AN INVESTIGATION OF A METHOD OF DETERMINING THE APPARENT MASS OF A TWO-DIMENSIONAL SLOTTED PISTON IN A CLOSED CYLINDER ACCELERATING IN INCOMPRESSIBLE POTENTIAL FLOW

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LIST OF SYMBOLS

C	Space Curve
D	Cylinder Height
d	Orifice Height
F	Force
K.E.	Kinetic Energy
L	Length
l	Piston Length
M, m	Mass
M_a	Apparent Mass
M_b	Mass of Body
n	Direction
R	Distance from filament to a point
s	Area
T	Time
u	Velocity in x direction
U_∞	Characteristic Velocity
V	Velocity
v	Velocity in y direction
x	Horizontal Direction
x_o	Piston Position
x'	Filament Coordinate
y	Vertical Direction
y'	Filament Coordinate

Γ	Constant
γ	Filament Strength
γ_{∞}	Characteristic Vorticity
Δ	Increment
θ	Angle Between Reference Line and Line Connecting Filament with Point at Which Velocity Is Induced
ρ	Fluid Mass Density
ϕ	Velocity Potential
∇^2	Laplacian Operator

SUMMARY

This study is concerned with a method for calculating the apparent mass of a body accelerating in an incompressible potential flow while confined to a finite fluid field. To facilitate dealing in specifics, the problem is reduced to attempting to make an analytic determination of the apparent mass of a two-dimensional slotted piston accelerating in incompressible potential flow in a closed cylinder. The ultimate objective of the study is to determine if apparent mass is a function of piston position in the cylinder.

An outline of the concept of apparent mass of a body moving in an incompressible potential flow is presented. The kinetic energy relations required to derive an analytic expression for the apparent mass of such a body are presented, and from these relations, an analytic expression for apparent mass is developed.

The equations used to describe the flow conditions in the cylinder are developed from the Law of Biot and Savart, and the boundary conditions applicable to potential flow and the chosen problem are discussed and applied to these equations.

In using the Law of Biot and Savart, it is necessary to replace velocity discontinuities in the cylinder with vortex sheets. In order not to exceed the capacity of the available computing equipment, a finite number of vortex filaments are placed in the cylinder in place of the vortex sheets. These filaments approximate the vortex sheets which they replace.

A discussion and analysis of the results to be expected and a method for arriving at these results are presented. Some possible indications of having arrived at solutions other than the desired solution are discussed, and it is pointed out that an analogy may be drawn between this problem and the classic thin airfoil problem.

The results of some calculations using the procedure outlined in the paper are presented and are used to show that the procedure is a reasonable one and that further computations using the procedure are justified.

Recommendations are made as to suggested further calculations using the method outlined in the paper. A possible method for altering the plan of attack is suggested.

CHAPTER I

INTRODUCTION

It has been experimentally observed that when a piston with an orifice operates in a closed cylinder, the time required for the piston to translate a certain distance in the cylinder when subjected to a uniform acceleration is affected by whether or not there is a vibratory motion imposed on the piston in addition to the uniform acceleration. This phenomenon is difficult to explain analytically, and it is considered that the difficulty may lie at least partially in the fact that available theory does not account for the apparent mass of the piston. Since apparent mass is a quantity that does not exist physically, but acts as though it does, a few words to explain the concept appear to be in order.

As a body moves at a constant velocity through a void, there is no force to hinder its motion. Further, should the body be immersed in a perfect, incompressible fluid, there is no force exerted to hinder its motion, once the motion has been started. This fact is derivable from Bernoulli's Equation, which, when properly integrated, shows that while a body moving in a perfect incompressible flow will in general have a resultant couple acting on it, there is in no case a resultant force. This fact is provable for all simply connected finite bodies, and is frequently referred to as D'Alemberts' Paradox[1].

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1. Von Mises, Theory of Flight, p. 224.

However, should this same body be accelerating in the fluid, there will be a requirement for a net force acting on the body in order to sustain the acceleration. The force required to sustain the acceleration is that needed to accelerate the mass of the body plus some mass of affected fluid. It is the determination of this mass (solid mass plus mass of affected fluid) that is the subject of this investigation.

Since each element of fluid is not necessarily accelerated in the direction of motion of the body, the mass of fluid affected need not necessarily be equal to the mass of fluid displaced. As an example, the apparent additional mass of a sphere is only one-half the mass of the displaced fluid[2].

2. Towsley, Apparent Additional Mass, p. 1.

CHAPTER II

KINETIC ENERGY EQUATIONS

As a starting point, it appears appropriate to present a development of the basic equation that will be used in this paper to determine the apparent mass of the body in question. In order to give an insight into the physical mechanisms involved, the development will be presented using a form of dimensional analysis.

As a body moves through an incompressible fluid, the flow pattern about it is established as if by impulse. This behavior is an outgrowth of the flow being incompressible, where an incompressible fluid is defined as one in which the speed of sound is infinite. Thus, any change of boundary conditions is transmitted instantaneously throughout the field to all fluid particles. An impulse is imparted to the fluid by each increment of surface area, and the direction of this impulse is normal to the surface imparting it. Further, as the body moves through the field, it pushes fluid particles out of its way, and, as each of these fluid particles has mass, work is thereby done on each fluid particle.

From a consideration of the units of the quantities involved, it can be seen that the impulse and the momentum of a fluid particle are equatable, or,

$$F \cdot \Delta T = MV \tag{1}$$

Equation (1) can be seen to be dimensionally correct.

It can now be shown that the kinetic energy of the particle is proportional to the product of the velocity of the particle in the direction of motion (which is normal to the solid surface) and the impulse. Checking units:

$$\frac{ML^2}{T^2} \propto \frac{L}{T} \cdot \frac{ML}{T^2} \cdot T$$

(Kinetic Energy) (Velocity) (Force) (Time)

However, the kinetic energy of the body is $\frac{1}{2} mV^2$, where m is the body mass and V the body velocity. If this expression for the kinetic energy is set equal to the product of the velocity times an impulse, and the resulting equation divided by $\frac{1}{2} V^2$, the equation will have units of mass. This mass then, is the apparent mass which this paper seeks to determine. It is now desirable to find an analytic expression whereby this apparent mass may be calculated.

As a starting point, the intuitive relation equating the kinetic energy to the product of a normal velocity and an impulsive pressure will be used.

$$\frac{1}{2} mV^2 = (F)\cdot(\Delta T)\cdot(V) \tag{2}$$

An analytic expression will be determined for the right hand side of this equation.

Lamb[3] points out that any actual state of motion of a fluid for which there exists a single valued velocity potential can be produced instantaneously from rest by a properly chosen system of impulsive

3. Lamb, Hydrodynamics, p. 18.

pressures, and that this system is $\rho\phi$, where ρ is the fluid mass density and ϕ the velocity potential. $\rho\phi$ can be shown to be an impulsive pressure by examining the units. The impulsive force may thus be found by multiplying $\rho\phi$ by an increment of surface area, Δs.

The normal velocity required in the right hand side of Equation (2) may be found from the velocity potential ϕ defining the flow. This is accomplished by differentiating the velocity potential with respect to the direction normal to the body surface. That this will yield a velocity can be seen by analyzing the units of any simple problem.

A proposition in dynamics[4] states that the work done by an impulse is equal to the impulse times one-half the sum of the initial and final velocities. Since for this case, the motion was produced in a fluid initially at rest, the average velocity will be $\frac{1}{2} \frac{\partial\phi}{\partial n}$, and from Equation (2) the kinetic energy ΔK.E. imparted to the fluid by an increment of surface area is

$$\Delta \text{ K.E.} = \left(\rho\phi\right)\left(\frac{1}{2} \frac{\partial\phi}{\partial n}\right)\Delta s \tag{3}$$

The Δ K.E.'s due to all the increments of surface area may be summarized by taking the surface integral of Equation (3)

$$\text{K.E.} = -\frac{\rho}{2}\int\int_S \phi \frac{\partial\phi}{\partial n} \, ds \tag{4}$$

where the minus sign is used to designate the inner normal to the surface.

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4. Lamb, Hydrodynamics, p. 46.

Substituting Equation (4) into Equation (2) and rearranging:

$$M_a = \frac{2(K.E.)}{V^2} + M_b \tag{5}$$

where M_a is the apparent mass of the body accelerating in the fluid, and K.E. is the kinetic energy of the fluid particles.

It should be noted at this point that while Equation (4) has been derived on an intuitive basis, it can be derived in a completely mathematical fashion. The method of attack in this case is to use Green's Theorem and combine it with Laplace's Equation $\nabla^2\phi = 0$. Using the proper notation, the result is idential to Equation (4). The proof is given in detail by Lamb[5].

This derivation presupposes the fluid to be initially at rest and the body to be moving through the fluid. Although the apparent mass would be the same for the case of the body being at rest and an infinite field of fluid flowing past it, the kinetic energy of such a field is obviously infinite and the same approach as used above could not be used to determine the apparent mass of the body. For such a case, the key to finding the apparent mass is recognition of the fact that introducing the body into the fluid produces a change of kinetic energy in the fluid field, and it is the calculation of this change of kinetic energy that leads to an expression for the apparent mass.

Some excellent examples of the application of Equation (4) are given in Reference 3.

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5. Lamb, Hydrodynamics, art. 44.

CHAPTER III

FLOW EQUATIONS

Problem Outline

Consider a piston orifice operation in a closed circular cylinder (see Figure 1).

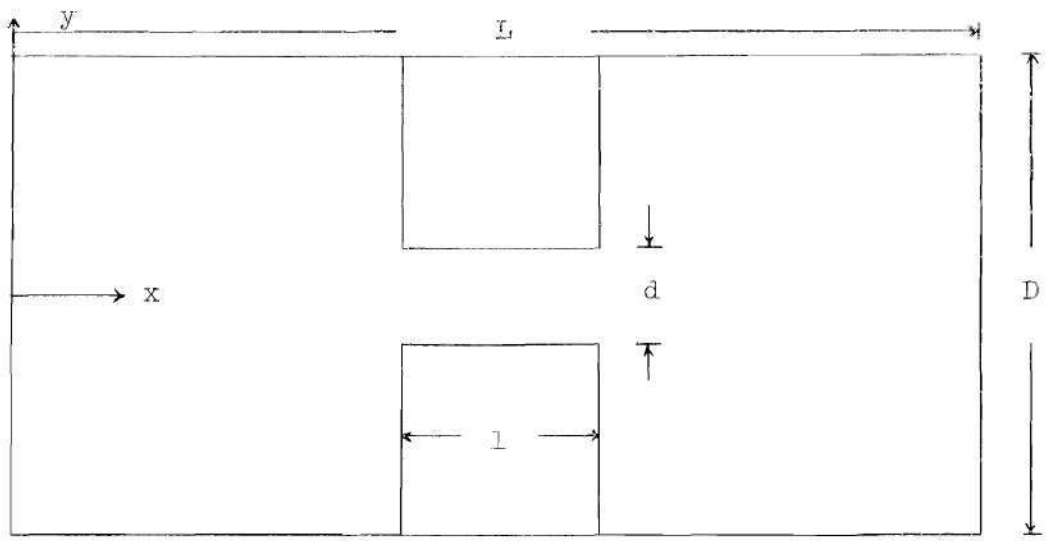

Figure 1. Piston Orifice in a Closed Cylinder

The piston is free to slide in the horizontal direction and the cylinder is filled with an incompressible fluid. A device such as this is useful as an acceleration integrator, and might also be used as a switching mechanism. When this device is subjected to a uniform acceleration, the time required for the piston to close on the end wall of the cylinder is predictable. However, when in addition to the uniform acceleration the piston is subjected to a vibratory motion, the time for the piston to close on the end wall has been observed to vary significantly from that predicted by available theory. It is this, at present unexplained, deviation from the computed time to close which has led to the present investigation.

Problem Analysis

The first step of the analysis is to attempt to build a mathematical model of the piston which will lend itself to solution and which shows promise of giving some insight into the nature of the mechanism causing this time deviation. Analyses to date have neglected apparent mass, and it is felt that this may be a serious omission.

This problem is basically three-dimensional. However, for reasons of simplification whose details will become apparent throughout the development, the initial simplifying assumption that will be made in this analysis is to reduce the problem to two dimensions. This is accomplished mathematically by imagining the cylinder of Figure 1 to extend to infinity in both directions in the plane normal to the paper. As is usual for this type of simplification, whenever a volume calculation is needed in the analysis, a unit depth will be taken in this direction.

A second simplification to be introduced at this point is that classic potential flow theory will be used in the analysis. The principal restrictions of this assumption are the relaxation of the "no slip" condition at the solid surfaces and the disregarding of any viscosity effects within the fluid itself. At first, this might appear to be a severe restriction on the physical interpretation of the results obtained, but it should be remembered that for small amplitude and high frequency the flow about an oscillating body approaches potential flow[6]. Further, use of potential theory will permit the replacement of velocity discontinuities along solid surfaces with vortex sheets. As will be seen, vortex theory is one of the principal tools used to solve the problem.

A third assumption will be that there is no friction between the wall and the piston. This is the most easily relaxed of the assumptions, as the simplification involved is only one of calculation, and not of theory.

Using these three principal assumptions, the scope of the problem for this paper will be to make an analytic determination of the apparent mass of a two-dimensional slotted frictionless piston in a closed cylinder accelerating in incompressible potential flow. The objective is to determine if there is a variation of apparent mass as the piston translates in the cylinder.

Two factors in this analysis are of interest. First, an analytic determination of the apparent mass of a body moving in a finite fluid field is an infrequent calculation. Second, the use of classic vortex theory in the determination of apparent mass is apparently little used. Although the theory of vortices as used in this paper is not difficult, the computations are lengthy and require the use of digital computing techniques.

The two-dimensional model constructed with the listed assumptions is shown in Figure 2.

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6. Towsley, <u>Apparent Additional Mass</u>, p. 42.

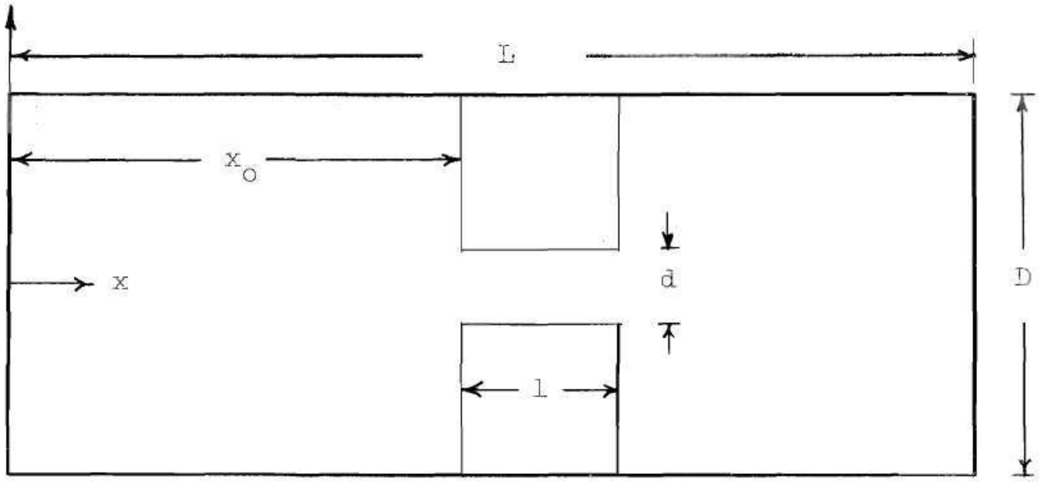

Figure 2. Model Piston Orifice Arrangement

x_o distance of piston from cylinder end wall

l piston length

L cylinder length

D cylinder height

d piston slot height

In order to derive equations describing the flow conditions within the cylinder, the velocity discontinuities existing along the solid surfaces will be replaced with vortex sheets whose filament axes are normal to the plane of the paper. The strength distribution and the slope of the strength distribution may be specified in order to meet the boundary conditions associated with the problem.

As noted below, the limitations imposed on this analysis by the available computing equipment require the replacement of the vortex sheets with a finite number of filaments. For this reason, the development which follows will be based on the velocity induced at a point by a vortex filament of finite strength[7]. The strength of certain of these filaments will be specified in order to meet certain boundary conditions of the system.

Using the Law of Biot and Savart we can compute the velocities that exist anywhere in the field due to one of these vortex filaments. One form of this law is[8]

$$V = \frac{\gamma}{2\pi R} \tag{6}$$

where V is the velocity induced in the field by the vortex filament, γ the strength of the filament, and R the distance from the filament to the point at which the velocity is induced (See Figure 3).

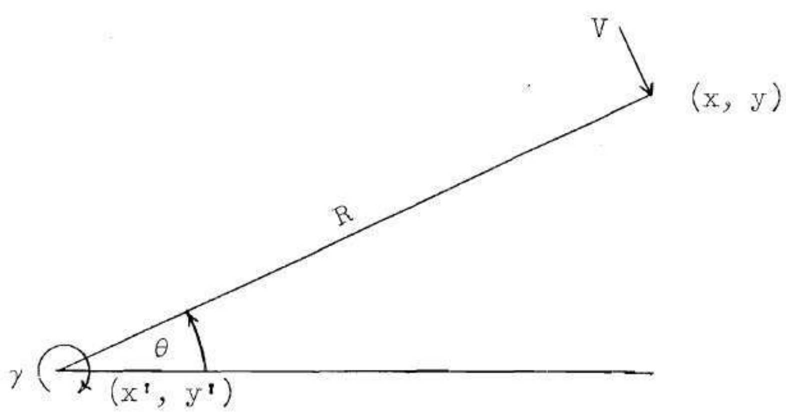

Figure 3. Law of Biot and Savart

7. The extension of the analysis to a vortex sheet is immediate. To accomplish this, the differential form of the Law of Biot and Savart is used. This relation is

$$dV = \frac{\gamma ds}{2\pi R}$$

where dV is the induced velocity at a point due to an increment of sheet width ds. The sheet strength is γ and R the distance from the incremental length to the point. The remainder of the development follows as in the main analysis, with an integration process replacing the summation.

8. Prandtl and Tietjens, <u>Fundamentals of Hydro- and Aeromechanics</u>, p. 206.

(x', y') are the coordinates of a vortex filament of strength γ, and (x, y), the coordinates of any point in the field, and $\overline{V}$, the velocity induced by γ at (x, y). Relations for u and v may be derived from the geometry of the situation.

$$u = V \sin\theta = \frac{\gamma \sin\theta}{2\pi \left[(x - x')^2 + (y - y')^2 \right]^{\frac{1}{2}}} \tag{7}$$

$$v = -V \cos\theta = -\frac{\gamma \cos\theta}{2\pi \left[(x - x')^2 + (y - y')^2 \right]^{\frac{1}{2}}}$$

Also from the geometry of Figure 3,

$$\cos\theta = \frac{(x - x')}{\left[(x - x')^2 + (y - y')^2 \right]^{\frac{1}{2}}}$$

$$\sin\theta = \frac{(y - y')}{\left[(x - x')^2 + (y - y')^2 \right]^{\frac{1}{2}}}$$

Substituting these relations into Equations (7):

$$u = \frac{1}{2\pi} \left\{ \frac{\gamma (y - y')}{(x - x')^2 + (y - y')^2} \right\} \tag{8}$$

$$v = -\frac{1}{2\pi} \left\{ \frac{\gamma (x - x')}{(x - x')^2 + (y - y')^2} \right\}$$

If there are n such filaments in a field, the induced velocity at any point in the field may be found by summing the contributions of the individual vortex filaments.

$$u = \frac{1}{2\pi} \sum_{1}^{n} \frac{\gamma_n (y - y'_n)}{(x - x'_n)^2 + (y - y'_n)^2} \tag{9}$$

$$v = -\frac{1}{2\pi} \sum_{1}^{n} \frac{\gamma_n (x - x'_n)}{(x - x'_n)^2 + (y - y'_n)^2}$$

Equations (9) are the basic relations which will be used to determine the flow pattern about the mathematical model described in Figure 2.

Before proceeding with the solution, Equations (9) will be non-dimensionalized. To accomplish this, the following non-dimensional quantities are defined:

$$\bar{u} = \frac{u}{U_\infty} \qquad \bar{v} = \frac{v}{V_\infty}$$

$$\bar{y} = \frac{y}{D} \qquad \bar{x} = \frac{x}{L}$$

$$\bar{y}'_n = \frac{y'_n}{D} \qquad \bar{x}'_n = \frac{x'_n}{L}$$

$$\bar{x}_o = \frac{x_o}{L} \qquad \bar{d} = \frac{d}{D} \qquad \bar{l} = \frac{l}{L}$$

where γ_∞ and U_∞ are at present defined only as a characteristic vortex strength and a characteristic velocity associated with the flow and used here for non-dimensional purposes.

Substituting these relations into Equation (9) and rearranging, the equations become

$$\bar{u} = \frac{\gamma_\infty}{2\pi D U_\infty} \sum_{1}^{n} \frac{\gamma_n (\bar{y} - \bar{y}'_n)}{\frac{L^2}{D^2} (\bar{x} - \bar{x}'_n)^2 + (\bar{y} - \bar{y}'_n)^2} \tag{10}$$

$$\bar{v} = - \frac{\gamma_\infty}{2\pi D U_\infty} \sum_{1}^{n} \frac{\gamma_n (\bar{x} - \bar{x}'_n)}{\frac{L^2}{D^2} (\bar{x} - \bar{x}'_n)^2 + (\bar{y} - \bar{y}'_n)^2}$$

and a graphic portrayal of the non-dimensional model is shown in Figure 4.

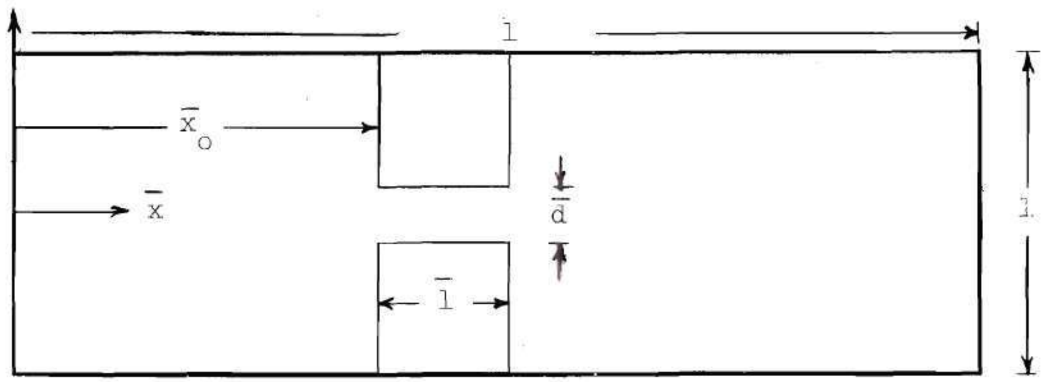

Figure 4. Non-Dimensional Piston Orifice Arrangement

From an examination of Equations (10) and Figure 4, it is apparent that the flow pattern existing in any closed cylinder for which this mathematical model is a fair representation is completely defined in terms of three geometrical parameters, $\bar{l}$, $\bar{d}$, and $\frac{L}{D}$, and one dynamic similarity parameter, $\frac{\gamma_\infty}{2\pi D U_\infty}$. In order to determine numerical results using these equations, it will be necessary to select values for the geometrical parameters involved. The values selected for use in this analysis will be discussed later in the paper.

Boundary Conditions

The boundary conditions to be imposed on Equations (10) in order to determine the flow pattern will be applied by specifying the strength of certain of the vortex filaments to be placed in the mathematical model. In general, the conditions imposed will be:

(1) No velocity component normal to any cylinder wall at the wall.

(2) The flow in the orifice must satisfy continuity.

(3) The fluid velocity component normal to the vertical piston surface must equal the piston velocity.

To satisfy these listed conditions, the distribution of the strength of the sheet along the vertical cylinder walls (Figure 5 - surfaces 1 and 7) must have zero strength and zero slope in the corners of the cylinder. These conditions are necessary to provide for no flow in the corner (point C) and to permit a smooth transition in the strength distribution from the vertical to the horizontal walls. Further, at the mid point (point A) of the vertical walls, the strength and slope of the distribution must be zero by reason of symmetry. This in effect makes this point a stagnation point in the flow.

Along the horizontal cylinder walls (surfaces 2, 6, 8, 12) at the cylinder ends, the strength and slope of the distribution must be zero. This follows directly from the argument of the preceding paragraph. At the piston (point B), this distribution should merge continuously into the constant sheet strength distribution between the piston and the wall (surfaces 13 and 14).

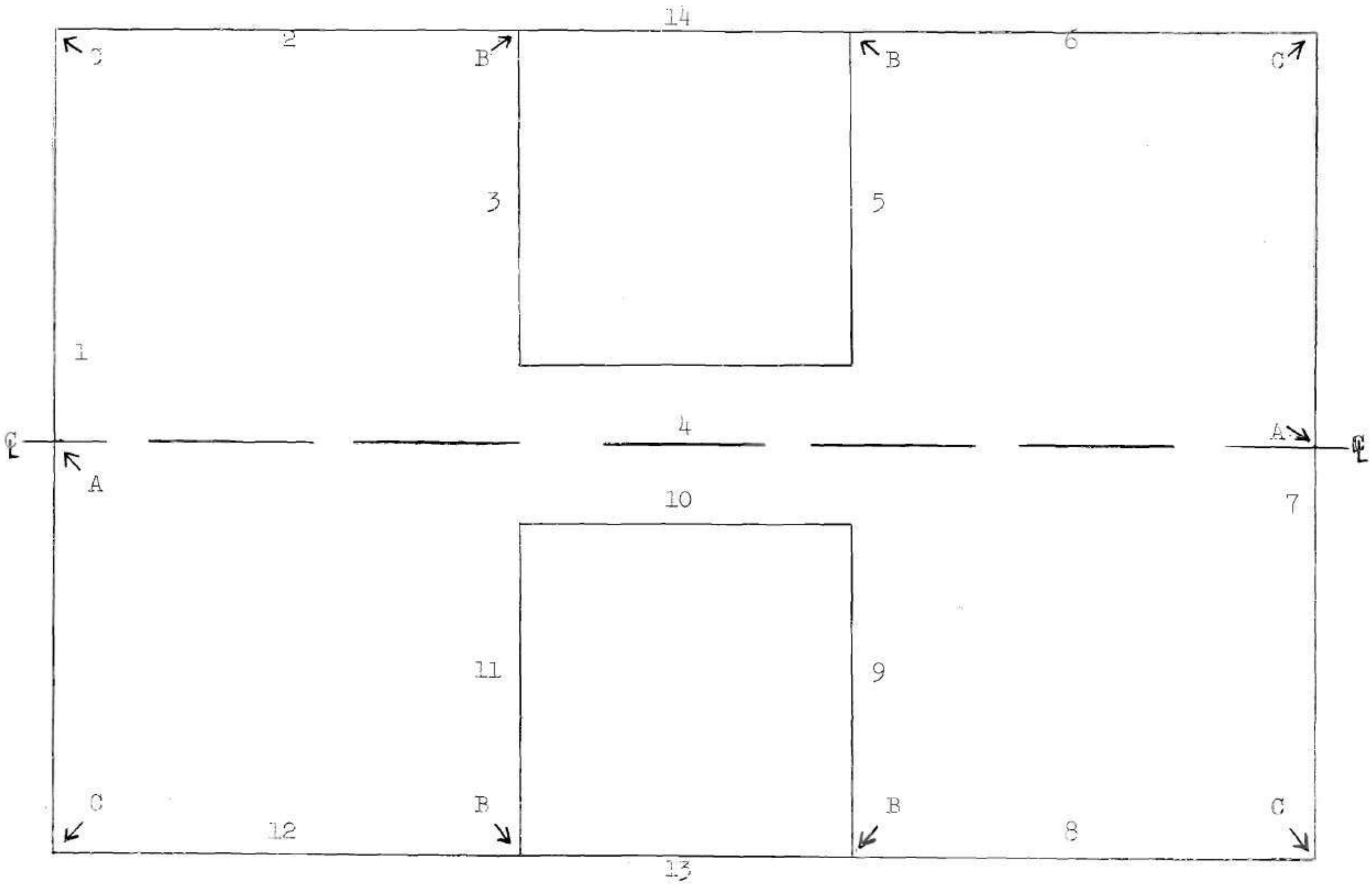

Figure 5. Piston and Cylinder Surfaces

Along the vertical piston faces (surfaces 3, 5, 9, 11), the strength and slope of the sheet must vanish at the wall (point B) since the fluid velocity in the piston-wall corner must be equal and parallel to the piston velocity.

Along the horizontal piston surfaces (surfaces 4, 10), the strength of the sheet must be a constant and equal in magnitude to the velocity discontinuity across the piston surface.

If the corners between surfaces 3 and 4, 4 and 5, 9 and 10, and 10 and 11 are assumed to be slightly rounded, then the vortex sheet strength on the piston surfaces 3, 5, 9, and 11 should merge continuously into the vortex sheet strength along surfaces 4 and 10. Making use of this requirement and the discussion of the previous paragraphs, the following relationships between the strengths of the finite vortex filaments which approximate the vortex sheet may be obtained.

$$\sum_{n=1}^{5} \frac{(-1)^n \ \gamma_n}{\Delta x_n} = 0$$

$$\sum_{n=6}^{15} \frac{(-1)^n \ \gamma_n}{\Delta x_n} = - \frac{L}{4\pi D}$$

$$\sum_{n=16}^{25} \frac{(-1)^n \ \gamma_n}{\Delta x_n} = + \frac{5L}{\pi D}$$

$$\sum_{n=30}^{39} \frac{(-1)^n \ \gamma_n}{\Delta x_n} = + \frac{5L}{\pi D}$$

$$\sum_{n=44}^{53} \frac{(-1)^n \ \gamma_n}{\Delta x_n} = - \frac{L}{4\pi D}$$

$$\sum_{n=54}^{58} \frac{(-1)^n \ \gamma_n}{\Delta x_n} = 0$$

A last requirement is that of Kelvin[9] who showed that the kinetic energy for the irrotational motion of a fluid occupying a simply-connected region was less than that of any other motion consistent with the same normal motion of the boundary. Thus the one solution of the many possible solutions is that one which makes the kinetic energy a minimum, and permits the use of Equation (4) for calculating purposes.

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9. Lamb, Hydrodynamics, p. 47

CHAPTER IV

PROBLEM SETUP

Due to computing equipment limitations, the solid surfaces of the model will be replaced with a finite number of vortex filaments. One hundred and sixteen filaments will be arranged in the cylinder. The exact locations of these filaments are shown in Table 1. They have been arbitrarily placed in their locations, with an effort being made to keep the actual numbers describing their coordinates as exact as possible. This is necessary as the available computing equipment is able to carry only eight significant figures, and if the coordinate position numbers were excessively long, considerable accuracy might be lost in solving the system of forty simultaneous equations which will be developed.

In Table 1, the strength of certain of these filaments has been designated. This is an outgrowth of the application of the boundary conditions previously noted.

As an example of determining the strength of certain filaments, consider the requirement that continuity must be satisfied within the orifice:

$$2\,(\bar{1})\,(1)\,\frac{(1-\bar{d})}{2}\,(u_{piston}) = (u_{orifice})\,(1)\,(\bar{d})\,(\bar{1})$$

$$u_{orifice} = \frac{(1-\bar{d})\,(u_{piston})}{\bar{d}}$$

Due to the flow being incompressible, the u component of velocity in the orifice will be everywhere constant, and equal to the value just derived. The strength of a vortex sheet must equal the magnitude of the velocity discontinuity which it represents, so the strength of the filaments placed in the orifice will be

$$\gamma_{orifice} = \frac{(1 - \bar{d})\ (u_{piston})}{\bar{d}} + u_{piston}$$

where the second term is required because all points within the piston boundary are moving with the velocity u_{piston}.

The filaments nearest the horizontal center line (Figure 5--point A) have been specified to have zero strength. This specifies no flow across the end wall at this point, as well as specifying no flow across the x axis, which is a requirement due to this axis being an axis of symmetry.

The filaments nearest the cylinder corners (point C) have been specified to be of zero strength in order to meet the condition of no flow in the corners. In order to meet the condition that the slope must vanish, these filaments have been spaced a finite distance from the corner.

Filaments in the space between the wall and the piston (surfaces 13, 14) have been assigned a strength equal to the piston velocity, representing the velocity discontinuity between the wall and piston.

The filaments immediately fore and aft of the piston along the cylinder walls (surfaces 2, 6, 8, 12--near point B) have also been assigned a strength equal to the piston velocity, accounting for the fact that the fluid immediately adjacent to the piston must have a velocity magnitude and direction equal to that of the piston. Also, the filament along the vertical piston surfaces nearest the cylinder wall (surfaces 3, 5, 9, 11--near point B) is fixed at zero strength, indicating a zero fluid velocity relative to the piston. As may be noted from examining Table 1, this leaves eighty filaments of undetermined strength. Due to the symmetry of the problem, only forty magnitudes must be determined, since, as shown in Table 1,

$$\gamma_1 = -\gamma_{59}$$
$$\gamma_2 = -\gamma_{60}$$
$$\vdots$$
$$\gamma_{58} = -\gamma_{116}$$

It should also be noted that this specification has the effect of making the circulation around the exterior of the model equal to zero, as one would naturally expect from the physical considerations involved.

In order to determine the strengths of the unknown γ_n's, the piston will be "frozen" in position instantaneously and the existing flow pattern studied for varying distances of the piston from the wall. At each piston position, or x_o, selected, forty computing points will be chosen on the various surfaces of the model and the equation for the velocity component normal to the surface at that point written[10]. This

10. Computing points may not coincide with filament locations, as this would yield infinite velocities. Table 2 contains the coordinates of the computing points.

will yield a system of forty equations in forty unknowns which may be solved simultaneously to obtain the unknown filament strengths.

As noted in Chapter III, there are still seven conditions remaining that must be satisfied by the flow. These conditions are the relations regarding the strengths of the unknown filaments, and the condition that the kinetic energy of the flow must be a minimum. Once the forty simultaneous equations that have been written at the various computing points have been determined to be independent, these additional relations may be taken into account by arbitrarily removing certain of the computing point equations.

The form of the equations involving the unknown filament strengths has been indicated in Chapter III, and the relation for the total circulation about one-half of the symmetrical vortex system may be written

$$\sum_{1}^{58} \gamma_n = \Gamma$$

where Γ is some arbitrary constant. Since the value of Γ for which the kinetic energy of the flow is a minimum is unknown, it appears necessary to make a trial and error solution to determine this value.

The trial and error solution to determine the value of Γ for which the kinetic energy of the flow is a minimum may be accomplished by assuming a value of Γ, solving the resulting equations, and then using a numerical process to determine the kinetic energy of the flow associated with that particular value of Γ. After performing this task for various values of Γ, a plot of Γ versus the kinetic energy may be made and the value of Γ that minimizes the kinetic energy may be selected.

This value may then be placed in the equations describing the system, and the unknown γ's describing the physical flow may be determined.

With these values of γ_n, the flow for the particular piston position is completely determined, within the limitations that are imposed by the derivation of Equations (10) and the fact that, due to the use of a finite number of vortex filaments, the equations describing the flow can be made to satisfy the boundary conditions at only a finite number of points, rather than at all points in the flow[11].

Once the equations of the flow are obtained, the velocity potential of the flow may be obtained by simple integration around the surface of the piston:

$$\phi = \int_c (udx + vdy)$$

This integration yields the velocity potential as the product of some constant times the instantaneous piston velocity $\bar{u}_{piston}$.

Having the velocity potential now determined, use may be made of Equation (4).

$$K.E. = -\frac{\rho}{2} \iint_s \phi \cdot \frac{\partial \phi}{\partial n} ds \qquad (4)$$

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11. There appears to be a valid criticism of this last mentioned limitation, and it is certainly a point which should be kept foremost in the mind when interpreting this paper. In defense of the limitation, it can be pointed out that any errors introduced by it will quite probably be of the same magnitude for each piston position, since the various ratios used in the computations remain the same for each piston position. If each piston position has the same error, the effects of the error should be self cancelling insofar as determining a variation of apparent mass with varying piston position.

This equation can be integrated over the surface of the piston, and thus the kinetic energy of the flow is determined. For this specific problem, the instantaneous piston velocity is treated as a constant. $\frac{\partial \phi}{\partial n}$ is the piston velocity, u_{piston}, along the vertical piston surfaces, and is zero elsewhere.

As expected, when this expression is set equal to $\frac{1}{2} mu^2$, the instantaneous kinetic energy of the piston, the velocities divide out, and what remains is a number having the units of mass. This is exactly as predicted by Equation (5). This process is repeated for each piston position selected, and a plot of apparent mass versus piston position may be made from the data thus obtained.

CHAPTER V

CONCLUDING REMARKS

Magnitude Analysis

In order to demonstrate that the magnitude of the apparent mass is an appreciable quantity, it is possible to compute the kinetic energy of the fluid in the orifice. As noted previously, the velocity in the orifice may be expressed as follows:

$$u_{orifice} = \frac{(1 - \bar{d})}{\bar{d}} u_{piston}$$

The contribution of this velocity to the velocity potential may be found and used in Equation (4). Upon carrying out the indicated integration, it is seen that the ratio $\frac{1 - \bar{d}}{\bar{d}}$ appears in the final expression for the kinetic energy of the fluid. Neglecting the contribution of the vertical piston faces to the kinetic energy, which can only add to the total kinetic energy, it is clearly seen that, for small orifice openings, the apparent mass effects are considerable. For arrangements where the fluid density and the piston density are of the same order of magnitude, the apparent mass of the piston can be many times its actual mass. These conclusions are borne out by Lamb. In solving the problem of determining the motion produced in a fluid contained between a moving solid sphere and a fixed concentric spherical boundary, he points out that the introduction of the finite external boundary "acts as a constraint increasing the kinetic energy" of the fluid[12].

Physical Insight

In the piston and cylinder arrangement analyzed in this study, the streamlines of the flow in the top half of the cylinder would be expected to have the general shape shown in Figure 6.

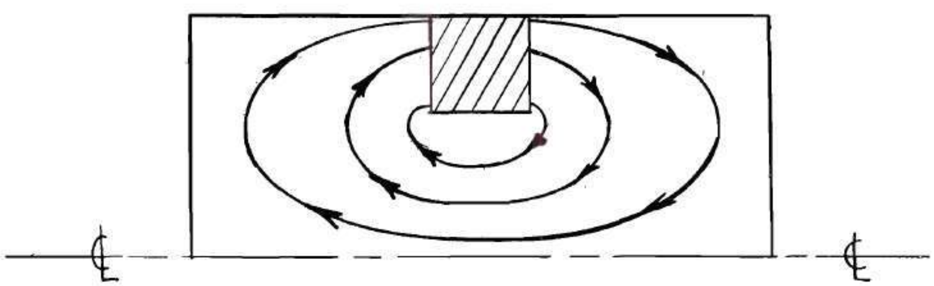

Figure 6. Streamline Shapes

From a study of these streamline shapes, it can be seen that the slope of the streamlines as they intersect the piston is indicative of the relative magnitude and sign of the filament located on the wall at the point of intersection. Intuitively, then, it would appear that all the filaments along the vertical piston faces should have the same sign and should increase in magnitude as the distance from the horizontal wall increases. Were the slopes of the streamlines to be of varying sign in this region, a rotational flow would be indicated. Any solution to the problem which displayed a condition of this type would not be the desired solution, since it would not be the desired minimum energy solution. By way of analogy, this problem might be compared to the classic thin airfoil problem, where a possible solution exists for each value of circu-

12. Lamb, Hydrodynamics, p. 125

lation about the airfoil. To obtain the physical solution to the problem, it is necessary to fix the value of the circulation at the trailing edge at zero, and to prescribe a value of the circulation about the entire airfoil that insures a stagnation point at the trailing edge.

Computations

In order to establish that the outlined procedure is reasonable, some calculations have been performed on the Burroughs 220 Electronic Data Processing System.

Initially, the equations describing the flow and satisfying the boundary conditions as outlined in Table 2 of the Appendix were programmed for solution. After some initial programming errors were corrected, it was found that the forty equations as posed were independent and could be made to yield a solution. From an analysis of the program and accuracy checks described in the Appendix, it was determined that the solutions obtained were accurate to three significant figures. As the machine is capable of carrying only eight significant figures, it is felt that the loss of accuracy in the solution must be attributed to the round-off error accumulated in the machine during the inversion and multiplication routines. Further, it is felt that this order of accuracy is within engineering tolerance, and that further computations are justified.

Having determined that the equations were independent, the six equations developed in Chapter III relating the unknown filaments were used to replace six of the computing-point equations. When it was attempted to solve this set of equations, it was found that the matrix of the coefficients could not be inverted. After careful examination, it appeared that the equations should be independent, and the failure to invert was attributed to the large number of zeroes which appeared in the matrix when the six equations relating the unknowns were used. Although this presents no theoretical difficulty to matrix inversion, it does present some practical difficulties in the numerical process used by the machine to invert the matrix of coefficients.

To overcome the problem of the large number of zeroes in the matrix, the six equations relating the unknowns were added to form one equation, and five of the original computing-point equations were restored, thus maintaining a system of forty equations in forty unknowns. These equations were then solved and it was noted that while the equation obtained by adding the six equations relating the unknowns was satisfied to the same degree of accuracy of three significant figures, the individual equations of which it was composed could not be satisfied with the values determined for the unknowns. Further, the predicted filament strengths along the piston faces were of varying sign. These two facts would seem to indicate that the previous analysis is correct and that the equations must be examined for varying values of the sum of the unknowns in order to determine the one for which the kinetic energy of the system is a minimum and thus determine the potential flow solution.

CHAPTER VI

RECOMMENDATIONS

1. As computer facilities become available, it is felt that further calculations to establish the value of Γ for which the kinetic energy of the flow is a minimum are justified and should be made. This will require considerable machine time, since with the computing equipment now available, the time necessary to determine the unknowns for just one value of Γ is thirty-three minutes. Add to this the task of then computing the kinetic energy of the flow for that value of Γ, and it is readily seen that the project is time consuming.

2. As a possible alternative to the approach used in this study, a method of attack which appears to offer promise in the solution of this problem is to place a vortex sheet of unknown strength along the various surfaces of the cylinder and then assume a Fourier Series distribution of the sheet strength. Using this approach, it would not be necessary to replace the vortex sheet with finite vortex filaments. Instead, it would be possible to solve for the unknown coefficients of the Fourier Series. Depending on the number of terms chosen in the series distribution, this would permit the boundary conditions imposed on the flow to be met more exactly.

APPENDIX

PROCEDURE FOR NUMERICAL CALCULATIONS

In order to compute numerical examples using the derived relations, it is necessary to select numbers for the geometric parameters involved in Equations (10). Although once these selections are made the computations are straightforward, they are cumbersome and time consuming, even with the aid of high-speed computing equipment. For this reason it is inconvenient to calculate numbers for large ranges of the various geometric parameters. For the numerical example to be computed here, the values of the geometric parameters are taken from a device in which the effects discussed in the problem outline have been noted. These values are:

$$\frac{L}{D} = 2.5$$

$$\overline{d} = .05$$

$$\overline{l} = .20$$

Using these values, the flow conditions may be completely determined in terms of the instantaneous piston velocity, u_{piston}, and the dynamic similarity parameter, $\frac{\gamma_\infty}{2\pi D U_\infty}$.

Once these values of the geometric parameters were selected, the equations were programmed for solution with the Burroughs 220 Electronic Data Processing System. This programming consisted of the three basic steps outlined here.

Step 1. Compute the coefficients associated with each unknown.

Step 2. Invert the matrix of the coefficients.

Step 3. Multiply the inverse obtained by a column vector in order to obtain the unknowns.

In addition to these steps, additions were made to the program to multiply the matrix of coefficients by its inverse. This step should yield a unit matrix and serves as a check on the accuracy of the inversion. A further check made in the program was to multiply the matrix of coefficients by the solution vector. This should yield the original column vector if the operations have been performed accurately.

Table 1. Filament Coordinates

$\bar{\ }_n$	$\bar{x}'$	$\bar{y}'$	$\bar{\ }_n$ value
$\bar{\gamma}_1$ (59)	0	(-) + .05	0
$\bar{\gamma}_2$ (60)	0	(-) + .10	
$\bar{\gamma}_3$ (61)	0	(-) + .20	
$\bar{\gamma}_4$ (62)	0	(-) + .30	
$\bar{\gamma}_5$ (63)	0	(-) + .40	0
$\bar{\gamma}_6$ (64)	$.05\ \bar{x}_o$	(-) + .50	0
$\bar{\gamma}_7$ (65)	$.15\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_8$ (66)	$.25\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_9$ (67)	$.35\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_{10}$ (68)	$.45\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_{11}$ (69)	$.55\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_{12}$ (70)	$.65\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_{13}$ (71)	$.75\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_{14}$ (72)	$.85\ \bar{x}_o$	(-) + .50	
$\bar{\gamma}_{15}$ (73)	$.95\ \bar{x}_o$	(-) + .50	$\frac{1}{10x_o}\ \frac{L}{2\pi D}\ u_{piston}$
$\bar{\gamma}_{16}$ (74)	$\bar{x}_o$	(-) + .485	0
$\bar{\gamma}_{17}$ (75)	$\bar{x}_o$	(-) + .435	

The subscript in parentheses denotes the corresponding anti-symmetric filament in the lower half of the model. The sign of its $\bar{y}'$ coordinate and the sign of the $\bar{\gamma}_n$ are noted in parentheses. The $\bar{x}'$ coordinate and $\bar{\gamma}$ value are the same.

Table 1 (continued)

$\bar{\ }_n$	$\bar{x}'$	$\bar{y}'$	$\bar{\ }_n$ value
$\bar{\gamma}_{18}$ (76)	$\bar{x}_o$	(-) + .385	
$\bar{\gamma}_{19}$ (77)	$\bar{x}_o$	(-) + .335	
$\bar{\gamma}_{20}$ (78)	$\bar{x}_o$	(-) + .285	
$\bar{\gamma}_{21}$ (79)	$\bar{x}_o$	(-) + .235	
$\bar{\gamma}_{22}$ (80)	$\bar{x}_o$	(-) + .185	
$\bar{\gamma}_{23}$ (81)	$\bar{x}_o$	(-) + .135	
$\bar{\gamma}_{24}$ (82)	$\bar{x}_o$	(-) + .085	
$\bar{\gamma}_{25}$ (83)	$\bar{x}_o$	(-) + .035	
$\bar{\gamma}_{26}$ (84)	$\bar{x}_o$ + .04	(-) + .025	$\frac{L}{2\pi D} u_{piston}$
$\bar{\gamma}_{27}$ (85)	$\bar{x}_o$ + .08	(-) + .025	$\frac{L}{2\pi D} u_{piston}$
$\bar{\gamma}_{28}$ (86)	$\bar{x}_o$ + .12	(-) + .025	$\frac{L}{2\pi D} u_{piston}$
$\bar{\gamma}_{29}$ (87)	$\bar{x}_o$ + .16	(-) + .025	$\frac{L}{2\pi D} u_{piston}$
$\bar{\gamma}_{30}$ (88)	$\bar{x}_o$ + .20	(-) + .035	
$\bar{\gamma}_{31}$ (89)	$\bar{x}_o$ + .20	(-) + .085	
$\bar{\gamma}_{32}$ (90)	$\bar{x}_o$ + .20	(-) + .135	
$\bar{\gamma}_{33}$ (91)	$\bar{x}_o$ + .20	(-) + .185	
$\bar{\gamma}_{34}$ (92)	$\bar{x}_o$ + .20	(-) + .235	
$\bar{\gamma}_{35}$ (93)	$\bar{x}_o$ + .20	(-) + .285	
$\bar{\gamma}_{36}$ (94)	$\bar{x}_o$ + .20	(-) + .335	
$\bar{\gamma}_{37}$ (95)	$\bar{x}_o$ + .20	(-) + .385	
$\bar{\gamma}_{38}$ (96)	$\bar{x}_o$ + .20	(-) + .435	
$\bar{\gamma}_{39}$ (97)	$\bar{x}_o$ + .20	(-) + .485	0

Table 1 (continued)

$\bar{\ }_n$		$\bar{x}'$	$\bar{y}'$	$\bar{\ }_n$ value
$\bar{\gamma}_{40}$	(98)	$\bar{x}_o + .04$	$(-) + .50$	$\frac{.04L}{2\pi D} u_{piston}$
$\bar{\gamma}_{41}$	(99)	$\bar{x}_o + .08$	$(-) + .50$	$\frac{.06L}{2\pi D} u_{piston}$
$\bar{\gamma}_{42}$	(100)	$\bar{x}_o + .12$	$(-) + .50$	$\frac{.06L}{2\pi D} u_{piston}$
$\bar{\gamma}_{43}$	(101)	$\bar{x}_o + .16$	$(-) + .50$	$\frac{.04L}{2\pi D} u_{piston}$
$\bar{\gamma}_{44}$	(102)	$\bar{x}_o + .2 + .05(.8 - \bar{x}_o)$	$(-) + .50$	$\frac{1}{10x_o} \frac{L}{2\pi D} u_{piston}$
$\bar{\gamma}_{45}$	(103)	$\bar{x}_o + .2 + .15(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{46}$	(104)	$\bar{x}_o + .2 + .25(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{47}$	(105)	$\bar{x}_o + .2 + .35(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{48}$	(106)	$\bar{x}_o + .2 + .45(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{49}$	(107)	$\bar{x}_o + .2 + .55(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{50}$	(108)	$\bar{x}_o + .2 + .65(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{51}$	(109)	$\bar{x}_o + .2 + .75(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{52}$	(110)	$\bar{x}_o + .2 + .85(.8 - \bar{x}_o)$	$(-) + .50$	
$\bar{\gamma}_{53}$	(111)	$\bar{x}_o + .2 + .95(.8 - \bar{x}_o)$	$(-) + .50$	0
$\bar{\gamma}_{54}$	(112)	1	$(-) + .40$	0
$\bar{\gamma}_{55}$	(113)	1	$(-) + .30$	
$\bar{\gamma}_{56}$	(114)	1	$(-) + .20$	
$\bar{\gamma}_{57}$	(115)	1	$(-) + .10$	
$\bar{\gamma}_{58}$	(116)	1	$(-) + .05$	0

Table 2. Computing Point Coordinates

Point	$\bar{x}$	$\bar{y}$	Action
1.	0	.025	Set $u = 0$
2.	0	.075	Set $u = 0$
3.	0	.15	Set $u = 0$
4.	0	.25	Set $u = 0$
5.	0	.35	Set $u = 0$
6.	$.10\ \bar{x}_o$	.50	Set $v = 0$
7.	$.30\ \bar{x}_o$	.50	Set $v = 0$
8.	$.50\ \bar{x}_o$	.50	Set $v = 0$
9.	$.70\ \bar{x}_o$	.50	Set $v = 0$
10.	$.90\ \bar{x}_o$	.50	Set $v = 0$
11.	$\bar{x}_o$	.460	Set $u = u_{piston}$
12.	$\bar{x}_o$	.360	Set $u = u_{piston}$
13.	$\bar{x}_o$	.260	Set $u = u_{piston}$
14.	$\bar{x}_o$	.160	Set $u = u_{piston}$
15.	$\bar{x}_o$	.060	Set $u = u_{piston}$
16.	$\bar{x}_o + .02$	.025	Set $v = 0$
17.	$\bar{x}_o + .06$	.025	Set $v = 0$
18.	$\bar{x}_o + .10$	.025	Set $v = 0$
19.	$\bar{x}_o + .14$	.025	Set $v = 0$
20.	$\bar{x}_o + .18$	.025	Set $v = 0$

Table 2. (continued)

Point	$\bar{x}$	$\bar{y}$	Action
21.	$\bar{x}_o + .20$	.060	Set $u = u_{piston}$
22.	$\bar{x}_o + .20$	.160	Set $u = u_{piston}$
23.	$\bar{x}_o + .20$	.260	Set $u = u_{piston}$
24.	$\bar{x}_o + .20$	.360	Set $u = u_{piston}$
25.	$\bar{x}_o + .20$	.460	Set $u = u_{piston}$
26.	$\bar{x}_o + .02$	.5	Set $v = 0$
27.	$\bar{x}_o + .06$	.5	Set $v = 0$
28.	$\bar{x}_o + .10$	.5	Set $v = 0$
29.	$\bar{x}_o + .14$	.5	Set $v = 0$
30.	$\bar{x}_o + .18$	.5	Set $v = 0$
31.	$\bar{x}_o + .2 + .10(.8 - \bar{x}_o)$	.5	Set $v = 0$
32.	$\bar{x}_o + .2 + .30(.8 - \bar{x}_o)$	.5	Set $v = 0$
33.	$\bar{x}_o + .2 + .50(.8 - \bar{x}_o)$	.5	Set $v = 0$
34.	$\bar{x}_o + .2 + .70(.8 - \bar{x}_o)$	.5	Set $v = 0$
35.	$\bar{x}_o + .2 + .90(.8 - \bar{x}_o)$	.5	Set $v = 0$
36.	1	.35	Set $u = 0$
37.	1	.25	Set $u = 0$
38.	1	.15	Set $u = 0$
39.	1	.075	Set $u = 0$
40.	1	.025	Set $u = 0$

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