CASCADE SYNTHESIS
OF RLC DRIVING-POINT IMPEDANCES

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CASCADE SYNTHESIS
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SUMMARY

A new procedure is developed for synthesis of certain rational, positive-real functions as driving-point impedances in a structure comprising a cascade of elementary sections. The procedure utilizes an elementary cascade section described by a compact open-circuit impedance matrix having elements that are suitably chosen fourth-degree rational functions of the complex frequency variable. Under certain conditions the cascade section, which is not lossless in general, is realizable in an unbalanced two-terminal-pair network containing neither ideal transformers nor mutual inductances. The locations of the zeros of transmission through the cascade section play an important role in the synthesis procedure; however, they are related to the driving-point impedance to be synthesized and may not be chosen arbitrarily.

Equivalent circuits for the cascade section are derived to make evident certain properties of the section.

Although the cascade synthesis procedure is not applicable to all rational, positive-real impedance functions, it includes techniques which, in principle, will determine whether one step of the procedure may be effected.

An advantage of the cascade synthesis procedure in cases where it is applicable is that only one remainder impedance function is required.

CHAPTER I

INTRODUCTION

Brief statement of the problem.--This study concerns a method for synthesis of certain rational, positive-real functions as driving-point impedances in a structure comprising a cascade of elementary sections, without the use of mutual inductive coupling.

The basic approach to this problem is to specify the form of an elementary cascade section and to determine its parameters so that removal of the cascade section leaves a remainder impedance function that is positive-real and simpler than the given driving-point impedance. Each cascade section realizes one or more pairs of zeros of transmission; however, these transmission zeros are not independent of the given impedance function.

Background of the cascade synthesis problem.--The origin of this problem concerns the properties of the available synthesis methods for RLC two-terminal impedances, i.e., driving-point impedances containing resistors, inductors, and capacitors. The earliest general method was announced in 1931 by Brune.[1] This method is characterized by a cascade network structure in which the number of elements required is approximately proportional to the degree of the driving-point impedance function. Brune's

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[1] O. Brune, "Synthesis of a finite two-terminal network whose driving-point impedance is a prescribed function of frequency," Journal of Mathematics and Physics, vol. 10, pp. 191-236; 1931.

method has the disadvantage that it may require the use of unity-coupled coils.

Eighteen years after Brune's method was published, a driving-point impedance synthesis procedure involving no transformers or mutual inductances was developed by Bott and Duffin.[2] This method is characterized by a tree-like network structure in which the number of elements required may increase exponentially with the degree of the given impedance function.

Other methods of synthesis are available; however, they usually possess to some extent the undesired features of the Brune or Bott-Duffin procedures, i.e., either mutual inductances or large numbers of redundant elements are required. Thus, an important theoretical problem is to investigate means of transformerless synthesis requiring fewer redundant elements than the Bott-Duffin technique or its recent variants.[3] This problem assumes a greater significance as the degree of the given driving-point impedance increases. This problem has been discussed by Darlington[4] and has been investigated recently by Kim.[5] Kim's results, which are

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[2] R. Bott and R. J. Duffin, "Impedance synthesis without use of transformers," Journal of Applied Physics, vol. 20, p. 816; August, 1949.

[3] R. M. Foster, "Passive network synthesis," Proceedings of the Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn, vol. 5, pp. 3-9; April, 1955.

[4] S. Darlington, "A survey of network realization techniques," IRE Transactions on Circuit Theory, vol. CT-2, pp. 291-297; December, 1955.

[5] Wan Hee Kim, "A new method of driving-point function synthesis," Interim Technical Report No. 1, Contract No. DA-11-022-ORD-1983, Engineering Experiment Station, University of Illinois; Urbana, Illinois; April, 1956.

obtained by topological methods, include non-series-parallel networks with fewer elements than the corresponding Brune networks.

The considerations outlined above indicate the desirability of a transformerless synthesis procedure in which the number of elements required is approximately proportional to the degree of the given impedance function. This requirement suggests the use of a cascade type of network structure. A cascade synthesis method employing no ideal transformers or mutual inductances has been developed by Guillemin.[6] However, Guillemin's method, which is based on a lossless cascade section, is not directly applicable to minimum functions. In common with the procedure to be presented here, Guillemin's method has the limitation that it cannot be applied to all positive-real impedance functions.

General approach.--The basic requirements pertinent to the cascade synthesis problem concern a single cascade section, shown in Fig. 1. Regarded as a two-terminal-pair network, the cascade section denoted by z in Fig. 1 is described by its open-circuit impedance matrix

$$z = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \qquad (1)$$

in which

$$z_{12} = z_{21} \qquad (2)$$

for the bilateral networks considered here. A simple calculation shows that the equation

[6] E. A. Guillemin, "New methods of driving-point and transfer impedance synthesis," Proceedings of the Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn, vol. 5, pp. 131-144; April, 1955.

$$(z_{11} - Z_1)\ (z_{22} + Z_2) = z_{12}^2 \tag{3}$$

governs the relation between the two-terminal impedances Z_1 and Z_2 of Fig. 1. The impedance function $Z_1(s)$ may be referred to as the "datum" impedance function and $Z_2(s)$ as the "remainder" impedance function.

In order for one step of a cascade synthesis procedure to be effected, it is necessary to select an appropriate open-circuit impedance matrix z for the cascade section. The approach employed here involves initially the specification of the matrix z in terms of certain parameters and subsequently the determination of these parameters so that the remainder impedance function, $Z_2(s)$, is positive-real and simpler than the datum impedance function, $Z_1(s)$. After an appropriate open-circuit impedance matrix has been obtained, it remains to develop a means of realizing the cascade section represented by the matrix. This realization can be effected without the use of mutual inductive coupling under certain conditions which are derived in a later section of this report.

Although the synthesis method to be described here does not include a complete specification of the defining characteristics of the function class or the network class for which the cascade synthesis procedure is applicable, it is possible at a given stage in the development to determine whether a single cascade section can be removed.

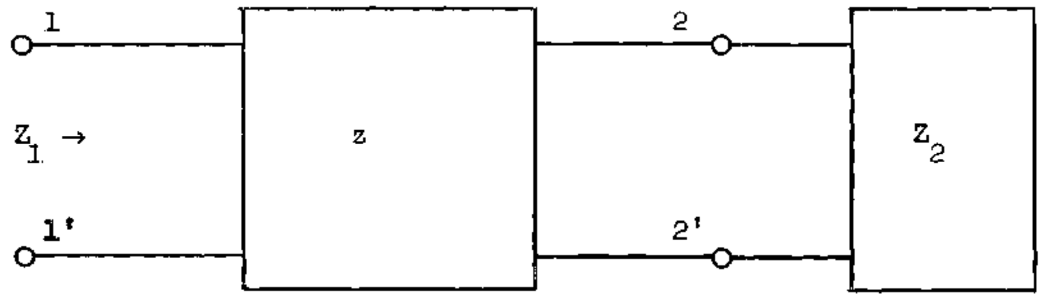

Figure 1. Cascade Section and Remainder Impedance.

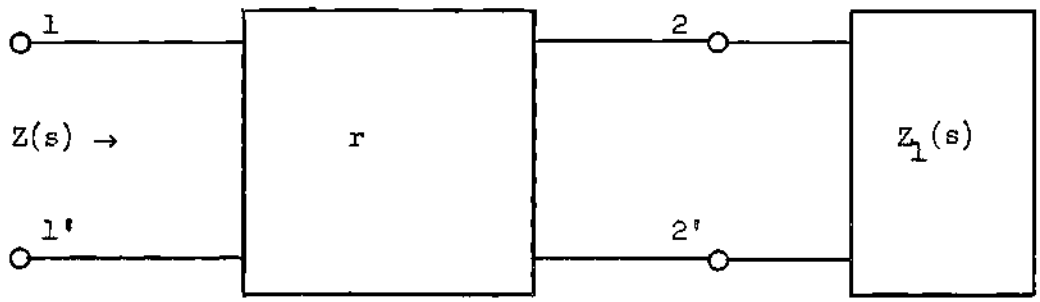

Figure 2. Reduction to Minimum-Reality.

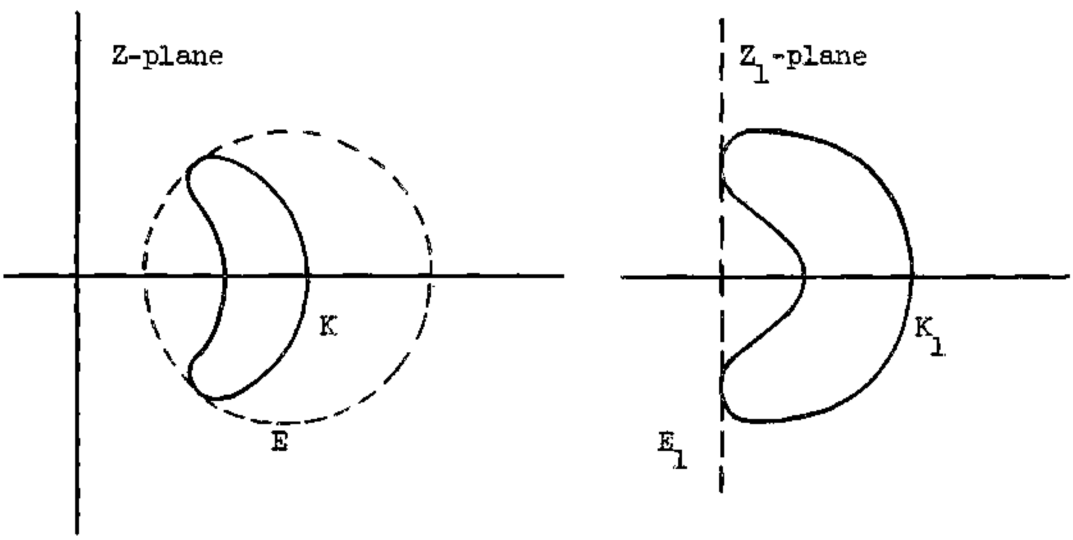

Figure 3. Impedance Plane Transformation.

CHAPTER II

PRELIMINARY CONSIDERATIONS

Reduction of positive-real functions to minimum functions.--The datum impedance functions to be considered in succeeding chapters will generally be assumed to be minimum functions of the complex frequency variable s, i.e., rational, positive-real functions that are minimum-reactive, minimum-susceptive, and minimum-real.[7] For convenience, the complex frequency variable will usually be normalized so that a zero of the real part of the datum impedance function falls at $s = j1$. Thus, if the datum impedance function is denoted by $Z_1(s)$, the value of $Z_1(j1)$ will be imaginary and non-zero.

The process of reducing a given rational, positive-real impedance function to a minimum function is well-known in network synthesis theory and will not be discussed in detail here. However, it will be useful to consider two ways to effect the reduction to minimum-reality of a rational, positive-real function which is already minimum-reactive and minimum-susceptive.

One method for reduction to minimum-reality involves the transformation of the given impedance function $Z(s)$ into another function $Z_1(t)$ by means of a bilinear transformation of the complex frequency variable s into another variable t. The bilinear transformation relating t and s

[7] D. F. Tuttle, Jr., "Network Synthesis," John Wiley and Sons, Inc., New York, N.Y., pp. 368-381; 1958.

may be written as

$$t = \frac{as + b}{cs + d} , \tag{4}$$

where a, b, c, and d are real, non-negative constants. An appropriate (not necessarily unique) choice of the constants a, b, c, and d produces a function $Z_1(t)$ that is minimum-real. Subsequent realization of $Z_1(t)$ by a method employing no mutual inductance implies realization of $Z(s)$ if each inductor and capacitor in the network for $Z_1(t)$ is replaced by the proper RL or RC network. This method, which is not new, is discussed by Nijenhuis[8] and Westcott.[9] It may be employed in conjunction with the synthesis technique to be described here in order to avoid the use of many lossless elements in the cascade sections.

A second method for reduction to minimum-reality involves a bilinear transformation of the impedance plane, rather than the frequency plane. Mathematically this procedure strongly resembles the first method; however the interpretation of the transformation in terms of an electric network is different. In this method the given impedance function $Z(s)$ is reduced to minimum-reality by the removal of a cascade section containing only resistors. Thus, the reduction may be represented as in Fig. 2, where the open-circuit impedance matrix r, describing the cascade section as a two-terminal-pair network, has real, positive elements, r_{ij},

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[8] W. Nijenhuis, "Impedance synthesis distributing available loss in the reactance elements," Phillips Research Reports, vol. 5, pp. 288-302; 1950.

[9] J. H. Westcott, "Driving-point impedance synthesis using maximally lossy elements," Proceedings of the Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn, vol.5, pp. 63-78; April, 1955.

and a non-negative determinant, $d(r)$. The relation between the impedance functions $Z(s)$ and $Z_1(s)$ implied by Fig. 2 may be expressed as

$$Z_1(s) = \frac{r_{22}Z(s) - d(r)}{r_{11} - Z(s)} \quad . \tag{5}$$

Some pertinent properties of this well-known transformation are shown in Fig. 3. The image of the imaginary axis of the Z_1-plane is a circle E in the Z-plane, lying in the right half-plane (but possibly tangent to the imaginary axis). Let K be the locus of $Z(s)$ for imaginary values of s, and let K_1 be the Z_1-plane image of this locus. Then $Z_1(s)$ is clearly minimum-real if K lies tangent to, but not outside of, E. For a given impedance function $Z(s)$ and the corresponding locus K, any circle centered on the positive real axis which encloses and is tangent to K, but lies within the right half-plane, defines two of the three constants required to determine the open-circuit impedance matrix r. In its simplest form this method consists of the conventional subtraction of an appropriate real, positive constant from $Z(s)$ or its reciprocal. The somewhat more general form of this method is discussed here because of a later application.

<u>A modification of the Brune process</u>.--The synthesis as a driving-point impedance of a minimum function, say $Z_1(s)$, was first effected by Brune,[1] using a cascade network structure like that shown in Figs. 4 and 5.[10] The Brune network represented in these figures is realizable in the form of

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[10]Element values in all figures are given in terms of ohms, henrys, and farads, except that admittances are denoted by the letter Y.

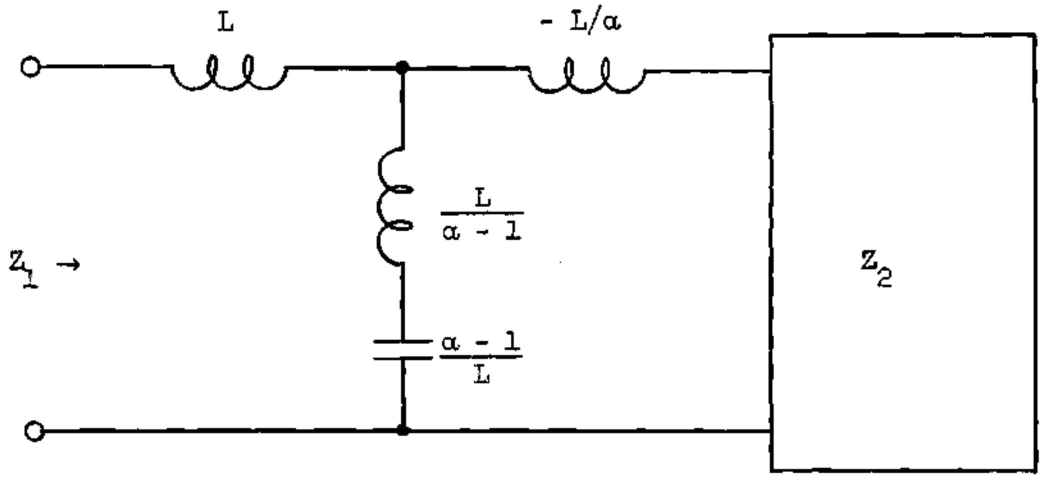

Figure 4. Brune Network.

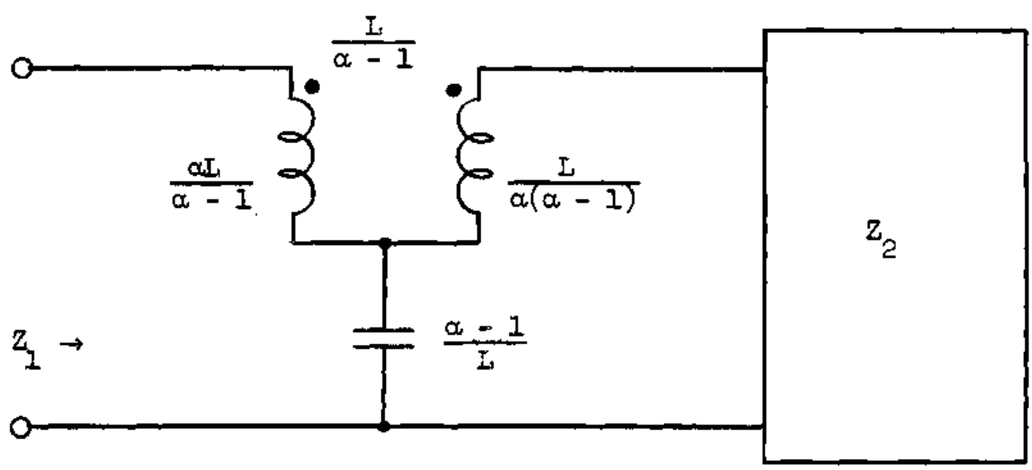

Figure 5. Brune Network.

Fig. 5 (with the use of unity-coupled coils) if the real constants L and $(\alpha - 1)$ have the same sign. It realizes $Z_1(s)$ if the parameters L and α are properly chosen. The choice of the parameters L and α depends only upon the value of $Z_1(s)$ and its first derivative at the frequencies of the transmission zeros of the Brune network. These transmission zeros must be chosen to coincide with a pair of imaginary-axis zeros of the real part of $Z_1(s)$. It is assumed in Figs. 4 and 5 and in the development to follow that the transmission zeros fall at $s = \pm j1$. Let

$$Z_1(j1) = jx_1 \tag{6}$$

and

$$Z_1'(j1) = r_1' \; , \tag{7}$$

where x_1 is non-zero.[11] Then it may be shown,[12] that r_1' is a real, positive constant subject to the inequality

$$r_1' \geq |x_1| \quad , \tag{8}$$

and that the parameters L and α should be determined by the equations

$$L = x_1 \tag{9}$$

and

$$\alpha = \frac{r_1' + x_1}{r_1' - x_1} \tag{10}$$

if $Z_2(s)$ is to be a positive-real function of lower degree than $Z_1(s)$.

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[11] Primes are used to denote differentiation.

[12] Tuttle, *op. cit.*, pp. 513-534.

Consideration of Fig. 5 shows that the parameter α, when regarded as a property of $Z_1(s)$ determined by (10),[13] fixes the turns ratio of the transformer in the Brune network. Thus this constant, which is positive by virtue of (8),[14] plays an important role in the Brune synthesis procedure. The value of α is greater than unity if L is positive and less than unity if L is negative, since (8) implies that

$$\frac{L}{\alpha - 1} = \frac{r_1' - x_1}{2} \tag{11}$$

is positive.

Suppose that

$$x_1 = L < 0 \ , \tag{12}$$

so that a series inductor can be removed from $Z_1(s)$ to leave a positive-real remainder function $Z_a(s)$, where

$$Z_a(s) = Z_1(s) - Ls \tag{13}$$

and

$$Z_a(\pm\, j1) = 0 \ . \tag{14}$$

After removal of the series inductor, the positive-real remainder admittance $1/Z_a(s)$ has a pole pair at $s = \pm\, j1$ with a residue of

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[13] Numbers enclosed in parentheses refer to designated mathematical expressions. When prefixed by A, the numbers refer to expressions in the appendix.

[14] The trivial case where $r_1' = |\, x_1 |$ is easily disposed of. See Tuttle, op. cit., p. 518.

$$K_1 = \frac{1}{Z_a'(j1)} = \frac{1}{r_1' - x_1} = \frac{\alpha - 1}{2L} > 0 \tag{15}$$

at each pole. Thus, a positive-real admittance of the form

$$y = \frac{2Ks}{s^2 + 1} \tag{16}$$

may be removed from $1/Z_a(s)$ to leave a positive-real remainder admittance if and only if

$$0 \leq 2K \leq 2K_1 = \frac{\alpha - 1}{L} . \tag{17}$$

Let the constant β be defined by the equation

$$2K = \frac{\beta - 1}{L} . \tag{18}$$

Then (17) will be satisfied if and only if

$$1 \geq \beta \geq \alpha . \tag{19}$$

The remainder impedance after removal of the admittance given by (16) is

$$Z_b(s) = \frac{1}{\frac{1}{Z_a(s)} - \frac{2Ks}{s^2 + 1}} . \tag{20}$$

This impedance function has a simple pole at infinity; its residue in that pole is made evident by the equation

$$\lim_{s \to \infty} \frac{Z_b(s)}{s} = \frac{L}{-\beta} . \tag{21}$$

The remainder impedance after removal of the pole at infinity from $Z_b(s)$ is

$$Z_c(s) = Z_b(s) + \frac{L}{\beta} s \quad . \tag{22}$$

This impedance function is a positive-real function of s having the same degree as $Z_1(s)$ unless β is chosen equal to α.

If

$$x_1 = L > 0 \ , \tag{23}$$

essentially the same result can be proved by using the dual form of the Brune network.[15] In this case, $Z_c(s)$ is positive-real and has the same degree as $Z_1(s)$ if and only if

$$1 \leq \beta < \alpha \ . \tag{24}$$

If β is chosen equal to α, $Z_c(s)$ is positive-real and of degree lower than $Z_1(s)$.

The relation between $Z_1(s)$ and $Z_c(s)$ is illustrated by Figs. 4 and 5 if Z_2 and α in these figures are replaced by Z_c and β, respectively.

To state the result of this section concisely, let the constant α determined from (10) be called the "Brune characteristic" of the datum impedance at the frequency of its real-part zero. Further, let β, the turns ratio of a Brune section satisfying (9), be called the "characteristic" of the Brune section. Then, if α is the Brune characteristic of $Z_1(s)$, a Brune section having any characteristic, β, lying in the interval

[15] The desired result may also be proved without reference to the sign of L by manipulation of the Brune network itself or by consideration of the relation between the even parts of $Z_1(s)$ and $Z_c(s)$.

between 1 and α (open at α and closed at 1) may be removed from $Z_1(s)$ to leave a remainder impedance that is positive-real and of the same degree as $Z_1(s)$. Since $(\alpha - 1)$ and $(\beta - 1)$ have the same sign for values of β in the interval stated, the Brune section is itself realizable also. The remainder impedance function $Z_c(s)$ is a minimum function satisfying the equation

$$Z_c(j1) = j\,\frac{L}{\beta}\ . \tag{25}$$

If β is chosen equal to unity, the Brune section degenerates to a direct connection and the remainder impedance becomes identical with $Z_1(s)$. If β is chosen equal to α, the resulting cascade section and remainder impedance become those obtained with the normal Brune procedure.

CHAPTER III

SELECTION OF THE CASCADE SECTION

Algebraic requirements.--Consider again the network structure indicated by Fig. 1 and mathematically characterized by (1). The equation

$$Z_2 = - z_{22} + \frac{z_{12}^2}{z_{11} - Z_1} = \frac{z_{22}Z_1 - d(z)}{z_{11} - Z_1} \quad , \tag{26}$$

where $d(z)$ is the determinant of the matrix z, is easily derived from (1). Let $z_{12}(s)$ be a transfer impedance function having zeros at $s = \pm j1$ and at two additional points in the complex frequency plane (s-plane). Either these additional points will be conjugate complex points or they will both be real. Suppose also that each element of the second-order square matrix z is a fourth-degree impedance function, i.e., each $z_{ij}(s)$ has (the same) four poles in the s-plane. Poles of $Z_2(s)$ are generated by the poles of z_{12} (or z_{22}) and the zeros of $z_{11} - Z_1$, unless cancellations occur in (26). Thus, if $Z_1(s)$ has degree n, $Z_2(s)$ has degree $n + 8$. However, if $d(z)$ has only simple poles at the poles of z_{12}, Z_2 will not contain the poles of z_{12} (barring some special circumstances). In this case the matrix z is said to be compact.[16] Only compact z-matrices will be admitted henceforth; thus, the degree of Z_2 will be $n + 4$.

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[16] This terminology was introduced by Dasher in reference to RC two-terminal-pair networks. See B. J. Dasher, "Synthesis of RC transfer functions as unblanced two terminal-pair networks," Technical Report No. 215, Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts; November, 1951.

Poles of Z_2 produced by zeros of $z_{11} - Z_1$ may also be reduced in number if the equation

$$z_{11} = Z_1 \tag{27}$$

is satisfied at some or all of the zeros of z_{12}. If (27) is satisfied at a pair of zeros of z_{12}, the degree of Z_2 is reduced by two. An additional reduction by two is obtained when

$$z'_{11} = Z'_1 \tag{28}$$

at the same pair of zeros of z_{12}. If the pair of zeros is a conjugate complex pair, satisfaction of (27) or (28) at one member of the pair implies its satisfaction at the other member, since Z_1, Z_2, and z_{ij} are all real for real values of s. Thus, by employing a cascade section having a compact z-matrix, and by requiring that

$$z_{11}(j1) = Z_1(j1) \tag{29}$$

and

$$z'_{11}(j1) = Z'_1(j1) \ , \tag{30}$$

a remainder impedance, Z_2, of degree n is obtained. A remainder impedance of degree n - 2 is obtained if (27) is satisfied at the additional zeros of z_{12}. In this event the additional zeros of z_{12} may be referred to as "surplus" transmission zeros, since the loaded transfer impedance,

$$Z_{21} = \frac{z_{21} Z_2}{z_{22} + Z_2} \ , \tag{31}$$

is not zero at the additional zeros of z_{12}.

Mathematical description of the cascade section.--In keeping with the compactness requirement discussed above, and in order to provide transmission zeros at $s = \pm j1$, let the elements of the open-circuit impedance matrix describing the cascade section be

$$z_{11} = \frac{1}{1 + c} (z_T + cz_R) \ , \tag{32}$$

$$z_{12} = \frac{1}{1 + c} (z_T - z_R) = z_{21} \ , \tag{33}$$

and

$$z_{22} = \frac{1}{1 + c} (z_T + \frac{z_R}{c}) \ , \tag{34}$$

where

$$z_T = \frac{L}{b - 1} \left[\frac{b^2Ts^2 + (b - 1)^2 s + bT}{s^2 + bTs + b} \right] \tag{35}$$

and

$$z_R = \frac{L}{a - 1} \left[\frac{a^2Rs^2 + (a - 1)^2 s + aR}{s^2 + aRs + a} \right] \ . \tag{36}$$

The real constant c is assumed to satisfy the inequality

$$c \geq 0 \ , \tag{37}$$

and a, b, L, T, and R are real non-negative constants subject to the inequalities

$$\frac{L}{a - 1} \geq 0 \tag{38}$$

and

$$\frac{L}{b - 1} \geq 0 \ . \tag{39}$$

The impedance functions $z_T(s)$ and $z_R(s)$ are positive-real, minimum-real, second-degree functions such that

$$z_T(j1) = z_R(j1) = jL \ , \tag{40}$$

$$z_T'(j1) = L(\frac{b + 1}{b - 1}) \ , \tag{41}$$

and

$$z_R'(j1) = L(\frac{a + 1}{a - 1}) \ . \tag{42}$$

It is evident that (33) and (40) imply that

$$z_{12}(j1) = 0 \ . \tag{43}$$

The transfer impedance function $z_{12}(s)$ may be expressed in factored form as

$$z_{12}(s) = \frac{abL}{(1+c)(a-1)(b-1)} \left[\frac{(s^2+1)(\theta_2 s^2+\theta_1 s+\theta_0)}{(s^2+bTs+b)(s^2+aRs+a)} \right] , \qquad (45)$$

where

$$\theta_2 = bT - aR + \frac{a}{b} R - \frac{b}{a} T \ , \tag{45}$$

$$\theta_1 = (b - a) \left[(\frac{a - 1}{a})(\frac{b - 1}{b}) - RT \right] \ , \tag{46}$$

and

$$\theta_0 = aT - bR + R - T \; . \tag{47}$$

It is interesting to note that θ_1 vanishes if

$$a = b \tag{48}$$

or if

$$RT = \left(\frac{a - 1}{a}\right)\left(\frac{b - 1}{b}\right) \; . \tag{49}$$

In either of these cases the surplus transmission zeros, defined by the zeros of

$$\theta(s) = \theta_2 s^2 + \theta_1 s + \theta_0 \; , \tag{50}$$

fall on the imaginary axis of the complex frequency plane. If (48) is satisfied, then θ_2 and θ_0 are equal and the surplus zeros of transmission occur at $s = \pm j1$. The asymptotic case in which R approaches zero and T approaches infinity, but in such a way that the product RT remains finite, is also interesting, since these conditions correspond to a lossless cascade section. Of course the transmission zeros must be on the imaginary axis of the s-plane in this case, since each $z_{ij}(s)$ is of fourth degree, and one pair of zeros of $z_{12}(s)$ is already constrained to fall at $s = \pm j1$. The transfer impedance function that results from the limiting process is

$$\lim_{\substack{R \to 0 \\ T \to \infty \\ RT = \text{positive constant}}} z_{12}(s) = \frac{Lb}{(1 + c)(b - 1)} \left[\frac{(s^2 + 1)(s^2 + a/b)}{s(s^2 + a)}\right] \; . \tag{51}$$

In the general case, whether the cascade section is lossy or lossless, (1) implies that the surplus transmission zeros must not lie in the right-half s-plane. Otherwise $z_{22} + Z_2$, which must be positive-real if $Z_2(s)$ is to be positive-real, would have at least one zero in the right-half s-plane. This situation is impossible; therefore, a necessary condition for a positive-real remainder impedance function is that $\theta(s)$ have no right-half plane roots (unless (27) and (28) should both be satisfied at the surplus transmission zeros). This requirement means that θ_2, θ_1, and θ_0 must have the same sign.

Since the two special cases where $R = 0$ and $1/T = 0$ will be exploited later in order to simplify the realization of the cascade section, it is appropriate to determine the locations of the surplus transmission zeros in these cases. If

$$R = 0 \ , \tag{52}$$

the roots of $\theta(s)$ are determined by

$$s^2 + \frac{a}{b}\left[\frac{(b-1)(b-a)}{Tab}\right] s + \frac{a}{b} = 0 \ . \tag{53}$$

The condition barring right-half plane surplus zeros is thus

$$(b-1)(b-a) \geq 0 \ . \tag{54}$$

Because (38) and (39) require that $(a - 1)$ and $(b - 1)$ have the same sign, it is clear that

$$b \geq a \tag{55}$$

or

$$b \leq a \tag{56}$$

is required, accordingly as L is positive or negative, respectively. Similarly, for

$$\frac{1}{T} = 0 \ , \tag{57}$$

the surplus zeros are defined by

$$s^2 + \frac{a}{b} \left[\frac{a - b}{a - 1} \right] Rs + \frac{a}{b} = 0 \ , \tag{58}$$

and the condition barring right-half plane zeros is

$$\frac{a - b}{a - 1} \geq 0 \ . \tag{59}$$

Thus the necessary requirement is

$$a \geq b \tag{60}$$

or

$$a \leq b \ , \tag{61}$$

accordingly as L is positive or negative, respectively.

Determination of constants.--In order to determine the appropriate values of the constants which enter as parameters in the mathematical description of the cascade section, it is necessary to return to (29) and (30), relating z_{11} and Z_1. Previous equations concerning $z_{11}(s)$ and the notations introduced in Chapter II allow (29) and (30) to be written as

$$x_1 = L \tag{62}$$

and

$$r_1' = \frac{L}{1 + c}\left[\left(\frac{b + 1}{b - 1}\right) + c\left(\frac{a + 1}{a - 1}\right)\right] \quad . \tag{63}$$

Implicit in these equations is the requirement that the datum impedance function $Z_1(s)$ be a minimum function with real-part zeros at $s = \pm j1$. In view of the discussion in Chapter II, this requirement is not restrictive. From (10) of Chapter II it may be deduced that

$$\frac{r_1'}{x_1} = \frac{\alpha + 1}{\alpha - 1} \quad . \tag{64}$$

This equation and (62) may be employed to reduce (63) to the form

$$(1 + c)\left(\frac{\alpha + 1}{\alpha - 1}\right) = \left(\frac{b + 1}{b - 1}\right) + c\left(\frac{a + 1}{a - 1}\right) \tag{65}$$

or to

$$c = \left(\frac{a - 1}{b - 1}\right)\left(\frac{b - \alpha}{\alpha - a}\right) \quad . \tag{66}$$

Equations (62) and (66) may be considered as solutions for the constants L and c if suitable values for the parameters a and b can be found.

The constant c fixes the impedance level disparity between right and left sides of the quasi-symmetric network which will be used to realize the z-matrix of the cascade section. It will be seen later that a non-negative value of c is necessary in order for the realization process to succeed. Therefore, it is important to observe that c is non-negative if a and b lie on opposite sides of α on the real line. Stated another way, since $(a - 1)$ and $(b - 1)$ have the same sign, c will be non-negative if

$$a \leq \alpha \leq b \tag{67}$$

or if

$$a \geq \alpha \geq b \ . \tag{68}$$

If neither (67) nor (68) is satisfied, a negative value of c results from (66). The inequalities of (67) and (68) will also prove pertinent to the problem of realizing the cascade section in an unbalanced network without mutual inductance and to the requirement that the remainder impedance function be positive-real.

CHAPTER IV

UNBALANCED FORM OF THE CASCADE SECTION

Procedure for realizing the cascade section.--The cascade network represented by the matrix z, with elements given by (32)-(34), can be realized in an unbalanced form by the following procedure if each step of the procedure succeeds:[17]

A symmetric two-terminal-pair network is constructed having

$$z_{och} = z_T \tag{69}$$

and

$$z_{sch} = z_R \,, \tag{70}$$

where z_{och} and z_{sch} represent, respectively, the open-circuit and short-circuit impedances of half of the bisected symmetric network. This construction may be accomplished by realizing first a symmetric lattice network with horizontal arms equal in impedance to z_{sch} and diagonal arms equal in impedance to z_{och}. The lattice network is then unbalanced in a step-by-step manner. The resulting symmetric two-terminal-pair network is bisected, the right half is multiplied in impedance level by the constant $1/c$, and the network is re-connected at the bisection plane.

---

[17] The procedure outlined here was given by Dasher, in connection with RC transfer function synthesis. It is treated in greater detail by Guillemin. See Dasher, op. cit., pp. 11-23 and E. A. Guillemin, "Synthesis of Passive Networks," John Wiley and Sons, Inc., New York, N.Y., pp. 207-210; 1957.

Network configuration.--The symmetric lattice network employed as an intermediate stage in the unbalancing process is shown in Fig. 6. In the case where L is positive, z_T may be represented by the Bott-Duffin network of Fig. 7. The Bott-Duffin network for z_R may be obtained from Fig. 7 by substituting a for b and R for T.[18] The parameters L_b and C_b are given by

$$L_b = \frac{Lb^2T^2}{(b - 1)\ [b^2T^2 + (b - 1)^2]} \tag{71}$$

and

$$C_b = \frac{b - 1}{L[b^2T^2 + (b - 1)^2]} \quad . \tag{72}$$

The lattice network of Fig. 6 may be represented as the parallel combination of the two component lattice networks of Figs. 8 and 9. The parameters L_a and C_a in these figures are defined analogously to L_b and C_b; expressions for them may be obtained from (71) and (72) by substituting a for b and R for T.[19] Component lattice network I has an unbalanced representation if

$$L_b \geq L_a \tag{73}$$

and

$$\frac{TL}{b - 1} \geq \frac{RL}{a - 1} \quad . \tag{74}$$

---

[18] The Bott-Duffin network for z_R is also shown in Fig. 32 in the appendix.

[19] Expressions for L_a and C_a are also given by (A6) and (A7) in the appendix.

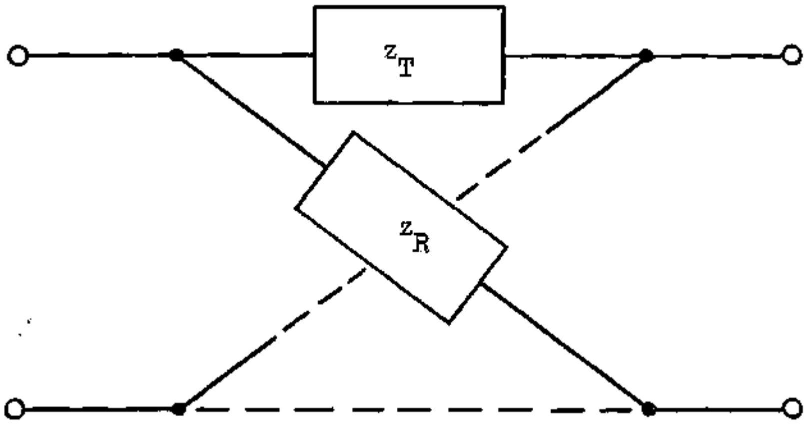

Figure 6. Symmetric Lattice Network.

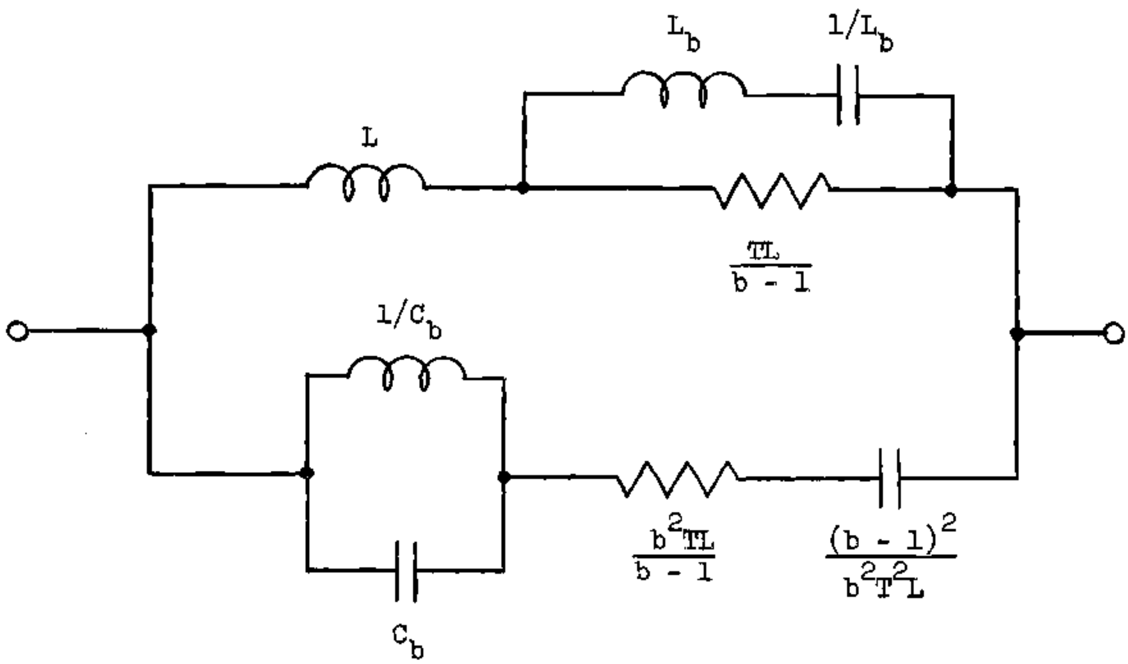

Figure 7. Bott-Duffin Network for z_T, $L \geq 0$.

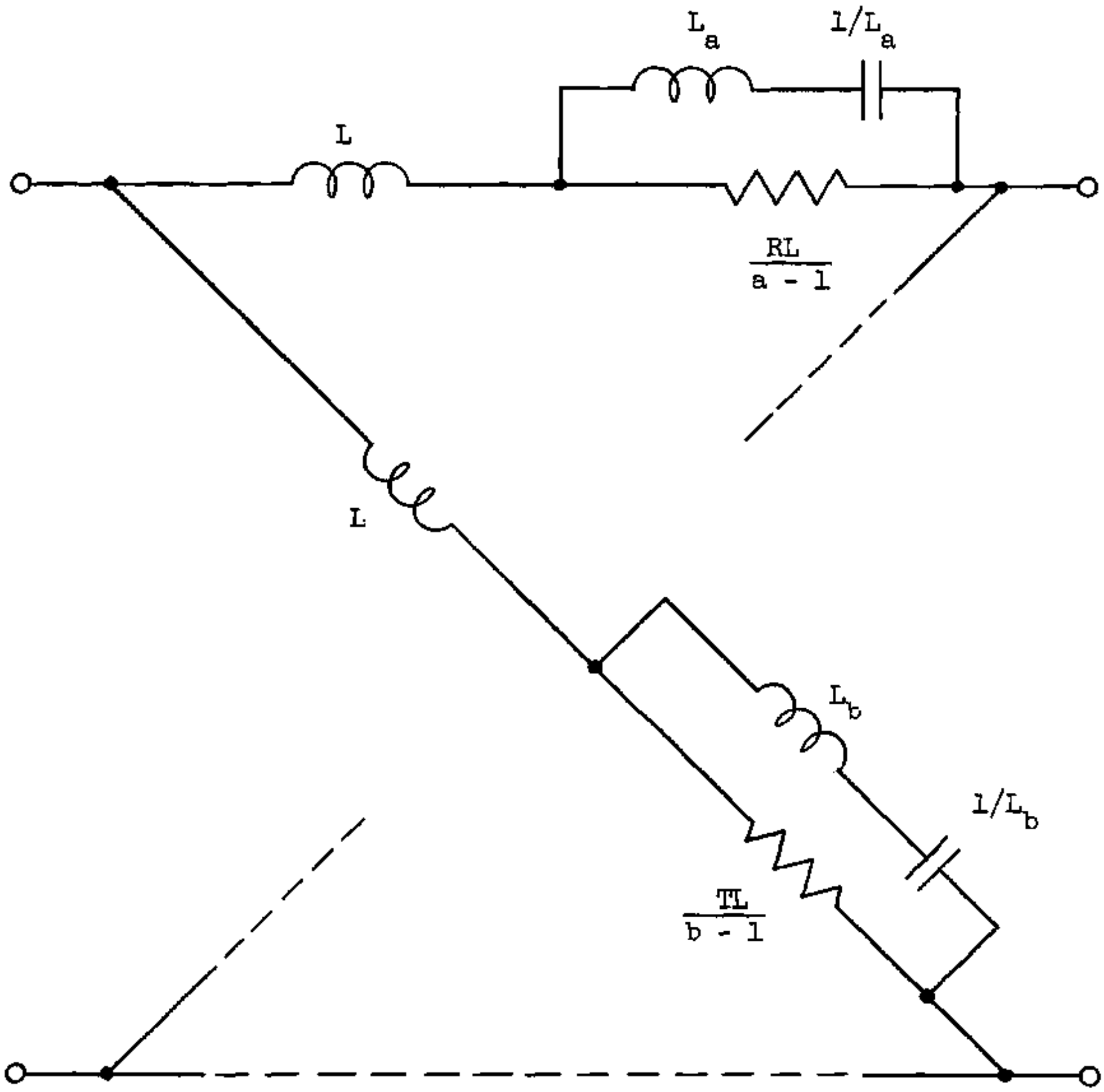

Figure 8. Component Lattice Network I, $L \geq 0$.

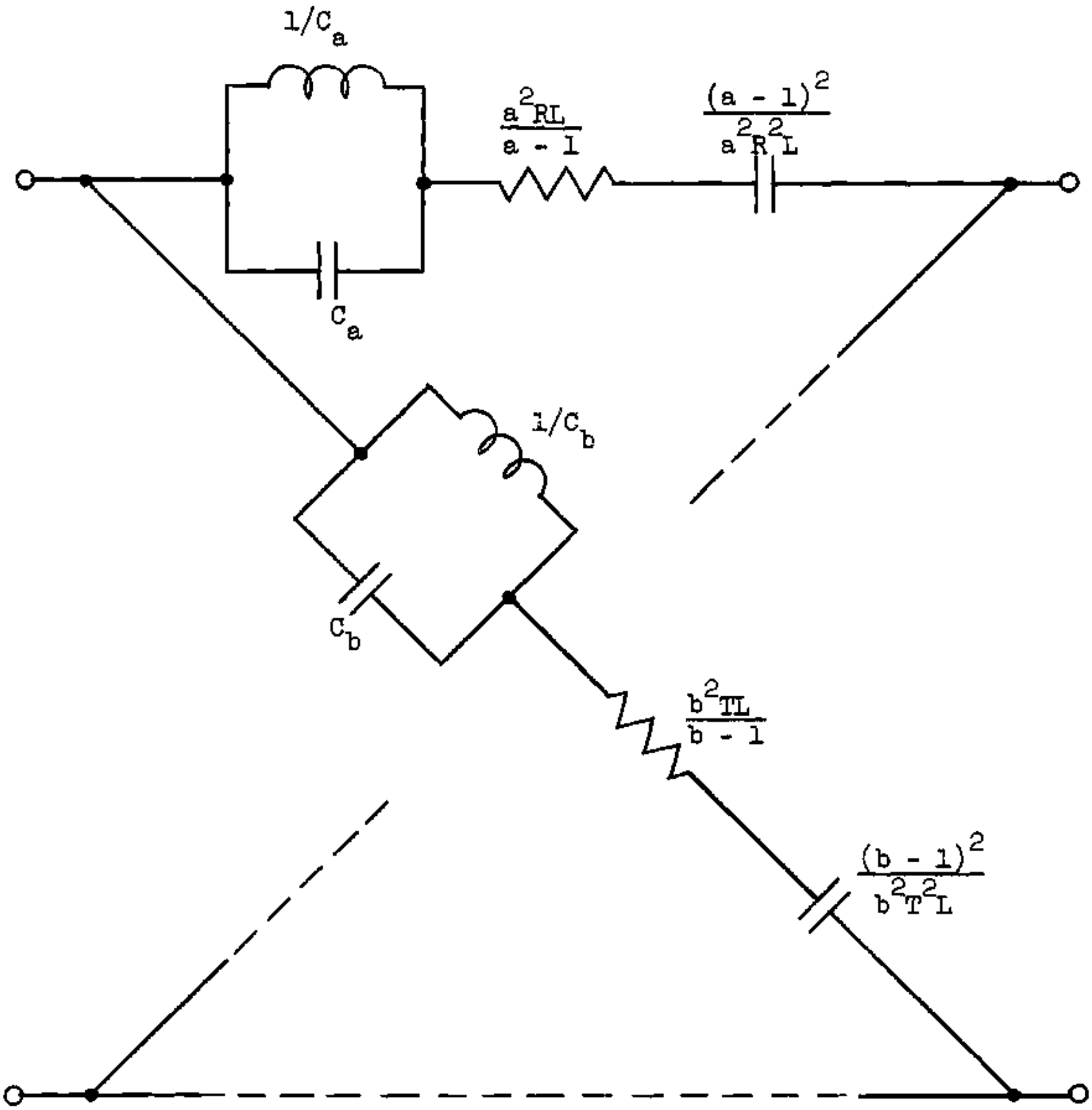

Figure 9. Component Lattice Network II, L $\geq$ 0.

Component lattice network II has an unbalanced representation if

$$\frac{b^2T^2L}{(b - 1)^2} \geq \frac{a^2R^2L}{(a - 1)^2} \quad , \tag{75}$$

$$\frac{b^2TL}{b - 1} \geq \frac{a^2RL}{a - 1} \quad , \tag{76}$$

and

$$C_a \geq C_b \quad . \tag{77}$$

If the unbalanced forms of the two component lattice networks are bisected and the impedance level of the right half of each is multiplied by $1/c$, the two component networks of Figs. 10 and 11 result. The two-terminal-pair network formed by the parallel combination of the component-networks of Figs. 10 and 11 is the desired unbalanced form of the cascade section described by the matrix z. The configuration of this network is shown in Fig. 12. Of course alternate forms of the final network may be obtained through the use of Tee-Pi transformations and two-terminal network equivalences.

The final network described above requires a total of 23 elements, of which 6 are resistors. The number of elements required is reduced to 11 reactors and 2 resistors if either of the restrictions

$$R = 0 \tag{78}$$

or

$$1/T = 0 \tag{79}$$

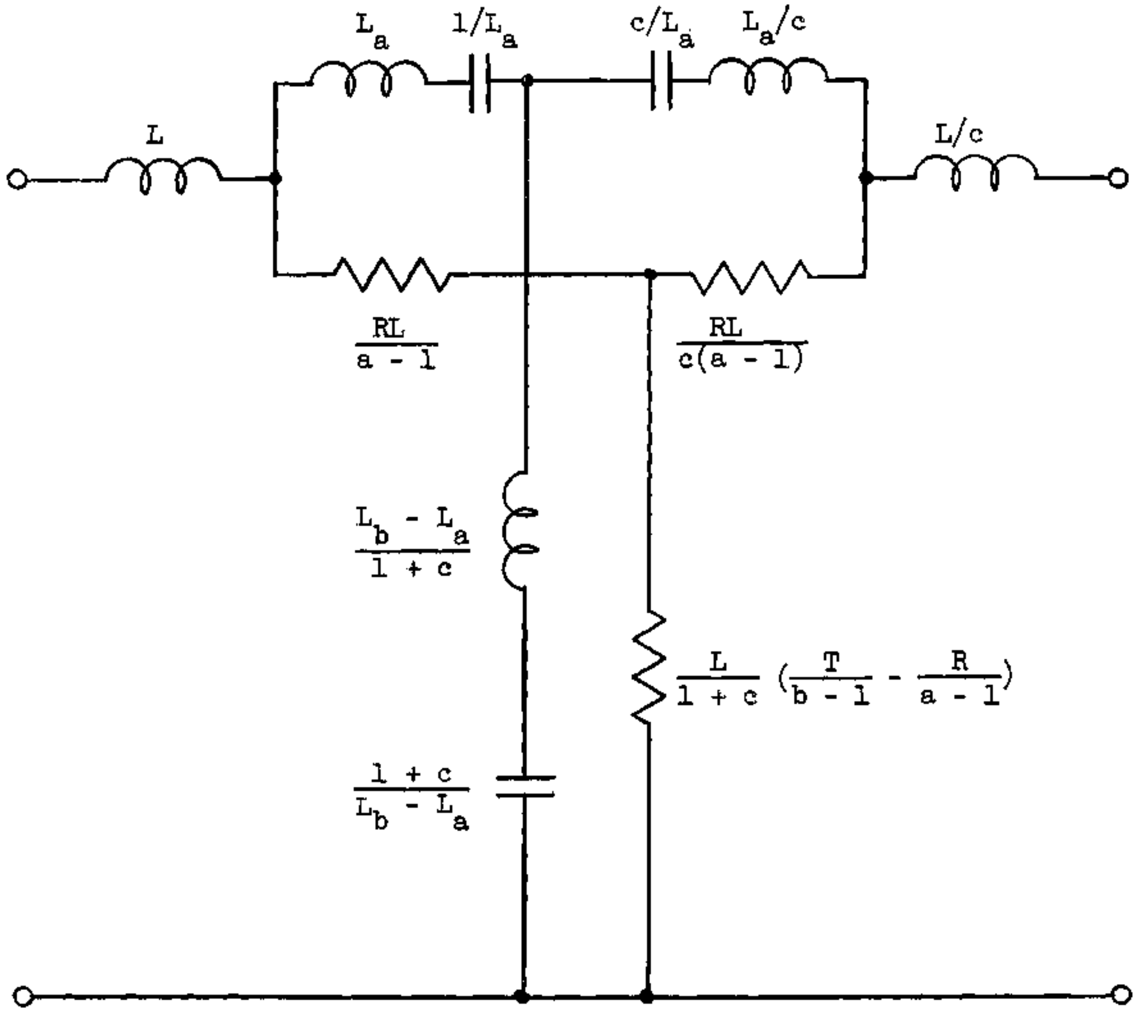

Figure 10. Component Network I, $L \geq 0$.

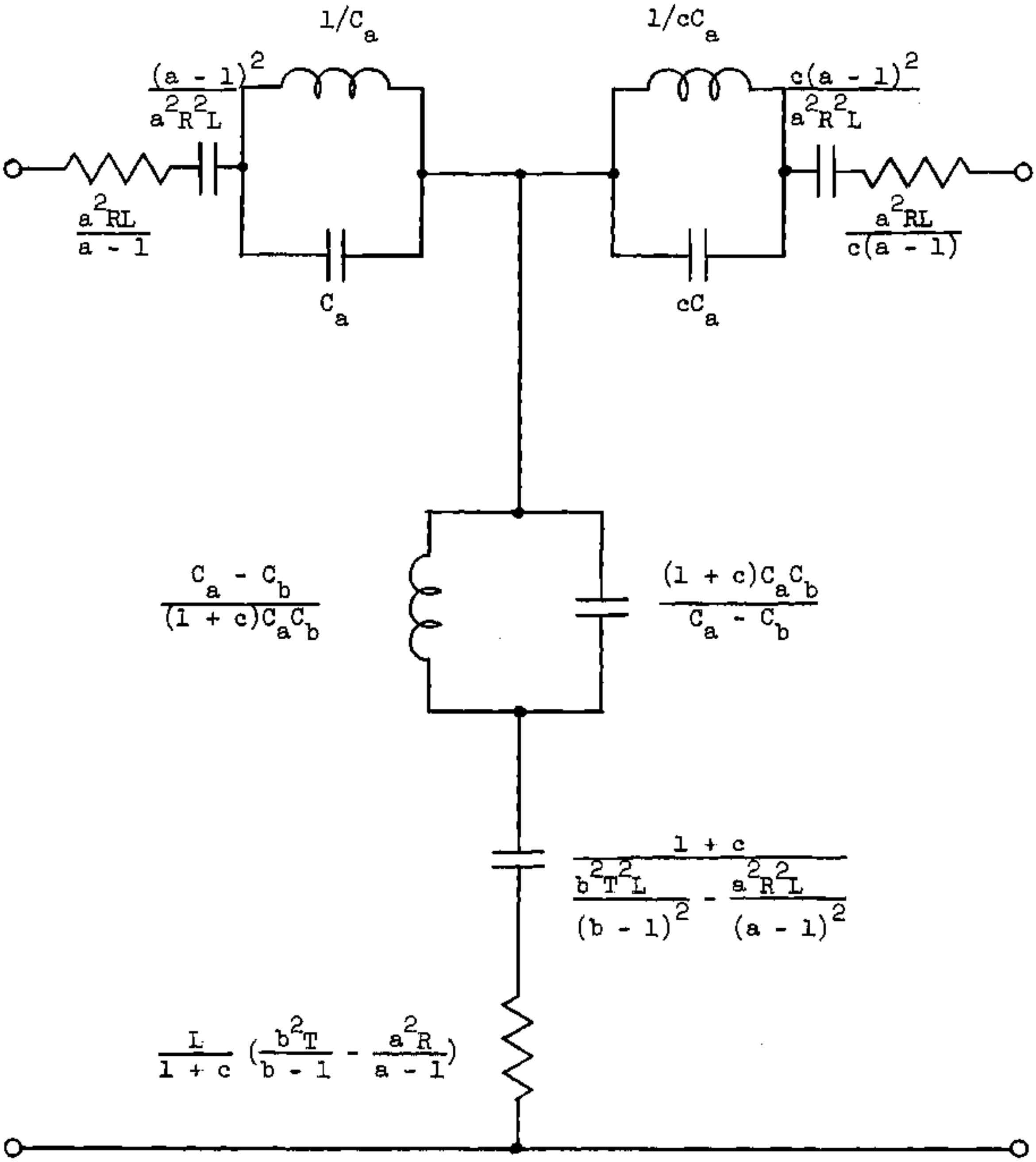

Figure 11. Component Network II, $L \geq 0$.

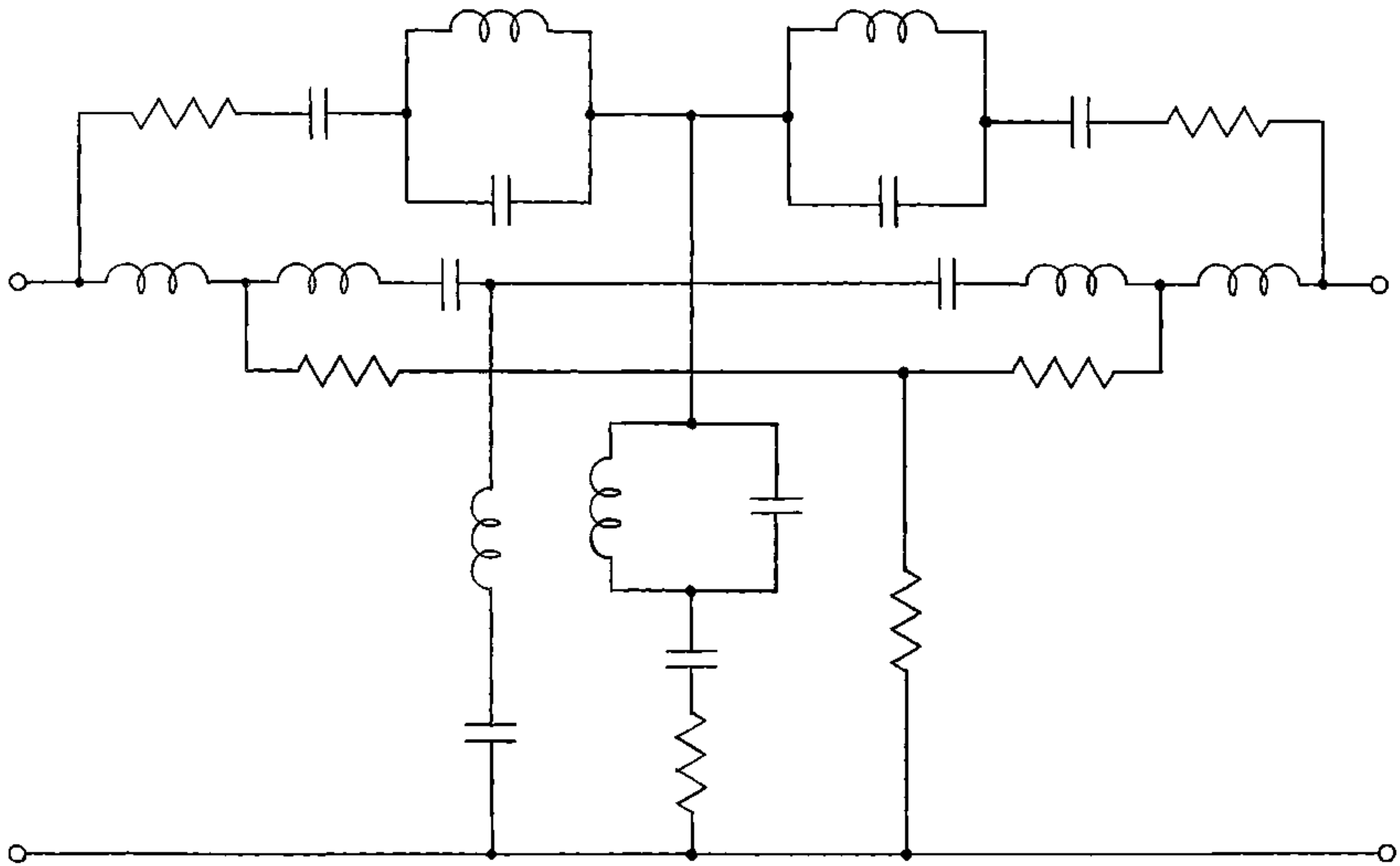

Figure 12. Configuration of Cascade Section, $L \geq 0$.

is enforced.[20] Primary attention will be given here and in succeeding chapters to the two cases where R and T are chosen to satisfy (78) or (79). However certain mathematical relations concerning the unbalanced form of the general cascade section are discussed in the Appendix.

It may be shown[21] that satisfaction of either of the inequalities

$$\frac{(a - 1)^4}{a^2R^2} - \frac{(b - 1)^4}{b^2T^2} \geq (b - 1)^2 - (a - 1)^2 \geq 0 \tag{80}$$

or

$$b^2T^2 - a^2R^2 \geq (a - 1)^2 - (b - 1)^2 \geq 0 \tag{81}$$

is a sufficient condition for transformerless realization of the cascade section when L is positive.[22] Inequality (80) implies (73)-(77), as well as

$$b \geq a \tag{82}$$

and

$$T \geq R . \tag{83}$$

Similarly, (81) implies (73)-(77), as well as

$$a \geq b . \tag{84}$$

---

[20] Equation (78) should more properly be considered as a short notation for a limiting process, i.e., the limits of the various element values of the cascade section are to be taken as T approaches infinity.

[21] See appendix.

[22] It is assumed that $a \geq 1$ and $b \geq 1$.

The choice $R = 0$ leads to the cascade section shown in Fig. 13. In this case it is evident from (80) that (82) is a sufficient condition for realization of the cascade section in the form shown by Fig. 13.

Similarly, the choice $1/T = 0$, i.e., T approaches infinity, leads to the cascade section shown in Fig. 14. In this case (81) reduces to (83), which is thus a sufficient condition for realization in the form shown by Fig. 14.

The development of a network representation of the matrix z follows an analogous pattern when L is negative. Again the general network contains 23 elements, and specialization to the two cases $R = 0$ and $1/T = 0$ reduces the number of elements required to 13. If $R = 0$, (84) becomes a sufficient condition for realization of the cascade section shown in Fig. 15. If $1/T = 0$, (82) is a sufficient condition for realization in the form shown by Fig. 16. Further details concerning realization of the cascade section when L is negative are contained in the Appendix.

A lossless cascade section results from the limiting process in which R approaches zero and RT remains finite and non-zero. In this case the cascade section assumes the form shown in Fig. 17 or Fig. 18 as L is positive or negative, respectively. The lossless cascade section has certain special properties, some of which are discussed in Chapter VII.

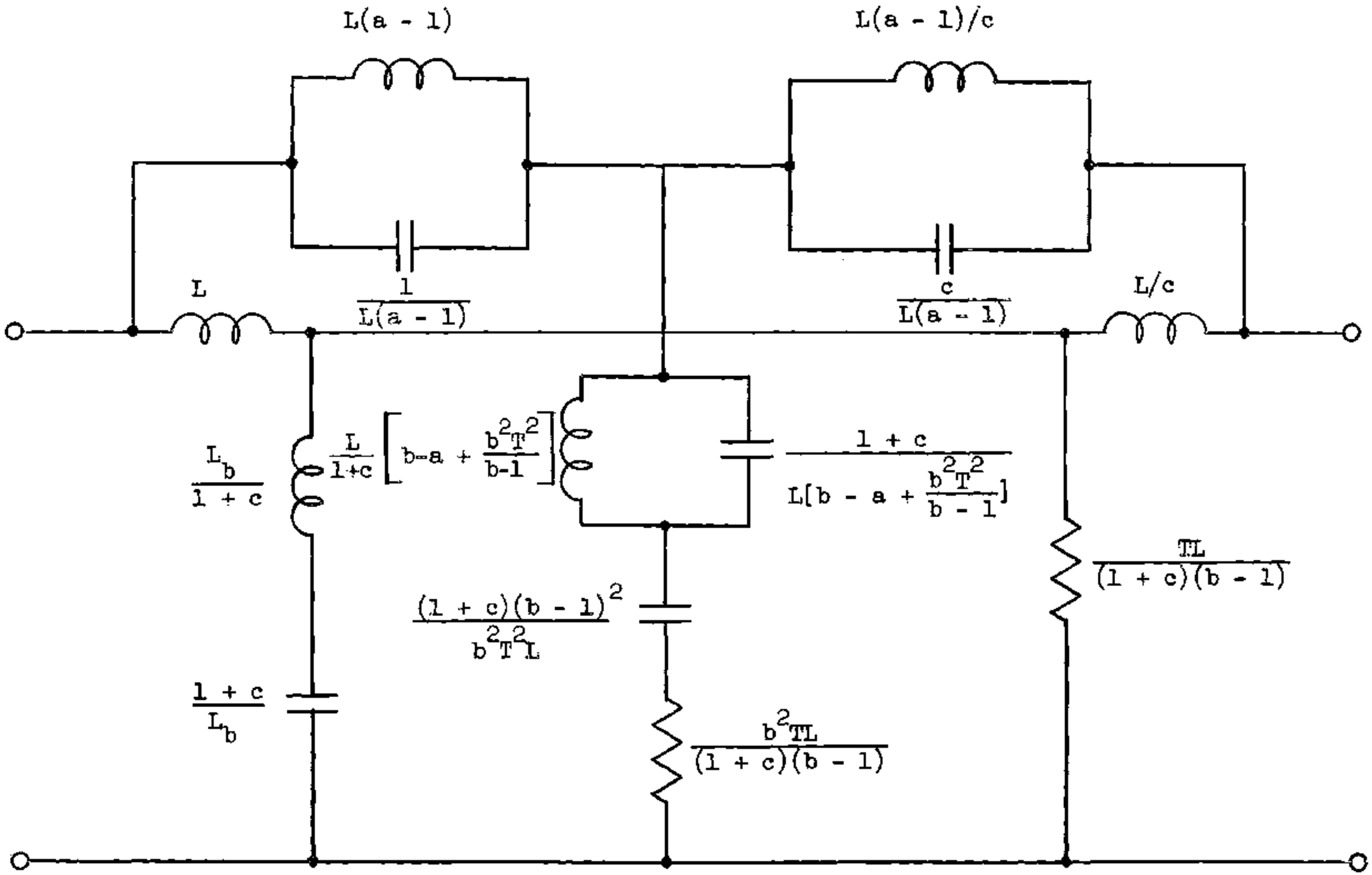

Figure 13. Cascade Section for $R = 0$, $L \geq 0$.

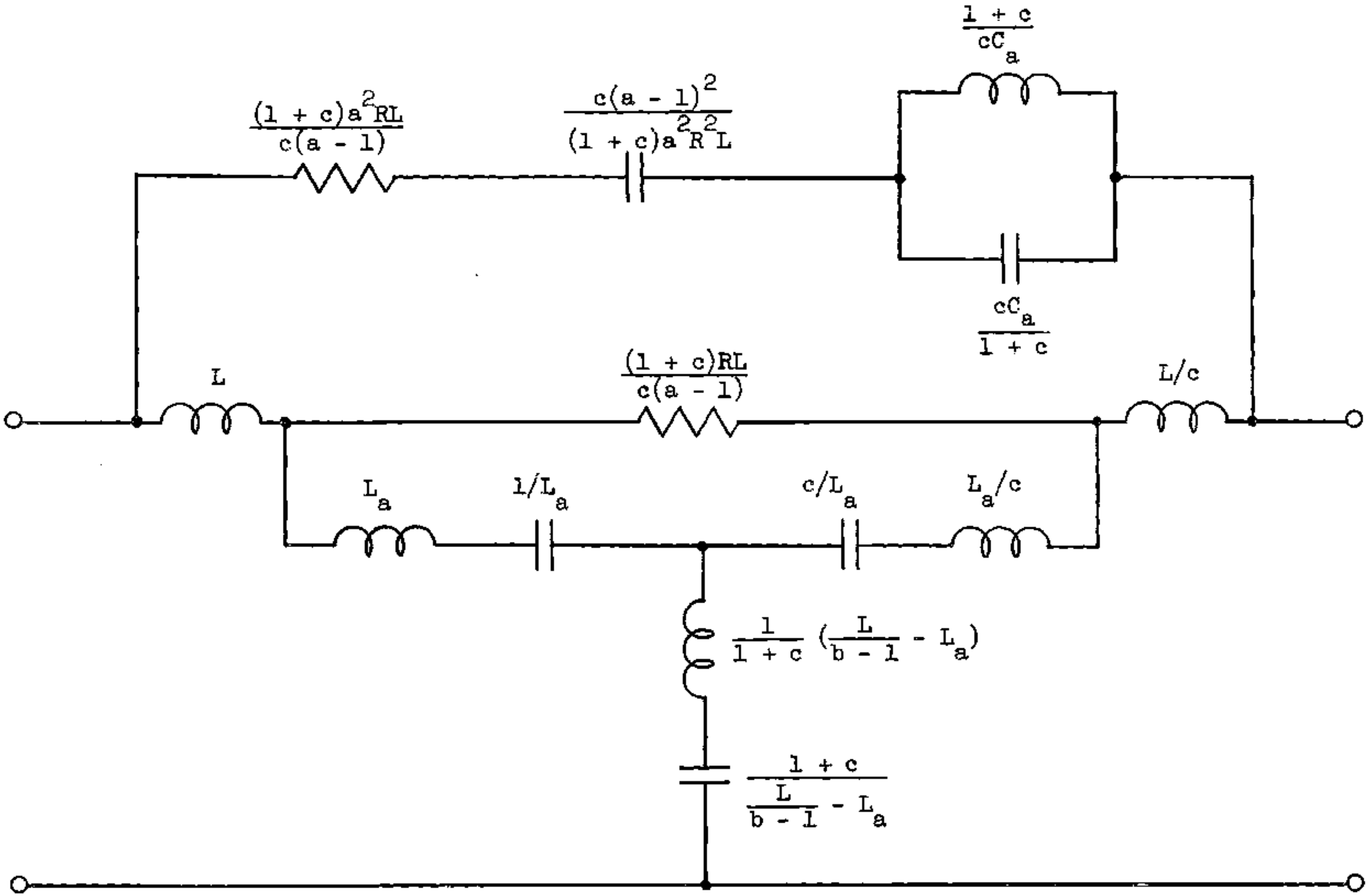

Figure 14. Cascade Section for $1/T = 0$, $L \geq 0$.

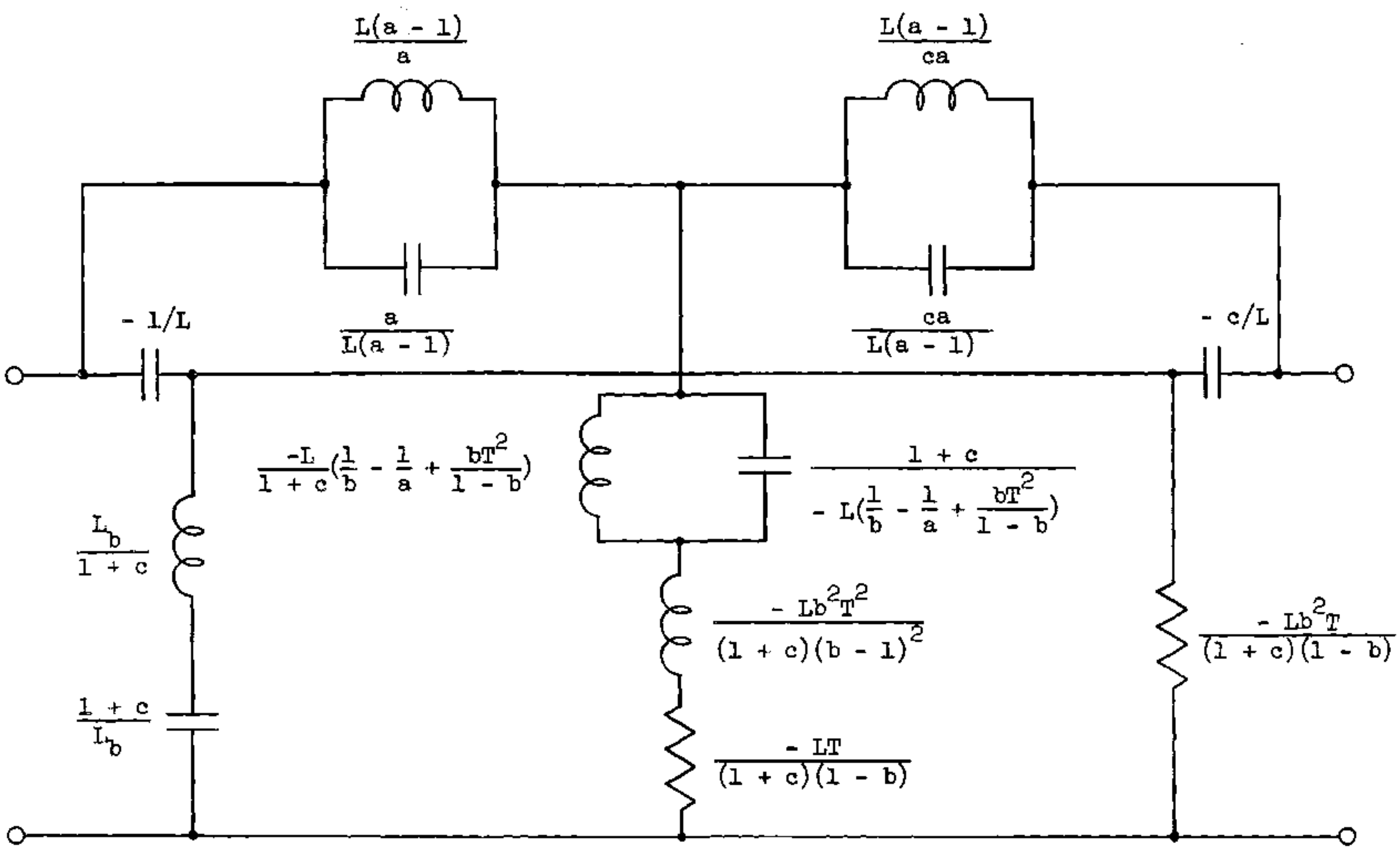

Figure 15. Cascade Section for $R = 0$, $L \leq 0$.

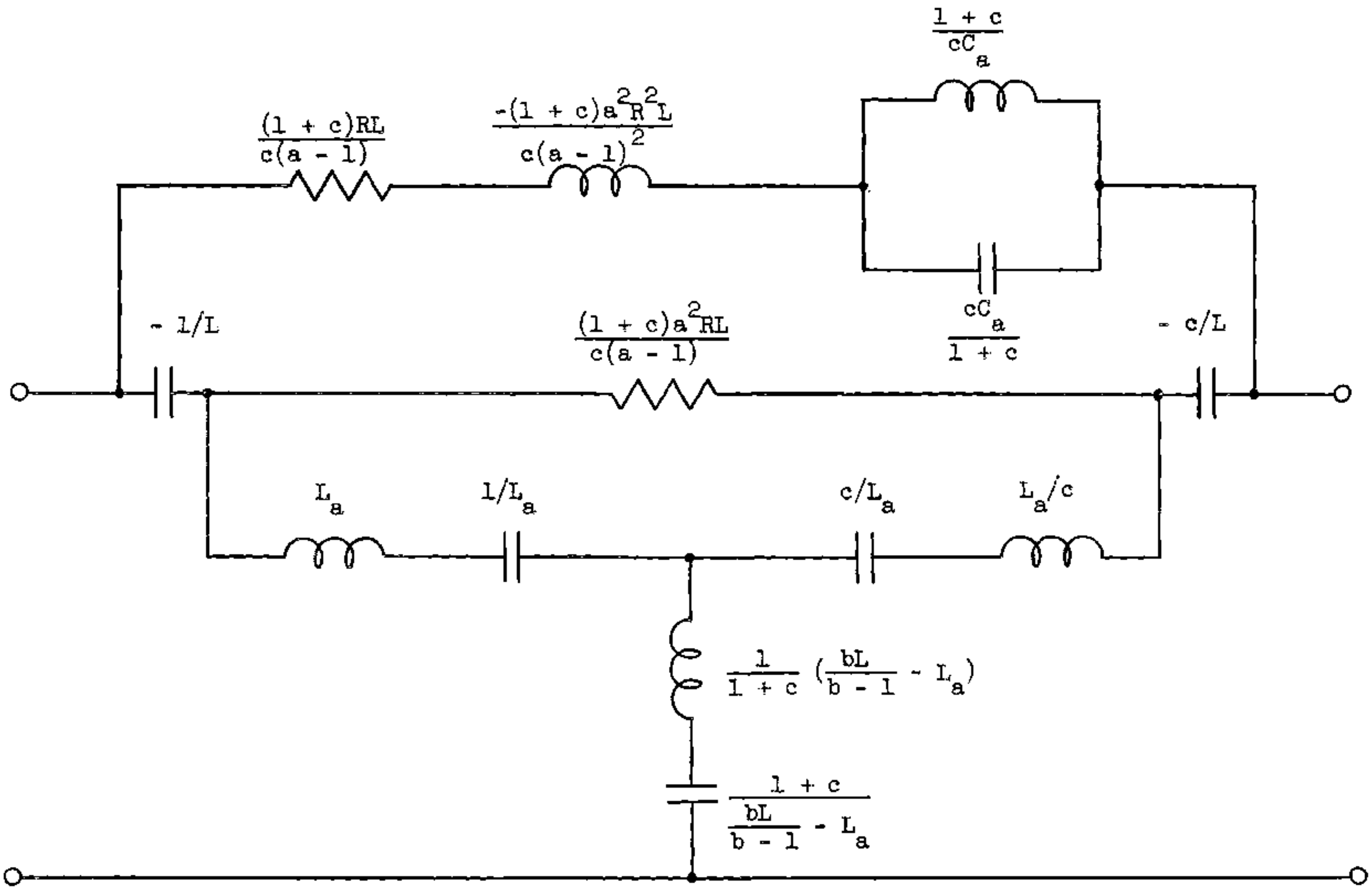

Figure 16. Cascade Section for $1/T = 0$, $L \leq 0$.

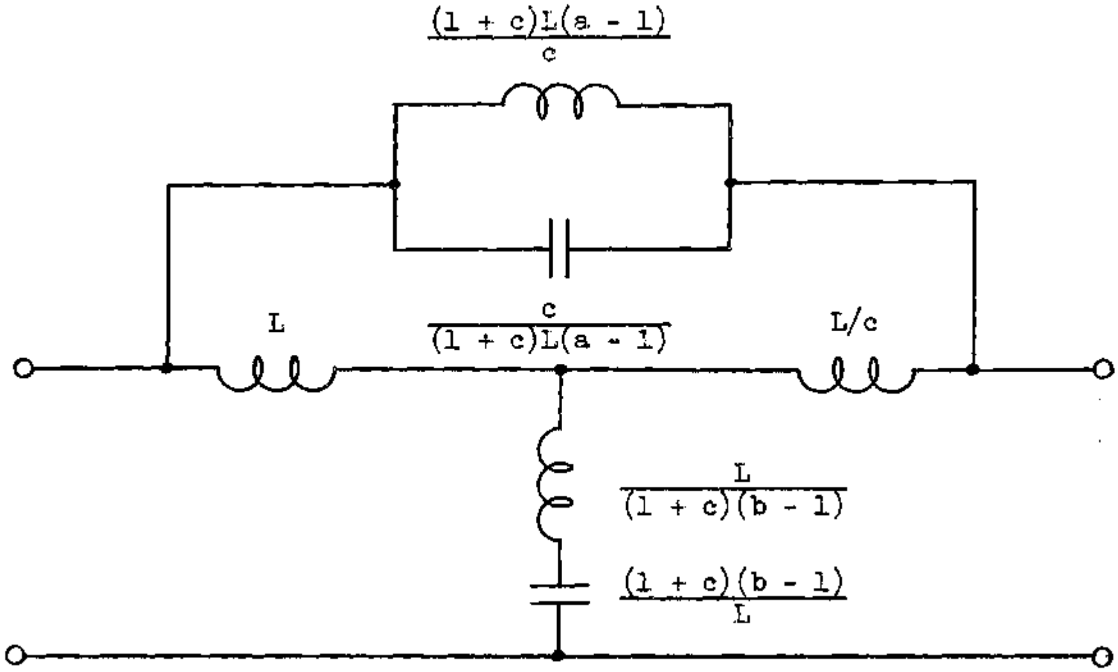

Figure 17. Lossless Cascade Section, $L \geq 0$.

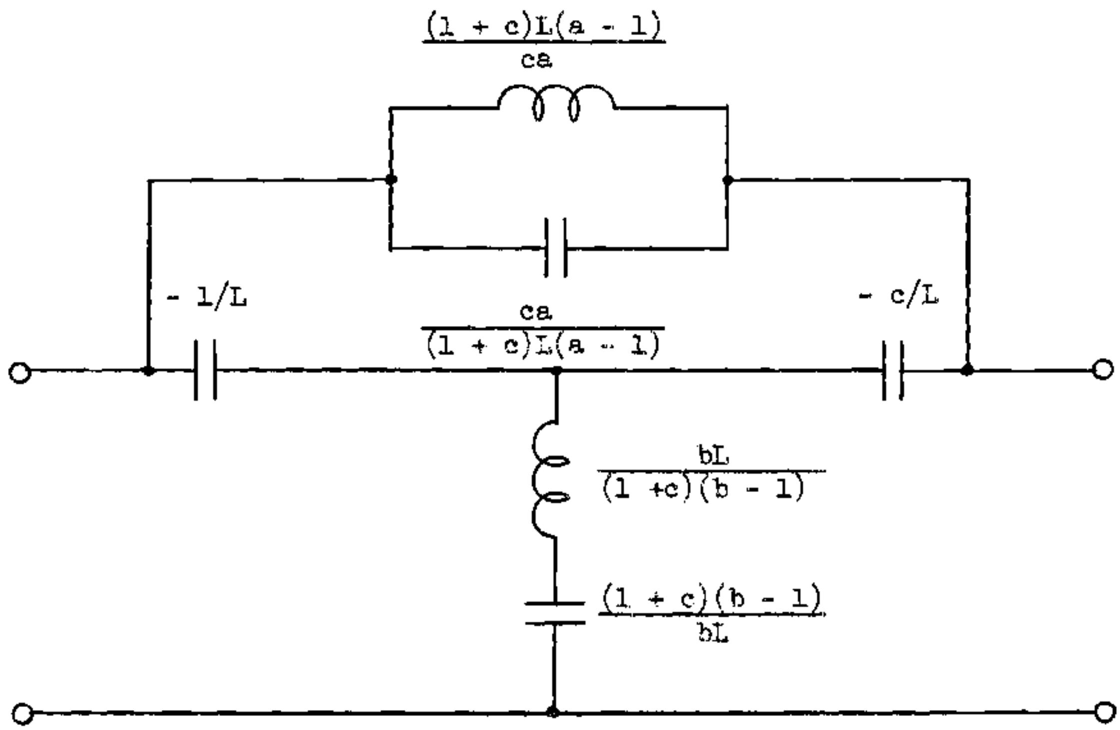

Figure 18. Lossless Cascade Section, $L \leq 0$.

CHAPTER V

REMOVAL OF THE CASCADE SECTION

<u>A pseudo-Brune development of the cascade section</u>.--In Chapter IV an unbalanced two-terminal-pair network realizing the open-circuit impedance matrix of the cascade section was developed. In order to control the characteristics of the remainder impedance in a step-by-step manner, it is expedient to develop another network representation of the cascade section. This representation, which resembles a cascade of two Brune sections, may be referred to as a "pseudo-Brune development" of the cascade section.

The procedure for obtaining the pseudo-Brune development of the cascade section follows the general pattern outlined in Chapter IV, except that z_T and z_R are represented by their Brune networks. The Brune network for z_R is shown by Fig. 30 in the appendix; the Brune network for z_T may be obtained from Fig. 30 by substituting b for a and T for R. The symmetric lattice network of Fig. 6, containing z_T and z_R as diagonal-arm and horizontal-arm impedances, is equivalent as a two-terminal-pair network to the network shown in Fig. 19. This network is obtained by removal of common series and shunt impedances from z_T and z_R. The two-terminal impedance Z_o is given by

$$Z_o(s) = \frac{L}{a-b}\left[\frac{(\frac{a}{b})^2(\frac{a-b}{a-1})Rs^2 + (\frac{a}{b} - 1)^2 s + \frac{a}{b}(\frac{a-b}{a-1})R}{s^2 + \frac{a}{b}(\frac{a - b}{a - 1})Rs + \frac{a}{b}}\right] . \qquad (85)$$

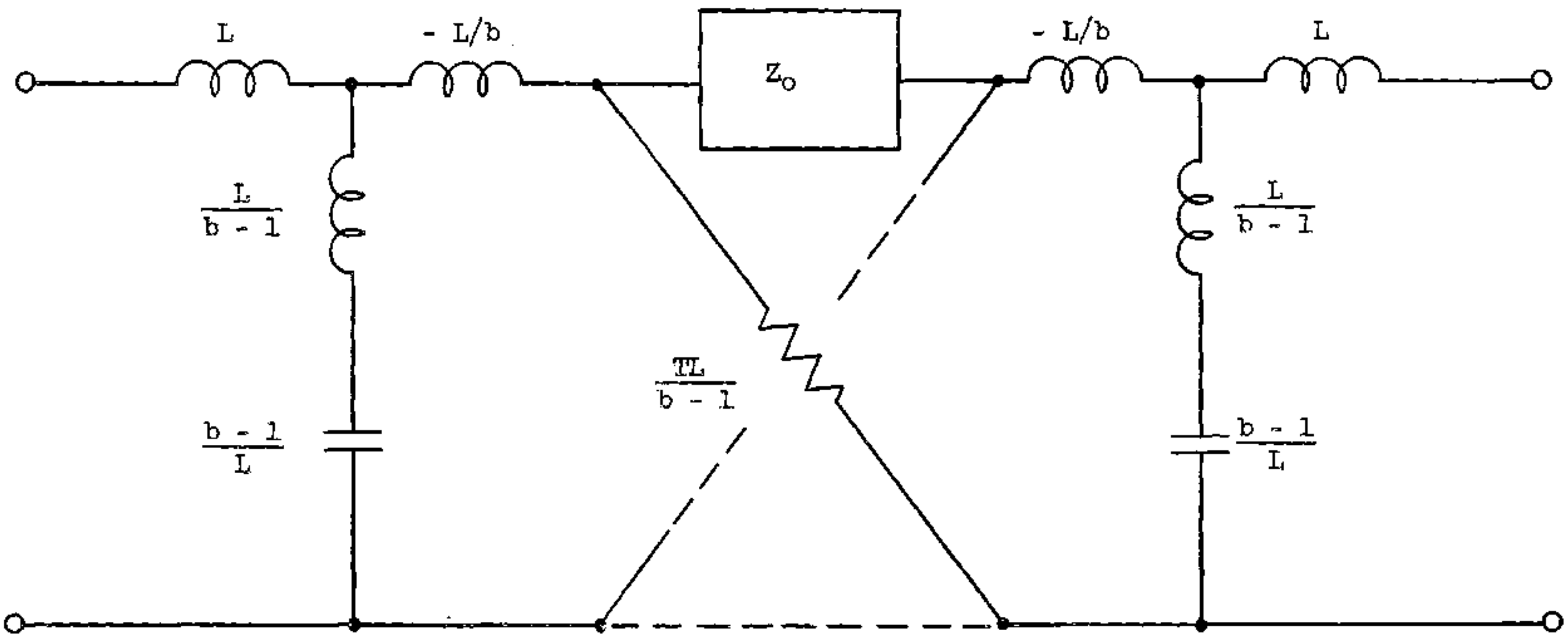

Figure 19. Reduction of the Symmetric Lattice Network.

The symmetric lattice internal to the circuit of Fig. 19 may be represented as a two-terminal-pair network by a Tee network. With the use of this representation, the complete network may be bisected. If the right half of the network is then multiplied in impedance level by the factor 1/c and the network is re-connected at the bisection plane, the cascade section shown in Fig. 20 results. This representation is called the pseudo-Brune development of the cascade section since it contains two Brune sections in cascade, separated by a Tee network. The left and right Brune sections have characteristics b and 1/b, respectively.

A procedure dual to that outlined above permits an alternate pseudo-Brune development of the cascade section in which the Brune sections have characteristics a and 1/a. In this procedure it is convenient to employ the dual forms of the Brune networks for z_T and z_R. Thus z_R is represented as in Fig. 31 in the Appendix, and the representation for z_T is obtained from the network for z_R by substituting b for a and T for R. The alternate pseudo-Brune development of the cascade section is shown in Fig. 21, where Y_o is an admittance given by the equation

$$Y_o(s) = \frac{ab}{L(b-a)} \left[\frac{\left(\frac{a}{b}\right)^2 \frac{(b-a)(b-1)}{Tab} s^2 + \left(\frac{a}{b} - 1\right)^2 s + \left(\frac{a}{b}\right) \frac{(b-a)(b-1)}{Tab}}{s^2 + \left(\frac{a}{b}\right) \frac{(b-a)(b-1)}{Tab} s + \frac{a}{b}} \right] . \quad (86)$$

<u>Location of surplus transmission zeros</u>.--It may be seen from Figs. 20 and 21 that the pseudo-Brune development makes evident the transmission zeros of the cascade section. In fact, the surplus transmission zeros are clearly the roots of

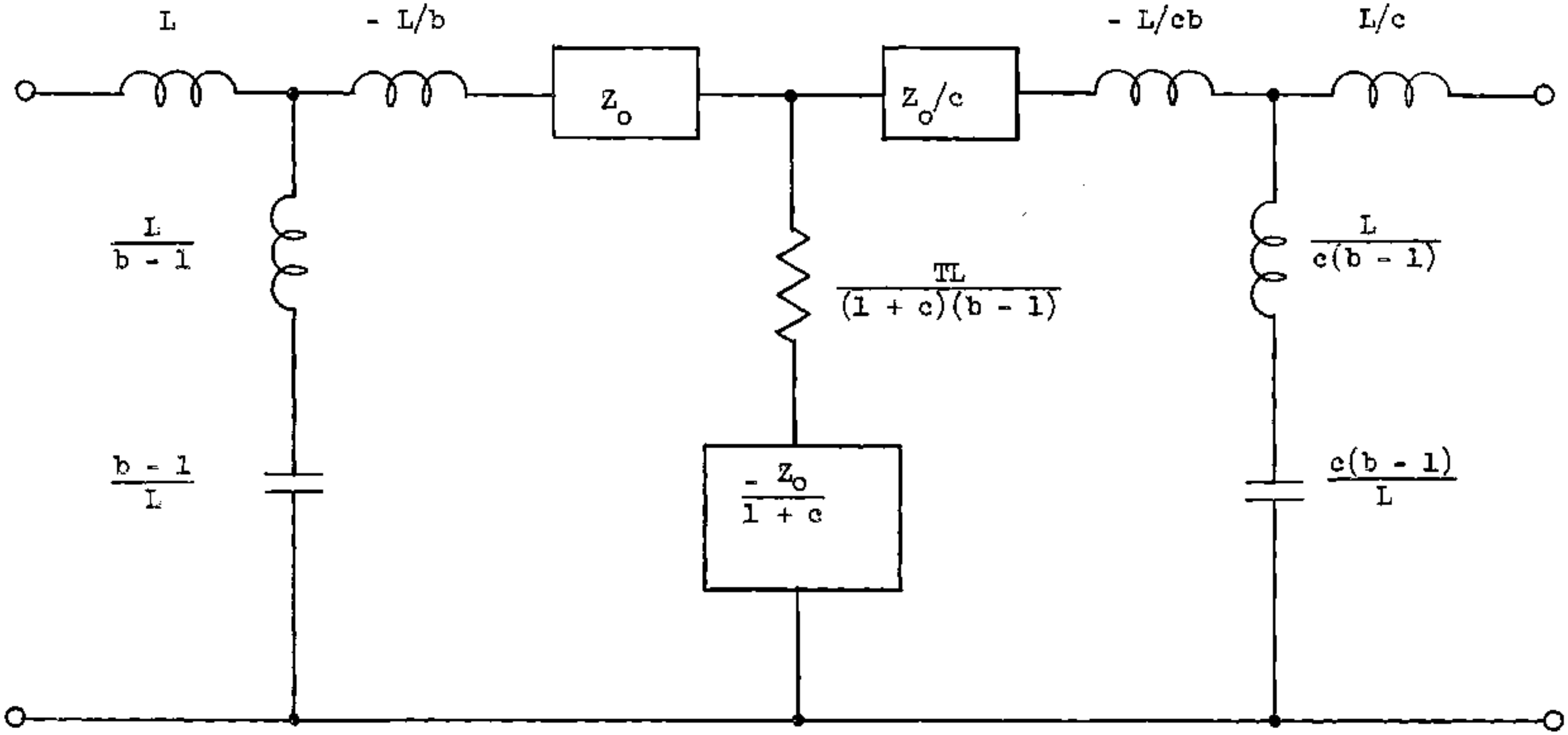

Figure 20. Pseudo-Brune Development of the Cascade Section.

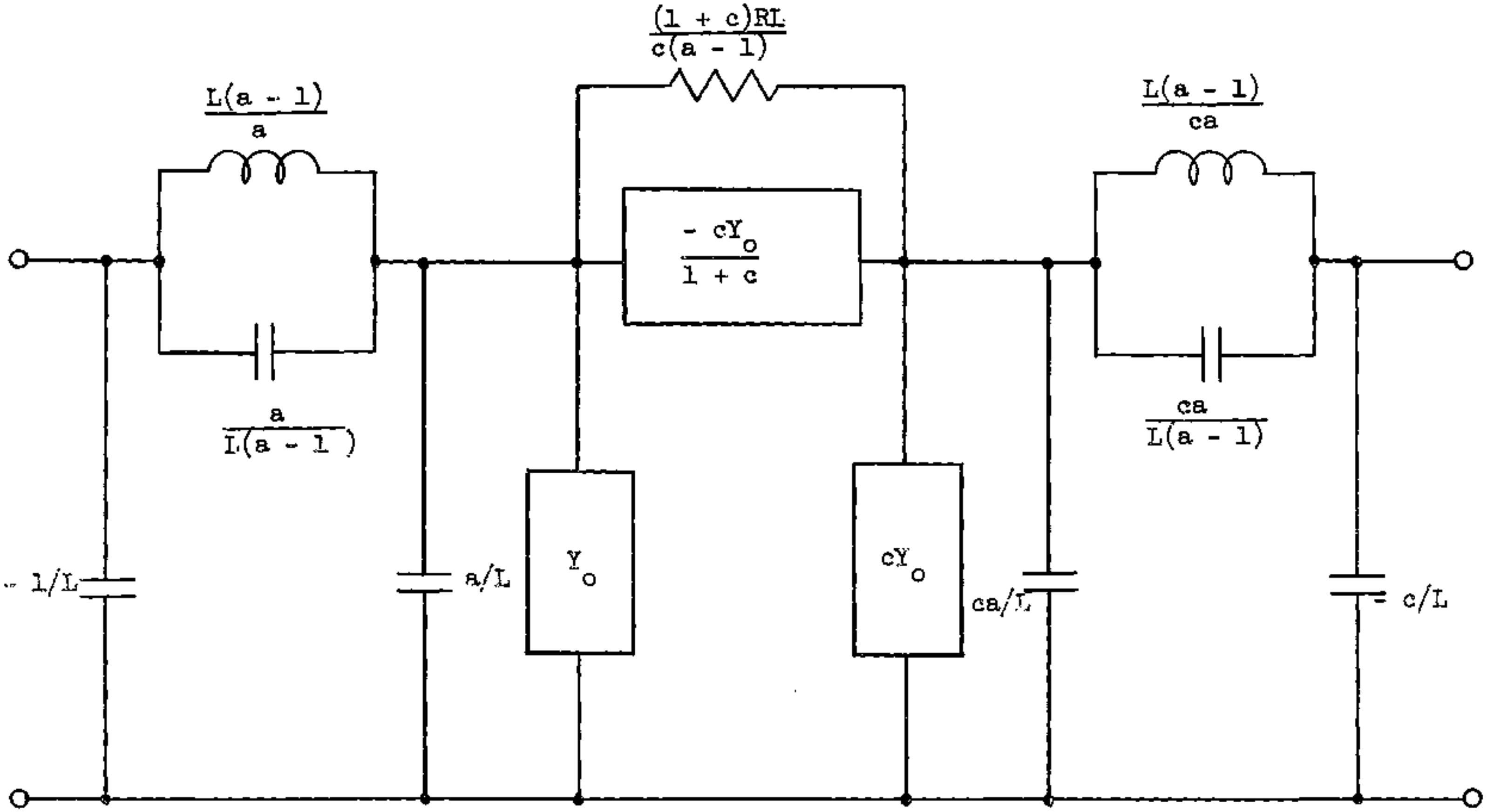

Figure 21. Pseudo-Brune Development of the Cascade Section.

$$Z_o(s) = \frac{TL}{b - 1} \tag{87}$$

or

$$Y_o(s) = \frac{a - 1}{RL} \ . \tag{88}$$

The two cases corresponding to the restrictions $R = 0$ and $1/T = 0$ are of interest here since the number of elements required in the transformerless realization is reduced by these restrictions. Moreover, as will be seen below, these cases allow some simplification in determining whether removal of the cascade section from the datum impedance leaves a positive-real remainder impedance function. Thus, except for the discussion of certain special cases in Chapter VI, one of the restrictions $R = 0$ or $1/T = 0$ will always be imposed in this and succeeding chapters. The restriction $1/T = 0$ causes the Tee network between the Brune sections of Fig. 20 to reduce to a single series impedance of $(1 + 1/c)Z_o(s)$, as shown by Fig. 22. Similarly, for $R = 0$ the Pi network between the Brune sections of Fig. 21 reduces to a single shunt admittance of $(1 + c)Y_o(s)$, as shown by Fig. 23. In Figs. 22 and 23 the networks denoted by $B(a)$ and $B(b)$ represent Brune sections with characteristics a and b, respectively, while $\frac{1}{c}B(a)$ and $\frac{1}{c}B(b)$ represent the geometrical images of $B(a)$ and $B(b)$ multiplied in impedance level by $1/c$.

It is indicated by Figs. 22 and 23 that the cascade section may be regarded as transforming the datum impedance $Z_1(s)$ into the remainder impedance $Z_2(s)$ by the following series of three steps:

(1) A modified Brune section is removed from $Z_1(s)$ to produce the impedance function $Z_u(s)$, which is positive-real and has the same degree as $Z_1(s)$ if the conditions outlined in Chapter II are satisfied.

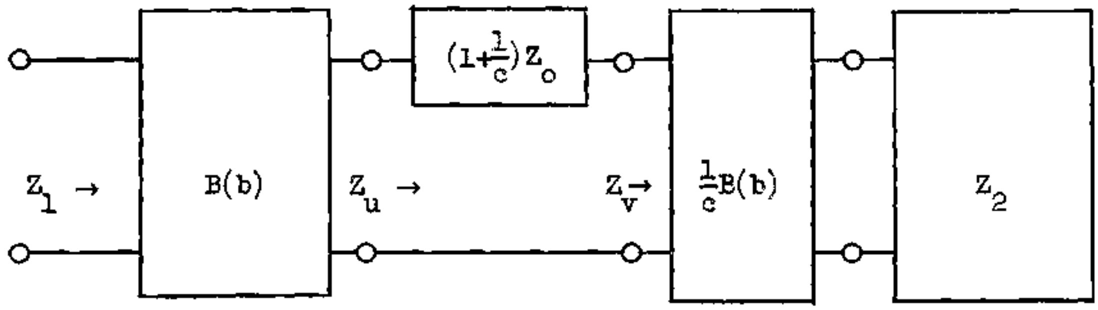

Figure 22. Transformation from Datum Impedance to Remainder Impedance, $1/T = 0$.

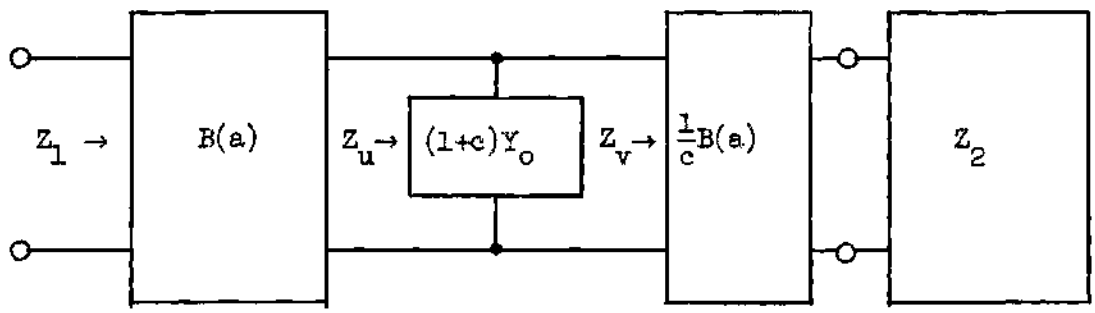

Figure 23. Transformation from Datum Impedance to Remainder Impedance, $R = 0$.

Table 1. Necessary Conditions for Realizability.

L	R or 1/T	Condition
$L \geq 0$	$R = 0$	$b \geq \alpha \geq a \geq 1$
$L \geq 0$	$1/T = 0$	$a \geq \alpha \geq b \geq 1$
$L \leq 0$	$R = 0$	$b \leq \alpha \leq a \leq 1$
$L \leq 0$	$1/T = 0$	$a \leq \alpha \leq b \leq 1$

(2) A series impedance or shunt admittance is removed from $Z_u(s)$ or its reciprocal. If the surplus transmission zeros of the cascade section can be located properly, this step produces an impedance function $Z_v(s)$ of the same degree as $Z_u(s)$. When the condition $R = 0$ is imposed, both (88) and Fig. 23 show that the surplus transmission zeros are simply the poles of $Y_o(s)$. Thus $Z_v(s)$ has the same degree as $Z_u(s)$ if the poles of $Y_o(s)$ are constrained to be zeros of $Z_u(s)$. It should be observed that the locations of these zeros are functions of the parameter a. Similarly, when $1/T = 0$ is required, (87) and Fig. 22 show that the surplus transmission zeros are the poles of $Z_o(s)$. Thus $Z_v(s)$ has the same degree as $Z_u(s)$ if the poles of $Z_o(s)$ are also poles of $Z_u(s)$. In either of the two cases $R = 0$ or $1/T = 0$, the proper choice of the poles of $Y_o(s)$ or $Z_o(s)$ insures that (27) is satisfied at the surplus transmission zeros. Thus, if the parameters L and c of the cascade section are determined according to (62) and (66), the remainder impedance $Z_2(s)$ must be lower in degree than $Z_1(s)$ by two.

(3) Impedance function $Z_2(s)$ is obtained from $Z_v(s)$ by the removal of a second Brune section. Since $Z_1(s)$, $Z_u(s)$, and $Z_v(s)$ have the same degree, it is clear that this step must effect a reduction in degree by two in transforming from $Z_v(s)$ to $Z_2(s)$. Therefore, the second Brune section must be identical with the Brune section that would be obtained if $Z_v(s)$ were synthesized by the Brune procedure. Evidently, $Z_v(s)$ must have real-part zeros at $s = \pm j1$. Moreover, $Z_u(s)$ and $Z_o(s)$ also have real-part zeros at these frequencies. Since the transformation from $Z_v(s)$ to $Z_2(s)$ is effected by the conventional Brune process, $Z_2(s)$ is positive-real if $Z_v(s)$ is positive-real.

Requirements for a positive-real remainder impedance.--The usefulness of the pseudo-Brune development of the cascade section is that it allows step-by-step control of the remainder function at each stage in the removal of the cascade section. If each of the three steps outlined in the previous section produces a positive-real remainder impedance, the cascade synthesis procedure succeeds, provided that the cascade section is itself realizable.

Removal of the cascade section from the datum impedance $Z_1(s)$ will leave a positive-real remainder impedance function $Z_2(s)$ lower in degree by two than $Z_1(s)$ if the following three conditions are satisfied:

(1) The pertinent inequality in Table 1 is satisfied. The inequalities shown in this table are necessary and sufficient conditions for c to be non-negative and $Z_u(s)$ to be positive-real. They are sufficient conditions for realization of the cascade section without the use of mutual inductive coupling.

(2) The poles of Z_o or Y_o are contained in Z_u or $1/Z_u$, as appropriate.

(3) The inequality

$$\mathrm{Re}\left\{Z_u(j\omega) - (1 + 1/c)\ Z_o(j\omega)\right\} \geq 0 \tag{89}$$

or

$$\mathrm{Re}\left\{1/Z_u(j\omega) - (1 + c)\ Y_o(j\omega)\right\} \geq 0\ , \tag{90}$$

as appropriate, is satisfied for all real values of ω.

That condition (1) is necessary and sufficient for a non-negative value of c and a positive-real impedance function $Z_u(s)$ has been shown in Chapters II and III. It has also been shown in Chapter IV that condition

(1) is sufficient to allow realization of the cascade section without the use of mutual inductive coupling. Since conditions (2) and (3) are interdependent, they must be considered together in determining whether removal of the cascade section is possible. The principle of duality allows the arbitrary choice of the case where $1/T = 0$ in the discussion to follow, because inversion of $Z_1(s)$ is equivalent to choosing the case where $R = 0$.

The transformation from $Z_1(s)$ to $Z_u(s)$ may be written as

$$Z_u(s) = \frac{(1 + \frac{s^2}{b})P(s) - L(\frac{b-1}{b})sQ(s)}{(1 + bs^2)Q(s) - (\frac{b-1}{L})sP(s)} , \tag{91}$$

where $P(s)$ and $Q(s)$ are the n-th degree numerator and denominator polynomials of

$$Z_1(s) = \frac{P(s)}{Q(s)} . \tag{92}$$

The impedance function $Z_u(s)$ given by (91) is also a quotient of polynomials, say P_b and Q_b. These polynomials have degree $n + 2$. However, the degree of $Z_u(s)$ is only n, since P_b and Q_b have the common factor $(s^2 + 1)$. Thus $Z_u(s)$ may be written as

$$Z_u(s) = \frac{P_b}{Q_b} = \frac{(s^2 + 1)P_u}{(s^2 + 1)Q_u} = \frac{P_u}{Q_u} . \tag{93}$$

Of course, if b is chosen equal to α, P_u and Q_u also have a common factor $(s^2 + 1)$. This possibility is considered in Chapter VI and will not be discussed here. Upon writing

$$P(s) = m_1 + n_1 \tag{94}$$

and

$$Q(s) = m_2 + n_2 \ , \tag{95}$$

where m_1 and m_2 are even and n_1 and n_2 are odd polynomials, the even part of $Z_1(s)$ may be written as

$$M_1 = \mathrm{Ev}\left\{Z_1\right\} = \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2} \quad . \tag{96}$$

It may be shown that the even-part zeros of $Z_1(s)$ are preserved in $Z_u(s)$ unless $b = \alpha$. Moreover, examination of (91) and (93) shows that

$$P(0) = P_u(0) \tag{97}$$

and

$$Q(0) = Q_u(0) \quad . \tag{98}$$

Thus the even part of $Z_u(s)$ may be written as

$$M_u = \mathrm{Ev}\left\{Z_u\right\} = \frac{m_1 m_2 - n_1 n_2}{Q_u(s) \cdot Q_u(-s)} \quad . \tag{99}$$

The transformation from $Z_u(s)$ to $Z_v(s)$ is

$$Z_v(s) = Z_u(s) - (1 + 1/c) Z_o(s) \ . \tag{100}$$

Since the poles of $Z_o(s)$ are the surplus transmission zeros, it must be required that the quadratic polynomial

$$s^2 + \frac{a}{b}\left(\frac{a - b}{a - 1}\right) Rs + \frac{a}{b} \tag{101}$$

be a factor of $Q_u(s)$. Let

$$Q_u(s) = (s^2 + ps + q)\ B(s)\ , \tag{102}$$

where B(s) is a polynomial of degree n - 2. Then parameters a, b, and R must be constrained to satisfy the equations

$$\frac{a}{b} = q \tag{103}$$

and

$$\frac{a}{b}\left(\frac{a - b}{a - 1}\right)R = p\ . \tag{104}$$

From these equations it is evident that

$$a = qb \tag{105}$$

and that $(1 + 1/c)\ Z_o(s)$ may be written as

$$(1+1/c)\ Z_o(s) = \frac{L(\alpha - 1)}{(qb-1)(\alpha-b)} \left[\frac{pqs^2 + (q-1)^2\ s + p}{s^2 + ps + q} \right] . \tag{106}$$

The parameter c has been eliminated from the right member of this equation by the use of (66).

Let

$$(1 + 1/c)\ Z_o(s) = \frac{t(s)}{u(s)}\ , \tag{107}$$

where

$$t(s) = t_o + t_1 s + t_2 s^2 \tag{108}$$

and

$$u(s) = u_o + u_1 s + u_2 s^2 = q + ps + s^2 \ . \qquad (109)$$

Then the even part of $(1 + 1/c)\ Z_o(s)$ is given by

$$M_o = \frac{pqL(\alpha - 1)}{(qb - 1)(\alpha - b)} \left[\frac{(s^2 + 1)^2}{u(s) \cdot u(-s)} \right] \ . \qquad (110)$$

Since

$$B(s) = Q_u(s) \cdot u(s) \ ,$$

it is clear that M_o can also be written as

$$M_o = \frac{pqL(\alpha - 1)}{(qb - 1)(\alpha - b)} \left[\frac{(s^2 + 1)^2\ B(s) \cdot B(-s)}{Q_u(s) \cdot Q_u(-s)} \right] \ . \qquad (111)$$

Moreover, since even-part zeros of a positive-real function which lie on the imaginary axis of the complex frequency plane must be even in order, M_u may be expressed as

$$M_u = \frac{(s^2 + 1)^2\ A(s) \cdot A(-s)}{Q_u(s) \cdot Q_u(-s)} \ , \qquad (112)$$

where $A(s)$ has no right-half-plane roots. Thus the even part of $Z_v(s)$ is

$$M_v = \mathrm{Ev}\left\{Z_v\right\} = \frac{(s^2 + 1)^2\ \phi(b, -s^2)}{Q_u(s) \cdot Q_u(-s)} \ , \qquad (113)$$

where

$$\phi(b, -s^2) = A(s) \cdot A(-s) - \frac{pqL(\alpha - 1)}{(qb - 1)(\alpha - b)}\ B(s) \cdot B(-s) \ . \qquad (114)$$

Now $Z_v(s)$ is regular in the right half-plane if $Z_u(s)$ is positive-real. Moreover, the even part of $Z_v(s)$ will be non-negative on the imaginary axis if $\phi(b, x)$ is non-negative for all real, positive values of x. Thus the inequality

$$\phi(b, x) \geq 0 \text{ for } x \geq 0 \tag{115}$$

is a necessary requirement if $Z_v(s)$ (and thereby $Z_2(s)$, also) is to be positive-real. This requirement is also sufficient if $Z_u(s)$ is positive-real, unless Z_o has j-axis poles. This possibility, which is excluded from consideration here, is discussed in Chapter VII. An alternate form of (115) is

$$\frac{pqL(\alpha - 1)}{(qb - 1)(\alpha - b)} \left| \frac{B(j\omega)}{A(j\omega)} \right|^2 \leq 1 , \tag{116}$$

where ω is real.

At this point the problem of determining whether removal of a cascade section is possible reduces to determining whether a value of b can be found for which (115) is satisfied and

$$qb \geq \alpha \geq b \geq 1 \tag{117}$$

or

$$qb \leq \alpha \leq b \leq 1 \tag{118}$$

is satisfied, accordingly as $L \geq 0$ or $L \leq 0$, respectively. These conditions cannot be satisfied for all datum impedance functions. However, they can be satisfied for certain datum impedance functions. In cases where an appropriate value of b can be determined, the cascade section can be removed to leave a remainder impedance function of degree $n - 2$. Moreover, the conditions mentioned above also permit the realization of the cascade section without the use of mutual inductive coupling.

The difficulty of selecting an appropriate value for b increases with the degree of the datum impedance, since the quantities p and q are multiple-valued functions of b. The number of branches of these functions clearly increases with increasing n. This difficulty appears to be a fundamental limitation imposed by algebraic considerations. It may be observed that the transformation from $Z_1(s)$ to $Z_u(s)$ is actually a zero-shifting procedure. Since the surplus transmission zeros are required to be roots of the equation

$$Z_1(s) = \frac{L(1 + bs^2)}{(b - 1)s} , \tag{119}$$

it is possible to plot the locus of possible transmission zeros by plotting the locus of the roots of (119) as b varies between 1 and α. For $b = 1$, the roots of (119) are simply the poles of $Z_1(s)$. For $b = \alpha$, (119) has one pair of roots at $s = \pm j1$. A typical root-locus diagram for (119) is shown in Fig. 24 for a fourth-degree datum impedance function.

In principle, a direct approach to determining whether an appropriate value of b exists for a given datum impedance function can be formulated. Such an approach involves minimizing $\phi(b, x)$ with respect to x to obtain a function $\phi_0(b)$. This minimization process is complicated by the functional dependence of p and q on b. In general, no explicit

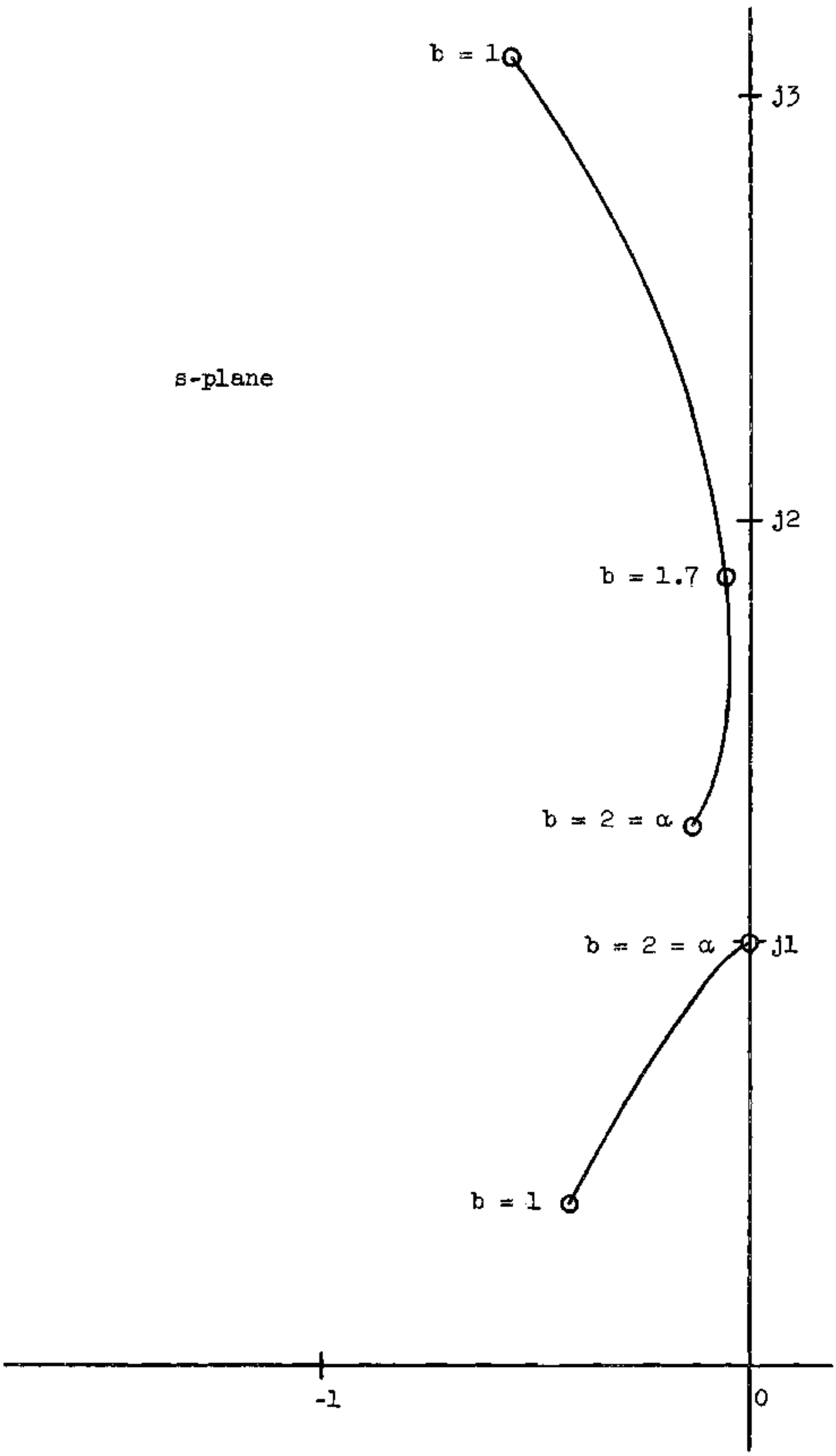

Figure 24. Locus of Surplus Transmission Zeros.

closed algebraic solution for p and q in terms of b is possible. Nevertheless, the function $\phi_o(b)$ exists in an abstract sense, and the roots of the equation

$$\phi_o(b) = 0 \tag{120}$$

define the end-points of the intervals in which b must lie if (115) is satisfied. Similarly, the equation

$$qb = \alpha \tag{121}$$

also defines the end-points of the intervals in which b must lie in order to satisfy (117) or (118). If, for a chosen branch of the functions relating p and q to b, the intervals defined above have any common subintervals lying between b = 1 and b = α, the remainder impedance function is positive-real and the cascade section is realizable without the use of mutual inductive coupling for values of b within such a common subinterval. In general, the existence of one appropriate value for b implies the existence of an appropriate range of values of b. However, under certain circumstances discussed below and in Chapter VII, only discrete values of b are appropriate. Also, as indicated above, there may be no values of b for which the cascade synthesis procedure succeeds.

The discussion above indicates that it is possible, in principle, to determine whether one step of the cascade synthesis procedure can succeed and to determine an appropriate range of values of b in this case. After removal of one cascade section and subsequent reduction of the remainder impedance to a minimum function, it is necessary to test again to determine whether another cascade section can be removed. It should be observed that the synthesis of a second-degree minimum function by the method discussed here reduces to the conventional Bott-Duffin synthesis. Thus, nothing is gained by employing the method unless the degree of the datum impedance is greater than two.

One special case concerning the properties of the function $\phi(b, x)$ which has not been discussed above is the case where p or q is zero. Since $A(s) \cdot A(-s)$ is non-negative for all imaginary values of s, $\phi(b,x)$ will be non-negative if p or q is zero. However, if q is zero, it is evident from Fig. 22 that $Z_1(s)$ must have a pole at $s = 0$. If p is zero, $Z_o(s)$ has poles at $s^2 = -q$. Impedance function $Z_u(s)$ must also have poles at $s^2 = -q$. Thus $s^2 = -q$ must be a root of (119), and $Z_1(s)$ must have real-part zeros at $s = \pm j\sqrt{q}$ (as well as $s = \pm j1$). Conversely, if $Z_1(s)$ has zeros of real part at $s = \pm j\omega_o$, then $A(s)$ is zero there, and p must vanish if (115) is to be satisfied, unless $B(j\omega_o)$ or q should coincidentally vanish. In order that $Z_v(s)$ (and thereby $Z_2(s)$, also) be positive-real, the resulting poles of $Z_o(s)$ on the j-axis of the complex frequency plane and the j-axis poles of $Z_u(s)$ must coincide. Thus q must be chosen equal to ω_o^2 . The case where $p = 0$ is discussed further in Chapter VII.

CHAPTER VI

SPECIAL CASES

Special values of the parameters a and b.--Certain cases involving special values of the parameters a and b have been omitted from consideration in previous chapters to avoid unduly lengthening the exposition of the general case. A discussion of these special cases includes the consideration of the limiting case where b or a approaches unity and of the case where a and b are equal. The analysis of both of the cases mentioned appears to yield some insight into the general properties of the cascade section. Moreover, the cascade section is reduced in complexity in the first case. Thus, the consequences of allowing b or a to approach unity and of setting a and b equal are discussed in this chapter.

The case where b or a approaches unity.--Reference to (65) or (66) shows that allowing b to approach unity implies that c approaches infinity. However, (66) may be written as

$$(b - 1)c = (a - 1)\left(\frac{\alpha - b}{a - \alpha}\right) , \qquad (122)$$

and this equation indicates that the product $(b - 1)c$ may be considered to remain finite and non-zero as $b - 1$ approaches zero. In fact, from (122),

$$\lim_{b \to 1} (b - 1)c = (a - 1)\left(\frac{\alpha - 1}{a - \alpha}\right) . \qquad (123)$$

The quantity $\alpha - 1$ cannot vanish if the datum impedance function $Z_1(s)$ is a minimum function. Similarly, $a - 1$ cannot approach zero simultaneously with $b - 1$ unless L also approaches zero, which violates (62) if $Z_1(s)$ is a minimum function. The possibility that $a - \alpha$ vanishes is considered in the next section. Thus the right member of (123) may be assumed to be finite and non-zero.

Since c approaches infinity as b approaches unity, the elements of the z-matrix of the cascade section approach

$$\lim_{b\to 1} z_{11} = z_L + z_R , \tag{124}$$

$$\lim_{b\to 1} z_{12} = z_L , \tag{125}$$

and

$$\lim_{b\to 1} z_{22} = z_L , \tag{126}$$

where

$$z_L = \lim_{b\to 1} (z_T/c) . \tag{127}$$

The last limit is

$$z_L = \lim_{b\to 1} \frac{L(a - \alpha)}{(\alpha - 1)(a - 1)} \left[\frac{b^2Ts^2 + (b - 1)^2 s + bT}{s^2 + bTs + b} \right] \tag{128}$$

or

$$z_L = \frac{L(a - \alpha)}{(\alpha - 1)(a - 1)} \left[\frac{Ts^2 + T}{s^2 + Ts + 1} \right] . \tag{129}$$

Thus (124)-(126) represent the network of Fig. 25.

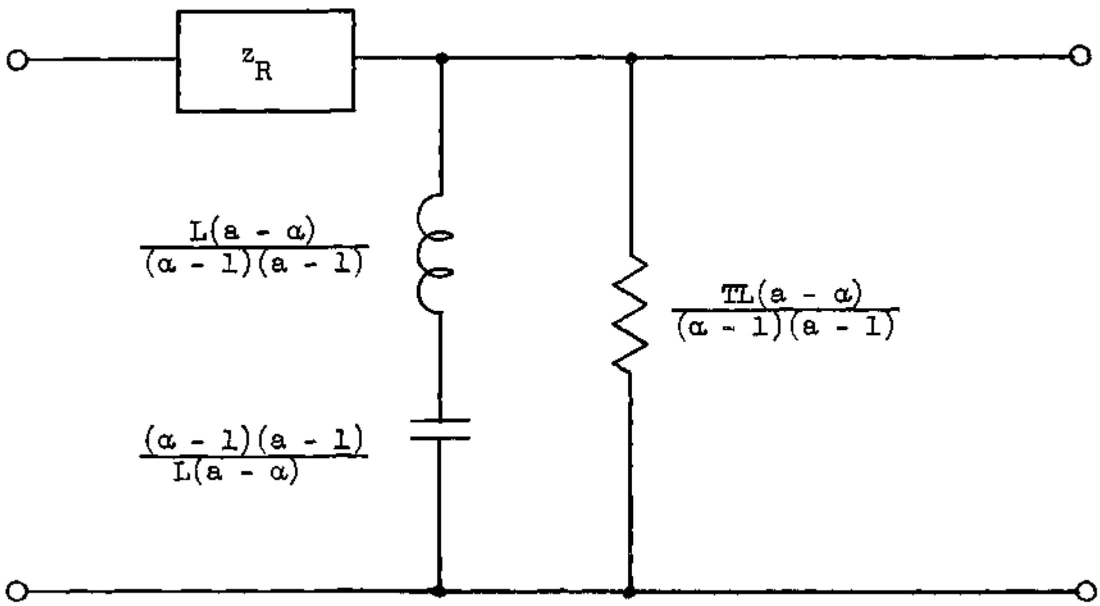

Figure 25. Cascade Section as $b \rightarrow 1$.

The quantities L and $\alpha - 1$ have the same sign; hence, the circuit elements in Fig. 25 are realizable if z_R is positive-real and

$$\frac{a - \alpha}{a - 1} \geq 0 \ . \tag{130}$$

This inequality requires that

$$a \geq \alpha \tag{131}$$

or

$$a \leq \alpha \tag{132}$$

accordingly as $\alpha - 1$ is positive or negative, respectively.

From Fig. 25 the procedure for testing to determine whether allowing b to approach unity permits the removal of a cascade section from the datum impedance $Z_1(s)$ is evident. The series impedance z_R is first removed from $Z_1(s)$, thereby shifting a pair of zeros of impedance to $s = \pm j1$. An impedance function

$$Z_v = Z_1 - z_R \tag{133}$$

results from this step. The reciprocal of Z_v has poles at $s = \pm j1$. These poles, with appropriate residues, are realized by the series LC circuit in Fig. 25, since (30) is satisfied by the limit of z_{11} given by (124). It may be seen from Fig. 25 that there is no advantage to be gained by considering a non-zero value of 1/T when b approaches unity.

The selection of z_R in the special case $b \rightarrow 1$ is similar to the selection of Z_o discussed in Chapter V. The impedance function $z_R(s)$ is completely determined by the parameter L and the locations of the poles of $z_R(s)$. In order that removal of the cascade section produce a remainder impedance of lower degree than $Z_1(s)$ the poles of $z_R(s)$ must be contained in $Z_1(s)$. Thus, a set of possible impedance functions $z_R(s)$ may be determined from the locations of the poles of $Z_1(s)$. The cascade synthesis procedure succeeds for the case where $b \to 1$ if one of these impedance functions $z_R(s)$ allows the simultaneous satisfaction of (130) and the inequality

$$\operatorname{Re}\left\{Z_v(j\omega)\right\} \geq 0 \ , \tag{134}$$

where ω is real.

The special case where a approaches unity results in a cascade section dual in form to the network of Fig. 25. Hence, the details of this case need not be discussed. However, it should be observed that a relation such as

$$R = K(a - 1)^2 \ , \tag{135}$$

where K is a non-negative constant, must be employed in the limiting process to cause R to approach zero as $(a - 1)^2$.

<u>The case where a and b are equal</u>.--If the parameters a and b are arbitrarily assumed to satisfy the equation

$$a = b \ , \tag{136}$$

the relation expressed by (65) becomes

$$(1 + c)\left(\frac{\alpha + 1}{\alpha - 1}\right) = (1 + c)\left(\frac{a + 1}{a - 1}\right) = (1 + c)\left(\frac{b + 1}{b - 1}\right). \tag{137}$$

This equation is consistent for non-negative values of c only if

$$a = \alpha = b. \tag{138}$$

Thus, (136) and (137) imply (138), and (137) cannot be used to determine c. However, it is possible to determine an appropriate value of c by a procedure to be given below. It may be observed here that the arbitrary choice of $a = \alpha$ or $b = \alpha$ also leads to (138).

It was shown in Chapter III that the restriction imposed by (136) causes the surplus transmission zeros of the cascade section to fall at $s = \pm j1$. Hence $z_{12}(s)$ has second-order zeros at $s = \pm j1$ in the special case considered here.

Reference to (85) and Fig. 20 shows that the pseudo-Brune development of the cascade section assumes the form shown by Fig. 26 when $a = \alpha = b$. The resistances r_1 and r_2 in Fig. 26 are given by

$$r_1 = \frac{RL}{\alpha - 1} \tag{139}$$

and

$$r_2 = \frac{L(T - R)}{(1 + c)(\alpha - 1)} \quad . \tag{140}$$

In the notation of Chapter II and Fig. 2 the following identifications may also be written:

$$r_1 = r_{11} - r_{12} \ , \tag{141}$$

$$r_2 = r_{12} \ , \tag{142}$$

and

$$r_1/c = r_{22} - r_{12} \ . \tag{143}$$

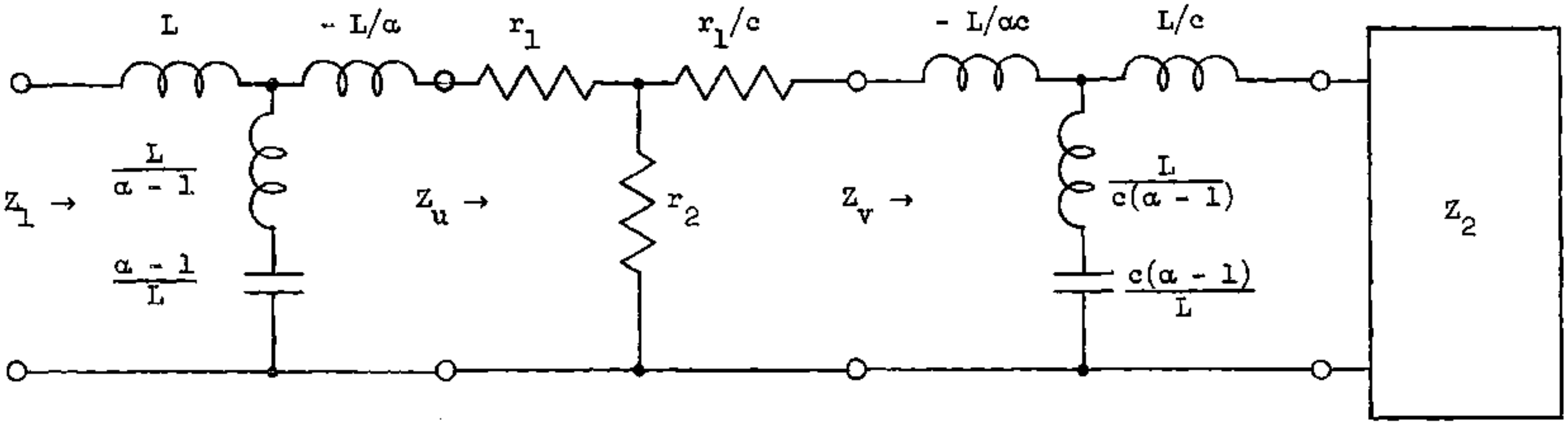

Figure 26. Pseudo-Brune Development for $a = b = \alpha$.

The real constants r_{11}, r_{12}, and r_{22} are the open-circuit impedance parameters of the Tee network internal to the circuit of Fig. 26. From (81) if $L > 0$ or from (A34) if $L < 0$ it may be seen that the cascade section has an unbalanced transformerless realization if

$$T \geq R \geq 0 \tag{144}$$

and

$$c \geq 0 \ . \tag{145}$$

The last inequality must be appended because it is no longer insured by the choice of a and b. The inequality in (144) is equivalent to requiring r_1 and r_2 to be non-negative.

Examination of Fig. 26 makes evident a procedure which may be used to determine the parameters c, R, and T and to determine whether a cascade section with $a = \alpha = b$ can be removed from the datum impedance $Z_1(s)$. The remainder impedance $Z_2(s)$ is obtained from $Z_1(s)$ by a series of transformations beginning with the transformation from $Z_1(s)$ to $Z_u(s)$. The impedance function $Z_u(s)$ is obtained from $Z_1(s)$ by the removal of a conventional Brune section. Thus, if $Z_1(s)$ is a minimum function of degree n, $Z_u(s)$ is a positive-real function of degree n - 2. Fig. 26 indicates that

$$Z_v(j1) = -\ jL/\alpha c \ . \tag{146}$$

Hence, the transformation from $Z_u(s)$ to $Z_v(s)$ by the Tee network of resistors r_1, r_2, and r_1/c must produce a minimum-real function $Z_v(s)$. Such a reduction to minimum-reality has been discussed in Chapter II.

The addition of some pertinent geometrical details to the Z-plane contours in Fig. 3 of that chapter results in Fig. 27. In this figure, only the portion of the circle E lying in the upper half of the complex plane need be shown. Obviously, $Z(s)$ of Fig. 3 must be identified with $Z_u(s)$ of Fig. 27.

The location and radius of the circle E, which is the locus of Z_u for which Z_v is imaginary, fix two of the three constants needed to determine the open-circuit impedance matrix r having elements r_{11}, r_{12}, and r_{22}. The impedance function $Z_v(s)$ will be positive-real if K, which is the locus of $Z_u(j\omega)$ for ω real, lies inside of or tangent to E. Since points where K and E are tangent correspond to values of $j\omega$ for which $Z_v(j\omega)$ is imaginary, (146) implies that E must be chosen to be tangent to K at $s = j1$. Let θ be the angle of inclination of the tangent to K at $s = j1$. Then θ is determined by

$$\tan\theta = \left.\frac{\dfrac{\partial X_u}{\partial\omega}}{\dfrac{\partial R_u}{\partial\omega}}\right|_{\omega = 1} = \frac{\operatorname{Re}\left\{Z_u'(j1)\right\}}{-\operatorname{Im}\left\{Z_u'(j1)\right\}} , \tag{147}$$

where R_u and X_u are the real and imaginary parts of Z_u, respectively. The center of the circle E lies at an abscissa of

$$F = R_o + X_o \tan\theta , \tag{148}$$

where

$$R_o + jX_o = Z_u(j1) . \tag{149}$$

The intersections of the circle E with the real axis occur at abscissae

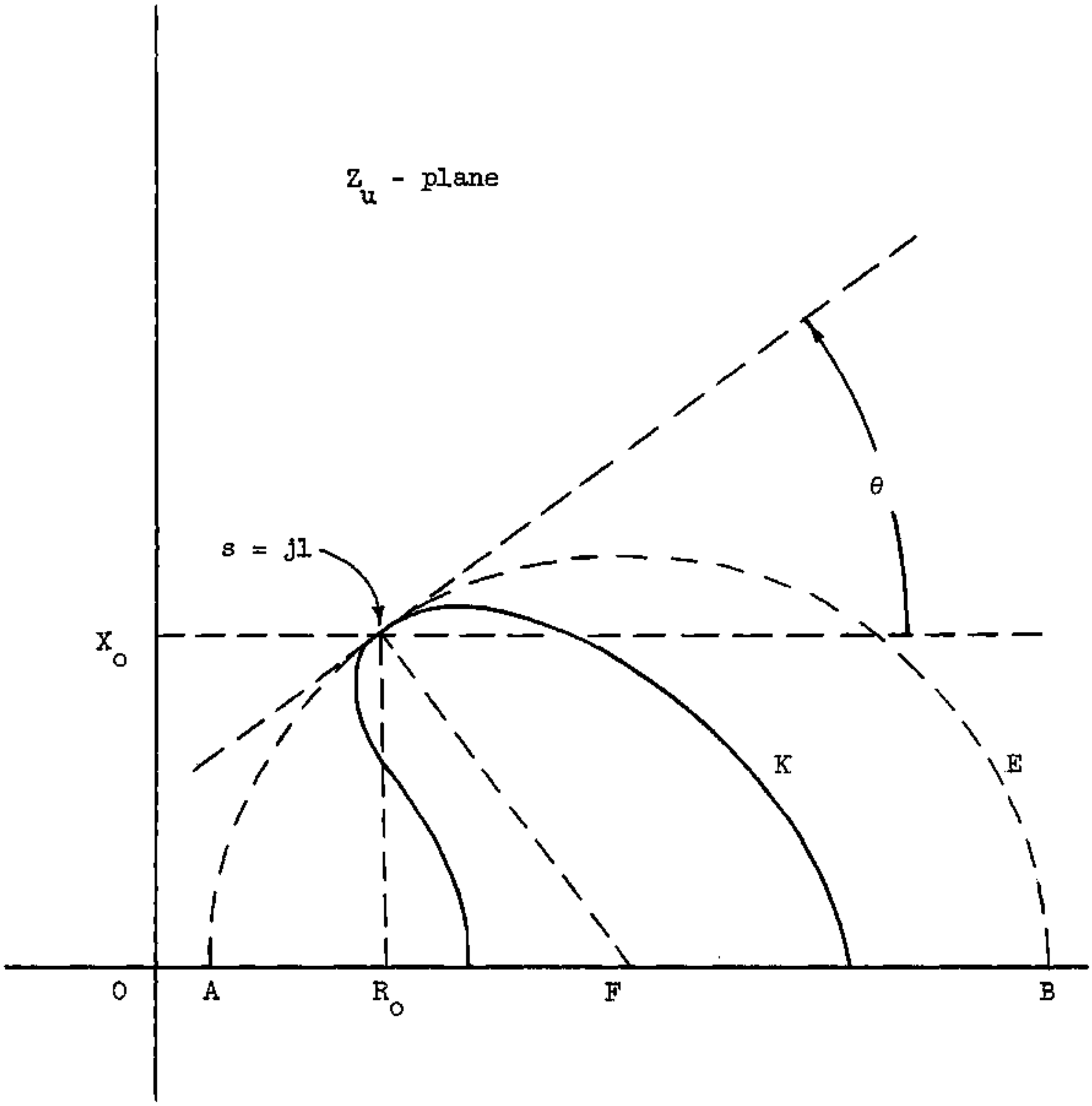

Figure 27. Locus of $Z_u(j\omega)$.

of A and B, where

$$A = F - |X_o/\cos\theta| \tag{150}$$

and

$$B = F + |X_o/\cos\theta| \; . \tag{151}$$

These intersections are related to the open-circuit impedance parameters by the equations

$$A = d(r)/r_{22} \tag{152}$$

and

$$B = r_{11} \; , \tag{153}$$

where

$$d(r) = r_{11}r_{22} - r_{12}^2 \; . \tag{154}$$

The relation between $Z_v(s)$ and $Z_u(s)$ may be written as

$$Z_v(s) = \frac{r_{22}Z_u(s) - d(r)}{r_{11} - Z_u(s)} \quad . \tag{155}$$

This equation may be combined with (146) to yield

$$r_{22}\left[\frac{Z_u(j1) - d(r)/r_{22}}{r_{11} - Z_u(j1)}\right] = - j\frac{L}{\alpha c} \quad . \tag{156}$$

The substitution of (149), (152), and (153) into (156) results in

$$r_{22}\left[\frac{R_o + jX_o - A}{B - R_o - jX_o}\right] = - j\frac{L}{\alpha c} \tag{157}$$

or

$$cr_{22} = \frac{L}{\alpha} \left[\frac{X_o + j(B - R_o)}{A - R_o - jX_o} \right] \quad . \tag{158}$$

This equation reduces to

$$cr_{22} = D, \tag{159}$$

where

$$D = \frac{L}{\alpha} \left(\frac{X_o}{A - R_o}\right) = \frac{L}{\alpha} \left(\frac{R_o - B}{X_o}\right) \quad , \tag{160}$$

since the expression in the brackets in (158) is real. Because cr_{22} must be non-negative, an obvious necessary condition for the success of the cascade synthesis procedure with $a = \alpha = b$ is that L and X_o have opposite signs.

The parameters r_{11}, r_{12}, r_{22}, and c may be determined from the equations developed above. The solution of (141), (143), (152), (153), and (159) results in

$$r_{11} = B \, , \tag{161}$$

$$r_{22} = D/c \, , \tag{162}$$

$$r_{12} = \sqrt{(B - A)D/c} \quad , \tag{163}$$

and

$$c = 1 + M \pm \sqrt{2M + M^2} \quad , \tag{164}$$

where

$$M = \frac{(B - D)^2}{2D(B - A)} \quad . \tag{165}$$

The parameter c given by (164) is a solution of the equation

$$c^2 - 2(1 + M)\, c + 1 = 0 \; . \tag{166}$$

Since M is non-negative if D is positive, this solution has two real, non-negative solutions for c. In general, one solution of (166) is greater than unity and the other solution is less than unity. One root of (166) is an extraneous value of c; it is necessary to select the root for c so that

$$c \leq 1 \text{ if } B \geq D \tag{167}$$

and

$$c \geq 1 \text{ if } B \leq D \; . \tag{168}$$

The Tee network of resistors internal to the circuit of Fig. 26 has positive elements and the cascade section has a transformerless realization if

$$D \geq (B - A)\; c \geq 0 \; , \tag{169}$$

as may be verified from (161)-(163).

The impedance function $Z_v(s)$ is positive-real if the locus K lies within or tangent to the circle E in the Z_u-plane. The positive-reality of $Z_v(s)$ is a necessary condition for a positive-real remainder impedance $Z_2(s)$. This requirement obviously cannot always be satisfied. In a given numerical case, it might be necessary to test $Z_v(s)$ for positive-reality directly from (155).

If $Z_v(s)$ is now assumed to be positive-real, a sufficient condition for positive-reality of $Z_2(s)$ may be stated by use of the results in Chapter II. Let β be defined by

$$\beta = \frac{Z_v'(j1) - jZ_v(j1)}{Z_v'(j1) + jZ_v(j1)} \quad . \tag{170}$$

Then the discussion of Chapter II indicates that $Z_2(s)$ will be positive-real and of no greater degree than $Z_v(s)$ if

$$1 \leq 1/\alpha \leq \beta \tag{171}$$

or

$$1 \geq 1/\alpha \geq \beta \tag{172}$$

accordingly as L is negative or positive, respectively. Since knowledge of $Z_u'(j1)$ is necessary in determining A, B, etc., the equation

$$Z_v'(s) = \frac{r_{12}^2 \; Z_u'(s)}{(Z_u - r_{11})^2} \tag{173}$$

may be conveniently employed in calculating β.

The preceding discussion indicates that the transformation from $Z_1(s)$ to $Z_u(s)$ is the only degree-reducing step in the transformation from $Z_1(s)$ to $Z_2(s)$, unless by coincidence $\alpha\beta = 1$. The transformation from $Z_u(s)$ to $Z_v(s)$ serves only to shift an even-part zero to $s = j1$. Clearly, this process will not always succeed. However, the conditions outlined above may be applied at a given stage in the synthesis of a driving-point impedance to determine whether a cascade section with $a = \alpha = b$ can be removed from the datum impedance. These conditions may be summarized as follows:

(1) Inequality (169) must be satisfied.

(2) Inequality (171) or (172) must be satisfied.

(3) Impedance function $Z_v(s)$ must be positive-real.

CHAPTER VII

THE LOSSLESS CASE

Conditions for removal of the cascade section.--It was shown in Chapter V that the surplus transmission zeros of the cascade section must lie at $s = \pm j\omega_o$ if the datum impedance function $Z_1(s)$ has real-part zeros at these frequencies. It was also shown in Chapter III that a lossless cascade section fulfills this requirement. The conditions for removal of the cascade section in this case are somewhat simpler than in the general case. Moreover, examination of the lossless case discussed here may help to give a qualitative insight into the conditions necessary for removal of a cascade section in the lossy case.

It has been indicated before that the cascade section represented by the open-circuit impedance matrix z becomes a lossless network in the limiting case where R and 1/T approach zero. The impedance functions z_T and z_R reduce to reactance functions given by

$$z_T = \frac{L(bs^2 + 1)}{(b - 1)s} \tag{174}$$

and

$$z_R = \frac{L(a - 1)s}{s^2 + a} \quad . \tag{175}$$

A bridged-Tee network realizing the lossless cascade section without transformers has already been given in Chapter IV by Fig. 17 or 18, as appropriate. One of the networks of Figs. 17 and 18 will always be realizable if $a - 1$ and $b - 1$ have the same sign as $\alpha - 1$ and the constant c is non-negative. A non-negative value of c results from (66) if (67) or (68) is satisfied, i.e., if a and b lie on opposite sides of α and on the same side of unity on the real line.

Since both of the conditions $R = 0$ and $1/T = 0$ are imposed in the lossless case, the two cases discussed in Chapter V, namely $R = 0$ and $1/T = 0$, lose their identity. However, it is still possible to represent the cascade section by a pseudo-Brune development. In fact, two such developments are possible and may be obtained directly from (174) and (175) or by the application of limiting techniques to (85), (86), and Figs. 20 and 21. Either procedure leads to networks having the form of Fig. 22 or Fig. 23, with

$$Z_o(s) = \frac{L(a - b)s}{b^2(s^2 + a/b)} \tag{176}$$

and

$$Y_o(s) = \frac{a(b - a)s}{bL(s^2 + a/b)} \quad . \tag{177}$$

The surplus transmission zeros of the cascade section lie at $s = \pm\sqrt{a/b}$, the locations of the poles of $Z_o(s)$ or $Y_o(s)$. Thus, if $Z_1(s)$ has real-part zeros at $s = \pm j\omega_o$ (in addition to those at $s = \pm j1$),[23] the parameters a and b must be constrained by the relation

$$a/b = \omega_o^2 \quad . \tag{180}$$

[23]It is assumed that $\omega_o \neq 1$.

In order that $z_{11} = Z_1$ at the surplus transmission zeros, the equation

$$\frac{L(bs^2 + 1)}{(b - 1)s} = Z_1(j\omega_o) \tag{181}$$

or

$$\frac{L(a - 1)s}{s^2 + a} = Z_1(j\omega_o) \tag{182}$$

must be satisfied. Since (180)-(182) are not independent equations, (182) may be discarded. Thus, (180) and (181) fix the following unique values for a and b:

$$b = \frac{\frac{X}{L} - \frac{1}{\omega_o^2}}{\frac{X}{L} - 1} \tag{183}$$

and

$$a = \omega_o^2 b \ , \tag{184}$$

where

$$Z_1(j\omega_o) = j\omega_o X \quad . \tag{185}$$

In deriving the conditions for successful removal of a cascade section in the lossless case it is convenient to make use of the equivalent circuits of Fig. 22 or Fig. 23 in somewhat the same manner as that employed in Chapter V. Although these two circuits are equivalent in the lossless case, it is expedient for analytical purposes to select the circuit for which $Z_u(s)$ is positive-real. The circuit of Fig. 22

should be selected, therefore, if

$$a \geq \alpha \geq b \geq 1 \tag{186}$$

or

$$a \leq \alpha \leq b \leq 1 \ , \tag{187}$$

while the circuit of Fig. 23 should be selected if

$$b \geq \alpha \geq a \geq 1 \tag{188}$$

or

$$b \leq \alpha \leq a \leq 1 \ . \tag{189}$$

If, for a given datum impedance function, none of the inequalities (186)-(189) is satisfied, the transformerless cascade synthesis procedure does not succeed when the even-part zeros of $Z_1(s)$ at $s = \pm j\omega_o$ are identified with the surplus transmission zeros.

The inequalities expressed by (186)-(189) are sufficient conditions for realizability of the cascade section. The relation between Z_u and Z_v in Fig. 22 or Fig. 23 must now be examined to determine the conditions under which $Z_2(s)$ is positive-real. As in the lossy case discussed in Chapter V, $Z_2(s)$ will be positive-real if $Z_v(s)$ is positive-real. It will be assumed in the following discussion that (186) or (187) is satisfied, and attention will be directed to the pseudo-Brune development of Fig. 22. The case where Fig. 23 is appropriate may be regarded as the dual of the case discussed here.

The conditions expressed by (186)-(189) involve the value of $Z_1(s)$ at $s = j\omega_o$, but they do not involve the derivative of $Z_1(s)$ at that frequency. However, it is necessary to consider the derivative of $Z_1(s)$ at

$s = j\omega_o$ in evaluating the residues of Z_u at the surplus transmission zeros. The relation between Z_u and Z_1 may be written as

$$Z_u = -W_{22} + \frac{W_{12}^2}{W_{11} - Z_1} \quad , \tag{190}$$

where

$$W_{11} = \frac{L}{b-1}(bs + 1/s) \quad , \tag{191}$$

$$W_{12} = \frac{L}{b-1}(s + 1/s) \quad , \tag{192}$$

and

$$W_{22} = \frac{L}{b-1}(s/b + 1/s) \quad . \tag{193}$$

Thus the residue of $Z_u(s)$ at $s = j\omega_o$ is

$$K = \frac{W_{12}^2(j\omega_o)}{W_{11}'(j\omega_o) - Z_1'(j\omega_o)} \quad , \tag{194}$$

and an expansion of $Z_u(s)$ in partial fractions would contain the term

$$\frac{2Ks}{s^2 + \omega_o^2} \quad . \tag{195}$$

Let γ be the Brune characteristic of $Z_1(s)$ at $s = j\omega_o$, i.e., let

$$\gamma = \frac{Z_1'(j\omega_o) + X}{Z_1'(j\omega_o) - X} \quad . \tag{196}$$

Then

$$Z_1'(j\omega_o) = X(\frac{\gamma + 1}{\gamma - 1}) \ , \tag{197}$$

and calculation of 2K from (194) yields

$$2K = \frac{\dfrac{-\ 2L(\omega_o^2 - 1)^2}{b - 1}}{(a - 1)\ [\ (\frac{a + 1}{a - 1}) - (\frac{\gamma + 1}{\gamma - 1})\ ]} \ . \tag{198}$$

The impedance function $Z_v(s)$ is related to $Z_u(s)$ by

$$Z_v(s) = Z_u(s) - (1 + 1/c)Z_o(s) \ . \tag{199}$$

Hence, $Z_v(s)$ is regular in the right half s-plane if $Z_u(s)$ is positive-real. But $Z_u(s)$ is positive-real if (186) or (187) is satisfied. Moreover Re $\left\{Z_v(j\omega)\right\}$ is non-negative for real values of ω if $Z_u(s)$ is positive-real, since $Z_o(s)$ represents the impedance of a lossless network. Therefore, $Z_v(s)$ will be positive-real if the residue of $Z_u(s)$ at $s = j\omega_o$ is not less than the residue of $(1 + 1\ c)Z_o(s)$ at the same pole, i.e., if

$$2K \geq \frac{(1 + c)(a - b)L}{cb^2} \ . \tag{200}$$

The right member of (200) is non-negative by virtue of (186) or (187). After some manipulation, (200) may be reduced to

$$\frac{2}{(b - 1)\ [(\frac{\gamma + 1}{\gamma - 1}) - (\frac{a + 1}{a - 1})]} \geq \frac{\alpha - 1}{\alpha - b} \ , \tag{201}$$

where it is assumed that (186) or (187) is valid. The steps leading from (200) to (201) do not change the algebraic sign of either member of the inequality. If (186) or (187) is satisfied, $Z_u(s)$ is positive-real, and 2K must be non-negative. Since the right member of (200) is also non-negative, (201) may also be written as

$$\left(\frac{b - 1}{2}\right) \left[\left(\frac{\gamma + 1}{\gamma - 1}\right) - \left(\frac{a + 1}{a - 1}\right) \right] \leq \frac{\alpha - b}{\alpha - 1} \quad . \tag{202}$$

The inequality expressed by (202) is a necessary and sufficient condition for obtaining a positive-real remainder impedance function when (186) or (187) is satisfied. When (188) or (189) is satisfied, the inversion of $Z_1(s)$ results in a function such that (186) or (187) is satisfied. If none of the conditions given by (186)-(189) is satisfied, or if (202) or its analogue in the dual case is not fulfilled, the cascade synthesis procedure does not succeed.

It may be observed from the preceding discussion that the conditions which determine whether a lossless cascade section may be removed from a given datum impedance are inequalities among various constants, rather than functions of frequency. Thus, the procedure of testing a datum impedance to determine whether a cascade section of the form discussed in this thesis may be removed is somewhat simpler when the datum impedance function has two real-part zeros on the $j\omega$-axis of the complex frequency plane than in the general case when the procedure of Chapter V must be employed. If desired, it is always possible to produce the doubly minimum-real condition required for a lossless cascade section by the removal in series or parallel of an appropriate second-degree minimum function.

This reduction process can be effected without increasing the degree of the driving-point impedance function by proper selection of the poles of the second-degree minimum function.

Equivalence to cascade Brune sections.--It is the purpose of this section to discuss the equivalence of the cascade section in the lossless case to a two-terminal-pair network consisting of two Brune sections in cascade. Although such an equivalence offers no advantage in the practical realization of driving-point impedances, the equivalent circuit containing Brune sections is an aid to understanding the conditions necessary for success of the cascade synthesis procedure of this thesis.

The lossless cascade section described by the open-circuit impedance matrix having elements given by (32)-(34) and (174)-(175) is equivalent as a two-terminal-pair network to the circuit of Fig. 28 if the parameters of Fig. 28 are related to the parameters of the matrix z by the equations

$$N = \frac{b(a - 1)}{ca(b - 1)} \quad , \tag{203}$$

$$\omega_o^2 = a/b \ , \tag{204}$$

and

$$\alpha = \frac{ca(b - 1) + b(a - 1)}{c(b - 1) + (a - 1)} \ . \tag{205}$$

The parameter ω_o is again the (angular) frequency of the surplus transmission zero. Since α given by (205) is a solution of (64), it must be equated with the Brune characteristic of the datum impedance at $s = j1$ if the cascade section is to be employed for synthesis purposes.

Similarly, L in Fig. 28 is the same parameter as the constant L occurring in earlier discussions.

The parameters a, b, and c of (32)-(34) and (174)-(175) are also related uniquely to the parameters N, ω_o, and α of Fig. 28 by the equations

$$a = \alpha\left(\frac{1 + \omega_o^2 N}{1 + N}\right) , \tag{206}$$

$$b = \frac{\alpha}{\omega_o^2}\left(\frac{1 + \omega_o^2 N}{1 + N}\right) , \tag{207}$$

and

$$c = \frac{1}{\omega_o^2 N}\left(\frac{a - 1}{b - 1}\right) . \tag{208}$$

Examination shows that the two cascade Brune sections in Fig. 28 are realizable if

$$\frac{L}{\alpha - 1} \geq 0 \tag{209}$$

and

$$N \geq 0 . \tag{210}$$

The parameter N always satisfies (210) when a and b are non-negative and the basic inequalities given by (37)-(39) are valid. Computation of $\alpha - 1$ from (205) yields

$$\alpha - 1 = \frac{(a - 1)(1 + 1/c)}{1 + \dfrac{a - 1}{c(b - 1)}} , \tag{211}$$

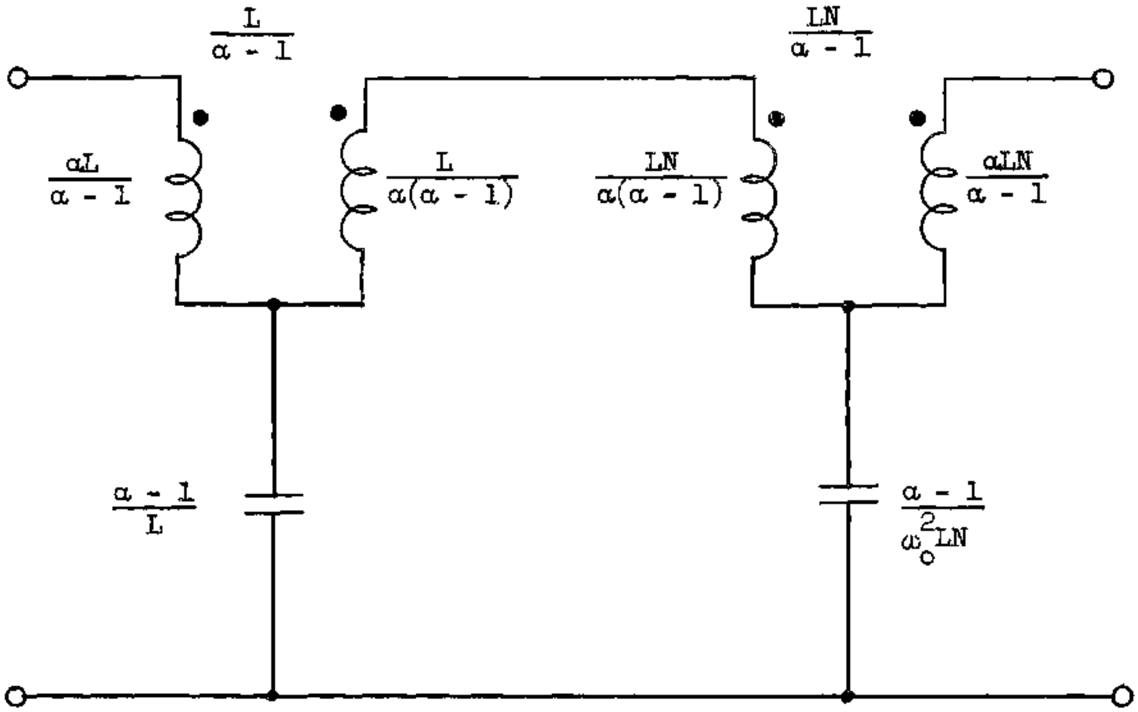

Figure 28. Equivalent Circuit for Lossless Cascade Section.

which indicates that (37)-(39) also imply (209). Thus, the equivalent circuit of Fig. 28 is always realizable (using unity-coupled coils) when the cascade section fulfills the conditions outlined in Chapter III. However, (209) and (210) are not sufficient conditions for the realization of the cascade section without the use of mutual inductive coupling.

The networks of Figs. 17 and 18 demonstrate that the cascade section is realizable without transformers when a, b, and c are non-negative constants such that a - 1 and b - 1 have the same sign as α - 1. It is evident from (208) that c is non-negative if (210) is satisfied and a - 1 and b - 1 have the same sign. Hence, in view of (211), the cascade section equivalent to the circuit of Fig. 28 is realizable without transformers if (210) is satisfied and

$$\frac{a - 1}{b - 1} = \frac{(\alpha - 1) + N(\alpha\omega_o^2 - 1)}{(\alpha/\omega_o^2 - 1) + N(\alpha - 1)} \geq 0 \ . \tag{212}$$

Since this inequality can also be written as

$$\omega_o^2 \left(\frac{a - 1}{a - \omega_o^2}\right) \geq 0 \tag{213}$$

it is evident that a - 1 and b - 1 have the same sign if and only if a does not lie within 1 and ω_o^2 on the real line. This condition may be related to the sign of the constant X/L by noting that (27), (32), and (33) imply that the equation

$$Z_1 = z_R \tag{214}$$

must be satisfied at $s = \pm j1$ and at $s = \pm j\omega_o$. But $\mathrm{Im}\left\{z_R\right\}$ does not change sign between $\omega = 1$ and $\omega = \omega_o$ if (213) is valid. Hence $\mathrm{Im}\left\{Z_1(j1)\right\}$ and $\mathrm{Im}\left\{Z_1(j\omega_o)\right\}$ have the same sign if and only if (213) is satisfied. Therefore, the inequality

$$X/L \geq 0 \tag{215}$$

is equivalent to (213) or (212).

The synthesis by Brune's procedure of a doubly minimum-real datum impedance function will result in a network containing only certain elements equal to those in Fig. 28, except in rather special circumstances. Let Fig. 29 represent the network resulting from the completion of two cycles in the Brune synthesis of a doubly minimum-real datum impedance function $Z_1(s)$. When the datum impedance function possesses real-part zeros at $s = \pm j1$ and $s = \pm j\omega_o$, where ω_o is real, two Brune developments of the datum impedance are possible. It is assumed in Fig. 29 that a Brune section associated with the real-part zeros at $s = \pm j1$ is removed from the datum impedance first. This selection will be seen to correspond to placing the surplus transmission zeros of the cascade section at $s = \pm j\omega_o$, rather than at $s = \pm j1$.

From Fig. 29 it is evident that (210) will not always be satisfied. In fact, comparison of Figs. 28 and 29 shows that the parameter N must be fixed by

$$M = - LN/\alpha \tag{216}$$

or

$$N = - \alpha M/L \ . \tag{217}$$

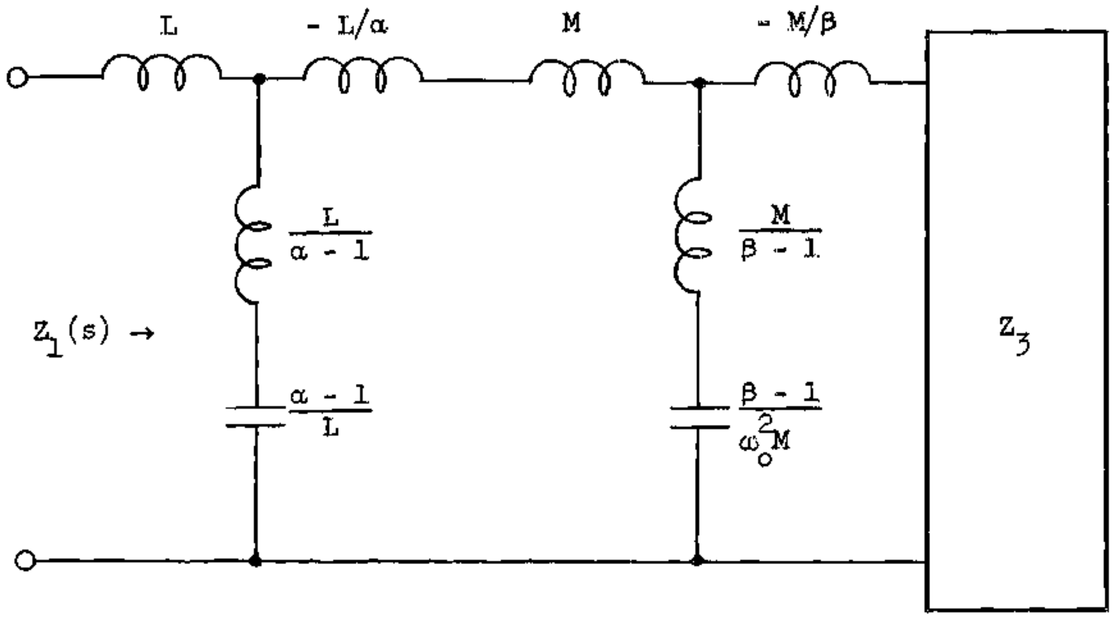

Figure 29. Brune Synthesis of $Z_1(s)$.

Since α is non-negative, N satisfies (210) if

$$- M/L \geq 0 \ , \tag{218}$$

i.e., if M and L have opposite signs.

Let it now be assumed that (210) and (212) are satisfied, so that the cascade section is realizable without transformers. Then further comparison of Figs. 28 and 29 and the discussion in Chapter II allows a necessary and sufficient condition for positive-reality of the remainder impedance $Z_2(s)$ to be stated. This condition is that $Z_2(s)$, the remainder impedance after removal of a lossless cascade section with parameters given by (206)-(208) from the doubly minimum-real datum impedance $Z_1(s)$, will be positive-real if

$$1 \leq 1/\alpha \leq \beta \tag{219}$$

or

$$1 \geq 1/\alpha \geq \beta \ . \tag{220}$$

The condition expressed by (219) or (220) is actually a relation between the characteristics, i.e., the turns ratios, of the Brune sections resulting from two cycles in the conventional Brune synthesis of $Z_1(s)$. If (219) or (220) is to be satisfied it is necessary that one Brune section have a step-up action and the other section a step-down action as s approaches infinity. This necessary condition is equivalent to (210).

The discussion above indicates that the satisfaction of (215) and (219) or (220) allows one step in the cascade synthesis of a doubly minimum-real function to be effected. Hence, these conditions may be considered as alternatives to the conditions stated in the first section of this chapter.

In the preceding discussion it has been assumed that the real-part zeros of $Z_1(s)$ at $s = \pm j\omega_o$ have been identified with the surplus transmission zeros of the cascade section. However, a frequency-scaling transformation will allow the roles of the real-part zeros at $s = \pm j1$ and $s = \pm j\omega_o$ to be interchanged. When this procedure leads to the successful removal of a cascade section, it may be employed to obtain a different network realizing the datum impedance function. It might also be employed when the first identification of the surplus transmission zero does not permit the cascade synthesis procedure to succeed. However, it is evident that the failure to satisfy (215) cannot be relieved by this device. Moreover, when the datum impedance is a doubly minimum-real function of fourth degree it may be shown that the conditions allowing the successful removal of a lossless cascade section are invariant to an interchange of the roles of the real-part zeros at $s = \pm j1$ and $s = \pm j\omega_o$.

Comparison with the lossy case.--The results of this chapter are applicable to the case of a lossless cascade section and a doubly minimum-real datum impedance function. However, they may be used to provide a qualitative insight into the conditions for cascade synthesis of certain datum impedance functions having only one pair of real-part zeros on the $j\omega$-axis of the complex frequency plane. In particular, if the datum impedance $Z_1(s)$ has real-part zeros at $s = \pm j1$ and a relative minimum of real part at $s = j\omega_o$, the synthesis of $Z_1(s)$ by the Brune procedure may lead to a network consisting of a cascade of two Brune sections separated by a series or shunt resistor and terminated by a remainder impedance. If the real part of $Z_1(j\omega_o)$ is sufficiently small in comparison with the general impedance level of $Z_1(s)$, it is reasonable to expect the transmission zeros of the second Brune section to lie relatively close to $s = \pm j\omega_o$. It is also to be expected that the series resistance or shunt conductance between the two Brune sections will be relatively small. If these conditions obtain, it is likely in many cases that a test based on disregarding the series resistance or shunt conductance will determine whether or not one step of the cascade synthesis procedure should be expected to succeed. Moreover, it is reasonable in such a case to suppose that the range of values of a and b for which the exact criteria of Chapter V are fulfilled will allow the approximate satisfaction of (204). These conjectures have been verified to a certain extent by numerical examples.

CHAPTER VIII

EVALUATION OF THE METHOD

Discussion.--The primary theoretical considerations pertinent to the problem of transformerless cascade synthesis using the cascade section proposed in this thesis have been discussed in preceding chapters. In particular, Chapters I through V form a treatment of this problem in the general case, while certain special cases have been examined in Chapters VI and VII. However, there remain some general remarks concerning the synthesis procedure described here. These remarks primarily deal with the applicability of the procedure.

It should be observed first that the method of driving-point impedance synthesis proposed here is not general, i.e., it is not universally applicable to rational positive-real impedance functions. In fact, the datum impedance functions to which the method is applicable are required *a priori* to be minimum functions. However, this requirement is not unduly restrictive, since the conventional preamble to the Brune synthesis procedure may be used to reduce a given driving-point impedance function to a minimum function, without the use of transformers. In addition, the methods of the first section in Chapter II may be utilized to perform the last step in such a reduction. Presumably, this reduction might be executed in a manner calculated to influence favorably the realization procedure at later stages. This possibility has not been exploited in conjunction with the synthesis procedure presented here because of what appear to be formidable mathematical difficulties. However, it is obvious that the well-known predistortion technique reviewed briefly in Chapter II, used with the present procedure, affords the possibility of avoiding the necessity for perfect lossless elements, except in the case where the datum impedance already is minimum-real. In the latter case, it is clear that there can be no power dissipation and that all lossy elements in any network realizing the datum impedance must be effectively decoupled from the input terminals at the (steady-state) frequency of the real-part zero.

A set of more subtle restrictions on the datum impedance arises as a result of the requirement that the cascade section to be removed at a given stage must be realizable without the use of transformers or mutual inductive coupling and the requirement that the remainder impedance function be positive-real and of lower degree than the datum impedance function. These restrictions have been set forth in Chapters IV and V for the general case. However, the discussion in Chapter V of the requirement that the remainder impedance function be positive-real does not include a treatment of the computational problems associated with this requirement. Aside from these difficulties, it should be observed that the tests outlined in Chapter V are applicable at each minimum-function stage in the synthesis of a driving-point impedance.

The special cases discussed in Chapters VI and VII occur in essentially different ways. The lossless cascade section discussed in Chapter VII is applicable when the datum impedance function has two pairs of real-part zeros on the $j\omega$-axis of the complex frequency plane. It must be employed in this case and only in this case. In contrast, one step of cascade synthesis using the special techniques of Chapter VI may always be attempted, except when the technique of Chapter VII applies. When such an attempt succeeds in the special case where the parameter a or b approaches unity, a saving in the number of elements required is effected, by comparison with the general case. However, the special case where the parameters a, b, and α are equal requires the use of a large number of elements. This case has been discussed primarily because of its theoretical interest.

The conditions that govern the possibility of success in the special synthesis procedures of Chapter VI are somewhat simpler than their counter-parts in the general case. Thus, it may be advantageous in a given numerical example to attempt to remove a cascade section of the special form before resorting to the more tedious general conditions of Chapter V. Similarly, it is also possible to force the datum impedance to be doubly minimum-real, and thus to permit the use of the simpler criteria of Chapter VII to determine whether removal of a cascade section may be effected. As discussed in Chapter VII, the criteria governing the possibility of removing a cascade section in the lossless case also may provide qualitative guidance in applying the exact conditions of Chapter V in the general case.

Precise mathematical statements defining a function class and network class for which the cascade synthesis procedure discussed here is universally applicable would serve to complete the theory associated with the particular form of the cascade section employed. However, although such definitions of the function class and network class are absent, the cascade synthesis procedure of this thesis demonstrates that a synthesis in cascade sections without mutual inductance can be employed in certain cases, including instances where the datum impedance is represented by a singly or doubly minimum-real function. Moreover, when the cascade synthesis procedure succeeds at a given stage, the degree of the driving-point impedance function is reduced without the creation of more than one remainder impedance function. Thus, it is conceivable that the successful application of the cascade synthesis procedure at an early stage in the transformerless synthesis of a driving-point impedance may reduce significantly the number of network elements required.

Conclusions.--A new cascade synthesis procedure appropriate to certain rational positive-real driving-point impedance functions has been developed. The procedure makes use of an elementary cascade section described by a compact open-circuit impedance matrix having elements that are suitably chosen fourth-degree rational functions of the complex frequency variable. Under certain conditions developed above, the cascade section, which is not lossless in general, is realizable in an unbalanced two-terminal-pair network utilizing neither ideal transformers nor mutual inductive coupling. The locations of the zeros of transmission through the cascade section play an important role in the synthesis procedure; however, they are related to the driving-point impedance to be synthesized and may not be chosen arbitrarily.

Equivalent circuits for the cascade section have been derived to make evident certain properties of the cascade section.

The class of impedance functions for which the synthesis procedure is applicable is a sub-class of the class of minimum functions. Although the cascade synthesis procedure is not always applicable, it includes techniques which, in principle, allow a determination of whether one step of the procedure may be effected. In certain special cases, the criteria for success of the cascade synthesis procedure reduce to inequalities among certain constants. In general, however, the criteria for successful removal of a cascade section involve inequalities containing functions of more than one variable.

An advantage of the cascade synthesis procedure in cases where it is applicable is that the degree of the driving-point impedance function is reduced by removal of the cascade section without the creation of more than one remainder impedance function.

The cascade synthesis procedure reduces to the Bott-Duffin synthesis method when applied to the synthesis of second-degree minimum functions as driving-point impedances. Thus, nothing is gained by application of the new method unless the driving-point impedance to be synthesized is of at least third degree.

APPENDIX

Properties of second-degree minimum functions.--In this section some well-known properties of second-degree minimum functions are collected for reference purposes, using the notations and terminology employed in the earlier sections of this report.

The most general second-degree minimum function may be written as

$$Z(s) = \frac{L}{a - 1}\left[\frac{a^2Rs^2 + (a - 1)^2 s + aR}{s^2 + aRs + a}\right] \tag{A1}$$

if the frequency scale is normalized so that

$$Z(j1) = jL \, . \tag{A2}$$

Parameters a, L, and R in (A1) must satisfy the inequalities

$$R \geq 0 \, , \tag{A3}$$

$$a \geq 0 \, , \tag{A4}$$

and

$$\frac{L}{a - 1} \geq 0 \tag{A5}$$

in order that Z(s) be positive-real.[24] The function given by (A1) may be realized as a driving-point impedance by the Brune networks shown in

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[24] The conditions R = 0 and a = 0 lead to reactance networks.

Figs. 30 and 31 and by one of the Bott-Duffin networks of Figs. 32 and 33. The network of Fig. 32 is appropriate if $L \geq 0$ and that of Fig. 33 is appropriate if $L \leq 0$. The parameters in Fig. 32 which have not already been defined are given by

$$L_a = \frac{La^2R^2}{(a - 1)\ [a^2R^2 + (a - 1)^2]} \tag{A6}$$

and

$$C_a = \frac{a - 1}{L[a^2R^2 + (a - 1)^2]} \quad . \tag{A7}$$

Similarly, the new L_a and C_a in Fig. 33 are given by

$$L_a = \frac{a^3R^2L}{(a - 1)\ [a^2R^2 + (a - 1)^2]} \tag{A8}$$

and

$$C_a = \frac{a(a - 1)}{L[a^2R^2 + (a - 1)^2]} \quad . \tag{A9}$$

In either of the Bott-Duffin networks, the terminal pair X may be open-circuited, short-circuited, or terminated by any arbitrary impedance.

Realization of the cascade section.--Certain details pertinent to the problem of realizing the cascade section, which are omitted from Chapter IV in order to avoid digressions, are considered here. These matters include the discussion of sufficient conditions for an unbalanced realization of the cascade section and a more complete exposition of the unbalancing process for the case where $L \leq 0$.

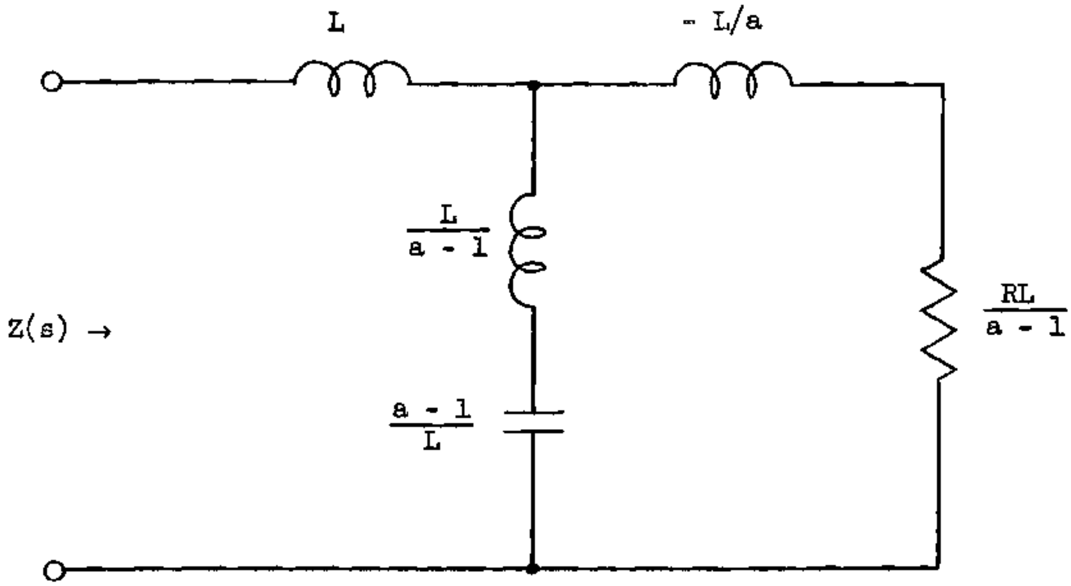

Figure 30. Second-Degree Brune Network.

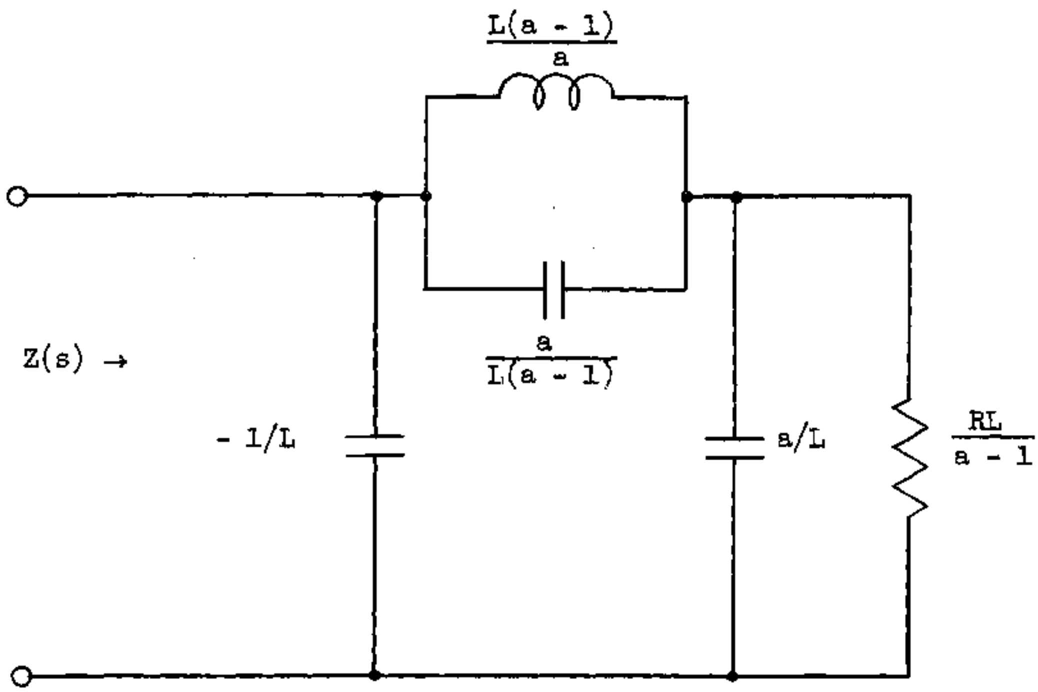

Figure 31. Second-Degree Brune Network.

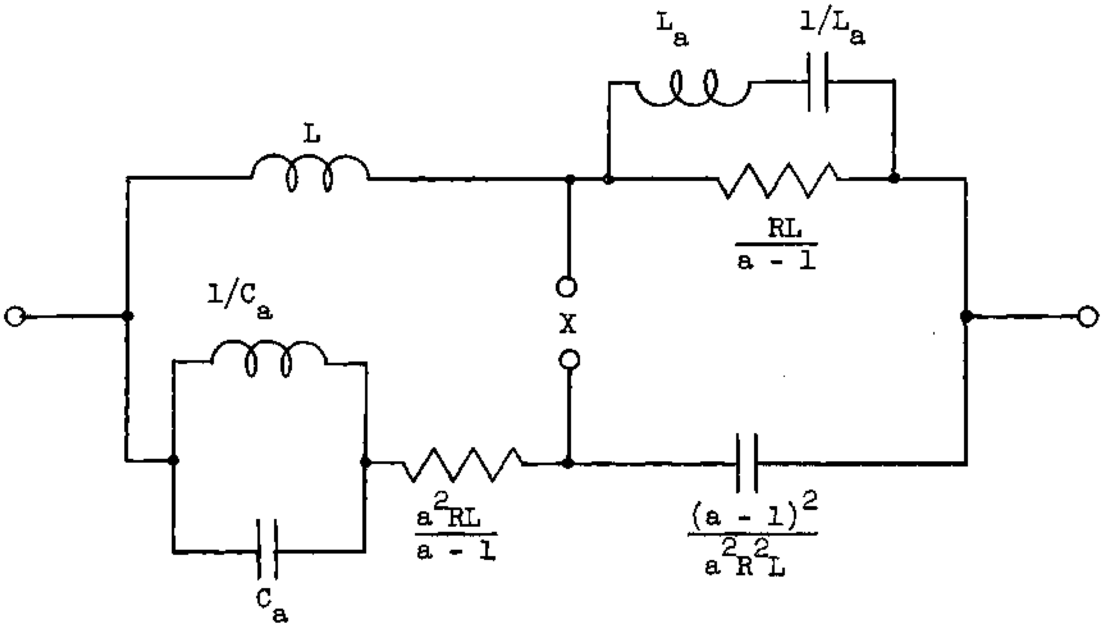

Figure 32. Second-Degree Bott-Duffin Network, $L \geq 0$.

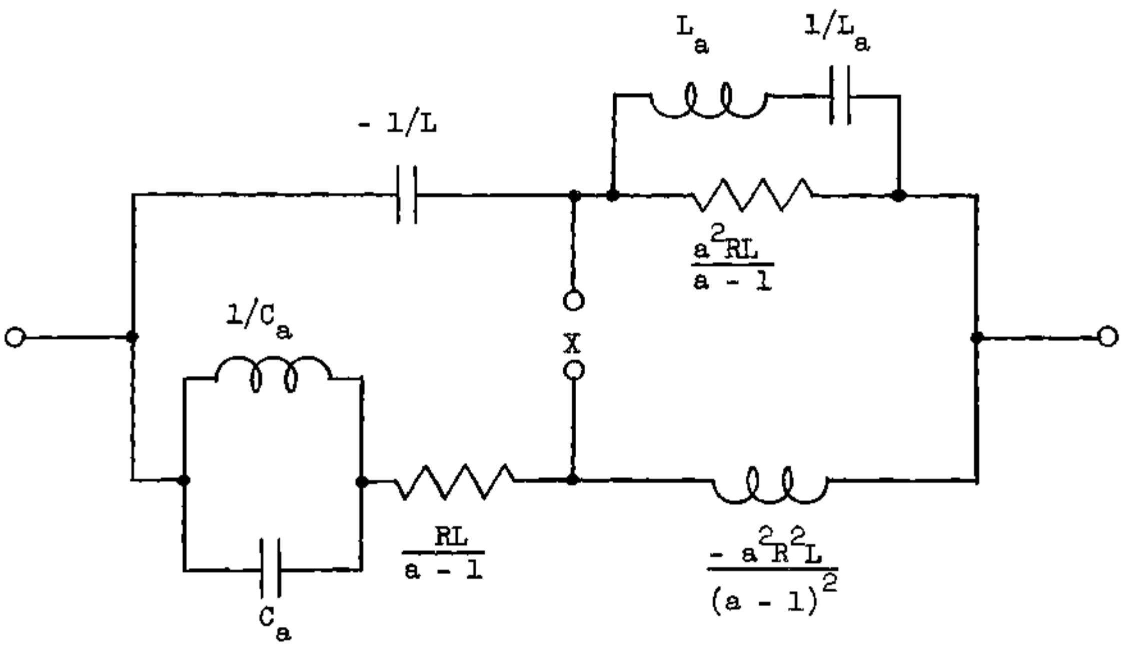

Figure 33. Second-Degree Bott-Duffin Network, $L \leq 0$.

Consider first the case where $L \geq 0$. It is stated in Chapter IV that the satisfaction of either (80) or (81) is a sufficient condition for the transformerless realization of the cascade section when L, (a - 1), and (b - 1) are non-negative. The proof of these sufficient conditions consists of a demonstration that (80) and (81) each imply (73)-(76). It will be shown here that (80) also implies (82) and (83). The basic inequality (80) is

$$\frac{(a-1)^4}{a^2R^2} - \frac{(b-1)^4}{b^2T^2} \geq (b-1)^2 - (a-1)^2 \geq 0 \; . \tag{A10}$$

The last part of this relation implies

$$b - 1 \geq a - 1 \tag{A11}$$

or

$$b \geq a \; , \tag{A12}$$

which is also (82). It is obvious that

$$-\frac{1}{b} \geq -\frac{1}{a} \; , \tag{A13}$$

which implies

$$\left(\frac{b-1}{b}\right)^2 \geq \left(\frac{a-1}{a}\right)^2 \quad , \tag{A14}$$

since (a - 1) and (b - 1) are non-negative. The first part of (A10) implies

$$\frac{b^2T^2}{(b-1)^4} \geq \frac{a^2R^2}{(a-1)^4} \quad . \tag{A15}$$

Combination of (A14) and (A15) yields

$$\frac{T^2}{(b - 1)^2} \geq \frac{R^2}{(a - 1)^2} \tag{A16}$$

or, since $L \geq 0$,

$$\frac{TL}{b - 1} \geq \frac{RL}{a - 1} \quad , \tag{A17}$$

which is also (74). Relations (A11) and (A17) imply

$$T \geq R \ , \tag{A18}$$

which is also (83). The inequalities of (A12) and (A17) imply

$$\frac{b^2 TL}{b - 1} \geq \frac{a^2 RL}{a - 1} \quad , \tag{A19}$$

which is also (76). Since $L \geq 0$, (A17) and (A19) may be combined to yield

$$\frac{b^2 T^2 L}{(b - 1)^2} \geq \frac{a^2 R^2 L}{(a - 1)^2} \quad , \tag{A20}$$

which is also (75). The inequalities of (A11), (A12), and (A19) imply

$$b - 1 + \frac{b^2 T^2}{b - 1} \geq a - 1 + \frac{a^2 R^2}{a - 1} \tag{A21}$$

or, since $L \geq 0$,

$$C_b = \frac{b - 1}{L[b^2 T^2 + (b - 1)^2]} \leq \frac{a - 1}{L[a^2 R^2 + (a - 1)^2]} = C_a, \tag{A22}$$

which is also (77). Rewriting (A10) results in

$$(b - 1)^2 + \frac{(b - 1)^4}{b^2T^2} \leq (a - 1)^2 + \frac{(a - 1)^4}{a^2R^2} \quad . \tag{A23}$$

Combination of (A11) and (A23) yields

$$\frac{1}{(b-1)} \left[(b-1)^2 + \frac{(b-1)^4}{b^2T^2} \right] \leq \frac{1}{(a-1)} \left[(a-1)^2 + \frac{(a-1)^4}{a^2R^2} \right] \tag{A24}$$

or

$$(b-1) \left[\frac{b^2T^2 + (b-1)^2}{b^2T^2} \right] \leq (a-1) \left[\frac{a^2R^2 + (a-1)^2}{a^2R^2} \right] . \tag{A25}$$

Since $L \geq 0$,

$$L_b = \frac{Lb^2T^2}{(b-1)[b^2T^2 + (b-1)^2]} \geq \frac{La^2R^2}{(a-1)[a^2R^2 + (a-1)^2]} = L_a , \tag{A26}$$

which is also (73). Thus (80) implies (73)-(77), (82), and (83). A proof of the sufficiency of (81) is not given here, since it is quite similar to the proof for (80).

Now consider the case where $L \leq 0$. The symmetric lattice network from which the cascade section may be derived is shown in Fig. 6. The horizontal-arm impedance z_R may be represented by the Bott-Duffin network of Fig. 33. The Bott-Duffin network for the diagonal-arm impedance z_T may be obtained from Fig. 33 by substituting b for a and T for R.

The lattice network of Fig. 6 may be represented as the parallel combination of the two component lattice networks of Figs. 34 and 35.

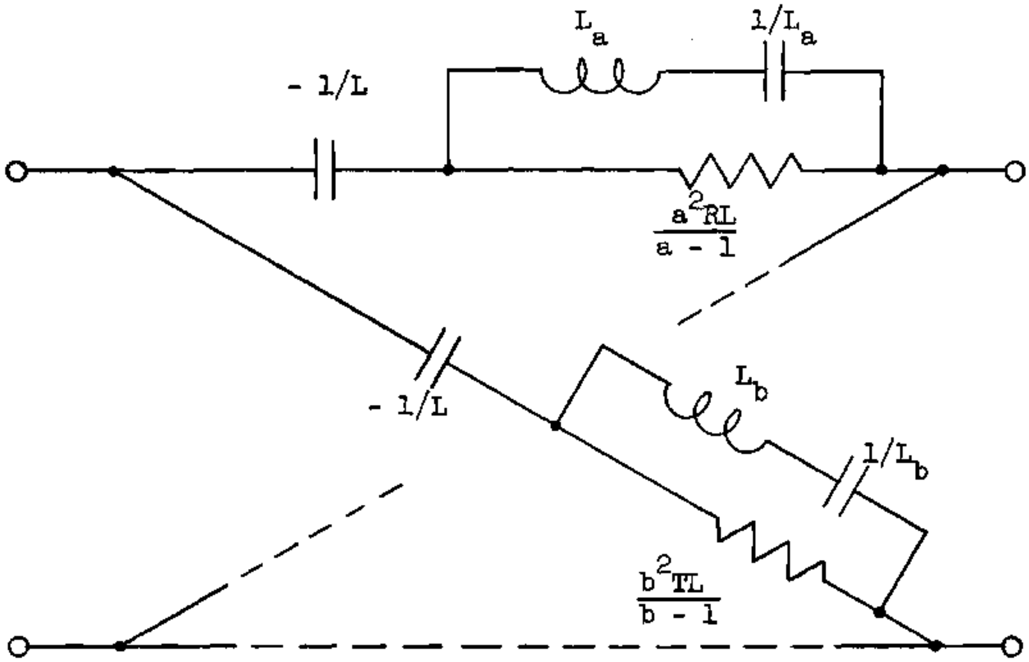

Figure 34. Component Lattice Network I, L $\leq$ 0.

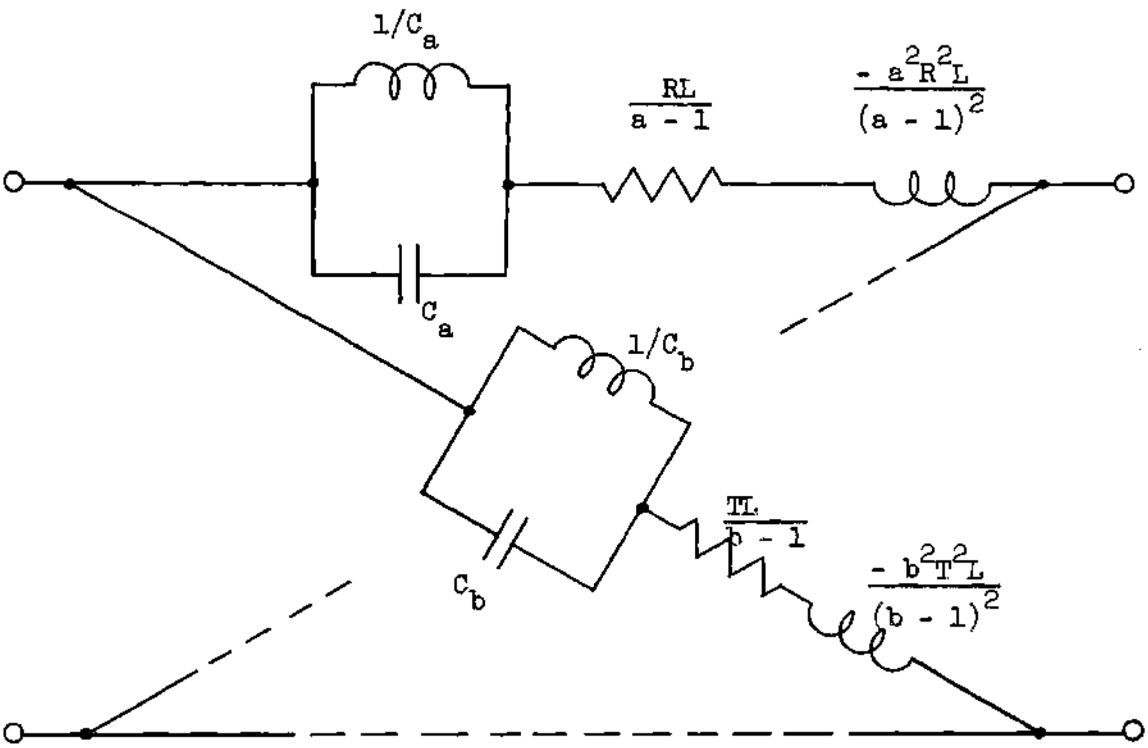

Figure 35. Component Lattice Network II, L $\leq$ 0.

The parameters L_b and C_b in these figures are given by

$$L_b = \frac{b^3T^2L}{(b - 1)\ [b^2T^2 + (b - 1)^2]} \tag{A27}$$

and

$$C_b = \frac{b(b - 1)}{L[b^2T^2 + (b - 1)^2]} \quad . \tag{A28}$$

The parameters L_a and C_a are defined analogously to L_b and C_b; expressions for them are given in (A7) and (A8). Component lattice network I has an unbalanced realization if

$$L_b \geq L_a \tag{A29}$$

and

$$\frac{b^2TL}{b - 1} \geq \frac{a^2RL}{a - 1} \quad . \tag{A30}$$

Component lattice network II has an unbalanced representation if

$$\frac{-\ b^2T^2L}{(b - 1)^2} \geq \frac{-\ a^2R^2L}{(a - 1)^2} \quad , \tag{A31}$$

$$\frac{TL}{b - 1} \geq \frac{RL}{a - 1} \quad , \tag{A32}$$

and

$$C_a \geq C_b \ . \tag{A33}$$

If the unbalanced forms of the two component lattice networks are bisected and the impedance level of the right half of each network is multiplied by 1/c, the two component networks of Figs. 36 and 37 result. The two-terminal-pair network formed by the parallel combination of the component networks of Figs. 36 and 37 is the desired unbalanced form of the cascade section. The configuration of this network is shown in Fig. 38. As in the case where $L \geq 0$, alternate forms of the final network are possible.

In a manner analogous to that applied when $L \geq 0$, it may be shown that the satisfaction of either of the inequalities

$$T^2 - R^2 \geq \left(\frac{1 - a}{a}\right)^2 - \left(\frac{1 - b}{b}\right)^2 \geq 0 \tag{A34}$$

or

$$\frac{(1 - a)^4}{a^4 R^2} - \frac{(1 - b)^4}{b^4 T^2} \geq \left(\frac{1 - b}{b}\right)^2 - \left(\frac{1 - a}{a}\right)^2 \geq 0 \tag{A35}$$

is a sufficient condition for transformerless realization of the cascade section when $L \leq 0$. Condition (A34) implies (A29)-(A33), as well as

$$b \geq a \tag{A36}$$

and

$$T \geq R . \tag{A37}$$

Similarly, (A35) implies (A29)-(A33), as well as

$$a \geq b . \tag{A38}$$

The final network described above requires a total of 23 elements, of which 6 are resistors. The number of elements required is again

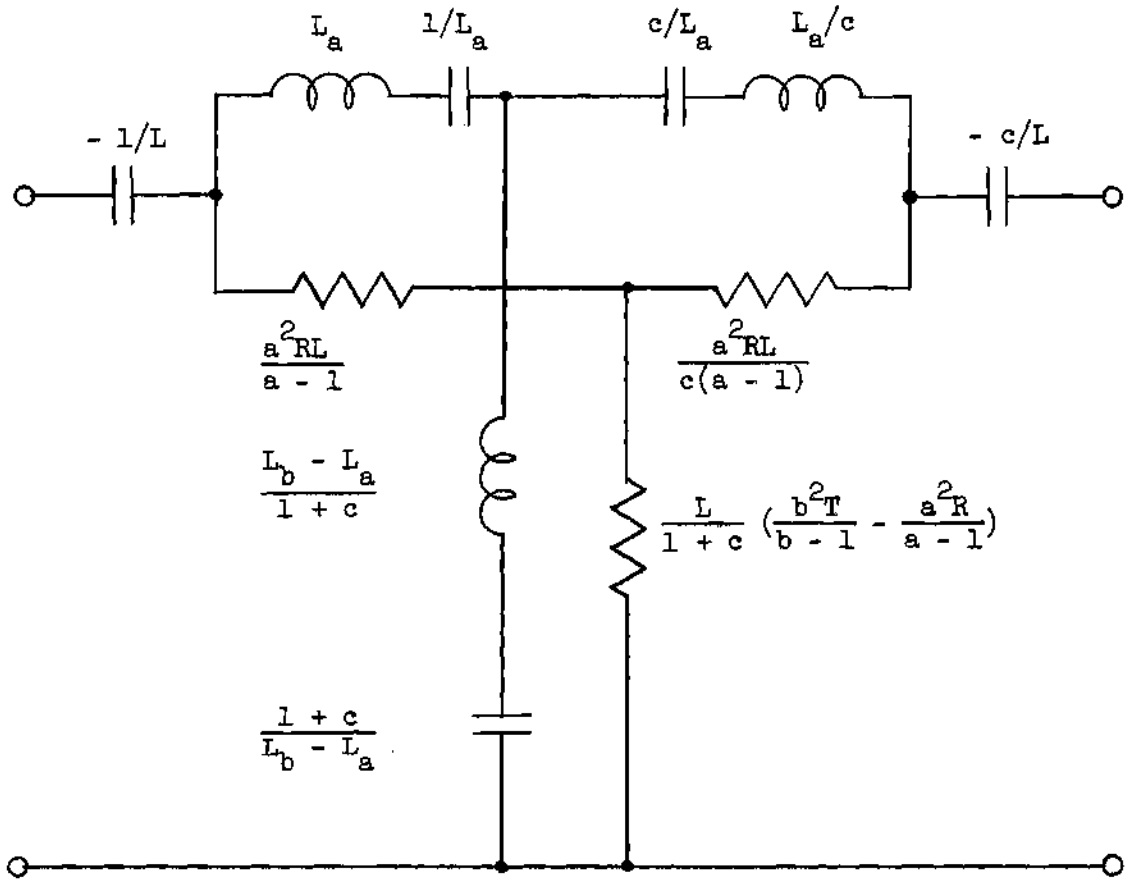

Figure 36. Component Network I, $L \leq 0$.

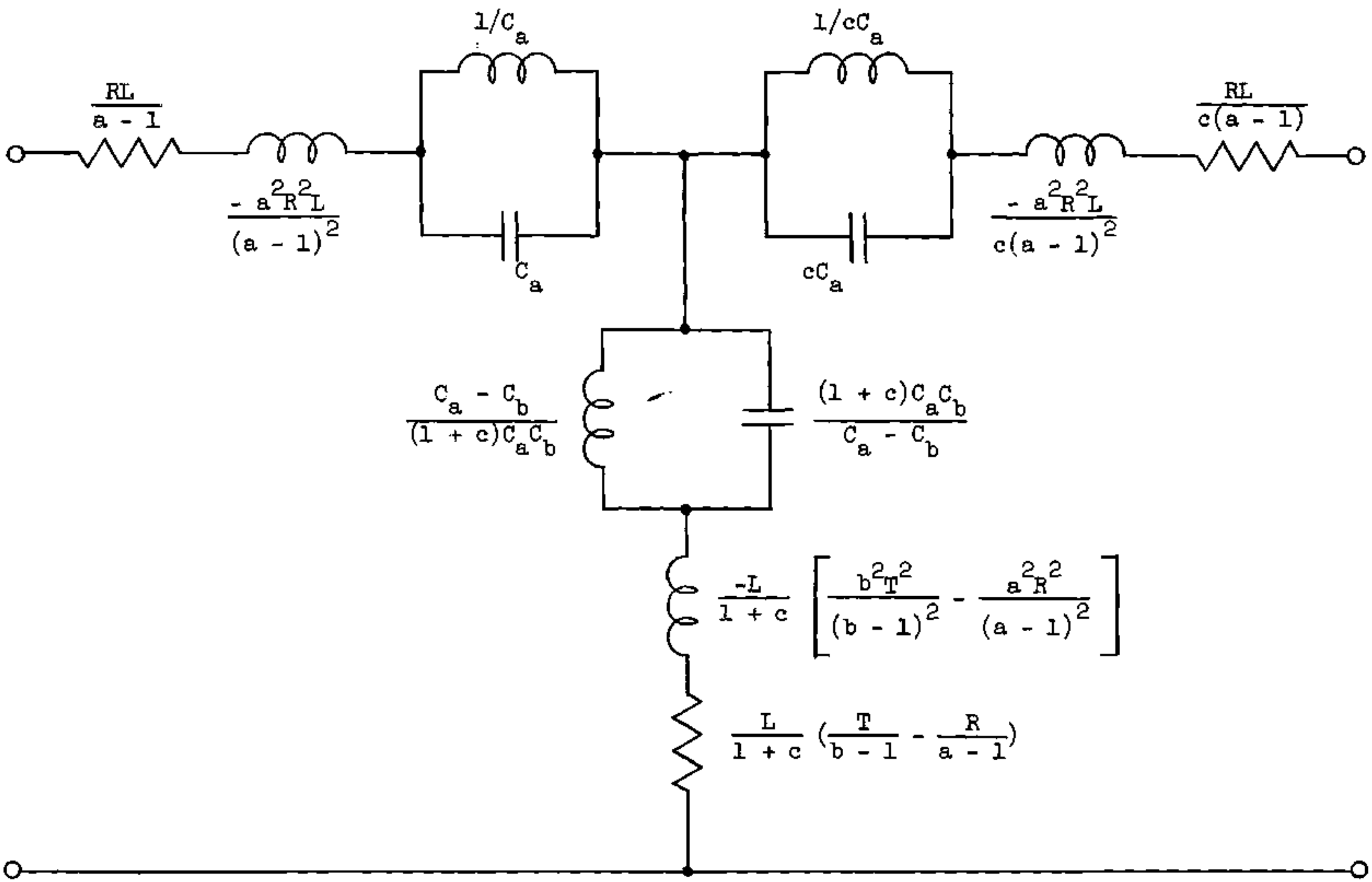

Figure 37. Component Network II, $L \leq 0$.

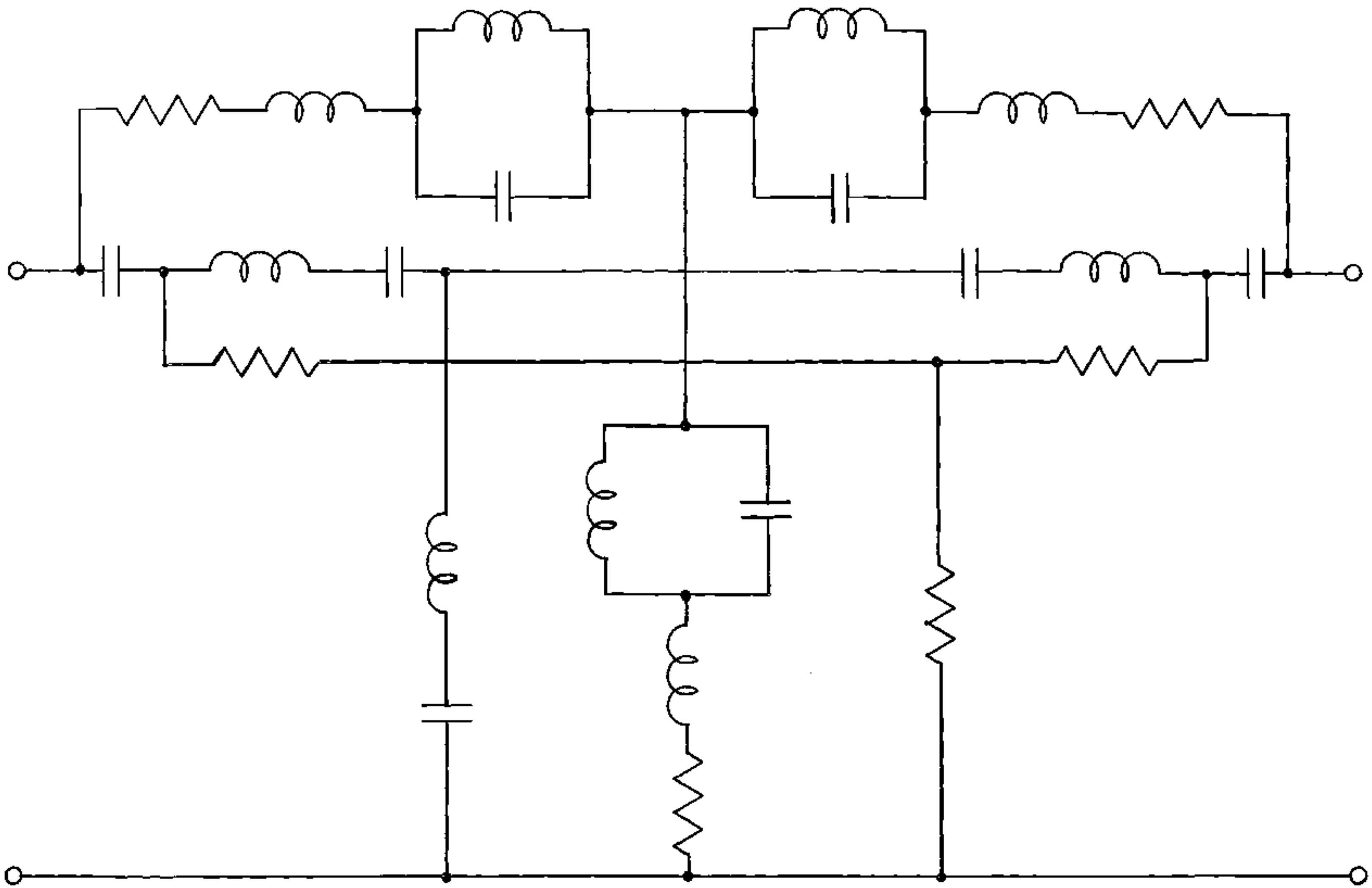

Figure 38. Configuration of Cascade Section, $L \leq 0$.

reduced to 11 reactors and 2 resistors if either of the two restrictions

$$R = 0 \tag{A39}$$

and

$$1/T = 0 \tag{A40}$$

is enforced. The choice $R = 0$ leads to the cascade section shown in Fig. 15. In this case it is evident from (A35) that (A38), or (84), is a sufficient condition for realization of the cascade section in the form shown by Fig. 15. Similarly, if the choice $1/T = 0$ is made, (A34) shows that (A36), or (82), is a sufficient condition for realization of the cascade section in the form shown by Fig. 16.

Numerical examples.--The following numerical examples illustrate the cascade synthesis procedure:

(1) Consider the fourth-degree minimum function

$$Z_1(s) = \frac{6.56s^4 + 32.08s^3 + 35.92s^2 + 18.24s + 16.}{s^4 + 1.96s^3 + 11.12s^2 + 8.64s + 3.2}, \tag{A41}$$

for which

$$Z_1(j1) = j2 \tag{A42}$$

and

$$\alpha = 2 . \tag{A43}$$

For this datum impedance, the locus of the roots of (119) are shown in Fig. 24. The function $Z_v(s)$ obtained by (100) is positive-real for b = 1.7 if the surplus transmission zeros are chosen to correspond to the upper branch of the locus in Fig. 24. The parameters of the cascade section corresponding to this value of b are

$$L = 2\ , \tag{A44}$$

$$1/T = 0\ , \tag{A45}$$

$$R = 0.039090607\ , \tag{A46}$$

$$a = 5.9656356\ , \tag{A47}$$

$$b = 1.7000000\ , \tag{A48}$$

and

$$c = 0.53664274\ . \tag{A49}$$

The remainder impedance is

$$Z_2(s) = \frac{4.9555409s^2 + 2.8537204s + 15.855733}{s^2 + 0.061897579s + 3.2000000}\ . \tag{A50}$$

The network realizing $Z_1(s)$ is shown in Fig. 39.

The computations necessary in performing the numerical work for this example were carried out to eight significant figures. Of course, this procedure does not insure eight figure accuracy.

(2) Let the minimum function

$$Z_1(s) = \frac{4s^4 + 104s^3 + 82s^2 + 68s + 32}{16s^4 + 9s^3 + 66s^2 + 32s + 32} \tag{A51}$$

be the datum impedance function to be realized. This function is doubly minimum-real, and the method of Chapter VII is therefore appropriate.

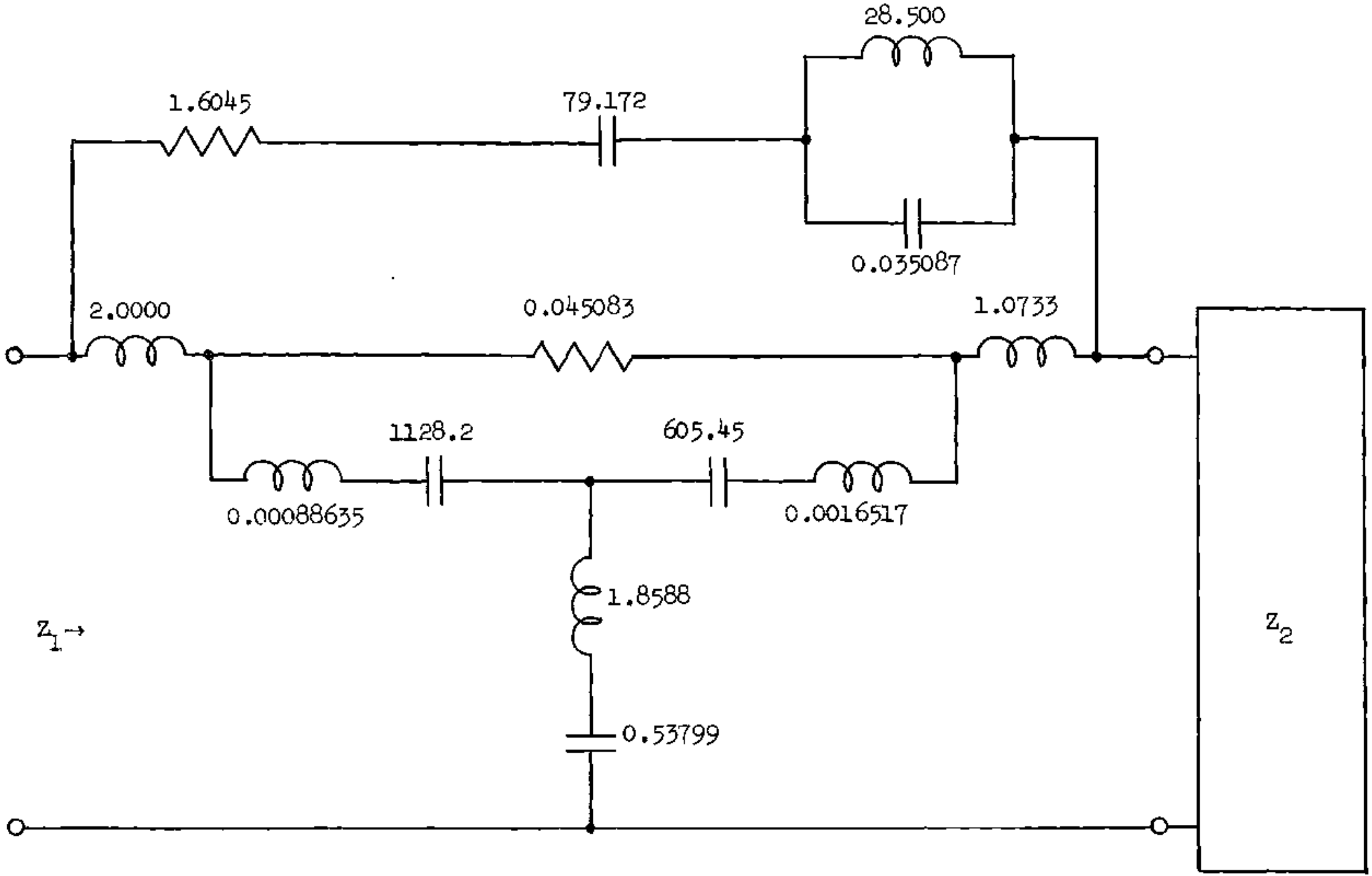

Figure 39. Network Realizing $Z_1(s)$ of Example (1).

From the equations

$$Z_1(j1) = j2 \tag{A52}$$

and

$$Z_1(j2) = -j29 , \tag{A53}$$

it is evident that (215) cannot be satisfied. Thus, the cascade synthesis procedure of Chapter VII cannot be applied successfully.

(3) Consider the doubly minimum-real function

$$Z_1(s) = \frac{4s^4 + 176s^3 + 74s^2 + 104s + 32}{16s^4 + 5s^3 + 84s^2 + 24s + 32} , \tag{A54}$$

for which

$$\omega_o = 2 , \tag{A55}$$

$$L = 2 , \tag{A56}$$

$$\alpha = 2 , \tag{A57}$$

and

$$X = 25/2 . \tag{A58}$$

From (183), (184), and (66) the parameters

$$b = 8/7 , \tag{A59}$$

$$a = 32/7, \tag{A60}$$

and

$$c = 25/3 \tag{A61}$$

may be obtained. Since $a - 1$, $b - 1$, and $\alpha - 1$ have the same sign and c is positive, the cascade section is realizable in the network shown by Fig. 17. From (196), γ may be calculated to be

$$\gamma = 146/121 \tag{A62}$$

and computation shows that (202) is satisfied. Alternately, the parameter β of Fig. 29 may be calculated to be

$$\beta = 1/4 \; , \tag{A63}$$

and (220) is satisfied. Thus the cascade synthesis procedure is successful in this case. The remainder impedance is given by

$$Z_2(s) = 3 \left[\frac{s^2 + 3s + 2}{12s^2 + 2s + 6} \right] \; . \tag{A64}$$

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