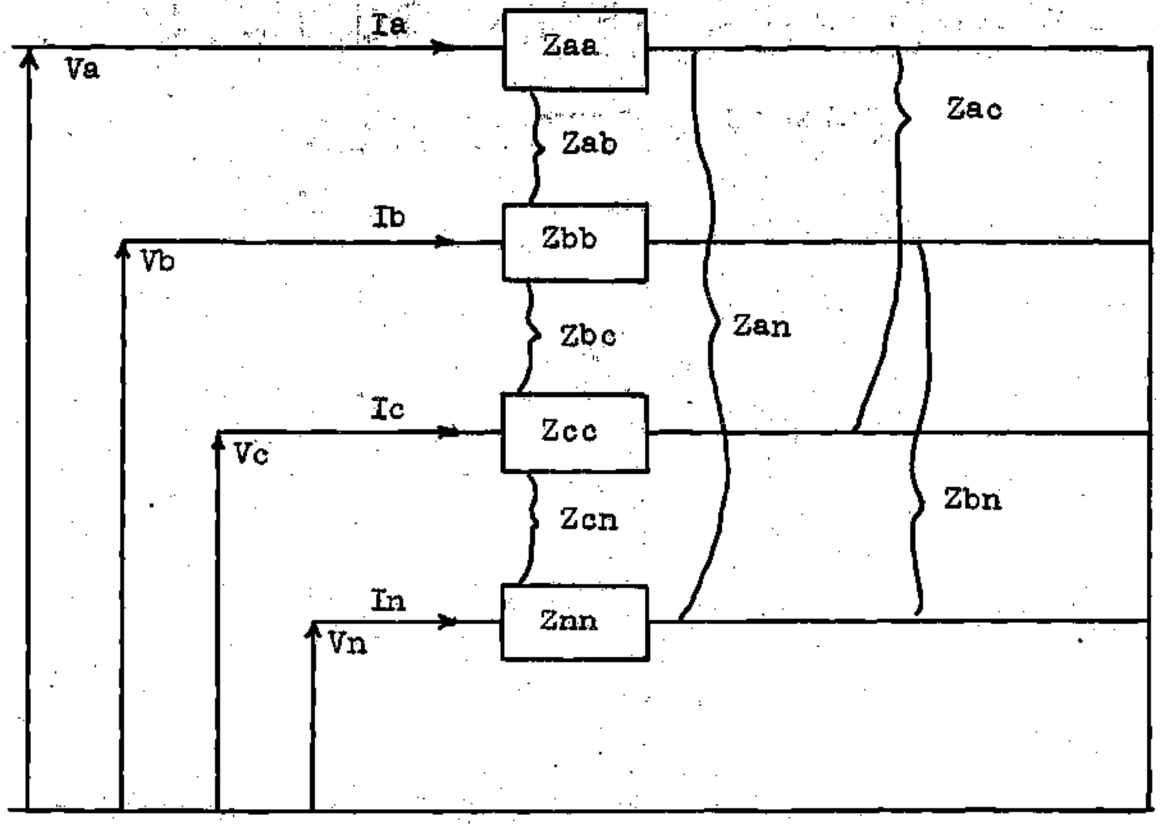

Fig. 3

Series four-wire three-phase system

and

$$O_{(1)}\cdot Z = \begin{bmatrix} a & b & c & g \\ 0 & -\frac{b^2}{a}+d & -\frac{bc}{a}+f & -\frac{gb}{a}+h \\ c & f & e & k \\ g & h & k & n \end{bmatrix} \tag{56}$$

Let the next operation place a zero in the first column and in the third row.

$$O_{(2)} = O_{31}(-c/a)\cdot U = O_{31}(k_2)\cdot U \tag{57}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -c/a & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$O_{(2)}\cdot O_{(1)}\cdot Z = \begin{bmatrix} a & b & c & g \\ 0 & -\frac{b^2}{a}+d & -\frac{bc}{a}+f & -\frac{bg}{a}+h \\ 0 & -\frac{bc}{a}+f & -\frac{c^2}{a}+e & -\frac{cg}{a}+k \\ g & h & k & n \end{bmatrix} \tag{58}$$

Similarly

$$O_{(3)} = O_{41}(-g/a)\cdot U = O_{41}(k_3)\cdot U = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{g}{a} & 0 & 0 & 1 \end{bmatrix} \tag{59}$$

introduces a zero in the first column, fourth row.

$$O(4) = O_{32}\left(- \frac{-\frac{bc}{a}+f}{-\frac{bb}{a}+d} \right) \cdot U = O_{32}(k_4) \cdot U = \tag{60}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & - \dfrac{-\frac{bc}{a}+f}{-\frac{bb}{a}+d} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

introduces a zero in the second column, third row of Z. Next

$$O(5) = O_{42}\left(- \frac{-\frac{bg}{a}+h}{-\frac{bb}{a}+d} \right) \cdot U = O_{42}(k_5) \cdot U = \tag{61}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & - \dfrac{-\frac{bg}{a}+h}{-\frac{bb}{a}+d} & 0 & 1 \end{bmatrix}$$

Let the last operation be

$$O(6) = O_{43}\left(- \frac{-\dfrac{\left(-\frac{gb}{a}+h\right)\left(-\frac{bc}{a}+f\right)}{-\frac{bb}{a}+d} - \frac{gc}{a}+k}{-\dfrac{\left(-\frac{bc}{a}+f\right)^2}{-\frac{bb}{a}+d} - \frac{c^2}{a}+e} \right) \cdot U = O_{43}(k_6) \cdot U \tag{62}$$

So Z' is triangularized by

$$O_{(6)} \cdot O_{(5)} \cdot O_{(4)} \cdot O_{(3)} \cdot O_{(2)} \cdot O_{(1)} \cdot Z = \tag{63}$$

$$\begin{bmatrix}
a & b & c & g \\
0 & -\frac{bb}{a}+d & -\frac{bc}{a}+f & -\frac{bg}{a}+h \\
0 & 0 & -\frac{\left(-\frac{bc}{a}+f\right)^2}{-\frac{bb}{a}+d} - \frac{c^2}{a}+e & -\frac{\left(-\frac{bc}{a}+f\right)\left(-\frac{bg}{a}+h\right)}{-\frac{bb}{a}+d} - \frac{gc}{a}+k \\
0 & 0 & 0 & -\frac{\left(-\frac{\left(-\frac{bc}{a}+f\right)\left(-\frac{bg}{a}+h\right)}{-\frac{bb}{a}+d} - \frac{gc}{a}+k\right)^2}{-\frac{\left(-\frac{bc}{a}+f\right)^2}{-\frac{bb}{a}+d} - \frac{cc}{a}+e} - \frac{\left(-\frac{bg}{a}+h\right)^2}{-\frac{bb}{a}+d} - \frac{g^2}{a}+n
\end{bmatrix}$$

Therefore, P is developed and is defined as

$$P = 0_{(6)} \cdot 0_{(5)} \cdot 0_{(4)} \cdot 0_{(3)} \cdot 0_{(2)} \cdot 0_{(1)} \tag{64}$$

or letting the elements of P which are not zero or unity be written in terms of the k_p of Eqs. 55, 57, 59, 60, 61, and 62

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ k_1 & 1 & 0 & 0 \\ k_1k_4+k_2 & k_4 & 1 & 0 \\ k_3+k_1k_4k_6+k_2k_6+k_5k_1 & k_4k_6+k_5 & k_6 & 1 \end{bmatrix} \tag{65}$$

It will be noted that the submatrix formed by the first three rows of the first three columns of Eq. 65 is the same matrix of Eq. 26. The k_p are the same.

The solution of the four-wire system can now be carried out in exactly the same manner as was illustrated for the three-wire system. The transpose of P can be more easily obtained from Eq. 65 rather than using the column operations. The inverse of P_t can be obtained by the procedure with which Eq. 45 was developed. The transformation equations are the same as in the three-wire system.

CHAPTER VI

COMPLEX-SIMILARITY TRANSFORMATIONS OF THREE-PHASE SYSTEMS

The complex-similarity transformation is a general type of transformation which actually encompasses the methods of all of the presently used components. However, the algebraic difficulties which are encountered in the application of this method to obtain exact transforming equations for the general three-phase system renders this impractical without the aid of a digital computer. This chapter will discuss this transformation and outline the methods whereby it may be utilized for a specific system (one in which the algebra has been performed).

The complex-similarity transformation acts to diagonalize a symmetric matrix Z by the following transformation equation

$$Z' = T^{-1} \cdot Z \cdot T \qquad (66)$$

where T is a matrix which is developed from the impedance matrix Z and acts to produce the diagonal matrix Z'.

The development of T from Z involves the use of the following features of the matrix Z:

(a) the characteristic determinant of Z which is denoted as

$$|Z - XU| \tag{67}$$

where U is the unit matrix and Z is any one of the roots (called the characteristic roots) of the equation

$$|Z - XU| = 0 \tag{68}$$

and (b) the characteristic vectors N are obtained from the equation

$$[Z - XU] \cdot N = 0 \tag{69}$$

where N is a column matrix whose elements form the characteristic vectors N.

The number of characteristic roots will be the same as the order of the matrix Z, and there will be one N for each characteristic root. These characteristic vectors are used to form the columns of the matrix T. The action of the matrix T which is obtained in this manner to diagonalize Z will not be affected by an interchange in the columns of T; hence, the column matrices may be placed in any order.

The evaluation of T depends on the determination of N which, in turn, depends on the evaluation of the characteristic roots. For a three-phase system the evaluation of the characteristic roots requires the solution of a third-order (or a fourth-order, in the case of a four-wire system) complex algebraic equation. The solution for these roots, as stated before, is practical only with the use of a computer.

The proof that the use of the above prescribed matrix T will indeed diagonalize a symmetrical matrix is shown in Reference (7), and will not be presented here.

This procedure will be illustrated by applying it to a balanced three-wire system for which the characteristic roots are known. The impedance Z will be

$$Z = \begin{bmatrix} Z & Z_m & Z_m \\ Z_m & Z & Z_m \\ Z_m & Z_m & Z \end{bmatrix} \tag{70}$$

where: Z is the self-impedance in each phase and

Z_m is the impedance between phases.

From Eq. 68,

$$\left| Z - XU \right| = \begin{vmatrix} Z-X & Z_m & Z_m \\ Z_m & Z-X & Z_m \\ Z_m & Z_m & Z-X \end{vmatrix} = 0 \tag{71}$$

The values of X which satisfy Eq. 71 are

$$\left.\begin{aligned} X_1 &= Z - Z_m \\ X_2 &= Z - Z_m \\ X_3 &= Z + 2Z_m \end{aligned}\right| \tag{72}$$

For the root $X_1 = Z-Z_m$, the characteristic vector N will have components n_1, n_2, and n_3. From Eq. 21,

$$\begin{bmatrix} Z-(Z-Z_m) & Z_m & Z_m \\ Z_m & Z-(Z-Z_m) & Z_m \\ Z_m & Z_m & Z-(Z-Z_m) \end{bmatrix} \cdot \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = 0 \tag{73}$$

From Eq. 73,

$$n_1+n_2+n_3 = 0 \tag{74}$$

or

$$n_1 = 1,\ n_2 = a,\ n_3 = a^2$$

The second characteristic vector N will have components n_4, n_5, and n_6. Obviously, its components will be the same as for the first root. It should be noted that T must have a nonzero determinant; therefore, two columns of T can not be identical. Thus,

$$n_4 = 1,\ n_5 = a^2,\ n_6 = a \tag{75}$$

For the third root,

$$\begin{bmatrix} Z-(Z+2Z_m) & Z_m & Z_m \\ Z_m & Z-(Z+2Z_m) & Z_m \\ Z_m & Z_m & Z-(Z+2Z_m) \end{bmatrix} \cdot \begin{bmatrix} n_7 \\ n_8 \\ n_9 \end{bmatrix} = 0 \tag{76}$$

$$n_7 + n_8 - 2n_9 = 0 \tag{77}$$

$$n_7 = n_8 = n_9 = 1$$

The matrix T will be formed by these characteristic vectors

$$T = \begin{bmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & a^2 & a \end{bmatrix} \tag{78}$$

which will be recognized as the C transformation matrix of Eq. 5. The only difference in the two is the constant factor of 1/3.

This method is applicable to unbalanced systems as well as balanced. It is interesting to note that alpha, beta, zero components (8) can be derived in a like manner. Alpha, beta, zero components will diagonalize an impedance matrix with an unbalance represented by changing the impedance of one of the main-diagonal elements of Eq. 70.

The complex-similarity transformation is the general transformation which includes all transformations (excepting congruent) of the symmetrical impedance matrix Z into a diagonal matrix Z'. It will be noted that symmetrical components is a special case of the general transformation.

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