Institute of Paper Science and Technology
Atlanta, Georgia

IPST Technical Paper Series Number 757

Numerical Evaluation for Various Upwinding Schemes

X. Wang

October 1998

Submitted to
Transactions of the ASME Journal of Fluids Engineering

INSTITUTE OF PAPER SCIENCE AND TECHNOLOGY
PURPOSE AND MISSIONS

The Institute of Paper Science and Technology is an independent graduate school, research organization, and information center for science and technology mainly concerned with manufacture and uses of pulp, paper, paperboard, and other forest products and byproducts. Established in 1929, the Institute provides research and information services to the wood, fiber, and allied industries in a unique partnership between education and business. The Institute is supported by 52 North American companies. The purpose of the Institute is fulfilled through four missions, which are:

- to provide a multidisciplinary education to students who advance the science and technology of the industry and who rise into leadership positions within the industry;
- to conduct and foster research that creates knowledge to satisfy the technological needs of the industry;
- to serve as a key global resource for the acquisition, assessment, and dissemination of industry information, providing critically important information to decision-makers at all levels of the industry; and
- to aggressively seek out technological opportunities and facilitate the transfer and implementation of those technologies in collaboration with industry partners.

ACCREDITATION

The Institute of Paper Science and Technology is accredited by the Commission on Colleges of the Southern Association of Colleges and Schools to award the Master of Science and Doctor of Philosophy degrees.

NOTICE AND DISCLAIMER

Numerical Evaluation for Various Upwinding Schemes

by

Xiaodong Wang, Assistant Professor of Engineering

Institute of Paper Science and Technology

500 10th Street, N.W., Atlanta, GA 30318

Abstract

Different upwinding schemes in the context of finite element, finite volume, finite difference methods are discussed. Numerical tests are presented to identify numerically whether or not, for the solutions of multi-dimensional convection-diffusion systems, given upwinding schemes combine improved stability with high-order accuracy.

1998 ASME FLUIDS ENGINEERING FEDSM98-4926

Transactions of the ASME Journal of Fluids Engineering (submitted)

1 Introduction

Great advances have been accomplished during recent years in the analyses of general fluid flows, heat transfer, and their structural interactions. The use of finite element methods has made it possible to analyze such problems with complex geometries, and to integrate with many mature finite element packages for solids and structures. Nevertheless, convective terms (hyperbolic in nature) introduce nonsymmetry into the discretized coefficient matrix and remain the source of non-physical oscillatory solutions, which often occur along the sharp internal or boundary layers and are very similar to Gibbs' phenomenon [1].

To circumvent the skew matrix derived with the standard Galerkin finite element, finite volume or central difference methods, researchers have proposed various discretization procedures (often called upwinding schemes). The basic idea of upwinding, which was discussed by Courant *et al.* [2] in 1952, is to assign more weight to the nodal solution in the upstream direction than in the downstream direction. The initial upwinding finite element schemes were outlined by Christie *et al.* [3] and Heinrich *et al.* [4] [5]. Noble control volume finite element upwinding formulations include the quadrilateral elements by Schneider and Raw [6] and the 4/3-c triangular and 5/4-c tetrahedral elements by Bathe *et al.* [7]. Some other upwinding approaches, such as the Lax-Wendroff/Taylor-Galerkin formulation [8], the Galerkin least squares method [9], and the Galerkin method with bubble functions [10], were also developed recently. Most importantly, the well-known Streamline Upwind Petrov/Galerkin (SUPG) method, originally developed by Brooks and Hughes [11], was studied and analyzed extensively [12] [13].

Although it is possible to achieve exact nodal solutions for the one-dimensional model problem, and this idea has been widely used in various upwinding schemes such as the exponential schemes developed in control volume finite difference procedures (see Spalding [14], Patankar [15], and Minkowycz *et al.* [16]), solutions for multi-dimensional cases, in general, exhibit either excessive diffusion or oscillatory behavior. In fact, all upwinding schemes are, in essence, equivalent to the standard Galerkin or central difference method with a so-called artificial diffusivity.

In this paper, starting with the one-dimensional convection-diffusion model problem, we compare various upwinding schemes and propose a numerical test to investigate whether any given upwinding scheme is as accurate as the standard Galerkin formulation with sufficiently refined meshes, and/or as oscillation free as the monotonic classic upwinding approach. This test is applied to two generic two-dimensional convection-diffusion examples and used to explain the solution behaviors with various discretizations.

2 One-Dimensional Model

For the one-dimensional convection-diffusion model, the governing differential equation can be written as follows:

$$v\frac{d\theta}{dx} - \alpha\frac{d^2\theta}{dx^2} = 0 \tag{1}$$

with the boundary conditions

$$\begin{aligned} \theta &= 0 \qquad \text{at x=0} \\ \theta &= 1 \qquad \text{at x=1} \end{aligned}$$

where α is the thermal diffusivity and v is the prescribed velocity.

Although Eq. (1) is a simple constant coefficient ordinary differential equation with the exact solution $\theta = (e^{vx/\alpha} - 1)/(e^{v/\alpha} - 1)$, we recognize that the basic observations of discretization procedures are applicable to the solution of multi-dimensional cases and to the Navier-Stokes equations. In this paper, we elaborate the inner relationships among various upwinding schemes with the analogously simple form of Eq. (1).

The typical ith finite difference equation, with central differencing for both the convective and diffusive terms, takes the form

$$-\left(\frac{v}{2}+\frac{\alpha}{h}\right)\theta_{i-1}+2\frac{\alpha}{h}\theta_i+\left(\frac{v}{2}-\frac{\alpha}{h}\right)\theta_{i+1}=0 \tag{2}$$

where h denotes the mesh size. The oscillatory nature of Eq. (2), when the element Peclet number $Pe^e = v\alpha/h > 2$, has been widely reported. To illustrate the remedy designed by Courant *et al.* [2], we assume without loss of generality that v is positive. If we discretize the convective term with a backward Euler scheme (a so-called classical upwinding scheme), we arrive at the following ith equation

$$-\left(v+\frac{\alpha}{h}\right)\theta_{i-1}+\left(v+2\frac{\alpha}{h}\right)\theta_i-\frac{\alpha}{h}\theta_{i+1}=0. \tag{3}$$

Using Eq. (3), the oscillatory solution behavior is no longer present. It is not difficult to identify that in order to get Eq. (3), we, in fact, add an artificial diffusivity to Eq. (2),

$$\frac{v}{2}(\theta_{i-1}-2\theta_i+\theta_{i+1}) \tag{4}$$

and Eq. (3) corresponds to a modified problem

$$v\frac{d\theta}{dx}=\left(\alpha+\frac{vh}{2}\right)\frac{d^2\theta}{dx^2}. \tag{5}$$

The control volume finite element method is a rather straightforward approach for this one-dimensional problem. The ith equation corresponds to satisfying the equilibrium of the flux $v\theta - \alpha \frac{d\theta}{dx}$ for the ith control volume between the stations $i - 1/2$ and $i + 1/2$. In the standard control volume method (without upwinding), with

$$\begin{aligned} \theta_{i-1/2} &= (\theta_i + \theta_{i-1})/2 & (6) \\ \theta_{i+1/2} &= (\theta_i + \theta_{i+1})/2 & (7) \end{aligned}$$

we find that the ith equation is exactly the same as Eq. (2); while in the control volume method with upwinding, assuming v is positive, with

$$\begin{aligned} \theta_{i-1/2} &= \theta_{i-1} & (8) \\ \theta_{i+1/2} &= \theta_i & (9) \end{aligned}$$

we obtain Eq. (3). Note that, in both cases (with and without upwinding), we apply

$$\begin{aligned} \left.\frac{d\theta}{dx}\right|_{i-1/2} &= (\theta_i - \theta_{i-1})/h & (10) \\ \left.\frac{d\theta}{dx}\right|_{i+1/2} &= (\theta_{i+1} - \theta_i)/h. & (11) \end{aligned}$$

In the standard Galerkin finite element formulation, the same trial functions are employed to express the weighting and the solution. However, in principle, different functions may be used in the variational formulation. The modified weight function in the SUPG method includes a first derivative term and can be written, in the one-dimensional case as

$$\delta\bar{\theta} = \delta\theta + \frac{v\xi h}{2}\frac{d\delta\theta}{dx} \tag{12}$$

where ξ is a parameter to be adjusted. For a typical two-node element, we obtain the following stiffness matrix:

$$\mathbf{K}_e = \begin{bmatrix} -\frac{v}{2} + (\frac{\alpha}{h} + \frac{v\xi}{2}) & \frac{v}{2} - (\frac{\alpha}{h} + \frac{v\xi}{2}) \\ -\frac{v}{2} - (\frac{\alpha}{h} + \frac{v\xi}{2}) & \frac{v}{2} + (\frac{\alpha}{h} + \frac{v\xi}{2}) \end{bmatrix} \tag{13}$$

from which, after the element assemblage, the ith equation becomes

$$-(\frac{v}{2} + (\frac{\alpha}{h} + \frac{v\xi}{2}))\theta_{i-1} + (2\frac{\alpha}{h} + v\xi)\theta_i + (\frac{v}{2} - (\frac{\alpha}{h} + \frac{v\xi}{2}))\theta_{i+1} = 0. \tag{14}$$

We note that with $\xi = 0$, the standard Galerkin finite element equation is recovered; when $\xi = 1$, the classic upwinding scheme is obtained. In particular, for this one-dimensional model problem, ξ can be evaluated such that nodal exact values are obtained for all values of Pe^e,

$$\xi = coth(\frac{Pe^e}{2}) - \frac{2}{Pe^e}. \tag{15}$$

An ad hoc generalization is applied to multi-dimensional elements based on directional Peclet numbers. In the numerical implementation of Eq. (15), doubly asymptotic approximations or critical approximations are often used:

$$\xi = \begin{cases} Pe^e/6 & -6 \le Pe^e \le 6 \\ sgn(Pe^e) & |Pe^e| > 6 \end{cases}$$

or

$$\xi = \begin{cases} -1 - 2/Pe^e & Pe^e < -1 \\ 0 & -1 \le Pe^e \le 1 \\ 1 - 2/Pe^e & Pe^e > 1. \end{cases}$$

In fact, Eq. (14) represents a general form of upwinding schemes for this one-dimensional model problem, for example, the Galerkin least squares method results in the same ξ as in Eq. (15). With Eq. (15), Eq. (14) becomes equivalent to Eq. (3), in the hyperbolic limit as $\alpha \to 0$.

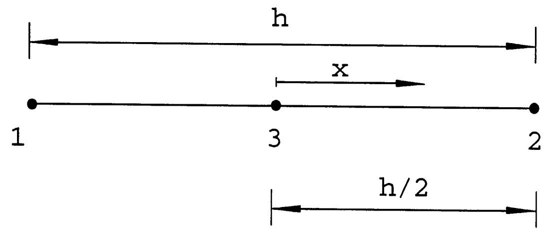

Figure 1: One-dimensional discretization of the convection-diffusion equation with a bubble function ($h_1 = 1/2 - x/h$, $h_2 = 1/2 + x/h$, and $\tilde{h}_3 = 1 - 4x^2/h^2$).

If we consider the type of elements shown in Fig. 1, it is not difficult to prove that the bubble function $\tilde{h}_3$, in essence, introduces an artificial diffusivity with $\xi = vh/6\alpha$ (this magic number is in fact the same as in the doubly asymptotic approximation of the SUPG method). A comprehensive mathematical study of the general connection between the standard Galerkin method with bubble functions and the SUPG method is available from Brezzi *et al.* [10].

To study the stability of Eqs. (2), (3), and (14), we solve the corresponding constant-coefficient homogeneous difference equations. Assume $\theta_i = CG^i$ with the exponent i; we have for Eq. (14), the following quadratic characteristic equation,

$$-\left(\frac{v}{2} + \left(\frac{\alpha}{h} + \frac{v\xi}{2}\right)\right) + \left(2\frac{\alpha}{h} + v\xi\right)G + \left(\frac{v}{2} - \left(\frac{\alpha}{h} + \frac{v\xi}{2}\right)\right)G^2 = 0 \tag{16}$$

with two roots

$$\begin{aligned} G_1 &= 1 \\ G_2 &= \frac{\alpha/h + (v + v\xi)/2}{\alpha/h - (v - v\xi)/2}. \end{aligned} \tag{17}$$

In the standard Galerkin finite element method, or the central difference method ($\xi = 0$), since $G_2 < 0$, if $Pe^e > 2$, it is obvious that the solution of Eq. (2) contains oscillations; in particular, if $\alpha \to 0$, i.e., $G_2 \to -1$, we observe the sawtooth profile (similar to the checkerboard pressure modes for incompressible analyses [17]).

Furthermore, we notice that the coefficient matrix based on Eq. (2), satisfying the consistency of course, is not diagonally dominant for $Pe^e > 2$. In the classical up-winding method ($\xi = 1$), G_2 is always positive for all Pe^e, and the coefficient matrix based on Eq. (3) is diagonally dominant (though not strictly diagonally dominant). Therefore, the solution of Eq. (3) does not contain non-physical oscillations.

It is also interesting to point out that if $0 < \xi < 1$, for $Pe^e > 2/(1-\xi)$, the solution retains oscillations and the corresponding coefficient matrix is not diagonally dominant.

3 General Convection-Diffusion Model

To introduce finite element approximations in multi-dimensional cases, we consider the homogeneous convection-diffusion problem

$$\mathbf{v} \cdot \boldsymbol{\nabla}\theta = \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla}\theta) + f \quad \text{in} \quad \Omega \tag{18}$$

$$\theta = 0 \quad \text{on} \quad \Gamma \tag{19}$$

where for the bounded n-dimensional domain $\Omega \subset \mathcal{R}^n$ with a Lipschitz continuous boundary Γ, the given data are the source function f, the velocity field $\mathbf{v}$ with $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, and the diffusivity α. With the following Sobolev space:

$$H^1_{0,\Gamma}(\Omega) = \{\theta \quad | \quad \theta \in H^1(\Omega), \quad \theta|_\Gamma = 0\}$$

the standard variational form of Eq. (18) can be defined as:

Find $\theta \in V = H^1_{0,\Gamma}(\Omega)$ such that

$$(\mathbf{v} \cdot \boldsymbol{\nabla}\theta, \delta\theta) + (\alpha \boldsymbol{\nabla}\theta, \boldsymbol{\nabla}\delta\theta) = (f, \delta\theta), \quad \forall \delta\theta \in V \tag{20}$$

where

$$(\mathbf{v}\cdot\boldsymbol{\nabla}\theta,\delta\theta) = \int_\Omega \delta\theta\mathbf{v}\cdot\boldsymbol{\nabla}\theta d\Omega, \quad (\alpha\boldsymbol{\nabla}\theta,\boldsymbol{\nabla}\delta\theta) = \int_\Omega \alpha\boldsymbol{\nabla}\delta\theta\cdot\boldsymbol{\nabla}\theta d\Omega;$$
$$(f,\delta\theta) = \int_\Omega \delta\theta f d\Omega.$$

Given a sequence of finite dimensional subspaces $V_h \subset V$, we can obtain the following finite dimensional approximation:

$$(\mathbf{v}\cdot\boldsymbol{\nabla}\theta_h,\delta\theta_h)+(\alpha\boldsymbol{\nabla}\theta_h,\boldsymbol{\nabla}\delta\theta_h)=(f,\delta\theta_h), \quad \forall\delta\theta_h\in V_h \tag{21}$$

where h is the mesh size parameter indicating the length of the side of a generic element or the diameter of a circle encompassing that element [18]. Moreover, we can derive from Eqs. (20) and (21) the relation

$$(\mathbf{v}\cdot\boldsymbol{\nabla}(\theta-\theta_h),\delta\theta_h)+(\alpha\boldsymbol{\nabla}(\theta-\theta_h),\boldsymbol{\nabla}\delta\theta_h)=0, \quad \forall\delta\theta_h\in V_h. \tag{22}$$

To establish an error estimate, we employ $|\delta\theta_h|^2_{1,\Omega}=(\boldsymbol{\nabla}\delta\theta_h,\boldsymbol{\nabla}\delta\theta_h)$, and the fact that (refer to Appendix A)

$$\alpha|\delta\theta_h|^2_{1,\Omega}=(\alpha\boldsymbol{\nabla}\delta\theta_h,\boldsymbol{\nabla}\delta\theta_h)-(\mathbf{v}\cdot\boldsymbol{\nabla}\delta\theta_h,\delta\theta_h). \tag{23}$$

Define $I_h\theta$ as the interpolation function of θ, i.e., an element in V_h that, at the finite element nodes, has the exact value of the unknown solution θ and geometrically corresponds to a function close to θ; we obtain the estimates of the interpolation errors,

$$|\theta-I_h\theta|_{0,\Omega} \le C_1h^{k+1}||\theta||_{k+1} \tag{24}$$
$$|\theta-I_h\theta|_{1,\Omega} \le C_2h^{k}||\theta||_{k+1} \tag{25}$$

where C_1 and C_2 are constants independent of h, and k is the order of interpolation functions [18] [19].

Let us choose $\delta\theta_h = \theta_h - I_h\theta$, and apply Eqs. (22) and (23); we obtain, based on the Cauchy-Schwarz inequality (refer to Appendix B) the estimate

$$|\theta_h - I_h\theta|_{1,\Omega} \leq |\theta - I_h\theta|_{1,\Omega} + \frac{||\mathbf{v}||}{\alpha}|\theta - I_h\theta|_{0,\Omega}. \tag{26}$$

Therefore, an error estimate is established in the semi-norms of $H^1_{0,\Gamma}$ of the form

$$\begin{aligned} |\theta - \theta_h|_{1,\Omega} &\leq |\theta - I_h\theta|_{1,\Omega} + |\theta_h - I_h\theta|_{1,\Omega} \\ &\leq 2|\theta - I_h\theta|_{1,\Omega} + \frac{||\mathbf{v}||}{\alpha}|\theta - I_h\theta|_{0,\Omega} \\ &\leq (2C_2 + \frac{C_1 h||\mathbf{v}||}{\alpha})h^k||\theta||_{k+1}. \end{aligned} \tag{27}$$

Note that, if $\alpha \to 0$, the inequality (27) does not yield convergence; however, for a finite α, with a sufficiently refined mesh $h \simeq O(\alpha)$, the inequality (27) guarantees the convergence of θ_h to θ. In fact, all upwinding schemes introduce the notion of an artificial diffusivity α^*; for instance, in the classic upwinding scheme, the artificial diffusivity $\alpha^* = C_0||\mathbf{v}||h$, and we have, with $\alpha \to 0$,

$$|\theta - \theta_h|_{1,\Omega} \leq (2C_2 + \frac{C_1}{C_0})h^k||\theta||_{k+1}. \tag{28}$$

In general, the monotonic and often *over diffusive* classical upwind scheme with $O(h)$ can be written in the form

$$(\mathbf{v}\cdot\nabla\theta_h, \delta\theta_h) + ((\alpha + C_o||\mathbf{v}||h)\nabla\theta_h, \nabla\delta\theta_h) = (f, \delta\theta_h), \quad \forall\delta\theta_h \in V_h \tag{29}$$

while the non-monotonic and *not too diffusive* SUPG formulation can be written as

$$(\mathbf{v}\cdot\nabla\theta_h, \delta\theta_h) + (\alpha\nabla\theta_h, \nabla\delta\theta_h) + \sum_{\Omega_e}(\mathbf{v}\cdot\nabla\theta_h - \nabla\cdot(\alpha\nabla\theta_h), \tau\mathbf{v}\cdot\nabla\delta\theta_h)$$

$$= (f, \delta\theta_h) + \sum_{\Omega_e}(f, \tau \mathbf{v} \cdot \boldsymbol{\nabla}\delta\theta_h), \quad \forall \delta\theta_h \in V_h \tag{30}$$

where $\tau = \dfrac{h_e \xi_e}{2\|\mathbf{v}\|}$ and the element subdomain Ω_e satisfies $\bigcup_e \Omega_e = \Omega$ and $\bigcap_e \Omega_e = \oslash$.

4 Proposed Numerical Tests

We recognize that to *completely* eliminate non-physical oscillatory solutions, we need to use monotonic schemes. Nevertheless, in order to achieve high-order accuracy for all ranges of finite diffusivities, upwinding schemes should approach the standard Galerkin formulation with sufficiently refined meshes. Although many of the upwinding schemes work well for selected examples, it is often the case that, in solving practical problems with distorted meshes, the stability and accuracy of such formulations need to be verified. The proposed numerical test in this paper will be used to compare any given upwinding scheme (in this paper, we select the SUPG formulation as an example) with two limit cases, i.e., the classical upwinding schemes (such as the Schneider and Raw scheme [6] for two-dimensional cases, and the Courant's scheme [2] for one-dimensional cases), and the standard Galerkin formulation. In order to measure the difference between a given upwinding scheme and the standard Galerkin formulation with a finite diffusivity, we construct the following semi-norms:

$$|\theta_h^c - \theta_h^g|_{1,\Omega} = (\boldsymbol{\nabla}(\theta_h^c - \theta_h^g), \boldsymbol{\nabla}(\theta_h^c - \theta_h^g)) \tag{31}$$

$$|\theta_h^c - \theta_h^g|_{0,\Omega} = (\theta_h^c - \theta_h^g, \theta_h^c - \theta_h^g) \tag{32}$$

where the superscripts c and g stand for the solutions of a given upwinding scheme and the standard Galerkin formulation, respectively. It is noteworthy that we can use the same approach to compare any given upwinding scheme with the SUPG method. In addition, to estimate the oscillatory behaviors, a generic eigenvalue test is designed to check whether the coefficient matrix is diagonally dominant for the hyperbolic limit.

Although numerical tests are not as affirmative as analytical proofs, in practice, a properly designed numerical evaluation is very likely to be effective. Similar ideas are used when analytical evaluations are not achieved in the studying of incompatible displacement formulations, the effects of element geometric distortions, and, most recently, the inf-sup condition of incompressible analyses. The following steps are designed in the proposed numerical test:

1) Loop every row of the assemblaged coefficient matrix and select the rows satisfying $\sum_{j=1}^{N} K_{ij} = 0$, where N is the number of nodal unknowns and K_{ij} stands for the ijth element of the matrix;

2) Check whether the selected rows (assume N_d such rows) are diagonally dominant;

3) If the selected rows are not diagonally dominant, check whether or not the ratio $F_d = \sqrt{\sum_{i=1}^{N_d} f_i^2}$ approaches 1, as $\alpha \to 0$, where $f_i = |K_{ii}| / \sum_{j=1, j\neq i}^{N} |K_{ij}|$;

4) Normalize the coefficient matrix, for every $1 \leq i \leq N$, $K'_{ij} = K_{ij}/max_{j=1}^{N}(K_{ij})$, and calculate the maximum and minimum moduli of the eigenvalues of the normalized matrix; and

5) Select a proper finite value of diffusivity ($Pe^e \simeq O(1)$), and study the rate of convergence of the semi-norms defined in Eqs. (31) and (32) with a proper sequence of meshes (the finer meshes should contain the coarser meshes).

Step 1 is used to separate the boundary effects from the upwinding scheme, whereas *Steps* 2, 3 and 4 are designed to evaluate whether or not the given upwinding scheme is monotonic for the hyperbolic limit. It is known that with a finite diffusivity, at best, different upwinding schemes can converge as fast as the corresponding standard Galerkin formulation, provided a sufficiently refined mesh is taken. Therefore, *Step* 5 is introduced to evaluate the accuracy of the given upwinding scheme by comparing it with the standard Galerkin formulation.

In this paper, we study the SUPG, the Schneider and Raw, and the standard Galerkin methods for the solution of two generic two-dimensional examples illustrated in Fig. 2.

In the diagonal flow problem, the flow is uniform ($v_1 = 1.0$ and $v_2 = 1.0$) in the diagonal direction; while in the rotating cosine hill problem, the flow is rotational ($v_1 = -x_2$ and $v_2 = x_1$). Figures 3 and 4 give the typical results from the standard Galerkin, SUPG, and Schneider and Raw formulations. It is clearly shown that the SUPG formulation retains spatial oscillations, while the Schneider and Raw scheme performs perfectly well for the diagonal flow problem but introduces excessive diffusion in the rotating cosine hill problem. This crosswind diffusion, of course, decreases as the mesh is refined.

The results in Figs. 5 and 6 show the changes in the diagonal and off-diagonal ratio F_d. It is not surprising to find that in the Schneider and Raw scheme, $F_d = 1$ holds for all ranges of finite diffusivities, which means the solutions are smooth and monotonic. In addition, Figures 5 and 6 indicate that the standard Galerkin formulation gradually loses the diagonal dominance as the diffusivity approaches zero, and that the SUPG method, with $F_d \simeq 1$, should produce solutions between the two limit cases. Of course, the distribution of $1 - f_i$, corresponding to the nodal unknown θ_i, implies possible non-physical spatial oscillations along node i.

The rates of convergence of the 1-norm and 0-norm defined in Eqs. (31) and (32) are shown in Figs. 7 and 8. It is not difficult to infer that the SUPG method has higher accuracy than the Schneider and Raw scheme, although the latter provides smoother solutions.

Figures 9 and 10 give the results of the proposed normalized eigenvalue problem with different diffusivities, whereas Figures 11 and 12 show the standard eigenvalues of the coefficient matrix. Note that with the normalization of the coefficient matrix, the lowest and highest moduli of the eigenvalues of the SUPG method are between the results of the standard Galerkin formulation and the Schneider and Raw scheme; however, the standard eigenvalues do not exhibit such relationships. It is obvious that Figures 9 and 10 match well with Figures 5 and 6, and that the normalized eigenvalue test again measures the likelihood of non-physical spatial oscillations.

5 Conclusion

It is possible to improve upwinding schemes, although for convection-diffusion problems, it appears that there is always a trade-off between the order of accuracy and stability. We conclude that the numerical evaluation proposed in this paper may be of important value with respect to determining the advantages and disadvantages of any upwinding discretizations and may eventually help in the development of optimal upwinding schemes. The proposed cheaply computable and very effective numerical test compares with two extreme cases (with and without distorted elements): firstly, the monotonic classical upwinding approach for the hyperbolic limit, and secondly, the standard Galerkin formulation with a finite diffusivity.

Along with the diagonal and off-diagonal ratio convergence test, the proposed normalized eigenvalue problem will help to identify whether or not a given upwinding scheme is monotonic. The convergence of two proposed semi-norms can be used to evaluate the accuracy of given upwinding schemes. For the coefficient matrices derived from specific approaches such as wavelets and other interpolation functions, this numerical test is still applicable.

Acknowledgment

The author would like to thank the Institute of Paper Science and Technology and its member companies for their support. The author is also grateful to Professors K.J. Bathe and F. Bloom for their helpful suggestions.

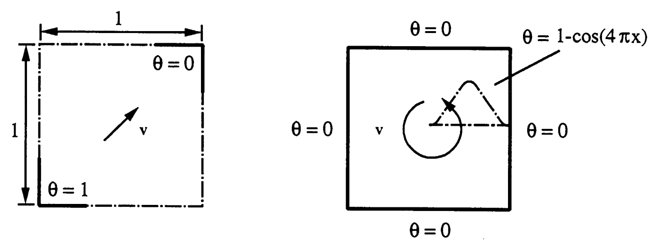

Figure 2: Two convection-dominated heat transfer problems.

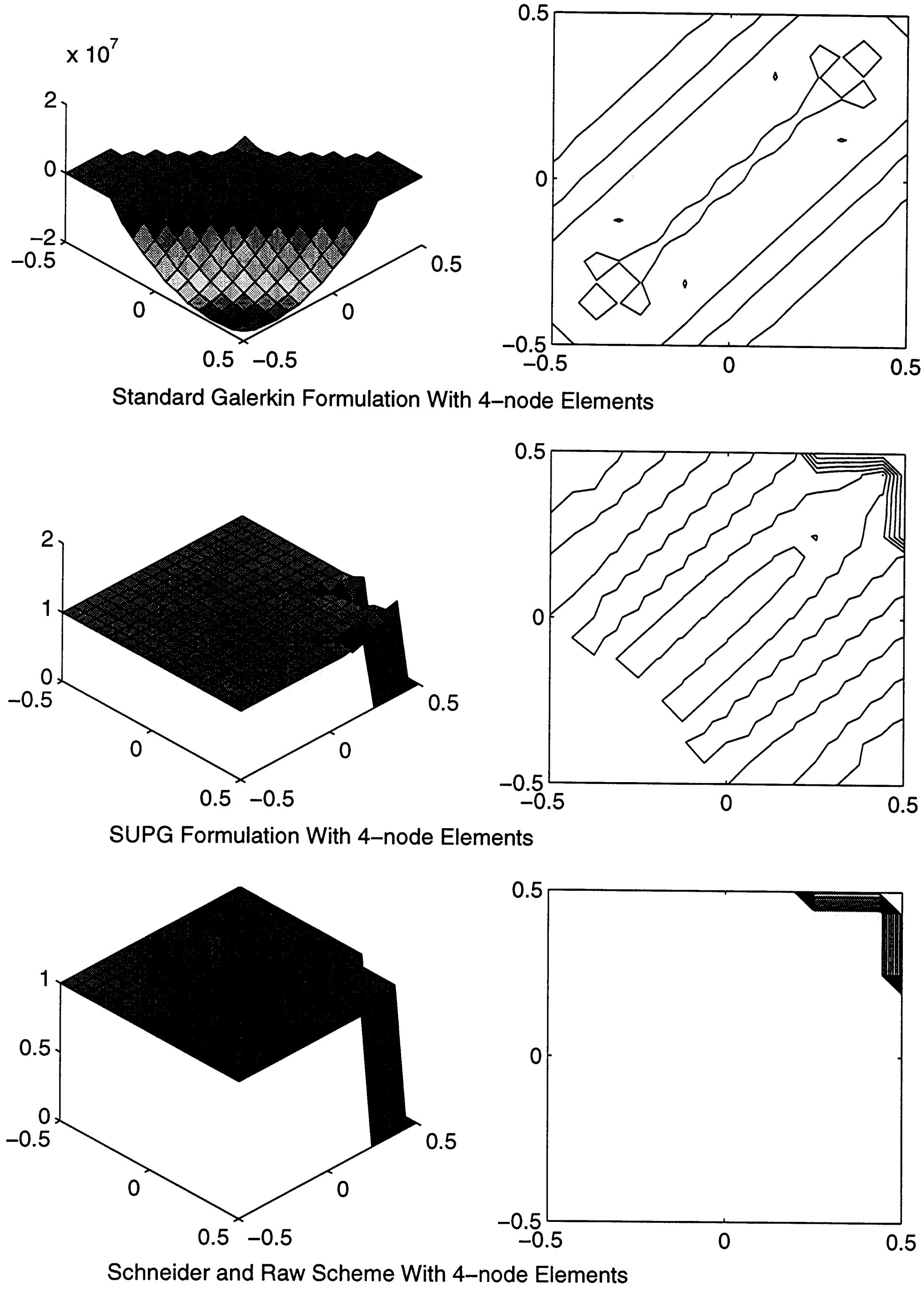

Figure 3: Solutions of the diagonal flow problem ($\alpha = 10^{-8}$).

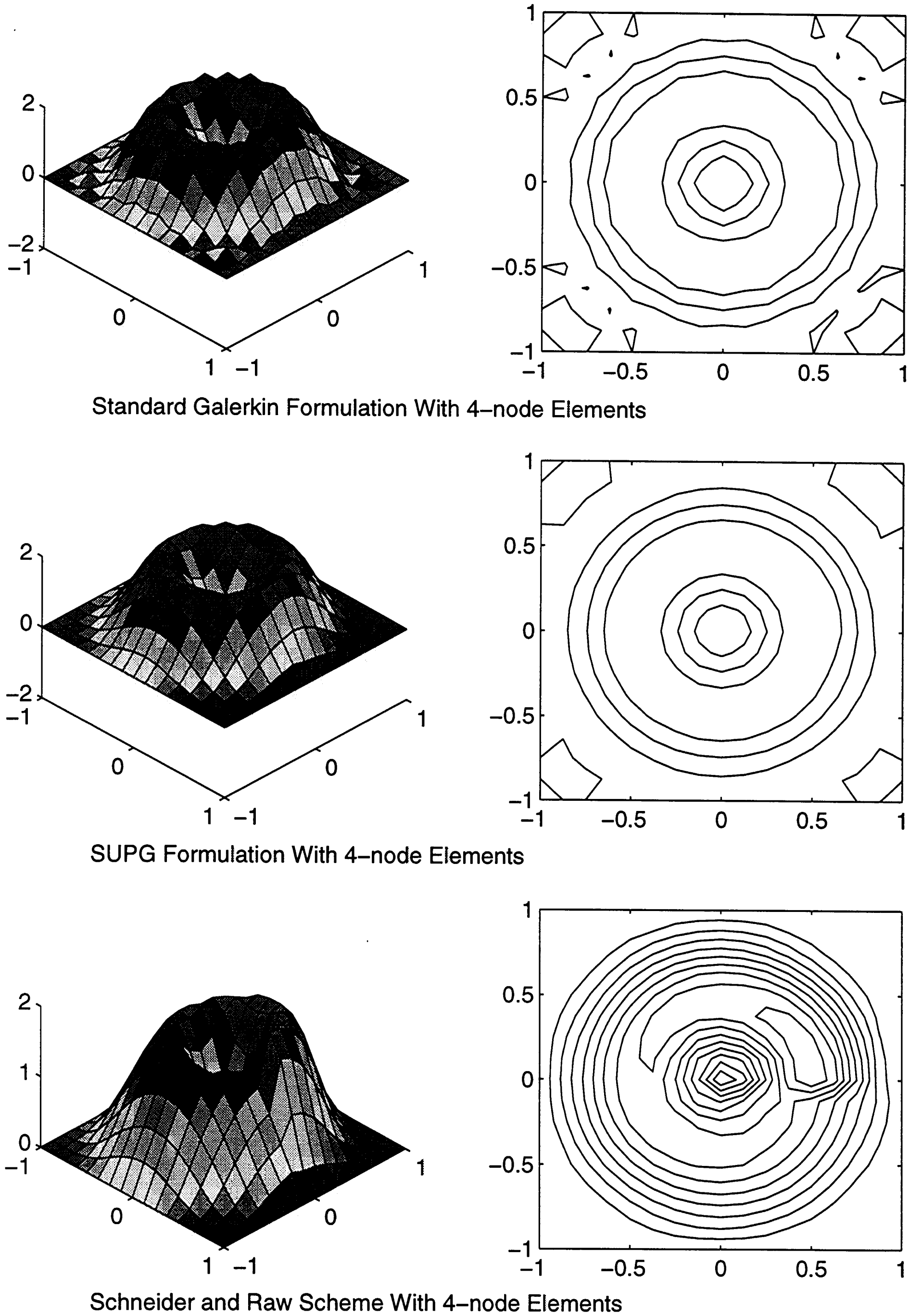

Figure 4: Solutions of the rotating cosine hill problem ($\alpha = 10^{-8}$).

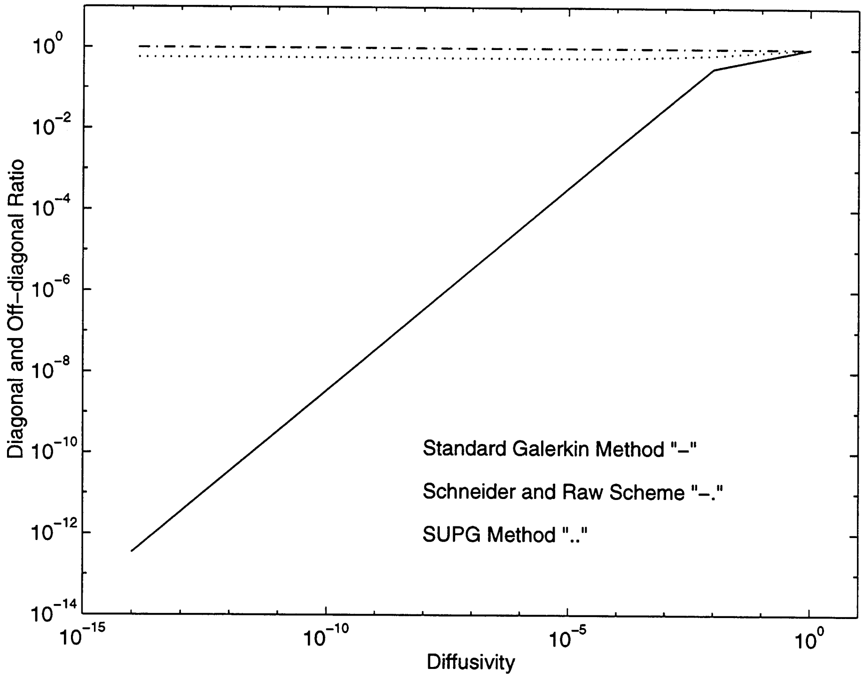

Figure 5: The numerical test results of diagonal and off-diagonal ratio of the diagonal flow problem.

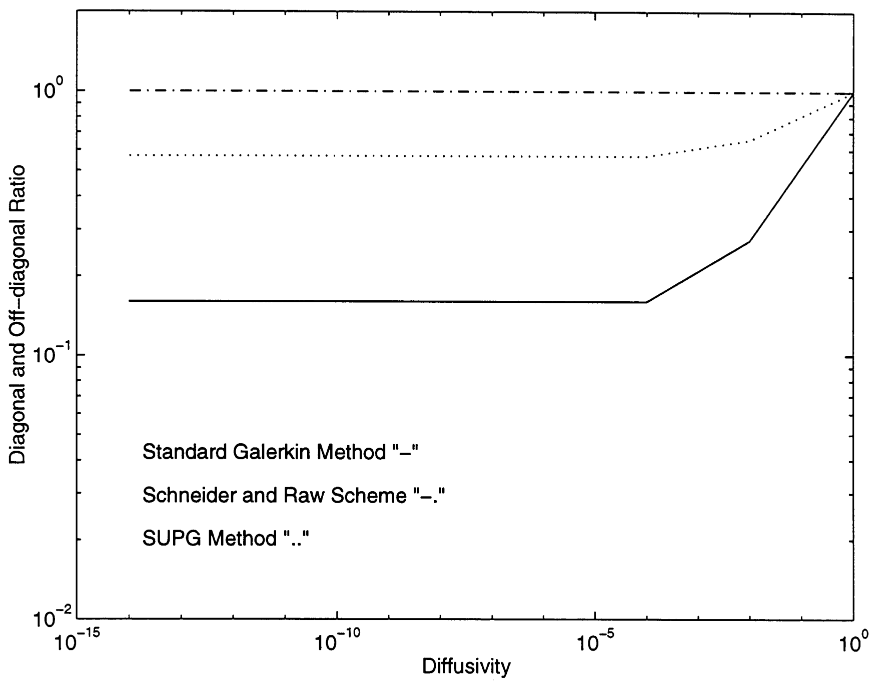

Figure 6: The numerical test results of diagonal and off-diagonal ratio of the rotating cosine hill problem.

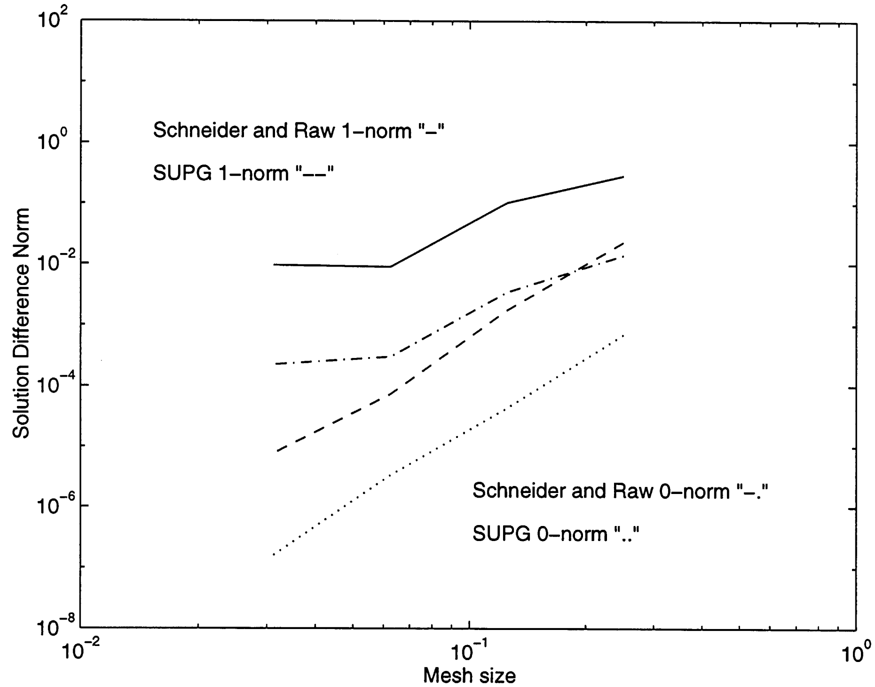

Figure 7: The proposed 1-norm and 0-norm convergence test of the diagonal flow problem ($\alpha = 0.2$).

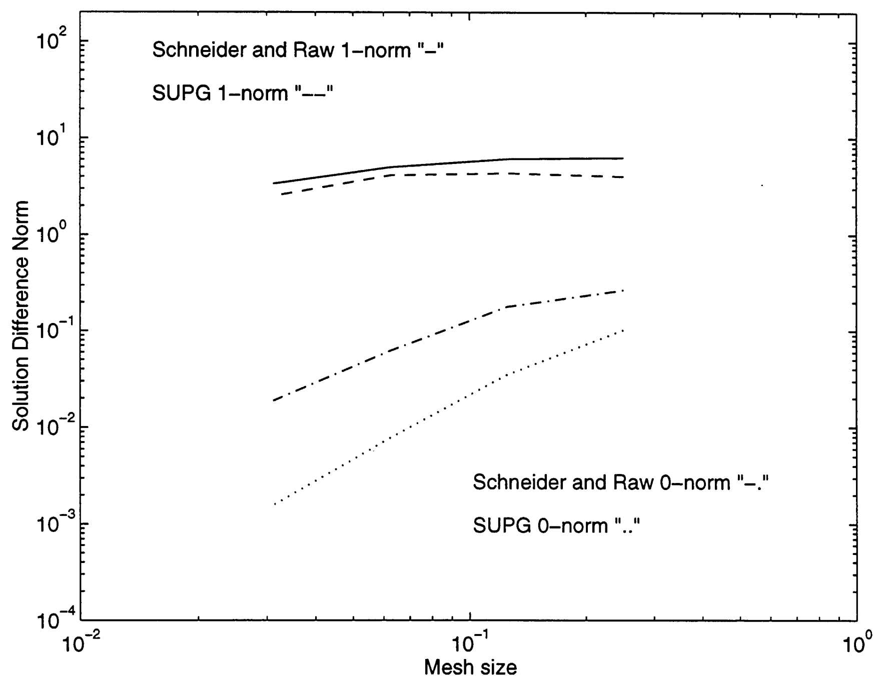

Figure 8: The proposed 1-norm and 0-norm convergence test of the rotating cosine hill problem ($\alpha = 0.005625$).

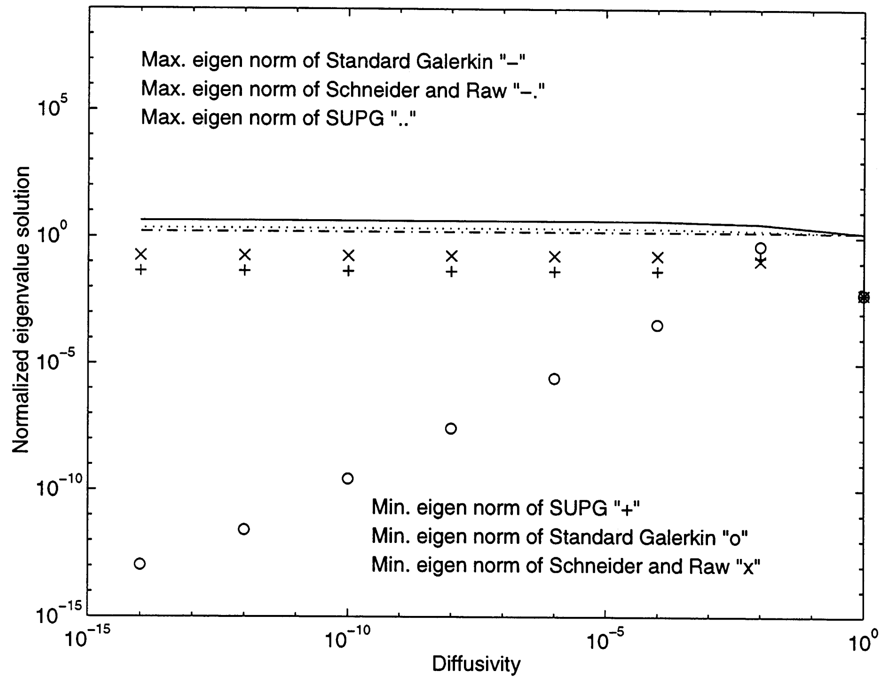

Figure 9: The results of the proposed normalized eigenvalue test of the diagonal flow problem.

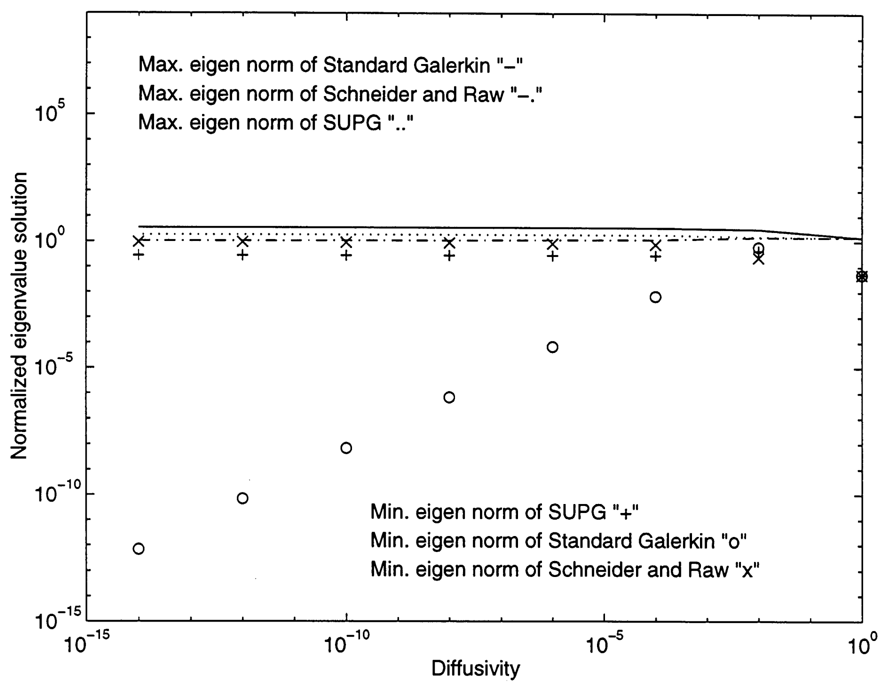

Figure 10: The results of the proposed normalized eigenvalue test of the rotating cosine hill problem.

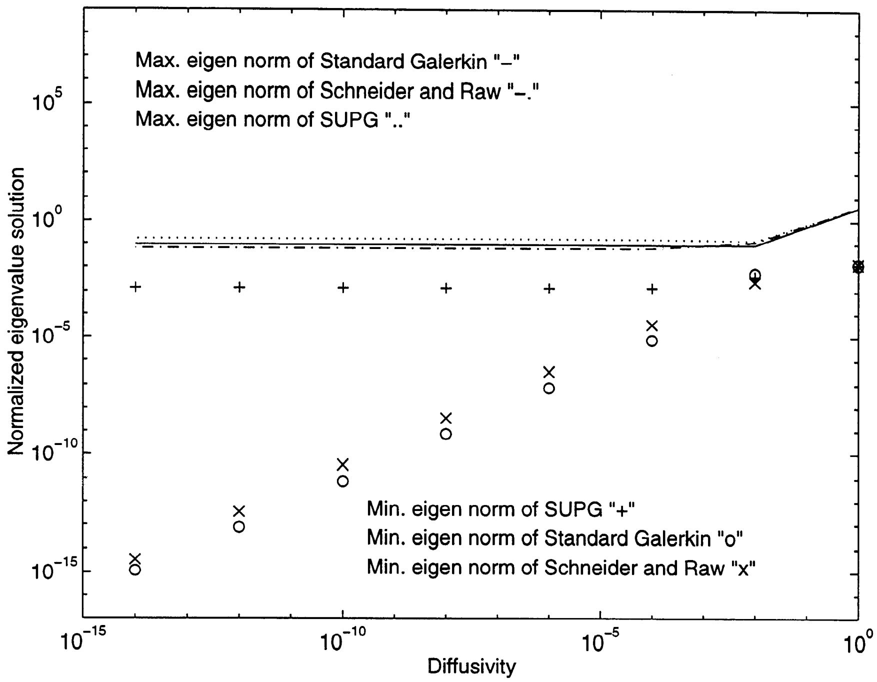

Figure 11: The results of the standard eigenvalue test of the diagonal flow problem.

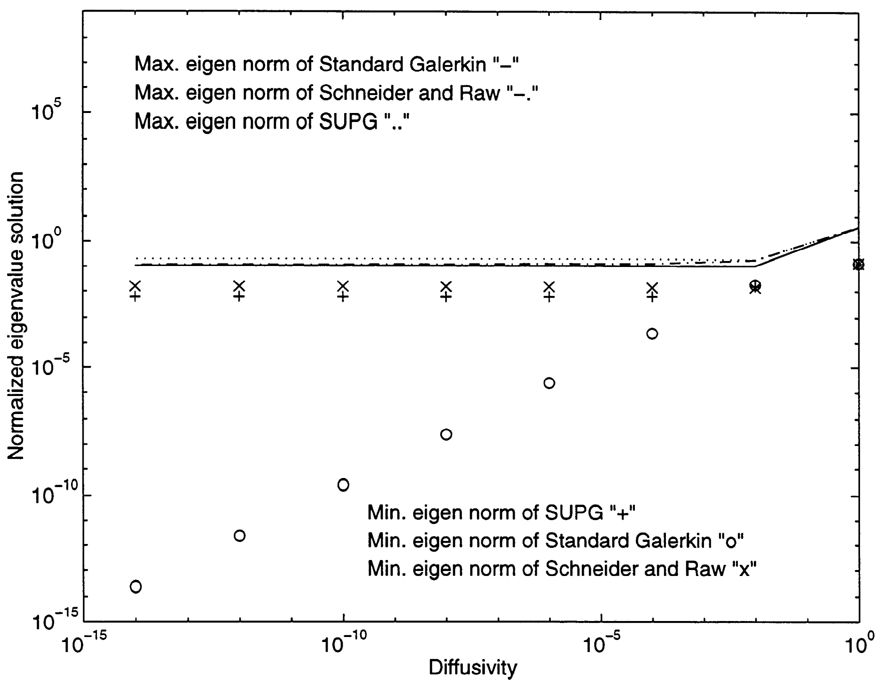

Figure 12: The results of the standard eigenvalue test of the rotating cosine hill problem.

Appendix A

Since $\delta\theta_h$ is zero on Γ and $\nabla \cdot \mathbf{v} = 0$ in Ω, it follows that

$$\begin{aligned}
(\mathbf{v} \cdot \nabla \delta\theta_h, \delta\theta_h) &= \int_\Omega \delta\theta_h \mathbf{v} \cdot \nabla \delta\theta_h d\Omega \\
&= \int_\Omega \mathbf{v} \cdot \nabla (\frac{\delta\theta_h^2}{2}) d\Omega \\
&= \int_\Omega \{\nabla \cdot (\mathbf{v}\frac{\delta\theta_h^2}{2}) - \frac{\delta\theta_h^2}{2} \nabla \cdot \mathbf{v}\} d\Omega \\
&= \int_\Gamma \frac{\delta\theta_h^2}{2} \mathbf{v} \cdot \mathbf{n} d\Gamma - \int_\Omega \frac{\delta\theta_h^2}{2} \nabla \cdot \mathbf{v} d\Omega \\
&= 0.
\end{aligned} \tag{A.1}$$

Therefore, Eq. (23) holds based on the semi-norm definition

$$|\theta|_{m,\Omega} = (\sum_{|k|=m} \int_\Omega \left|\partial^k \theta\right|^2 d\Omega)^{1/2}. \tag{A.2}$$

Appendix B

Substituting $\delta\theta_h = \theta_h - I_h\theta$ into Eqs. (22) and (23), we have

$$(\mathbf{v} \cdot \nabla(\theta - \theta_h), \theta_h - I_h\theta) + (\alpha \nabla(\theta - \theta_h), \nabla(\theta_h - I_h\theta)) = 0 \tag{B.1}$$

and

$$\alpha|\theta_h - I_h\theta|_{1,\Omega}^2 = (\alpha\nabla(\theta_h - I_h\theta), \nabla(\theta_h - I_h\theta)) - (\mathbf{v} \cdot \nabla(\theta_h - I_h\theta), \theta_h - I_h\theta). \tag{B.2}$$

The first term in Eq. $(B.1)$ can be rewritten as follows:

$$
\begin{aligned}
(\mathbf{v}\cdot\nabla(\theta-\theta_h),\theta_h-I_h\theta) &= \int_\Omega(\theta_h-I_h\theta)\mathbf{v}\cdot\nabla(\theta-\theta_h)d\Omega \\
&= \int_\Omega \nabla\cdot(\mathbf{v}(\theta-\theta_h)(\theta_h-I_h\theta))d\Omega \\
&- \int_\Omega(\theta_h-I_h\theta)(\theta-\theta_h)\nabla\cdot\mathbf{v}d\Omega \\
&- \int_\Omega(\theta-\theta_h)\mathbf{v}\cdot\nabla(\theta_h-I_h\theta)d\Omega \\
&= -(\mathbf{v}\cdot\nabla(\theta_h-I_h\theta),\theta-\theta_h)
\end{aligned}
\tag{B.3}
$$

and therefore, combining Eqs. $(B.1)$, $(B.2)$ and $(B.3)$ yields

$$
\alpha|\theta_h-I_h\theta|^2_{1,\Omega}=(\alpha\nabla(\theta-I_h\theta),\nabla(\theta_h-I_h\theta))-(\mathbf{v}\cdot\nabla(\theta_h-I_h\theta),\theta-I_h\theta). \tag{B.4}
$$

Then, the inequality (26) can be derived by using the Cauchy-Schwarz inequality

$$
\int_\Omega fgd\Omega \leq (\int_\Omega f^2d\Omega)^{1/2}(\int_\Omega f^2d\Omega)^{1/2}. \tag{B.5}
$$

References

[1] W.McC. Siebert. *Circuits, Signals, and Systems.* The MIT Press and McGraw-Hill Book Company, fourth edition, 1989.

[2] R. Courant, E. Isaacson, and M. Rees. On the solution of nonlinear hyperbolic differential equations by finite differences. *Communications on Pure and Applied Mathematics*, 5:243–255, 1952.

[3] I. Christie, D.F. Griffiths, A.R. Mitchell, and O.C. Zienkiewicz. Finite element methods for second order differential equations with significant first derivatives. *International Journal for Numerical Methods in Engineering*, 10:1389–1396, 1976.

[4] J.C. Heinrich, P.S. Huyakorn, O.C. Zienkiewicz, and A.R. Mitchell. An 'upwind' finite element scheme for two-dimensional convective transport equation. *International Journal for Numerical Methods in Engineering*, 11:131–143, 1977.

[5] J.C. Heinrich and O.C. Zienkiewicz. Quadratic finite element schemes for two-dimensional convective-transport problems. *International Journal for Numerical Methods in Engineering*, 11:1831–1844, 1977.

[6] G.E. Schneider and M.J. Raw. A skewed, positive influence coefficient upwinding procedure for control-volume-based finite-element convection-diffusion computation. *Numerical Heat Transfer*, 9:1–26, 1986.

[7] K.J. Bathe, H. Zhang, and M.H. Wang. Finite element analysis of incompressible and compressible fluid flows with free surfaces and structural interactions. *Computers & Structures*, 56(2/3):193–213, 1995.

[8] J. Donea. A Taylor-Galerkin method for convective transport problems. *International Journal for Numerical Methods in Engineering*, 20:101–119, 1984.

[9] T.J.R. Hughes, L.P. Franca, and G.M. Hulbert. A new finite element formulation for computational fluid dynamics: VIII. The Galerkin/least squares method for advective-diffusive equations. *Computer Methods in Applied Mechanics and Engineering*, 73:173–189, 1989.

[10] F. Brezzi, M. Bristeau, L.P. Franca, M. Mallet, and G. Rogé. A relationship between stabilized finite element methods and the Galerkin method with bubble functions. *Computer Methods in Applied Mechanics and Engineering*, 96:117–129, 1992.

[11] A.N. Brooks and T.J.R. Hughes. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. *Computer Methods in Applied Mechanics and Engineering*, 32:199–259, 1982.

[12] C. Johnson, U. Nävert, and J. Pitkäranta. Finite element methods for linear hyperbolic problems. *Computer Methods in Applied Mechanics and Engineering*, 45:285–312, 1984.

[13] T.J.R. Hughes and M. Mallet. A new finite element formulation for computational fluid dynamics:g III. The generalized streamline operator for multidimensional advective-diffusive systems. *Computer Methods in Applied Mechanics and Engineering*, 58:305–328, 1986.

[14] D.B. Spalding. A novel finite difference formulation for differential expressions involving both first and second derivatives. *International Journal for Numerical Methods in Engineering*, 4:551–559, 1972.

[15] S.V. Patankar. *Numerical Heat Transfer and Fluid Flow.* Hemisphere Publishing, 1980.

[16] W.J. Minkowycz, E.M. Sparrow, G.E. Schneider, and R.H. Pletcher. *Handbook of Numerical Heat Transfer.* John Wiley & Sons, 1988.

[17] F. Brezzi and M. Fortin. *Mixed and Hybrid Finite Element Methods.* Springer-Verlag, 1991.

[18] K.J. Bathe. *Finite Element Procedures.* Prentice Hall, Englewood Cliffs, N.J., 1996.

[19] P.G. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, 1978.