NONLINEAR OSCILLATIONS AND MULTISCALE DYNAMICS IN A CLOSED CHEMICAL REACTION SYSTEM

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Abstract. We investigate the oscillatory chemical dynamics in a closed isothermal reaction system described by the reversible Lotka-Volterra model. This is a three-dimensional, dissipative, singular perturbation to the conservative Lotka-Volterra model, with the free energy serving as a global Lyapunov function. We will show that there is a natural distinction between oscillatory and non-oscillatory regions in the phase space, that is, while orbits ultimately reach the equilibrium in a non-oscillatory fashion, they exhibit damped, oscillatory behaviors as interesting intermediate dynamics.

Key words. Closed chemical reaction, Reversible Lotka-Volterra system, Multi-scale dynamics, Nonlinear oscillation, Singular perturbation.

AMS subject classifications. 34C15, 34E15, 37L45, 92E20

1. Introduction.

Since the discovery of the Belousov-Zhabotinsky (BZ) reaction and the "Oregonator" mechanism ([4, 16, 23]), many new studies in cell biology have also indicated the importance of chemical oscillations and it is well-believed that these oscillations can emerge as the collective dynamic behavior of interacting components in the cell. But little understanding is known about the mechanisms and underlying principles of such oscillations, for which modeling and analysis are essential.

There are essentially two types of reaction systems in which chemical oscillations can be observed. One is *open* system where exchange of matter and/or energy with the surroundings is allowed, and the other is *closed* system without such exchange. Following the pioneer work of Belousov-Zhabotinsky, open systems have been widely studied for the understanding chemical oscillations (see [7, 9, 14, 15, 16, 18] and references therein) and related dynamical issues (see e.g., [8] for composite double oscillations, [6] for the existence of limit cycles, [21] for chaos, [2, 10, 11] for wave motions, and [13, 22, 24] for period-doubling bifurcations). To the contrary, little is known about the nature of chemical oscillations in a closed system. While sustained oscillations are typically observed in an open system, the reaction in a closed system should go to the equilibrium state ([17]) and yield transitory, quasi-stationary oscillation ([4]) according to the laws of non-equilibrium thermodynamics ([1]). In fact, our recent work [12] on a reversible Lotka-Volterra (LV) reaction system has shown that chemical oscillations in a closed system exhibit a unique dynamical behavior differing from that of the traditionally studied nonlinear oscillations arising in mechanical and electrical systems.

The present paper is a continuation of our previous study in [12] on reversible Lotka-Volterra (LV) reaction system in which detailed dynamical behaviors of the underlying chemical oscillations will be analyzed. More precisely, consider the reversible

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Lotka-Volterra (LV) reaction

$$A + X \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} 2X, \quad X + Y \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} 2Y, \quad Y \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} B, \tag{1.1}$$

where X, Y, A and B denote the concentrations of the four species in the chemical reaction, k_1, k_2 and k_3 are forward reaction rates and k_{-1}, k_{-2} and k_{-3} are reverse reaction rates. The law of mass action then yields the following differential equations

$$\begin{cases} \dfrac{dx}{dt} &= k_1 c_A x - k_{-1}x^2 - k_2 xy + k_{-2}y^2, \\ \dfrac{dy}{dt} &= k_2 xy - k_{-2}y^2 - k_3 y + k_{-3}c_B, \\ \dfrac{dc_A}{dt} &= -k_1 c_A x + k_{-1}x^2, \\ \dfrac{dc_B}{dt} &= k_3 y - k_{-3}c_B \end{cases} \tag{1.2}$$

on the concentration rates of X, Y, A, B respectively. Under the rescaling

$$\begin{array}{c} u = \frac{k_2}{k_3}x, \quad v = \frac{k_2}{k_3}y, \quad w = \frac{k_1}{k_3}c_A, \quad z = \frac{k_2}{k_3}c_B, \\ \tau = k_3 t, \quad \varepsilon = \frac{k_{-1}}{k_1} = \frac{k_{-2}}{k_2} = \frac{k_{-3}}{k_3}, \quad \sigma = \frac{k_1}{k_2}. \end{array} \tag{1.3}$$

the system (1.2) has the following dimensionless form

$$\begin{cases} \dfrac{du}{d\tau} &= u(w-v) - \varepsilon(\sigma u^2 - v^2) \\ \dfrac{dv}{d\tau} &= v(u-1) - \varepsilon v^2 + \varepsilon z \\ \dfrac{dw}{d\tau} &= -\sigma(wu - \varepsilon\sigma u^2), \\ \dfrac{dz}{d\tau} &= v - \varepsilon z. \end{cases} \tag{1.4}$$

We assume that the backward reaction is much slower than the forward reaction and the first forward reaction is slower than the second forward reaction, and the ratios σ and ε satisfy $0 < \varepsilon \ll \sigma \ll 1$. Using the conservation of total concentration $\xi = u + v + \frac{w}{\sigma} + z$, system (1.4) is further reduced into the following three-dimensional system

$$\begin{cases} \dfrac{du}{d\tau} &= u(w-v) - \varepsilon(\sigma u^2 - v^2) \\ \dfrac{dv}{d\tau} &= v(u-1) - \varepsilon v^2 + \varepsilon\left(\xi - u - v - \frac{w}{\sigma}\right) \\ \dfrac{dw}{d\tau} &= -\sigma(wu - \varepsilon\sigma u^2). \end{cases} \tag{1.5}$$

The system (1.5) is dissipative, and, in fact, a two-scale singular perturbation of Hamiltonian systems. As $\varepsilon = \sigma = 0$, the unperturbed systems become a family of standard conservative LV systems which bear Hamiltonian structures by appropriate transformations. Each unperturbed system admits a family of periodic solutions with distinct frequencies. But when the perturbation is tuned on, dissipation will eliminate all periodic motions. Instead, as what we will show, the system (1.5) admits a unique globally attracting equilibrium, and moreover, the attraction is non-oscillatory in a *non-oscillating zone* near the equilibrium, simply because the equilibrium is an asymptotically stable node. Still, away from the equilibrium, we will also show that there is an *oscillating zone* in which solutions will oscillate in the (u, v)-direction around a central axis while moving downward in the w-direction, with decreasing oscillating diameters in response to the damping. In fact, each round of oscillation around the central axis consists of two parts: a "horizontal" portion with nearly unchanged energy and w-value which shadows a periodic orbit of the unperturbed system on the same energy level, and a "vertical" portion with exponentially decay energy and w-values. Such an oscillatory phenomenon well fits in the physical intuition predicted by the Second Law of Thermodynamics that chemical oscillation is far away from the equilibrium and the reaction eventually approaches the chemical equilibrium state as free energy dissipates.

The system (1.5) is also interesting from a pure dynamics point of view. First of all, it is a dissipative perturbation to conservative systems. This is an area for which little is known in dynamical systems theory. The second, for such a system, although long term dynamics are simple, the intermediate, finite-time ones are of the most theoretical and physical interests. Again, intermediate dynamics have not been much explored in dynamical systems theory. Finally, each orbit in the system admits a mixture of conservative, dissipative, and monotone natures, corresponding to the shadowing of conservative oscillations, jumping due to dissipation, and non-oscillatory convergence, respectively. This is also a new phenomenon in dynamical systems theory.

Detail contents of the paper is as follows. Section 2 is devoted for the description of general global dynamics of (1.5), in which we show the dissipative property of the system and global attraction of the equilibrium. In Section 3, we use the integral manifolds theory and Poincaré maps to describe oscillations away from the equilibrium in the oscillating zone by considering the existence of central axis and nature of oscillations. In Section 4, we apply the geometric theory of singular perturbations to describe the nature of convergence in the non-oscillating zone near the equilibrium. Some discussions on transit dynamics in the *transition zone* in between the oscillating and the non-oscillating zones will be made in Section 5.

2. Global Dynamics.

In this section, we will consider the general global dynamics of the reversible LV system (1.5).

2.1. Invariance.

Because each state variable in system (1.5) represents the concentration of the respective chemical reactant, it only makes sense that each variable remains positive during the time evolution. The following theorem actually shows that our model preserves this property.

THEOREM 2.1. *For any fixed $\xi > 1$, the tetrahedron*

$$\mathcal{T} = \left\{ (u, v, w) \in \mathbb{R}^3, u, v, w \geq 0, \text{ and } u + v + \frac{w}{\sigma} \leq \xi \right\}.$$

is positively invariant under the flow induced by (1.5).

Proof. Write $\partial \mathcal{T} = S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$S_1 = \{(u, v, w) \in \mathcal{T}, \ \ u = 0\}, \quad S_2 = \{(u, v, w) \in \mathcal{T}, \ \ v = 0\},$$
$$S_3 = \{(u, v, w) \in \mathcal{T}, \ \ w = 0\}, \quad S_4 = \left\{(u, v, w) \in \mathcal{T}, \ \ u + v + \tfrac{w}{\sigma} = \xi\right\}.$$

We denote by $V(X)$ the vector field on the right hand side of (1.5) with $X = (u, v, w)$, and by $\vec{n}_i$, $i = 1, 2, 3, 4$, the outer normal vector to ∂S_i, respectively. Since

$$\begin{array}{lll} V(X)\cdot\vec{n}_1 = -\varepsilon v^2 < 0, & \vec{n}_1 = (-1,0,0)^\top, & \text{on} \quad S_1,\\ V(X)\cdot\vec{n}_2 = -\varepsilon(\xi - u - \frac{w}{\sigma}) < 0, & \vec{n}_2 = (0,-1,0)^\top, & \text{on} \quad S_2,\\ V(X)\cdot\vec{n}_3 = -\varepsilon\sigma^2 u^2 < 0, & \vec{n}_3 = (0,0,-1)^\top, & \text{on} \quad S_3,\\ V(X)\cdot\vec{n}_4 = -v < 0, & \vec{n}_4 = (1,1,\frac{1}{\sigma})^\top, & \text{on} \quad S_4, \end{array}$$

the flow-invariant set theorem ([19]) implies that $\mathcal{T}$ is positively invariant. □

We refer $\mathcal{T}$ as the *reaction zone*, over which all the analysis in the rest of the paper will be carried on.

2.2. Long Time Behavior. Our next result says that the long term dynamics of system (1.5) are simple.

THEOREM 2.2. *Denote $r(\varepsilon) = 1+\varepsilon+\varepsilon^2+\varepsilon^3$. The following holds in the reaction zone:*

a) The system (1.5) is dissipative.
b) The system (1.5) has a unique interior equilibrium point $P_* = (u^*, v^*, w^*)^\top$, where

$$u^* = \frac{\varepsilon^2\xi}{r(\varepsilon)}, \quad v^* = \frac{\varepsilon\xi}{r(\varepsilon)}, \quad w^* = \frac{\sigma\varepsilon^3\xi}{r(\varepsilon)},$$

which is a global attractor.

Proof. It follows from an elementary calculation that P_* is the unique interior equilibrium of (1.5) in $\mathcal{T}$.

To show the dissipative property, we trace back to the original four-dimensional system (1.4). Let $X = (u, v, w, z)$ where $z = \xi - u - v - \frac{w}{\sigma}$, and consider the function

$$L(X) = u\ln\left(\frac{u}{u^*}\right) + v\ln\left(\frac{v}{v^*}\right) + \frac{w}{\sigma}\ln\left(\frac{w}{w^*}\right) + z\ln\left(\frac{z}{z^*}\right),$$

where $z^* = \xi - u^* - v^* - \frac{w^*}{\sigma}$. Then

$$L'(X) \quad = \quad uwp\left(\frac{\varepsilon\sigma u}{w}\right) + uvp\left(\frac{\varepsilon v}{u}\right) + vp\left(\frac{\varepsilon z}{v}\right),$$

where $p(x) = (1-x)\ln x$. It is clear that $L'(X) \le 0$ for all $X \in \mathbb{R}^{4+}$, i.e., L is a Lyapunov function and (1.4) is dissipative. Now, $L'(X) = 0$ only if

$$p\left(\frac{\varepsilon\sigma u}{w}\right) = p\left(\frac{\varepsilon v}{u}\right) = p\left(\frac{\varepsilon z}{v}\right) = 0 \iff \frac{\varepsilon\sigma u}{w} = \frac{\varepsilon v}{u} = \frac{\varepsilon z}{v} = 1,$$

or, $w = \sigma\varepsilon u$, $u = \varepsilon v$, $v = \varepsilon z$, which, under the constraint $u + v + \frac{w}{\sigma} + z = \xi$ yields that $u = u^*$, $v = v^*$, $w = w^*$ and $z = z^*$. Hence the equilibrium $X_* = (u^*, v^*, w^*, z^*)$ is the global attractor. □

We remark that the Lyapunov function L is precisely the free energy in thermodynamics. Thus the conclusion of the theorem above agrees with the second Law of Thermodynamics that the free energy must be decreasing to reach its minimum. However, because system (1.5) is not a gradient system, at least not the gradient of L, the solution curve does not go along with the direction of $-\nabla L$, i.e., the reaction in our system does not proceed along the deepest decent direction of the free energy. This is somewhat surprising from a usual chemistry view point.

3. Oscillating Zone. Although each orbit of (1.5) in $\mathcal{T}$ ultimately reaches the equilibrium P_*, we will show in this section that there is an oscillating zone away from the equilibrium in which all solutions oscillate around a central axis.

3.1. Central Axis. In the case $\varepsilon = 0$, the system (1.5) reduces to the following partially perturbed system

$$\left\{ \begin{array}{rcl} \dfrac{du}{d\tau} & = & u(w-v), \\ \dfrac{dv}{d\tau} & = & v(u-1), \\ \dfrac{dw}{d\tau} & = & -\sigma uw. \end{array} \right. \tag{3.1}$$

It is easy to see that

$$W_s^o = \left\{(u,v,w),\ u = \mu_\sigma,\ v = w\right\},$$

where $\mu_\sigma = \frac{1}{1+\sigma}$, is an one-dimensional, asymptotically stable, invariant manifold of (3.1) on which $v = w \sim e^{-\sigma\mu_\sigma t}$. As ε is a regular perturbation parameter, our next theorem shows that a similar invariant curve exists for the full system (1.5).

THEOREM 3.1. *Away from the equilibrium, there exists an one-dimensional asymptotically stable, smooth, locally invariant manifold $W^o_{\sigma,\varepsilon}$ with the following properties:*

a) For fixed $\sigma \ll 1$,

$$\begin{array}{rl} W^o_{\sigma,\varepsilon} = \{(u,v,w) \in \mathcal{T}: & u = \mu_\sigma + \varepsilon f_1(w) + O(\varepsilon^2), \\ & v = w + \varepsilon g_1(w) + O(\varepsilon^2),\ w \geq \sigma^2\}, \end{array}$$

where

$$\begin{array}{rcl} f_1 & = & \pi\left[-f_{11}\displaystyle\int f_{12}(w)p(w)dw + f_{12}\int f_{11}(w)p(w)dw\right], \\ g_1 & = & \sigma\pi w\left[-f'_{11}\displaystyle\int f_{12}(w)p(w)dw + f'_{12}\int f_{11}(w)p(w)dw\right] - \sigma\mu_\sigma + \dfrac{w^2}{\mu_\sigma}, \end{array}$$

with $p(w) = c_2 + \frac{c_3}{w} + \frac{c_4}{w^2}$, $f_{11} = \sqrt{w}J_1(2\sqrt{c_1 w})$ and $f_{12} = \sqrt{w}Y_1(2\sqrt{c_1 w})$, and J_1 and Y_1 being the Bessel's functions of first and second kinds, respectively, and

$$c_1 = \left(\frac{1+\sigma}{\sigma}\right)^2, \quad c_2 = \frac{1-\sigma^2}{\sigma^2}, \quad c_3 = \frac{(1+\sigma)^2}{\sigma^3}, \quad c_4 = \frac{1+\sigma}{\sigma^2}(\xi - \mu_\sigma).$$

b) Each solution on $W^o_{\sigma,\varepsilon}$ decays exponentially with rate $\sim \sigma\mu_\sigma$.

Proof. We note that under the transformation

$$u \to x + \mu_\sigma, \qquad v \to y + w, \qquad w \to w,$$

the system (1.5) becomes

$$\left\{ \begin{array}{rcl} \frac{dX}{d\tau} & = & A_\sigma(w)X + G(X,w,\sigma,\varepsilon) \\ \frac{dw}{d\tau} & = & \sigma F(X,w,\sigma,\varepsilon) \end{array} \right. \tag{3.2}$$

where

$$A_\sigma(z) = \begin{bmatrix} 0 & -\mu_\sigma \\ (1+\sigma)w & -\sigma\mu_\sigma \end{bmatrix}, \quad F(X,w,\sigma,\varepsilon) = -(x+\mu_\sigma)\left[w - \varepsilon\sigma(x+\mu_\sigma)\right],$$
$$G(X,w,\sigma,\varepsilon) = \begin{bmatrix} -xy - \varepsilon\sigma(x+\mu_\sigma)^2 + \varepsilon(y+w)^2 \\ xy - \varepsilon\left[\sigma^2(x+\mu_\sigma)^2 + (y+w)^2 + (x+\mu_\sigma) + (y+w) + \frac{w}{\sigma} - \xi\right] \end{bmatrix}.$$

As $w \geq \frac{\sigma^2\mu_\sigma^2}{4}$, $A_\sigma(w)$ has two complex eigenvalues with negative real parts $-\frac{\sigma\mu_\sigma}{2}$ and hence the freezed system $\frac{dX}{d\tau} = A_\sigma(w)X$ is uniformly exponentially stable with exponent $\nu = \frac{\sigma\mu_\sigma}{2}$.

Let f be a smooth cutoff function such that $f(w) = 1$ for $w \in I = [\sigma^2, \sigma\xi]$, $f(w) = 0$ for $w \in (0, \frac{\sigma^2\mu_\sigma^2}{4}]$ and $w \in (\sigma\xi + 1, \infty)$, $0 \leq f(w) \leq 1$ and $|f'(w)| \leq 2$ otherwise. Define

$$\tilde{F}(X,w,\sigma) = F(X,w,\sigma)f(w)$$

and consider the modified system

$$\begin{cases} \frac{dX}{d\tau} &= A_\sigma(w)X + G(X,w,\sigma,\varepsilon) \\ \frac{dw}{d\tau} &= \sigma\tilde{F}(X,w,\sigma,\varepsilon). \end{cases} \tag{3.3}$$

We note that for fixed $\sigma \ll 1$, ε introduces a regular perturbation when $w \geq \sigma^2$, and $G = O(|X|^2 + \varepsilon)$. It is then easy to see that the proof in [20, 26] for the existence, asymptotic stability, and smoothness of integral manifolds holds true for system (3.3), yielding an one-dimensional, smooth, asymptotically stable, invariant manifold $\tilde{W}_{\sigma,\varepsilon}$ for (3.3). In fact, for any continuous function $X(t)$, let $W(w,X)(t)$ denotes the solution of the second equation of (3.3) with initial value $w \in I$. Then it follows from the proof in [20, 26] that there exists a sufficiently small constant $\varepsilon_1 > 0$ such that the mapping

$$\mathcal{F}(X)(t) = \int_{-\infty}^{t} \Phi_\sigma(t,s,W(w,X)(s))G(X,W(w,X)(s),s)ds$$

is well defined in an appropriate function space containing $\{|X| < \varepsilon_1\}$, and is also a uniformly contraction mapping. Moreover, if $X(t)$ is the fixed point, then the graph of $X(w,\sigma,\varepsilon) = X(0)$ defines the invariant manifold $\tilde{W}_{\sigma,\varepsilon}$. In vitro of the cutoff function, it is now clear that

$$W^o_{\sigma,\varepsilon} = \left\{(u,v,w) \in \tilde{W}_{\sigma,\varepsilon}, \quad w \in I\right\}$$

is the desired one-dimensional, locally invariant manifold for system (3.2).

To prove a), we use the fact that ε is a regular perturbation parameter and expand $X(w,\sigma,\varepsilon)$, for fixed σ, into the following Taylor series

$$u = \mu_\sigma + \sum_{k=1}^{\infty} f_k(w)\varepsilon^k, \qquad v = w + \sum_{k=1}^{\infty} g_k(w)\varepsilon^k.$$

Since $W^o_{\sigma,\varepsilon}$ is the graph of $X(w,\sigma,\varepsilon)$, b) is then proved by substituting the Taylor series into (1.5) and comparing the coefficients.

As the restricted equation on $W^o_{\sigma,\varepsilon}$ is a regular perturbation of that on W^o_σ, b) follows from the exponential decay property of solutions on W^o_σ. □

3.2. Oscillatory Behavior. Let ϕ_t denote the flow induced by system (1.5) and $B_\delta\left(W^o_{\sigma,\varepsilon}\right)$ be a δ-neighborhood of $W^o_{\sigma,\varepsilon}$. For a fixed $1<\alpha<2$ but sufficiently close to 2, we refer the set

$$\mathcal{T}^o = \mathcal{T} \bigcap \left(\bigcup_{t\le 0} \phi_t\left(B_\delta\left(W^o_{\sigma,\varepsilon}\right)\right)\right) \bigcap \{(u,v,w)\in\mathcal{T} : w \ge \sigma^\alpha\}$$

as the *oscillating zone*. From the construction of $W^o_{\sigma,\varepsilon}$ in the proof of Theorem 3.1, we see that as $w \ge \frac{\sigma^2\mu_\sigma^2}{4}$, the linearized matrix $A_\sigma(w)$ is exponentially stable with complex eigenvalues. This suggests the possibility of oscillations around the central axis W^o_σ. In fact, as described in the theorem below, oscillations around the central axis exhibit both regular and singular behaviors.

Let

$$\begin{aligned}
\mathcal{T}_1^o &= \{(u,v,w)\in\mathcal{T} : w \ge \sigma\}, \\
\mathcal{T}_2^o &= \left\{(u,v,w)\in\mathcal{T} : w \ge \sigma^2\right\}, \\
\Omega_- &= \{(u,v,w)\in\mathcal{T} : v < w\}, \\
\Omega_+ &= \{(u,v,w)\in\mathcal{T} : v > w\}.
\end{aligned}$$

Theorem 3.2. *The following holds as $\varepsilon \ll \sigma \ll 1$ and for some $\alpha > 1$.*

a) $\mathcal{T}_1^o \subset \mathcal{T}^o \subset \mathcal{T}_2^o$.

b) Each orbit starting in $\mathcal{T}^o$ oscillates a finite number of times around the central axis $W^o_{\sigma,\varepsilon}$, with decreasing distance to the axis, and moreover, there is a constant $c_0 > 0$ such that, the oscillation number of each orbit with initial energy E_0^σ is less than $-c_0(\alpha-1)\frac{\ln\sigma}{\mu_\sigma E_0^\sigma}$.

c) Each round of oscillation around $W^o_{\sigma,\varepsilon}$ consists of two parts: a "horizontal" portion in Ω_+ with nearly unchanged energy and w-value which shadows a periodic orbit of the unperturbed system on the same energy level, and a "vertical" portion in Ω_- with exponentially decay energy and w-values.

d) There is a constant $c > 0$ such that at the bottom of the oscillating zone $\mathcal{T}^o$ where $w \sim \sigma^\alpha$, the energy E^σ is less than or equal to $c(2-\alpha)\sigma^\alpha\ln\sigma$.

Proof. Since w is monotonically decreasing when $w \ge \sigma\varepsilon\xi$, part a) of the theorem follows from the definition of $\mathcal{T}^o$ and the construction of W^o_σ in the proof of Theorem 3.1.

As ε is a regular perturbation parameter, it is sufficient to prove the remaining parts of the theorem for the partially perturbed system (3.1). Consider the following transformation

$$\begin{aligned}
&(1+\sigma)\left[(u-\mu_\sigma) + \mu_\sigma \ln\left(\frac{\mu_\sigma}{u}\right)\right] = E^\sigma\cos^2\theta, \\
&(v-w) + w\ln\left(\frac{w}{v}\right) = E^\sigma\sin^2\theta,
\end{aligned}$$

where θ is the (counterclockwise) rotation angle around $W^o_\sigma = \{(u,v,w),\ u=\mu_\sigma,\ v=w\}$ which is an approximation of $W^o_{\sigma,\varepsilon}$, and E^σ is the translated total free energy for the partially perturbed system (3.1) which is proportional to the distance to W^o_σ. Under this transformation the system (3.1) becomes

$$\left\{\begin{array}{lcl}
\dot{E}^\sigma &=& -\sigma u E^\sigma \sin^2\theta, \\
\dot{w} &=& -\sigma u w, \\
\dot{\theta} &=& \Theta_\sigma(E^\sigma, w, \theta),
\end{array}\right. \tag{3.4}$$

where

$$\Theta_\sigma(E^\sigma, w, \theta) = \begin{cases} (-1)^k(1+\sigma)\dfrac{(u-\mu_\sigma)\sqrt{w}}{\sqrt{2E^\sigma}}, & \theta = k\pi; \\ (-1)^k(1+\sigma)\dfrac{(v-w)}{\sqrt{2E^\sigma}}, & \theta = k\pi + \frac{\pi}{2}; \\ \dfrac{(1+\sigma)}{2E^\sigma}\left(\dfrac{u-\mu_\sigma}{\cos\theta}\right)\left(\dfrac{v-w}{\sin\theta}\right) - \dfrac{\sigma u}{2}\sin\theta\cos\theta, & \text{otherwise.} \end{cases}$$

Clearly, $\dot{E^\sigma} \leq 0$.

Consider the Poincarè section $\Sigma^\sigma = \{(u,v,w), u > 0, u \neq \mu_\sigma, v = w \geq \sigma^\alpha\}$ and let

$$\Sigma_+^\sigma = \{(u,v,w) \in \Sigma^\sigma, u > \mu_\sigma\}, \qquad \Sigma_-^\sigma = \{(u,v,w) \in \Sigma^\sigma, 0 < u < \mu_\sigma\}.$$

It is easy to see that the flow ϕ_t^σ generated from the partially perturbed system (3.1) crosses Σ transversely except when $u = \mu_\sigma$, and, those start from Σ_+ (resp. Σ_-) will travel in Ω_+ (resp. Ω_-) then reach Σ_- (resp. Σ_+). Let $\pi : \Sigma^\sigma \to \Sigma^\sigma$: $P \mapsto \phi_{t(P)}(P)$, be the Poincaré map, where $t(P)$ is the first return time.

It is easy to see that there is a constant $c > 0$ such that $\dot\theta \geq c$ as $w > \sigma$. Hence each orbit starting in $\mathcal{T}_1^o$ oscillates around W_σ^o. Starting from a $P_0 \in \Sigma_+^\sigma \cap \mathcal{T}_1^o$, we consider a sequence of points $\{P_n = (E_n^\sigma, w_n, \theta_n)\} \subset \mathcal{T}^o$ such that $P_{n+1} = \pi(P_n)$, $n = 0, 1, \cdots$. For each n, write $w_n = c_n\sigma$. Then $c_n > \sigma^{\alpha-1}$, and in particular, $c_n = O(1)$ if $P_n \in \mathcal{T}_1^o$. Denote $t_n = t(P_n)$, $n = 0, 1, \cdots$. Since

$$\left[\ln(vw^{\frac{1}{\sigma}})\right]' = \frac{\dot v}{v} + \frac{1}{\sigma}\frac{\dot w}{w} = (u-1) - u = -1,$$

we have

$$t_n = \ln\left(v_n w_n^{\frac{1}{\sigma}}\right) - \ln\left(v_{n+1} w_{n+1}^{\frac{1}{\sigma}}\right) = \frac{1}{\sigma\mu_\sigma}\left(\ln w_n - \ln w_{n+1}\right). \tag{3.5}$$

We now consider in the above sequence three consecutive points $P_{2k}, P_{2k+1}, P_{2k+2}$. Then $P_{2k}, P_{2k+2} \in \Sigma_+$ and $P_{2k+1} \in \Sigma_-$. We note that Θ_σ can be rewritten as

$$\Theta_\sigma = \left(\frac{u-\mu_\sigma}{2\cos\theta}\right)\left[\left(\frac{v-w}{E^\sigma\sin\theta}\right) + \sigma\left(\frac{w\ln\frac{v}{w}}{E^\sigma\sin\theta} + \sin^3\theta\right)\right] - \frac{\sigma\mu_\sigma}{2}\sin\theta\cos\theta.$$

Let $(u(t), v(t), w(t))$ be the portion of the orbit from P_{2k} to P_{2k+2}. Since both $\dfrac{u-\mu_\sigma}{\cos\theta} > 0$ and $\dfrac{v-w}{\sin\theta} > 0$ and $\dfrac{v-w}{\sqrt{E^\sigma}\sin\theta} \geq \sqrt{2w_{2k+1}}$ as $\theta \in [2k\pi, (2k+1)\pi]$, there exists a constant c_* such that

$$\Theta_\sigma > c_*\sqrt{2w_{2k+1}} - \frac{\sigma}{4} > 0. \tag{3.6}$$

It follows that

$$t_{2k} = \int_{2k\pi}^{(2k+1)\pi} \frac{1}{\Theta_\sigma} d\theta < \frac{4\pi}{4c_*\sqrt{2w_{2k+1}} - \sigma}.$$

Hence by (3.5),

$$1 < \frac{w_{2k}}{w_{2k+1}} = e^{\sigma\mu_\sigma t_{2k}} < \exp\left(\frac{4\mu_\sigma\pi\sqrt{\sigma}}{4c_*\sqrt{2c_{2k+1}} - \sqrt{\sigma}}\right) \sim 1,$$

i.e., $w_{2k+1} = (1 + O(\sigma^{1-\frac{\alpha}{2}}))w_{2k}$. From the first two equation in (3.4), we also have

$$\frac{dE^\sigma}{dw} = \frac{E^\sigma}{w}\sin^2\theta \leq \frac{E^\sigma}{w},$$

which implies that

$$1 > \frac{E^\sigma_{2k+1}}{E^\sigma_{2k}} \geq \frac{w_{2k+1}}{w_{2k}} \sim 1.$$

Hence $E^\sigma_{2k+1} = (1 + O(\sigma^{1-\frac{\alpha}{2}}))E^\sigma_{2k}$. From the monotonicity of E^σ and w, we also see that for this portion of the orbit, E^σ and w remain near constants, for which the last equation of the system (3.4) becomes a regular perturbation to the corresponding one in the unperturbed system (i.e., when $\sigma = 0$ in (3.4)). It follows that this portion of the orbit shadows the (periodic) solution of the unperturbed system with the same initial value (see e.g., [25]).

On the other hand, as $\theta \in [(2k+1)\pi, (2k+2)\pi]$, we have $v \leq w \leq w_{2k+1}$. Then

$$t_{2k+1} = \int_{u_{2k+1}}^{u_{2k+2}} \frac{1}{u(w-v)}du > \int_{u_{2k+1}}^{u_{2k+2}} \frac{1}{uw}du > \int_{u_{2k+1}}^{u_{2k+2}} \frac{1}{uw_{2k+1}}du = \frac{1}{w_{2k+1}}\ln\frac{u_{2k+2}}{u_{2k+1}}.$$

Since $u_{2k+1} < \mu_\sigma e^{-E^\sigma_{2k+1}}$ and $u_{2k+2} > \mu_\sigma$, we have

$$\frac{w_{2k+2}}{w_{2k+1}} = e^{-\sigma\mu_\sigma t_{2k+1}} < \left(\frac{u_{2k+2}}{u_{2k+1}}\right)^{-\frac{\sigma\mu_\sigma}{w_{2k+1}}} < \exp\left(-\frac{\sigma\mu_\sigma E^\sigma_{2k+1}}{w_{2k+1}}\right).$$

But

$$\frac{d}{dt}\left(\frac{E^\sigma}{w}\right) = \frac{\dot{E}^\sigma w - E^\sigma \dot{w}}{w^2} = \frac{-\sigma u E^\sigma w\sin^2\theta + \sigma u E^\sigma w}{w^2} = \sigma u\cos^2\theta\frac{E^\sigma}{w} \geq 0,$$

that is, $\frac{E^\sigma}{w}$ is increasing in time. Thus,

$$\frac{w_{2k+2}}{w_{2k+1}} < \exp\left(-\frac{\sigma\mu_\sigma E^\sigma_0}{w_0}\right) = \exp\left(-\frac{\mu_\sigma E^\sigma_0}{c_0}\right). \tag{3.7}$$

To estimate $\frac{E_{2k+2}}{E_{2k+1}}$, we let

$$\tilde{\Theta}_\sigma = \frac{\Theta_\sigma}{u} = p(\theta)q(\theta) - \frac{\sigma}{2}\sin\theta\cos\theta,$$

where

$$p(\theta) = \frac{v-w}{\sqrt{2E^\sigma}\sin\theta} = \sqrt{w}\frac{1-\frac{v}{w}}{\sqrt{\frac{2E^\sigma}{w}}|\sin\theta|}, \quad q(\theta) = \frac{1-\frac{\mu_\sigma}{u}}{\sqrt{2E^\sigma}\cos\theta}.$$

Then

$$\ln\left(\frac{E^\sigma_{2k+2}}{E^s_{2k+1}}\right) = -\sigma\int_{(}2k+1)\pi^{2k+2)\pi}\frac{\sin^2\theta}{\tilde{\Theta}_\sigma}d\theta.$$

As $\theta \in [(2k+1)\pi, (2k+\frac{3}{2})\pi]$, since both $\frac{E^\sigma}{w}$ and $|\sin\theta|$ are increasing, $p(\theta)$ is decreasing. While as as $\theta \in [(2k+\frac{3}{2})\pi, 2(k+1)\pi]$,

$$\frac{d}{d\theta}\left(\frac{E^\sigma}{w}\sin^2\theta\right) = \frac{E^\sigma}{w}\left(2 + \frac{\sigma u\sin\theta\cos\theta}{\Theta_\sigma}\right)\sin\theta\cos\theta \leq 0,$$

that is, p is increasing. Thus for $\theta \in [(2k+1)\pi, (2k+2)\pi]$,

$$p(\theta) \le \max\{p((2k+1)\pi), p(2(k+1)\pi)\} = \max\{\sqrt{w_{2k+1}}, \sqrt{w_{2(k+1)}}\} \le \sqrt{w_{2k+1}}.$$

Similarly, $q(\theta)$ is decreasing for $\theta \in [(2k+1)\pi, (2k+2)\pi]$. Using the fact that if $f(x) = x - 1 - \ln x = a^2$ with $x < 1$ then $x \ge e^{-1-a^2}$, we have

$$q(\theta) \le \frac{\frac{\mu_\sigma}{u_{2k+1}} - 1}{\sqrt{2E^\sigma_{2k+1}}} \le \frac{e^{1+E^\sigma_{2k+1}} - 1}{\sqrt{2E^\sigma_{2k+1}}}.$$

It follows that as $\theta \in [(2k+1)\pi, (2k+2)\pi]$,

$$\tilde{\Theta}_\sigma \le (1+\sigma)\left(e^{1+E^\sigma_{2k+1}} - 1\right)\sqrt{\frac{w_{2k+1}}{2E_{2k+1}}} + \frac{\sigma}{4} \le (1+\sigma)\left(e^{1+E^\sigma_0} - 1\right)\sqrt{\frac{w_0}{2E_0}} + \frac{\sigma}{4}.$$

Consequently,

$$\ln\left(\frac{E^\sigma_{2k+2}}{E^\sigma_{2k+1}}\right) \le \frac{-2\sigma\mu_\sigma\pi}{4\left(e^{1+E^\sigma_0} - 1\right)\sqrt{\frac{w_0}{2E^\sigma_0}} + \sigma\mu_\sigma},$$

or equivalently,

$$\frac{E^\sigma_{2k+2}}{E^\sigma_{2k+1}} \le \exp\left(\frac{-2\sigma\mu_\sigma\pi}{4\left(e^{1+E^\sigma_0} - 1\right)\sqrt{\frac{w_0}{2E^\sigma_0}} + \sigma\mu_\sigma}\right).$$

This inequality shows that if $\frac{w_0}{E_0} \sim \sigma^2$, then the decay rate of the energy E^σ will be more or less a constant independent of σ, resulting a jump of energy during the second half of each oscillation. However, if $\frac{w_0}{E_0} \sim \sigma^\beta$ with $0 < \beta < 2$, then the decay rate will be in the order $\exp\left(-\sigma^{1-\frac{\beta}{2}}\right)$, resulting only a small jump of energy. The proof of c) is now complte.

It also follows from (3.7) that

$$\frac{w_{2n}}{w_0} = \prod_{k=1}^n \frac{w_{2k}}{w_{2k-2}} < \prod_{k=1}^n \frac{w_{2k}}{w_{2k-1}} < \exp\left(-\frac{n\mu_\sigma E^\sigma_0}{c_0}\right),$$

and consequently

$$n < \frac{c_0 \ln\left(\frac{w_0}{w_{2n}}\right)}{\mu_\sigma E^\sigma_0}.$$

We note that n is precisely the number of oscillations of the orbit $\phi^\sigma_t(P_0)$ ending at P_{2n} around the central axis. We note that if P_{2n} reaches the the bottom of the oscillating zone $\mathcal{T}^o$, then $w_{2n} \ge \sigma^\alpha$, from which b) easily follows.

By the inequality $\Theta_\sigma \ge c_*\sqrt{2w_{2n-1}} - \frac{\sigma}{4}$ as $\theta \in [2(n-1)\pi, (2n-1)\pi]$, we know that $\Theta_\sigma > 0$, that is, the oscillation proceeds in $\Omega_+ \cap \mathcal{T}^o_2$. While if $\theta \in [(2n-1)\pi, 2n\pi]$, then we have $\frac{v-w}{\sqrt{E^\sigma}\sin\theta} \ge \sqrt{2v} \ge \sqrt{2v_{2n-\frac{1}{2}}}$ and $\Theta_\sigma \ge c_*\sqrt{2v_{2n-\frac{1}{2}}} - \frac{\sigma}{4}$, where $v_{2n-\frac{1}{2}}$, $w_{2n-\frac{1}{2}}$ and $E^\sigma_{2n-\frac{1}{2}}$ (associated to $\theta = 2n\pi - \frac{\pi}{2}$) satisfy

$$v_{2n-\frac{1}{2}} - w_{2n-\frac{1}{2}} - w_{2n-\frac{1}{2}} \ln\left(\frac{v_{2n-\frac{1}{2}}}{w_{2n-\frac{1}{2}}}\right) = E^\sigma_{2n-\frac{1}{2}}$$

and hence

$$v_{2n-\frac{1}{2}} < w_{2n-\frac{1}{2}} \exp\left(-\frac{E^\sigma_{2n-\frac{1}{2}}}{w_{2n-\frac{1}{2}}}\right) < w_{2n-1} \exp\left(-\frac{E^\sigma_{2n-1}}{w_{2n-1}}\right).$$

To proceed the oscillation to reach P_{2n} which is close to the bottom of the oscillation zone $\mathcal{T}^o$, it is necessary that $\Theta_\sigma > 0$ for which it is sufficient to have

$$\frac{\sigma^2}{32c_*^2} \le v_{2n-\frac{1}{2}} < w_{2n-1} \exp\left(-\frac{E^\sigma_{2n-1}}{w_{2n-1}}\right)$$

or

$$E^\sigma_{2n} < E^\sigma_{2n-1} \le w_{2n-1} \ln\left(\frac{32c_*^2 w_{2n-1}}{\sigma^2}\right) \sim (2-\alpha)\sigma^\alpha \ln \sigma.$$

$\square$

According to numerical simulations, it seems that for two orbits initiated in $\Omega_+ \cap \mathcal{T}_1^o$ with the same w-initial value, the one with smaller initial distance to $W^o_{\sigma,\varepsilon}$ has a less number of oscillations around it in $\mathcal{T}^o$. As the number of jumps made by an orbit in $\mathcal{T}^o$ is approximately equal to its number of oscillations, a possible mechanism behind this could be that the one with smaller initial distance to $W^o_{\sigma,\varepsilon}$ would have bigger total length of jumps in $\mathcal{T}^o$ (hence a less number of jumps which would imply a less number of total oscillations). Indeed, by the estimate (3.7), this is true for the first jumps made by the two orbits.

The above theorem is stated for a fixed ξ. If we allow ξ to depend on σ in the scale of $\frac{1}{\sigma}$, then $w = O(1)$ can be considered. In this case, it is not hard to see that σ can be also treated as a regular parameter, and hence all oscillations in the oscillating zone are regular without significant jumps in w-values. The numerical plugs in Figure 3.1 below give comparisons between the regular and singular oscillatory behaviors.

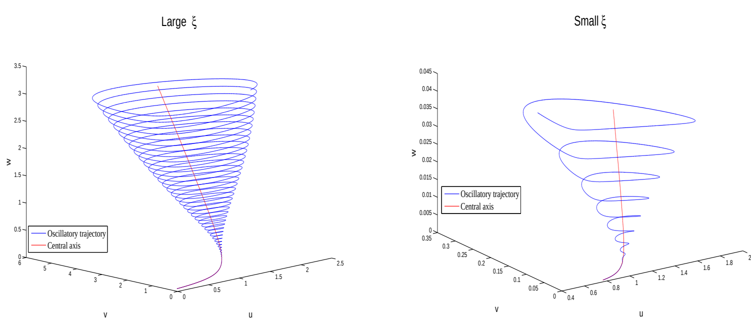

FIG. 3.1. *Oscillation zone. In the left panel,* $\xi = 500, \sigma = 0.01, \varepsilon = \sigma^2$ *with initial* $(2, 2, 3)$. *In the right panel,* $\xi = 10, \sigma = 0.005, \varepsilon = \sigma^2$ *with initial* $(0.5, 0.1, 0.045)$.

4. Non-oscillating Zone. In this section, we show the existence of a non-oscillating zone near the the equilibrium and study its dynamical behaviors.

THEOREM 4.1. *The unique interior equilibrium point P_* is an asymptotically stable node.*

Proof. Consider the Jacobi matrix at P_*,

$$J(P_*) = \frac{1}{r(\varepsilon)} \begin{pmatrix} -\varepsilon(1+\sigma\varepsilon^2)\xi & \varepsilon^2\xi & \varepsilon^2\xi \\ -\varepsilon r(\varepsilon) + \varepsilon\xi & -(1+\varepsilon)r(\varepsilon) - \varepsilon^2\xi & -\dfrac{\varepsilon r(\varepsilon)}{\sigma} \\ \sigma^2\varepsilon^3\xi & 0 & -\sigma\varepsilon^2\xi \end{pmatrix}.$$

The characteristic polynomial of $J(P_*)$ is $P(\lambda) = \lambda^3 + a\lambda^2 + b\lambda + c$, where

$$\begin{aligned} a &= \tfrac{1}{r(\varepsilon)}(1+\varepsilon)[r(\varepsilon) + \varepsilon(1+\sigma\varepsilon)\xi] > 0, \qquad c = \tfrac{1}{r(\varepsilon)}\sigma\varepsilon^3\xi^2 > 0, \\ b &= \tfrac{1}{r(\varepsilon)}[r(\varepsilon) - 1 + \sigma\varepsilon(r(\varepsilon)-1) + \sigma\varepsilon^3)]\xi + \tfrac{\sigma\varepsilon^2}{r^2(\varepsilon)}(r(\varepsilon)-1)\xi^2 > 0. \end{aligned}$$

Since $a \sim 1$, $b \sim \varepsilon\xi$, and $c \sim \sigma\varepsilon^3\xi^2$, it is not hard to see that all eigenvalues of $J(P_*)$ are real and negative. Thus P_* is an asymptotically stable node. $\square$

The eigenvalues $\lambda_1, \lambda_2, \lambda_3$ of $J(P_*)$ satisfies the following estimate

$$\lambda_1 \sim -(1+\varepsilon) + O(\varepsilon^2), \qquad \lambda_2 \sim -\varepsilon\xi + O(\varepsilon^2), \qquad \lambda_3 \sim -\sigma\varepsilon^2\xi + O(\sigma^2\varepsilon^3)$$

associated with eigenvectors

$$\vec{v}_1 \sim [0,1,0]^\top, \quad \vec{v}_2 \sim [1,(\xi-1)\varepsilon,0]^\top, \quad \vec{v}_3 \sim [0,0,1]^\top$$

respectively.

It then follows from standard invariant manifolds theory that near P_* there is a two-dimensional strongly stable manifold which is tangent to v_2, v_3 at P_* and contains an one-dimensional stable manifold along with stable foliations. Such a two-dimensional invariant manifold thus provides a barrel for any oscillation to take place near P_*.

We now apply the geometric theory of singular perturbation to show that there actually exists a non-oscillating zone in a bigger region near P_*.

THEOREM 4.2. *There exists a positively invariant set $\mathcal{T}^n$ containing P_* with the following properties:*

1. $\{(u,v,w) \in \mathcal{T}, w \leq \sigma\varepsilon\} \subset \mathcal{T}^n \subset \{(u,v,w) \in \mathcal{T}, w \leq \sigma\}$.
2. *There exists a two-dimensional, exponentially stable, positively invariant manifold $M^2_{\sigma,\varepsilon}$ in $\mathcal{T}^n$ containing P_* along with global one-dimensional stable foliations.*
3. *On $M^n_{\sigma,\varepsilon}$, there exists an one-dimensional, exponentially stable, positively invariant manifold $M^1_{\sigma,\varepsilon}$ containing P_* along with global one-dimensional stable foliations.*

Proof. In vitro of Theorem 3.2, we consider the following rescaling

$$u \to u, \qquad v \to \varepsilon v, \qquad w \to \sigma\varepsilon w, \tag{4.1}$$

by which equation (1.5) becomes

$$\begin{cases} \dfrac{du}{d\tau} &= \varepsilon\left[u(\sigma w - v) - (\sigma u^2 - \varepsilon^2 v^2)\right] \\ \dfrac{dv}{d\tau} &= v(u-1) - \varepsilon^2 v^2 + (\xi - u - \varepsilon v - \varepsilon w) \\ \dfrac{dw}{d\tau} &= -\sigma u(w-u). \end{cases} \tag{4.2}$$

When $\sigma = 0$ (hence $\varepsilon = 0$), system (4.2) has a two-dimensional reduced manifold

$$M_0^\eta = \left\{(u,v,w), \quad v = h_0(u,w) = \frac{\xi - u}{1-u},\ u, w \in [0,\eta]\right\}$$

for some fixed $\eta < 1$, consisting of equilibria, which is normally, exponentially stable.

By employing a cutoff function near the boundary ∂M_0^η, we have by the geometric theory of singular perturbation ([5]) that under the perturbation M_0^η gives rise to a slightly deformed, two-dimensional, normally exponentially stable, positively invariant manifold

$$M^2_{\sigma,\varepsilon} = \{(u,v,w), \quad v = h_{\sigma,\varepsilon}(u,w),\ u,w \in [0,\eta]\},$$

where $h_{\sigma,\varepsilon}(u,w) = h_0(u,w) + O(\sigma)$. Since the existence of global attractor P_* implies that there is no periodic orbit in $\mathcal{T}$, the compact invariant manifold $M^2_{\sigma,\varepsilon}$ must contain at least one equilibrium point. In fact, it is easy to see that P_* is the only equilibrium contained in $M^2_{\sigma,\varepsilon}$. The geometric theory of singular perturbations also asserts the existence of one-dimensional, global, stable foliations $W^s_{\sigma,\varepsilon}(p)$, $p \in M^2_{\sigma,\varepsilon}$, in $\mathcal{T}$.

Now, the restricted flow on $M^2_{\sigma,\varepsilon}$ is described by the system

$$\begin{cases} \dfrac{du}{d\tau_1} &= \delta\left[u(\sigma w - h_{\sigma,\varepsilon}(u,w)) - (\sigma u^2 - \sigma^2\delta^2 h^2_{\sigma,\varepsilon}(u,w))\right] \\ \dfrac{dw}{d\tau_1} &= -u(w-u), \end{cases} \tag{4.3}$$

where $\tau_1 = \sigma\tau$ and $\delta = \frac{\varepsilon}{\sigma}$. When $\delta = 0$, the system has an one-dimensional, normally exponentially stable, reduced manifold

$$M_*^\eta = \{(u,w), \quad u = w \in [0,\eta]\}$$

consisting of equilibria. When $\delta \neq 0$, the geometric theory of singular perturbation again leads to a slightly deformed, one-dimensional, normally exponentially stable, positively invariant manifold

$$M^1_{\sigma,\varepsilon} = \{(u,w),\ w = u + O(\delta),\ u \in [0,\eta]\}$$

containing P_*, along with global, one-dimensional stable foliations.

Let $\mathcal{T}^n$ be defined by $\cup_{p\in M^2_{\sigma,\varepsilon}} W^s_{\sigma,\varepsilon}(p)$ with respect to the original scales. It follows from the above and the actual construction of the stable foliations ([5]) that

$$\{(u,v,w) \in \mathcal{T}, w \leq \sigma\varepsilon\} \subset \mathcal{T}^n \subset \{(u,v,w) \in \mathcal{T}, w \leq \sigma\}.$$

□

According to the theorem, all orbits on $M^2_{\sigma,\varepsilon}$ converge to P_* exponentially following the one-dimensional stable manifold $M^1_{\sigma,\varepsilon}$ and each orbit in $\mathcal{T}^n \setminus M^2_{\sigma,\varepsilon}$ converges to P_* exponentially following an orbit on $M^2_{\sigma,\varepsilon}$, it is clear that orbits in $\mathcal{T}^n$ do not oscillate in any way. We thus refer $\mathcal{T}^n$ as the *non-oscillating zone.*

Figure 4.1 is a numerical demonstration of dynamics in the non-oscillating zone $\mathcal{T}^n$. The left panel of Figure 4.1 shows how orbits follow the two-dimensional strongly stable invariant manifold $M^2_{\sigma,\varepsilon}$, and the right panel shows how orbits in $M^2_{\sigma,\varepsilon}$ follow the one-dimensional stable manifold $M^1_{\sigma,\varepsilon}$.

5. Transition Zone. Let

$$\mathcal{T}^t = \mathcal{T} \setminus \{\mathcal{T}^o \cup \mathcal{T}^n\}.$$

Then $\mathcal{T}^t$ is the *transition zone* in which transitions from oscillating to non-oscillating dynamical behaviors of orbits take place. In this section, we give some numerical evidences on what would happen to the dynamics in this zone.

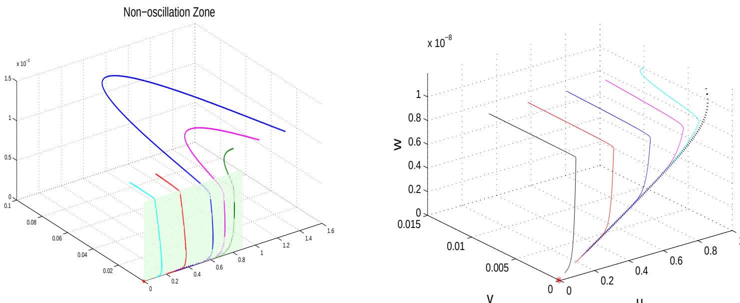

FIG. 4.1. *Nonoscillation zone. In both panel,* $\xi = 100; \sigma = 0.01, \varepsilon = \sigma^3$. *In the left one, the initials are* $(u_0, 0.02, 0.001)$ *with* $u_0 = 0.1, 0.333, 1.033, 1.267, 1.5$. *In the right one, the initials are* $(u_0, \sigma, \sigma\varepsilon)$ *with* $u_0 = 0.1, 0.325, 0.55, 0.775, 1$.

In the left panel of Figure 5.1, three particular orbits in the transition zone are plotted. An interesting phenomenon we observe numerically is that, although an orbit in the oscillating zone has a decreasing oscillating diameters to the central axis and the diameter is nearly zero at the bottom of the oscillating zone, it can actually reassume some oscillations in the transition zone with increasing diameters, but as its w value decreases, its number of complete oscillations also decreases. Once the orbit exits the transition zone, it immediately follows the two-dimensional strong stable manifold in the non-oscillating zone in a monotone fashion.

The right panel of Figure 5.1 is a MatLab simulation of dynamics in the transition zone. We observe that there is a unique, narrow, 'Tornado"-looking pathway in $\mathcal{T}^t$ through which the transition occurs. Moreover, no matter where they start in the oscillating zone $\mathcal{T}^o$, orbits eventually cluster in the pathway and then follow the "Tornado tail" to enter the non-oscillating zone $\mathcal{T}^n$. We note that for an orbit $(u(t), v(t), w(t))$ entering in the transition zone, the initial entry value u_0 of $u(t)$ is of the scale $\mu_\sigma < 1 < \xi$. It then follows from (1.5) that $u(t) \le (\xi - 1)^{1-\xi} e^{-\xi t}$, that is, $u(t)$ starts to decay exponentially during the transition. This explains the formation of the narrow pathway in the transition zone.

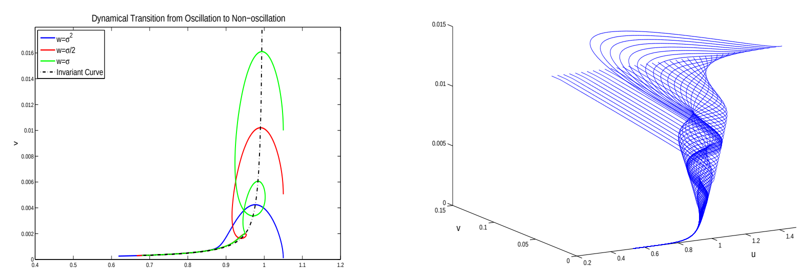

FIG. 5.1. *Dynamical Transition from* $\mathcal{T}^o$ *into* $\mathcal{T}^n$, *where* $\xi = 100, \sigma = 0.01, \varepsilon = \sigma^3$. *In the left panel, the initials values are* $(1.05, v_0, w_0)$ *with* $v_0 = w_0 = \sigma^2, \frac{\sigma}{2}, \sigma$, *respectively.*

MatLab simulations also suggest that the backward extension of the one-dimensional stable manifold in the non-oscillating zone makes a "turn" in the pathway of the transition zone then merges with the central oscillating axis in the oscillating zone, and the backward extension of the two-dimensional strongly stable manifold into the oscillating zone spirals around the central axis $W^o_{\sigma,\varepsilon}$ (see Figure 5.2).

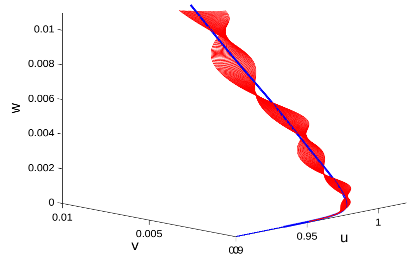

FIG. 5.2. *Backward extension of the portion of the 2d stable manifold.*

Because orbits in the transition zone can behave in either oscillatory or non-oscillatory way, there should exist a surface layer to separate these distinct behaviors. Indeed, this is the case as indicated in our numerical simulation showing in Figure 5.3. In the figure, the red surface is the separating surface, inside which (the green region) oscillations proceed accompanying with jumps, and outside which (the blue region) oscillations are blocked and orbits jump to the bottom quickly.

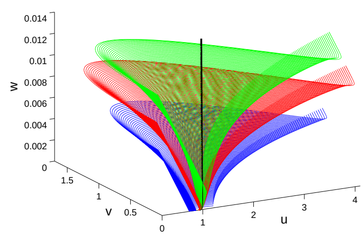

FIG. 5.3. *Separation of Oscillation and Non-oscillation in the Transition Zone.*

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