AN ANALYSIS OF THE THERMAL STABILITY OF

THE SOIL ENVIRONMENT OF UNDERGROUND ELECTRICAL CABLES

A THESIS

Presented to

The Faculty of the Division of Graduate Studies

By

James Gary Hartley

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Mechanical Engineering

Georgia Institute of Technology

August, 1977

AN ANALYSIS OF THE THERMAL STABILITY OF THE SOIL

SOIL ENVIRONMENT OF UNDERGROUND ELECTRICAL CABLES

Approved:

_____

W. Z. Black, Chairman

_____

W. F. Ames, School of Math

_____

H. C. Ward, School of Chemical Engineering

_____

R. D. Barksdale, School of Civil Engineering

_____

M. A. Martin, Georgia Power Company

Date approved by Chairman: _8/4/77_

ACKNOWLEDGMENTS

I wish to thank the members of my thesis committee, Dr. W. F. Ames, Dr. R. D. Barksdale, Dr. H. C. Ward, and Mr. M. A. Martin. I would also like to thank Richard Bush and Gerritt Handgraaf for making their experimental results available to me.

Special appreciation is expressed to my thesis advisor Dr. W. Z. Black for suggesting this area of research, for his skilled guidance, and for his interest and support.

I am especially indebted to my wife, Julene, for her encouragement, patience, and understanding.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SUMMARY

Direct burial of underground electrical cables is becoming more and more prominent in the scheme of electrical power transmission and distribution. Such systems represent a large financial investment and in order to fully exploit their current-carrying capacities without threat of thermal failure, accurate predictions of the thermal behavior of the systems are essential.

The purposes of this study are to (1) determine the soil characteristics and cable system parameters which have the most significant influence on the heat transfer and moisture movement in the soil, and (2) attempt to define limits for those parameters in order to insure a thermally stable operation.

The cable-soil system has been modeled by a cylindrical heat source embedded in an infinite homogeneous medium. The thermal and hydraulic properties of two soils, Plainfield sand and "Georgia red clay" (a sandy silt), were used in this study. The two coupled, second-order partial differential equations which describe the heat transfer and moisture movement in the soil were first simplified through an order of magnitude analysis and subsequently further simplified through examination of the influence of the remaining coefficients in numerical simulations.

The equations were solved numerically using an implicit finite difference scheme. Surface heat transfer rates and initial soil conditions were varied. Analytical predictions were checked with experimental results whenever possible.

The model predicts a two-stage process of drying in the vicinity of the heat source. During the first stage the rate of moisture movement away from the heat source decreases until a critical moisture content is reached. Thereafter, the rate of moisture movement increases until complete drying of the soil adjacent to the heat source occurs. The value of the moisture content, X_{crit}, at which the drying enters stage two is essentially independent of the surface heat transfer rate, and is a function of the initial moisture content of the soil for a given porosity. The magnitude of X_{crit} decreases as the initial moisture content decreases.

There appears also to be a critical maximum value of the surface heat transfer rate below which significant drying of the soil will be considerably delayed.

For a soil of fixed porosity, the surface heat transfer rate and the initial moisture content are the most significant parameters affecting the moisture movement. Given a complete set of thermal and hydraulic properties the model can predict reasonably well the time at which significant drying will commence as well as the time at

which complete drying of the soil adjacent to the surface of the source will occur.

The results of comparisons between model predictions and experimental data indicate that the needle-probe method of measuring thermal conductivity might also be used as a method of evaluating the thermal stability limits of a soil.

CHAPTER I

INTRODUCTION

The process of simultaneous heat and mass transfer in soils is an important area of interest mainly because there are many practical examples of heat dissipation in the earth and because so little is known about the complex movement of heat and moisture throughout the earth in the vicinity of a heat source. Several areas in which the transport of heat and mass in soils is important are:

1. Heat dissipation from underground electrical power cables.

2. Heat transfer to and from buried pipelines.

3. Heat transfer to and from heat pump coils.

4. Waste heat rejection from power plant condenser water used to extend the growing season in cold regions.

5. Rejection of heat generated in buried radioactive wastes.

6. Storage of solar energy in underground tanks.

7. Energy conservation in the underground portion of buildings.

8. Recovery of geothermal energy.

Heat transfer in soils is complicated by the presence of solid, liquid and gaseous phases. Since soil is a porous

medium, heat must be conducted through a system of series and parallel paths. Thermal gradients induce moisture movement which, in turn, creates moisture gradients.

The thermal conductivity of soil depends on the soil constituents, the soil porosity and the relative amount of moisture in the soil. At elevated temperatures, the moisture in the soil adjacent to a heat source can be driven away resulting in a zone of low moisture content soil in contact with the source. Since thermal conductivity decreases as the moisture content decreases, the dry zone creates an insulating blanket which further inhibits the dissipation of heat from the source. Thermally induced moisture movement is thereby increased.

The possibility exists that the moisture level in the soil surrounding the source may increase with time due to return flow of water by capillary action. This phenomenon results in an increased ability to transport heat. Such moisture migration would be dependent upon the changes in the hydraulic and thermal properties of the soil. If the moisture content does not increase with time, the capability of the source to dissipate heat through the dry, low thermal conductivity soil is greatly reduced. Being able to identify factors which cause this type of soil behavior is extremely important, particularly when the heat source is required to transfer large amounts of heat. Such would be the case for underground electrical cables and buried heat

pump coils, for example.

The movement of moisture in soils occurs in both the liquid and vapor phases. In a completely saturated soil the movement of liquid is described by Darcy's law and characterized through the saturated hydraulic conductivity. As the moisture content decreases the hydraulic conductivity also decreases and becomes essentially zero at a point where the continuity of the liquid no longer exists. The remaining liquid is held in the soil by strong molecular forces or retained as a hygroscopic layer on the surface of the soil particles. Thus, at moisture contents below that for which liquid continuity is maintained the soil behaves essentially as a dry medium.

At low moisture contents the thermal conductivity of the soil reaches a minimum value and heat dissipation can be severely inhibited. Furthermore, when liquid continuity breaks down rewetting of the soil is much more difficult than when the soil is moist.

The precise conditions under which significant moisture migration will occur are not known, and it is therefore not possible to fully exploit the current-carrying capacities of buried cables with respect to the heat transfer characteristics of soils. Many cases of so-called "thermal runaway" or "thermal instability" have also been reported. In such cases a quick rise of cable temperature occurs due to migration of moisture from the surrounding

soil and the formation of a dry zone around the cable [100].

Some evidence [5] suggests that drying is not usually associated with cables subject to normal cyclic loading. However, it has been reported [3] that cables which have operated normally for considerable lengths of time can suddenly experience rapid increases in temperature due to significant migration of moisture. Furthermore, complete drying may occur at temperatures well below the boiling point of water.

The purpose of this research is twofold. The first phase consists of identifying the soil properties and cable system operating characteristics which have the most significant influence on thermal stability. The second phase consists of correlating the results of the analysis so that the conditions which lead to thermal instability in soils can be predicted.

The buried cable system is modeled by a cylindrical heat source in an infinite, homogeneous, isotropic medium. The mathematical formulation of the problem in cylindrical coordinates has rotational and axial symmetry since the effects of gravity, surface phenomena (e.g. heat flux and moisture movement at the earth's surface) and location of the water table are neglected in the analysis.

In the formulation of the model it is assumed that hysteretic characteristics of the soil are absent. This is

tantamount to assuming monotonic drying of the soil.

The governing equations are nondimensionalized, and an order-of-magnitude analysis is performed to determine those dimensionless groups which have the most significant effect on the heat transfer and moisture movement in the soil surrounding the heated cable.

Because of the complexity of the problem, the transient form of the governing equations is solved by an implicit finite difference scheme using the CDC-7400 computer.

CHAPTER II

DEFINITIONS

The study of heat transfer and moisture movement in soils draws upon the literature of the thermal sciences as well as soil science. Some terms may be unfamiliar in one field or the other or may even have slightly different connotations. Therefore the definitions of certain selected terms are given in this chapter.

1. Capillary Moisture. Water which is held in the soil pore spaces by surface tension forces.

2. Diffusivity. Unless otherwise stated, the term diffusivity usually refers to the isothermal liquid diffusivity; i.e. the product of the unsaturated hydraulic conductivity, K, and the slope of the soil moisture characteristic curve, $\partial \Psi / \partial \theta$. In general, there are four moisture diffusivity terms:

a) $D_{\theta l}$ and $D_{\theta v}$ are isothermal liquid and vapor diffusivities, respectively. They relate the fluxes of liquid and vapor to the gradient of the moisture content.

b) D_{Tl} and D_{Tv} are thermal liquid and vapor diffusivities, respectively. They relate the fluxes of liquid and vapor to the gradient of temperature.

3. Dry Density. The density, ρ_o, of the soil with

all moisture removed.

4. Hygroscopic Coefficient. The soil suction value at which the soil moisture is in equilibrium with an atmosphere of 98 percent relative humidity. At moisture contents below that which corresponds to the hygroscopic coefficient, the soil behaves as a dry medium.

5. Moisture Content. The volumetric moisture content is the volume of water per unit bulk volume of soil. In the literature of the power cable industry, moisture content is usually expressed as a percent of the dry weight or density of the soil. The volumetric moisture content is related to the moisture content by weight through the relation

$$\theta \text{(by volume)} = \text{(dry density/liquid density)} \times$$
moisture content by weight(expressed as a decimal fraction)

6. Porosity. The soil pore volume per unit bulk volume of soil. The soil porosity, ε, is related to the dry density through the relation

$$\varepsilon = 1 - \rho_o / \rho_{solid}$$

where ρ_{solid} is the density of the solid soil particles, usually about 2.65 g/cm^3.

7. Soil Moisture Characteristic. The relationship

between soil moisture suction and moisture content. The characteristic curve is obtained by plotting soil suction against soil moisture content. The soil suction is zero at complete saturation and increases continuously to a very large value when the soil is oven dry.

8. Soil Moisture Potential (Soil Suction). The soil moisture potential, Ψ, usually includes matric, osmotic, and pneumatic potentials (see Total Potential) so that total soil moisture potential is the sum of the gravitational potential and the soil moisture potential, i.e. $\bar{\Phi} = \Psi + \Psi_g$. In the absence of dissolved salts, the osmotic potential is zero, and in most cases air pressure differences are nonexistent or are negligibly small and the moisture potential can be taken to be the matric potential. The terms soil moisture potential and soil suction are sometimes used interchangeably even though the soil moisture potential is a negative quantity and the soil suction is a positive quantity.

9. Total Potential. The total potential, $\bar{\Phi}$, is the work required per unit mass of pure water to remove, reversibly and isothermally, an increment of water from the soil at some equilibrium state. The total potential is divided into four components [8]:

a) Gravitational potential, Ψ_g, which is related to elevation relative to some arbitrary reference;

b) Matric Potential, Ψ_m, which is related to the

adsorption forces between solid surfaces and water, including the effect of cohesive forces between molecules;

c) Osmotic potential, Ψ_0, which is related to the forces of attraction between ions and water molecules; and

d) Pneumatic potential, Ψ_p, which is related to the forces arising from unequal pressures in the gaseous phase.

9. Unsaturated Hydraulic Conductivity. The unsaturated hydraulic conductivity, K, is the proportionality factor in Darcy's law for unsaturated soil which relates moisture flux to the gradient of the total moisture potential, i.e. $\vec{J}_2/\rho_2 = -K(\theta)\nabla\vec{\phi}$. Various names have been proposed for this factor, among them "permeability." Childs [23] suggests the use of "hydraulic conductivity" following the recommendation of the Subcommittee on Permeability and Infiltration of the Soil Science Society of America.)

CHAPTER III

REVIEW OF LITERATURE

Perhaps the earliest experiments directed toward determining the influence of temperature gradients on moisture movement in soils were conducted by Bouyoucos [12]. His findings indicated that there was a moisture content which resulted in a maximum amount of thermally induced moisture movement. Smith [135] verified the existence of a "critical moisture content" and observed that little moisture migration occurred at low or high moisture contents. Instead, the amount of moisture movement reached a maximum value at some intermediate moisture content. Although this characteristic was seemingly independent of the physical condition of the soil, the magnitude of the moisture migration changed considerably as soil conditions were altered. Smith concluded that vapor diffusion was negligibly small and proposed capillary movement induced by vapor condensation as a mechanism of moisture movement. This conclusion was also supported by Philip and de Vries [115], Woodside and Kuzmak [161], and Derjaguin and Melnikova [35].

It is now generally accepted that movement of soil moisture in the liquid phase predominates if liquid

continuity in the soil is maintained, but that vapor transfer is of comparable or greater magnitude when liquid continuity breaks down [57]. The overall process of moisture movement in response to a temperature gradient has been described at length by Milne and Mochlinski [100]:

The movement in the liquid phase is a temperature-controlled stable process caused by the surface tension of water decreasing with increasing temperature, weakening the forces retaining liquid water in the higher-temperature regions and causing it to move away towards the cooler regions. The time constants of the process are controlled by the hydraulic gradient resulting from the temperature gradient and by the permeability of the soil to liquid.

The water-vapor pressure in the pore spaces in the higher-temperature regions exceeds that in the lower-temperature regions. Water vapor therefore moves along the temperature gradient and, evaporating in the higher-temperature region, creates a hydraulic gradient by providing a surplus to hydraulic requirements in the lower-temperature region and a deficit in the higher. The evaporation system may remain in continuous dynamic equilibrium by a constant transfer back of liquid under this hydraulic gradient; or, alternatively, if the ratio of liquid permeability to vapor permeability of the soil is too low, the system may remain unstable, the moisture-movement process ceasing only when all capillary moisture has been removed from those higher-temperature regions in which balance is not achieved. These are thus dried out.

In a given soil, with increase of moisture, permeability to liquid increases, while permeability to vapor decreases. Large interconnected voids unfilled with moisture offer good permeability to vapor diffusion through soil. On the other hand, in soil in which empty voids are saturated with moisture, the vapor movement is restricted and evaporation is not expected to take place at temperatures normally encountered on the cable surface.

In a saturated soil, the vapor-movement process will not take place until the temperature approaches the boiling point of water.

Once the capillary moisture has all been removed,
breaking the capillary channels, the permeability of the
soil to the liquid becomes very low indeed.   In
consequence, even when the source of heat is removed, a
zone once dried out may be a very long time in regaining
its normal moisture content, either by capillary action
or by percolation of gravity water.

Movement of Moisture in Soils

Although there are several proposed explanations for

the various mechanisms of moisture migration, most

researchers agree that Fick's law consistently underpredicts

observed water vapor movement [18,22,45,59,115,122].

Observed rates of movement greatly exceed those predicted by

the molecular diffusion equation modified to take account of

the reduction of diffusion cross-section by soil and liquid

water, and of the tortuous diffusion path in the soil [115].

Philip and de Vries [115] introduced a factor to account for

the difference in average temperature gradient across

air-filled pores in the porous medium.   This resulted in

calculated vapor movements which were from two to eight

times greater than those predicted with the usual diffusion

equation and corresponded well with previous experiments.

This idea of large microscopic thermal gradients across air

spaces is supported by Cary [18], Woodside and Kuzmak [161]

and Hadas [61] and the experimental work of Cassel, et al

[22].

The flow of liquids in saturated porous media is

described by Darcy's law

$$\vec{I}_2 / \rho_2 = -K \, \nabla\tilde{\phi} \qquad (1)$$

where $\tilde{\phi}$ is the total potential. Childs and Collis-George [25] confirmed that Darcy's law is also applicable to unsaturated systems if expressed in a more general form in which the unsaturated hydraulic conductivity, K, is a function of the moisture content. In this case $\tilde{\phi} = \Psi + \Psi_g$ where the energy state of the water is denoted by Ψ. In the literature Ψ is termed either moisture potential, capillary potential, moisture tension, moisture suction, or negative pressure. If Ψ is expressed in units of length, then $g\Psi$ may be associated with the specific Gibbs function of the soil water [114]. The gravitational potential is denoted by Ψ_g.

The moisture potential, Ψ, is generally determined by measuring the suction pressure which is required to remove water from the soil. The curve obtained by plotting the moisture potential against the moisture content is called the "soil moisture characteristic." [22] Such curves exhibit the same general shape for all soils as well hysteretic characteristics. That is, the location of the curve depends upon whether measurements were made during wetting or drying of the soil sample. For the same moisture content, the moisture potential for the drying curve is larger than that for the wetting curve.

At a fixed temperature, Ψ is a well-defined function of moisture content, provided there is monotonic wetting or

drying, for any given soil, but the variation of Ψ with temperature has not been definitely established [108,110,114]. Philip and de Vries [115] describe the variation of Ψ with temperature at a given moisture content through the temperature coefficient of the surface tension of water, but some evidence suggests that the temperature coefficient of Ψ may be higher than can be explained by the influence of surface tension changes alone [48,49,51,73,74].

Analytical Descriptions

Analytical descriptions of the soil moisture phenomenon generally fall into methods involving irreversible thermodynamics or the application of the basic conservation equations. The principles of irreversible thermodynamics [17-21,74,75,108,140] relate the entropy change of the soil system to the rates of the system (i.e. the heat transfer rate, the rate of pressure redistribution, and the rate of moisture movement). Thus the heat and moisture fluxes can be related to the driving forces, or gradients, in the system by certain phenomenological coefficients. This method is particularly suited for steady-state models and has been employed by Cary and Taylor [20], Jury and Miller [73], and Kay and Groenevelt [75].

The analysis of the heat and moisture processes can also be conducted through the use of the basic equations for the conservation of mass, momentum, and energy

[94,112,115,122,148]. The most widely accepted conservation formulation is that of Philip and de Vries [115]. Even though the formulation in its most general form is valid for the transient three-dimensional case, attempts at experimental verification of the model have been limited primarily to the steady-state, one-dimensional case.

Several portions of the two theories have been tested and compared [52,72,123,124,154] and have led to varying conclusions. The results of steady-state experiments by Gee [52] showed that the Philip-de Vries theory underestimated moisture flux by a factor of two to three whereas the Cary-Taylor equation overestimated the flow slightly. Dirksen and Miller [36], who investigated transient moisture movements, found that the results of the Philip-de Vries theory compared reasonably well for the conditions tested while concluding that the Cary-Taylor theory could not adequately describe the behavior of the transient system. Cassel,et al [22] compared the moisture flux equations of the two theories to the data of their steady-state experiments and found that the Philip-de Vries theory yielded values approximately equal to those measured. The Cary-Taylor equation predicted values that were from 10 to 40 times smaller that those observed experimentally.

Application of these models in even the most simplified cases requires a knowledge of various soil properties such as thermal conductivity, the liquid and

vapor diffusion coefficients, the hydraulic conductivity and their dependence on temperature, moisture content, density, and other factors. De Vries [149] has proposed a method of calculating the thermal conductivity and specific heat of soil based on the composition and moisture content. Diffusion coefficients could be determined through a knowledge of the functional relations between soil suction and moisture content and between hydraulic conductivity and moisture content [148]. Cary [18] and Haridasan and Jensen [63] suggest that the hydraulic conductivity of soils should be considered to have the same temperature dependence as the viscosity of water. Farouki [42] has proposed a comprehensive equation for the thermal conductivity of a general soil which takes into account the moisture migration due to heat transfer. Gohar [53] has presented a formula for effective thermal conductivity of soil due to moisture migration. The results of experiments conducted by Jackson [69] led to the conclusion that the temperature dependence of the ratio of surface tension to viscosity appeared to be the dominant factor in describing the temperature dependence of soil-water diffusivity.

The various diffusivities which appear in the governing equations can be calculated once the functional relationships $\Psi(\theta)$ and $K(\theta)$ are known. In the absence of measured values of $K(\theta)$ over the entire range of moisture contents, the unsaturated hydraulic conductivity can be

calculated from a knowledge of the moisture characteristic curve and one measured value of K, usually the value at saturation [13,14,16,23,51,60,70,80,84,99,119]. Thermal conductivity and specific heat of the soil can be determined as functions of temperature and moisture content once the composition and porosity of the soil are known. Several methods of estimating the thermal conductivity of soils can be found in the literature [42,107,133,139,145,149].

The analytical and numerical treatment of the problem of heat transfer in soils has largely neglected the effects of the accompanying thermally induced moisture movement [15,27,47,54,55,66,129,152,153,158]. The conditions which give rise to significant migration of moisture have not been studied in any detail. In past models it has generally been assumed that the soil maintains a constant thermal conductivity but a few analyses have included the possibility of regions of different, but constant, thermal conductivities (e.g. underground cables surrounded by special backfill materials). Although considerable experimental work has been conducted to better define the exact nature of the phenomenon of moisture migration [18,20,21,26,45,59,62,73,81,82,122-124,135,154,161,162], the actual mechanisms which cause the migration of soil moisture, which may occur in both the liquid and vapor phases, have not been precisely determined. The influence of such migration on the thermal conductivity of soil,

however, is well established [7,42,45,73,107,120,149,159, 161], and there is a substantial amount of experimental data and field experience to show that serious moisture migration away from a heat source can occur [2,5,45,100,159,160].

Stability Considerations

Of primary interest to this study are those experimental results [2,5,45,62,130] which suggest that there may be certain "critical" values of the soil properties or system operating characteristics which have significant influence on heat transfer in soils. Hadley and Eisenstadt [62] reported a moisture content of four percent, below which drying would be observed in the two different soil media studied. On the basis of experimental work with underground cable systems Arman, et al [5] concluded that there is a certain maximum cable surface temperature above which dangerous migration of moisture may occur possibly causing complete drying of the surrounding soil. Bauer and Nease [7] concluded that while the drying rate depends somewhat on the surface heat flux, the factor of primary importance is initial moisture content. Milne and Mochlinski [100] and Radhakrishna [117] made similar observations. Adams and Baljet [2] reported that for all soils, including clays and sands, no migration will occur above a certain critical moisture content. Schmill [130], from a rather simplified analytical treatment, suggested

that a critical temperature or a critical heat flux should exist below which a buried cable system would be thermally stable. However, Fink [43] has indicated that a limiting cable surface temperature does not exist below which moisture migration will not take place.

Experimental studies have been carried out in an effort to establish the relation between thermal conductivity and soil moisture content and moisture movement under thermal gradients. These studies have been conducted primarily on steady-state models with an imposed linear temperature gradient [18,20,21,59,62,73,76,88,122,154]. It has not been verified that results from such models apply to different gradients or temperature levels.

CHAPTER IV

SOIL PROPERTIES

The basic soil properties needed to describe the heat transfer and moisture movement in a soil include the thermal conductivity, the hydraulic conductivity, the soil moisture characteristic, and the specific heat. Even though this study is not devoted to the measurement of the soil properties, a discussion of their general characteristics and their dependence on temperature and moisture content is necessary at this point.

A survey of the published literature reveals that the soil property data is not readily available, and it is unlikely that all of the necessary data for even one soil could be compiled from the literature. It therefore becomes necessary to rely upon empirical correlations and theoretical analyses to extend the available data. The purpose of this section is to outline what procedures are available and to discuss possible alternatives.

Moisture Characteristic

Several methods of measuring the soil moisture potential as a function of moisture content θ have been developed [108]. Such measurements provide the data necessary to construct the soil moisture characteristic

curve, $\Psi(\theta)$. In order to use the results in numerical or analytical studies it is convenient to fit the data to some empirical equation, preferably an equation which works well not only for a specific soil but also for a number of soils.

Toward this end many forms of equations for $\Psi(\theta)$ have been proposed [6,13,78,127], some of them requiring complicated calculation procedures to extract values for the constants which characterize the behavior of a specific soil. The characteristic shape of the $\Psi(\theta)$ curve suggests that any proposed equation should satisfy two basic requirements. They are

(1) $\Psi \to \infty$ as $\theta \to 0$ or alternatively as $S \to S_{min}$ where S is the relative saturation θ/ϵ.

(2) $\Psi \to 0$ as $\theta \to \theta_{sat}$ or as $S \to 1$

Rubin, et al [127] used the equation

$$\Psi = c_1 + c_2/\theta + c_3 EXP(c_4\theta) + EXP(c_5\theta + c_6)$$

which can be made to satisfy the stated conditions with proper selection of the six unknown constant coefficients.

In 1964, Brooks and Corey [13] proposed a relationship of the following form

$$S_e = \begin{cases} (\Psi_b/\Psi)^\lambda & \text{for } \Psi > \Psi_b \\ 1 & \text{for } \Psi < \Psi_b \end{cases} \tag{2}$$

where $S_e = (S-S_r)/(1-S_r)$ is called the effective saturation, Ψ_b the bubbling pressure, and λ the pore-size distribution index.

As $\Psi \to \infty$ in the above expression, $S_e \to 0$ so that the saturation, S, approaches a finite value, S_r, termed the residual saturation. For $\Psi < \Psi_b$, $S_e = 1$ or $S = 1$. The values of λ and Ψ_b would be determined by plotting $Log(S_e)$ versus $\Psi/\rho_2 c_2$.

Brooks and Corey found that such a curve has a negative slope over a very large range of S_e. By extending the straight-line portion to $S_e = 1$, the value of the intercept is taken to be $\Psi_b/\rho_2 c_2$.

The residual saturation, S_r, needed in the definition of S_e is determined by using a graphical technique requiring a trial and error approach. The resulting power-law expression for S_e works best for intermediate moisture contents and may not adequately describe the moisture characteristic at very low moisture contents, i.e. very high values of Ψ.

King [78] proposed the use of an equation written here as

$$S = \delta \left[\frac{Cosh[(\Psi/\Psi_0)^\beta + \varepsilon] - \gamma}{Cosh[(\Psi/\Psi_0)^\beta + \varepsilon] + \gamma} \right] \qquad (3)$$

where Ψ_0, β, γ, δ and ε are constants for a particular soil

such that $\Psi_o > 0$, $\beta < 0$, $0 < \gamma \leqslant \cosh \epsilon$, $0 < \delta \leqslant 1$ and $\epsilon \geqslant 0$. As approaches zero, S approaches and

$$\lim_{\Psi \to \infty} S = \delta(\cosh \epsilon - \gamma)/(\cosh \epsilon + \gamma) = S_{min}$$

Therefore $S \leqslant \delta$ and S_{min} is similar to the residual saturation of Brooks and Corey [13]. While providing a rather good fit over the entire range of moisture contents, fitting experimental data to this equation is complicated. As King [78] stated, "The only good estimate is for δ which is 1 for drainage from complete saturation...By adopting $\epsilon = 0$ as a first estimate, the initial fitting must still be accomplished in a three-parameter space."

Ahuja and Swartzendruber [6] developed a form of $\Psi(\theta)$ by first proposing for $D_{\theta \ell}$ the equation

$$D_{\theta \ell} = a\theta^n/(\theta_c - \theta)^m \qquad (4)$$

where a, n, m and θ_c are constants. Employing a power-law form for the hydraulic conductivity

$$K(\theta) = B\theta^N \quad ; \quad B,N \quad \text{constants}$$

they obtained

$$d\Psi/d\theta = a/[B\theta^{n-m}(\theta_c - \theta)^m] \qquad (5)$$

The values of the constants were restricted such that N=n>0 and 0<m<1. Furthermore, by assuming that N=n+m=2 it was possible to integrate the above expression to obtain

$$\Psi(\theta) = [a/(B\theta_c(1-m))] [(\theta_c-\theta)/\theta]^{1-m} \qquad (6)$$

where $\theta=\theta_c$ when $\Psi=0$.

With this definition of θ_c, the equation may be written

$$\Psi(S) = A[(1-S)/S]^{1-m} \qquad (7)$$

which has the desirable feature that there are only two constants which must be determined from the experimental data. This form also satisfies the two basic requirements for a $\Psi(\theta)$ or $\Psi(S)$ curve stated previously. However, the author's experience with equations of this form has been that a single equation can seldom adequately describe the moisture characteristic over the entire range of moisture contents.

The moisture characteristics of most soils have some features in common. First, the moisture potential Ψ becomes very large, not at zero saturation, but at some finite small value of S, called the residual saturation, S_r, by Brooks and Corey [13]. Secondly, the moisture potential approaches zero as S approaches unity. And finally, in the range of

moisture contents $S_r < S < 1$. the slope of the moisture characteristic decreases sharply and thereafter increases as Ψ approaches zero at complete saturation ($S=1$).

Thus it is convenient to choose $\Psi = \Psi(S_e)$ rather than $\Psi = \Psi(S)$ where S_e is the effective saturation defined by Brooks and Corey [13]. The author proposes the following new form for $\Psi(S_e)$

$$\Psi(S_e) = aS_e^b(1-S_e)^c[\coth(\pi S_e)]^d \qquad (8)$$

where a, b, c, and d are constants.

Although the form of the expression appears complicated, especially since it contains five unknowns (a, b, c, d, S_r), it is particularly well suited to a curve fit of data by the method of least squares [137]. In this instance the least squares process requires only the solution of four simultaneous linear algebraic equations. The application of equation (8) is explained fully in Appendix A.

In addition to being straightforward in its application, the author has found that equation (8) can adequately describe many moisture characteristics over the entire range of moisture contents.

The slope of the moisture characteristic is called the differential (or specific) water capacity [57]

$$C_\theta = d\theta/d\Psi$$

If equation (8) is used to describe the moisture characteristic the reciprocal of the specific water capacity is

$$d\Psi/d\theta = [a\Psi/(1-S_r)][b/S_e - c/(1-S_e) - 2\pi d/Sinh(2\pi S_e)] \quad (9)$$

Hydraulic Conductivity

The unsaturated hydraulic conductivity (hereinafter called simply hydraulic conductivity) is also sometimes termed the permeability of the medium. It is the coefficient in Darcy's law for unsaturated flow [121] which, when multiplied by the gradient of the total moisture potential, gives the flux of liquid through the porous medium.

$$\vec{J}_2/\rho_2 = -K(\Psi)\nabla\bar{\Phi}$$

The "specific" permeability ordinarily used in petroleum engineering is related to the hydraulic conductivity by [90]

$$K = (\rho_\ell/\mu) K'$$

where ρ_ℓ is the density of the liquid, μ is the viscosity of

the liquid, and K' is the specific permeability (a function of the medium only and not of the flowing liquid).

The hydraulic conductivity is highly hysteretic when expressed as a function of the moisture potential, $K(\Psi)$, but if expressed as a function of the moisture content, $K(\theta)$, it is less affected by hysteresis [142]. The problems encountered with the former may, however, be avoided by limiting consideration to essentially monotonic wetting or drying of the medium [65].

As in the case of the moisture characteristic curve, many expressions have been proposed for use in relating hydraulic conductivity to either moisture content or soil moisture potential.

King [78] proposed the relation

$$K = \sigma K_s \left\{ Cosh[(\Psi/\Psi_2)^\eta] - 1 \right\} / \left\{ Cosh[(\Psi/\Psi_2)^\eta] + 1 \right\} \qquad (10)$$

where Ψ_2, η and σ are constants and $\eta < 0$, $\Psi_2 > 0$, $0 < \sigma \leqslant 1$. Evaluation of the constants is difficult in practice.

The shape of the curve represented by

$$K = K_s / [(\Psi/a)^b + 1] \qquad (11)$$

proposed by Gardner [50] is nearly identical with that of King's equation [78] above, but this expression is much easier to fit to experimental data. Furthermore, this form

adequately fits much measured hydraulic conductivity-soil potential data [82].

Other forms include

$$K = K_s / (a\Psi^b + 1)^c$$

used by Ahuja and Swartzendruber [6] and

$$K = K_s (\Psi_b / \Psi)^\eta$$

proposed by Brooks and Corey [13]. Here, η is related to λ of equation (2) by $\eta = 2 + 3\lambda$.

Measurements of hydraulic conductivity over a wide range of moisture contents is difficult and time consuming and theories have been developed which enable one to calculate hydraulic conductivity values from known moisture retention data.

Jackson, et al [70] have evaluated three methods of calculating hydraulic conductivities from a soil's moisture characteristic. They found that the method of Childs and Collis-George [25], even as modified by Marshall [91], did not predict the proper shape of the hydraulic conductivity curve. The method of Millington and Quirk [98,99] did predict the correct shape of the curve but did not reproduce actual measured values of hydraulic conductivity. When a matching factor was used the method gave good results over a

very wide range of moisture contents, even those at which measurements are often very difficult to obtain. The matching factor is obtained by matching calculated and measured values of hydraulic conductivity at saturation. This method, then, is called the modified Millington-Quirk method.

Bruce [14] made comparisons of the above three methods as well as the method of Laliberte, et al [84]. His results generally confirmed the findings of Jackson, et al [70]. However, the results of his calculations showed that the modified Millington-Quirk method gave better results when matched with a measured hydraulic conductivity in the 0.1 to 0.3 bar range of soil suction rather than with the value at saturation.

Gardner [51] explained the limitations inherent in the capillary models used to predict the hydraulic conductivity from soil moisture characteristics but conceded that they do give a reasonable shape for the hydraulic conductivity curve and that the methods are still some of the best available.

The modified Millington-Quirk method was selected for use in this study. Details of the procedure are contained in Appendix B.

Measured moisture retention data are fitted to equation (8) and hydraulic conductivity values are calculated using the results of the curve fit and the

modified Millington-Quirk method. Finally, the conductivity data are fitted to equation (11) for use in the analytical phase of the study.

Specific Heat

The specific heat of moist soil, neglecting the contribution due to the presence of dry air, is calculated from [149]

$$C = C_0 + c_2 \rho_2 \theta \qquad cal/cm^3 - {}^o C \qquad (12)$$

where C_0 is the specific heat of the dry soil, defined by

$$C_0 = 0.46x_m + 0.6x_0 \qquad cal/cm^3 - {}^o C$$

and x_m and x_0 are the volume fractions of soil minerals and of organic materials, respectively.

Thermal Conductivity

The effective thermal conductivity which appears in the conservation of energy equation (see Chapter V) is the sum of two components [109,148,149]

$$\lambda = \lambda_* + \lambda_v$$

The first component is usual thermal conductivity, λ_*, of the moist medium and the second is an apparent

thermal conductivity, λ_v, which is due to vapor movement.

The definition of λ_v is given in Chapter V. Our concern here is the thermal conductivity λ_*.

Sinclair, et al [133] give a review of several empirical and theoretical relations for λ_* that are available in the literature. Among those mentioned for moist soils are Mickley's equation [95], Gemant's equation [133], the Kersten formula [75b], and the Van Rooyen formula [145]. Sinclair, et al state that their best agreement with experimental data was obtained with the Kersten and Van Rooyen formulas. Most cable engineers prefer the Kersten model because of its relative simplicity, even though the Van Rooyen formula produced the best overall results.

The Van Rooyen formula [145], used in this study, expresses the soil thermal resistivity as

$$1/\lambda_* = 10^{A-Bm}+S \qquad {}^\circ C\text{-}cm/W \qquad (13)$$

where m is the moisture content as a fraction of the voids filled with water, B and S are linear functions of the density of the dry soil, ρ_o, and A is a quadric function of ρ_o. Furthermore, A, B, and S contain constants which are functions of the soil composition and granulometry. A detailed description of the Van Rooyen formula is contained in Appendix C.

Property Data

Two soils, a sandy soil and a sandy silt, were modeled in this study. The Plainfield sand used by Jury [73] was chosen for the sandy soil primarily because a complete set of thermal and hydraulic properties were available for it. The property data for this soil were taken from curves presented by Jury and are shown in dimensionless form in Figure 1.

A soil commonly called "Georgia red clay" was used as the sample of the sandy silt. Thermal and hydraulic properties of this soil were measured by Bush [14a] and are shown in dimensionless form in Figure 2.

Other pertinent characteristics of the two soils are summarized in Table 1.
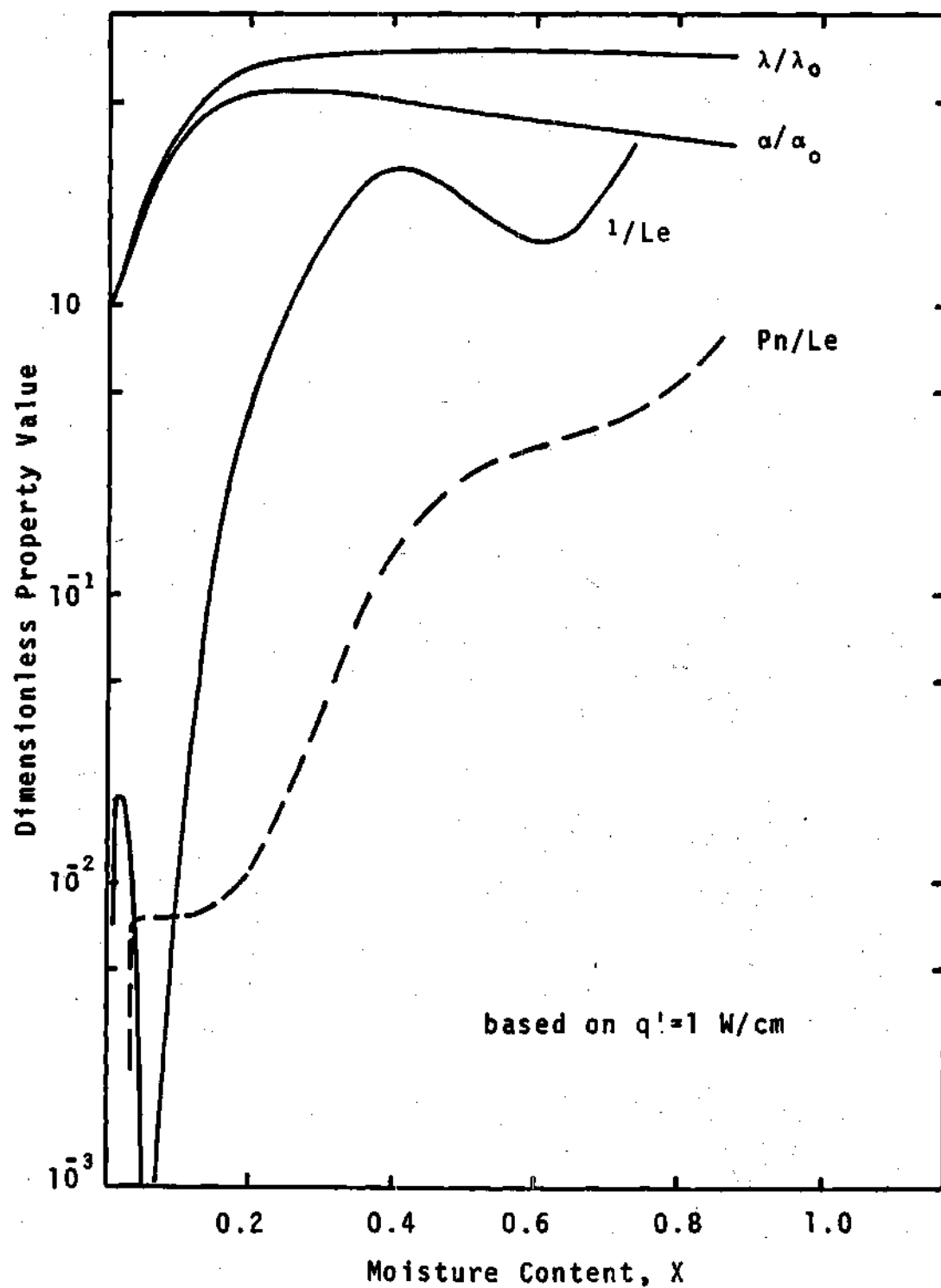
Figure 1. Dimensionless Thermal and Hydraulic
Properties of Plainfield Sand
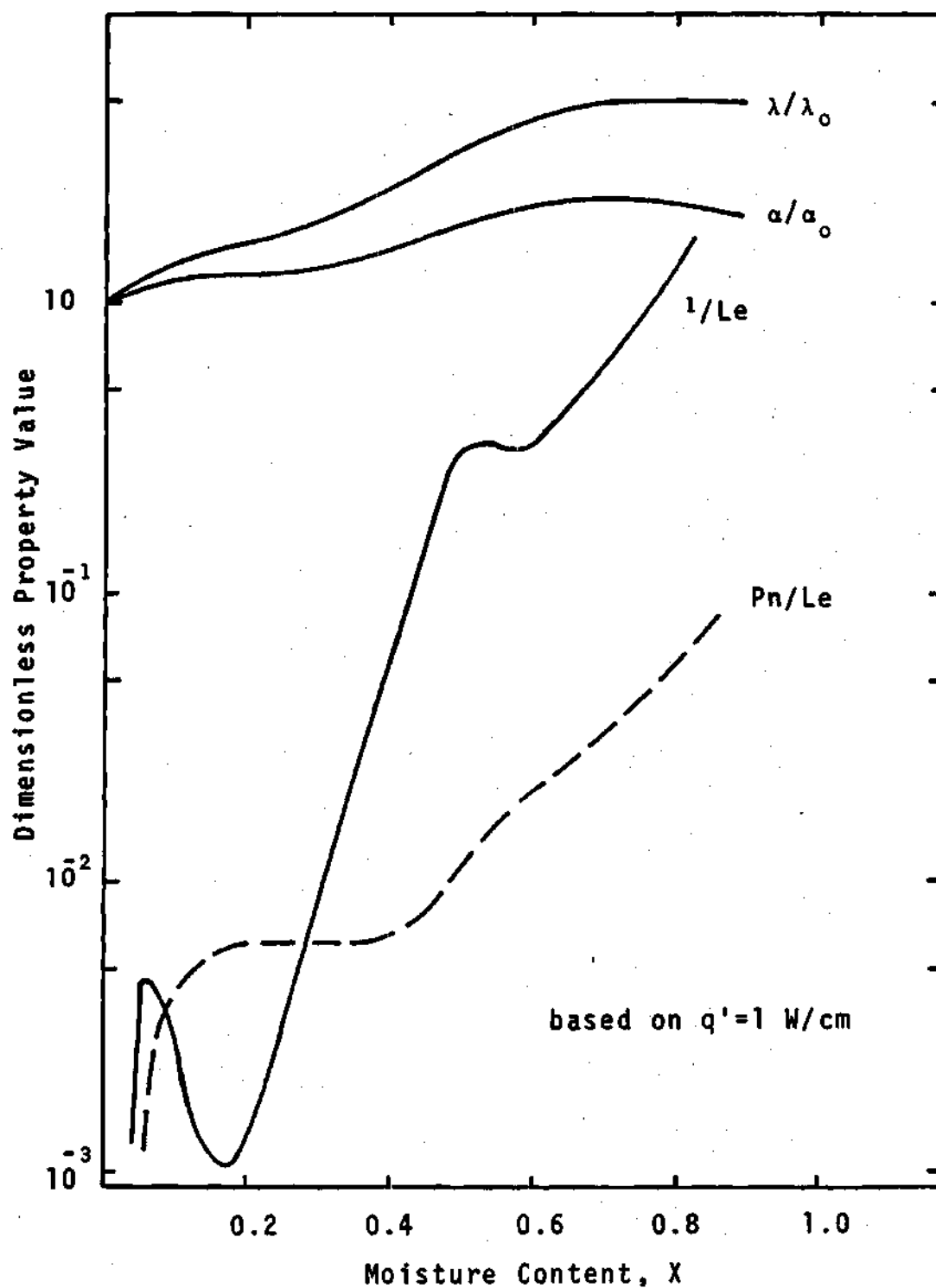
Figure 2. Dimensionless Thermal and Hydraulic
Properties of Sandy Silt

Table 1. Characteristics of Soils Used in Analytical Study

Quantity	Plainfield Sand	Red Sandy Silt
λ_o W/cm-^{o}C	4.47×10^{-3}	4.42×10^{-3}
c_o J/cm^3-^{o}C	1.25	0.943
ε	0.35	0.51
K_s cm/s$_3$	2.43×10^{-3}	3.0×10^{-3}
ρ_o g/cm	1.67	1.28

CHAPTER V

GOVERNING EQUATIONS FOR HEAT TRANSFER AND MOISTURE MOVEMENT IN SOILS

In this section the equations which govern the heat transfer and moisture movement in soils are considered. The analysis begins with statements of the basic laws of conservation of mass and conservation of energy.

Conservation of Mass

The conservation of mass equation for a porous medium as developed by Luikov [85-89] is

$$\partial(\rho_2\theta_i)/\partial t = -\nabla\cdot\vec{J}_i + I_i, \qquad i=1,2,3 \qquad (14)$$

Here, $\vec{J}_i$ is the flux of component i based on a mean velocity, the mean being taken over a volume containing many grains of the porous medium. In what follows the subscript $i=1$ refers to water in the vapor phase, $i=2$ refers to the liquid phase, $i=3$ refers to air and the subscript zero refers to the porous medium (soil) in its dry state. The presence of ice is not considered here.

The symbol I_i designates the rate of conversion of component i. The dry air is considered to be inert and no chemical conversion occurs so that $I_3 = 0$. The only other

phase conversions considered are from liquid to vapor or from vapor to liquid and hence $I_2 = -I_1$.

In the literature of soil science the moisture content is generally expressed in terms of the volumetric moisture content, θ. The volumetric liquid content θ_2, is

$$\theta_2 = (\text{volume occupied by liquid})/(\text{total volume}) = V_2/V.$$

The volumetric vapor content, θ_1, is the volume of "precipitable" water [148] per unit volume,

$$\theta_1 = (\text{volume of vapor "as liquid"})/(\text{total volume})$$
$$= V_1(\rho_1/\rho_2)/V$$

For the range of temperatures encountered the changes in the density of the liquid are small and the liquid density may be taken as constant. Defining $E = I_1/\rho_2$ the conservation of mass equations may be written

$$\partial\theta_1/\partial t = -\nabla\cdot(\vec{j}_1/\rho_2) + E$$

and

$$\partial\theta_2/\partial t = -\nabla\cdot(\vec{j}_2/\rho_2) - E$$

Adding these two equations, the conservation of mass

equation for soil moisture becomes

$$\partial(\theta_1 + \theta_2)/\partial t = -\nabla \cdot [(\vec{j}_1 + \vec{j}_2)/\rho_2]$$

or

$$\partial\theta/\partial t = -\nabla \cdot (\vec{j}_m/\rho_2) \tag{15}$$

where $\theta = \theta_1 + \theta_2$ and $\vec{j}_m = \vec{j}_1 + \vec{j}_2$.

Conservation of Energy

The conservation of energy equation for the porous medium is

$$\partial U/\partial t = -\nabla \cdot \vec{q}''' \tag{16}$$

The internal energy per unit volume, U, is

$$U = \rho_o u_o + \rho_2 \theta_1 u_1 + \rho_2 \theta_2 u_2 \tag{17}$$

$$= \rho_o u_o + \rho_1 (\epsilon - \theta_2) u_1 + \rho_2 \theta_2 u_2$$

and the heat flux $\vec{q}'''$ is given by

$$\vec{q}''' = -\lambda_* \nabla T + h_1 \vec{j}_1 + h_2 \vec{j}_2 \tag{18}$$

Here the first term on the right hand side of (18)

represents the heat transfer by conduction, λ_* being the thermal conductivity for the case of no moisture movement. The second and third terms represent the fluxes of enthalpy due to movement of water vapor and liquid water, respectively.

In equations (17) and (18) the contributions due to the presence of air have been neglected.

Substituting (17) and (18) into (16) and expanding, one obtains

$$C\partial T/\partial t + (\rho_2 u_2 - \rho_1 u_1)\partial\theta_2/\partial t = \nabla\cdot(\lambda_*\nabla T) - h_1\nabla\cdot\vec{j}_1 \quad (19)$$

$$- h_2\nabla\cdot\vec{j}_2 - \vec{j}_1\cdot\nabla h_1 - \vec{j}_2\cdot\nabla h_2$$

where the volumetric heat capacity of the moist soil, C, has been defined as

$$C = \rho_0 c_{v_0} + \rho_2\theta_1 c_{v1} + \rho_2\theta_2 c_{v2}$$

Upon adding and subtracting the term $h_2\nabla\cdot\vec{j}_1$ on the right hand side of the energy equation, defining the enthalpy of vaporization,

$$h_{21} = h_1 - h_2$$

and using the definition of $\vec{j}_m$, one obtains

$$C \partial T/\partial t + (\rho_2 u_2 - \rho_1 u_1)\partial\theta_2/\partial t = \nabla\cdot(\lambda_* \nabla T) - h_{21}\nabla\cdot\vec{j}_1$$

$$- h_2\nabla\cdot\vec{j}_m - \vec{j}_1\cdot\nabla h_1 - \vec{j}_2\cdot\nabla h_2$$

The second term on the right hand side may be rearranged as

$$h_{21}\nabla\cdot\vec{j}_1 = \nabla\cdot(h_{21}\vec{j}_1) - \vec{j}_1\cdot\nabla h_{21}$$

$$= \nabla\cdot(h_{21}\vec{j}_1) - \vec{j}_1\cdot\nabla h_1 + \vec{j}_1\cdot\nabla h_2$$

Using this result the energy equation becomes

$$C \partial T/\partial t + (\rho_2 u_2 - \rho_1 u_1)\partial\theta_2/\partial t = \nabla\cdot(\lambda_* \nabla T)$$

$$- \nabla\cdot(h_{21}\vec{j}_1) - h_2\nabla\cdot\vec{j}_m - \vec{j}_m\cdot\nabla h_2$$

Upon substituting equation (14) and noting that $h_2 \doteq u_2$, the energy equation may be further simplified to

$$C \partial T/\partial t - \rho_1 u_1 \partial\theta_2/\partial t - \rho_2 u_2 \partial\theta_1/\partial t$$

$$= \nabla\cdot(\lambda_* \nabla T) - \nabla\cdot(h_{21}\vec{j}_1) - c_2\vec{j}_m\cdot\nabla T$$

Since $\rho_1(\epsilon-\theta_2) = \rho_2\theta_1$ the energy equation becomes

$$C \partial T/\partial t + \rho_2 u_{21}\partial\theta_1/\partial t = \nabla\cdot(\lambda_* \nabla T) - \nabla\cdot(h_{21}\vec{j}_1) - c_2\vec{j}_m\cdot\nabla T \quad (20)$$

The term $\partial\theta_1/\partial t$ may be replaced by terms involving $\partial\theta_2/\partial t$ and $\partial T/\partial t$ in the following manner.

Assuming that the vapor and liquid phases are in equilibrium,

$$\theta_1 = \rho_1(\epsilon-\theta_2)/\rho_2 = \rho_s h(\epsilon-\theta_2)/\rho_2$$

where ρ_s is the density of saturated water vapor and h is the relative humidity of the air-water vapor mixture in the soil pore spaces. Taking the time derivative of θ_1,

$$\partial\theta_1/\partial t = - (\rho_s h/\rho_2)\partial\theta_2/\partial t + [h(\epsilon-\theta_2)/\rho_2](d\rho_s/dt)\partial T/\partial t$$

$$+ [\rho_s(\epsilon-\theta_2)/\rho_2](dh/d\theta_2)\partial\theta_2/\partial t$$

since $\partial h/\partial T$ may be taken to be zero (see section entitled Vapor Flux). Also, since

$$dh/d\theta_2 = (hg/RT)\,\partial\Psi/\partial\theta$$

the above equation becomes

$$\partial\theta_1/\partial t = \{[(\epsilon-\theta_2)\rho_s hg/\rho_2 RT]\partial\Psi/\partial\theta) - \rho_s h/\rho_2\}\partial\theta_2/\partial t$$

$$+ [h(\epsilon-\theta_2)/\rho_2](d\rho_s/dT)\partial T/\partial t$$

and therefore

$$\partial\theta/\partial t = \partial\theta_1/\partial t + \partial\theta_2/\partial t = [1+[(\epsilon-\theta_2)\rho_s hg/(\rho_2 RT)]\partial\Psi/\partial\theta_2 \quad (21)$$

$$- \rho_s h/\rho_2]\partial\theta_2/\partial t + [h(\epsilon-\theta_2)/\rho_2](d\rho_s/dT)\partial T/\partial t$$

The energy equation may be rewritten as

$$[C + u_{21}h(\epsilon-\theta_2)(d\rho_s/dT)]\partial T/\partial t \quad\quad (22)$$

$$+ \{[u_{21}(\epsilon-\theta_2)\rho_s hg/(\rho_2 RT)\partial\Psi/\partial\theta \quad - u_{21}\rho_s h]\partial\theta_2/\partial t$$

$$=\nabla\cdot(\lambda_* \nabla T) - \nabla\cdot(h_{21}\vec{J}_1) - c_{p2}\vec{J}_m \cdot\nabla T$$

Liquid Flux

The flux of liquid in a porous medium is governed by Darcy's law

$$\vec{J}_2 = -\rho_2 K\nabla\bar{\Phi} \quad\quad\quad (23)$$

where K is the unsaturated hydraulic conductivity and $\bar{\Phi}$ is the total soil potential. The total soil potential, being the sum of a pressure component, Ψ, and a gravitational component, may be written $\bar{\Phi} = \Psi$ when the effects of gravity are negligible.

The hydraulic conductivity and the soil potential are functions of temperature and moisture content. Therefore it is desirable to rewrite (23) as the sum of fluxes due to gradients of temperature and moisture content by expanding

$\nabla \vec{\phi} = \nabla \Psi$ by the chain rule.

$$\vec{J}_2/\rho_2 = -K\partial\Psi/\partial\theta)_T \nabla\theta - K\partial\Psi/\partial T)_\theta \nabla T$$

In the notation of Klute [80] and Philip [112,113],

$$\vec{J}_2/\rho_2 = -D_{\theta\ell}\nabla\theta - D_{T\ell}\nabla T \qquad (24)$$

where $\vec{J}_2$ is the liquid flux and the diffusivity terms are

$D_{\theta\ell}$ = Isothermal liquid diffusivity = $K\partial\Psi/\partial\theta)_T$

$D_{T\ell}$ = Thermal liquid diffusivity = $K\partial\Psi/\partial T)_\theta$

Vapor Flux

The appropriate form of Fick's law, which describes the flux of vapor in the porous medium, is [10]

$$\vec{J}_1 = -\rho D_{13}\nabla(\rho_1/\rho) \qquad (25)$$

Here, ρ_1 and ρ are the density of the water vapor and the water vapor-air mixture, respectively, and D_{13} is the molecular diffusivity of water vapor in air modified so that it applies to the porous medium.

Expanding (25) and using the fact that $\rho = \rho_3 + \rho_1$, where ρ_3 is the density of the air,

$$\vec{J}_1 = -D_{13}[(\rho_3/\rho)\nabla\rho_1 - (\rho_1/\rho)\nabla\rho_3]$$

Assuming that the air, water vapor, and air-water vapor mixture behave as ideal gases, we can write

$$\nabla \rho_3 = \nabla(P_3 M_3/RT) = (M_3/R)[(1/T)\nabla P_3 - (P_3/T^2)\nabla T]$$

Furthermore, since the total pressure, $P = P_1 + P_3$, remains constant, $\nabla P_3 = -\nabla P_1$, so that

$$\nabla \rho_3 = (M_3/R)[-(1/T)\nabla P_1 - (P_3/T^2)\nabla T]$$

But

$$\nabla P_1 = (R/M_1)\nabla(\rho_1 T) = (R/M_1)[\rho_1\nabla T + T\nabla\rho_1]$$

So that

$$\nabla \rho_3 = (M_3/R)[-(R/M_1 T)(\rho_1\nabla T + T\nabla\rho_1) - (P_3/T^2)\nabla T] \qquad (26)$$

$$= - (M_3/M_1)\nabla\rho_1 - (PM_3/RT^2)\nabla T$$

Substituting this result, equation (11) becomes

$$\vec{j}_1 = -D_{13}[[(\rho_3 M_1 + \rho_1 M_3)/\rho M_1]\nabla\rho_1 + (\rho_1 M_3 P/\rho RT^2)\nabla T]$$

But $1/M = (M_1\rho_3 + M_3\rho_1)/\rho M_1 M_3$, therefore,

$$\vec{J}_1 = -(D_{13}M_3/M)[\nabla\rho_1 + (\rho_1/T)\nabla T] = -(D_{13}M_3/MT)\nabla(\rho_1 T) \quad (27)$$

The difficulty associated with using (27) is that the molecular weight of the air-water vapor mixture varies as the composition of the mixture varies. M may be eliminated by noting that

$$1/M = (P_3+P_1)/(P_3M_3+P_1M_1) \cong v/M_3$$

Eliminating M from (27),

$$\vec{J}_1 = -(D_{13}v/T)\nabla(\rho_1 T)$$

The density of the water vapor may be written as

$$\rho_1 = h\rho_S \quad (28)$$

where ρ_S is the density of saturated water vapor, a function of temperature only, and h is the relative humidity of the porous medium which Philip and de Vries [115] have shown to be largely independent of temperature, being determined mainly by the moisture content. Thus,

$$\nabla\rho_1 = \nabla(h\rho_S) = h(d\rho_S/dT)\nabla T + \rho_S(dh/d\theta)\nabla\theta$$

With this result it is possible, as in the case of the

liquid flux, to express the vapor flux as the sum of fluxes due to gradients of temperature and moisture content. Substituting into (28) and rearranging, one obtains

$$\vec{J}_1/\rho_2 = -D_{\theta v}\nabla\theta - D_{Tv}\nabla T \tag{29}$$

where

$D_{\theta v}$ = Isothermal vapor diffusivity = $(D_{13}\nu\rho_s/\rho_2)dh/d\theta$

D_{Tv} = Thermal vapor diffusivity = $(D_{13}\nu h/\rho_2 T)d(\rho_s T)/dT$

Evaluation of Diffusivity Terms

The essence of the Philip-de Vries theory [115] is in the method of evaluating three of the four diffusivity terms which appear in equations (24) and (29). These terms will be discussed in the following paragraphs.

Thermal Liquid Diffusivity

Philip and de Vries [115] have shown that in the range of moisture contents where liquid movement occurs (K>0) the potential is determined by capillarity, Ψ being proportional to the surface tension of water, σ, or

$$\Psi = c\sigma$$

so that

$$\partial\Psi/\partial T = c\ d\sigma/dT = \gamma\Psi(T) \tag{30}$$

where $\gamma = (1/\sigma)d\sigma/dT$ is called the temperature coefficient of the surface tension of water and is essentially constant over a wide range of temperatures. Since the moisture characteristic is usually measured at only one temperature T_R it is necessary to correct the measured value for temperature variations. Noting that

$$\Psi(T) = c\sigma(T) \quad \text{and} \quad \Psi(T_R) = c\sigma(T_R)$$

then

$$\Psi(T) = \Psi(T_R)[\sigma(T)/\sigma(T_R)]$$

and therefore

$$\partial\Psi(T)/\partial T = \gamma\Psi(T_R)[\sigma(T)/\sigma(T_R)] \tag{31}$$

Equation (31), therefore, relates $\partial\Psi/\partial T$ at any temperature T to the moisture characteristic measurements at temperature T_R.

The observed increase in the hydraulic conductivity as temperatures increase can be adequately accounted for by the decrease in the viscosity of water [63,69], or

$$K(T) = c/\mu(T)$$

Thus, if hydraulic conductivity measurements are made at temperature T_R,

$$K(T) = K(T_R)[\mu(T_R)/\mu(T)]$$ (32)

Substituting equations (31) and (32) into the definition of the thermal liquid diffusivity, one obtains

$$D_{T\ell} = K\partial\Psi/\partial T = D_{T\ell}(T_R)[\mu(T_R)\sigma(T)/\sigma(T_R)\mu(T)]$$ (33)

where $D_{T\ell}(T_R) = \gamma K(T_R) \Psi(T_R)$.

Therefore, the variation of the thermal liquid diffusivity with temperature can be determined by the temperature dependence of the surface tension-viscosity ratio. This has been verified by Jackson [69].

Isothermal Vapor Diffusivity

The relative humidity, h, of the air-water vapor mixture in the spaces between soil particles is related to the potential, Ψ, and temperature by [32]

$$h = Exp(\Psi g/RT)$$

As noted earlier, $\partial h/\partial T \doteq 0$, so that

$$dh/d\theta = (g/RT)Exp(\Psi g/RT)\partial\Psi/\partial T)_T = (hg/RT)\partial\Psi/\partial\theta)_T$$

and therefore

$$D_{\theta v} = (D_{13} v h \rho_s \, g / \rho_2 RT) \partial \Psi / \partial \theta)_T$$

where $\partial \Psi / \partial \theta)_T$ is the slope of the moisture characteristic curve at temperature T.

The modified molecular diffusivity D_{13} is the product of the diffusivity of water vapor in air, D_{atm}, the volumetric air content of the medium, a, and a tortuosity factor, α.

$$D_{13} = \alpha a D_{atm}$$

Based on kinetic theory and a survey of experimental data, de Vries and Kruger [150] proposed the use of

$$D_{atm} = 0.217 (P_o / P)(T/T_o)^{1.88} \ cm^2/s \qquad for \ 273K < T < 373K \quad (34)$$

for the diffusivity of water vapor in air. Here $T_o = 273K$ and $P_o = 1$ atm. Since the total pressure in the medium is assumed constant, D_{atm} is a function of temperature only.

The dependence of the rate of diffusion upon porosity is accounted for by the factor αa. Penman [111] and van Bavel [144] concluded that there exists a constant ratio between the diffusivity of gases and vapors in a porous medium and the diffusivity in a gaseous phase. For a

porosity, ϵ, less than about 0.7 the numerical value of the constant is approximately $\alpha = 0.66$. Thus in a dry porous medium

$$D_{13} = \alpha \epsilon D_{atm},$$

but if the medium is partially saturated ϵ must be replaced by the apparent porosity, i.e. the volumetric air content, a. Hence

$$D_{13} = \alpha a D_{atm}.$$

The factor α is called the tortuosity since it accounts for the extra length traveled by gases or vapors due to the tortuous path in the porous medium.

Philip and de Vries [115], Rollins, et al [122] and others also include a term $P/(P-P_1)$ called the "mass flow factor" which is derived on the basis of steady diffusion in a closed system between an evaporating source and a condensing sink. However, use of this factor would lead to an unreasonable result in the range of temperatures close to 373K. This term is not included in $D_{\theta v}$ because, as Luikov [86] has stated,

In the expression for vapor diffusion flow no correction is introduced into the molar transfer (Stefan flow) since the conditions for diffusion of vapor in a capillary-porous body are different from the conditions for diffusion of vapor in a boundary layer with

evaporation of liquid from a free surface.

The isothermal vapor diffusivity is, therefore,

$$D_{\theta v} = (\alpha a \nu D_{atm} \rho_s h/\rho_2 RT) \partial \Psi/\partial \theta)_T \qquad (35)$$

where D_{atm} is defined by (34).

With the results of the preceeding paragraphs the thermal vapor diffusivity can be written as

$$D_{Tv} = (\alpha a D_{atm} \nu h/\rho_2 T) d(\rho_s T)/dT \qquad (36)$$

Philip and de Vries [115] regard moisture movement due to a temperature gradient as a series-parallel flow through regions of vapor and liquid. In describing this process they choose to replace a by the product $\zeta f(\theta)$, where

$$\zeta = \frac{\text{average temperature gradient in air-filled pores}}{\text{overall temperature gradient}}$$

and

$$f(\theta) = \begin{cases} \varepsilon & \text{for } \theta < \theta_K \\ a[1+\theta/(\varepsilon-\theta_K)] & \text{for } \theta < \theta_K \end{cases}$$

No tortuousity is included here because the effect of

tortuosity is accounted for in the factor ζ.

Hence, the thermal vapor diffusivity is defined as

$$D_{Tv} = f(\theta)(\zeta D_{atm} vh/\rho_2 T) d(\rho_s T)/dT \tag{37}$$

The factor ζ can be calculated by the method of de Vries [148,149].

Summary of Governing Equations

At this point it is helpful to summarize the the results of the preceding paragraphs.

Conservation of Mass

Upon substituting equations (21), (24) and (29) into equation (15), the conservation of mass equation for soil moisture becomes

$$\{1+[(\epsilon-\theta)\rho_s hg/(\rho_2 RT)](\partial\Psi/\partial\theta)_T-\rho_s h/\rho_2\}\partial\theta_2/\partial t \tag{38}$$

$$+ [h(\epsilon-\theta_2)/\rho_2](d\rho_s/dT)\partial T/\partial t$$

$$= \nabla\cdot[D_\theta\nabla\theta_2+D_T\nabla T]$$

Conservation of Energy

Similarly, the conservation of energy equation becomes

$$[C+h_{21}(\epsilon-\theta_2)h\ d\rho_s/dT]\ \partial T/\partial t \tag{39}$$

$$+\{[h_{21}(\epsilon-\theta_2)\rho_s hg/RT](\partial\Psi/\partial\theta)_T-h_{21}\rho_s h\}\partial\theta/\partial t$$

$$= \nabla\cdot(\lambda\nabla T) + (\rho_2 h_{21}D_{\theta v}\nabla\theta) + c_2\rho_2(D_\theta\nabla\theta+D_T\nabla T)\cdot\nabla T$$

where D_θ = Isothermal moisture diffusivity = $D_{\theta\ell} + D_{\theta v}$

D_T = Thermal moisture diffusivity = $D_{T\ell} + D_{Tv}$

$D_\theta = K(\partial\Psi/\partial\theta)_T$

$D_T = K(\partial\Psi/\partial T)_\theta$

$D_{\theta v} = (\alpha a \nu D_{atm} h \rho_s g/\rho_2 RT)\partial\Psi/\partial\theta)_T$

$D_{Tv} = [f(\theta)\zeta\nu D_{atm}h/\rho_2 T]d(\rho_s T)/dT$

λ = Effective thermal conductivity = $\lambda_* + \rho_2 h_{21}D_{Tv}$

Boundary Conditions

The boundary conditions for the problem under discussion consist of the following:

(a) The temperature and moisture content of the soil very far removed from the cable surface are essentially unaffected. These conditions are

$$T(r,t) = T_{init} \quad \text{at} \quad r=r_{inf} \tag{40}$$

$$\theta(r,t) = \theta_{init} \quad \text{at} \quad r=r_{inf} \tag{41}$$

(b) The cable surface is assumed to be impermeable

to moisture so that the net flux of moisture at the cable surface, r_c, is taken to be zero.

$$\vec{j}_m = -\rho_2(D_\theta\nabla\theta + D_T\nabla T) = 0 \qquad \text{at } r=r_c \qquad (42)$$

(c) The heat transfer rate, $q^\bullet$, is specified at the cable surface and is related to the heat flux at the surface by

$$\vec{q}^\bullet = 2\pi r \vec{q}^{\prime\prime\prime} \qquad \text{at } r=r_c \qquad (43)$$

From equation (18) $\vec{q}^{\prime\prime\prime}$ is given by

$$\vec{q}^{\prime\prime\prime} = -\lambda_*\nabla T + h_1\vec{j}_1 + h_2\vec{j}_2$$

$$= -\lambda_*\nabla T + h_{21}\vec{j}_1 + h\,\vec{j}_m$$

At the cable surface, the last term in the above equation is zero as a result of (42). Substituting the definition of $\vec{j}_1$ from equation (29) one obtains

$$\vec{q}^{\prime\prime\prime} = -(\lambda_* + \rho_2 h_{21}D_{Tv})\,\nabla T - \rho_2 h_{21}D_{\theta v}\nabla\theta$$

$$= -\lambda\nabla T - \rho_2 h_{21}D_{\theta v}\nabla\theta$$

Therefore

$$q^* = -2\pi r[\ \lambda \nabla T + \rho_2 h_{21} D_{\theta v} \nabla \theta] \qquad at\ r = r_c \qquad (44)$$

Interface Conditions

When the soil adjacent to the surface of the heat source dries out a drying front will begin to propagate into the surrounding soil. The location of the front at any time, t, separates a moist region from a region which is considered to be dry. The moisture content of the dry region will not be zero but some minimum value θ_{min} due to the presence of adsorbed surface water. This condition is written as

$$\theta(r,t) = \theta_{min} \qquad r_c < r < r_{front} \qquad (45)$$

The heat transfer in the dry region is therefore due to conduction only, and the last term on the right hand side of (44) is zero.

Two conditions may be derived for the interface to complete the specification of the problem with a moving boundary present.

Referring to Figure 3, the net flux of moisture into the control volume must equal the net flux of moisture out of the control volume plus the time rate of change of moisture in the control volume, m_{cv}, or

$$\vec{j}_{m,in} = \vec{j}_{m,out} + [1/(2\pi L r_{front})]\partial(m_{cv})/\partial t$$
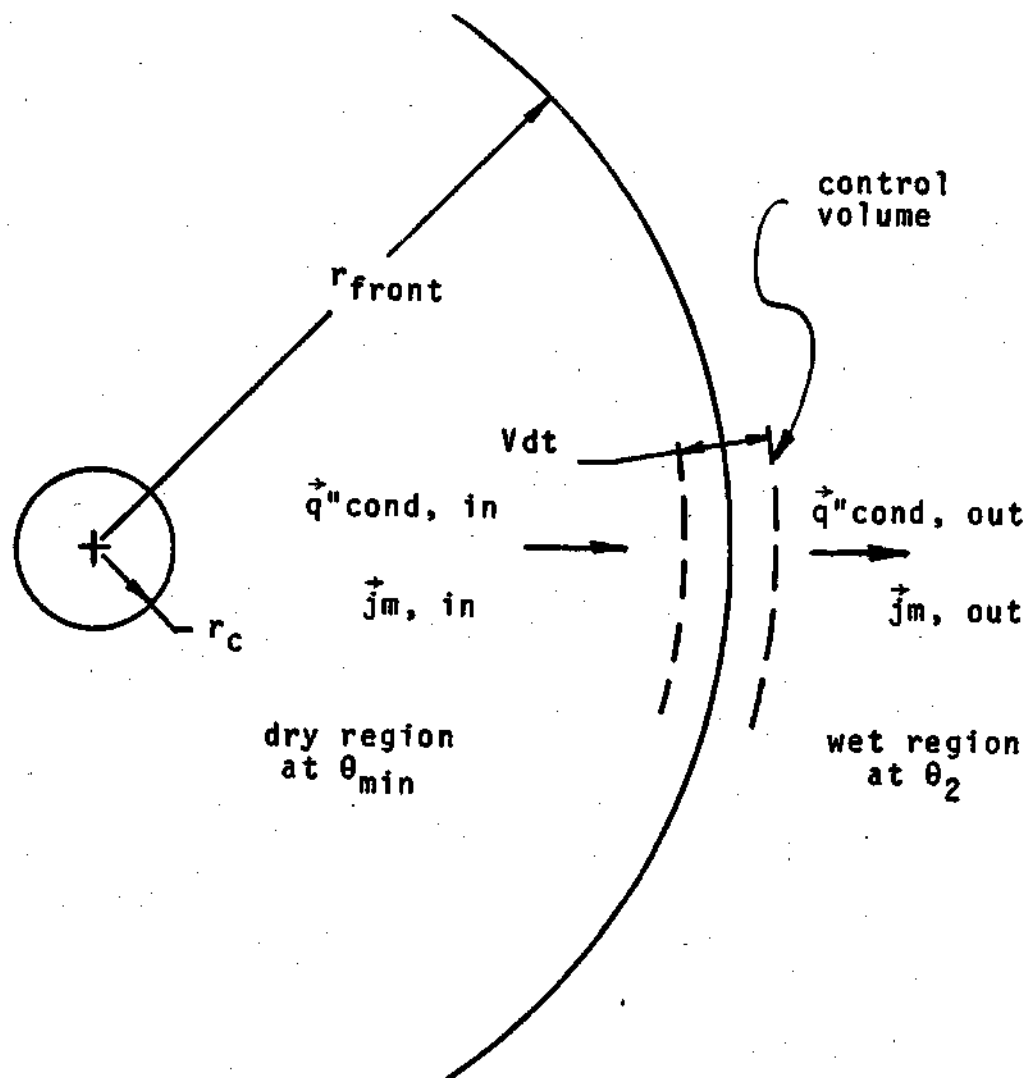
Figure 3.   Control Volume for Interface Conditions

Here,

$$[1/(2\pi L r_{front})]\partial(m_{cv})/\partial t = (m_{cv,t+dt} - m_{cv,t})/(2\pi L r_{front}dt)$$

$$= Vdt[\rho_2\theta_{min} + \rho_1(\epsilon - \theta_{min}) - [\rho_2\theta_2 + \rho_1(\epsilon - \theta_2)]]/dt$$

$$= -(\rho_2 - \rho_1)V(\theta_2 - \theta_{min})$$

The moisture flux in the dry region is taken to be zero so that

$$\vec{j}_{m,out} = V(\theta_2 - \theta_{min})(\rho_2 - \rho_1)$$

where V represents the velocity of propagation of the front. With the definition of $\vec{j}_m$, this condition may be written

$$-\vec{j}_m/\rho_2 = D_\theta\nabla\theta + D_T\nabla T = -V(\theta_2 - \theta_{min})(1 - \rho_1/\rho_2) \quad \text{at } r = r_{front} \quad (45)$$

An energy balance for the control volume states that the heat conducted into the control volume is equal to the sum of the heat conducted out of the control volume and the time rate of change of energy of the control volume. Thus,

$$\vec{q}''_{cond,in} = \vec{q}''_{cond,out} + h_1\vec{j}_{1,out} + h_2\vec{j}_{2,out}$$

$$+ [(1/2\pi L r_{front})] \partial(um_{cv})/\partial t$$

and

$$[1/(2\pi L r_{front})]\partial(um_{cv})/\partial t = V[u_2\rho_2\theta_{min}+u_1\rho_1(\varepsilon-\theta_{min})$$

$$-[u_2\rho_2\theta_2+u_1\rho_1(\varepsilon-\theta)]\}$$

$$= -V(\theta_2-\theta_{min})(u_2\rho_2-u_1\rho_1)$$

Substituting into the energy balance equation and eliminating V with the use of (46) one obtains

$$\vec{q}''_{cond,in} = \vec{q}''_{cond,out}+h_1\vec{j}_{1,out}+h_2\vec{j}_{2,out}$$

$$-\vec{j}_{m,out}(h_2\rho_2-h_1\rho_1)/(\rho_2-\rho_1)$$

$$= \vec{q}''_{out}+h_2\vec{j}_{m,out}-\vec{j}_{m,out}(h_2\rho_2-h_1\rho_1)/(\rho_2-\rho_1)$$

$$= \vec{q}''_{out}+\vec{j}_{m,out}(\rho_1 h_{21}/(\rho_2-\rho_1))$$

where $h_2 \doteq u_2$ and $h_1 \doteq u_1$ have been used. Thus

$$\lambda\nabla T)_{r=r^-_{front}} = \qquad\qquad (47)$$

$$[\lambda\nabla T+(\rho_2\rho_1 h_{21}/(\rho_2-\rho_1))(D_\theta\nabla\theta+D_T\nabla T)]_{r=r^+_{front}}$$

When a drying front is present in the soil equations (45) and (47) replace equations (42) and (44) at the location of the front. Once the temperature and moisture content distributions have been determined for time t, the velocity of the front at time t can be calculated from equation (46).

CHAPTER VI

NONDIMENSIONALIZED EQUATIONS AND
ORDER OF MAGNITUDE ANALYSIS

In this section the governing equations (38) and (39) are nondimensionalized in their general form. They are then simplified through an order-of-magnitude analysis in a later section.

The moisture content is normalized with respect to the moisture content at saturation $\theta_s = \epsilon$.

$$X = \theta_2(r,t)/\theta_s = \theta_2(r,t)/\epsilon$$

With this definition, the normalized moisture content ranges from zero at extreme dryness to unity at saturation.

To simplify the form of the heat flux boundary condition, the temperature is nondimensionalized with respect to the heat flux at the cable surface, the initial temperature, and the thermal conductivity of the dry soil.

$$T^* = 2\pi\lambda_0 (T(r,t) - T_i)/q^*$$

The dry thermal conductivity, λ_0, is of the order of 1 mcal/cm-s-^{0}C. With $T_i = 293K$ and $T_{max}(r,t) = 373K$, the

dimensionless temperature in the range of interest for moisture movement is

$$T_{max}^* = 2.37/q^o$$

where q^o has units of W/cm. The heat flux levels of interest will generally be in the range of .5 to 2 W/cm [14a] so that

$$1 \le T_{max}^* \le 5$$

and therefore T^* ranges from zero initially to T_{max}^*.

The radial coordinate, r, is nondimensionalized with respect to r_{inf}, the location at which the temperature and moisture content remain essentially unaffected by the presence of the heat source,

$$R = r/r_{inf}$$

In addition, the Fourier number for the problem under discussion is defined as

$$Fo \equiv \alpha_o t/r_{inf}^2$$

where α_o is the thermal diffusivity of the dry porous medium.

Since only radial variations are considered, the del operator becomes

$$\nabla = (r_{inf}/r_{inf})\partial/\partial r = (1/r_{inf})\partial/\partial R = (1/r_{inf})\nabla$$

and terms such as $\nabla\cdot(K\nabla(\ ))$ become

$$\nabla\cdot(K\nabla(\ )) = (1/r)\frac{\partial}{\partial R}(rK\partial/\partial r) = (1/r_{inf}^2 R)\frac{\partial}{\partial R}(RK\partial/\partial R)$$

Conservation of Mass

The conservation of mass equation (38) is of the form

$$A_1\partial\theta_2/\partial t + A_2\partial T/\partial t = \nabla\cdot(A_3\nabla\theta + A_4\nabla T)$$

Introducing the nondimensionalized variables, one obtains

$$(\alpha_o A_1\epsilon/r_{inf}^2)\partial X/\partial Fo + (\alpha_o A_2 q^*/(2\pi\lambda_o r_{inf}^2))\partial T^*/\partial t$$

$$= (1/r_{inf}^2)\nabla\cdot(A_3\epsilon\nabla X + (A_4 q^*/2\pi\lambda_o)\nabla T^*)$$

or defining $Q = q^*/2$

$$A_1\partial X/\partial Fo + (A_2 Q/\epsilon)\partial T^*/\partial Fo \qquad (48)$$

$$= \nabla\cdot((A_3/\alpha_o)\nabla X + (A_4 Q/\alpha_o\epsilon)\nabla T^*)$$

Equation (48) is the dimensionless form of the conservation of mass equation, the coefficients of which will be discussed subsequently.

Conservation of Energy

The energy equation (39) is of the form

$$B_1 \partial T / \partial t + B_2 \partial \theta_2 / \partial t = \nabla \cdot (B_3 \nabla T)$$

$$+ \nabla \cdot (B_4 \nabla \theta) + [B_5 \nabla \theta + B_6 \nabla T] \cdot \nabla T$$

In terms of the dimensionless variables, this becomes

$$(B_1 Q \alpha_0 / r_{inf}^2) \partial T^* / \partial Fo + (B_2 \epsilon \alpha_0 / r_{inf}^2) \partial X / \partial Fo$$

$$= (1/r_{inf}^2) \nabla \cdot (B_3 Q \nabla T^*) + (1/r_{inf}^2) \nabla \cdot (B_4 \epsilon \nabla X)$$

$$+ (Q/r_{inf}^2)[B_5 \epsilon \nabla X + B_6 Q \nabla T^*] \cdot \nabla T^*$$

Upon multiplying each term by $r_{inf}^2 / Q\lambda_0$ one obtains

$$(B_1/C_0) \partial T^* / \partial Fo + (B_2 \epsilon / C_0 Q) \partial X / \partial Fo \qquad (49)$$

$$= \nabla \cdot [(B_3/\lambda_0) \nabla T^*] + \nabla \cdot [(B_4 \epsilon / \lambda_0 Q) \nabla X]$$

$$+ [(B_5 \epsilon / \lambda_0) \nabla X + (B_6 Q / \lambda_0) \nabla T^*] \cdot \nabla T^*$$

Equation (49) is the dimensionless form of the energy equation.

Dimensionless Groups

Some of the dimensionless groups which appear in equations (48) and (49) with the A_i and θ_i from equations (38) and (39) are defined as follows:

$Le = \alpha_0/D_\theta$ — Lewis number based on the isothermal moisture diffusivity

$Le^* = \alpha_0/D_{\theta v}$ — Lewis number based on the isothermal vapor diffusivity

$Ko = h_{21}\rho_1\epsilon/QC_o$ — Kossovich number [86]

$Pn = D_T Q/D_\theta\epsilon$ — A form of the Posnov number [86] based on the thermal and isothermal moisture diffusivities

Unnamed Dimensionless Groups

$\eta = D_{\theta v}/(\alpha v D_{atm}) - \rho_1/\rho_2$

$\xi = [hQ(\epsilon-\theta_2)/\rho_2] dp_s/dT$

$\delta = \rho_2 c_{p2}\epsilon/C_o$

Summary of Equations

With these definitions, equations (48) and (49) may be rewritten as

$$(1+\eta)\partial X/\partial Fo + \xi\partial T^*/\partial Fo = \nabla\cdot[(\nabla X + Pn\nabla T^*)/Le] \quad (50)$$

$$(C/C_o + Ko\xi)\partial T^*/\partial Fo + Ko\eta\partial X/\partial Fo = \nabla\cdot[(\lambda/\lambda_o)\nabla T^*] \quad (51)$$

$$+ \nabla\cdot[(Ko/Le^*)\nabla X] + (\delta/Le)(\nabla X + Pn\nabla T^*)\cdot\nabla T^*$$

Order of Magnitude Analysis

In this section an order of magnitude analysis is performed on equations (50) and (51). The purpose of the analysis is to determine which of the terms and dimensionless groups have the most significant effect on the heat transfer and moisture movement in the soil. In what follows, the notation $O(Z)$ denotes "the order of magnitude of Z."

The order of magnitude of the terms appearing in equations (50) and (51) will be discussed next, with equation (51) being considered first.

Coefficient C/C_0

The volumetric specific heat of the moist soil, neglecting the contribution due to air, is defined as

$$C = C_0 + c_1 \rho_2 \theta_1 + c_2 \rho_2 \theta_2$$

$$= C_0 + c_2 \rho_2 \theta + (c_1 - c_2) \rho_2 \theta_1$$

Since c_1 is less than c_2 an upper limit on the specific heat is obtained by writing

$$C = C_0 + c_2 \rho_2 \theta$$

The value of $c_2 \rho_2$ is approximately 1 cal/cm^3-$^\circ$C so that

$$C/C_0 = 1 + \theta/C_0$$

when C_0 is in cal/cm^3-°C.

An approximate expression for the specific heat of dry soil is given by de Vries [149]

$$C_0 = 0.46x_m + 0.60x_0 \qquad \text{cal/cm}^3\text{-°C}$$

where x_m is the volume fraction of soil minerals and x_0 is the volume fraction of organic material.

When the soil is completely dry $x_0 + x_m = 1 - \varepsilon$ where ε is the porosity of the soil. Generally the amount of organic matter present in the soil is small so that the specific heat of dry soil can be approximated by

$$C_0 = 0.46(1 - \varepsilon)$$

and therefore

$$C/C_0 = 1 + \theta/(0.46(1 - \varepsilon))$$

If the soil is completely dry $\theta = 0$ and $C/C_0 = 1$. If the soil is completely saturated $\theta = \varepsilon$ and

$$C/C_0 = 1 + \varepsilon/(0.46(1 - \varepsilon))$$

Maximum values of soil porosity are seldom greater that 0.7 so that

$$C/C_o)_{max} \approx 6.87$$

Thus C/C_o is $O(1)$ in the completely dry state and $O(10)$ if the soil is saturated.

Kossovich Number

The Kossovich number is defined as

$$Ko = h_{21}\rho_2\epsilon/(QC_o)$$

The density of the liquid is about 1 g/cm^3 and the latent heat h_{21} is approximately $2.4\cdot10^6$ J/kg or 570 cal/g over the range of temperatures of interest. Using $C_o=0.46(1-\epsilon)$ cal/$cm^3-{}^oC$, one obtains

$$Ko = O[1240\epsilon/(Q(1-\epsilon))] \quad {}^oC^{-1}$$

Since q^* will generally be in the range $0.5<q^*<2$, it follows that the range of Q is $20<Q<100$ and thus $3<Ko<15$ for low porosity soils and $30<Ko<150$ for very porous soils.

Coefficient ξ

The dimensionless parameter ξ is defined as

$$\xi = [hQ(\epsilon-\theta_2)/\rho_2\epsilon]d\rho_s/dT$$

In the range of temperatures $273K < T < 373K$ the maximum value (see Appendix E) of $d\rho_s/dT = 0.02$ kg/m^3-K at 373K and the minimum value of $d\rho_s/dT = 3 \cdot 10^{-4}$ kg/m^3-K at 273K.

The relative humidity of the soil ranges from zero at extreme dryness to unity at moderately moist conditions. But $(\epsilon - \theta_2)$ attains its maximum value of ϵ at extreme dryness when $h = 0$ and the minimum value of zero after h has attained a value of unity.

Even though the maximum values will not occur simultaneously, a limiting value of ξ may be calculated using the maximum possible values of the parameters,

$$\xi_{max} = 0\left[\frac{(1)(\epsilon)(.02kg/m^3-K)(100^0 C)(10^3 g/kg)}{(1 \ g/cm^3) \ \epsilon \ (10^6 cm^3/m^3)}\right] = 0(2 \cdot 10^{-3})$$

Of course $\epsilon_{min} = 0(0)$.

With these results, the relative orders of magnitude of the terms in the coefficient of $\partial T^*/\partial Fo$ in the energy equation can be compared.

$$[(C/C_0)/(Ko\xi)]_{max} = (C/C_0)_{max}/(Ko\xi)_{min} \rightarrow \infty$$

and

$$[(C/C_0)/(Ko\xi)]_{min} = (C/C_0)_{min}/(Ko\xi)_{max} = 0(3)$$

Normal porosities are usually about $\epsilon=0.35$ for which

$$[(C/C_o)/(Ko\xi)]_{min} = O(15)$$

This leads to the conclusion that $C/C_o \gg Ko\xi$ for most cases or $Ko\xi$ is negligible compared to C/C_o.

Coefficient η

The dimensionless parameter η is defined as

$$\eta = D_{\theta v}/(\alpha v D_{atm}) - \rho_1/\rho_2$$

Typical maximum values of $D_{\theta v}$ are summarized in Table 2. These maximum values of $D_{\theta v}$ occur in a very narrow range of moisture contents in the dry region. The maximum value of $D_{\theta v}$ is taken to be $D_{\theta v,max}=O(10^{-3} \text{ cm}/s)$.

The minimum value of D_{atm} from equation (34) at 273K is 0.22 cm^2/sec. The factor v has been defined as

$$v = P/[(P-P_1)+P_1 M_1/M_3]$$

and the minimum value of v is unity. Taking $\alpha=0.66$, the maximum value of the first term of η is

$$[D_{\theta v}/(\alpha v D_{atm})]_{max} = D_{\theta v,max}/(\alpha v D_{atm})_{min} = O(10^{-2})$$

At 273K, where $D_{atm,min}$ occurs, $\rho_1/\rho_2=O(10^{-5})$ and is

Table 2. Typical Maximum Values of $D_{\theta v}$

Source		Soil Type	$D_{\theta v, max}$ cm^2/s
Philip	[112]	Yolo light clay	$<10^{-4}$
deVries	[148]	medium sand	$<10^{-3}$
Gee	[52]	Palouse silt loam	$<10^{-4}$
Jury	[73]	Plainfield sand	$<10^{-4}$
Fritton, et.al	[48]	Webster silty clay loam	$<2 \times 10^{-3}$

negligible compared to the first term of η. Therefore $\eta_{max} = O(10^{-2})$ and $\eta_{min} \rightarrow 0$.

With this result and the preceding results, the order of magnitude of the terms on the left hand side of equation (51) can be compared.

$$O[(C/C_0 + Ko\xi)(\partial T^*/\partial Fo)/(Ko\eta \partial X/\partial Fo)]$$

$$= O(C/C_0) \; O(T^*)/[O(Ko\eta) \; O(X)]$$

From the results obtained thus far the minimum value of the term can be written as

$$O[(C/C_0)/(Ko\eta)]_{min} = O[(C/C_0)_{min}/(Ko\eta)_{max}] = O(70)$$

For normal porosity soils ($\epsilon = 0.35$)

$$O[(C/C_0)/(Ko\eta)]_{min} = O(3)$$

Thus for soils with $\epsilon < 0.4$ the order of magnitude of the above ratio is O(10) or less. In other words, for such soils the value of $Ko\eta$ will not exceed 10 percent of C/C_0 and under these conditions the second term on the left hand side of equation (51) can be neglected in comparison to the first term.

Among those coefficients on the right hand side of

equation (51) is λ/λ_0 which is expected to dominate the heat transfer process under normal conditions. The minimum value of this term occurs when extreme dryness is encountered and is

$$\lambda/\lambda_0)_{min} = 1$$

Furthermore, $\lambda_0 = 0(10^{-3}$ cal/cm-s-^{0}C) [149].

The remaining terms on the right hand side of (51) will be compared with the first.

$$0[\nabla \cdot ((\lambda/\lambda_0)\nabla T^*) + \nabla \cdot ((Ko/Le^*)\nabla X)] = 0[\lambda/\lambda_0 + Ko/Le^*]$$

To determine if the second term is significant the maximum value of Ko/Le^* can be compared to λ/λ_0.

Using results obtained previously

$$Ko/Le^*)_{max} = KoD_{\theta v}C_0/\lambda_0)_{max} = 0(570\varepsilon/Q) = 0(3)$$

This maximum value of Ko/Le^* occurs in the narrow range of moisture contents where $D_{\theta v}$ reaches a maximum. At higher or lower moisture contents, the value of Ko/Le^* drops sharply. Therefore, the term in equation (51) containing Ko/Le^* can be neglected except in a narrow range of moisture contents in the dry region.

Comparing the last term of equation (51)

$$O[\nabla \cdot (\lambda/\lambda_o) \nabla T^*] + (\delta/Le)(\nabla X + Pn \nabla T^*) \cdot \nabla T^*]$$

$$\approx O[\lambda/\lambda_o + (\delta/Le \ (1+Pn)]$$

so that for typical soils

$$Pn \approx D_T Q/(D_\theta \epsilon) = O(3QD_T/D_\theta)$$

Examination of typical curves of D_T and D_θ versus moisture content [48,52,73,112,148] reveals that $D_T \ll D_\theta$ except in a very small range of moisture contents within which a characteristic "dip" in the value of D_θ occurs. The dip separates what is essentially a liquid dominated range of moisture contents from one dominated by vapor. In the vicinity of the dip, D_T and D_θ have comparable orders of magnitude and the value of Pn reaches a maximum

$$Pn_{max} \approx O(3Q) = say \ O(300)$$

At greater or lesser moisture contents, Pn is always less than this value. Only at high moisture contents is Pn<<1 so that it becomes negligible compared to unity.

Coefficient δ/Le

The term δ/Le is defined as

$$\delta/Le = (\rho_2 c_{p2} \epsilon/C_o) \ (D_\theta/\alpha_o) = (\rho_2 c_{p2} \epsilon/\lambda_o) D_\theta$$

$$\approx 0(10^3 \epsilon D_\theta)$$

where D_θ has units of cm^2/sec.

Thus the maximum value of $(/Le)(1+Pn)$ occurs at the "dip" in the D_θ curve where, typically [52,73,112,148], $D_\theta = 0(10^{-1})$ and $Pn < 1$ so that

$$[(\delta/Le)Pn]_{max} = 0(10^4)$$

in a narrow range of low moisture contents. This value in equation (51) is multiplied by the square of the temperature gradient, the result being a large positive quantity which would cause a rapid temperature rise. This same term in equation (50), however, could produce a large negative quantity which would be associated with rapid drying of the medium.

At high moisture contents where Pn approaches zero, $Pn << 1$ and δ/Le attains a large value. But in this case the term is coupled with the product of the gradients of temperature and moisture content and therefore represents a negative quantity, or cooling. Likewise, the net effect in equation (50) would be that of an increase of moisture.

Thus it is concluded that in the range of moisture contents of greatest interest to this study the third and fourth terms on the right hand side of equation (51) must be retained, and the latter may have significant influence on

the heat transfer in the soil surrounding the heated cable.

Therefore, the order of magnitude analysis of the energy equation leads to the following simplified forms of equation (51)

$$(C/C_0) \partial T^*/\partial Fo = \nabla \cdot [(\lambda/\lambda_0) \nabla T^*] + \nabla \cdot [(Ko/Le) \nabla X] \qquad (52)$$

$$+ (\delta/Le)(\nabla X + Pn \nabla T^*) \cdot \nabla T^*$$

for the low moisture contents and

$$(C/C_0) \partial T^*/\partial Fo = \nabla \cdot [(\lambda/\lambda_0) \nabla T^*] + (\delta/Le) \nabla X \cdot \nabla T^* \qquad (53)$$

for very high moisture contents. Next the analysis is extended to the conservation of mass equation (50).

The conclusions reached about the behavior of Pn and 1/Le in the preceding paragraphs lead to the following simplified forms of the conservation of mass equation (50).

$$\partial X/\partial Fo + \xi \partial T^*/\partial Fo = \nabla \cdot [(1/Le)(\nabla X + Pn \nabla T^*)] \qquad (54)$$

for the low moisture contents and

$$\partial X/\partial Fo = \nabla \cdot [(1/Le) \nabla X] \qquad (55)$$

for very high moisture contents.

Equation (55) is the nondimensionalized form of the familiar Darcy equation.

CHAPTER VII

GENERAL NUMERICAL SCHEME

An implicit finite difference scheme was selected for the numerical solution of the governing equations.

Since the problem is formulated in cylindrical coordinates, terms such as

$$(1/R) \frac{\partial}{\partial R} [KR \, \partial V / \partial R]$$

arise in the governing equations. Here, K represents any of the variable properties and V represents either dimensionless temperature or moisture content. Such terms are expanded as follows before applying the difference equations:

$$(1/R) \frac{\partial}{\partial R} [KR \, \partial V / \partial R]$$

$$= K \partial^2 V / \partial R^2 + (1/R)[1 + (R/K) \partial K / \partial R] \partial V / \partial R \}$$

The usual three-point finite difference replacements for the derivatives are used:

$$\partial^2 V / \partial R^2 = (V_{j+1} - 2V_j + V_{j-1}) / \Delta R^2 + O(\Delta R^2) \qquad (56)$$

$$\partial V/\partial R = (V_{i+1} - V_{i-1})/2\Delta R + O(\Delta R^2) \qquad (57)$$

If we let

$$dK_i = 1 + (R_i/K_i)[K_{i+1} - K_{i-1}]/2\Delta R$$

the general term may be written in finite difference form as

$$(1/R)\frac{\partial}{\partial R}[KR\partial V/\partial R]_i = K_i\{(V_{i+1} - 2V_i + V_{i-1})/\Delta R^2 + O(\Delta R^2)$$

$$+ (V_{i+1} - V_{i-1})dK_i/(2R_i\Delta R) + O(dk_i\Delta R^2)\}$$

$$= (K_i/\Delta R^2)\{V_{i-1}(1 - \Delta RdK_i/2R_i) - 2V_i + V_{i+1}(1 + \Delta RdK_i/2R_i)\}$$

$$= (K_i/\Delta R^2)\{V_{i-1}(1 - C_{Ki}) - 2V_i + V_{i+1}(1 + C_{Ki})\} + O(K_i\Delta R^2)$$

where $C_{Ki} = \Delta RdK_i/2R_i$.

Governing Equations

With these definitions, the energy equation

$$A\partial T^*/\partial Fo = \nabla\cdot(B\nabla T^*) + \nabla\cdot(E\nabla X) + C(\nabla X + D\nabla T^*)\cdot\nabla T^*$$

where A, B, C, D, and E represent the coefficients in equation (52), is written in finite difference form as

$$A_i(T_i^* - T_i^{*0})/\Delta Fo = (B_i/\Delta R^2)\{T_{i-1}^*(1 - C_{Bi}) - 2T_i^* + T_{i+1}^*(1 + C_{Bi})\}$$

$$+ (E_i/\Delta R^2)\{X_{i-1}(1-C_{Ei})-2X_i+X_{i+1}(1+C_{Ei})\}$$

$$+ (C_i/4\Delta R2)\{X_{i+1}-X_{i-1}+D_i(T^*_{i+1}-T^*_{i-1})\}(T^*_{i+1}-T^*_{i-1})$$

$$+ O(A_i\,\Delta Fo + B_i\,\Delta R^2)$$

Defining $P = \Delta Fo/\Delta R^2$,

$$T^*_i = T^{*n}_i + (B_i P/A_i)\{T^*_{i-1}(1-C_{Bi})-2T^*_i+T^*_{i+1}(1+C_{Bi})\}$$

$$+ (E_i P/A_i)\{X_{i-1}(1-C_{Ei})-2X_i+X_{i+1}(1+C_{Ei})\}$$

$$+ (C_i P/4A_i)\{X_{i+1}-X_{i-1}+D_i(T^*_{i+1}-T^*_{i-1})\}(T^*_{i+1}-T^*_{i-1})$$

Upon rearranging this becomes

$$T^*_{i-1}\{(C_i D_i P/4A_i)(T^*_{i+1}-T^*_{i-1})-(B_i P/A_i)(1-C_{Bi})\} \qquad (58)$$

$$+ X_{i-1}\{(C_i P/4A_i)(T^*_{i+1}-T^*_{i-1})-(E_i P/A_i)(1-C_{Ei})\}$$

$$+ T^*_i\{1+2B_i P/A_i\} + X_i\{2E_i P/A_i\}$$

$$+ T^*_{i+1}\{-(C_i D_i P/4A_i)(T^*_{i+1}-T^*_{i-1})-(B_i P/A_i)(1+C_{Bi})\}$$

$$+ X_{i+1}\{-(C_i P/4A_i)(T^*_{i+1}-T^*_{i-1})-(E_i P/A_i)(1+C_{Ei})\} = T^{*n}_i$$

The finite difference form of the conservation of mass equation

$$\partial X/\partial Fo + H\partial T^*/\partial Fo = \nabla\cdot(F\nabla X) + \nabla\cdot(G\nabla T^*)$$

where H, F, and G represent the coefficients in equation (54), can be written as

$$T_{i-1}^*\{-G_i P(1-C_{Gi})\} + X_{i-1}\{-F_i P(1-C_{Fi})\} \tag{59}$$

$$+ T_i^*\{H_i + 2G_i P\} + X_i\{1+2F_i P\}$$

$$+ T_{i+1}^*\{-G_i P(1+C_{Gi})\} + X_{i+1}\{-F_i P(1+C_{Fi})\} = X_i^n + H_i T_i^{*n}$$

Equations (58) and (59) are coupled equations of the form

$$a_i^1 T_{i-1}^* + a_i^2 X_{i-1} + b_i^1 T_i^* + b_i^2 X_i + c_i^1 T_{i+1}^* + c_i^2 X_{i+1} = d_i^1 \tag{60}$$

$$a_i^3 T_{i-1}^* + a_i^4 X_{i-1} + b_i^3 T_i + b_i^4 X_i + c_i^3 T_{i+1}^* + c_i^4 X_{i+1} = d_i^2$$

N pairs of simultaneous equations such as those in (60) result when the finite difference equations are written for the N nodes at which solutions are desired. These N pairs of equations can be solved quickly using the bitridiagonal algorithm [145a].

Since property values must be calculated at unknown temperatures and moisture contents and since the term $(T_{i+1}^* - T_{i-1}^*)$ appears in the finite difference equations, it is necessary to use an iterative scheme for the solution of (58) and (59). Properties are first evaluated at temperatures T_i^{*n} and moisture contents X_i^n and next updated using calculated values T_i^* and X_i.

Boundary Conditions

The boundary conditions of equations (40), (41), (42) and (44) are written as

$$[F\nabla X + G\nabla T^*]_{R=R_c} = 0 \tag{61}$$

$$[B\nabla T^* + E\nabla X]_{R=R_c} = -1/R_c \tag{62}$$

$$T^*(1,Fo) = 0 \tag{63}$$

$$X(1,Fo) = \theta_{init}/\varepsilon \tag{64}$$

These conditions are put in finite difference form in the following manner.

The energy equation at the cable surface, node 1, is

$$A\partial T^*/\partial Fo)_1 = (1/R_1)\,\frac{\partial}{\partial R}(BR\partial T^*/\partial R)]_1 \tag{65}$$

$$+ (1/R_1)\frac{\partial}{\partial R}(ER\partial X/\partial R)]_1$$

$$+ (\delta/Le)(\partial X/\partial R + Pn\partial T^*/\partial R)]_1\,(\partial T^*/\partial R)]_1$$

The last term in this equation is zero as a result of (61).

Expanding $BR\partial T^*/\partial R)_{1\frac{1}{2}} + ER\partial X/\partial R)_{1\frac{1}{2}}$ about node 1, one obtains,

$$BR\partial T^*/\partial R)_{1\frac{1}{2}} + ER\partial X/\partial R)_{1\frac{1}{2}} = BR\partial T^*/\partial R)_1 + ER\partial X/\partial R)_1$$

$$+ (\Delta R/2)[\frac{\partial}{\partial R}(BR\partial T^*/\partial R)]_1 + \frac{\partial}{\partial R}(ER\partial X/\partial R)]_1] + O(\Delta R^2)$$

Using (62) and (65) this becomes

$$B\partial T^*/\partial R)_{1\frac{1}{2}} + ER\partial X/\partial R)_{1\frac{1}{2}} = -1 + (\Delta R/2)[R_1 A\partial T^*/\partial Fo)_1] + O(\Delta R^2)$$

Inserting the usual finite difference replacements for the derivatives, the result is

$$(BR)_{1\frac{1}{2}}(T_2^* - T_1^*) + (ER)_{1\frac{1}{2}}(X_2 - X_1) = -\Delta R$$
$$+ (R_1 A/2P)(T_1^* - T_1^{*n}) + O(\Delta R^3)$$

Rearranging, the final form is

$$T_1^* [1 + 2P(BR)_{1\frac{1}{2}}/R_1 A_1] - T_2^* [2P(BR)_{1\frac{1}{2}}/R_1 A_1] \qquad (66)$$
$$+X_1 [2P(ER)_{1\frac{1}{2}}/R_1 A_1] - X_2 [2P(ER)_1/R_1 A_1] = T_1^{*n} + 2P\Delta R/R_1 A_1$$

The finite difference form of the conservation of mass equation at the cable surface is obtained by first expanding $FR\partial X/\partial R)_{1\frac{1}{2}} + GR\partial T^*/\partial R)_{1\frac{1}{2}}$ about node 1.

$$FR\partial X/\partial R)_{1\frac{1}{2}} + GR\partial T^*/\partial R)_{1\frac{1}{2}} = FR\partial X/\partial R)_1 + GR\partial T^*/\partial R)_1$$
$$+ (\Delta R/2)[\frac{\partial}{\partial R}(FR\partial X/\partial R)]_1 + \frac{\partial}{\partial R}(GR\partial T^*/\partial R)]_1] + O(\Delta R^2)$$

Using (61) and the conservation of mass equation this becomes

$$FR\partial X/\partial R)_{1\frac{1}{2}} + GR\partial T^*/\partial R)_{1\frac{1}{2}} = (\Delta R/2)[R_1\partial X/\partial Fo)_1] + O(\Delta R^2)$$

Inserting the finite difference replacements and combining terms, the final form is

$$T_1^* \quad [2P(GR)_{1\frac{1}{2}}/R_1] - T_2^* \quad [2P(GR)_{1\frac{1}{2}}/R_1] \qquad (67)$$

$$+ X_1 \quad [1+2P(FR)_{1\frac{1}{2}}/R] - X_2 \quad [2P(FR)_{1\frac{1}{2}}/R_1]=X_1^n +O(\Delta R \cdot \Delta Fo)$$

Error Analysis

A brief discussion of the consistency, stability and convergence of the finite difference scheme is appropriate at this point.

In the development of the finite difference forms of the energy equation (58) and the conservation of mass equation (59) the truncation errors of the finite difference equations were shown to be, respectively,

$$O[A_1\Delta Fo+B_1\Delta R^2] \quad \text{and} \quad O[\Delta Fo+\Delta R^2(F_1+G)_1]$$

Since the truncation error tends to zero as $\Delta Fo, \Delta R \rightarrow 0$ in both cases, the implicit replacement analogs are consistent with the governing equations.

As a matter of convenience, the discussion of stability will be limited to the simplified forms of equations (58) and (59). These simplified forms (see

Results and Discussion) are

$$T^*_{i-1}[-(B_i P/A_i)(1-C_{Bi})] + T^*_i(1+2B_i P/A_i) \qquad (68)$$

$$+ T^*_{i+1}[-(B_i P/A_i)(1+C_{Bi})] = T^{*n}_i$$

and

$$T^*_{i-1}[-G_i P(1-C_{Gi})] + T^*_i(2G_i P) + T^*_{i+1}[-G_i P(1+C_{Gi})] \qquad (69)$$

$$+ X_{i-1}[-F_i P(1-C_{Fi})] + X_i(1+2F_i P)$$

$$+ X_{i+1}[-F_i P(1+C_{Fi})] = X^n_i$$

The method of analysis is that of von Newmann as discussed by Carnahan, et al [15a]. We assume that the variables are separable so that $T^{*n}_i = U(t)e^{j\beta r}$ and $X^n_i = V(t)e^{j\beta r}$ and that at any time t a Fourier expansion, with typical term $e^{j\beta r}$, of the solutions can be made. Substituting these expressions into equations (68) and (69), the forms of U and V can be determined and stability criteria can be established.

Denoting T^{*n+1}_i by $U \cdot (t)e^{j\beta r}$, equation (68) becomes

$$U \cdot \{e^{-j\beta \Delta r}[-(B_i P/A_i)(1-C_{Bi})] + (1+2B_i P/A_i)$$

$$+ e^{j\beta \Delta r}[-(B_i P/A_i)(1+C_{Bi})]\} = U$$

After simplifying, this can be written as

$$U^* [1 + (2B_i P/A_i) [2 \sin^2 (\beta \Delta R/2) - j C_{Bi} \sin(\beta \Delta R)]] = U \qquad (70)$$

Similarly, denotng x_i^{n+1} by $V^* (t) e^{j\beta r}$, equation (59) can be simplified to the following

$$U^* [2G_i P [2 \sin^2 (\beta \Delta R/2) - j C_{Gi} \sin(\beta \Delta R)]] \qquad (71)$$

$$+ V^* [1 + 2F_i P [2 \sin^2 (\beta \Delta R/2) - j C_{Fi} \sin(\beta \Delta R)]] = V$$

Equations (70) and (71) are of the form

$$a_1 U^* \qquad = U$$

$$b_1 U^* + b_2 V^* = V$$

and can be solved fo U^* and V^* to obtain

$$U^* = (1/a_1) U$$

$$V^* = -(b_1/a_1 b_2) U + (1/b_2) V$$

or

$$\begin{bmatrix} U^* \\ V^* \end{bmatrix} = \begin{bmatrix} 1/a_1 & 0 \\ -b_1/a_1 b_2 & 1/b_2 \end{bmatrix} \begin{bmatrix} U \\ V \end{bmatrix}$$

which is of the form $\alpha' = A\alpha$ where   is the column vector whose elements are U and V. A is called the amplification matrix [15a] and for stability, the modulus of each eigenvalue λ_1 and λ_2 of the matrix A must not exceed unity. Thus, for stability we require,

$$|\lambda_1| \leq 1 \quad \text{and} \quad |\lambda_2| \leq 1 \quad \text{for all } \beta.$$

The eigenvalues of the amplification matrix are

$$\lambda_1 = 1/a_1 = \{1 + (2B_i P/A_i)[2\sin^2(\beta\Delta R/2) - j C_{Bi} \sin(\beta\Delta R)]\}^{-1}$$

and

$$\lambda_2 = 1/b_2 = \{1 + 2F_i P[2\sin^2(\beta\Delta R/2) - j C_{Fi} \sin(\beta\Delta R)]\}^{-1}$$

For $|\lambda_1| \leq 1$ we require

$$|[1 + (4B_i P/A_i)\sin^2(\beta\Delta R/2)] - j C_{Bi}(2B_i P/A_i)\sin(\beta\Delta R)| \geq 1$$

or

$$\{[1 + (4B_i P/A_i)\sin^2(\beta\Delta R/2)]^2 + [(2B_i P/A_i)C_{Bi}\sin(\beta\Delta R)]^2\}^{\frac{1}{2}} \geq 1$$

Both terms in brackets are positive definite, and $(4B_i P/A_i)\sin^2(\beta\Delta R/2) \geq 0$ so that $|\lambda_1| \leq 1$ for all β.

Similarly, it can be shown that $|\lambda_2| \leqslant 1$ for all β and stability of equations (68) and (69) is ensured.

Therefore, stability and consistency of the finite difference forms of the simplified governing equations have been demonstrated. This, at least in the linear case, ensures convergence of the numerical scheme [85a].

Several different finite difference replacements of the boundary condition (62) were tested and (66) was found to give the best numerical results. Equation (66) has a truncation error of $O[R_1 A_1 \Delta R \Delta Fo + \Delta R^2]$ which goes to zero as ΔFo, $\Delta R \rightarrow 0$.

The accuracy of the numerical scheme can be judged to a certain degree by comparing the numerical solution of the equation

$$\partial T^*/\partial Fo = (1/R) \frac{\partial}{\partial R}(R \partial T^*/\partial R) \qquad (72)$$

with the exact solution (Appendix D).

The equivalent forms of (68) and (66) were used for this comparison and results are presented in Table 3 as percent error of the exact solution for various $Fo = \alpha t/R^2$, $P = \Delta Fo/\Delta R$ and combinations of ΔR and R_c. The best overall accuracy was obtained with $P=2$ and $\Delta R = R_c$.

For long-time solutions, r_{inf} must be located sufficiently far away from the inner boundary so that the location does not significantly influence the solution.

Table 3. Comparison of Numerical and Exact Solutions of
Equation (72) as Percent Error

	Node	$F_0=$ $\alpha t/r^2$ 2×10^{-3}	1×10^{-2}	2×10^{-2}	1×10^{-1}
P=2 $\Delta R=R_c$	1	0.004	-0.90	-0.96	-0.90
	2	1.33	0.01	-0.16	-0.26
	3	1.89	0.27	0.05	-0.11
	4	2.43	0.40	0.15	-0.04
P=2 $\Delta R=.5R_c$	1	1.95	0.15	-0.07	-0.21
	2	2.47	0.36	0.11	-0.08
	3	2.85	0.47	0.19	-0.02
	4	3.19	0.55	0.24	0.003
P=2 $\Delta R=2R_c$	1	-3.95	-3.08	-2.82	-2.35
	2	0.79	0.66	0.56	0.41
	3	2.46	1.72	1.45	1.04
	4	3.92	2.43	2.01	1.39
P=.5 $\Delta R=R_c$	1	0.84	-0.76	-0.90	-0.89
	2	2.58	0.18	-0.09	-0.25
	3	3.51	0.48	0.14	-0.09
	4	4.33	0.65	0.25	-0.03
P=5 $\Delta R=R_c$	1	-1.70	-1.16	-1.08	-0.92
	2	-1.20	-0.35	-0.32	-0.29
	3	-1.29	-0.16	-0.13	-0.13
	4	-1.19	-0.10	-0.06	-0.07

CHAPTER VIII

NUMERICAL SCHEME WITH DRYING FRONT PRESENT

When a drying front occurs in the soil it is desirable to follow the progress of the front and it becomes necessary to apply an energy balance and a mass balance at the front in order to match the dry region and wet region solutions.

The general procedure is as follows:

(a) Determine the time at which the soil adjacent to the cable surface reaches the minimum or dry moisture content and calculate the velocity of the ensuing front.

(b) The front velocity is assumed to be constant throughout the next small time interval and the front is stepped forward into the dry medium a fixed small increment. With the new front location known, the time required to reach that location is calculated and used in step (c).

(c) With the moisture content known and assuming that negligible moisture changes occur in the dry region, solve the pure conduction equation in the dry region and the coupled equations in the wet region, matching the two regions through application of the interface conditions.

(d) Once the solutions for both the wet region and dry region are known, the new velocity of the front may be

calculated. Return to step (b) to continue calculations.

Since the location of the moving boundary will not, in general, coincide with one of the nodes of the finite difference grid, a special set of difference equations must be developed for the nodes adjacent to the front. The equations for the dry region and wet region will be treated separately.

Dry Region

Since the front progresses by fixed spatial increments, only two different situations may arise. The front will either be located between two nodes in the grid or it will coincide with a node location. The special fractional-step finite difference equations, then, are required only in the first instance. The fractional-step equations are derived as follows:

Referring to Figure 4, $S\Delta R$ denotes the distance between node i and the moving boundary at time level n+1.

The first partial derivative of the dependent variable V is evaluated by expanding V_{i+s} and V_{i-1} in a Taylor's series in R about node i.

$$V_{i+s} = V_i + S\Delta R(\partial V/\partial R)_i + (S^2\Delta R^2/2)(\partial^2 V/\partial R^2)_i + O(S^3 \partial R^3) \quad (73)$$

$$V_{i-1} = V_i - \Delta R(\partial V/\partial R)_i + (\Delta R^2/2)(\partial^2 V/\partial R^2)_i + O(\Delta R^3) \quad (74)$$

where V_{i+s} represents V evaluated at the moving boundary.
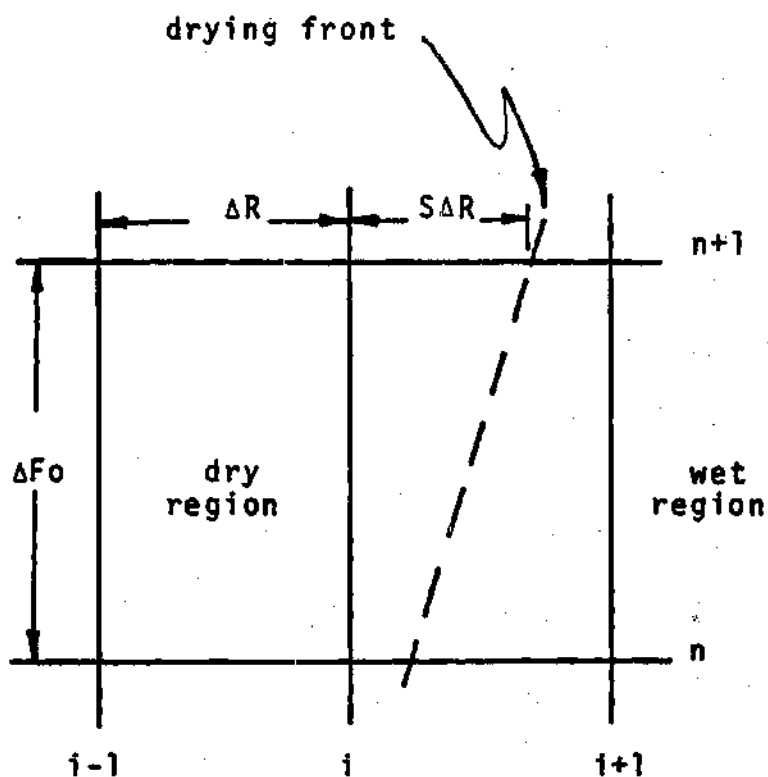
Figure 4.   Grid Division for Fractional-Step Difference
Equations in Dry Region

Multiplying equation (74) by S^2, subtracting the result from equation (73) and rearranging, one obtains

$$\partial V/\partial R)_i = [V_{i+s} - (1-S^2)V_i - S^2 V_{i-1}]/[S(1+S)\Delta R] + 0(S\Delta R^2) \quad (75)$$

The second partial derivative of V is evaluated by multiplying equation (74) by S and adding the result to equation (73). After rearranging, one obtains

$$\partial^2 V/\partial R^2)_i = [V_{i+s} - (1+S)V_i + SV_{i-1}]/[S(S+1)\Delta R^2/2] + 0(\Delta R) \quad (76)$$

Equations (75) and (76), with V replace by T^*, are the special finite difference equations required for the application of the conduction equation at a node adjacent to the moving boundary in the dry region.

Wet Region

In the wet region the fractional step $S\Delta R$ is shown in Figure 5. Expanding V_{i+1} and V_{i-s} in a Taylor's series

$$V_{i+1} = V_i + \Delta R \partial V/\partial R)_i + (\Delta R^2/2)\partial^2 V/\partial R^2)_i + 0(\Delta R^3) \quad (77)$$

$$V_{i-s} = V_i - S\Delta R \partial V/\partial R)_i + (S^2 \Delta R^2/2)\partial^2 V/\partial R^2)_i + 0(\Delta R^3) \quad (78)$$

where V_{i-s} represents V evaluated at the moving boundary.

Multiplying equation (77) by S^2, subtracting equation
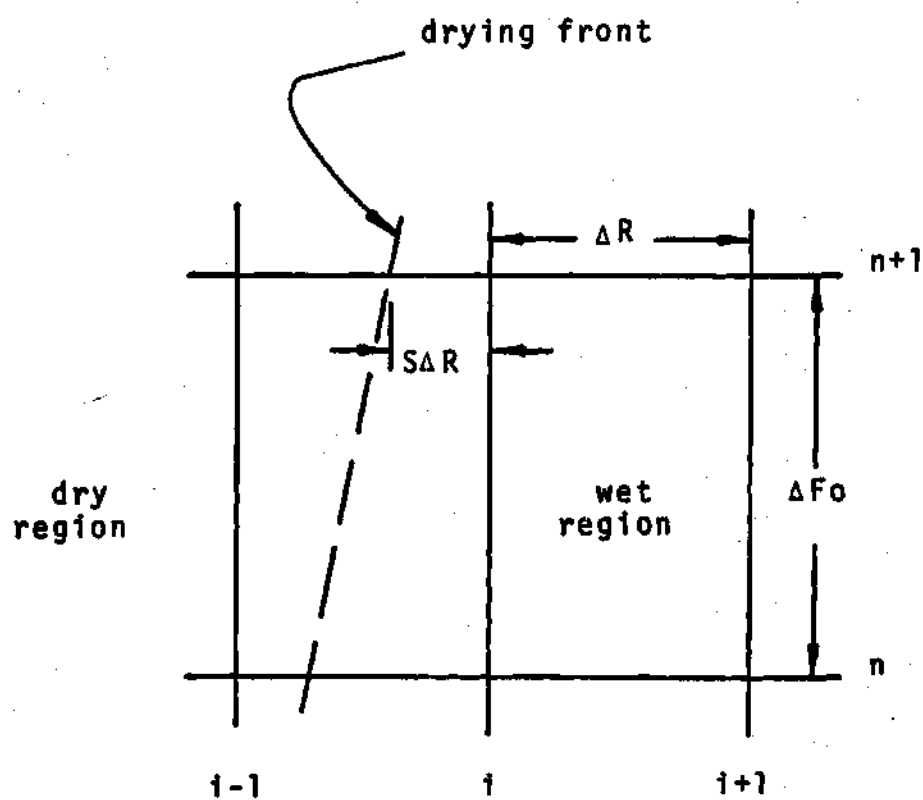
Figure 5. Grid Division for Fractional-Step Difference Equations in Wet Region

(78) and rearranging one obtains

$$\partial V/\partial R)_i = [S^2 V_{i+1} - (S^2 - 1)V_i - V_{i-s}]/[S(S+1)\Delta R] \qquad (79)$$

Multiplying equation (72) by S, adding equation (73) and rearranging, one obtains

$$\partial^2 V/\partial R^2)_i = [SV_{i+1} - (S+1)V_i + V_{i-s}]/[S(S+1)\Delta R^2/2] \qquad (80)$$

When the finite difference forms of the governing equations are used at the node adjacent to the moving boundary in the wet region, equations (79) and (80) replace equations (56) and (57), respectively, with V replaced by T^* or X, as appropriate.

If the location of the front should coincide with a node, then equations (80), (76), (79),and (75) are still used and the coefficient S is assigned the value of unity.

Interface Conditions

The interface equation, (47), is used to supply the additional equation required since T^*_{front} is unknown. The finite difference equations (75) and (79), as appropriate, are used to replace the first derivatives appearing in the equation. In this instance, subscript 1 would refer to the location of the front.

Finally, equation (46) is used to calculate the new velocity of the moving boundary.

CHAPTER IX

RESULTS AND DISCUSSION

Numerical simulations were conducted with the nondimensionalized form of the governing equations to determine the influence of the various coefficients. The simulations, which were conducted for the sandy and sandy silt soils, consisted of two phases. First, the values of selected coefficients were reduced to zero individually. Secondly, the coefficients were, again individually, multiplied by a factor of ten.

The numerical simulations were conducted for various initial moisture contents and surface heat transfer rates. Typical results are shown as percent error in Tables 4 and 5 for sand and Tables 6 and 7 for the sandy silt. The symbol E_T is the percent error in calculated surface temperature for the condition stated relative to the temperature calculated with the coefficients unchanged. The symbol E_X is the percent error in calculated moisture content. The tables show that reducing Ko/Le^* to zero had no significant effect on either the moisture content distribution or the temperature distribution. Likewise, reducing δ/Le or Pn to zero in the energy equation or the coefficient ξ in the conservation of mass equation had no significant effect for

Table 4. Effect of Property Variations - Plainfield Sand

Percent Error for	$X_{init}=0.1429$ $q°=0.4$ W/cm		$X_{init}=0.2143$ $q°=1.5$ W/cm		$X_{init}=0.5714$ $q°=1.5$ W/cm	
Coefficient	E_T	E_X	E_T	E_X	E_T	E_X
$δ/Le=0$ In Eq. (52)	0.04	-0.11	0.23	-2.22	1.00	-0.12
$Pn=0$ In Eq. (52)	-0.04	0.12	-0.17	1.40	-1.10	0.13
$Ko/Le^*=0$ In Eq. (52)	0.	0.	0.	0.	0.	0.
$ξ=0$ In Eq. (54)	-0.02	0.16	-0.01	0.23	0.	0.
All as above=0	-0.01	0.17	0.03	0.05	0.08	0.

Table 5. Effect of Property Variations - Plainfield Sand

Percent Error for	$X_{init}=0.1429$ $q°=0.4$ W/cm		$X_{init}=0.2143$ $q°=1.5$ W/cm		$X_{init}=0.5714$ $q°=1.5$ W/cm	
Coefficient	E_T	E_X	E_T	E_X	E_T	E_X
$δ/Le·10$ In Eq. (52)	-0.37	0.98	-0.12	2.57	-8.30	0.99
$Pn·10$ In Eq. (52)	0.36	-1.18	4.55	*	11.17	-1.26
$Ko/Le^*·10$ In Eq. (52)	0.	-0.05	0.	-0.01	0.	0.
$ξ·10$ In Eq. (54)	0.17	-1.55	0.11	-2.76	0.	0.

*Convergence difficulties with iterative solution

Table 6. Effect of Property Variations - Sandy Silt

Percent Error for	$X_{init}=0.3984$ $q^*=0.5$ W/cm		$X_{init}=0.4942$ $q^*=1.2$ W/cm	
Coefficient	E_T	E_X	E_T	E_X
$\delta/Le=0$ in Eq. (52)	0.10	-2.40	0.49	-2.23
$Pn=0$ in Eq. (52)	0.	0.	0.	0.
$Ko/Le^*=0$ in Eq. (52)	0.93	1.30	0.	0.02
$\xi=0$ in Eq. (54)	0.	0.	-0.03	0.23
All as above=0	1.10	-0.20	0.45	-1.91

Table 7. Effect of Property Variations - Sandy Silt

Percent Error for	$X_{init}=0.3984$ $q^*=0.5$ W/cm		$X_{init}=0.4942$ $q^*=1.2$ W/cm	
Coefficient	E_T	E_X	E_T	E_X
$\delta/Le \cdot 10$ in Eq. (52)	-1.22	11.60	0.49	-2.23
$Pn \cdot 10$ in Eq. (52)	0.	0.	0.	0.
$Ko/Le^* \cdot 10$ in Eq. (52)	-10.56	-10.10	0.03	-0.18
$\xi \cdot 10$ in Eq. (54)	-0.19	-8.39	0.26	-2.41
All as above$\cdot 10$	-8.22	3.66	-2.88	7.74

any conditions tested for either soil. The maximum percent
error encountered with the coefficients multiplied by a
factor of ten was about ten percent. This is a qualitative
measure of the effect of an error in property measurements
and is seen to have minimal effect on calculated temperature
and moisture content distributions for the conditions
indicated.

It should be noted, however, that the calculated
moisture content distribution is sensitive to changes in the
values of the coefficients appearing in the conservation of
mass equation. Numerical simulations indicate that the
relative magnitudes of those two coefficients are quite
important. The Posnov number, the dimensionless ratio of
thermal diffusivity to liquid diffusivity, is a direct
measure of the relative magnitudes of the coefficients.
For the buried cable environment, temperature gradients are
generally larger than gradients of moisture content, and a
Posnov number near unity or larger results in net migration
of moisture away from the cable.

On the strength of these results the energy equation
was simplified to the following form

$$(C/C_0) \partial T^* / \partial Fo = \nabla \cdot [(\lambda/\lambda_0) \nabla T^*]$$

which is the conduction equation with an effective
thermal conductivity that depends upon the moisture content

of the soil. Therefore, the energy equation is essentially
uncoupled from the conservation of mass equation. It should
be possible, then, to model the heat transfer and moisture
movement in soil with a numerical scheme which steps forward
the solutions of the two governing equations alternately.
Assuming moderate changes in the temperature distribution
from one time step to the next, the moisture content
distribution could be determined from a solution of the
conservation of mass equation and that distribution used to
calculate the properties needed in the solution of the
energy equation.

Since the temperature has been normalized with
respect to the surface heat transfer rate, a plot of T^*
versus ln Fo is a straight line and represents a universal
curve for specified initial soil conditions in the absence
of property variations caused by moisture movement.

Typical numerical simulations for sand are shown in
Figure 6. The slope of the straight-line portion of the
curve is a measure of the thermal resistivity (reciprocal of
thermal conductivity) of the soil evaluated at the initial
conditions. As the soil begins to dry out next to the heat
source, the curve deviates from this straight line. When
the soil adjacent to the heat source is completely dry a
second straight-line portion with a steeper slope is
achieved. This steeper slope is a measure of the thermal
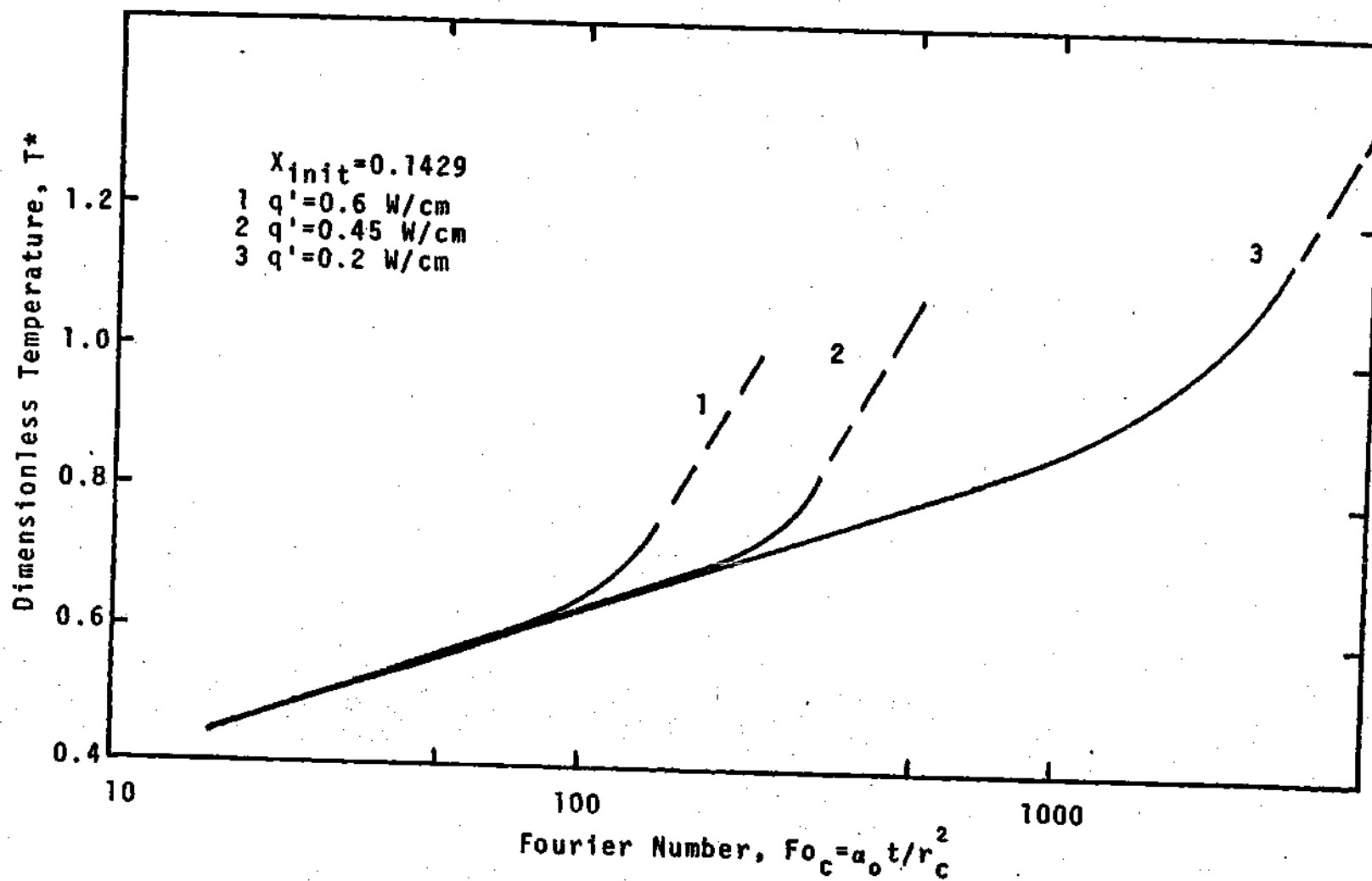resistivity of the dry soil. In both instances, the thermal

Figure 6. Typical Dimensionless Temperature-Time Response
for Plainfield Sand

resistivity can be calculated by applying the simple probe solution, equation (D-5). This is discussed further in later paragraphs.

Figure 6 shows that significant drying occurs at smaller values of the Fourier number as the surface heat transfer rate is increased. The curves of Figure 6 have been extrapolated from the point of complete drying at the surface of the heat source since the numerical solution initially gives unrealistically large temperature increases when the numerical scheme for the moving boundary is used. The large predictions for temperature are caused either by the method used to start the scheme, or by the formulation of the boundary conditions at the moving boundary, or by the rapid changes in property values in the range of moisture contents near θ_{min}.

From the standpoint of drying of the medium, examination of the variation in the rate of moisture movement at the surface of the heat source reveals some distinguishing characteristics. Typical results for the sand and sandy silt with different initial moisture contents are shown in Figures 7 through 10. The figures show that there is a distinct minimum rate of moisture movement for each value of the surface heat transfer rate. The magnitude of this minimum value decreases with decreasing surface heat transfer rate, but the minimum value occurs at essentially the same surface moisture content regardless of the heat
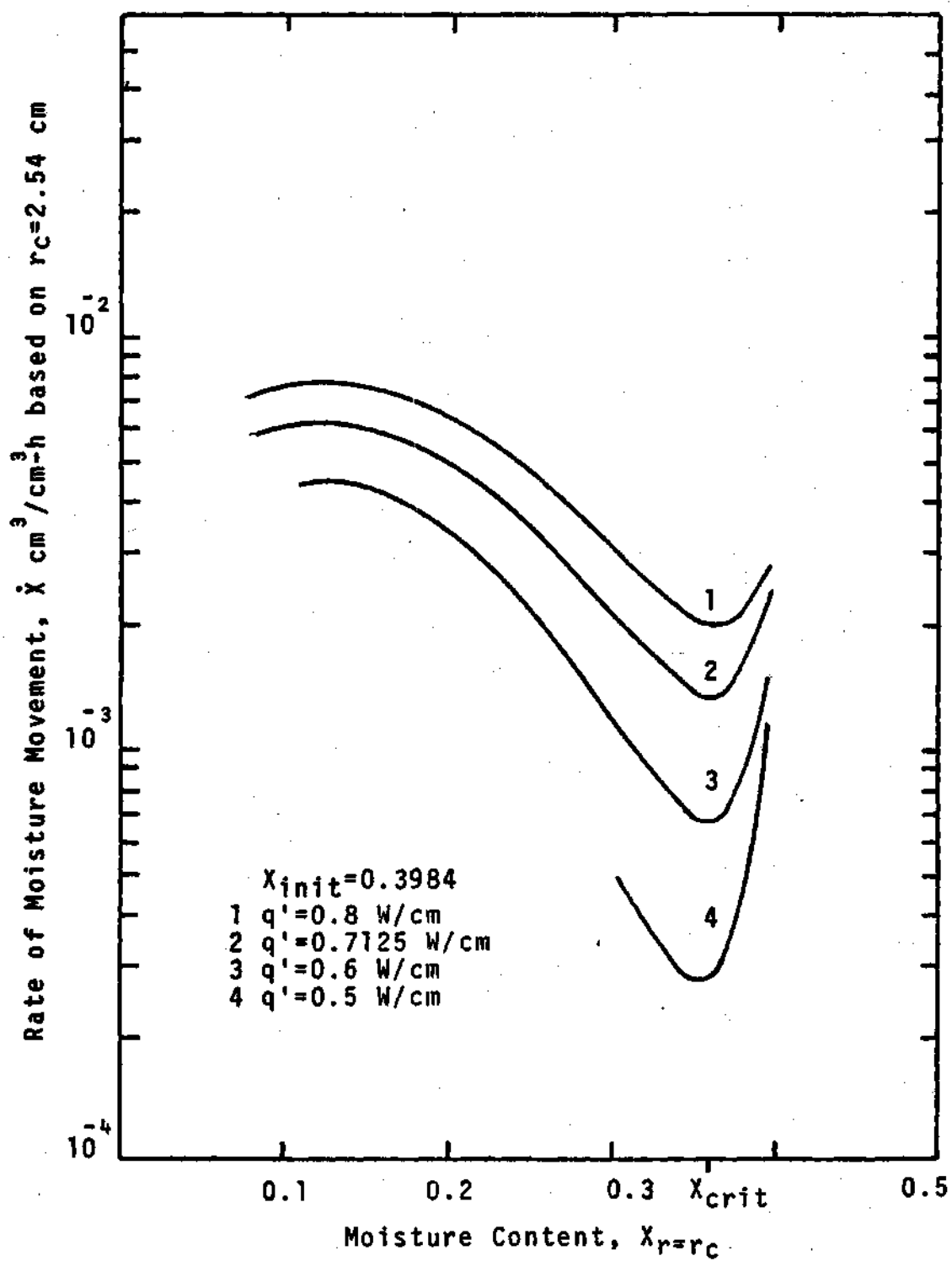
Figure 7. Rate of Moisture Movement at Cable
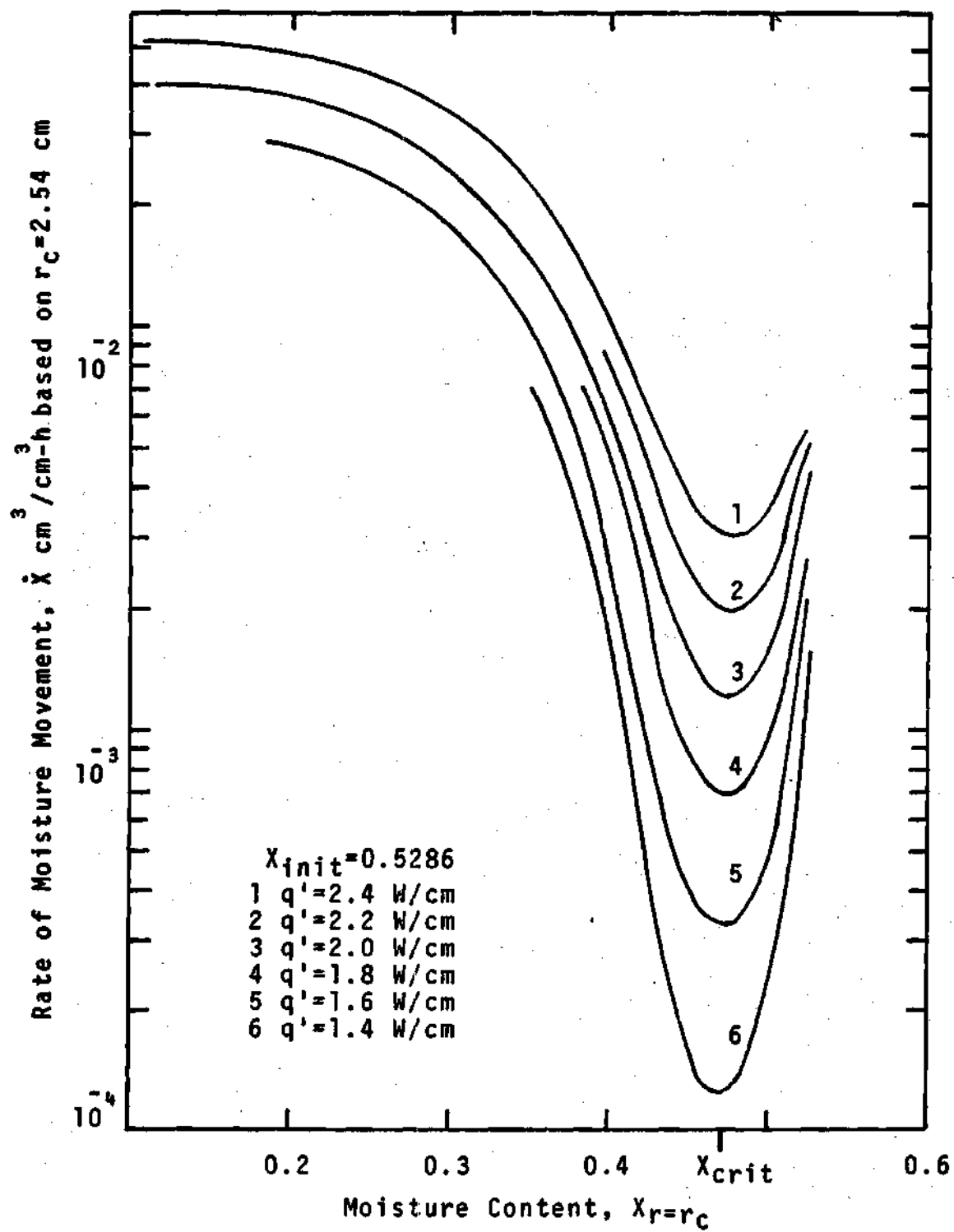Surface for Sandy Silt

Figure 8. Rate of Moisture Movement at Cable
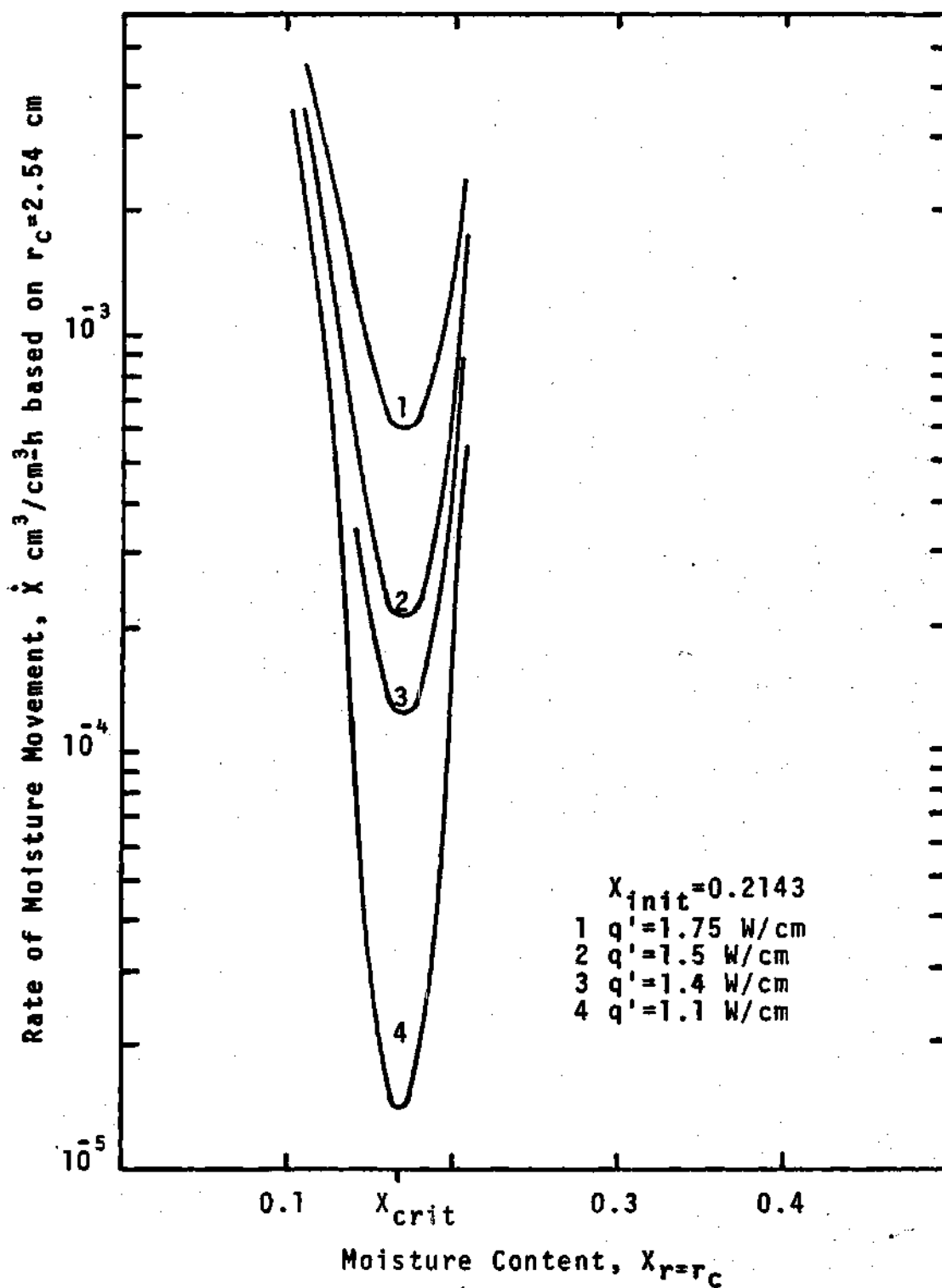Surface for Sandy Silt

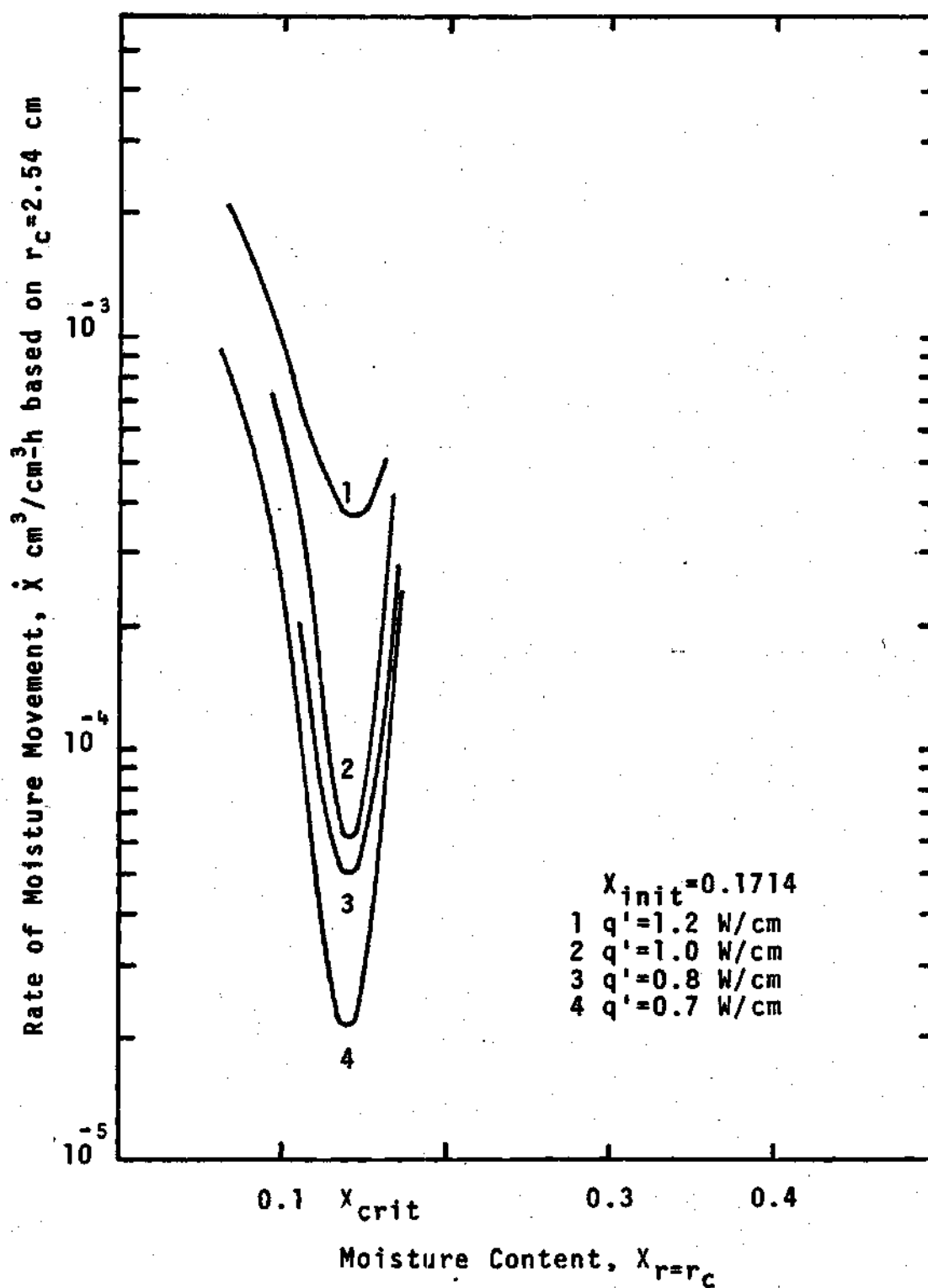Figure 9.   Rate of Moisture Movement at Cable
Surface for Plainfield Sand

Figure 10. Rate of Moisture Movement at Cable
Surface for Plainfield Sand

transfer rate. Furthermore, the value of the moisture content, X_{crit}, at which the minimum rate of moisture movement occurs decreases as the initial moisture content of the medium decreases.

These results are shown more clearly in Figures 11 through 14. The variation of X_{crit} with the initial moisture content, X_{init}, is shown in Figure 14.

Figures 11 and 12 display the value of the Fourier number based on the radius of the heat source at which X_{crit} occurs for various surface heat transfer rates and initial moisture contents. This Fourier number increases as the surface heat transfer rate is decreased. There appears to be a limiting value of the surface heat transfer rate for each initial moisture content for which the time to reach X_{crit} becomes very large. This value will be called the critical heat transfer rate, q^*_{crit}, and is a function of the initial moisture content for a given soil of specified porosity. Figure 13 shows the value of Fo_c for the sandy silt at which complete drying occurs in the soil adjacent to the heat source.

When the surface heat transfer rate is maintained above the value for q^*_{crit} the rate of moisture movement continually decreases until X_{crit} is reached. Thereafter, the rate of moisture movement generally increases continuously until the soil adjacent to the source is completely dried out, provided that a constant surface heat
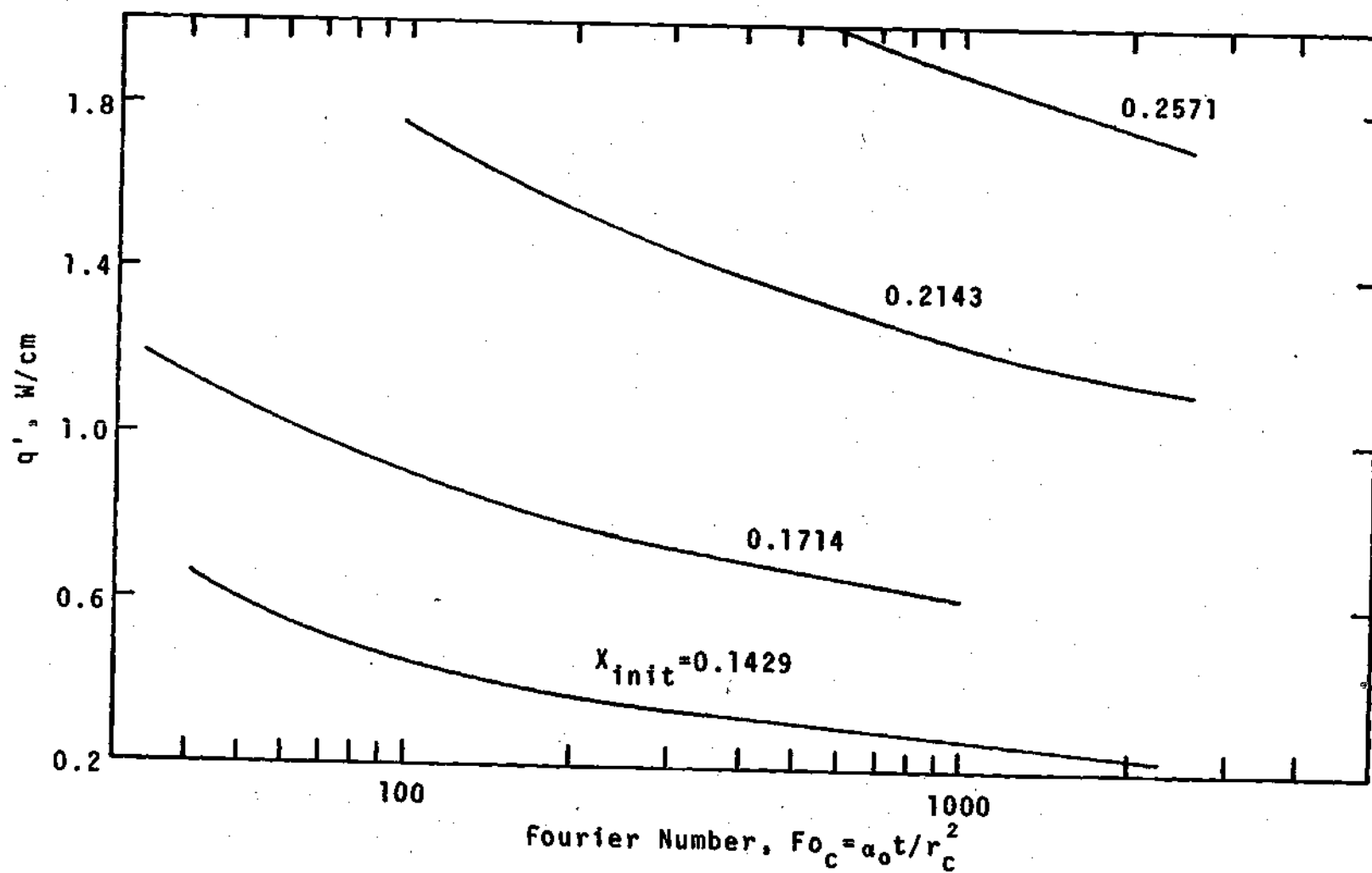
Figure 11. Fourier Number at Attainment of X_{crit} for Plainfield Sand
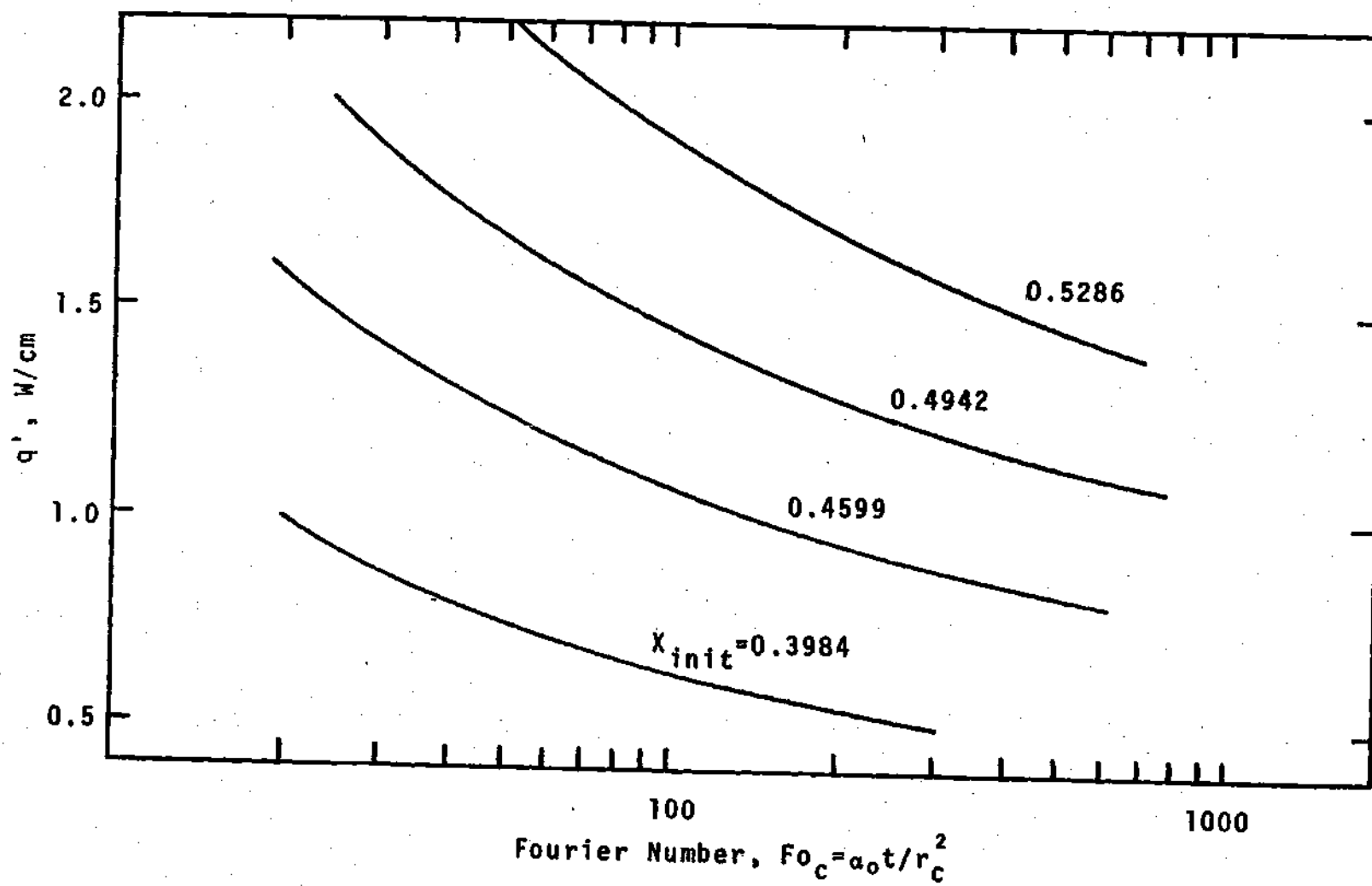
Figure 12. Fourier Number at Attainment of X_{crit} for Sandy Silt

The chart's vertical axis is labeled q', W/cm with values 0.5, 1.0, 1.5, 2.0. The horizontal axis is labeled Fourier Number, $Fo_c = \alpha_o t / r_c^2$ with values 100 and 1000. The curves are labeled 0.5286, 0.4942, 0.4599, and $X_{init} = 0.3984$.

Figure 13. Fourier Number at Complete Dryout at Cable Surface for Sandy Silt
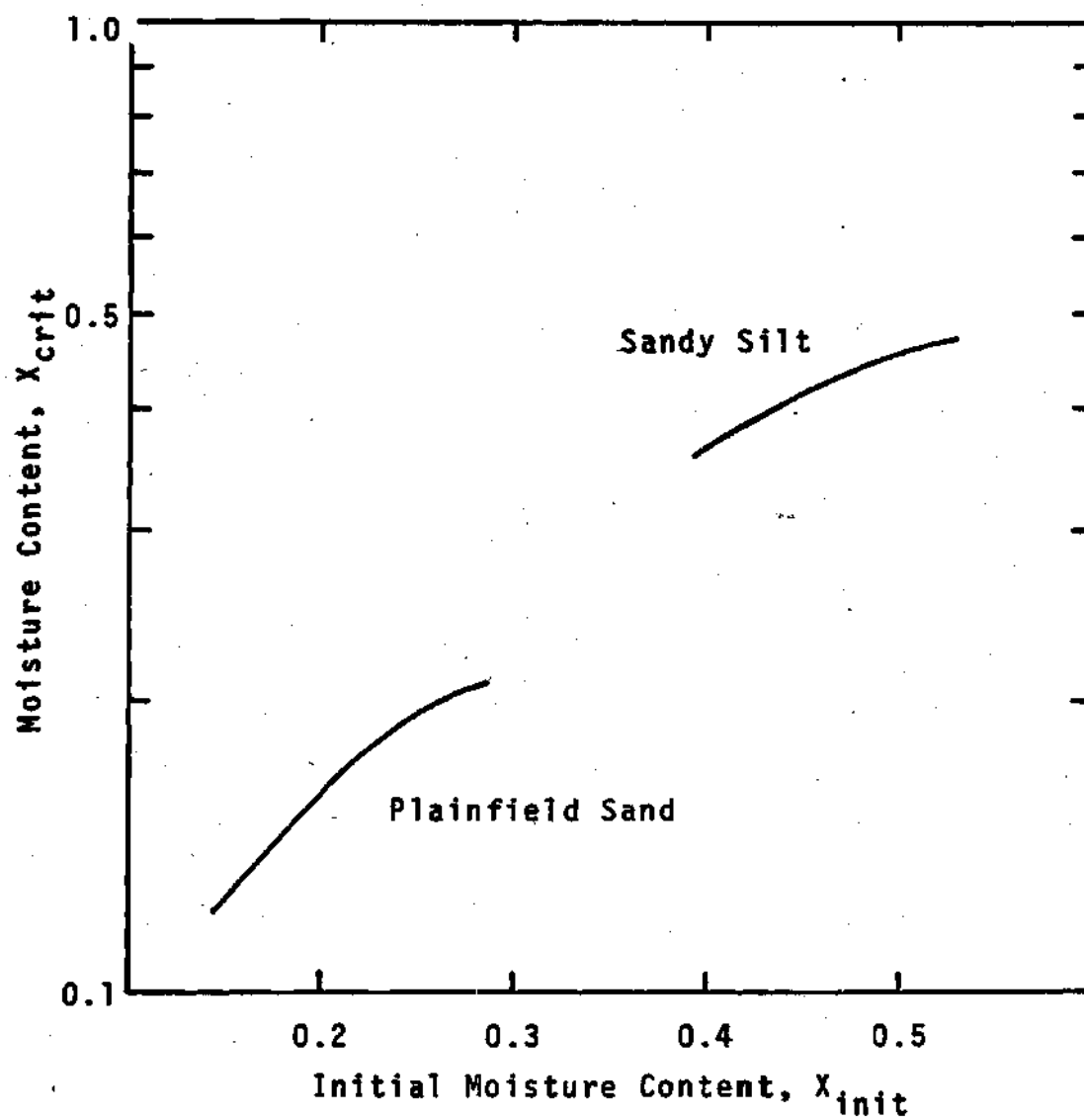
Figure 14.   Dependence of X_{crit} on Initial
Moisture Content

transfer rate is maintained. If the surface heat transfer rate is maintained below the critical value, $q^\bullet_{crit}$, the rate of moisture movement decreases continuously and complete drying of the soil is not expected to occur for a very long time.

With these results it appears to be possible to predict when significant drying will commence if the surface heat transfer rate is specified. Or, alternately, for given soil initial conditions one could prescribe the surface heat transfer rate, $q^\bullet_{crit}$, below which the soil will remain thermally stable.

Cable surface temperatures calculated based on the moisture content X_{crit} would therefore predict conservative values which would not be exceeded during operation if the surface heat transfer rate is maintained below $q^\bullet_{crit}$.

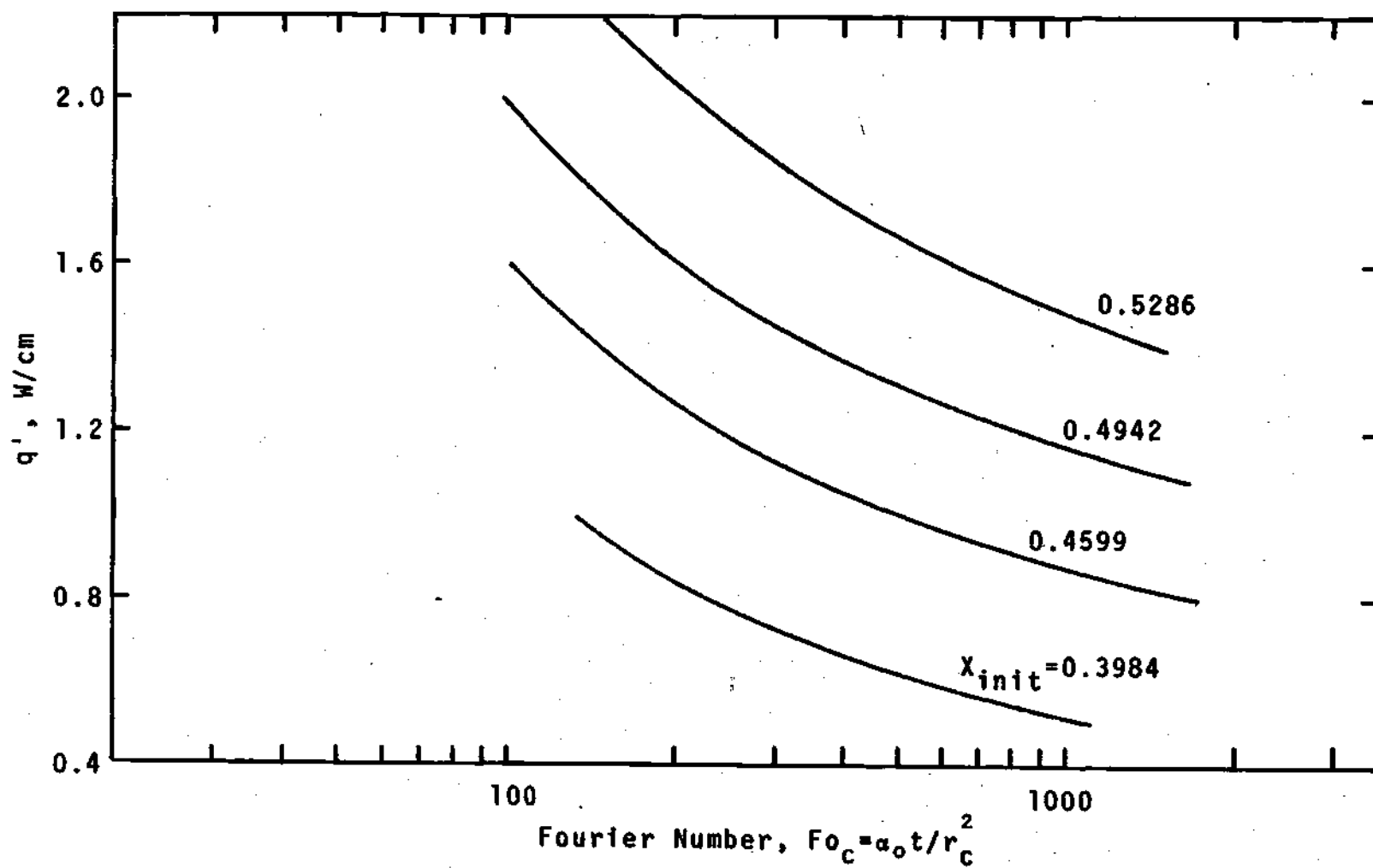The results presented for sand are based upon hydraulic and thermal properties of Plainfield sand measured by Jury [73]. No experimental verification of the numerical simulations for the Plainfield sand was performed, but the trends of the numerical results are meaningful in that they closely resemble those of the sandy silt. Furthermore, some preliminary experimental verification for the model predictions based on the sandy silt have been made with two very different experiments.

In the first experiment, Bush [14a] used a 1.83 m length of heated cable (2.54 cm outside radius) buried in a

plexiglas tank (.91 m by .91 m by 1.83 m) containing the sandy silt. The sandy silt had a dry density of 1.28 gm/cm^3 (80 lb/ft^3) and an initial moisture content of about 21 percent by dry weight (X=.53). Two experiments were conducted with this soil. In the first, the surface heat transfer rate was maintained at approximately 0.433 W/cm. No difference in moisture content was discernable after 400 hours (Fo$_c$= 1000) of operation. Figure 12 indicates that this heat transfer rate is well below q$^*_{crit}$ for the prevailing initial moisture content, so that significant differences in the surface moisture content should not be expected.

The surface heat transfer rate for the second test [14a] was selected such that significant moisture movement would occur. Therefore, 1.77 W/cm was used and was maintained constant for about 20 hours at which time it increased to an average value of 1.95 W/cm. Figure 12 shows that for a heat transfer rate of 1.95 W/cm, q$^*_{crit}$ is exceeded and X$_{crit}$=0.475 (19 percent moisture by dry weight) should be reached at a Fourier number of about 95 (37 hours). The moisture content near the cable surface measured by Bush [14a] was 19.9 percent by dry weight at 33 hours and 18.3 percent at 53 hours. The measured moisture content distribution [14a] at the end of 78 hours, when the test was terminated, is shown in Figure 15 along with the distribution predicted with the numerical model. The
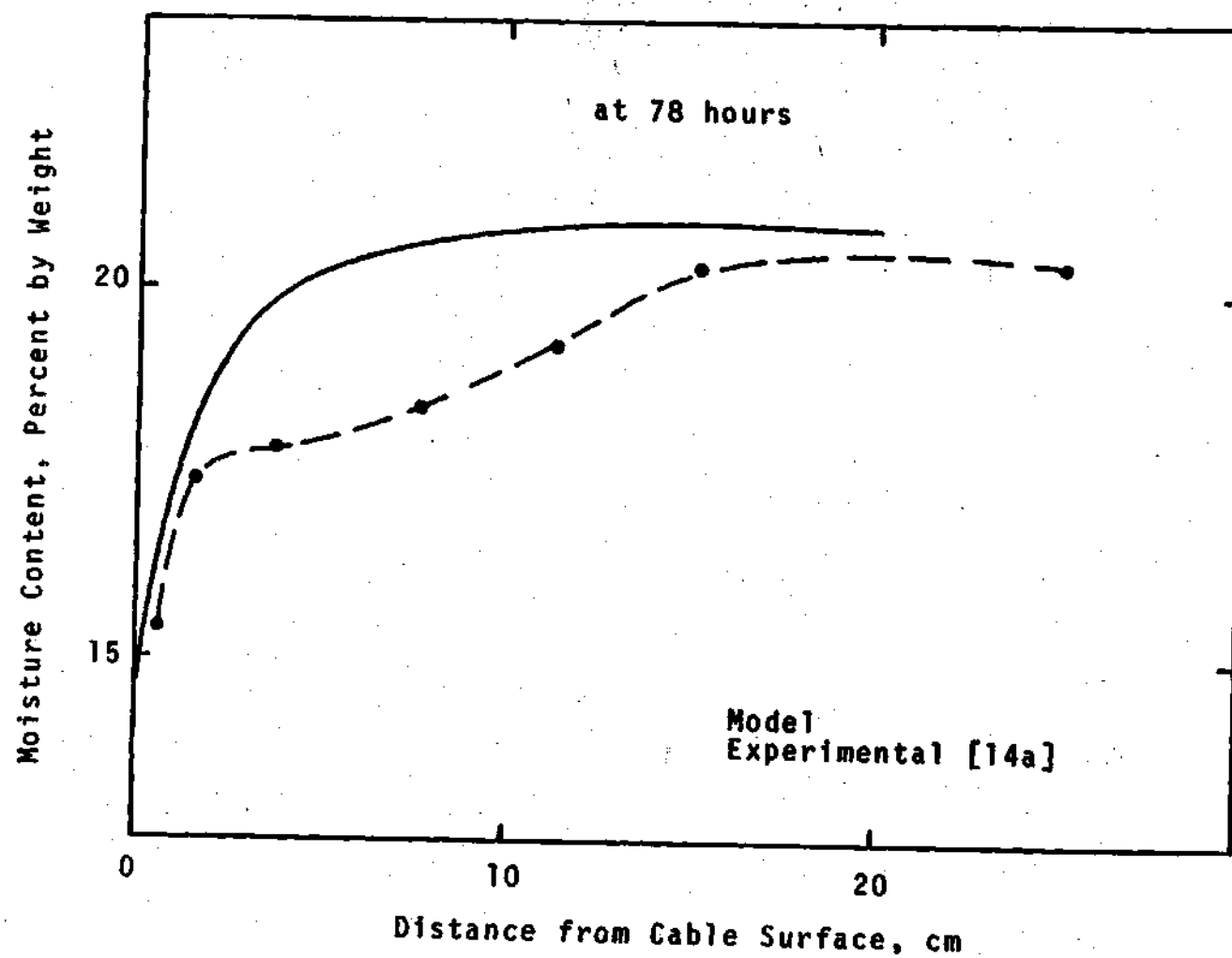
Figure 15.   Comparison of Predicted and Observed
Moisture Content Distributions

moisture content at the surface agrees well with experimental observations but model predictions of moisture content further away are higher than observed experimentally.

In a second experimental check of the model, Handgraaf [62a] measured the thermal conductivity of the sandy silt under consideration using the probe method [33,83,92,93]. The results of several tests on the sandy silt with a dry density of 1.28 g/cm^3 (80 lb/ft^3) are shown in Figures 16 through 18. Power inputs to the probe were selected such that drying would occur. The figures indicate the extent of drying and, in fact, drying of the specimen was visibly evident in each case. The predictions of the mathematical model are lower than observed probe temperatures in each case. This is due to the fact that the thermocouple used in the probe is located not at the probe surface but at a location equal to approximately one-half of the probe radius. Although moisture content measurements were not made during the period of the test, it is clear that the point at which the model predicts deviations from the straight-line portion due to moisture content changes agrees very well with the experimental curves. As discussed previously, such deviations are a measure of the influence of moisture movement. These curves also show the secondary straight line which is indicative of the thermal conductivity of the dry soil. The model generally predicts
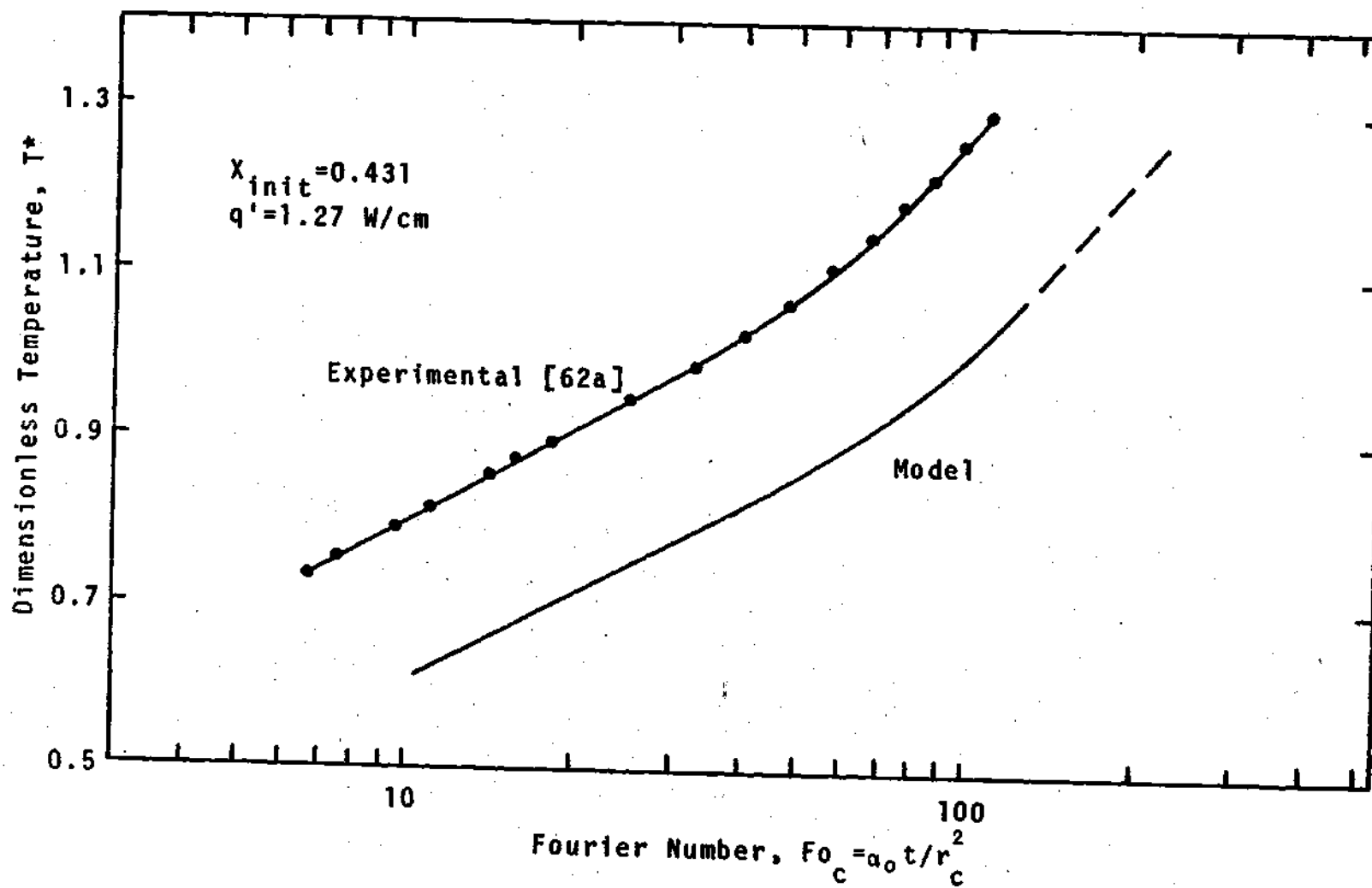
Figure 16. Comparison of Predicted and Observed
Onset of Drying of Sandy Silt
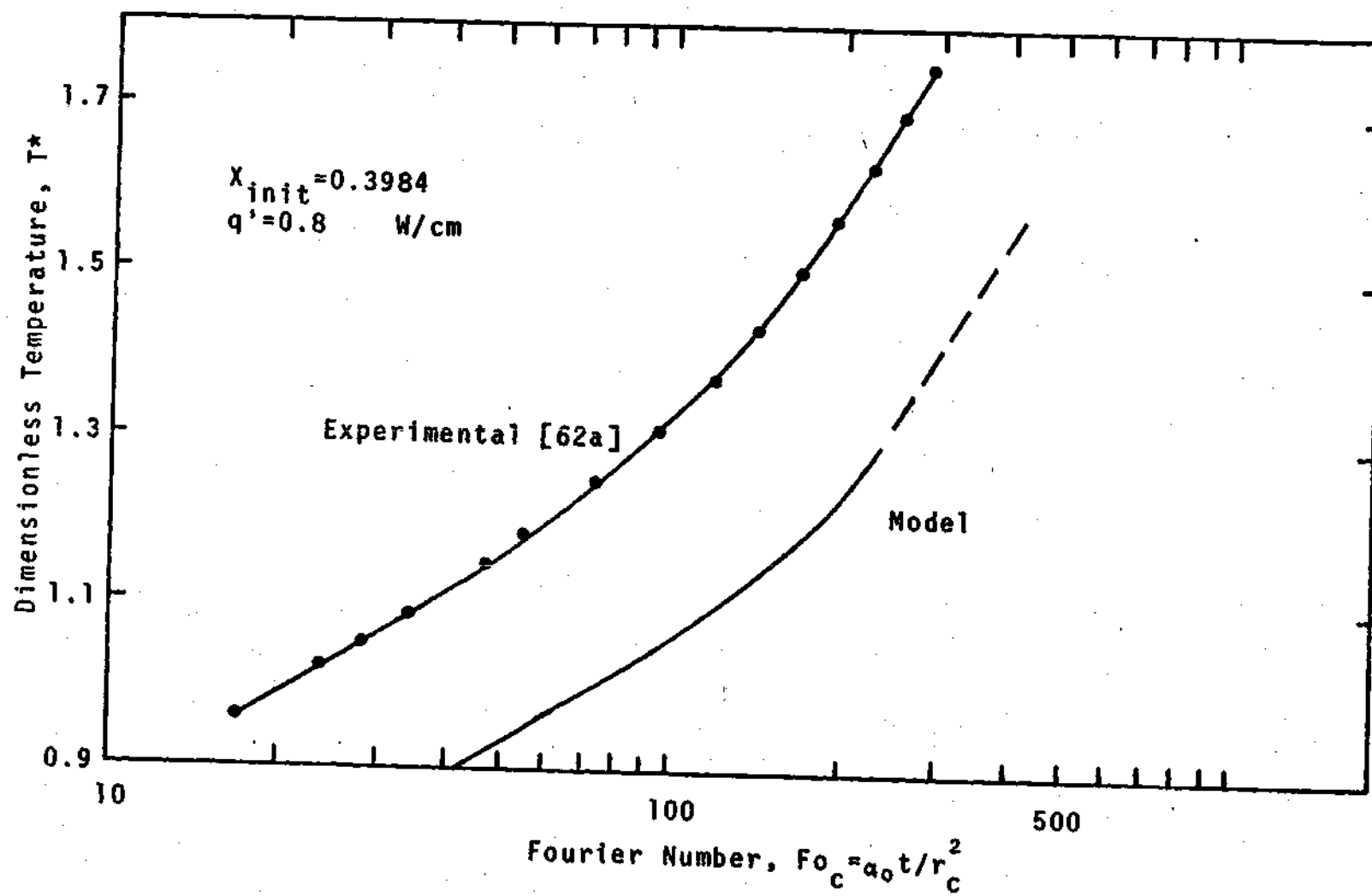
Figure 17. Comparison of Predicted and Observed
Onset of Drying of Sandy Silt
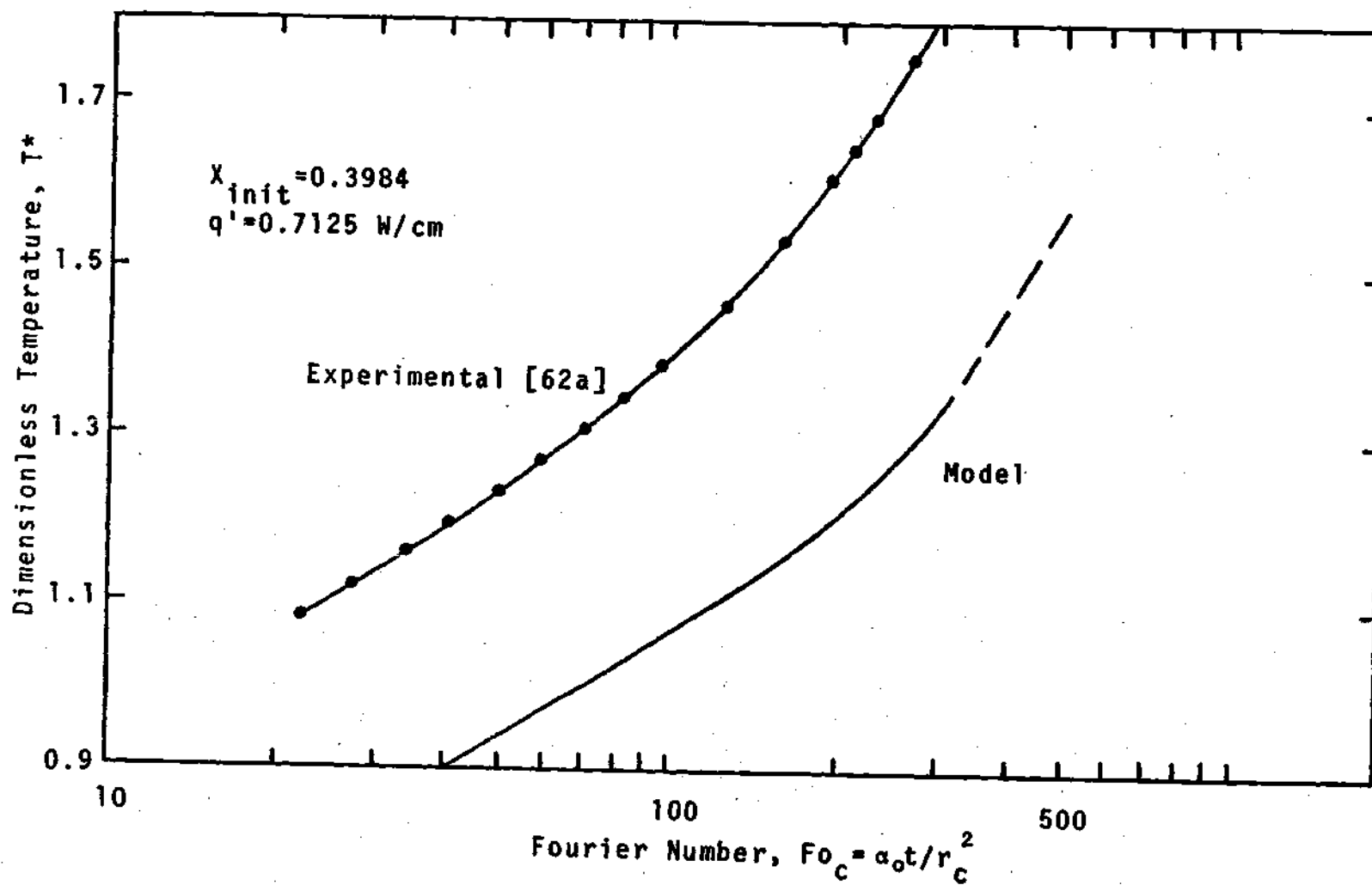
Figure 18.   Comparison of Predicted and Observed
Onset of Drying of Sandy Silt

that drying occurs somewhat sooner than actually observed experimentally.

From a single needle probe test, therefore, it is possible to

(a) Determine the thermal conductivity of the moist soil,

(b) Determine the thermal conductivity of the dry soil,

(c) Determine the approximate time for the onset of drying adjacent to the surface of the source, and

(d) Determine when drying is complete in the soil adjacent to the source.

The latter two factors have been discussed, an example of the first two is now given. Figure 19 shows experimental results of thermal conductivity measurements made by Bush [14a] with a needle probe on the sandy silt over a wide range of moisture contents. The solid line is a fourth-order least-squares curve fit of the experimental data. Figure 17 shows the results of independent needle-probe measurements on a sample of the sandy silt with the same dry density and an initial moisture content of 15.9 percent by dry weight $(X=0.3984)$. For this test the heat dissipation rate of the probe was selected such that drying of the soil would occur. The thermal conductivity of the moist soil from the primary straight-line portion was calculated to be 0.01077 W/cm-$^{\circ}$C using equation (D-5). Using the secondary straight line, which is indicative of
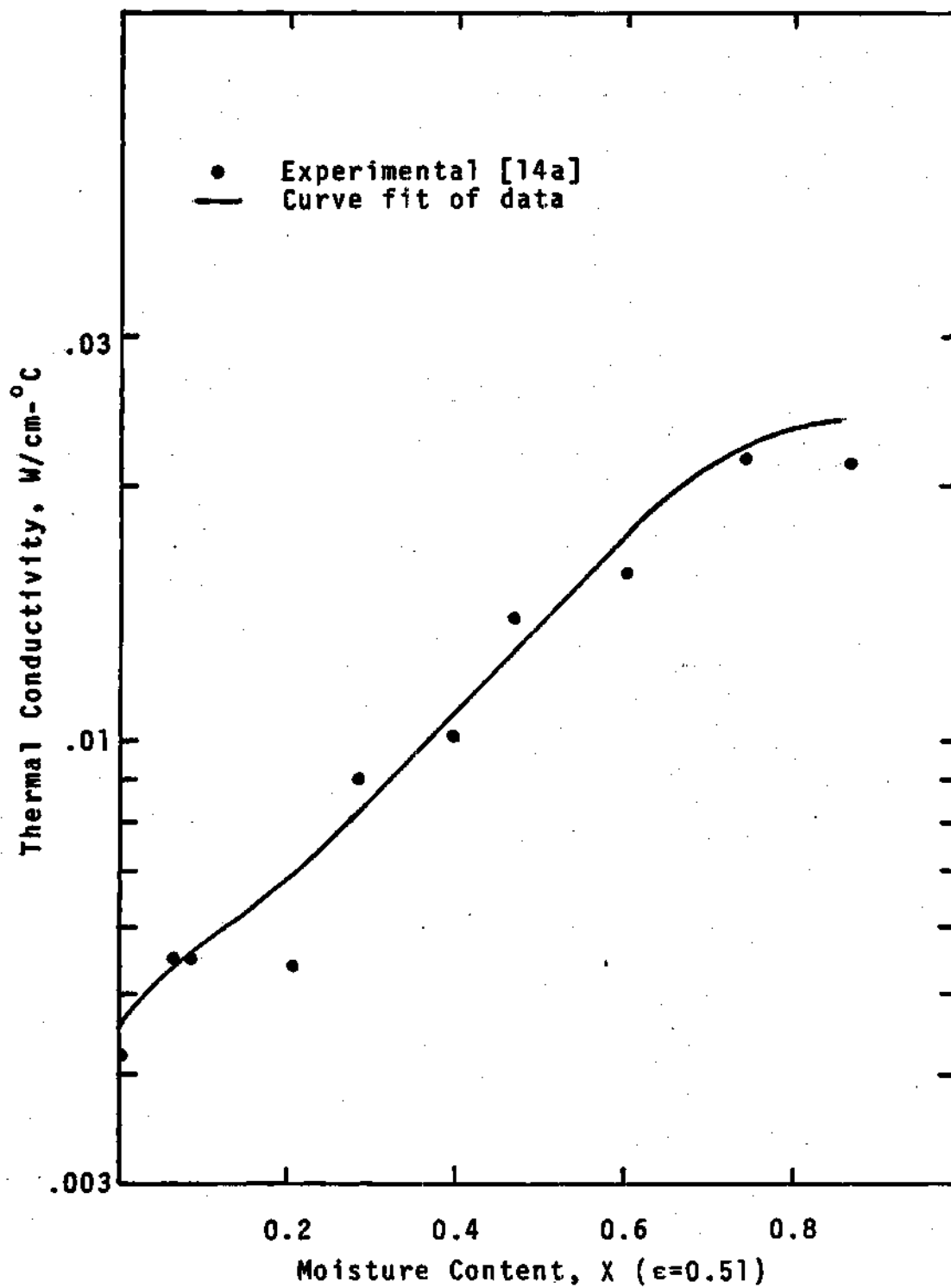
Figure 19. Variation of Thermal Conductivity with Moisture Content for Sandy Silt

complete drying adjacent to the source, the thermal conductivity of the dry soil was calculated to be 0.00422 W/cm-^{o}C. The corresponding values from the least-squares fit of the data of Figure 19 are 0.01076 W/cm-^{o}C and 0.00456 W/cm-^{o}C, respectively. The thermal conductivity of a dry sample of this soil is reported by Bush [14a] to be 0.00425 W/cm-^{o}C.

As a practical application of these results one could survey the soil in a proposed cable route using a needle probe. By examining the temperature-time response of the probe for given heat dissipation rates, the time, t_p, required for the onset of significant drying in the soil adjacent to the probe can be determined. The response of a cable buried in the same soil with the same heat dissipation rates would be related to the probe response through the Fourier number. That is, significant drying at the cable would be expected to occur at the time $t_c = t_p (r_c/r_p)^2$, where the subscripts c and p refer to the cable and probe, respectively.

If q* is fixed by cable design and the thermal response of the native soil proved unacceptable, the response of the soil at higher densities could be investigated. One could thus determine if proper compaction of the backfill would enhance the thermal response of the soil. Otherwise, special backfills or soil additives should be considered.

The numerical scheme presented in Chapter VII is not capable of simulating the effect of surface evaporation or infiltration, daily temperature variations, the influence of the presence of a water table, or gravity effects. In order to study such phenomena the numerical scheme must be extended to the two-dimensional, transient formulation.

Since a constant heat flux was used in all numerical simulations, the results apply to actual cyclical load operation only in the sense of an average heat flux at a load factor of about 100 percent. To determine the qualitative effect of a change in load factor, one simulation was conducted with sandy silt (X_{init}=.5285, Figure 8) at q"=1.8 W/cm initially. When the moisture content was within 10 percent of X_{crit}, q" was reduced to 1.44 W/cm. The rate of moisture movement decreased rapidly and thereafter followed approximately the same X curve as if the heat flux were initially at 1.44 W/cm. The time to reach X_{crit} was 60 percent longer than if q" had been maintained at the initial value. Further work must be done, however, before any general conclusions can be made.

Finally, it should be noted that substantial scatter in the data can occur if a very small diameter probe is used in a very porous (low density) soil. At high porosities the air spaces between the soil particles can approach the same size as the diameter of the probe, and use of a larger probe diameter may eliminate some of the scatter in the data.

CHAPTER X

CONCLUSIONS

This analytical study of the heat transfer and moisture movement in the soil surrounding a cylindrical heat source has shown that:

(1) The coupled equations used to describe the temperature and moisture content distributions in the soil can be simplified somewhat by determining the influence of the dimensionless coefficients. The conduction equation with moisture-dependent thermal conductivity is adequate to determine the temperature distribution.

(2) The drying of the soil adjacent to the heat source occurs in two distinct stages. During the first stage the rate of moisture movement decreases. This decrease continues until a moisture content, X_{crit}, is reached after which the rate generally increases until complete drying of the soil adjacent to the heat source has occurred.

(3) The value of the moisture content X_{crit} is essentially independent of the surface heat transfer rate, and is a function of the initial moisture content of the soil for a given porosity. The magnitude of X_{crit} decreases as the initial moisture content decreases.

(4) The attainment of X_{crit} and the point of drying out predicted by the numerical model agree favorably with preliminary experimental observations for one soil.

(5) For a given soil (specified porosity and initial moisture content) the model indicates that there is a surface heat transfer rate below which significant drying will be delayed considerably, possibly indefinitely. If the surface heat transfer rate is sufficiently high a needle-probe test of a soil sample can provide information concerning the thermal conductivity of the moist soil, the thermal conductivity of the dry soil, the onset of drying at the surface of the probe, and the time when drying at the surface of the probe is complete. From this information, it should be possible to predict the behavior of the same soil in the vicinity of a larger buried source. The corresponding times, being related through the Fourier number, are in inverse proportion to the square of the diameters. Thus the needle-probe test could also be a test for the thermal stability of the soil.

(6) Accurate descriptions of the thermal and hydraulic properties of the soil over the entire range of moisture contents are necessary for the analytical model.

CHAPTER XI

RECOMMENDATIONS

The temperature and moisture content distributions in the soil surrounding a buried heat source can be predicted with the model if a complete set of property measurements are available for the soil of interest. The predictions of the attainment of X_{crit} could be used to select the limit for the surface heat transfer rate or to select the proper density and moisture content to be used with cable backfill materials.

Before this can be done, however, the results of this study should be extended to include the effect of the porosity (or dry density). Further efforts might also include determination of the effect of surface phenomena, or the presence of a water table. With further experimental verification of the predictions, the simplest and most economical method of providing a profile of the thermal stability of a soil may be with a comprehensive program of needle-probe tests. The effort involved with such tests is much less than would be expanded in developing the thermal and hydraulic properties needed for the analytical model. Even though the probe tests do not lend themselves to determination of detailed temperature and moisture content

distributions, they do appear to be able to provide
essential information, i.e. the time required for
significant drying to occur, as well as the time required
for complete drying. Accurate prediction of those times for
a cable-soil system whose initial and operating conditions
are known is, after all, the objective as far as thermal
stability is concerned. Tests should also be conducted with
cyclically varying surface heat transfer rates in order to
simulate actual cable operating conditions. Finally, the
influence of soil additives and special backfills should be
studied using the same procedures.

APPENDIX A

CORRELATION OF MOISTURE RETENTION DATA

The soil moisture characteristic shown in Figure A-1 displays several traits which are common to most soils:

(a) $\Psi \to \infty$ as $S_e \to 0$

(b) $\Psi \to 0$ as $S_e \to 1$

(c) A region of relatively constant slope at intermediate values of the effective saturation $S_e = (S-S_r)/(1-S_r)$, where S_r is called the residual saturation [13]. The residual saturation is that value of S at which effectively goes to infinity.

One function which is able to reproduce this typical shape, and which is proposed for the correlation of moisture retention data, is

$$\Psi = aS_e^b(1-S_e)^c[Coth(\pi S_e)]^d \qquad (A-1)$$

The first step necessary in the correlation procedure used to determine the constants a,b,c,d, and S_r is to plot $\Psi(S)$ from moisture retention data as shown in Figure A-1.

From this curve, estimate the value of S at which approaches a vertical asymptote. This value of S is the approximate value of the residual saturation S_r. (While
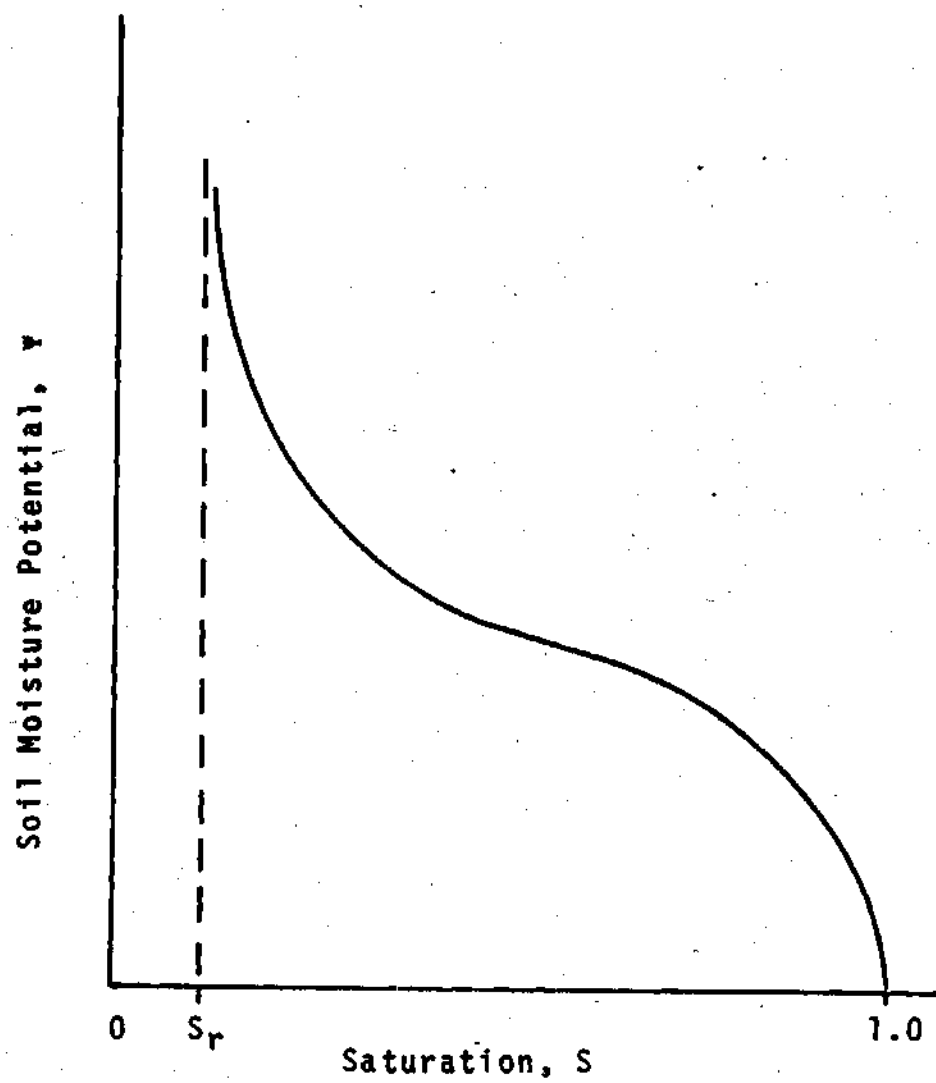
Figure A-1. Typical Soil Moisture Characteristic Curve

this step requires a visual estimate of S_r, the actual value of S_r is not critical to the procedure. An approximate value is all that is necessary). This first step is identical to that used by Brooks and Corey [13].

Next, with S_r thus determined, calculate the effective saturation values. The (Ψ, S_e) data pairs thus obtained are used in a least-squares curve fit.

APPENDIX B

CALCULATION OF HYDRAULIC CONDUCTIVITY

The modified Millington-Quirk method [70] of calculating hydraulic conductivity from soil mositure retention data was selected for use in this study.

The general equation used to calculate the hydraulic conductivity is [51,57,70]

$$K(\theta)_i = (K_s/K_{sc})[30\gamma^2\varepsilon^p/(\rho g n n^2)]\sum_{j=1}^{n}[(2j+1-2i)h_j^{-2}] \qquad (B-1)$$
$$\text{for } i=1,2,\ldots,n$$

where

K(θ)$_i$ is the calculated hydraulic conductivity (cm/min) for a specified moisture content θ_i,

i denotes the last moisture content class on the wet end, for example i=1 identifies the pore class corresponding to the moisture content at saturation, and i=n identifies the pore class corresponding to the lowest moisture content for which hydraulic conductivity is calculated [51],

K_s/K_{sc} is the matching factor (ratio of measured hydraulic conductivity at saturation to calculated hydraulic conductivity at saturation),

γ, ρ and η are the surface tension, density and viscosity, respectively, of water,

ε is the porosity of the soil, and

h is the soil moisture potential.

In the Millington-Quirk method $p=4/3$ and n is the number of pore classes between $\theta=0$ and saturation. The constant term $30\gamma^2/\rho g \eta$ has a value of 1.884 $cm\text{-}mb^2/min$ at 27°C. When h is given in units of millibars, then $K(\theta)_i$ has units of cm/min.

The computer program used to calculated $K(\theta)$ values is a slightly modified form of the one used by Reisenaur [119].

The values of $K(\theta)$ calculated using the modified Millington-Quirk method were used, together with (ψ,θ) data to fit the hydraulic conductivity curve to the form

$$K = K_s / [(\psi/a)^b + 1] \qquad (8\text{-}2)$$

proposed by Gardner [50].

APPENDIX C

THERMAL CONDUCTIVITY OF MOIST SOIL

The thermal conductivity of the moist soil, λ_*, can be calculated from the Van Rooyen formula [145]

$$1/\lambda_* = 10^{A-Bm} + S \qquad C-cm/W \qquad (C-1)$$

where m is the moisture content as a fraction of the voids filled with water, B and S are linear functions of the density of the dry soil, ρ_0, and A is a quadric function of ρ_0.

The quantity m is related to the volumetric moisture content, θ, and the porosity, ε, by

$$m = \theta/\varepsilon \qquad (C-2)$$

The dry density can be calculated from the porosity and the density of the solid material, ρ_{solid}, using the relation

$$\rho_0 = (1-\varepsilon)\rho_{solid}$$

For example, the dry density of sand whose solid component is quartz ($\rho_{solid} = 2.66 \text{ g/cm}^3$) is

$$\rho_0 = 2.66(1-\epsilon) \qquad g/cm^3$$

The quantity S is related to the dry density, ρ_0, by

$$S = S_1 - S_2\rho_0$$

where S_1 and S_2 are functions of the type of material. For sand $S_2=47.5$ and $S_1\doteq113.3$.

The quantity B is calculated from

$$B = b_1 - b_2\rho_0$$

where b_1 and b_2 are functions of the soil composition and granulometry. Van Rooyen found that $b_2\doteq5.5$ for all soils tested and b_1 is about 15-20 for sand.

The quanity A is calculated from

$$A = a_1 + a_2\rho_0 + a_3\rho_0^2$$

Van Rooyen found that $a_2\doteq0$ and $a_3=-0.44$ for all materials tested. For sand, the value of a_1 is about 3.6.

Therefore, the thermal conductivity of a sand can be calculated as (using $b_1=18.71$) a function of moisture content and porosity using

$$1/\lambda_* = 10^{\{3.6-.44\rho_0^2-\theta(18.71-5.5\rho_0)/\epsilon\}}+113.3-47.5\rho_0$$

where $\rho_0 = 2.66(1-\epsilon)$ for sand.

To account for the variation of s_1, s_2, a_1 and b_1 with different materials and soils, additional parameters used are the percent quartz, q, percent clay, c, and the surface area per unit volume of material larger than clay size, r.

Thus for soils other than sand the following relations apply:

(a) For quartz pecentage q > 75%

$$S_2 = -47.5, \text{ and } S_1 = 200-0.94q$$

For quartz percentage 20% < q < 75%

$$S_2 = 2q-200, \text{ and } S_1 = 435-4.07q$$

(b) $a_1 = 16r^{-.9} + 3.4 \cdot 10^{-.008c}$

(c) $b_1 = 5.6 \cdot 10^{-.04c} + 9.58$

APPENDIX D

PROBE SOLUTION

The exact solution for the problem involving a cylindrical heat source embedded in an infinite medium having constant thermal properties is given by Ingersoll, et al [66] as

$$T - T_i = \Delta T = (q'/2\pi\lambda) \int_x^\infty (e^{-\beta^2}/\beta) \, d\beta \qquad (D-1)$$

where $x = r/\sqrt{4\alpha t} = L/2Z^{1/2}$, $L = r/R'$, $Z = t/R^2$, and R' is some characteristic dimension.

If we let $u = \beta^2$ then $du/u = 2d\beta/\beta$ and

$$\lambda \Delta T/q' = (1/2\pi) \int_{x^2}^\infty (e^{-u}/2u) \, du = E_1(x^2)/4\pi \qquad (D-2)$$

where $E_1(x^2)$ is the exponential integral.

For values of $0 \leqslant x \leqslant 1$, Abramowitz and Stegun [1] give the following series expansion for $E_1(x)$:

$$E_1(x) + \ln x = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \epsilon(x) \qquad (D-3)$$
$$|\epsilon(x)| < 2 \cdot 10^{-7}$$

where $\quad a_0 = -0.57721566 \qquad a_3 = +0.05519968$

$$a_1 = +0.99999193 \qquad a_4 = -0.00976004$$

$$a_2 = -0.24991055 \qquad a_5 = +0.00107857$$

For values of x>1, the series expansion is [1]

$$e^X \ x \ E_1(x) = (x^2 + a_1 x + a_2)/(x^2 + b_1 x + b_2) + \epsilon(x)$$

$$|\epsilon(x)| < 5 \cdot 10^{-5}$$

where

$$a_1 = +2.334733 \qquad b_1 = +3.330657$$

$$a_2 = +0.250621 \qquad b_2 = +1.681534$$

Equation (D-2) may be written in terms of the variables which appear in Chapter VI by making the following definitions.

For $Z = \alpha t/R^2$, let $R = r_{inf}$ the "unaffected" radius. Then

$$Z = (\alpha t/r_\infty^2)(\alpha_0/\alpha_0) = \alpha^* Fo$$

Then $L = r/R' = r/r_\infty = R$, and $L^2/4Z = R^2/(4\alpha^* Fo)$.

Since $T^* = 2\pi\lambda_0\Delta T/q'$ equation (D-2) becomes

$$T = (\lambda/\lambda_0)E_1(L^2/4Z) = (1/2B)E_1(L^2/4Z) \qquad (D-4)$$

Equation (D-4) represents the solution for the constant property problem. In the actual case, property variations occur as the temperature and moisture content of

the medium change. However, if the properties of the medium are evaluated at the initial moisture content and initial temperature, and used in equation (D-4) the results can be compared to those of the variable property case to determine the relative importance of property variations.

For nearly all conditions of practical interest for the buried cable problem, the parameter $x = r/\sqrt{4_\alpha t}$ is less than unity. Furthermore, the author has found that using only two terms of equation (D-3) gives sufficient accuracy for most calculations. Therefore, equation (D-4) may be approximated by

$$T^* = (1/2B)[\ln(4\alpha^* Fo/R^2) - 0.5772] \qquad (D-5)$$

APPENDIX E

PROPERTY DATA CURVE FITS

Property data for water taken from the sources indicated were curve fitted using the least-squares method for use in the computer program.

(a) Surface tension of water [70a]

$$\sigma = 72 \text{Exp}(0.796494 - 0.002657T) \quad \text{dynes/cm for } 273K < T < 373K$$

where the temperature coefficient of the surface tension is

$$\gamma = (1/\sigma)d\sigma/dT = -0.002657 \quad K^{-1}$$

(b) Density of saturated water vapor [75a]

$$\rho_S = \text{Exp}[12.430038 - 4834.12174/T] \quad \text{kg/m}^3 \text{ for } 273K < T < 373K$$

(c) Heat of vaporization of water [75a]

$$h_{21} = 2896784.15 - 712.526093T - 2.28578216T^2 \quad \text{J/kg for } 273K < T < 373K$$

(d) Viscosity of water [70a]

$$\mu = 5.5869309 \times 10^{-24} \text{Exp}[-1.2103154 \times 10^{-4}T^2 + 0.1322294 8T$$

$$+ 7.3816066 \times 10^3/T] \quad \text{centipoise for } 273K<T<373K$$

(e) Saturation pressure for water [75a]

$$P_s = -438.469849 + 4.69673651T - 0.0202296871T^2$$

$$+ 4.38335472 \times 10^{-5}T^3 - 4.78239174 \times 10^{-8}T^4$$

$$+ 2.10382906 \times 10^{-11}T^5 \quad \text{psia for } T \text{ in } {}^oR$$

APPENDIX F

LIST OF SYMBOLS

A_i general coefficients

a volumetric air content, cm^3/cm^3

B_i general coefficients

C volumetric heat capacity, cal/cm^3-^oC

c specific heat, cal/g

D_{atm} diffusion coefficient of water vapor in air, cm^2/s

D_T thermal moisture diffusivity, $cm^2/s-^oC$

D_θ isothermal moisture diffusivity, cm^2/s

E rate of evaporation, s^{-1}

Fo Fourier number, dimensionless; $Fo = \alpha_0 t/r_\infty^2$, $Fo_c = \alpha_0 t/r_c^2$

f pore geometry factor, dimensionless

g acceleration of gravity, cm/s^2

h relative humidity, dimensionless

h enthalpy(with subscript), cal/g

I rate of conversion, g/cm^3-s

$\vec{j}$ mass flux, g/cm^2-s

K unsaturated hydraulic conductivity, cm/s

Ko Kossovich number, dimensionless

Le, Le^* Lewis numbers, dimensionless

M molecular weight, $g/gmol$

N number of finite difference nodes, dimensionless

P   absolute pressure, atm

P $(=\Delta Fo/\Delta R^2)$, dimensionless

Pn   Posnov number, dimensionless

Q $(=q^*/2\pi\lambda_0)$, ^{o}C

q^* heat transfer rate per unit length, W/cm

q''' heat flux, cal/cm^2-s

R   universal gas constant, cal/gmol-K

R $(=r/r_{inf})$ radial position, dimensionless

r   radial position, cm

S $(=\theta/\varepsilon)$, saturation, dimensionless

T   absolute temperature, K

T^* temperature, dimensionless

t   time, s

U   internal energy per unit volume, cal/cm^3

u   specific internal energy, cal/g

V   volume, cm^3 or velocity cm/s

W   differential heat of wetting, cal/g

X   moisture content, dimensionless

z   vertical coordinate, positive upward, cm

Greek Symbols

α tortuosity, dimensionless or thermal
diffusivity, cm^2/s

β $(=d_{\rho_S}/dT)$, g/cm^3-^{o}C

γ temperature coefficient of surface tension, $^{o}C^{-1}$

δ unnamed parameter, dimensionless

ε porosity, cm^3/cm^3

ζ temperature gradient factor, dimensionless

η unnamed parameter, dimensionless

θ volumetric moisture content, cm^3/cm^3

λ thermal conductivity, cal/cm-s-^{0}C

μ viscosity, centipoise

ν $\{=P/[(P-P_1)+P_1 M_1/M_3]\}$, dimensionless

ξ unnamed parameter, dimensionless

ρ density, g/cm^3

σ surface tension, dynes/cm

Φ total soil moisture potential, cm

Ψ soil moisture potential, cm

∇ del operator, 1/cm

∇ del operator, dimensionless

Subscripts

atm   atmospheric

b   bubbling pressure

c   cable

e   effective

front   moving boundary location

g   gravitational

i   species
        =0   soil in dry state
        =1   water vapor
        =2   liquid water
        =3   air

i   node location in finite difference grid

inf   infinity

init	initial value
K	denotes value of moisture content below which liquid continuity is not maintained
l	liquid
m	matric, moisture, or minerals
o	osmotic, organic, or reference state
p	pneumatic
R	reference state
r	residual
s	saturation
T	temperature
v	vapor
θ	moisture
*	excluding vapor movement

Superscripts

n	time level

BIBLIOGRAPHY

1. Abramowitz, M. and Stegun, I. A., eds., Handbook of
    Mathematical Functions, Dover Publications, New York,
    1974.

2. Adams, J. I., and Baljet, A. F., "Discussion on
    Influence of Moisture Content of Soil on Buried-Cable
    Ratings," Proc. IEE, Vol. III, 1964, pp.
    2081-2095.

3. Adams, J. I., and Baljet, A. F., "The Thermal Behavior
    of Cable Backfill Materials," IEEE Trans., Vol. 87,
    No. 4, April 1968, pp. 1149-1161.

4. AIEE Committee Report, "Soil Thermal Characteristics
    in Relation to Underground Power Cables," AIEE Trans.,
    Dec. 1966, pp. 792-856.

5. Arman, A. N., Cherry, D. M., Gosland, L.,
    Hollingsworth, P. M., "Influence of Soil-Moisture
    Migration on Power Rating of Cables in H.V.
    Transmission Systems," Proc. IEE, Vol. III, 1964,
    pp. 1000-1016.

6. Ahuja, L. R., and Swartzendruber, D., "An Improved
    Form of Soil-Water Diffusivity Function," Soil Sci.
    Soc. Amer. Proc., Vol. 36, 1972, pp. 9-14.

7. Bauer, C. A., and Nease, R. J., "Soil Factors
    Affecting Buried-Pipe-Cable Temperatures," AIEE
    Trans., pt. III, Vol. 76, 1957, pp. 257-268.

8. Baver, L. D., Gardner, W. H., and Gardner, W. R., Soil
    Physics, John Wiley & Sons, New York, 1972.

9. Beck, A. E., Anglen, F. M., and Sass, J. H., "Analysis
    of Heat Flow Data - In situ Thermal Conductivity
    Measurements," Can. Jour. of Earth Sciences, Vol.
    8, Jan. 1971, pp. 1-19.

10. Bird, R. B., Stewart, W. E., and Lightfoot, E. N.,
    Transport Phenomena, John Wiley & Sons, New York,
    1960.

11. Bouwer, H., "Unsaturated Flow in Ground Water
    Hydraulics," Journal of the Hydraulics Division, ASCE,

Paper No. HY5, Vol. 90, 1964, p. 121.

12. Bouyoucos, G. J., "Effect of Temperature on the Movement of Water Vapor and Capillary Moisture in Soils," J. Agr. Res., Vol. 5, 1915, pp. 141-172.

13. Brooks, R. H., and Corey, A. T., "Hydraulic Properties of Porous Media," Hydrology Paper No. 3, Colorado State Univ., Ft. Collins, Colorado, 1964.

14. Bruce, R. R., "Hydraulic Conductivity Evaluation of the Soil Profile from Soil Water Retention Relations," Soil Sci. Soc. Amer. Proc., Vol. 36, 1972, pp. 555-560.

14a Bush, R. A., "An Empirical Study of the Conditions Leading to Thermal Instability of Soils Used as Cable Backfill," Georgia Power Company Research Report, 1977.

15. Carlson, J. H., and Outcalt, S., "The Integrated Simulation of Soil Heat - Water Flow, Ice Segregation, and Surface Energy Transfer Regime," in Workshop on Heat and Mass Transfer in Porous Media, Case Western Reserve Univ, 1975.

15a Carnahan, B., Luther, H. A., and Wilkes, J. O., Applied Numerical Methods, John Wiley and Sons, Inc., New York, 1969.

16. Campbell, G. S., "A Simple Method for Determining Unsaturated Conductivity from Moisture Retention Data," Soil Sci., Vol. 117, 1974, pp. 311-314.

17. Cary, J. W., "Onsager's Relation and the Non-Isothermal Diffusion of Water Vapor," J. Phys. Chem., Vol. 67, 1963, pp. 126-129.

18. Cary, J. W., "Water Flux in Moist Soil: Thermal versus Suction Gradients," Soil Sci., Vol. 100, 1965, pp. 168-175.

19. Cary, J. W., "Soil Moisture Transport Due to Thermal Gradients: Practical Aspects," Soil Sci. Soc. Amer. Proc., Vol. 30, 1966, pp. 428-433.

20. Cary, J. W., and Taylor, S. A., "Thermally Driven Liquid and Vapor Phase Transfer of Water and Energy in Soil," Soil Sci. Soc. Amer. Proc., Vol. 26, 1962, pp. 417-420.

21. Cary, J. W., and Taylor, S. A., "The Interaction of the Simultaneous Diffusions of Heat and Water Vapor," Soil Sci. Soc. Amer. Proc., Vol. 26, 1962, pp. 413-416.

22. Cassel, D. K., Nielsen, D. R., and Biggar, J. W., "Soil-Water Movement in Response to Imposed Temperature Gradients," Soil Sci. Soc. Amer. Proc., Vol. 33, 1969, pp. 493-500.

23. Childs, E. C., An Introduction to the Physical Basis of Soil Water Phenomena, John Wiley & Sons, London, 1969.

24. Childs, E. C., "Concepts of Soil Water Phenomena," Soil Sci., Vol. 113, 1972, pp. 246-253.

25. Childs, E. C., and Collis-George, N., "The Permeability of Porous Materials," Proc. Roy. Soc., Vol. A201, 1950, pp. 392-405.

26. Cornish, P. M., Laryea, K. B. and Bridge, B. J., "A Nondestructive Method of Following Moisture Content and Temperature Changes in Soils Using Thermistors," Soil Sci., Vol. 115, 1973, pp. 309-314.

27. Cox, H. N., and Coates, R., "Thermal Analysis of Power Cables in Soils of Temperature-Responsive Thermal Resistivity," Proc. IEE, Vol. III, 1965, pp. 2275-2283.

28. Crank, J., "Two Methods for the Numerical Solution of Moving-Boundary Problems in Diffusion and Heat Flow," Quart. J. Mech. and Applied Math., Vol. X, pt. 2, 1957, pp. 220-231.

29. Crank, J., The Mathematics of Diffusion, 2nd Ed., Oxford University Press, London, 1975.

30. Crank, J., and Gupta, R. S., "A Moving Boundary Problem Arising from the Diffusion of Oxygen in Absorbing Tissue," J. Inst. Maths. Applics., Vol. 10, 1972, pp. 19-33.

31. Crank, J., and Gupta, R. S., "A Method for Solving Moving Boundary Problems in Heat Flow Using Cubic Splines or Polynomials," J. Inst. Math. and Its Appl., Vol. 10, 1972, pp. 296-303.

32. Croney, D., and Coleman, J. D., "Soil Thermodynamics Applied to the Movement of Moisture in Road

Foundations," Proc. 7th Int. Cong. for Appl. Mechanics, Vol. 3, 1948, pp. 163-177.

33. Cull, J. P., "Thermal Conductivity Probes for Rapid Measurement in Rock," Scientific Instruments, Vol. 7, 1974, pp. 771-774.

34. Dempsey, B. J., and Elzeftawy, A., "A Mathematical Model for Predicting Moisture Movement in Pavement Systems," private communication, 1976.

35. Derjaguin, B. V., and Melnikova, N. K., "Mechanism of Mosture Equilibrium and Migration in Soils," Water and Its Conduction in Soils, Highway Res. Bd. Sp. Rpt. 40, 1958, pp. 43-54.

36. Dirksen, C., and Miller, R. D., "Closed-System Freezing of Unsaturated Soil," Soil Sci. Soc. Amer. Proc., Vol. 30, 1966, pp. 168-173.

37. Douglas, J., and Gallie, T. M., "On the Numerical Integration of a Parabolic Differential Equation Subject to a Moving Boundary Condition," Duke Math. J., Vol. 22, 1955, pp., 557-571.

38. Douglas, J., "A Survey of Numerical Methods for Parabolic Differential Equations," Advances in Computers, Vol. 2, 1961, pp. 1-54.

39. Eckert, E., Bligh, T. P., and Pfender, E., "ENERGY CONSERVATION BY SUBSURFACE CONSTRUCTION Heat Transfer Studies in a Large Underground Building," First Annual Report of NSF(RANN) Contract No. SIA-75-03481, University of Minnesota, July 1976.

40. Ehrlich, L. W., "A Numerical Method of Solving a Heat Flow Problem with Moving Boundary," Assn. for Computing Machinery Journal, Vol. 5, 1958, pp. 161-176.

41. Elzeftawy, A., and Mansell, R. S., "Measured and Calculated Values of Hydraulic Conductivity of Partially Saturated Lakeland Sand," Soil Sci. Soc. Amer. Proc., Vol. 39, 1975, pp. 324-329.

42. Faroukl, O. T., "Physical Properties of Granular Materials with Reference to Thermal Resistivity," Highway Res. Record, Vol. 128, 1966, pp 25-44.

43. Fenger, M. J., "Thermal Soil-Resistivity Gives Cable Ampacity," Electrical World, Vol. 176, Nov. 1, 1973,

pp. 90-91.

44. Fink, L. H., "Control of the Thermal Environment of Buried Cable Systems," AIEE Trans., 1954, pp. 406-412.

45. Fink, L. H., "Soil Thermal Characteristics in Relation to Underground Power Cables Part III - Soil Moisture Characteristics," Trans. AIEE, pt. 3, 1960, pp. 803-819.

46. Fink, L. H., and Smerke, J. J., "Control of the Thermal Environment of Buried Cable Systems - Part II," AIEE Trans., June 1958, pp. 161-168.

47. Flatabo, N., "Transient Heat Conduction Problems in Power Cables Solved by the Finite Element Method," IEEE Trans. on Power Apparatus and Systems, Vol. 92, 1973, pp. 56-63.

48. Fritton, D. D., Kirkham, D., and Shaw, R. H., "Soil Water Evaporation, Isothermal Diffusion, and Heat and Water Transfer," Soil Sci. Soc. Amer. Proc., Vol. 34, 1970, pp. 183-189.

49. Gardner, R., "Relation of Temperature to Moisture Tension of Soil," Soil Sci., Vol. 79, 1955, pp. 257-264.

50. Gardner, W. R., "Laboratory Studies of Evaporation from Soil Columns in the Presence of Water Table," Soil Sci., Vol. 85, 1958, pp. 244-249.

51. Gardner, W. R., "The Permeability Problem," Soil Sci., Vol. 117, 1974, pp. 243-249.

52. Gee, G., "Water Movement in Soils as Influenced by Temperature Gradients," Ph. D. Thesis, Washington State Univ., 1966.

53. Gohar, M. K., "Effect of Moisture Migration on Temperature Rise in Underground Cables," Proc. IEEE, Oct. 1971, pp. 1540-1541.

54. Goldenberg, H., "Thermal Transients in Linear Systems with Heat Generation Linearly Temperature Dependent: Application to Buried Cables," Proc. IEEE, Vol. 114, 1967, pp. 375-377.

55. Goldenberg, H., "Transient Heating of Buried Power Cables," Proc. IEE, Vol. 114, 1967, pp. 809-914.

56. Gosland, L., and Hollingsworth, P. M., "Current Ratings of Buried Distribution Cables," Int. Conf. on Electricity Distribution, London, 1973, pp. 57-68.

57. Green, R. E., and Corey, J. C., "Calculation of Hydraulic Conductivity: A Further Evaluation of Some Predictive Methods," Soil Sci. Soc. Amer. Proc., Vol. 35, 1975, pp., 3-8.

58. Gupta, J. P., and Churchill, S. W., "Heat and Moisture Transfer in Wet Sand During Freezing," Environmental and Geophysical Heat Transfer, Winter Annual Mtg. of ASME, 1971, pp. 99-105.

59. Gurr, C. G., Marshall, T. J., and Hutton, J. T., "Movement of Water in Soil Due to a Temperature Gradient," Soil Sci.,Vol. 74, 1952, pp. 335-345.

60. Gurr, C. G., and Marshall, T. J., "Unsaturated Permeability - Its Measurement and Its Estimation from Other Properties of the Material," 7th Int. Congress of Soil Sci., Madison, Wisc., 1960, pp. 306-310.

61. Hadas, A., "Simultaneous Flow of Water and Heat Under Periodic Heat Fluctuations," Soil Sci. Soc. Amer. Proc., Vol. 32, 1968, pp. 297-301.

62. Hadley, W. A., and Eisenstadt, R., "A Critical Soil Moisture Condition Affecting Buried Transmission Cables," Trans. AIEE, pt. III, Vol. 72, 1953, pp. 849-851.

62a Handgraff, G., "Thermal Conductivity of Soil Additives and Ga. Red Clay," Report of Summer Special Problem, Ga. Institute of Technology, Aug. 1977.

63. Haridasan, M., and Jensen, R. D., "Effect of Temperature on Pressure Head-Water Content Relationship and Conductivity of Two Soils," Soil Sci. Soc. Amer. Proc., Vol. 36, 1972, pp. 703-708.

64. Hauk, R., "Soil Temperature and Moisture Fields Predicted by a System of Coupled Equations," Ph. D. Thesis, Cornell Univ., 1971.

65. Hillel, D., Soil and Water. Principles and Processes, Academic Press, New York, 1971.

66. Ingersoll, L. R., Adler, F. T., Plass, H. J., and

Ingersoll, A. C., "Theory of Earth Heat Exchangers for the Heat Pump," Heating, Piping and Air Conditioning, Vol. 22, 1950, pp. 113-122.

67. Irmay, S., "On the Hydraulic Conductivity of Unsaturated Soils," Trans. American Geophysical Union, Vol. 35, 1954, pp. 463-467.

68. Irmay, S., "Extension of Darcy Law to Unsteady Unsaturated Flow through Porous Media," Symposia Darcy, Int. Assn. Sci. Hydrology, Pub. No. 41, 1956, pp. 57-66.

69. Jackson, R. D., "Temperature and Soil-Water Diffusivity Relations," Soil Sci. Soc. Amer. Proc., Vol. 27, 1963, pp. 363-366.

70. Jackson, R. D., Reginato, R. J., and van Bavel, C. H. M., "Comparison of Measured and Calculated Hydraulic Conductivities of Unsaturated Soils," Water Resources Res., Vol. 1, 1965, pp. 375-380.

70a JANAF Thermochemical Tables, NSRDS-NBS-37, 1971.

71. Jones, H. E., and Kohnke, H., "The Influence of Soil Moisture Tension on Vapor Movement of Soil Water," Soil Sci. Soc. Amer. Proc., Vol. 16, 1952, pp. 245-248.

72. Joshua, W. D., and DeJong, E., "Soil Moisture Movement Under Temperature Gradients," Can. J. Soil Sci., Vol. 53, 1973, pp. 49-57.

73. Jury, W. A., "Simultaneous Transport of Heat and Moisture Through a Medium Sand," Ph. D. Thesis, Univ. of Wisconsin, 1973.

74. Jury, W. A., and Miller, E. E., "Measurement of the Transport Coefficients for Coupled Flow of Heat and Moisture in a Medium Sand," Soil Sci. Soc. Amer. Proc., Vol. 38, 1973, pp. 551-557.

75. Kay, B. D., and Groenevelt, P. H., "On the Interaction of Water and Heat Transport in Frozen and Unfrozen Soils, Parts I and II," Soil Sci. Soc. Amer. Proc., Vol. 38, 1974, pp. 395-404.

75a Keenan, J. H., Keyes, F. G., Hill, P. G., and Moore, J. G., Steam Tables, Wiley, New York, 1969.

75b Kersten, M. S., "Thermal Properties of Soils," Univ.

of Minnesota, Engr. Exp. Station, Bulletin No. 28, 1949.

76. Kimball, B. A., Jackson, R. D., Nakayama, F. S., Idso, S. B., and Reginato, R. J., "Soil-Heat Flux Determination: Temperature Gradient Method with Computed Thermal Conductivities," Soil Sci. Soc. Amer. Proc., Vol. 40, 1976, pp. 25-28.

77. Kimball, B. A., Jackson, R. D., Reginato, R. J., Nakayama, F. S., and Idso, S. B., "Comparison of Field-Measured and Calculated Soil-Heat Fluxes." Soil Sci. Soc. Amer. J., Vol. 40, 1976, pp. 18-24.

78. King, L. G., "Imbibition of Fluids by Porous Solids," Ph. D. Thesis, Colorado State Univ., Ft. Collins, Colorado, 1964.

79. King, L. G., "Description of Soil Characteristics for Partially Saturated Flow," Soil Sci. Soc. Amer. Proc., Vol. 24, 1965, pp. 99-105.

80. Klute, A., "The Determination of the Hydraulic Conductivity and Diffusivity of Unsaturated Soils," Soil Sci., Vol. 113, 1972, pp. 264-276.

81. Kunze, R. J., and Kirkham, D., "Capillary Diffusion and Self-Diffusion of Soil Water," Soil Sci., Vol. 97, 1964, pp. 145-151.

82. Kuzmak, J. M., and Sereda, P. J., "The Mechanism by Which Water Moves Through a Porous Material Subjected to a Temperature Gradient, Part I," Soil Sci., Vol. 84, 1957, pp. 291-299.

83. Lachenbrach, A. H., "A Probe for Measurement of Thermal Conductivity of Frozen Soils in Place," Trans. Amer. Geophysical Union, Vol. 38, No. 5, Oct. 1957, pp. 691-697.

84. Laliberte, G. E., Corey, A. T., and Brooks, R. H., "Properties of Unsaturated Porous Media," Hydrology Paper No. 17, Colorado State University, Ft. Collins, Colorado, 1966.

85. Lasseter, T. J., "The Numerical Simulation of Heat and Mass Transfer in Multi-Dimensional Two-Phase Geothermal Reservoirs," ASME Paper No. 75-WA/HT-71, Winter Annual Meeting, Heat Trans. Div. of ASME, 1975.

85a Lax, P. D., and Richtmyer, R. D., "Survey of the
     Stability of Linear Finite Difference Equations,"
     Comm. Pure Appl. Math., Vol. 9, 1956, pp.
     267-293.

85. Luikov, A. V., Heat and Mass Transfer in
     Capillary-Porous Bodies, Pergamon Press, London, 1966.

87. Luikov, A. V., and Mikhailov, Y. A., Theory of Energy
     and Mass Transfer, Pergamon Press, London, 1965.

88. Luikov, A. V., "Heat and Mass Transfer in
     Capillary-Porous Colloidal Bodies," Phenomenes De
     Transport Avec Changement De Phase Dans Les Milieux
     Poreux Ou Colloidaux, Paris, Editions Du Centre
     National De La Recherche Scientifique, No. 160.,
     1970, pp. 21-38.

89. Luikov, A. V., "Systems of Differential Equations of
     Heat and Mass Transfer in Capillary-Porous Bodies,"
     Int. J. Heat and Mass Transfer, Vol. 18, 1975, pp.
     1-14.

90. Lytton, R. L., "Theory of Moisture Movement in
     Expansive Clays," Research Report No. 118-1, Center
     for Highway Research, The University of Texas at
     Austin, Sept. 1969.

91. Marshall, T. J., "A Relation Between Permeability and
     Size Distribution of Pores," J. Soil Sci., Vol. 9,
     1958, pp. 1-8.

92. Mason, V. V., and Kurtz, M., "Rapid Measurement of the
     Thermal Resisitivity of Soil," AIEE Trans., Aug.
     1952, pp. 570-577.

93. Mench, A. F., and Evans, D. D., "Thermal Conductivity
     and Diffusivity of Soil Using a Cylindrical Heat
     Source," Soil Sci. Soc. Amer. Proc., 34:377-381,
     1970.

94. Mickley, A. S., "The Thermal Movement of Moisture in
     Soil," AIEE Trans., Vol. 68, pt.I, 1949, pp.
     330-335.

95. Mickley, A. S., "The Thermal Conductivity of Moist
     Soils," AIEE Trans., Vol. 70, 1951, pp. 1789-1797.

96. Mikhailov, M. D., "Exact Solution of Temperature and
     Moisture Distributions in a Porous Half-Space with
     Moving Evaporation Front," Int. J. Heat Mass

Transfer, Vol. 18, 1975, pp. 797-804.

97. Mikhailov, Y., "Highly Intensive Heat and Mass Transfer in Dispersed Media," Int. J. Heat Mass Transfer, Vol. 1, 1960, pp. 37-45.

98. Millington, R. J., and Quirk, J. P., "Transport in Porous Media," 7th Int. Congr. of Soil Sci., Madison, Wisc., 1960, pp. 97-106.

99. Millington, R. J., and Quirk, J. P., "Permeability of Porous Solids," Trans. Faraday Soc., Vol. 57, 1961, pp. 1200-1207.

100. Milne, A. G., and Mochlinski, K., "Characteristics of Soil Affecting Cable Ratings," Proc. IEE, Vol. III, 1964, pp. 1017-1039.

101. Min, K., and Emmons, H. W., "The Drying of Porous Media," Proc. Ht. Trans. and Fluid Mech. Inst., 1972, pp. 1-18.

102. Mole, G., "Discussion on Influence of Moisture Content of Soil on Buried-Cable Ratings," Proc. IEE, Vol. III, 1964, pp. 2081-2095.

103. Murray, W. D., and Landis, F., "Numerical and Machine Solutions of Transient Heat-Conduction Problems Involving Melting or Freezing Part I - Method of Analysis and Sample Solutions," J. Ht. Trans., Trans. ASME, 1959, pp 106-112.

104. Neher, J. H., "The Transient Temperature Rise of Buried Cable Systems," AIEE Trans., Feb. 1964, pp. 102-114.

105. Neher, J. H., "The Temperature Rise of Buried Cables and Pipes," AIEE Trans., Vol. 681, 1949, pp. 9-21.

106. Neher, J. H., and McGrath, M. H., "The Calculation of the Temperature Rise and Load Capacity of Cable Systems," AIEE Trans., Oct. 1959, pp. 752-772.

107. Nerpin, S. V., Physics of the Soil, Israel Program for Scientific Translations, Jerusalem, 1970.

108. Nielsen, D. R., Jackson, R. D., Cary, J. W., and Evans, D. D., eds., Soil Water, Am. Soc. of Agronomy and Soil Sci Soc. Amer., Madison, Wisc, 1972.

109. Pandey, G. N., Tek, M. R., and Katz, D. L., "Diffusion

of Fluids through Porous Media with Implications in Petroleum Geology," The Amer. Assn. of Petroleum Geologists Bulletin, Vol. 58, 1974, pp. 291-303.

110. Peck, A. J., "Change of Moisture Tension with Temperature and Air Pressure: Theoretical," Soil Sci., Vol. 89, 1960, pp. 303-310.

111. Penman, H. L., "Gas and Vapor Movement in Soil Water," J. Agricultural Sci., Vol. 30, 1940, pp. 437-462.

112. Philip, J. R., "The Concept of Diffusion Applied to Soil Water," Proc. Nat. Acad. Sci. India, Vol. 2A, 1955, pp. 93-104.

113. Philip, J. R., "Evaporation, and Moisture and Heat Fields in the Soil," J. Meteorology, Vol. 14, 1957, pp. 354-366.

114. Philip, J. R., "Physics of Water Movement in Porous Solids," Water and Its Conduction in Soils, Highway Res. Bd. Sp. Rpt. 40, 1958, pp. 147-163.

115. Philip, J. R., and de Vries, D. R., "Moisture Movement in Porous Materials Under Temperature Gradients," Trans. Am. Geophysical Union, Vol. 38, 1957, pp. 222-232.

116. Raats, P. A. C., "Transformation of Fluxes and Forces Describing the Simultaneous Transport of Water and Heat in Unsaturated Porous Media," Water Resources Research, Vol. 11, 1975, pp. 938-942.

117. Radhakrishna, H. S., "Heat Flow and Moisture Migration in Cable Backfills," Ontario Hydro Research Quarterly, Vol. 20, 1968, pp. 18-20.

118. Raudkivi, A. J., and Nguyen Van U'u, "Soil Moisture Movement by Temperature Gradient," J. Geotechnical Engr. Div., ASCE, Vol. GT12, pp. 1225-1244, 1976.

119. Reisenaur, A. E., "Calculation of Soil Hydraulic Conductivity from Soil-Water Retention Relationships," Battelle Pacific Northwest Laboratories, Report No. BNWL - 170, 1973.

120. Reno, W. H., and Winterkorn, H. F., "Thermal Conductivity of Kaolinite Clay as a Function of Exchange Ion, Density, and Moisture Content," Highway Research Record, Vol. 209, 1968, pp. 79-85.

121. Richards, L. A., "Capillary Conduction of Liquids in Porous Media," Physics, Vol. 1, 1931, pp. 318-333.

122. Rollins, R. L., Spangler, M .G., and Kirkham, D., "Movement of Soil Moisture Under a Thermal Gradient," Proc. Highway Res. Bd., Vol. 33, 1954, pp. 492-500.

123. Rose, D. A., "Water Movement in Porous Materials: Part 1 - Isothermal Vapor Transfer," Brit. J. Appl. Phys., Vol. 14, 1963, pp. 256-262.

124. Rose, D. A. , "Water Movement in Porous Materials: Part 2 - The Separation of the Components of Water Movement," Brit. J. Appl. Phys., Vol. 14, 1963, pp. 491-496.

125. Rose, M. E., "A Method for Calculating Solutions of Parabolic Equations with a Free Boundary," Mathematics of Compututlon, Vol. 14, 1960, pp. 249-256.

126. Rubin, A., and Schweitzer, S., "Heat Transfer in Porous Media with Phase Change," Int. J. Heat and Mass Transfer, Vol. 15, 1972, pp.43-60.

127. Rubin, J., Steinhardt, R., and Reiniger, P., "Soil Water Relations During Rain Infiltration II: Moisture Content Profiles During Rains of Low Intensities," Soil Sci. Soc. Amer. Proc., Vol. 28, 1964, pp. 1-5.

128. Saxena, S. K., Boersma, L., Lindstrom, F. T., and Young, J. L., "Effect of Pore Size on Diffusion Coefficients in Porous Media," Soil Sci., Vol. 117, 1974, pp. 80-86.

129. Schmill, J. V., "Mathematical Solution to the Problem of the Control of the Thermal Environment of Buried Cables," Trans. AIEE, Vol. 79, pt. III, 1960, pp. 175-180.

130. Schmill, J. V., "Discussion on Influence of Moisture Content of Soil on Buried-Cable Ratings, "Proc. IEE, Vol. III, 1964, pp. 2081-2095.

131. Schmill, J .V., "Variable Soil Resistivity - Steady State Analysis", IEEE Trans., Vol. 86, No. 2, Feb. 1967, pp. 215-223.

132. Simmons, Donald M., "Calculation of the Electrical Problems of Underground Cables," The Electric Journal,

Vol. 29, May-November 1932, pp. 523-530.

133. Sinclair, W. A., Bullen, F. H., and Benham, C. B., "Soil Thermal Resistivity; Typical Field Values and Calculating Formulas, Soil Thermal Characteristics in Relation to Underground Power Cables - Part IV," Trans. AIEE, pt. III, 1960, pp. 820-832.

134. Slaninka, P., "The External Thermal Resistance of H.V. Cable in Soil as Non-Linear Environment," Int. Conf. on Large High Voltage Electric Systems, Vol. 1, Paris, 1974, pp. 1-8.

135. Smith, W. O., "Thermal Transfer of Moisture in Soil," Trans. Am. Geophysical Union, Vol. 24, 1943, pp. 511-523.

136. Smirnov, M. S., "On a System of Differential Equations for Highly Intensive Heat and Mass Transfer," Int. J. Heat Mass Trans., Vol. 5, 1962, pp. 521-524.

137. Southworth, R. W. and Deleeuw, S. L., Digital Computation and Numerical Methods, Mcgraw-Hill Book Company, New York, 1965.

138. Sugawara, A., "Heat Transfer Accompanying Fluid Flow in Porous Materials," Aust. J. Appl. Sci., Vol. 14, 1963, pp. 109-122.

139. Tanayeva, S. A., "Thermal Conductivity of Moist Bulk Materials," Heat Transfer-Soviet Res., Vol. 6, 1974, pp. 107-111.

140. Taylor, S. A., and Cary, J. W., "Linear Equations for the Simultaneous Flow of Matter and Energy in a Continuous Soil System," Soil Sci. Soc. Amer. Proc., Vol. 28, 1964, pp. 167-172.

141. Taylor, S. A., and Stewart, G. L., "Some Thermodynamic Properties of Soil Water," Soil Sci. Sci. Amer. Proc., Vol. 24, 1970, pp. 243-247.

142. Topp, G. C. and Miller, E. E., "Hysteretic Moisture Characteristics and Hydraulic Conductivities for Glass Bead Media," Soil Sci. Soc. Amer. Proc., Vol. 30, 1966, pp. 156-162.

143. Trench, W. F., "On an Explicit Method for the Solution of a Stefan Problem," J. Soc. Indust. Appl. Math., Vol. 7, 1959, pp. 184-204.

144. van Bavel, C. H. M., "Gaseous Diffusion and Porosity in Porous Media," Soil Sci., Vol. 73, 1952, pp. 91-104.

145. Van Rooyen, M., "Soil Thermal Resistivity," Ph.D. Thesis, Princeton Univ., 1958.

145a von Rosenberg, D. U., Methods for the Numerical Solution of Differential Equations, American Elsevier Pub. Co., New York, 1969.

146. de Vries, D. A., "Some Remarks on Heat Transfer by Vapor Movement in Soils," Trans. 4th International Cong. of Soil Sci., Amsterdam, Vol. II, 1950, pp. 38-41.

147. de Vries, D. A., "Some Remarks on Gaseous Diffusion in Soils," Trans. 4th International Cong. of Soil. Sci., Amsterdam, Vol. II, 1950, pp. 41-43.

148. de Vries, D. A., "Simultaneous Transfer of Heat and Moisture in Porous Media," Trans. Am. Geophysical Union, Vol. 39, 1958, pp. 909-916.

149. de Vries, D. A., "Thermal Properties of Soil," in W. R. Van Wijk, ed., Physics of Plant Environment, North-Holland Publishing Co., Amsterdam, 1963.

150. de Vries, D. A., and Kruger, A. J., "On the Value of the Diffusion Coefficient of Water Vapor in Air," Phenomenes De Transport Avec Changement De Phase Dans Les Milieux Poreux Ou Colloidaux, Paris, 1967, pp. 61-69.

151. de Wet, J. A., "A Formulation for the Three-Dimensional Moisture Movement in a Partially Saturated Soil," Moisture Equilibria and Moisture Changes in Soils Beneath Covered Areas, A Symposia in Print, Butterworth, Sydney, 1965.

152. Weedy, B. M., "Thermal Aspects of Changes in the Environment of Underground Cables," Proc. IEE, Vol. 114, 1967, pp. 519-524.

153. Weedy, B. M., "Effect of Environment on Transient Thermal Performance of Underground Cables," Proc. IEE, Vol. 118, 1971, pp. 1534-1540.

154. Westcot, D. M., and Wierenga, P. J., "Transfer of Heat by Conduction and Vapor Movement in a Closed Soil System," Soil Sci. Soc. Amer. Proc., Vol. 38,

1974, pp. 9-14.

155. Whisler, F. D., "Calculating the Unsaturated Hydraulic Conductivity and Diffusivity," Soil Sci. Soc. Amer. Proc., Vol. 48, 1976, pp. 150-151.

156. Wierenga, P. J. and de Wit, C. T., "Simulation of Heat Transfer in Soils," Soil. Sci. Soc. Amer. Proc., Vol. 34, 1970, pp. 845-848.

157. Wilkinson, G. E., and Klute, A., "The Temperature Effect on the Equilibrium Energy Status of Water Held by Porous Media," Soil Sci. Soc. Amer. Proc., Vol. 26, 1962, pp. 326-329.

158. Winders, J. J., "Computer Program Analyzes Heat Flow from Cables Buried in Regions of Discontinuous Thermal Resistivity," IEEE Transactions of Power Apparatus and Systems, Vol. PAS-93, 1974, pp. 396-403.

159. Winterkorn, H. F., "Behavior of Moist Soils in a Thermal Energy Field," 9th Natl. Conf. on Clays and Clay Minerals, 1962, pp. 85-103.

160. Wong, C. M., Hoh, K. C., Hadley, W. A., and Baker, H. D., "Effect of Temperature Gradient on Moisture Migration in Granular Media," Trans. AIEE, pt. III, Vol. 80, 1961, pp. 1031-1036.

161. Woodside, W., and Kuzmak, J. M., "Effect of Temperature Distribution on Moisture Flow in Porous Materials," Trans. Am. Geophysical Union, Vol. , 1958, pp. 676-680.

162. Woodside, W., and de Bruyn, C. M. A., "Heat Transfer in a Moist Clay", Soil Sci., Vol. 87, 1959, pp. 166-173.

163. Workshop on Heat and Mass Transfer in Porous Media, Case Western Reserve University, Cleveland, October 1975, FTAS/TR, pp. 75-117.

VITA

James Gary Hartley was born in Salina, Kansas, on July 11, 1946, the son of Joseph O. and Eileen E. Hartley. He attended Bossier High School in Bossier City, Louisiana, and was graduated in May, 1964. In September of that year he entered Louisiana Polytechnic Institute. There he received the Bachelor of Science in Mechanical Engineering in March, 1969, and at that time he was commissioned a Second Lieutenant in the United States Air Force. He entered the Graduate School of Louisiana Polytechnic Institute in March, 1969, and received the Masters degree in Mechanical Engineering in March, 1970. After serving four and one-half years of active duty in the Air Force, he entered the Graduate School of the Georgia Institute of Technology in September, 1974. He married Julene Day on August, 1974.