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A HYPERCIRCLE METHOD FOR BOUNDING THE INFLUENCE COEFFICIENTS OF CIRCULAR CYLINDRICAL SHELLS

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A HYPERCIRCLE METHOD FOR BOUNDING THE

INFLUENCE COEFFICIENTS OF CIRCULAR CYLINDRICAL SHELLS

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

This study presents a method for determining upper and lower bounds for the influence coefficients of a thin circular cylindrical shell of variable wall thickness. The shell is considered to be stressed by a bending moment M_o and a shearing force H_o applied along its lower edge. The upper edge of the shell is not subjected to any external forces. Under the loading described, the lower edge of the shell undergoes a radial displacement u_o and a rotation β_o. The relations expressing u_o and β_o in terms of M_o and H_o are assumed to be linear and have the following form.

$$u_o = - C_{uH}H_o + C_{uM}M_o$$

$$\beta_o = C_{\beta H}H_o - C_{\beta M}M_o$$

The coefficients C_{uH}, C_{uM}, $C_{\beta H}$, $C_{\beta M}$ are called the influence coefficients of the shell under consideration. An exact determination of the influence coefficients for shells of variable wall thickness is in general impossible because of the difficulty of integrating the relevant differential equations; so methods such as the one described in this study must be used to find suitable approximate values.

The method of the hypercircle is based upon the concepts of function space. A function space is introduced in which the "points" are quadruples of functions representing the stress resultants and couples in the wall of the shell. The expression for the strain energy provides a metric in the function space, in which various planes and spheres are

defined by using the differential equations which govern the behavior of the shell. Ultimately it is shown that the actual solution to the differential equations is represented by a point which lies on a certain hypercircle in the function space. Upper and lower bounds on the strain energy associated with <u>any</u> point on the hypercircle are derived; and thus, in particular, upper and lower bounds on the strain energy associated with the actual solution are obtained.

Bounds on the influence coefficients are then deduced from the upper and lower bounds on the strain energy of the shell by using elementary results from the theory of quadratic forms, and a numerical example is presented which illustrates the calculations which must be performed in order to obtain numerical values for the various bounds. In the numerical example a shell is considered in which the wall thickness decreases exponentially with distance from the loaded edge:

$$h = h_r e^{-\rho\xi} .$$

Bounds obtained by using the method of the hypercircle are compared to results previously obtained by Reissner and Sledd by use of the variational principles of elasticity. In the particular example considered, the method of the hypercircle yields slightly better results than the variational method with approximately the same effort. The results obtained are presented in tabular and graphic form as functions of the parameter ρ, the thickness h_r of the shell wall at the loaded edge, and the radius $\underline{a}$ of the undeformed middle surface.

Chapter I summarizes the basic equations needed from the theory of elasticity and uses them to calculate the influence coefficients for shells of constant wall thickness. These constant-thickness coefficients subsequently serve as reference values to which the bounds for the influence coefficients of shells of non-constant wall thickness may be compared in order to obtain a quantitative indication of the effect of the thickness variation. In Chapter II the method of the hypercircle is presented in terms which facilitate an estimate of the amount by which the strain energy associated with an approximate solution differs from the strain energy of the actual solution. Chapter III applies the method of the hypercircle to the actual problem of obtaining bounds for the influence coefficients of thin circular cylindrical shells. In Chapter IV the numerical results obtained in this study are compared to the results previously obtained by Reissner and Sledd [2, 4]. Since successful use of the method of the hypercircle depends strongly on the selection of suitable functions to define the various approximate solutions needed, some comment on this point is included in Chapter III.

CHAPTER I

INTRODUCTION

In 1947 Prager and Synge [1] discussed the approximate solution of certain problems in the theory of elasticity by use of the method of the hypercircle, which is essentially a procedure for establishing upper and lower bounds of pertinent quadratic functionals. In 1957 Reissner and Sledd [2] computed upper and lower bounds for the influence coefficients of semi-infinite circular cylindrical shells of non-uniform wall thickness by using the variational principles of elasticity. The present study is an application of the method of the hypercircle to solve the problem considered by Reissner and Sledd. The work is also extended to shells of finite length. An interesting feature of the work is that bounds for all three of the influence coefficients are obtained simultaneously.

Definitions and Basic Equations

The cylindrical shell dealt with in this study is considered stressed by a radially directed shearing force H_o and a bending moment M_o applied uniformly to the circumference of the bottom edge of the shell and acting about a circumferential tangent to the lower edge of the middle surface (see Figure 1). The upper edge of the shell is not subjected to any external forces.

The governing equations for a shell with the geometry shown in Figure 1 were derived by Reissner [3] and are as follows (the prime

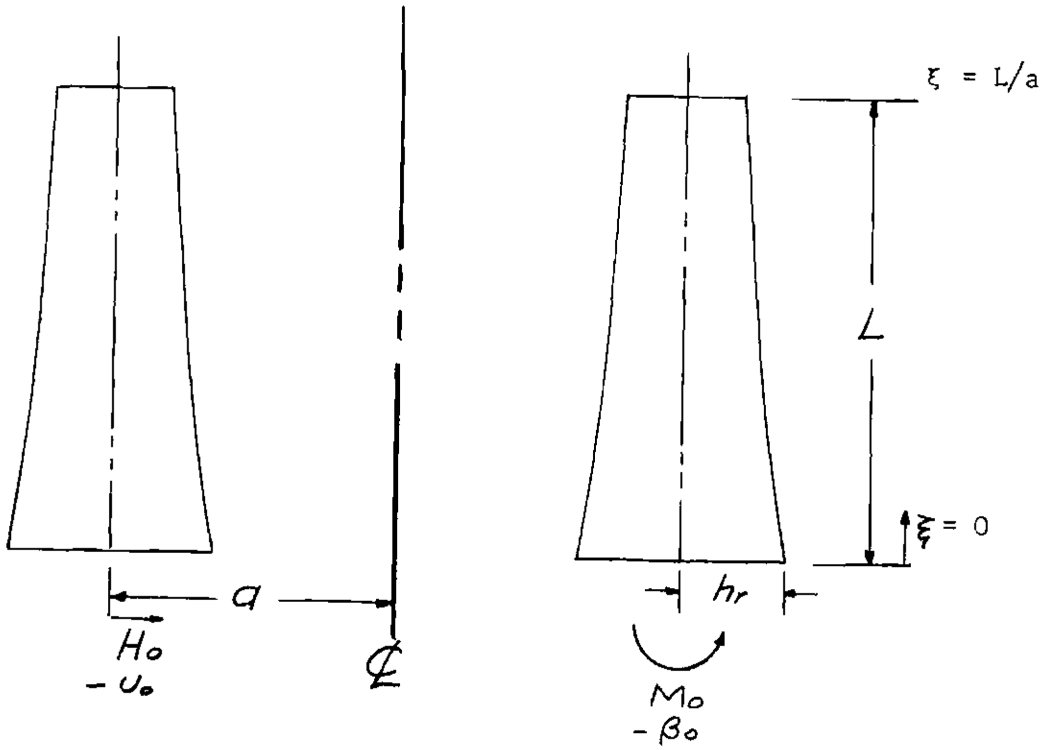


Figure 1. Geometry and Sign Conventions.

denotes differentiation with respect to ξ, the normalized coordinate in the axial direction).

1. The Equilibrium Equations

$$
\begin{aligned}
aH + M_\xi' &= 0 \\
N_\theta - H' &= 0 \\
N_\xi &= 0
\end{aligned}
\tag{1}
$$

2. The Boundary Conditions

$$
\begin{aligned}
M_\xi(0) &= M_o \\
H(0) &= H_o \\
M_\xi(L/a) &= 0 \\
H(L/a) &= 0
\end{aligned}
\tag{2}
$$

3. The Compatibility Equation

$$
u' = a\beta \tag{3}
$$

4. The Strain-Displacement Relations

$$
\begin{aligned}
\varepsilon_\theta &= u/a \\
\varepsilon_\xi &= -\nu u/a \\
K_\xi &= \beta'/a \\
K_\theta &= 0
\end{aligned}
\tag{4}
$$

5. The Stress-Strain Relations

$$
\begin{aligned}
C\varepsilon_\theta &= N_\theta - \nu N_\xi \\
C\varepsilon_\xi &= N_\xi - \nu N_\theta \\
DK_\xi &= (M_\xi - \nu M_\theta)/(1 - \nu^2) \\
DK_\theta &= (M_\theta - \nu M_\xi)/(1 - \nu^2)
\end{aligned}
\tag{5}
$$

Here

$$
\begin{aligned}
C &= Eh, \\
D &= Eh^3/12(1 - \nu^2) .
\end{aligned}
\tag{6}
$$

E is Young's modulus of the material and ν is Poisson's ratio; θ is the azimuthal angular coordinate; N_ξ is the meridional stress resultant in the shell wall (see Figure 2); N_θ is the hoop stress resultant; ε_θ is the hoop strain; K_ξ is the change in meridional curvature; and β is the decrease (under load) in the angle of inclination of the meridional tangent to the middle surface of the shell.

These equations can be solved exactly in the case h = constant; and the solution, which will be given in the next section of this chapter, can be used to determine exact values of the influence coefficients. These constant-thickness coefficients later serve as reference values with which to compare the results for shells of non-constant wall thickness. In the case of the thickness variation

$$h = h_r e^{-\rho \xi} ,$$

which will also be studied, the equations become too difficult to integrate;

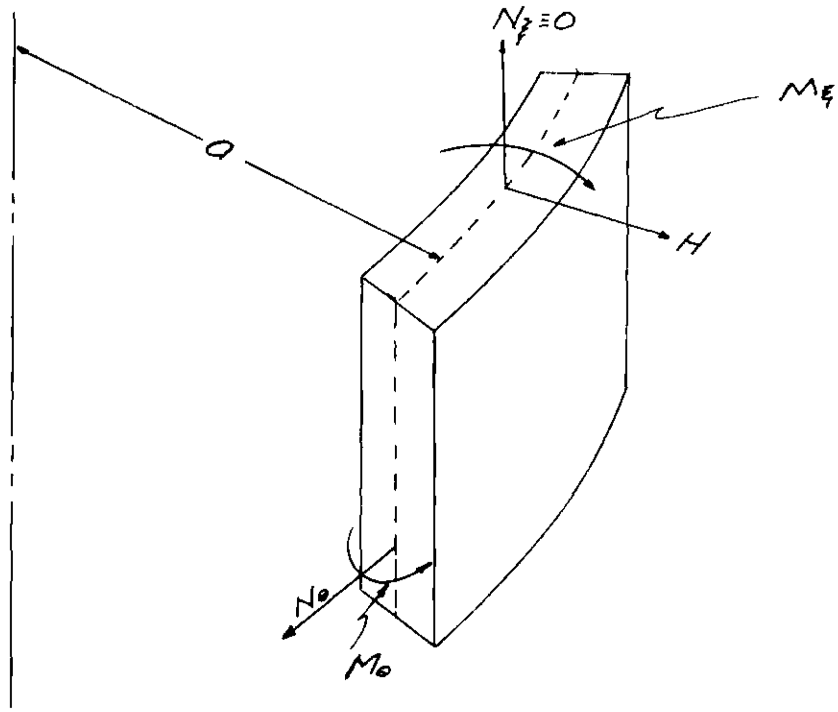


Figure 2. Stress Resultants and Couples Acting upon an Element of the Cylinder.

so an approximate method is introduced which yields upper and lower bounds on the influence coefficients.

Shells of Constant Thickness

If the thickness of the shell is constant, equations (1) - (5) can be summarized by the equation (Sledd, [4])

$$\beta^{(IV)} + 4\,\lambda^4 \beta = 0\ , \tag{7}$$

where

$$\lambda^2 = \sqrt{3(1 - \nu^2)}\ \frac{a}{h_r}\ . \tag{8}$$

The general solution of equation (7) is

$$\begin{aligned} \beta = {} & e^{-\lambda\xi}(c_1 \cos \lambda\xi + c_2 \sin \lambda\xi) \\ & + e^{\lambda\xi}(c_3 \cos \lambda\xi + c_4 \sin \lambda\xi). \end{aligned} \tag{9}$$

From equations (3) and (9) it follows that

$$u = \frac{a}{2\lambda}\left\{\begin{aligned} & e^{-\lambda\xi}[-(c_1 + c_2) \cos \lambda\xi + (c_1 - c_2) \sin \lambda\xi] \\ & + e^{\lambda\xi}[(c_3 - c_4) \cos \lambda\xi + (c_3 + c_4) \sin \lambda\xi] \end{aligned}\right\}. \tag{10}$$

Equations (9), (4), and (5) yield

$$M_\xi = \frac{\lambda D_r}{a}\left\{\begin{aligned} & e^{-\lambda\xi}[(c_2 - c_1) \cos \lambda\xi - (c_1 + c_2) \sin \lambda\xi] \\ & + e^{\lambda\xi}[(c_3 + c_4) \cos \lambda\xi + (c_4 - c_3) \sin \lambda\xi] \end{aligned}\right\}. \tag{11}$$

Equations (11) and (1) then give

$$H = \frac{2\lambda^2 D_r}{a^2} \left\{ \begin{aligned} & e^{-\lambda\xi}(c_2 \cos \lambda\xi - c_1 \sin \lambda\xi) \\ & + e^{\lambda\xi}(-c_4 \cos \lambda\xi + c_3 \sin \lambda\xi) \end{aligned} \right\} . \tag{12}$$

The coefficients c_1, c_2, c_3, and c_4 are to be determined from the boundary conditions, equations (2). In determining the values of these constants, it is convenient to consider two cases, $L/a = \infty$ and $L/a < \infty$.

<u>Case I: $L/a = \infty$</u>

Equations (2) yield $c_3 = c_4 = 0$ and

$$\frac{\lambda D_r}{a}(c_2 - c_1) = M_o , \tag{13}$$

$$\frac{2\lambda^2 D_r}{a^2} c_2 = H_o . \tag{14}$$

Hence

$$c_2 = \frac{a^2 H_o}{2\lambda^2 D_r} \tag{15}$$

and

$$c_1 = -\frac{aM_o}{\lambda D_r} + \frac{a^2 H_o}{2\lambda^2 D_r} . \tag{16}$$

These values for c_1 and c_2 yield

$$\beta = e^{-\lambda\xi}[M_o(-\frac{a}{\lambda D_r}) \cos \lambda\xi \tag{17}$$

$$+ H_o(\frac{a^2}{2D_r\lambda^2})(\sin \lambda\xi + \cos \lambda\xi)] ;$$

$$u = e^{-\lambda\xi}[M_o(\frac{a^2}{2D_r\lambda^2})(\cos\lambda\xi - \sin\lambda\xi) + H_o(\frac{-a^3}{2D_r\lambda^3})\cos\lambda\xi] \; . \quad (18)$$

When $\xi = 0$, equations (17) and (18) become

$$\beta_o = (-a/\lambda D_r)M_0 + (a^2/2D_r\lambda^2)H_0 \; , \quad (19)$$

$$u_o = (a^2/2D_r\lambda^2)M_o + (-a^3/2\lambda^3 D_r)H_o \; , \quad (20)$$

or, in terms of the influence coefficients,

$$\beta_o = -C_{\beta M}M_o + C_{\beta H}H_o \; , \quad (21)$$

$$u_o = C_{uM}M_o - C_{uH}H_o \; , \quad (22)$$

where

$$C_{\beta M} = a/\lambda D_r \; , \quad (23)$$

$$C_{uH} = a^3/2D_r\lambda^3 \; , \quad (24)$$

$$C_{\beta H} = C_{uM} = a^2/2D_r\lambda^2 \; . \quad (25)$$

Equations (21) and (22) define the influence coefficients C_{uH}, $C_{\beta M}$, and C_{uM} for <u>any</u> shell, whether it is cylindrical or not and whether it is of constant thickness or not. In the event that the shell is cylindrical, of constant thickness, and of infinite length, the values of the influence coefficients are given by equations (23), (24), and (25).

<u>Case II: $L/a < \infty$</u>

In the case of a shell of finite length, the boundary conditions

(2) yield

$$-c_1 + c_2 + c_3 + c_4 = aM_o/\lambda D_r \ ; \tag{26}$$

$$c_2 \qquad - c_4 = a^2H_o/2\lambda^2 D_r \ ; \tag{27}$$

$$-(\cos \lambda L/a + \sin \lambda L/a)e^{-\lambda L/a}c_1 + e^{-\lambda L/a}(\cos \lambda L/a$$

$$- \sin \lambda L/a)c_2 + e^{\lambda L/a}(\cos \lambda L/a - \sin \lambda L/a)c_3$$

$$+ e^{\lambda L/a}(\cos \lambda L/a + \sin \lambda L/a)c_4 = 0 \ ; \tag{28}$$

$$-e^{-\lambda L/a}(\sin \lambda L/a)c_1 + e^{-\lambda L/a}(\cos \lambda L/a)c_2$$

$$+ e^{\lambda L/a}(\sin \lambda L/a)c_3 - e^{\lambda L/a}(\cos \lambda L/a)c_4 = 0. \tag{29}$$

Equations (26) through (29) simplify to

$$-c_1 + c_2 + c_3 + c_4 = aM_o/\lambda D_r \ ; \tag{30}$$

$$c_2 - \qquad c_4 = a^2H_o/2\lambda^2 D_r \ ; \tag{31}$$

$$-(\cos 2\lambda L/a)c_1 + (1 - 2 \sin 2\lambda L/a)c_2 + e^{2\lambda L/a}(1-2 \sin 2\lambda L/a)c_3$$

$$+ e^{2\lambda L/a}(\cos 2\lambda L/a)c_4 = 0 \ ; \tag{32}$$

$$-(\sin \lambda L/a)c_1 + (\cos \lambda L/a)c_2 + e^{2\lambda L/a}(\sin \lambda L/a)c_3$$

$$- e^{2\lambda L/a}(\cos \lambda L/a)c_4 = 0. \tag{33}$$

These equations can be solved rather easily, although the general solution is too complicated to be of much instructive value. The equations are simpler in the case $\lambda L/a = 2n\pi$, and the solution then provides insight into the effects of the finite length of the shell upon the values of the influence coefficients. If $\lambda L/a = 2n\pi$ for some integer $n \geq 1$, equations (30) through (33) simplify to

$$-c_1 + c_2 + c_3 + c_4 = aM_o/\lambda D_r \ ; \tag{30}$$

$$c_2 \qquad - c_4 = a^2H_o/2\lambda^2 D_r \ ; \tag{31}$$

$$-c_1 + c_2 + e^{4n\pi}(c_3 + c_4) = 0 \ ; \tag{34}$$

$$c_2 - e^{4n\pi}c_4 = 0 \ . \tag{35}$$

These equations have the solution

$$c_1 = (a^2H_o/2\lambda^2 D_r - aM_o/\lambda D_r)\left(\frac{e^{-4n\pi}}{1 - e^{4n\pi}}\right) \tag{36}$$

$$c_2 = (a^2H_o/2\lambda^2 D_r)\left(\frac{-e^{4n\pi}}{1 - e^{4n\pi}}\right) \ ; \tag{37}$$

$$c_3 = (a^2H_o/2\lambda^2 D_r + aM_o/\lambda D_r)/(1 - e^{4n\pi}) \ ; \tag{38}$$

$$c_4 = \frac{-1}{1 - e^{4n\pi}} \ a^2H_o/2\lambda^2 D_r \ . \tag{39}$$

Now from equation (9)

$$\begin{aligned}\beta_o &= c_1 + c_3 \\ &= a^2H_o/2\lambda^2 D_r - a/\lambda D_r\left(1 + \frac{2}{-1 + e^{4n\pi}}\right)M_o \\ &= C_{\beta H}H_o - C_{\beta M}M_o \ ,\end{aligned} \tag{40}$$

where

$$C_{\beta H} = a^2/2\lambda^2 D_r \tag{41}$$

and

$$C_{\beta M} = (a/\lambda D_r)(1 + \frac{2}{-1 + e^{4n\pi}}) . \tag{42}$$

In a like manner it is found that

$$C_{uH} = (a^3/2\lambda^3 D_r)(\frac{1 + e^{4n\pi}}{-1 + e^{4n\pi}}) . \tag{43}$$

A comparison of equations (41), (42), and (43) with (25), (23), and (24) shows that the finite-length effects on $C_{\beta M}$ and C_{uH} are negligible for even moderate values of L/a. For the particular values of $\lambda L/a = 2n\pi$, there are no finite-length effects at all on $C_{\beta H}$.

The computations in this section indicate that for shells of constant thickness the effects of finite length upon the influence coefficients are negligible when the ratio of the length of the shell to the radius of its middle surface is greater than unity. In Chapter IV the results of evaluating the influence coefficients of shells of finite length and non-constant wall thickness are presented. Those results, which will be interpreted in detail when presented, bear out the conjecture suggested by the results of the present chapter — namely, that whether the wall thickness is constant or not, the influence coefficients for shells of finite length differ very little from those for shells of semi-infinite length provided the length of the finite shell exceeds the radius of the middle surface. That this fact is of computational importance is evident from the two cases discussed above, where the influence coefficients for the shell of semi-infinite length were obtained much more easily than those for the shell of finite length.

CHAPTER II

THE METHOD OF THE HYPERCIRCLE

In this chapter the method of the hypercircle is used to derive upper and lower bounds on the strain energy of a thin circular cylindrical shell subjected to the external forces described by equations (2). A procedure by which these bounds can be made successively better (that is, closer together) is also described.

After a function space has been introduced in which the "points" of the space are quadruples of functions representing the stress resultants and couples N_ξ, N_θ, M_ξ, M_θ, geometric intuition plays an important role in the discussion. In fact, much of the power of the method of the hypercircle lies in the analogy which may be drawn between the function space and ordinary three-dimensional Euclidean space. The analogy is helpful in suggesting useful relations, in motivating reasoning which might otherwise be obscure, and in interpreting complex analytical results which have simple geometric meanings. Distance in the function space is measured in terms of strain energy, and an inner product is obtained by generalizing the expression for strain energy.

Preliminaries

By a state of stress is meant a set of four functions N_ξ, N_θ, M_ξ, M_θ defined on the closed interval $0 \leq \xi \leq L/a$. The unstressed state $N_\xi = N_\theta = M_\xi = M_\theta = 0$ is represented by the origin in the function space. A state distinct from the origin may be thought of as a point P in the function space, or alternatively (and figuratively) as the position vector beginning at the origin and terminating at the point P.

Addition and subtraction of vectors and multiplication of a vector by a real number k are defined in the usual way. Thus, if

$$\bar{S}_1 = (N_{\xi 1}, N_{\theta 1}, M_{\xi 1}, M_{\theta 1})$$

and

$$\bar{S}_2 = (N_{\xi 2}, N_{\theta 2}, M_{\xi 2}, M_{\theta 2}),$$

then

$$\bar{S}_1 \pm \bar{S}_2 = (N_{\xi 1} \pm N_{\xi 2}, N_{\theta 1} \pm N_{\theta 2}, M_{\xi 1} \pm M_{\xi 2}, M_{\theta 1} \pm M_{\theta 2})$$

and

$$k\bar{S}_1 = (kN_{\xi 1}, kN_{\theta 1}, KM_{\xi 1}, kM_{\theta 1}) .$$

The inner product of two vectors $\bar{S}_1$ and $\bar{S}_2$ is defined by the relation

$$\bar{S}_1 \cdot \bar{S}_2 = 2\pi a^2 \int_o^{L/a} 1/2(\varepsilon_{\xi 1} N_{\xi 2} + \varepsilon_{\theta 1} N_{\theta 2} + K_{\xi 1} M_{\xi 2} + K_{\theta 1} M_{\theta 2}) d\xi ,$$

where $\varepsilon_{\xi 1}$, $\varepsilon_{\theta 1}$, $K_{\xi 1}$, $K_{\theta 1}$, are strains calculated from the stresses of the state $\bar{S}_1$ by using the stress-strain relations (5). The inner product defined above has the following properties.

(1) Linearity in the left-hand factor

$$(\bar{S}_1 + \bar{S}_2) \cdot \bar{S}_3 = \bar{S}_1 \cdot \bar{S}_3 + \bar{S}_2 \cdot \bar{S}_3$$

and

$$(k\bar{S}_1) \cdot \bar{S}_2 = k(\bar{S}_1 \cdot \bar{S}_2)$$

(2) Symmetry (this property is an expression of the reciprocity relation)

$$\bar{S}_1 \cdot \bar{S}_2 = 2\pi a^2 \int_0^{L/a} 1/2(\varepsilon_{\xi 1} N_{\xi 2} + \varepsilon_{\theta 1} N_{\theta 2} + \ldots)d\xi$$

$$= 2\pi a^2 \int_0^{L/a} 1/2[1/C(N_{\xi 1} - \nu N_{\theta 1})N_{\xi 2} + \frac{1}{C}(N_{\theta 1} - \nu N_{\xi 1})N_{\theta 2} + \ldots]d\xi$$

$$= 2\pi a^2 \int_0^{L/a} 1/2[1/C(N_{\xi 2} - \nu N_{\theta 2})N_{\xi 1} + \frac{1}{C}(N_{\theta 2} - \nu N_{\xi 2})N_{\theta 2} + \ldots]d\xi$$

$$= 2\pi a^2 \int_0^{L/a} 1/2[\varepsilon_{\xi 2} N_{\xi 1} + \varepsilon_{\theta 2} N_{\theta 1} + \ldots]d\xi$$

$$= \bar{S}_2 \cdot \bar{S}_1$$

(3) Positiveness

$$\bar{S}_1 \cdot \bar{S}_1 = 2\pi a^2 \int_0^{L/a} 1/2(\varepsilon_{\xi 1}N_{\xi 1} + \varepsilon_{\theta 1}N_{\theta 1} + K_{\xi 1}M_{\xi 1} + K_{\theta 1}M_{\theta 1})d\xi$$

$$= 2\pi a^2 \int_0^{L/a} 1/2[1/C(N_{\xi 1}^2 + N_{\theta 1}^2 - 2\nu N_{\xi 1}N_{\theta 1}) + \frac{1}{D(1-\nu^2)}(M_{\xi 1}^2 + M_{\theta 1}^2 - 2\nu M_{\theta 1}M_{\xi 1})d\xi$$

> 0 unless $\bar{S}_1 \equiv 0$, since $C > 0$, $D > 0$ and $0 < \nu < 1$

The length S or $|\bar{S}|$ of a vector $\bar{S}$ is defined as the non-negative square root of the inner product of $\bar{S}$ with itself. Thus,

$$S^2 = \bar{S} \cdot \bar{S} = 2\pi a^2 \int_0^{L/a} 1/2(\varepsilon_\xi N_\xi + \varepsilon_\theta N_\theta + \ldots)d\xi$$

$$= 2\pi a^2 \int_0^{L/a} W d\xi ,$$

where

$$W = 1/2(\varepsilon_\xi N_\xi + \varepsilon_\theta N_\theta + K_\xi M_\xi + K_\theta M_\theta)$$

is the strain energy function and $2\pi a^2 \int_0^{L/a} W d\xi$ is the strain energy of the state $\bar{S}$. Thus, the length of a vector is the non-negative square root of the strain energy associated with the state of stress which the vector represents.

Since the inner product has the three properties listed above, it follows that the space is Euclidean, and thus that the inner product just defined satisfies the Schwarz inequality

$$|\bar{S}_1 \cdot \bar{S}_2| \leq S_1 S_2$$

and the triangle inequality

$$|\bar{S}_1 + \bar{S}_2| \leq |\bar{S}_1| + |\bar{S}_2| \ .$$

If $\bar{S}_1$ and $\bar{S}_2$ are both non-zero states, then the angle θ between $\bar{S}_1$ and $\bar{S}_2$ may be defined by the relation

$$\theta = \cos^{-1}(\bar{S}_1 \cdot \bar{S}_2/S_1 S_2) \ .$$

The distance between two points $\bar{S}_1$ and $\bar{S}_2$ (or between the ends of the vectors $\bar{S}_1$ and $\bar{S}_2$) is defined as the length of the difference of the two points $|\bar{S}_1 - \bar{S}_2|$, so that

$$|\bar{S}_1 - \bar{S}_2| = 2\pi a^2 \int_0^{L/a} 1/2[(\varepsilon_{\xi 1} - \varepsilon_{\xi 2})(N_{\xi 1} - N_{\xi 2}) + \ldots]d\xi \ .$$

The distance so defined has the usual metric properties of ordinary space.

If the inner product of two states is zero, then the two states are said to be orthogonal. If the inner product of a state with itself is unity, the state is said to be normalized.

Of the theorems which can be carried over from the geometry of ordinary space, the Pythagorean theorem is particularly easy to verify. For, if $\bar{S}_1$ and $\bar{S}_2$ are any two vectors, then

$$|\bar{S}_1 - \bar{S}_2|^2 = (\bar{S}_1 - \bar{S}_2) \cdot (\bar{S}_1 - \bar{S}_2)$$

$$= S_1^2 - 2\bar{S}_1 \cdot \bar{S}_2 + S_2^2 .$$

If $\bar{S}_1$ and $\bar{S}_2$ are orthogonal, then

$$|\bar{S}_1 - \bar{S}_2|^2 = S_1^2 + S_2^2 .$$

States of Stress

The following is a list of symbols and definitions which are used in the ensuing discussion. Each symbol in the list is used throughout the rest of this chapter to denote a state of stress satisfying the indicated conditions or having the indicated properties.

Symbol	Name	Explanation
$\bar{S}^*$	complete equilibrium state	any state which satisfies the equilibrium equations (1), the boundary conditions (2), but not necessarily the compatibility equation (3)
$\bar{S}^*_p$	homogeneous equilibrium states ($p=1,\ldots,m$)	a set of m states each of which satisfies the equilibrium equations (1) and the homogeneous boundary conditions $\xi=0$, $M_\xi=H=0$; $\xi=L/a$, $M_\xi=H=0$, but not necessarily the compatibility equation (3)[the states $\bar{S}^*_p$ are assumed to be linearly independent]

$\bar{I}^*_p$	orthonormal homogeneous equilibrium states $(p=1,2,\ldots,m)$	a set of m orthonormal states, each of which is a linear combination of the states $\bar{S}^*_p$ [and therefore satisfies the same equilibrium equations and boundary conditions]
$*\bar{S}_q$	compatible states $(q=1,2,\ldots n)$	a set of n states each of which satisfies the compatibility equation (3), but not necessarily the equilibrium equations (1) nor any boundary conditions[the states $*\bar{S}_q$ are assumed to be linearly independent]
$*\bar{I}_q$	orthonormal compatible states $(q=1,2,\ldots,n)$	a set of n orthonormal states, each of which is a linear combination of the states $*\bar{S}_q$ [and therefore satisfies the compatibility equation (3)].
$\bar{S}$	actual state	the state actually existing within the shell. This state satisfies the equilibrium equations (1), the compatibility equation (3), and the boundary conditions (2).

The orthonormality of the states $\bar{I}^*_p$ and of the states $\bar{I}^*_q$ may be expressed by the relations

$$\bar{I}^*_p \cdot \bar{I}^*_r = \delta_{pr}(p,r = 1, 2, \ldots, m),$$

$$*\bar{I}_q \cdot *\bar{I}_s = \delta_{qs}(q,r = 1, 2, \ldots, n),$$

where δ_{jk}, the Kronecker delta, is defined by the relation

$$\delta_{jk} = \begin{cases} 1 \text{ if } j = k \\ 0 \text{ if } j \neq k \end{cases} .$$

The Hyperplane of Equilibrium and the Hyperplane of Compatibility

A large part of the power of the method of the hypercircle lies in the geometrical interpretations of the method. The analogy between the function space and the familiar three-dimensional physical space is helpful in suggesting interesting relations and inequalities which might otherwise be obscure and in suggesting lines of reasoning which might otherwise be poorly motivated.

In this geometric interpretation, the hyperplanes of equilibrium and of compatibility figure so prominently that they are discussed separately below before proceeding with an exposition of the method of the hypercircle.

By virtue of the linearity of the equilibrium equations (1) and the homogeneity of the boundary conditions satisfied by a homogeneous equilibrium state, any linear combination of homogeneous equilibrium states is itself a homogeneous equilibrium state. For the same reasons, if a complete equilibrium state be added to any homogeneous equilibrium state, the result is a complete equilibrium state. Thus the state

$$\bar{S}^* + \sum_{p=1}^{m} a_p^* \bar{S}_p^* \tag{44}$$

is a complete equilibrium state for any real values of the constants a_p^*. If these constants are allowed to assume arbitrary real values, the resulting class of vectors of the form (44) is an m-dimensional hyperplane L_m^* in the function space. This hyperplane is called the hyperplane of equilibrium.

Similarly, any linear combination of compatible states

$$\sum_{q=1}^{n} {}^*a_q \, {}^*\bar{S}_q \tag{45}$$

is also a compatible state, since the compatibility equation (3) is linear and no boundary conditions are imposed on the states $*\bar{S}_q$. If the constants $*a_q$ are allowed to assume arbitrary real values, the resulting class of vectors of the form (45) is an n-dimensional hyperplane $*L_n$, the hyperplane of compatibility.

If the two hyperplanes L_m^* and $*L_n$ have a point in common, then that point represents the actual state $\bar{S}$, since the equilibrium equations, the compatibility equation, and the boundary conditions are satisfied by the state represented by the common point. But since the two hyperplanes have only m+n dimensions, while the function space has infinitely many dimensions, the hyperplanes will not in general have a point of intersection. In either case, there is a point in L_m^* and a point in $*L_n$ at which the distance between the two hyperplanes is a minimum, and as will appear later, the location of these two points is a key step in finding upper and lower bounds for the strain energy of the actual state.

A property of the hyperplanes L_m^* and $*L_n$ which is used in the following discussion is their mutual orthogonality, which may be shown as follows.

Any vector $\bar{S}_E$ lying in the hyperplane of equilibrium L_m^* has the form (see Figure 3)

$$\bar{S}_E = \sum_{p=1}^{m} a_p *\bar{I}_p^* .$$

Any vector lying in the hyperplane of compatibility $*L_n$ has the form

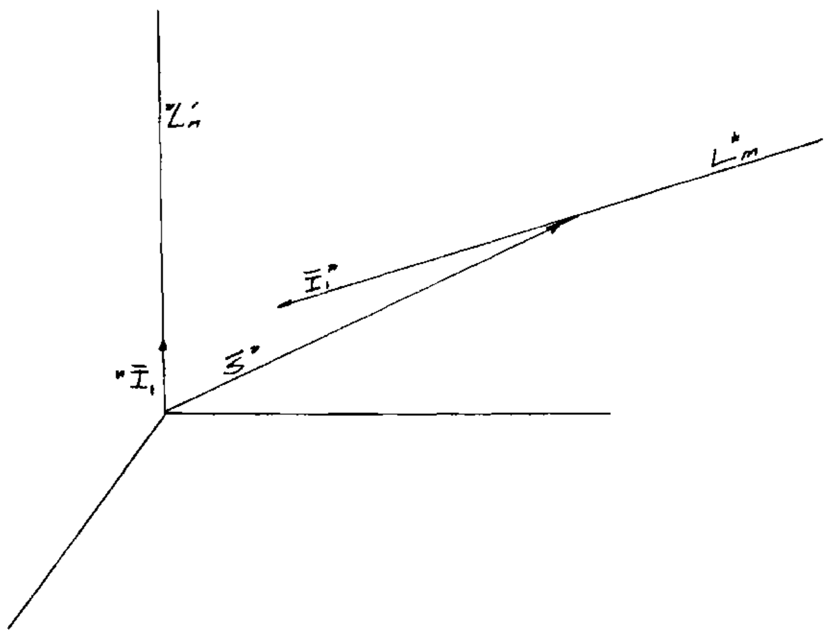


Figure 3. The Hyperplanes of Equilibrium and Compatibility for the Case m=n=1.

$$\bar{S}_C = \sum_{q=1}^{n} {}^*a_q {}^*\bar{I}_q \ .$$

Orthogonality of the two hyperplanes is established by showing that

$$\bar{S}_C \cdot \bar{S}_E = 0 \ . \tag{46}$$

From the definition of the inner product,

$$\bar{S}_C \cdot \bar{S}_E = 2\pi a^2 \int_0^{L/a} 1/2(\varepsilon_{\xi C} N_{\xi E} + \varepsilon_{\theta C} N_{\theta E} + K_{\xi C} M_{\xi E}$$

$$+ K_{\theta C} M_{\theta E}) d\xi$$

$$= 2\pi a^2/2 \int_0^{L/a} [(-\nu/a) u_C N_{\xi E} + (1/a) u_C N_{\theta E}$$

$$+ (1/a) \frac{d\beta_c}{d\xi} M_{\xi E} + 0 \cdot M_{\theta\xi}] d\xi$$

or, since $N_{\xi E} \equiv 0$ and $N_{\theta E} = dH_E/d\xi$,

$$\bar{S}_C \cdot \bar{S}_E = \pi a^2 \int_0^{L/a} (u_C dH_E/d\xi + d\beta_C/d\xi M_{\xi E}) d\xi \ .$$

An integration by parts gives

$$\bar{S}_C \cdot \bar{S}_E = a\pi \left\{ [u_C H_E]_0^{L/a} + [\beta_C M_{\xi E}]_0^{L/a} \right.$$

$$\left. - \int_0^{L/a} (H_E du_C/d\xi + \beta_C dM_\xi/d\xi) d\xi \right\} .$$

The third term of the above equation is zero since

$$H_E = -1/a \; dM_{\xi E}/d\xi$$

and

$$du_C/d\xi = a\beta_C \; .$$

Since $\bar{S}_E$ is a linear combination of homogeneous equilibrium states,

$$H_E(0) = M_{\xi E}(0) = H_E(L/a) = M_{\xi E}(L/a) = 0 \; ,$$

and so

$$\bar{S}_C \cdot \bar{S}_E = 0 \; ,$$

as was to be shown.

In particular, then, since $\bar{I}^*_p$ and $^*\bar{I}_q$ are vectors lying in the hyperplanes L^*_m and $^*L_n$ respectively,

$$^*\bar{I}_q \cdot \bar{I}^*_p = 0 \qquad (p = 1, 2, \ldots, m; \; q = 1, 2, \ldots, n) \; .$$

Figure 3 is a representation of the hyperplanes L^*_m and $^*L_n$ when m=n=1, in which case the hyperplanes are straight lines.

<u>An Outline of the Discussion to Follow</u>

The succeeding steps in establishing bounds on the strain energy of the actual state (i.e., on the square of the length of $\bar{S}$) may be described in geometrical terms as follows:

a) The actual state is located on a hypersphere, on m+n hyperplanes related to $^*L_n$ and L^*_m, and then on a hypercircle Γ which is the intersection of the m+n hyperplanes and the hypersphere.

b) The center $\bar{C}$ and the radius R of the hypercircle Γ are determined.

c) A minimal property of the hypercircle is verified and the extremities of a diameter of Γ are located (these extremities are the points in $*L_n$ and L^*_m at which the distance between the two hyperplanes is a minimum).

d) A proof is given of the fact that the squares of distances of these extremities from the origin are respectively lower and upper bounds on S^2.

Location of the Actual State on the Hypercircle

Let $\bar{S}$ be the actual state, and let $\bar{S}^*$ be any complete equilibrium state. Then, since $\bar{S}$ is itself a complete equilibrium state, $\bar{S} - \bar{S}^*$ is a homogeneous equilibrium state. Also $\bar{S}$ is a compatible state, and hence, by (46)

$$\bar{S} \cdot (\bar{S} - \bar{S}^*) = 0 . \tag{47}$$

Thus, $\bar{S}$ and $\bar{S} - \bar{S}^*$ are orthogonal, and so the end of $\bar{S}$ lies on a hypersphere of which $\bar{S}^*$ is a diameter. The center of the hypersphere is at $\frac{1}{2}\bar{S}^*$, and the radius is $\frac{1}{2} S^*$ (see Figure 4).

If $\bar{S}$ is the actual state (and hence a compatible state) and $\bar{S}_E$ is any homogeneous equilibrium state, then by (46)

$$\bar{S} \cdot \bar{S}_E = 0 .$$

In particular, since each $\bar{I}^*_p$ is a homogeneous equilibrium state, it follows that

$$\bar{S} \cdot \bar{I}_p^* = 0 . \qquad (p = 1, 2, \ldots, m) \tag{48}$$

Thus, $\bar{S}$ is orthogonal to the hyperplane of equilibrium and lies in each of the m hyperplanes orthogonal to the vectors $\bar{I}_p^*$ (see Figure 5).

If $\bar{S}$ is the actual state and $\bar{S}^*$ any complete equilibrium state, then $\bar{S} - \bar{S}^*$ is a homogeneous equilibrium state. It follows from (46) that if $\bar{S}_C$ is any compatible state, then

$$\bar{S}_C \cdot (\bar{S} - \bar{S}^*) = 0 . \tag{49}$$

Thus, the difference between the actual state $\bar{S}$ and any complete equilibrium state $\bar{S}^*$ is orthogonal to any compatible state. In particular, since each ${}^*\bar{I}_q$ is a compatible state,

$$(\bar{S} - \bar{S}^*) \cdot {}^*\bar{I}_q = 0$$

or

$$\bar{S} \cdot {}^*\bar{I}_q = \bar{S}^* \cdot {}^*\bar{I}_q . \qquad (q = 1, 2, \ldots, n) \tag{50}$$

This means that the difference $\bar{S} - \bar{S}^*$ is orthogonal to the hyperplane of compatibility (see Figure 6) and lies in each of the n hyperplanes which pass through the extremity of $\bar{S}^*$ and are orthogonal to each of the vectors ${}^*\bar{I}_q$.

Equation (47) locates the end of the vector $\bar{S}$ on a hypersphere; equation (48) locates it in each of the m hyperplanes passing through the origin and orthogonal to L_m^*; and equation (50) locates it in each of the n hyperplanes passing through the extremity of $\bar{S}^*$ and orthogonal to ${}^*L_n$.

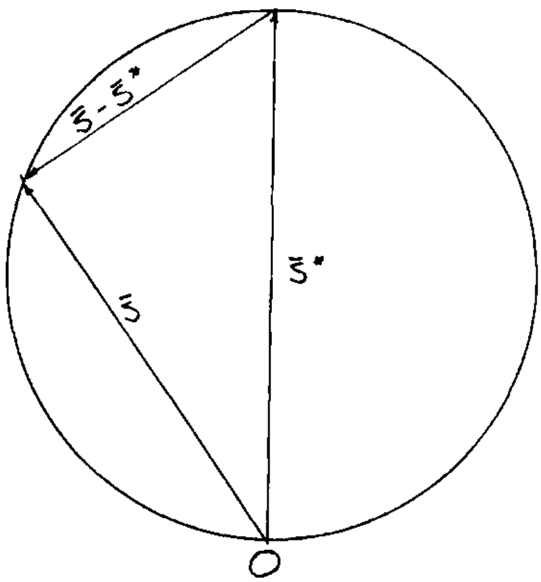


Figure 4. The Hypersphere of which $\bar{S}^*$ is a Diameter. Also shown are the actual state $\bar{S}$ and the difference $\bar{S}-\bar{S}^*$.

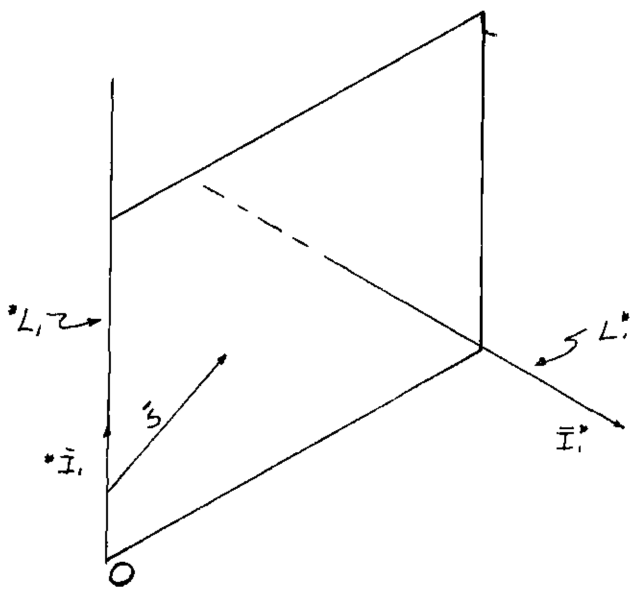


Figure 5. The Hyperplane Orthogonal to $\bar{I}^*$, Showing $\bar{S}$, $^*\bar{I}_1$ and $\bar{I}_1^*$ for the Case m=n=1.

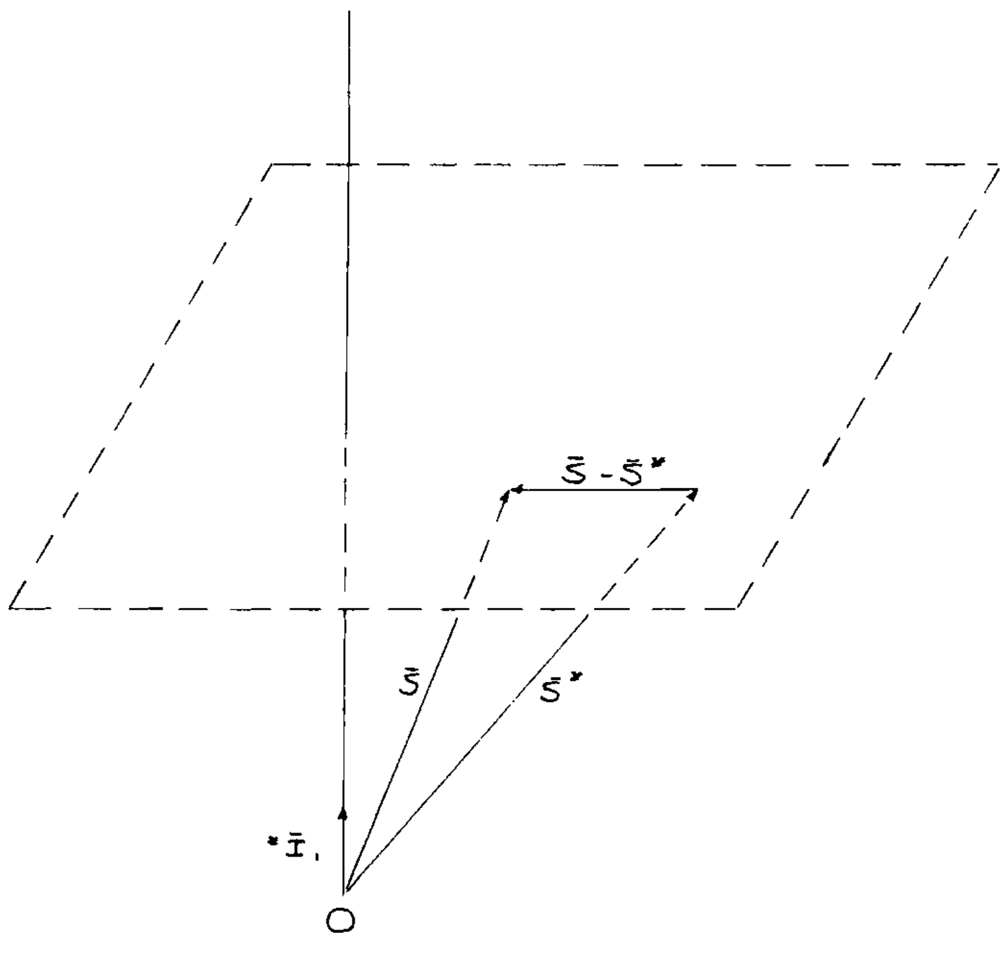


Figure 6. The Hyperplane Orthogonal to $^*\bar{I}_1$, Showing the Vectors $^*\bar{I}_1$, $\bar{S}^*$, and $\bar{S}$.

So the end of $\bar{S}$ lies on a hypercircle Γ which is the intersection of the hypersphere and the m+n hyperplanes just described.

Definition of the Center $\bar{C}$ and Determination of the Radius R of the Hypercircle Γ

The center $\bar{C}$ of the hypercircle Γ is defined to be the point

$$\bar{C} = 1/2 \left[\bar{S}^* - \sum_{p=1}^{m} \bar{I}_p^* (\bar{S}^* \cdot \bar{I}_p^*) + \sum_{q=1}^{n} {}^*\bar{I}_q (\bar{S}^* \cdot {}^*\bar{I}_q) \right] . \tag{51}$$

The radius of Γ may be found from the relation

$$R^2 = (\bar{S} - \bar{C}) \cdot (\bar{S} - \bar{C})$$

or

$$R^2 = 1/4 \left[S^{*2} - \sum_{p=1}^{m} (\bar{S}^* \cdot \bar{I}_p^*)^2 - \sum_{q=1}^{n} (\bar{S}^* \cdot {}^*\bar{I}_q)^2 \right] . \tag{52}$$

The preceding discussion may be summarized as follows.

The extremity of $\bar{S}$ (the actual state) is located on a hypercircle Γ whose equations are

$$\begin{aligned} &\bar{X} \cdot (\bar{X} - \bar{S}^*) = 0 , \\ &\bar{X} \cdot \bar{I}_p^* = 0 \qquad (p = 1, 2, \ldots, m) , \\ &\bar{X} \cdot {}^*\bar{I}_q = \bar{S}^* \cdot {}^*\bar{I}_q \quad (q=1, 2, \ldots, n) , \end{aligned} \tag{53}$$

where $\bar{X}$ is any point on Γ. The center C and radius R of Γ are given by equations (51) and (52) respectively.

Verification of a Minimal Property of the Hypercircle Γ and Location of the Extremities of a Diameter

The next step in the argument is motivated by the following observation: If $\bar{S}$ is the actual state, $\bar{S}^*$ any complete equilibrium state, and $\bar{S}_C$ any compatible state, then

$$(\bar{S} - \bar{S}^*) \cdot (\bar{S} - \bar{S}_C) = \bar{S} \cdot (\bar{S} - \bar{S}^*) - \bar{S}_C \cdot (\bar{S} - \bar{S}^*) = 0 .$$

The preceding equation may be verbalized as: the actual state lies on every hypersphere whose diameter is the line joining any point of the hyperplane of equilibrium to any point of the hyperplane of compatibility.

Now points ${}^*\bar{V}_n$ and $\bar{V}_m^*$ in ${}^*L_n$ and L_m^* respectively can be found for which the length of their difference is less than or equal to the length of the difference of any two points in ${}^*L_n$ and L_m^* respectively. Thus, ${}^*\bar{V}_n$ is the best approximation to $\bar{S}$ in ${}^*L_n$, and $\bar{V}_m^*$ is the best approximation to $\bar{S}$ in L_m^*. These two points may be found by using the following procedure.

The general point in L_m^* is

$$\bar{U}^* = \bar{S}^* + \sum_{p=1}^{m} b_p^* \bar{I}_p^* ,$$

and the general point in ${}^*L_n$ is

$${}^*\bar{U} = \sum_{q=1}^{n} {}^*b_q \, {}^*\bar{I}_q ,$$

where b_p^* and ${}^*b_q$ are arbitrary real numbers. The square of the distance between the two points is

$$D^2 = |\bar{U}^* - {}^*\bar{U}|^2 = (\bar{S}^* + \sum_{p=1}^{m} b_p^* \ \bar{I}_p^* - \sum_{q=1}^{n} {}^*b_q \ {}^*\bar{I}_q)^2$$

$$= S^{*2} + 2 \sum_{p=1}^{m} b_p^* \bar{S}^* \cdot \bar{I}_p^* - 2 \sum_{q=1}^{n} {}^*b_q \bar{S}^* \cdot {}^*\bar{I}_q$$

$$+ \sum_{p=1}^{m} b_p^{*2} + \sum_{q=1}^{n} {}^*b_q^2 \ .$$

The quantity D^2 (and hence D itself) is a minimum when the constants b_p^* and ${}^*b_q$ have the values

$$b_p^* = - \bar{S}^* \cdot \bar{I}_p^* \ ,$$

$${}^*b_q = \bar{S}^* \cdot {}^*\bar{I}_q \ ,$$

that is, when $\bar{U}^* - {}^*\bar{U}$ is orthogonal to both L_m^* and ${}^*L_n$. Thus,

$$\bar{V}_m^* = \bar{S}^* - \sum_{p=1}^{m} \bar{I}_p^* (\bar{S}^* \cdot \bar{I}_p^*) \tag{54}$$

and

$${}^*\bar{V}_n = \sum_{q=1}^{n} {}^*\bar{I}_q (\bar{S}_q^* \cdot {}^*\bar{I}_q) \ . \tag{55}$$

Comparing (54) and (55) with (51) and with (52) shows that

$$\bar{C} = 1/2(\bar{V}_m^* + {}^*\bar{V}_n)$$

and

$$R^2 = 1/4(\bar{V}_m^* - {}^*\bar{V}_n)^2 .$$

Thus, the hypercircle Γ previously constructed enjoys the property that the points ${}^*\bar{V}_n$ and $\bar{V}_m^*$, where the hyperplanes of compatibility and equilibrium are closest together, are the extremities of a diameter of Γ. Furthermore, since any chord of Γ may be represented by a vector $\bar{Y} = \bar{X}_1 - \bar{X}_2$, where $\bar{X}_1$ and $\bar{X}_2$ terminate on Γ, it follows from the last of equations (53) that

$$\begin{aligned}\bar{Y} \cdot {}^*\bar{I}_q &= (\bar{X}_1 - \bar{X}_2) \cdot {}^*\bar{I}_q \\ &= \bar{X}_1 \cdot {}^*\bar{I}_q - \bar{X}_2 \cdot {}^*\bar{I}_q \\ &= \bar{S}^* \cdot {}^*\bar{I}_q - \bar{S}^* \cdot {}^*\bar{I}_q \\ &= 0 \qquad\qquad (q = 1, 2, \ldots, n)\end{aligned}$$

and hence that

$$\bar{Y} \cdot {}^*\bar{V}_n = 0 .$$

That is, ${}^*\bar{V}_n$ is orthogonal to the plane of Γ.

In summary, the vertices $\bar{V}_m^*$ and ${}^*\bar{V}_n$ of the hyperplanes of equilibrium and of compatibility are the extremities of a diameter of the hypercircle Γ on which the actual state $\bar{S}$ lies. The position vector ${}^*\bar{V}_n$ of the vertex of the hyperplane of compatibility is orthogonal to every chord of Γ.

Proof that V_m^{*2} and ${}^*V_n^2$ are Respectively Upper and Lower Bounds on S^2

The summary just stated and an examination of Figure 7 suggest the inequalities

$$ {}^*V_n^2 \leq S^2 \leq V_m^2 \,. $$

A proof may be constructed as follows. By reason of equations (55), any point $\bar{X}$ on Γ satisfies the relations

$$ (\bar{X} - \bar{V}_m^*) \cdot (\bar{X} - {}^*\bar{V}_n) = 0 \tag{57} $$

and

$$ \bar{X} \cdot {}^*\bar{V}_n = \bar{S}^* \cdot {}^*\bar{V}_n \,. \tag{58} $$

The first of these relations is deduced as follows.

$$ (\bar{X} - \bar{V}_m^*) \cdot (\bar{X} - {}^*\bar{V}_n) $$

$$ = \left[\bar{X} - \bar{S}^* + \sum_{p=1}^{m} (\bar{S}^* \cdot \bar{I}_p^*)\bar{I}_p^*\right] \cdot \left[\bar{X} - \sum_{q=1}^{n} (\bar{S}^* \cdot {}^*\bar{I}_q)\, {}^*\bar{I}_q\right] $$

$$ = (\bar{X} - \bar{S}^*) \cdot \bar{X} + \bar{X} \cdot \sum_{p=1}^{m} (\bar{S}^* \cdot \bar{I}_p^*)\bar{I}_p^* $$

$$- (\bar{X} - \bar{S}^*) \cdot \sum_{q=1}^{n} (\bar{S}^* \cdot {}^*\bar{I}_q)\,{}^*\bar{I}_q$$

$$-\left(\sum_{p=1}^{m} (\bar{S}^* \cdot \bar{I}_p^*)\bar{I}_p^*\right) \cdot \left(\sum_{q=1}^{n} (\bar{S}^* \cdot {}^*\bar{I}_q)\,{}^*\bar{I}_q\right) = 0$$

by reason of equations (53) and (46). The second equation is implicit in the third of equations (53), since ${}^*\bar{V}_n$ is a linear combination of the ${}^*\bar{I}_q$. From (57),

$$\bar{X}^2 = \bar{V}_m^* \cdot \bar{X} + {}^*\bar{V}_n \cdot \bar{X} - (\bar{V}_m^* \cdot {}^*\bar{V}_n)$$

or, by use of equation (58),

$$\bar{X}^2 = \bar{V}_m^* \cdot \bar{X} + \bar{S}^* \cdot {}^*\bar{V}_n - (\bar{V}_m^* \cdot {}^*\bar{V}_n) \; . \tag{59}$$

But by virtue of equation (46)

$$\bar{V}_m^* \cdot {}^*\bar{V}_n = \bar{S}^* \cdot {}^*\bar{V}_n \; ,$$

and hence

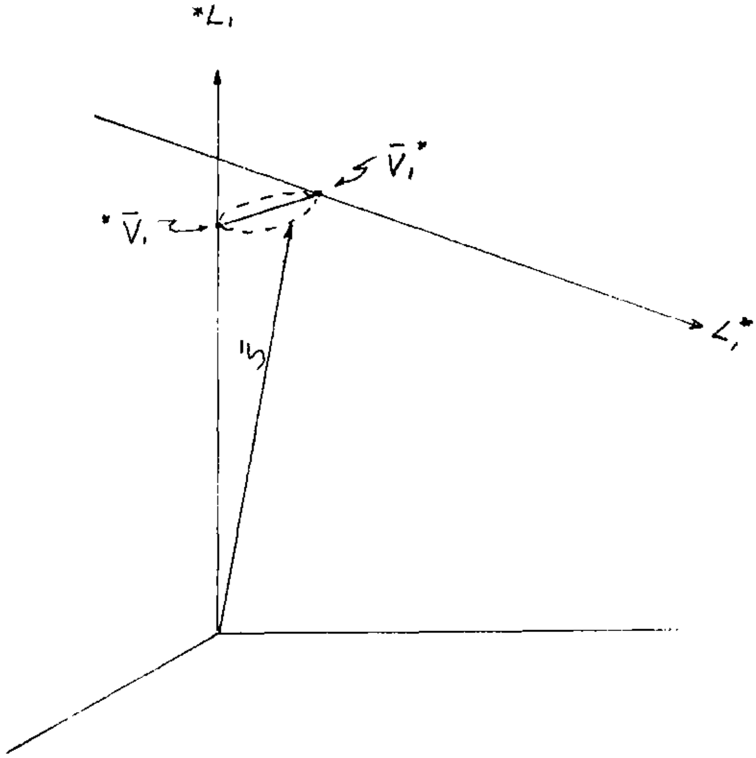
$$\bar{X}^2 = \bar{V}_m^* \cdot \bar{X} \; . \tag{60}$$

From the preceding equation, $\bar{X}^2$ has its maximum value when

$$\bar{X} = \bar{V}_m^* \; ,$$

since $\bar{X}$ must lie on Γ. Thus, V_m^* is the maximum of X.

On the other hand, equation (60) may be rewritten in the



NOTE: This figure is in some respects deceiving. Actually, Γ consists of the two points $\bar{V}_1^*$ and $^*\bar{V}_1$. But to make an illustration more suggestive of the general case, Γ is shown here also as a (hyper) circle.

Figure 7. The Hypercircle Γ, Showing $^*\bar{V}_n$, $\bar{V}_m^*$ and $\bar{S}$.

equivalent form

$$X^2 = V_m^* X \cos \Theta , \tag{61}$$

where Θ is the angle between $\bar{V}_m^*$ and $\bar{X}$. The left-hand side of (61) is small when $\bar{X}$ is small and Θ is large. The vector to the point ${}^*\bar{V}_n$ has the smallest length of any point on Γ and the angle between ${}^*\bar{V}_n$ and $\bar{V}_m^*$ is larger than the angle between $\bar{V}_m^*$ and any other vector terminating on Γ (since $\bar{V}_m^*$ and ${}^*\bar{V}_n$ terminate on the opposite ends of a diameter of Γ). Hence, the minimum of X occurs when $\bar{X} = {}^*\bar{V}_n$. Thus, for any point $\bar{X}$ on Γ,

$$({}^*\bar{V}_n)^2 \leq \bar{X}^2 \leq (\bar{V}_m^*)^2 . \tag{62}$$

Since $\bar{S}$ lies on Γ, inequality (56) is proved.

In summary, the distance of the actual state from the origin is bounded above by the distance from the origin to the vertex of the hyperplane of equilibrium and below by the distance from the origin to the vertex of the hyperplane of compatibility. That is,

$$({}^*V_n)^2 = \sum_{q=1}^{n} (\bar{S}^* \cdot {}^*\bar{I}_q)^2 \leq S^2 \leq (S^*)^2 \tag{63}$$

$$- \sum_{p=1}^{m} (\bar{S}^* \cdot \bar{I}_p^*)^2 = (V_m^*)^2 .$$

These inequalities bound the strain energy of the actual state $\bar{S}$ above and below.

CHAPTER III

BOUNDS ON THE INFLUENCE COEFFICIENTS

In this chapter, bounds on the influence coefficients C_{uH}, $C_{\beta M}$, and $C_{\beta H}$ are derived from inequality (62). A numerical example is presented which illustrates the procedure normally followed in obtaining values for these bounds, and some discussion is devoted to the problem of making astute choices of the several states necessary to carry out the computations.

The Strain Energy in Terms of the Influence Coefficients C_{uH}, $C_{\beta M}$, and $C_{\beta H}$

The strain energy of the actual state $\bar{S}$ is given by the relation

$$S^2 = 2\pi a^2 \int_0^{L/a} 1/2(\varepsilon_\xi N_\xi + \varepsilon_\theta N_\theta + K_\xi M_\xi + K_\theta M_\theta)d\xi \; . \qquad (64)$$

By equations (1), $N_\xi = 0$, and by equations (4) $K_\theta = 0$. Hence equation (64) may be written in the form

$$S^2 = \pi a^2 \int_0^{L/a} (\varepsilon_\theta N_\theta + K_\xi M_\xi)d\xi \; ,$$

which, when combined with the strain-displacement relations (4) and the equilibrium equations (1) becomes

$$S^2 = \pi a^2 \int_0^{L/a} \left(\frac{u}{a} H' + \frac{\beta'}{a} M_\xi\right) d\xi \quad . \tag{65}$$

Integrating (65) once by parts and applying equations (1) and (3) once again yields the relation

$$S^2 = \pi a[(uH + \beta M)]_0^{L/a} - \int_0^{L/a} (u'H + \beta M'_\xi) d\xi \tag{66}$$

$$= -\pi a(u_0 H_0 + \beta_0 M_0) - \pi a \int_0^{L/a} H(u' - a\beta) d\xi$$

$$= -\pi a(u_0 H_0 + \beta_0 M_0) \; ,$$

since $\bar{S}$ satisfies the relation $u' = a\beta$. If u_0 and β_0 are expressed in terms of the influence coefficients, then (66) becomes

$$S^2 = -\pi a[(-C_{uH} H_0 + C_{uM} M_0) H_0 + (C_{\beta H} H_0 - C_{\beta M} M_0) M_0] \tag{67}$$

or, since $C_{uM} = C_{\beta H}$,

$$S^2 = \pi a[C_{uH} H_0^2 - 2C_{\beta H} H_0 M_0 + C_{\beta M} M_0^2] \; . \tag{68}$$

Thus the strain energy of the actual state $\bar{S}$ can be expressed as a quadratic form in the quantities H_0 and M_0 with coefficients related to the influence coefficients as shown by equation (68).

Bounds on the Influence Coefficients

If S^2 is given by equation (68), and the equilibrium state $\bar{S}^*$ is chosen so that S^{*2} is also a quadratic form in M_0 and H_0, then the inequalities (62) take the form

$$
\begin{aligned}
a_1H_0^2 - 2a_2H_0M_0 + a_3M_0^2 & \\
\le C_{uH}H_0^2 - 2C_{\beta H}H_0M_0 + C_{\beta M}M_0^2 & \qquad (69) \\
\le b_1H_0^2 - 2b_2H_0M_0 + b_3M_0^2 \ , &
\end{aligned}
$$

where the numbers a_1, a_2, a_3, b_1, b_2, and b_3 are calculable. Each member of (69) is positive definite, and the inequalities are valid for all values of H_0 and M_0. In particular, then, if $H_0 = 1$ and $M_0 = 0$,

$$a_1 \le C_{uH} \le b_1 \ ; \tag{70}$$

if $H_0 = 0$ and $M_0 = 1$,

$$a_3 \le C_{\beta M} \le b_3 \ . \tag{71}$$

Bounds for $C_{\beta H}$ which are not particularly confining can be deduced from the fact that S^2 is positive definite, so that

$$(C_{\beta H})^2 - C_{uH}C_{\beta M} \le 0$$

or

$$|C_{\beta H}| \le \sqrt{C_{uH}C_{\beta M}} \ .$$

More exacting bounds can be obtained by the following reasoning. For values of H_0 and M_0 for which $H_0M_0 > 0$, the right inequality in (69) becomes, upon division by H_0M_0,

$$C_{\beta H} \geq b_2 - 1/2[b_1 - C_{uH}]H_0/M_0 - 1/2[b_3 - C_{\beta M}]M_0/H_0 \;. \qquad (72)$$

The exact values of C_{uH} and $C_{\beta M}$ are not known, but lower bounds, $\underline{C_{uH}}$ and $\underline{C_{\beta M}}$, can be found by inequalities (70) and (71). Using these bounds in (72) will not destroy the sense of the inequality, and so (72) may be written in the form

$$C_{\beta H} \geq b_2 - 1/2[b_1 - \underline{C_{uH}}]H_0/M_0 - 1/2[b_3 - \underline{C_{\beta M}}]M_0/H_0 \;. \qquad (73)$$

If the right side of (73) is thought of as a function of H_0/M_0 $(H_0/M_0 > 0)$, then the right member assumes its largest value when

$$H_0/M_0 = \sqrt{(b_3 - \underline{C_{\beta M}})/(b_1 - \underline{C_{uH}})} \;,$$

and hence inequality (73) in its most restrictive form is

$$C_{\beta H} \geq b_2 - \sqrt{(b_1 - \underline{C_{uH}})(b_3 - \underline{C_{\beta M}})} \;. \qquad (74)$$

For values of H_0 and M_0 for which $H_0M_0 < 0$, the right inequality in (69) becomes

$$C_{\beta H} \leq b_2 + 1/2[b_1 - C_{uH}](-H_0/M_0) + 1/2[b_3 - C_{\beta M}](-M_0/H_0) \;.$$

By reasoning as above, the inequality

$$C_{\beta H} \leq b_2 + \sqrt{(b_1 - \underline{C_{uH}})(b_3 - \underline{C_{\beta M}})} \qquad (75)$$

is obtained. The combining of (74) and (75) yields

$$b_2 - \sqrt{(b_1 - \underline{C}_{uH})(b_2 - \underline{C}_{\beta M})} \leq C_{\beta H} \leq b_2$$

$$+ \sqrt{(b_1 - C_{uH})(b_3 - C_{\beta M})} \quad . \tag{76}$$

The left inequality of (69) similarly yields

$$a_2 - \sqrt{(\overline{C}_{uH} - a_1)(\overline{C}_{\beta M} - a_3)} \leq C_{\beta H} \leq a_2$$

$$+ \sqrt{(\overline{C}_{uH} - a_1)(\overline{C}_{\beta M} - a_3)} \quad . \tag{77}$$

By collecting the inequalities (70), (71), (76), and (77), the following summary of bounds for the influence coefficients is obtained.

$$a_1 \leq C_{uH} \leq b_1$$

$$a_3 \leq C_{\beta M} \leq b_3$$

$$b_2 - \sqrt{(b_1 - \underline{C}_{uH})(b_3 - \underline{C}_{\beta M})} \leq \begin{Bmatrix} C_{\beta H} \\ C_{uM} \end{Bmatrix} \leq b_2$$

$$+ \sqrt{(b_1 - \underline{C}_{uH})(b_3 - \underline{C}_{\beta M})}$$

$$a_2 - \sqrt{(\overline{C}_{uH} - a_1)(\overline{C}_{\beta M} - a_3)} \leq \begin{Bmatrix} C_{\beta H} \\ C_{uM} \end{Bmatrix} \leq a_2$$

$$+ \sqrt{(\overline{C}_{uH} - a_1)(\overline{C}_{\beta M} - a_3)}$$

Here a_1, a_2, and a_3 are the coefficients of H_0^2, $-2H_0M_0$, and M_0^2 respectively in the expression $\frac{1}{\pi a}\ ({}^{*}V_n)^2$; b_1, b_2, and b_3 are the similar coefficients in the expression $\frac{1}{\pi a}\ (V_m^{*})^2$; and $\overline{C_{uH}}$, $\overline{C_{\beta M}}$, $\underline{C_{uH}}$, and $\underline{C_{\beta M}}$, are upper and lower bounds on the quantities C_{uH} and $C_{\beta M}$ respectively.

A Numerical Example

In this section the relations derived above are used to compute bounds on the influence coefficients for the shell of the type shown in Figure 1 described by the parameters a = 1, L = 3, and $h = h_r e^{-\rho\xi}$, where $h_r = 0.1$ and $\rho = 2.8$.

The choices of the functions which define the states ${}^{*}\bar{S}$ and $\bar{S}^{*}$ should be made in such a way that the strain energies of the two states are about equal to each other, and thus almost equal to the strain energy of the actual state. To insure that the strain energy of the equilibrium state $\bar{S}^{*}$ is almost equal to the strain energy of the actual state $\bar{S}$, it is necessary that the state $\bar{S}^{*}$ be chosen so that the stresses closely approximate those of the actual state. Likewise, the compatible state ${}^{*}\bar{S}$ should be chosen in such a way that the displacements closely approximate those of the actual state. Since the shell is subjected to external forces and moments only along the lower edge, it is physically reasonable to assume that the behavior of the shell will be determined almost entirely by the characteristics of the part of the shell near the lower edge. In particular, for small values of ρ, the strains and stresses within the shell should be nearly the same as those in a shell with a constant thickness equal to the lower edge thickness, or reference thickness, of the variable shell. Thus, it seems proper to chose the functions defining ${}^{*}\bar{S}$ and $\bar{S}^{*}$ to be of the same form as given in equations (9), (10), (11), and (12).

In choosing the state ${}^{*}\bar{S}_1$, which will then define the normalized state ${}^{*}\bar{I}_1 = {}^{*}\bar{S}_1/{}^{*}S_1$, the simplest procedure to follow is to assume a displacement u and use equations (3) and (4) to find the strains. Equation (5) may be used to express the inner product in the form

$$ {}^{*}S_1^2 = \pi a^2 \int_0^{L/a} [C\varepsilon_\theta^2 + DK_\xi^2]d\xi \tag{78}$$

$$ = \pi \int_0^{L/a} [Cu^2 + D(\beta')^2]d\xi \ .$$

u is assumed to be of the form

$$ u = a/2m \left\{ e^{-m\xi}[-(C_1 + C_2) \cos m\xi + (C_1 - C_2) \sin m\xi] \right\} \tag{79}$$

by analogy with equation (9), where m, C_1, and C_2 are considered to be optimizing parameters. Then β is given by the relation

$$ \beta = e^{-m\xi}(C_1 \cos m\xi + C_2 \sin m\xi) \ , \tag{80}$$

since equations (3) must be satisfied.

For a complete equilibrium state, a suitable function M_ξ can be chosen; then H and N_θ are determined from equations (1). Equations (5) may be used to express the inner product in the form

$$ S^{*2} = \pi a^2 \int_0^{L/a} (N_\theta^2/C + M_\xi^2/D)d\xi \ . \tag{81}$$

By analogy with equation (11), M_ξ is chosen to be of the form

$$M_\xi = mD_r/a\Big\{e^{-m\xi}[(C_2 - C_1)\cos m\xi - (C_1 + C_2)\sin m\xi] \tag{82}$$

$$+ e^{m\xi}[(C_3 + C_4)\cos m\xi + (C_4 - C_3)\sin m\xi]\Big\}.$$

Since $\bar{S}^*$ must satisfy equations (1), H must therefore satisfy the relation

$$H = 2m^2D_r/a^2\Big\{e^{-m\xi}[C_2 \cos m\xi - C_1 \sin m\xi] \tag{83}$$

$$+ e^{m\xi}[-C_4 \cos m\xi + C_3 \sin m\xi]\Big\} ,$$

where

$$D_r = D_{\xi=0} = Eh_r^3/[12(1-\nu^2)]$$

and C_1, C_2, C_3, and C_4 are chosen so that equations (2) are satisfied.

To compute lower bounds, ${}^*S_1^2$ is first computed by substitution of (79) and (80) into (78). This yields

$$^*S^2 = \pi\Big\{C_r a^2/4m^2 \; [(C_1 + C_2)^2 \int_0^{L/a} e^{-(2m+\rho)\xi} \cos^2 m\xi\, d\xi \tag{84}$$

$$-2(C_1^2 - C_2^2) \int_0^{L/a} e^{-(2m+\rho)\xi} \cos m\xi \sin m\xi\, d\xi$$

$$+ (C_1 - C_2)^2 \int_0^{L/a} e^{-(2m+\rho)\xi} \sin^2 m\xi\, d\xi\,]$$

$$+ m^2D_r \; [(C_1 - C_2)^2 \int_0^{L/a} e^{-(2m + 3\rho)\xi} \cos^2 m\xi\, d\xi$$

$$-2(C_2^2 - C_1^2) \int_0^{L/a} e^{-(2m+3\rho)\xi} \sin m\xi \cos m\xi \, d\xi$$

$$+ (C_1 + C_2)^2 \int_0^{L/a} e^{-(2m+3\rho)\xi} \sin^2 m\xi \, d\xi \Big] \Big\}$$

$$= X_1 C_1^2 + X_2 C_1 C_2 + X_3 C_2^2 .$$

By equation (63),

$$^*V_n^2 = (\bar{S}^* \cdot {}^*\bar{S}/{}^*S)^2 \tag{85}$$

$$= [-\pi a(u_0 H_0 + \beta_0 M_0)^2]/{}^*S^2$$

$$= \pi^2 a^2 [-(C_1 + C_2)H_0 + C_1 M_0]^2 / {}^*S^2 .$$

Equations (70) and (85) yield

$$\pi a (C_1 + C_2)^2 / {}^*S^2 \leq C_{uH} , \tag{86}$$

where C_1, C_2, and m may all be used to make the left-hand side of the inequality (86) as large as possible. The optimum value for C_2 with $C_1 = 1$ may be found by setting $C_1 = 1$, differentiating the left-hand side of inequality (86), and solving for the value of C_2 which makes the above derivative vanish. The results of these operations are found to be

$$\begin{aligned} C_1 &= 1, \\ C_2 &= (X_2 - 2X_1)/(X_2 - 2X_3) , \end{aligned} \tag{87}$$

where X_1, X_2, and X_3 are defined by equation (84). By trial and error, the maximizing value of m is found to be given approximately by the relation

$$m = (1 + 0.41\rho/\lambda)\lambda \ , \tag{88}$$

where λ is defined by equation (8).

For the particular case being considered, the results of the above computations are as follows.

$$X_1 = 703.55$$

$$X_2 = -366.02$$

$$X_3 = 564.04$$

$$C_1 = 1$$

$$C_2 = 0.36$$

$$^*S^2 = 1172.92$$

$$0.00013 \leq C_{uH} \tag{89}$$

By using equation (24), the value of C_{uH} for a shell of constant thickness h_r and infinite length is found to be

$$(C_{uH})_{CT} = 0.815 \times 10^{-4} \ ,$$

and hence (89) can be written in the form

$$1.6(C_{uH})_{CT} \leq C_{uH} \ . \tag{90}$$

For $C_{\beta M}$, the analogue to equation (88) is

$$m = (1 - 0.32\rho/\lambda)\lambda \quad . \tag{91}$$

By using this equation and performing calculations similar to those giving a lower bound on C_{uH}, the following results are obtained.

$$X_1 = 703.55$$

$$X_2 = -366.02$$

$$X_3 = 564.04$$

$$C_1 = 1$$

$$C_2 = 1.19$$

$$^*S^2 = 1063.54$$

$$0.0052 \leq C_{\beta M} \tag{92}$$

By equation (23),

$$(C_{\beta M})_{CT} = 0.0027 \ ,$$

and hence inequality (92) can be written in the form

$$1.95(C_{\beta M})_{CT} \leq C_{\beta M} \ . \tag{93}$$

Requiring equations (82) to satisfy the boundary conditions $H_0 = 1$, $M_0 = 0$ and solving for the constants C_1, C_2, C_3, and C_4 yields the following results.

$$C_1 = C_2 = 1.15 \times 10^{-4}$$

$$C_3 = C_4 \doteq 0 \tag{94}$$

Substituting equations (82) and (83) into equation (81) and using the values of the constants given by (94) yields the result

$$S^{*2} = 1.39 \times 10^{-4}\pi a$$

when the approximate minimizing value

$$m = 6.89$$

is used. Upon comparing this result with equation (69) and recalling that $M_0 = 0$ and $H_0 = 1$, it is seen that

$$C_{uH} \leq b_1 = 1.39 \times 10^{-4} . \tag{95}$$

By equation (24),

$$(C_{uH})_{CT} = 8.13 \times 10^{-5} ,$$

and so (95) can be rewritten in the form

$$C_{uH} \leq 1.70(C_{uH})_{CT} . \tag{96}$$

A similar computation with $M_0 = 1$, $H_0 = 0$, and $m = 6.85$ yields the results

$$S^{*2} = 6.75 \times 10^{-3}\pi a \tag{97}$$

and

$$C_{\beta M} \leq b_3 = 6.75 \times 10^{-3} . \tag{98}$$

By equation (23),

$$(C_{\beta M})_{CT} = 2.69 \times 10^{-3} ,$$

and so (89) can be rewritten in the form

$$C_{\beta M} \leq 2.51(C_{\beta M})_{CT} . \tag{99}$$

The following procedure is found to yield good upper and lower bounds on the influence coefficient $C_{uM}(=C_{\beta H})$. The values of a_1, a_2, and a_3 are computed from equation (85) by using the values for C_1 and C_2 given by equation (87) and the value of m given by equation (88). These values are found to be

$$a_1 = 8.477 \times 10^{-5} ,$$

$$a_2 = 6.638 \times 10^{-4} ,$$

$$a_3 = 5.198 \times 10^{-3} . \tag{100}$$

The above values and the upper bounds on C_{uH} and $C_{\beta M}$ given by (95) and (98), when substituted into equation (77), yield

$$6.22 \times 10^{-4} \leq C_{uM}(=C_{\beta H}) \leq 9.240 \times 10^{-4} . \tag{101}$$

The value of C_{uM} for a shell of constant thickness h_r and infinite length is

$$(C_{uM})_{CT} = 3.3 \times 10^{-4} ,$$

and so equation (101) can be written in the form

$$1.89 \begin{Bmatrix} (C_{uM})_{CT} \\ (C_{\beta H})_{CT} \end{Bmatrix} \leq \begin{Bmatrix} C_{uM} \\ C_{\beta H} \end{Bmatrix} \leq 2.81 \begin{Bmatrix} (C_{uM})_{CT} \\ (C_{\beta H})_{CT} \end{Bmatrix} . \qquad (102)$$

Inequalities (90), (93), (95), (98), and (102) can be summarized as follows.

$$\begin{aligned} 1.60(C_{uH})_{CT} &\leq C_{uH} \leq 1.70(C_{uH})_{CT} \\ 1.95(C_{\beta M})_{CT} &\leq C_{\beta M} \leq 2.51(C_{\beta M})_{CT} \\ 1.89(C_{uM})CT &\leq C_{uM} \leq 2.81(C_{uM})_{CT} \\ 1.89(C_{\beta H})_{CT} &\leq C_{\beta H} \leq 2.81(C_{\beta H})_{CT} \end{aligned} \qquad (103)$$

The resolution of these results as compared to results obtained by other methods is discussed in Chapter IV.

CHAPTER IV

RESULTS AND CONCLUSIONS

In this chapter bounds on the influence coefficients obtained by repeatedly applying the procedure of Chapter III are presented graphically and in tabular form and compared with similar results obtained by Sledd [4].

As is shown in Sledd [4] and in the appendix to this paper, the influence coefficients for semi-infinite cylinders depend only on the value of the parameter ρ/λ. Since the results obtained by the method of the hypercircle showed no appreciable effects due to finite length, the approximation was made that the influence coefficients are essentially functions only of the parameter ρ/λ. Confidence in this approximation may be gained by an examination of Table 1, which shows the bounds on the influence coefficients obtained by fixing the parameter ρ/λ and varying L/a.

In the computations yielding the upper bounds on the influence coefficients C_{uH} and $C_{\beta M}$, the parameter m is available as an optimizing parameter. The utilization of this parameter is absolutely necessary in order to obtain good (i.e., small) upper bounds, especially in the higher ranges of the parameter ρ/λ. By trial and error, the approximate minimizing values of m were found to satisfy the equations

$$m = (1 + 1.01\rho/\lambda)\lambda \qquad (104)$$

Table 1. Variation of Bounds on the Influence Coefficients C_{uH} and $C_{\beta M}$ with L/a for Fixed ρ/λ. (Values shown have been divided by the appropriate influence coefficient for a semi-infinite constant-thickness shell of the same reference thickness.)

L/a	$(C_{uH})_{upper}$	$(C_{uH})_{lower}$	$(C_{\beta M})_{upper}$	$(C_{\beta M})_{lower}$
		$\rho/\lambda = 0.25$		
1.0	1.03	1.025	1.03	1.03
2.0	1.03	1.025	1.03	1.03
4.0	1.03	1.025	1.03	1.03
		$\rho/\lambda = 0.49$		
1.0	1.50	1.41	1.91	1.62
2.0	1.50	1.41	1.91	1.62
4.0	1.50	1.41	1.91	1.62
		$\rho/\lambda = 0.74$		
1.0	1.76	1.57	2.68	1.93
2.0	1.76	1.57	2.69	1.93
4.0	1.76	1.57	2.69	1.93
6.0	1.76	1.57	2.69	1.93

and

$$m = (1 + 0.9\rho/\lambda)\lambda$$

for the computation of C_{uH} and $C_{\beta M}$ respectively. Values obtained from equations (104) were used in all the computations of this chapter as well as in the numerical example given in Chapter III.

In the computations yielding lower bounds, all maximizing parameters were chosen by the method outlined in Chapter III.

Table 2 shows the effect of the minimizing parameter m in the calculation of upper bounds by comparing results obtained by choosing m according to equations (104) and the results obtained by choosing $m = \lambda$.

Table 2. Effect of Utilizing the Minimizing Parameter m.

(Values shown have been divided by the corresponding influence coefficients for a semi-infinite shell of constant thickness.)

		$L/a = 3.0$		
p/λ	$(C_{uH})_{m=\lambda}$	$(C_{uH})_m$ by eqns. (104)	$(C_{\beta M})_{m=\lambda}$	$(C_{\beta M})_m$ by eqns. (104)
0.1	1.12	1.1	1.14	1.13
0.6	5.05	1.60	4.95	2.19
0.8	303.6	1.76	191.2	2.89

The results obtained by the method of the hypercircle are summarized in Table 3. This table gives the ratio of the upper and lower bounds for each influence coefficient to the appropriate influence coefficient of a semi-infinite shell of constant thickness equal to the reference thickness of the finite shell of variable thickness. All values were computed for a shell with a ratio of length to meridional radius (L/a) equal to 3.

Table 3. Bounds on the Influence Coefficients C_{uH}, $C_{\beta M}$, and $C_{uM}(=C_{\beta H})$.

$L/a = 3$

ρ/λ	$(C_{uH})_{upper}$	$(C_{uH})_{lower}$	$(C_{\beta M})_{upper}$	$(C_{\beta M})_{lower}$	$(C_{uM})_{upper}$	$(C_{uM})_{lower}$
0.05	1.05	1.05	1.06	1.06	1.06	1.06
0.1	1.10	1.10	1.13	1.12	1.13	1.12
0.2	1.20	1.19	1.29	1.25	1.29	1.24
0.5	1.50	1.44	1.91	1.66	1.91	1.57
0.8	1.81	1.67	2.89	2.11	2.81	1.89
1.0	2.03	1.82	3.84	2.44	3.29	2.09

Figures (8-10) show a comparison of the results obtained in this paper with similar results obtained by Sledd [4]. As can be seen by examining these figures, the results do not differ significantly except in the case of the coefficient C_{uH} (Figure 8), where the results obtained in this paper represent a significant improvement.

Based on the results presented in this chapter the following conclusions can be stated.

1. The method of the hypercircle provides an elegant method for bounding the influence coefficients of shells of varying wall thickness. The results obtained can be expected to be comparable to results obtained by the method of minimum potential and complementary energies as outlined by Sledd [4].

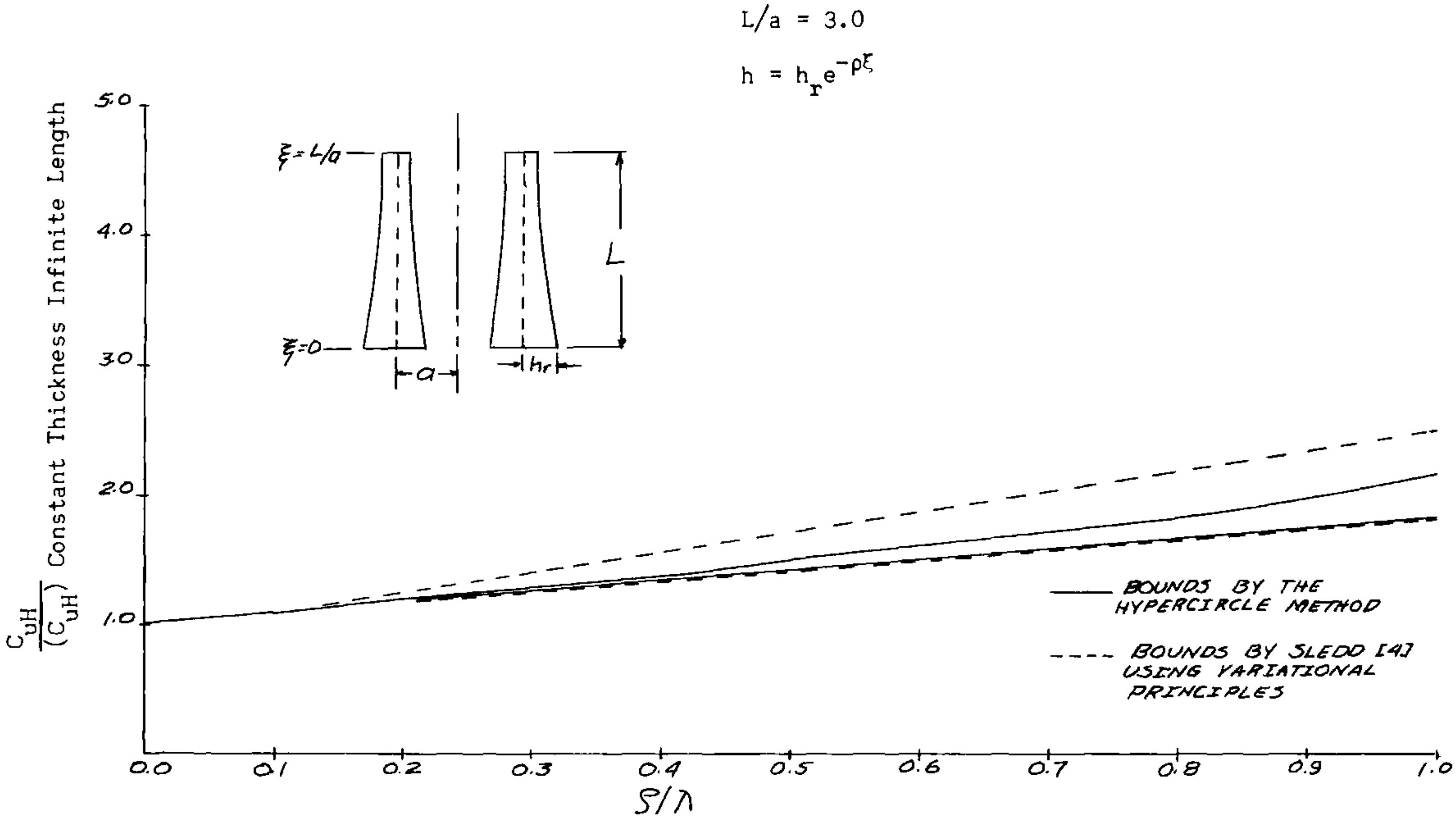


Figure 8. Graph of $C_{uH}/(C_{uH})$ Constant Thickness Infinite Length.

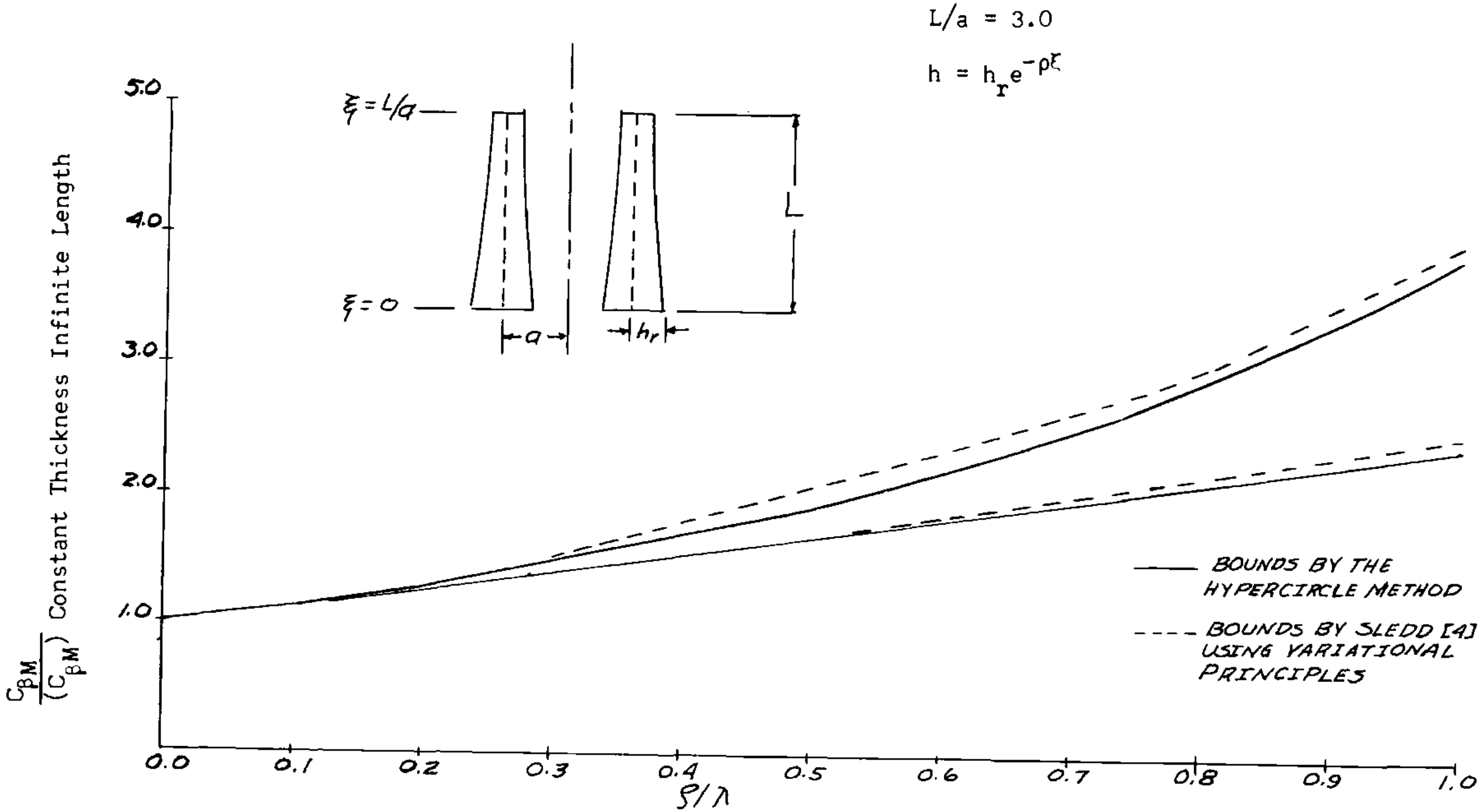


Figure 9. Graph of $C_{\beta M}/(C_{\beta M})$ Constant Thickness Infinite Length.

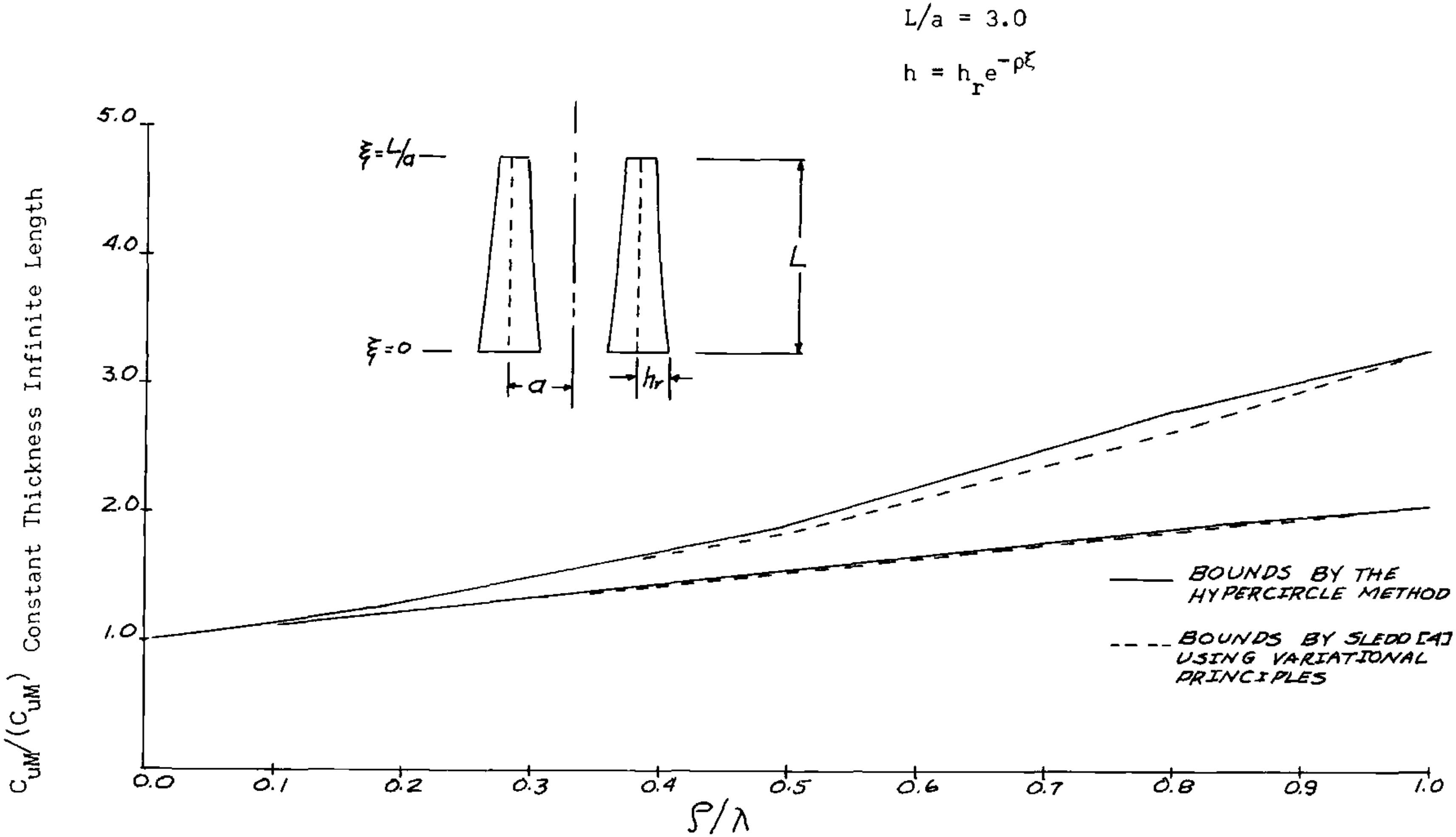


Figure 10. Plot of $C_{uM}/(C_{uM})$ Constant Thickness Infinite Length.

2. Whenever practical, approximating functions should be chosen which contain arbitrary parameters to be used to optimize the results obtained.

3. Influence coefficients of shells of even moderate length-to-radius ratio do not differ significantly from their infinite-length counter-parts. This fact is important computationally because it simplifies the boundary conditions which the approximating functions must satisfy and also simplifies the many integrals which must be evaluated in a computation of the strain energies from which the bounds on the influence coefficients are obtained. From an engineering point of view, it means that for practical application the influence coefficients depend only on the characteristics of that part of the shell in the immediate vicinity of the applied loads and that the influence coefficients of finite shells can be considered to be functions of the parameter ρ/λ alone.

Appendix

Bounds on the Influence Coefficients $C_{\beta M}$ and C_{uM} for a Shell of Infinite Length

In the case $L/a = \infty$, the boundary conditions which must be satisfied by the state $\bar{S}^*$ are

$$M_\xi(0) = M_0 \ ;$$

$$H(0) = H_0 \ ;$$

$$M_\xi \to 0 \text{ as } \xi \to \infty \ ; \qquad \text{A-1}$$

$$H \to 0 \text{ as } \xi \to \infty \ .$$

The functions

$$M_\xi = (mD_r/a)e^{-m\xi}\left[(C_2 - C_1)\cos m\xi - (C_1+C_2) \sin m\xi\right]$$

and

$$H = (am^2D_r/a)e^{-m\xi}(C_2 \cos m\xi - C_1 \sin m\xi)$$

with

$$C_1 = a^2H_0/2m^2D_r - aM_0/mD_r$$

and

$$C_2 = a^2H_0/2m^2D_r$$

define a suitable state $\bar{S}^*$.

The upper bound on $C_{\rho M}$ can be found from the strain energy of $\bar{S}^*$ with $H_0 = 0$ and $M_0 = 1$. In this case,

$$C_1 = -a/mD_r$$

and

$$C_2 = 0.$$

The strain energy of $\bar{S}^*$ is found by using equation (81),

$$\begin{aligned} S^{*2} &= \pi a^2 \int_0^\infty (N_\theta^2/C + M_\xi^2/D)d\xi \\ &= \pi a^2[1/C_r(4m^4/a^2)(1/(\rho - 2m)) \\ &\quad + \frac{1}{C_r}(-8m^4/a^2)m/((\rho-2m)^2 + 4m^2) \\ &\quad + (1/D_r)(1/(3\rho - 2m)) \\ &\quad + (1/D_r)m/((3\rho-2m)^2 + 4m^2)] \quad . \qquad \text{A-2} \end{aligned}$$

It follows from equations (6) and (8) that

$$C_r = 4D_r\lambda^4/a^2 \; ;$$

and by use of this relation equation A-2 may be rewritten in the form

$$\begin{aligned} S^{*2} &= \pi a(a/\lambda D_r)[(m/\lambda)^4 \cdot 1/(\rho/\lambda-2m/\lambda) - 2(m/\lambda)^5 \\ &\quad \cdot 1/((\rho/\lambda - 2m/\lambda)^2 + 4(m/\lambda)^2) + 1/(3\rho/\lambda-2m/\lambda) \\ &\quad + (m/\lambda)/((3\rho/\lambda - 2m/\lambda)^2 + 4(m/\lambda)^2)]. \end{aligned}$$

From equation (23),

$$(C_{\beta M})_{CT} = \frac{a}{\lambda D_r} ;$$

and from inequalities (63) and (69) with $M_0 = 1$ and $H_0 = 0$,

$$C_{\beta M} \leq \frac{S^{*2}}{\pi a} .$$

Thus,

$$\begin{aligned} C_{\beta M}/(C_{\beta M})_{CT} \leq [&(M/\lambda)^4 \cdot 1/(\rho/\lambda - 2m/\lambda) \\ &- 2(m/\lambda)^5 \cdot 1/((\rho/\lambda - 2m/\lambda)^2 \\ &+ 4(m/\lambda)^2) + 1/(3\rho/\lambda - 2m/\lambda) \\ &+ (m/\lambda)/((3\rho/\lambda - 2m/\lambda)^2 + 4(m/\lambda)^2)]. \end{aligned} \qquad \text{A-3}$$

Inequality A-3 gives an upper bound on $C_{\beta M}$ in terms of the parameter ρ/λ and the influence coefficient $(C_{\beta M})_{CT}$. The parameter m is to be assigned the value which makes the bound as small as possible.

Similar calculations with $M_0 = 0$, $H_0 = 1$ yield the following upper bound on the coefficient C_{uH}.

$$\begin{aligned} C_{uH}/(C_{uH})_{CT} \leq [&(m/\lambda)^2(1/(\rho/\lambda - 2m/\lambda)) \\ &+ (\rho/\lambda - 2m/\lambda)/((\rho/\lambda - 2m/\lambda)^2 + 4(m/\lambda)^2) \\ &+ (\lambda/m)^2(1/(3\rho/\lambda - 2m/\lambda)) \\ &- (3\rho/\lambda - 2m/\lambda)/((3\rho/\lambda - 2m/\lambda)^2 \\ &+ 4(m/\lambda)^2)] . \end{aligned} \qquad \text{A-4}$$

To find lower bounds on C_{uH} and $C_{\beta M}$, the state $^*\bar{S}$ may be defined by the relations

$$u = \frac{a}{2m} e^{-m\xi} [(C_1 + C_2) \cos m\xi + (C_2 - C_1) \sin m\xi] ,$$

$$\beta = e^{-m\xi}[C_1 \cos m\xi + C_2 \sin m\xi] \; .$$

The strain energy of ${}^*\bar{S}$ is given by the relation

$$^*S^2 = \pi \int_0^\infty [Cu^2 + D(\beta')^2]d\xi \; .$$

Performing the indicated integrations gives the following expressions for the terms X_1, X_2 and X_3 defined by equation (84).

$$\begin{aligned}
X_1 = {} & D_r\lambda^4/[m^2(2m + \rho)] + m^2D_r/(2m + 3\rho) \\
& - 2\lambda^4D_r/(m[(2m + \rho)^2 + 4m^2]) \\
& + 2m^3D_r/[(2m + 3\rho)^2 + 4m^2] \; ; \\
X_2 = {} & 2D_r(2m + \rho)/[(2m + \rho)^2 + 4m^2] \\
& - 2m^2D_r/[(2m + 3\rho)^2 + 4m^2] \; ; \\
X_3 = {} & D_r/[m^2(2m + \rho)] + m^2D_r/(2m + 3\rho) \\
& + 2\lambda^4D_r/m[(2m + \rho)^2 + 4m^2] \\
& - 2m^3D_r/[(2m + 3\rho)^2 + 4m^2] \; .
\end{aligned} \qquad \text{A-5}$$

The lower bounds can be found from equations A-5 using the procedure outlined in Chapter III.

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