Table of Contents

About This Document

This resource contains curriculum for the distance education version of a course offered at the Georgia Institute of Technology, Math 1502, in Fall 2014. This distance education course explored linear algebra, infinite series, and differential equation concepts during lectures and recitations. Recitations are synchronous sessions that offer students an opportunity to apply and review course concepts, which they have been exposed to in lectures.

Contained in this curriculum are materials for 26 recitations, available in PDF and presentation slide formats. The slide format is offered for teaching assistants to import directly into web-conferencing software. Slides contain activities that students would solve during recitations. The associated notes contain solutions to corresponding activities and are available in PDF format.

An similar version of this work, that corresponds to activities conducted in the Fall 2013 semester, is available through SMARTech at http://smartech.gatech.edu/handle/1853/51343

Copyright

For Further Information

Questions regarding this document can be directed to Greg Mayer (gsmayer@gmail.com), who would be happy to hear your suggestions on how to improve this document.

List of Topics

The following table presents a list of topics that were explored in the recitation activities. Numbers in brackets correspond to section numbers in the course textbook (Lay, D., Linear Algebra and its Applications, Fourth Edition).

Week	Recitation	Topics
1	1	Introduction to Math 1502, logistical matters, icebreaker
	2	Improper integrals, Numerical integration (8.7)
2	3	Improper integrals: comparison test, review of techniques of integration
	4	Separable & Linear DEs
3	5	Test for Divergence
	6	Infinite Series, Differential Equations
4	7	Quiz 1 Review
	8	No Recitation - Quiz 1
5	9	Power Series, Radius/Interval of Convergence and Absolute Convergence
	10	Taylor Polynomials and Series
6	11	l’Hospital’s Rule
	12	Lines, Planes, Dot Products
7	13	Quiz 2 Review
	14	No Recitation - Quiz 2
8	15	Solving linear systems of equations
	16	Span, Linear Dependence
9	17	No Recitation - Fall Break
	18	Span, Linear Dependence, Linear Transforms
10	19	Quiz 3 Review

	20	No Recitation - Quiz 3
11	21	Matrix Inverses, LU Decomposition
	22	Column Space and Null Space, LU Decomposition
12	23	Column and Null Space
	24	Determinants (3.1, 3.2), Diagonalization (5.3)
13	25	Quiz 4 Review
	26	No Recitation - Quiz 4
14	27	Diagonalization (5.3), Orthogonality (6.1, 6.2)
	28	Orthogonality (6.1, 6.2, 6.3)
15	29	Orthogonality (6.1 to 6.5)
	30	No Recitation - Thanksgiving
16	31	QR Decomposition, Orthogonality Review (6.1 to 6.5)
	32	Final Exam Review

Welcome to Your Distance Calculus Recitation!

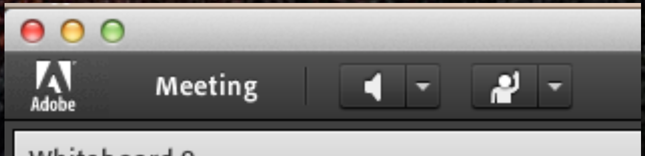
We'll get started at 8:05.

While we are waiting, see if you can use the chat window (bottom right) to join the discussion.

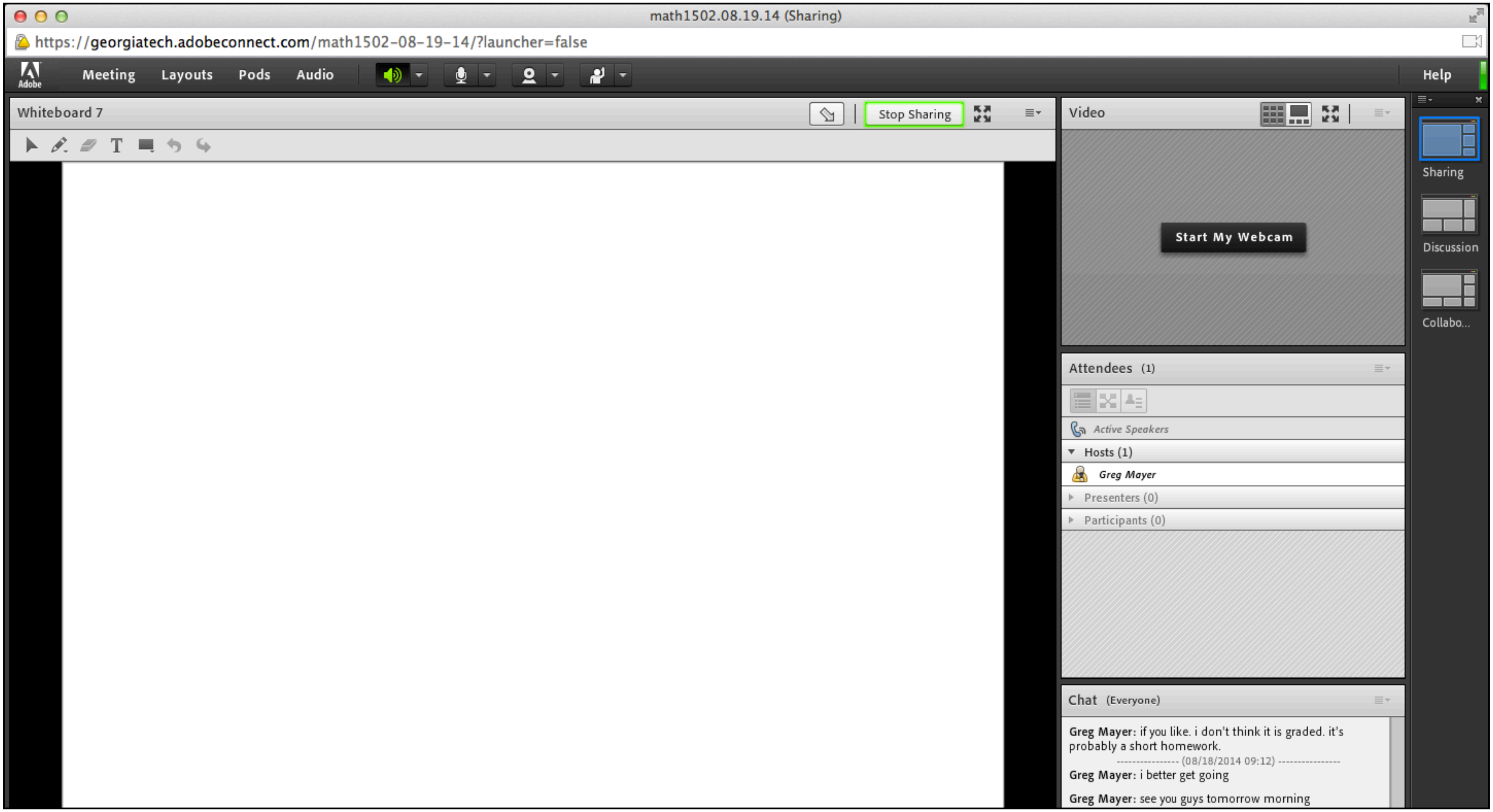
Today:

- Introduction to Adobe Connect
- What are Recitations?
- Icebreaker
- Numerical integration (8.7)

If you can't hear the TA, click the speaker icon



Adobe Connect



https://georgiatech.adobeconnect.com/math1502-08-19-14/?launcher=false

Greg Mayer: if you like. i don't think it is graded. it's probably a short homework.

---------------- (08/18/2014 09:12) ----------------

Greg Mayer: i better get going

Greg Mayer: see you guys tomorrow morning

Microphones, Webcams, Tablets

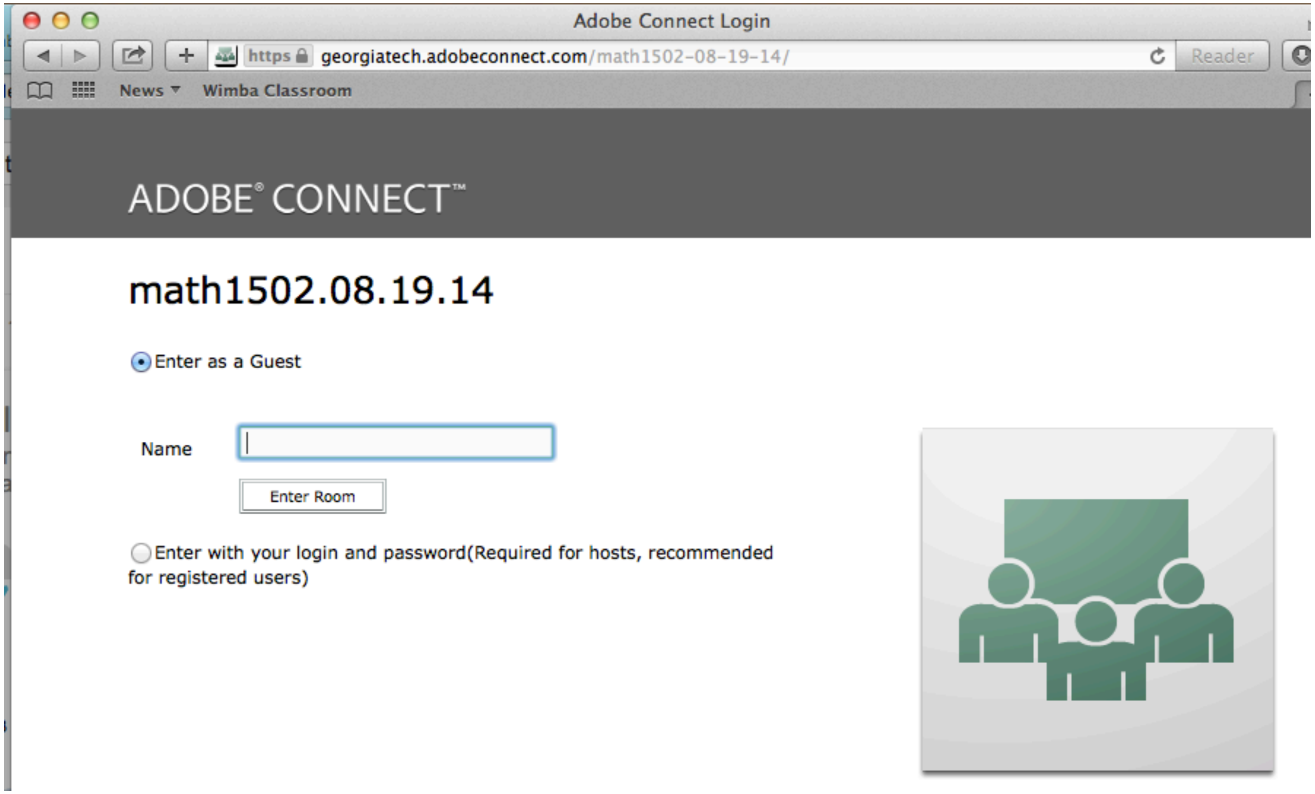
We can loan you a wacom bamboo tablet, if you'd like to borrow one please send me an email.

If you have a mic or a webcam, you are welcome to use them.

The Whiteboard

- You need to be a presenter to write on the board
- Only a host can chanage permission levels
- All writing on the board is anonymous
- Please respect other students taking this course (and your TA): you are responsible for your learning

Logging Into Adobe Connect for Recitations



math1502.08.19.14

● Enter as a Guest

Name

Enter Room

○ Enter with your login and password(Required for hosts, recommended for registered users)

Thursday’s recitation is at https://georgiatech.adobeconnect.com/math1502-08-19-14

Adobe Connect Technical Problems?

You can:

- reload your browser
- log in/out
- use a different web browser
- reboot
- get help from another student and/or your TA

I strongly recommend that, if possible, you use a wired connection.

What are Recitations?

- Our goal: help students understand course material so that they can complete assignments and prepare for quizzes and exams.
- please bring questions about the homework or lectures

Our Section in a Nutshell

- students in Math 1502 are divided into many sections
- ours is the only section that
  - doesn't have on-campus students
  - uses Connect for recitations
- Why Adobe Connect?
  - it's cheaper
  - you can interact with students at other schools

Tablets

- Students in our section can borrow tablets.
- If you already have a tablet you want to use, that's ok
- Equipment need to be returned to your facilitator
- If you don’t have a tablet and want to borrow one, email me
- Tablets (should) come with a CD, use it to configure tablet settings

Course Websites

- Recordings of recitations and lectures: tegrity.gatech.edu
- Discussion forum: piazza.com
- Live lectures: gtcourses.gatech.edu
- Textbook and homework: www.mymathlab.com

- First homework due ________________

Grading Weights

	Weight (%)
Homework	10
Final	25
Quizzes	60
Pop Quizzes	5
Total	100

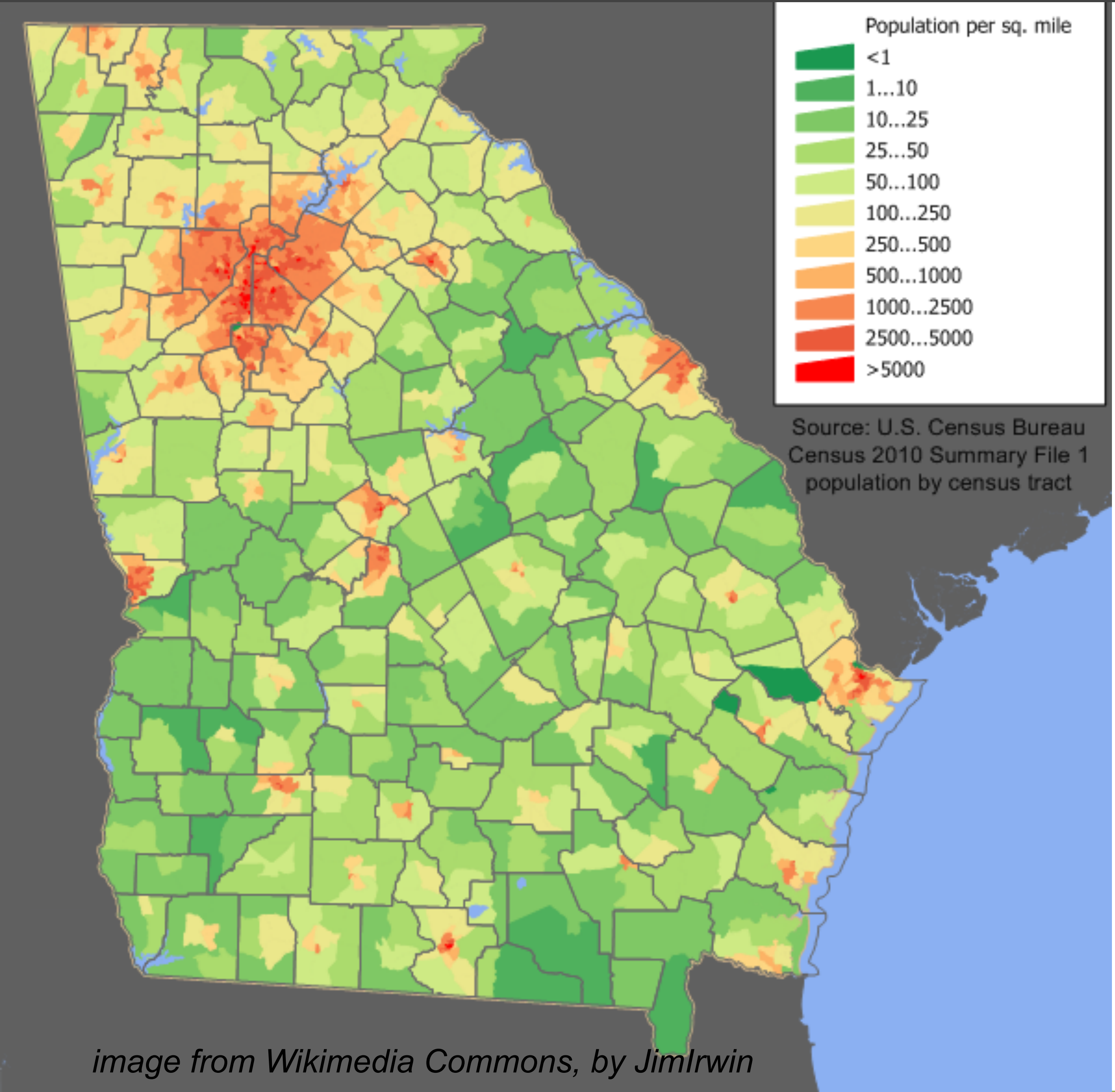
Grades will be made available through T-Square

Your TA: Greg

- email: greg.mayer@gatech.edu
- Text me or call me at: __________
- Skype: greg.s.mayer

- Canadian, eh
- alors mon français est tres mauvais
- moved to the US ~2 years ago
- post-doctoral fellow at GT

Icebreaker

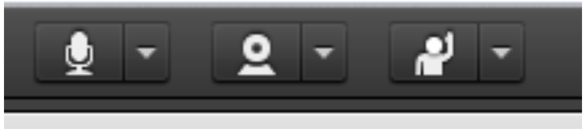


Source: U.S. Census Bureau
Census 2010 Summary File 1
population by census tract

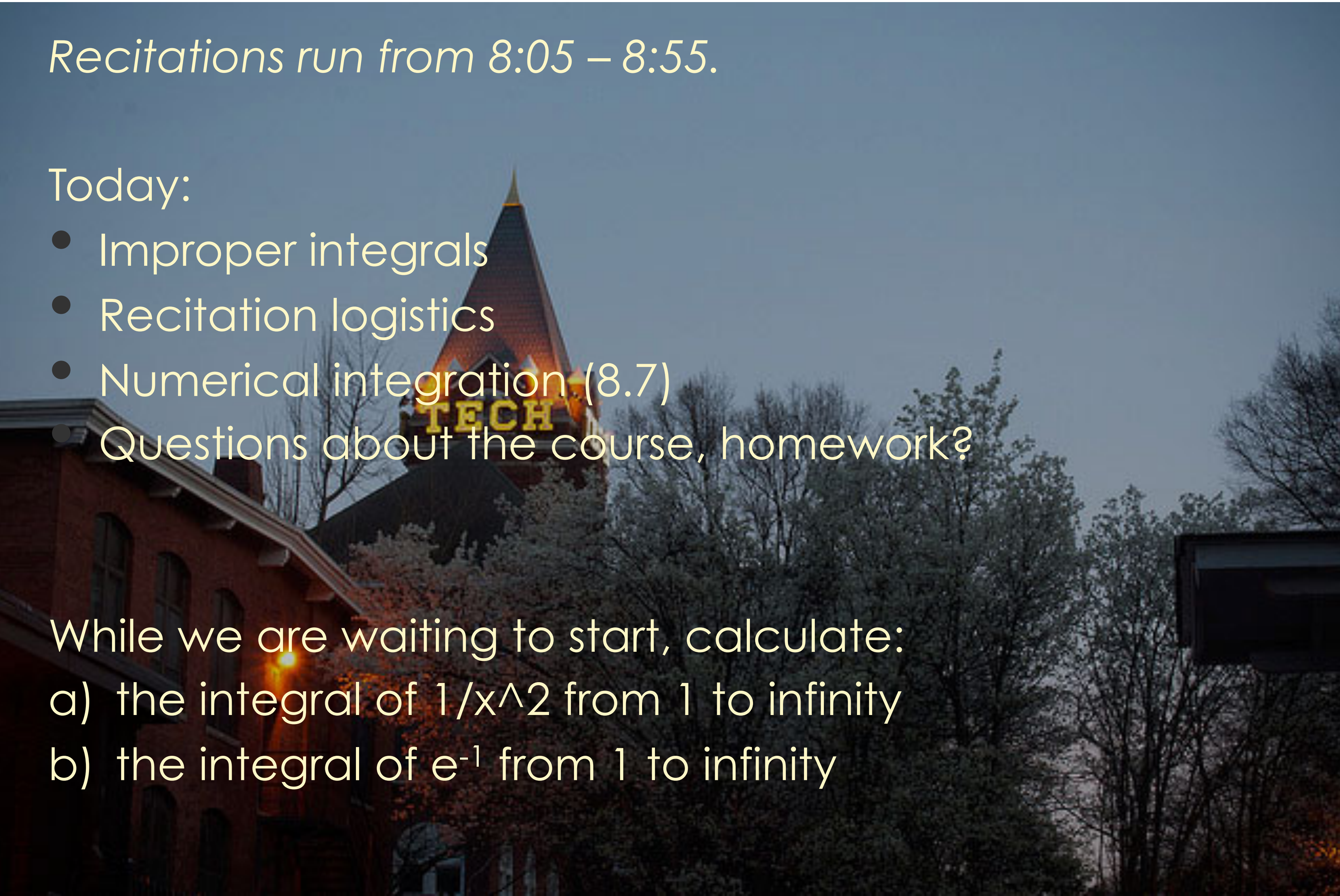
image from Wikimedia Commons, by JimIrwin

Everyone:

- type/say your name,
- one thing about yourself,
- place a dot on the map that approximates your current location.



Recitation 02



Recitations run from 8:05 – 8:55.

Today:

- Improper integrals
- Recitation logistics
- Numerical integration (8.7)
- Questions about the course, homework?

While we are waiting to start, calculate:

a) the integral of 1/x^2 from 1 to infinity

b) the integral of e^{-1} from 1 to infinity

Improper Integrals

$$\int_{1}^{\infty} \frac{1}{x^2} dx$$

$$\int_{1}^{\infty} e^{-x}\, dx$$

Logistics

HW1
send me questions: greg.mayer@gatech.edu, or skype _________

Chat Pod

- You can send private messages to students
- If we have a Q&A pod, do we need a chat pod?
- I'd like group discussion in chat pod to be:
    i. positive
    ii. respectful of others

Are there any other conditions that you would like to add?

Example: Integrate 1/x from 1 to 2

a) What is the exact answer?

b) Set up but don’t evaluate an expression for the area using Simpson’s Rule.

Example: Integrate 1/x from 1 to 2

c) Find the number of subintervals required for four digit accuracy using Simpson’s rule.

Recitation 03

Today:

1. Improper integrals: comparison test
2. A few announcements
3. Improper integrals: techniques of integration

Use the comparison test to determine whether the following integral converges.

$$\int_1^\infty \frac{1}{\sqrt{1+x^2}}\,dx$$

Announcements

HW2

- due tomorrow
- send me questions:

Invitation to Participate in a Study

- you're receiving snail mail soon!
- please review forms with your parents and send them back

Online Survey

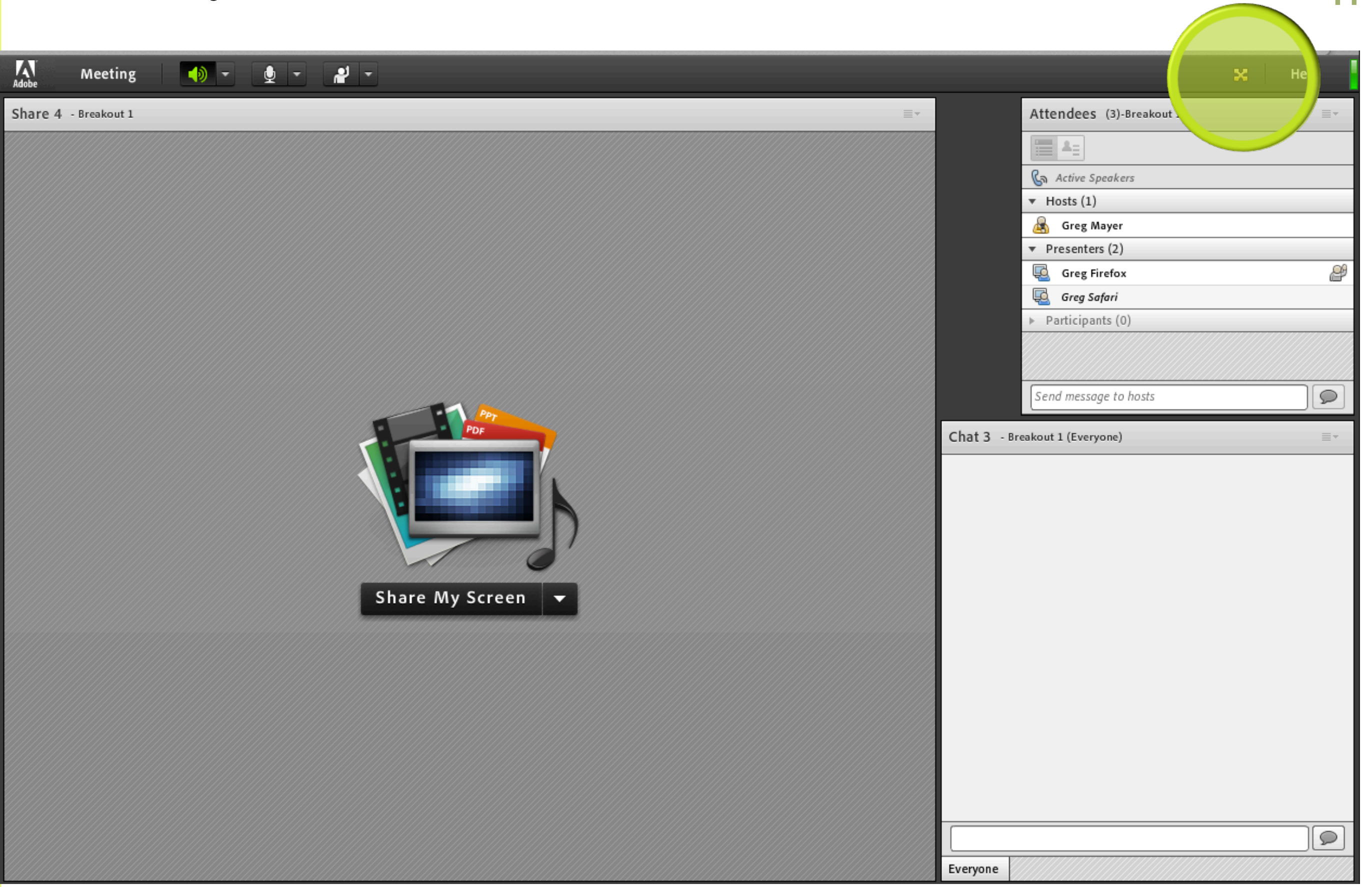
- anonymous!
- available at:
- please provide feedback on web cams, collaboration, the chat pod, etc

Chat Pod

- I'd like group discussion in chat pod to be:
    - i. positive
    - ii. respectful of others

Group Work

when you are in a breakout room:



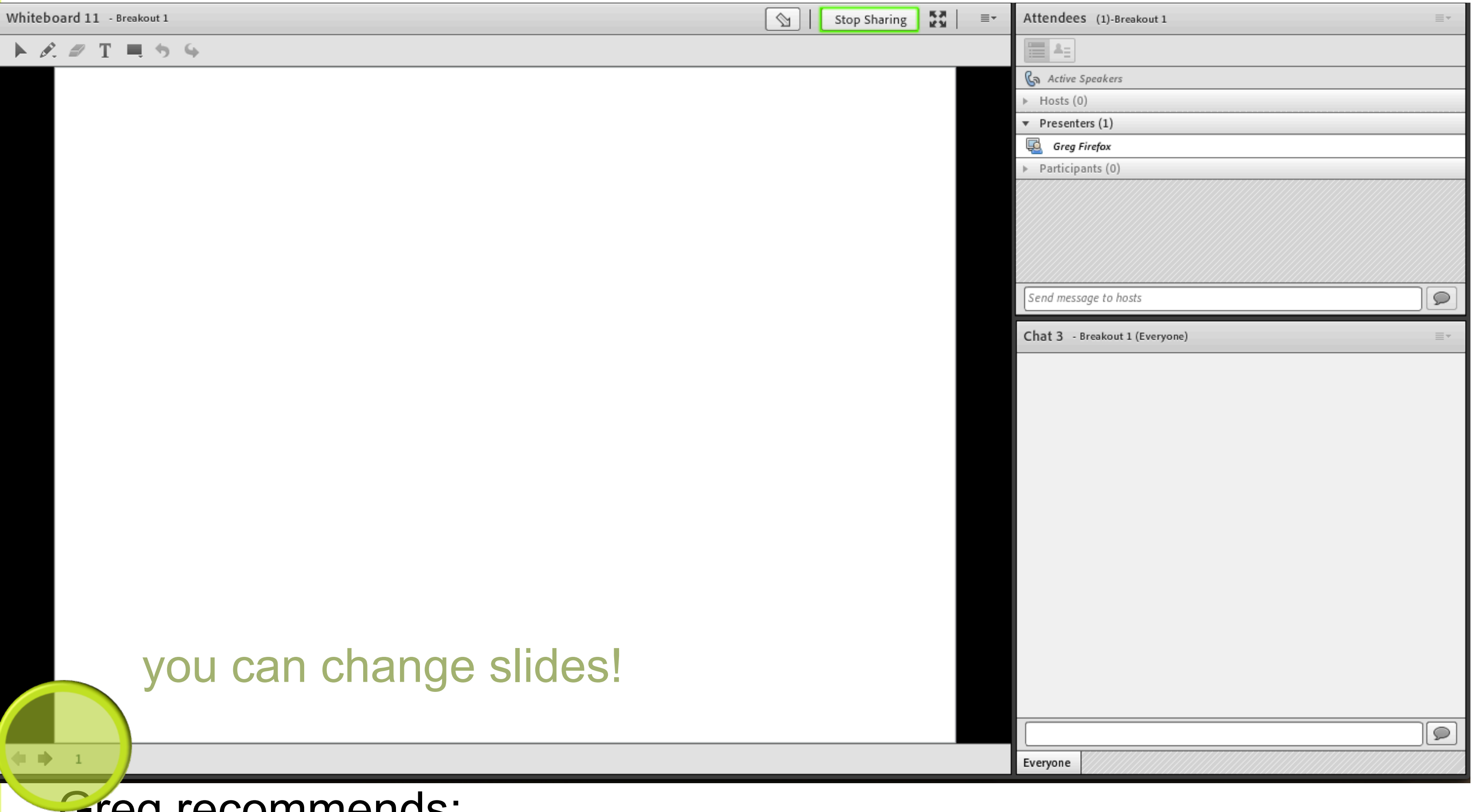
message TA

Greg recommends:

1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5

Group Work

when you are in a breakout room:



you can change slides!

Greg recommends:

1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5

Group Work Questions

Evaluate the following integrals.

4) $$\int_{5}^{\infty} \frac{1}{x^2+25}\,dx$$

5) $$\int_{0}^{144} \frac{1}{\sqrt{144-x}}\,dx$$

6) $$\int_{0}^{12} \frac{1}{\sqrt{144-x^2}}\,dx$$

7) $$\int_{0}^{\infty} \frac{1}{x^2+7x+6}\,dx$$

8) $$\int_{-\infty}^{\infty} \frac{Ax}{\left(x^2+B\right)^{12}}\,dx$$

9) $$\int x^3 \ln x\,dx$$

Recitation 04

Today:

1. 1st order linear DE
2. A few announcements
3. Group Work: Seperable & Linear DEs

Consider:

$$xy' - y = 2x\ln x$$

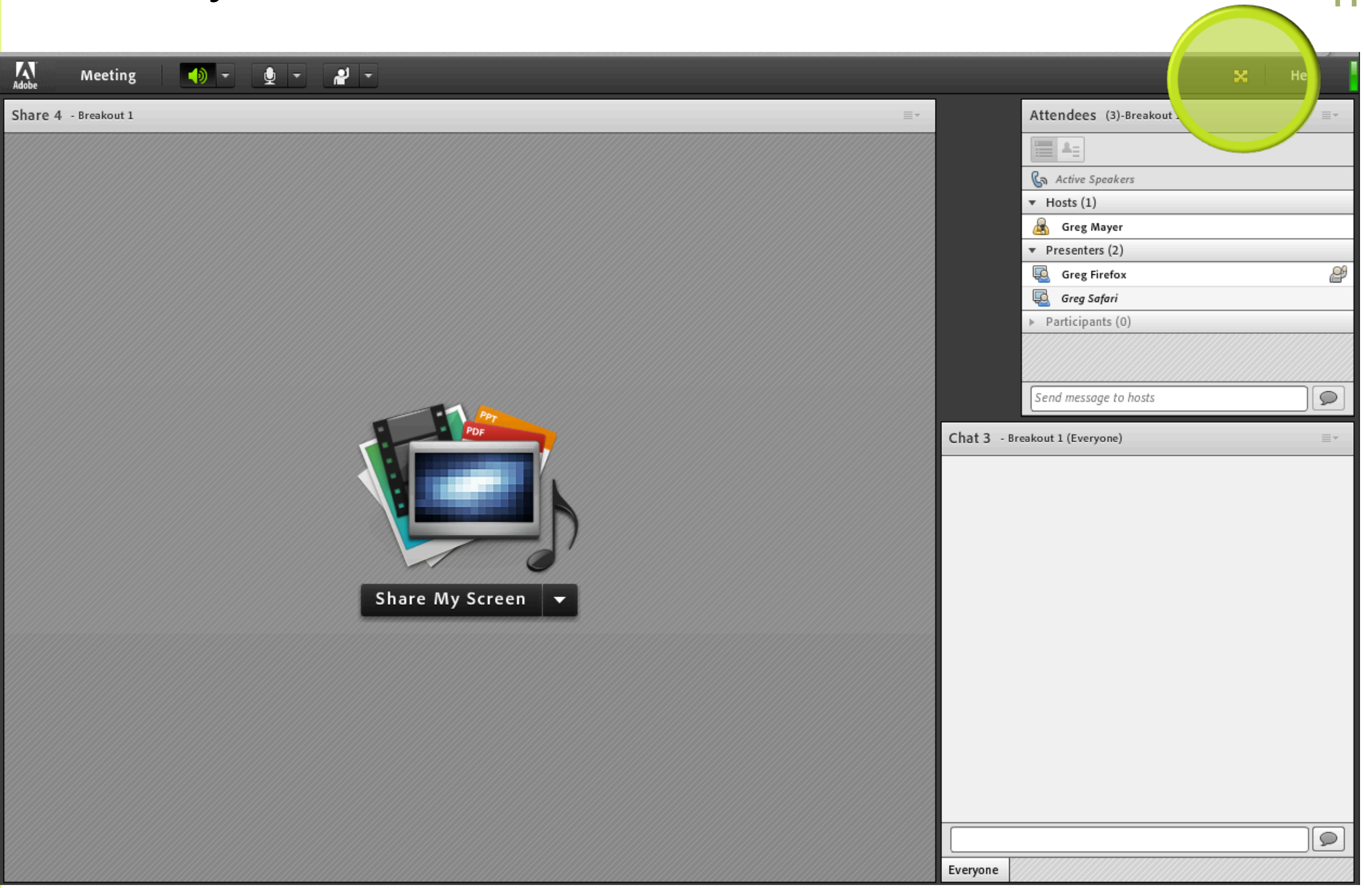
a) Is the DE seperable?
b) Is the DE 1st order linear in y(t)?
c) What is the integrating factor?

Solve the differential equation:

$$xy' - y = 2x\ln x$$

Group Work

when you are in a breakout room:



message TA

Greg recommends:

1) saying hello :D
2) decide who will activate whiteboard/document
3) discuss which question you want to start on

Group Work Questions

For each of the following DE’s,

a) State whether the DE is seperable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

$$1)\ \ xy' + 3y = 3 - \frac{4}{x}$$

$$2)\ \ xy' + y = (1 + x)e^{x}$$

$$3)\ \ \frac{1}{x}y' = ye^{x^2} + 2\sqrt{y}e^{x^2}$$

$$4)\ \ (\sin t)y' + (\cos t)y = \tan t$$

$$5)\ \ \cos(y) + (1 + e^{-x})(\sin y)y' = 0$$

Group Work Questions

For each of the following DE's,

a) State whether the DE is seperable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

$$1)\quad xy' + 3y = 3 - \frac{4}{x}$$

Group Work Questions

For each of the following DE's,

a) State whether the DE is seperable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

$$2)\quad xy' + y = (1+x)e^{x}$$

Group Work Questions

For each of the following DE's,

a) State whether the DE is seperable.
b) State whether the DE is 1^{st} order linear in y.
c) Solve the DE.

$$3)\quad \frac{1}{x}y' = ye^{x^2} + 2\sqrt{y}e^{x^2}$$

Group Work Questions

For each of the following DE's,

a) State whether the DE is seperable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

4) $(\sin t)y' + (\cos t)y = \tan t$

Group Work Questions

For each of the following DE's,

a) State whether the DE is seperable.
b) State whether the DE is 1st order linear in y.
c) Solve the DE.

5) $\cos(y) + (1 + e^{-x})(\sin y)y' = 0$

Recitation 05

Today:

1. Geometric Series
2. Group Work: Infinite Series

The sum of a geometric series is: $\sum_{k=1}^{\infty} ar^{k-1}$ =

The geometric series is convergent iff ______________ .

Express 1.7979797979 … as a rational number.

Express 1.7979797979 … as a rational number.

1.79797979797979

Input interpretation:

1.79797979797979

Rational approximation:

$$\frac{178}{99} = 1 + \frac{79}{99}$$

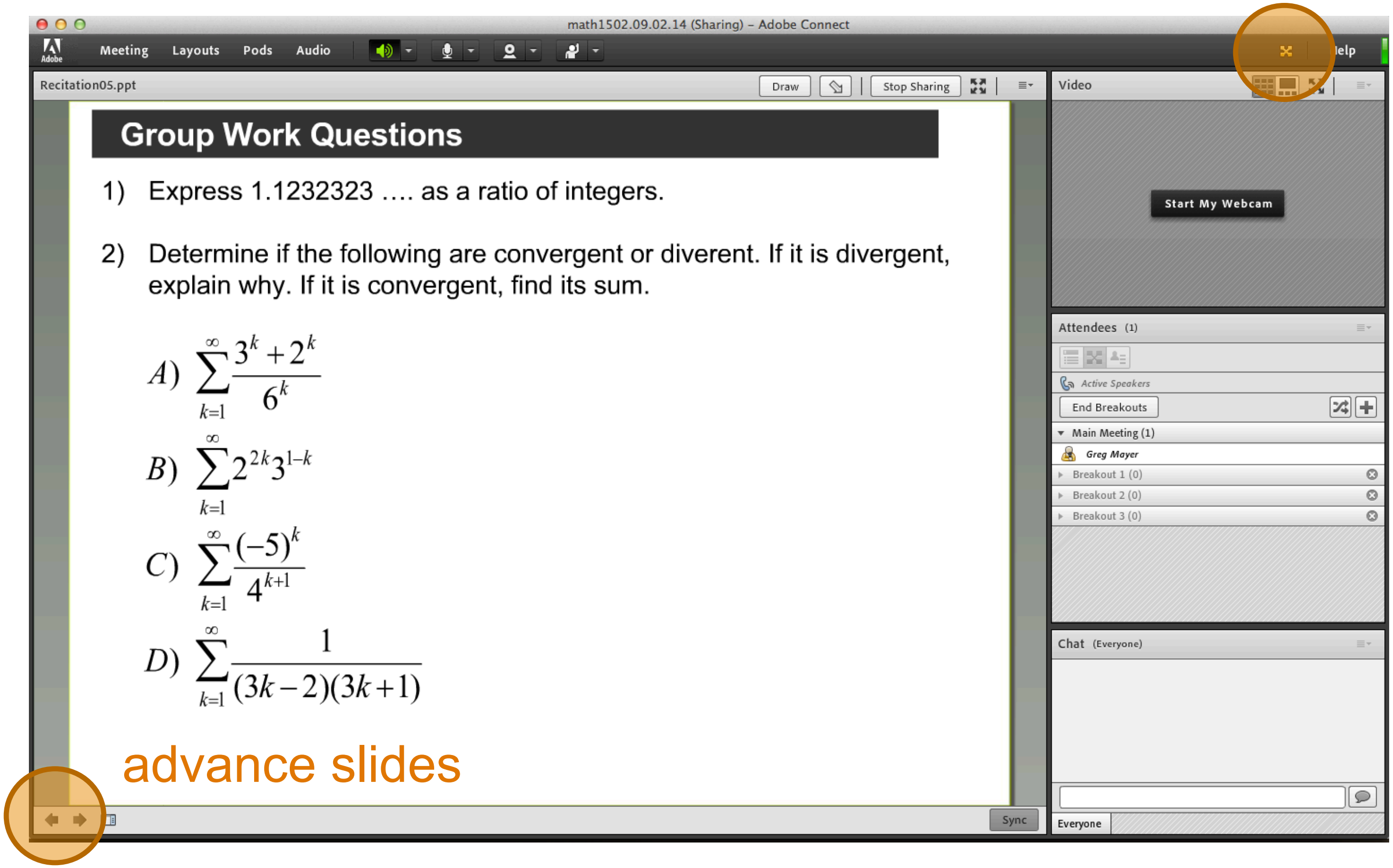
Determine if the following series convergent or divergent. If it is convergent, find its sum.

$$\sum_{k=1}^{\infty} \frac{4^{k+1}}{5^{k}}$$

Group Work

When you are in a breakout room:

message TA



Group Work Questions

1) Express 1.1232323 …. as a ratio of integers.

2) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty}\frac{3^k+2^k}{6^k}$$

$$B)\ \sum_{k=1}^{\infty}2^{2k}3^{1-k}$$

$$C)\ \sum_{k=1}^{\infty}\frac{(-5)^k}{4^{k+1}}$$

$$D)\ \sum_{k=1}^{\infty}\frac{1}{(3k-2)(3k+1)}$$

advance slides

Greg recommends:

1) saying hello :D
2) advance to the next slide with arrows located bottom left

Group Work Questions

1) Express 0.301301301 …. as a ratio of integers.

2) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k}$$

$$B)\ \sum_{k=1}^{\infty} 2^{2k} 3^{1-k}$$

$$C)\ \sum_{k=1}^{\infty} \frac{(-1)^k}{4^{k+1}}$$

$$D)\ \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$$

3) Express 1.1232323 …. as a ratio of integers.

Group Work Questions

1) Express 0.301301301 …. as a ratio of integers.

Group Work Questions

Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k}$$

Group Work Questions

Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$B)\ \sum_{k=1}^{\infty} 2^{2k} 3^{1-k}$$

Group Work Questions

Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$C) \quad \sum_{k=1}^{\infty} \frac{(-1)^k}{4^{k+1}}$$

Group Work Questions

Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$D)\ \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$$

Group Work Questions

3) Express 1.1232323 …. as a ratio of integers.

Recitation 06

Today:

1. Test for Divergence
2. Group Work: Infinite Series, Differential Equations

The **Test for Divergence (Theorem 7 from 10.2)**

$$\text{If } \lim_{k\to\infty} a_k \text{ is not zero or does not exist, then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

Note that:

$$\text{If } \lim_{k\to\infty} a_k \text{ is equal to zero, then } \underline{\hspace{5cm}}$$

Use the test for divergence to determine whether these series converge.

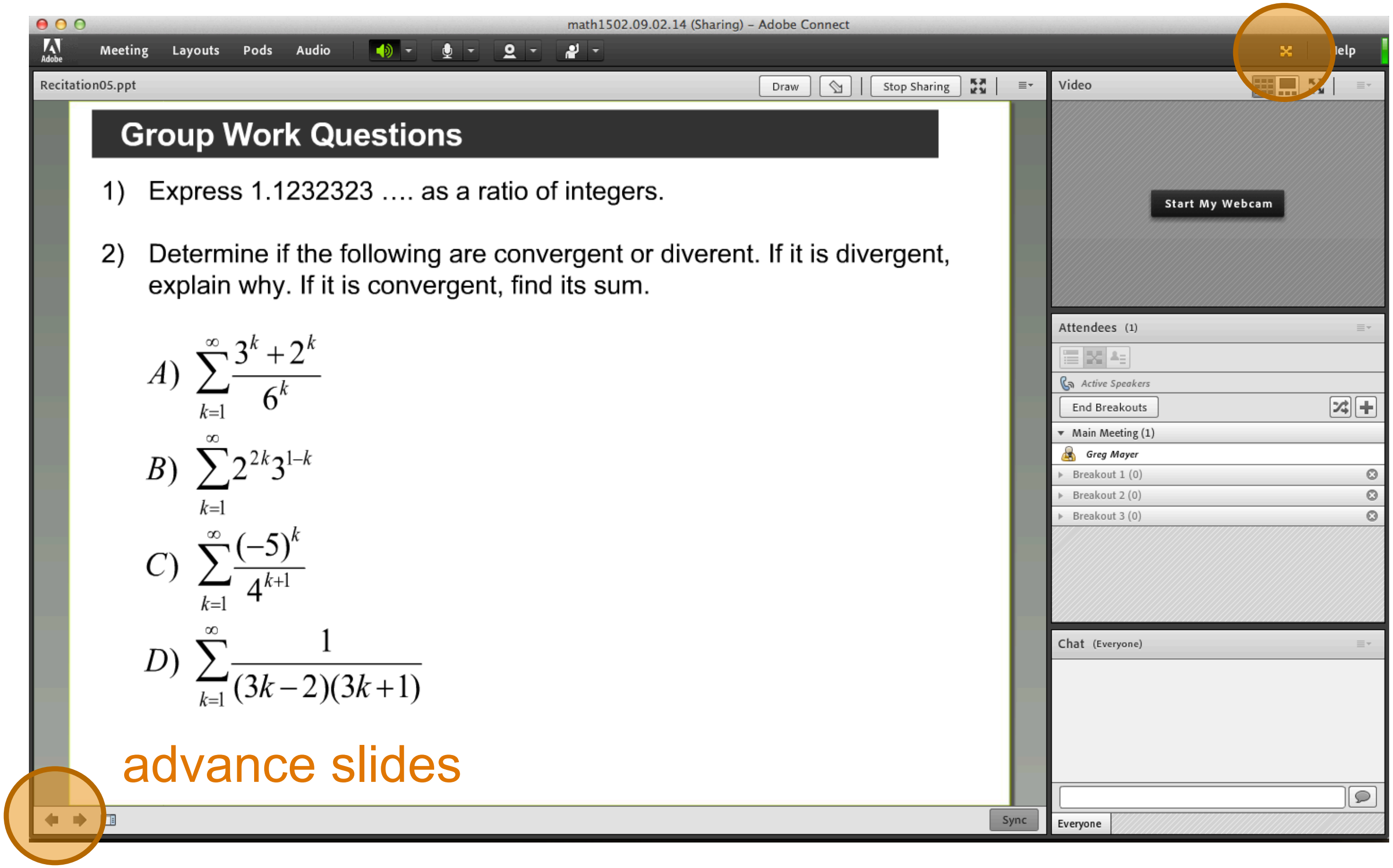
$$A) \quad \sum_{k=1}^{\infty} \frac{1}{k+2}$$

$$B) \quad \sum_{k=1}^{\infty} \frac{k}{k+2}$$

Group Work

When you are in a breakout room:

message TA



Group Work Questions

1) Express 1.1232323 …. as a ratio of integers.

2) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty}\frac{3^k+2^k}{6^k}$$

$$B)\ \sum_{k=1}^{\infty}2^{2k}3^{1-k}$$

$$C)\ \sum_{k=1}^{\infty}\frac{(-5)^k}{4^{k+1}}$$

$$D)\ \sum_{k=1}^{\infty}\frac{1}{(3k-2)(3k+1)}$$

advance slides

Greg recommends:

1) Saying hello :D
2) Each student write & chat in their favourite color
3) Advance to the next slide with arrows located bottom left

Group Work Questions

1) Express 0.002100210021 …. as a ratio of integers.

2) Solve the following DE for x > 0: $x^3 y' + (2 - 3x^2)y = x^3$

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$A)\ \sum_{k=1}^{\infty} \frac{k^2}{5k^2 + 5}$

$B)\ \sum_{k=1}^{\infty} \frac{1}{k(k+3)}$

$C)\ \sum_{k=1}^{\infty} 3(0.4)^k - 2(-0.1)^{k+1}$

$D)\ \sum_{k=1}^{\infty} \left(\sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right) \right)$

$E)\ \sum_{k=1}^{\infty} \arctan(k)$

4) Consider the DE $y' + y = y^2 e^x$
   a) Determine if the DE is seperable, and/or 1st order linear in y.
   b) Solve the DE for x > 0. *Hint: let z = 1/y.*

Group Work Questions

1) Express 0.002100210021 …. as a ratio of integers.

Group Work Questions

2) Solve the following DE for x > 0: $x^3 y' + (2 - 3x^2) y = x^3$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty} \frac{k^2}{5k^2+5}$$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$B)\ \sum_{k=1}^{\infty}\frac{1}{k(k+3)}$$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$C)\ \sum_{k=1}^{\infty} 3(0.4)^k - 2(-0.1)^{k+1}$$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$D)\ \sum_{k=1}^{\infty}\left(\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)\right)$$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$E)\ \sum_{k=1}^{\infty} \arctan(k)$$

Group Work Questions

4) Consider the DE $y' + y = y^2 e^x$

   a) Determine if the DE is seperable, and/or 1st order linear in y.

   b) Solve the DE for x > 0. *Hint: let z = 1/y.*

Recitation 07

Today:

1. Announcements
2. Quiz Review

For what values of x does the following converge? Why?

$$\sum_{k=1}^{\infty} x^{k-1} = 1 + x + x^2 + x^3 + \ldots$$

When the series converges, what is the series equal to?

Express the following as an infinite series.

$$\frac{1}{1-x^4}$$

How Quizzes Work

How Quizzes Work

1. Facilitator gets a copy of quiz
2. You write quiz on Thursday, during recitation (8:05 – 8:55)
3. Quiz is proctored (perhaps by your facilitator)
4. Give your completed quiz to your proctor.

Format of Quiz

- Are calculators allowed?
- Are textbooks and notes allowed?
- Quiz length: probably three pages, 1 question per page, questions can have multiple parts
- Don’t write on back of pages
- Leave a 1 inch margin around edges of page
- If run out of space, you can use extra pages

How Quizzes Work

If You Have Questions During the Quiz

1. If your proctor agrees, you can use to Adobe Connect to ask your TA questions (you'd need to be closely proctored)
2. Otherwise: call/text me at _______________

Before The Quiz

Find your facilitator, and discuss

- where you are writing the quiz, and
- whether you are connecting via Adobe Connect.

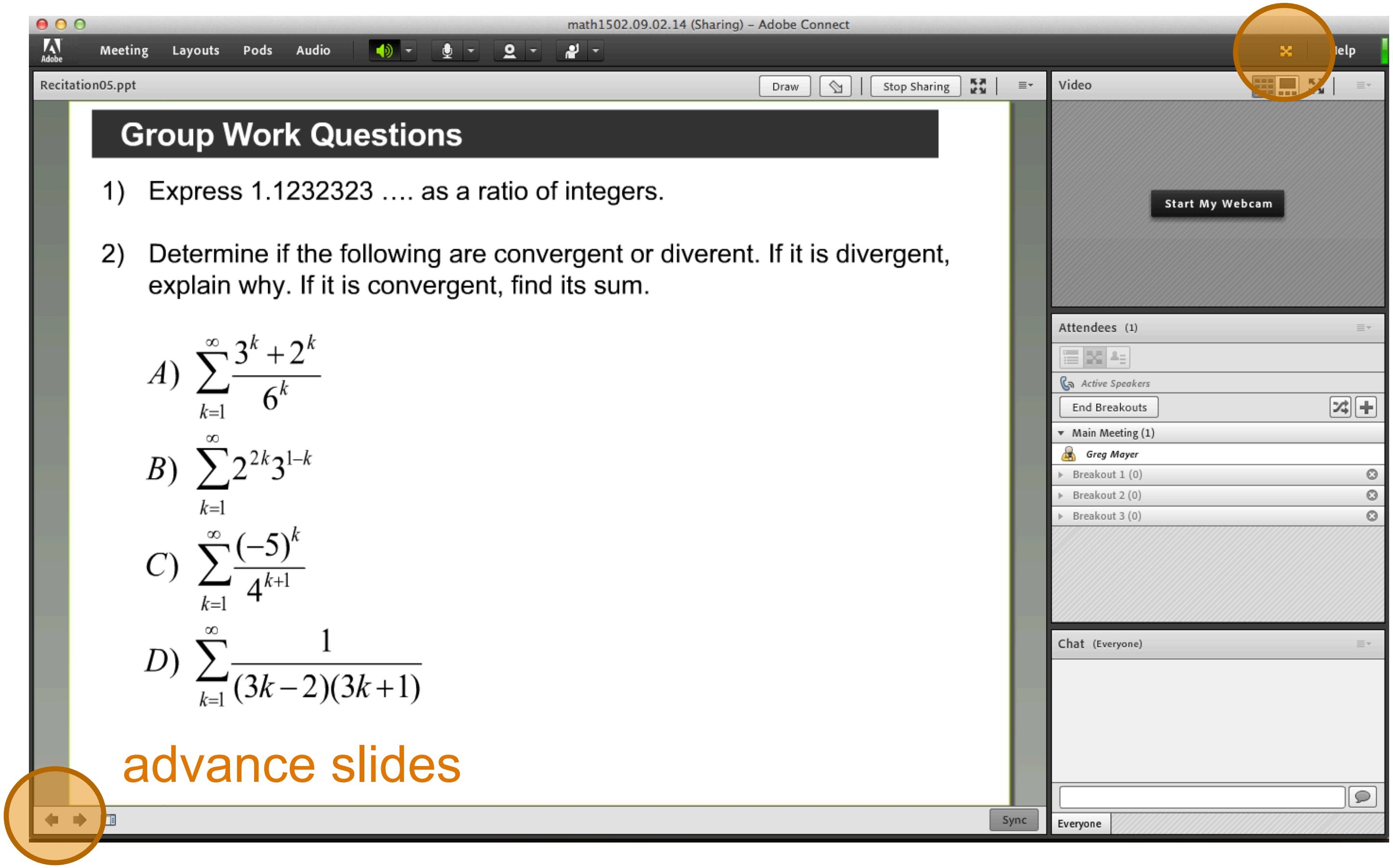
What Happens After The Quiz

1. Your facilitator scans/emails your quiz to GT,
2. they get graded in about a week or two,
3. your TA enters grades in T-Square,
4. your graded quiz is returned via email to you.

Group Work

When you are in a breakout room:

message TA



Group Work Questions

1) Express 1.1232323 …. as a ratio of integers.

2) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

A) $\sum_{k=1}^{\infty}\frac{3^k+2^k}{6^k}$

B) $\sum_{k=1}^{\infty}2^{2k}3^{1-k}$

C) $\sum_{k=1}^{\infty}\frac{(-5)^k}{4^{k+1}}$

D) $\sum_{k=1}^{\infty}\frac{1}{(3k-2)(3k+1)}$

advance slides

Greg recommends:

1) Saying hello :D
2) Each student write & chat in their favourite color
3) Advance to the next slide with arrows located bottom left

1) Solve $xy' + 2y = \cos(x) / x$, with $y(\pi) = 1$.

2) Find the sum of the following series.

$$\sum_{k=3}^{\infty} \frac{(-2)^{k+2}}{5^{k-1}}$$

3) Find the number of intervals, N, needed to ensure an accuracy of at least 0.01 using Simpson’s Rule for the following integral.

$$\int_{1}^{e} \ln x \, dx$$

4) Determine whether the following converge. Do not evaluate these integrals.

$$A)\ \int_{e}^{\infty} \frac{1}{x\ln x}\,dx$$

$$B)\ \int_{0}^{\infty} \cos(x)\sin(x)e^{-2x}\,dx$$

5) Express as an infinite series. Assume |x| < 1.

$$\frac{x}{1+x^2}$$

6) Find the sum of the following series.

$$\sum_{k=0}^{\infty} \frac{5}{(k+1)(k+5)}$$

7) Determine whether the following series converges.

$A)\quad \sum_{k=1}^{\infty} \frac{k^2+3}{k3^k}$

Recitation 09

Today: 10.7 (Power Series, Radius/Interval of Convergence and Absolute Convergence)

A) The interval of convergence is ______________________

B) If an infinite series Σa_n is called **absolutely convergent** if ___________

C) If a series is absolutely convergent, then it is ______________

D) A series is called **conditionally convergent** if it is _______________________

Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

Quiz 1 will be graded on Tuesday afternoon, grades entered Wednesday.

How can recitations be improved to better prepare you for quizzes?

Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k 2^k}$$

Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

1) Find the radius and interval of convergence.

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

2) Find the radius and interval of convergence.

$$\sum_{k=1}^{\infty}(-k)^{4k}x^{4k}$$

3) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$a)\quad \sum_{k=1}^{\infty} (-1)^k \frac{k}{k^2+1}$$

$$b)\quad \sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k}$$

4) Find the radius and interval of convergence. Then find the values of x for which series is

a) absolutely convergent
b) conditionally convergent
c) divergent.

$$1-\frac{x}{2}+\frac{2x^2}{4}-\frac{3x^3}{8}+\frac{4x^4}{16}-...$$

Recitation 10

Today: 10.8, 10.9 (Taylor Polynomials and Series)

Complete the Following Forumlas

The Nth order Taylor Polynomial about x = a is:

$$P_N\left(x\right) = \sum_{k=0} \frac{f^{(k)}(\quad)}{\qquad\qquad} (x$$

The Taylor Series about x = a is:

$$f(x) = \sum_{k=0} \frac{f^{(k)}(\quad)}{\qquad\qquad} x$$

The phrase "about x = a" means that ______________________________

Remainder formula for expansion about x = a: $\left|R_n(x)\right| \le \left(\max_{t\in[0,x]} \left| f^{(n+1)}(t) \right| \right) \frac{\left|x-a\right|^{n+1}}{(n+1)!}$

Quiz 1 has been graded, grades are in T-Square, you'll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.

Example

a) Plot a rough sketch of $f(x) = 1/(1 - x)^2$ for x between -1 and + 1.

b) Use the Taylor expansion for $1/(1 – x)$ to find $P_0(x)$, $P_1(x)$, and $P_2(x)$ of $f(x)$ about $x = 0$.

Hint: the Taylor expansion of $1 / (1 – x)$ about $x = 0$ is $\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + ...$

Example

a) Plot a rough sketch of $f(x) = 1/(1 - x)^2$ for x between -1 and + 1.

b) Use the Taylor expansion for $1/(1 – x)$ to find $P_0(x)$, $P_1(x)$, and $P_2(x)$ of f(x) about x = 0.

Hint: the Taylor expansion of 1 / (1 – x) about x = 0 is $\dfrac{1}{(1-x)} = 1 + x + x^2 + x^3 + \ldots$

Group Work Suggestions

- everyone pick a color to write in, and match it with their text
- use mics if you have them
- before moving to the next question:
  - ask if everyone agrees with answer
  - ask if everyone understands how to get the answer
- message me if your group gets stuck

Make sure you can solve Question #4 before the next quiz.

1) Find the Taylor expansion about x = 0 of $f(x) = \frac{2x}{1+x^2}$

Hint: write the Taylor expansion for 1/(1-x) at x = 0, and then apply suitable modifications to the expansion.

2) Fill in the blank: the Maclaurin series is just the Taylor series about the point _________ .

The Maclaurin series for e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{(e^x)^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Use this expansion to find the Maclaurin series for f(x), and $P_2(x)$ about x = 0, where:

$$f(x) = e^{-x^2} = \exp(-x^2)$$

3) Estimate the following integral to within 0.01 using series.
Hint: use the Alternating Series Remainder Theorem.

$$\int_0^1 x^4 e^{-x^2}\,dx$$

4) Estimate $e^{3/2}$ to within 10^{-4}.

Note: this is a question from a Math 1502 quiz from 2012.

Hint: use the Taylor Series Remainder Theorem, and approximate $e^{3/2}$ *with 5.*

5) Let $f(x) = (\exp(2x^2) - 1 - x^2) / x^4$. Note: $\exp(x) = e\wedge x = e^x$

a) Find $P_2(x)$ about $x = 0$
b) Use the result from part a) to evaluate the limit of $f(x)$ as x goes to zero.

Recitation 11

Today: 4.5 (l'Hospital's Rule)

Describe how you would evaluate the following limits.

$\lim_{x\to a}\frac{f(x)}{g(x)}$ is of the form 0/0 or ∞/∞, then we can:

$\lim_{x\to a} f(x)-g(x)$ is of the form ∞ − ∞, then we can:

$\lim_{x\to a} f(x)^{g(x)}$ is of the form 0^0, 1^∞, or ∞^0, then we can:

Quiz 1 has been graded, grades are in T-Square, you'll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.

POP QUIZ #1

- Start time: 8:10?
- Ends at: 8:25
- Pop quiz grading
  - 5 points: correct
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, choose any of the following:

  A. **work on whiteboard in breakout room**
     - type A in text chat so I know you want to work in breakout room
     - submit work by letting me know when done, and/or email me a screen capture of your work

  B. **work on paper and give work to facilitator**
     - type B in text chat so I know you're doing this
     - leave 2 inch margin on paper
     - write your name and QH8 at the top
     - facilitator is receiving instructions today on how to submit your work

  C. **work on paper and email a photo of your work to me**
     - type C in text chat so I know you are emailing your work to me
     - email your photo to me before 8:40

Find the Taylor series, centered at a = 0, of sin(5 x^2).

Evaluate: $\lim_{x\to\infty}\left(\cos\frac{1}{x}\right)^{x}$

Group Work Suggestions

- everyone pick a color to write in, and match it with their text
- use mics if you have them
- before moving to the next question:
    - ask if everyone agrees with answer
    - ask if everyone understands how to get the answer
- message me if your group gets stuck

1) Evaluate the following limit by using l'Hospital's rule, if possible.

Note: this limit is a well-known definition for an important number.

$$\lim_{x \to 1} x^{1/(x-1)}$$

2) Evaluate the following limit by using l'Hospital's rule, if possible.

$$\lim_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$$

3) Evaluate the following limit by using l'Hospital's rule, if possible.

$$\lim_{x \to 0^+} x(\ln x)^2$$

4) Evaluate the following limit by using l'Hospital's rule, if possible. *Hint: only one of these limits exists.*

a) $\lim_{x \to \infty} \frac{1}{x} \int_0^x e^{t^2}\, dt$

b) $\lim_{x \to 0} \left(\frac{1}{\ln(1+x)} - \frac{1}{x} \right)$

5) a) $\vec{v} \cdot \vec{v} = \left\| \vec{v} \right\|^2$, is the ________________ of vector **v**.

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

Recitation 12

Today: Lines, Planes, Dot Products

Suppose we have the points P(1,2,3), Q(1,3,4), R(2,2,2).

a) Find a vector that is normal to the plane that contains the three points.
b) Find an equation of the plane.

2) The equation Ax + By + Cz = D is a plane. A vector perpendicular to the plane is:

Find a parametrization of the line that is the intersection of the planes

P: $x - 2y + z = 3$

Q: $2x + y + z = 1$

Note: this is a question from a 2012 Math 1502 quiz.

Group Work Suggestions

- everyone pick a color to write in, and match it with their text

- use mics if you have them

- before moving to the next question:
    - ask if everyone agrees with answer
    - ask if everyone understands how to get the answer

- message me if your group gets stuck

1) Find the equation for the line that is perpendicular to the yz-plane, and also passes through P(1,4,3).

2) Do these lines intersect each other? Why/why not?

$x_1 = 1 + t$ $x_2 = 1 - u$

$y_1 = -1 - t$ $y_2 = 1 + 3u$

$z_1 = -4 + 2t$ $z_2 = -2u$

3) a) If $\mathbf{a}\times\mathbf{b} = 0$ and $\mathbf{a}\bullet\mathbf{b} = 0$, what can we conclude about vectors **a** and **b**? Explain your reasoning.

b) Which of the following make sense? Explain why/why not.

i. $\mathbf{a} \times (\mathbf{b} \bullet \mathbf{c})$

ii. $\mathbf{a} \bullet (\mathbf{b} \bullet \mathbf{c})$

iii. $\mathbf{a} \times (\mathbf{b}\times\mathbf{c})$

iv. $\mathbf{a} \bullet (\mathbf{b}\times\mathbf{c})$

4) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

P: **j** – **k**

Q: 3**i** – **j** + 2**k**

R: 3**i** – 2**j** + 3**k**

5) a) $\vec{v} \cdot \vec{v} = \left\|\vec{v}\right\|^2$, is the _______________ of vector **v**.

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

Recitation 13

Today: Quiz 2 Review

Estimate to within 0.0001 with a Taylor Polynomial.

$$\int_0^{1/2} \frac{\ln(1+x)}{x}dx$$

Office hours: Tuesday and Wednesday 7:30 to 8:30 pm
https://georgiatech.adobeconnect.com/distancecalculusofficehours

Group Work Suggestions

- before moving to the next question:
  - ask if everyone agrees with answer
  - ask if everyone understands how to get the answer

- everyone pick a color to write in, and match it with their text

- use mics if you have them

- message me if your group gets stuck

1) Evaluate by a) using power series, and b) l'Hopital's rule.

$$\lim_{x\to 0}\frac{\cos\left(x\right)-1}{x\sin x}$$

2) Find the interval of convergence.

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}(x+1)^k$$

3) Find the distance between planes P1 and P2.

P1: $x + 2y + z = 3$
P2: $x + 2y + z = 9$

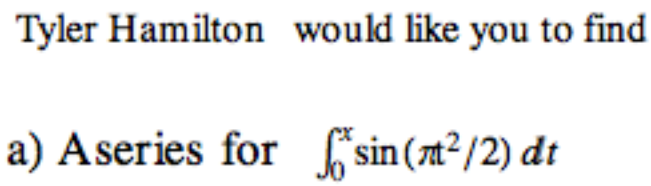
4) The line L is determined from P_1 and P_2. The plane Q is determined by Q_1, Q_2, Q_3.

Does L intersect Q? If so, where?

$P_1(1,-1,2)$
$P_2(-2,3,1)$

$Q_1(2,0,-4)$
$Q_2(1,2,3)$
$Q_3(-1,2,1)$

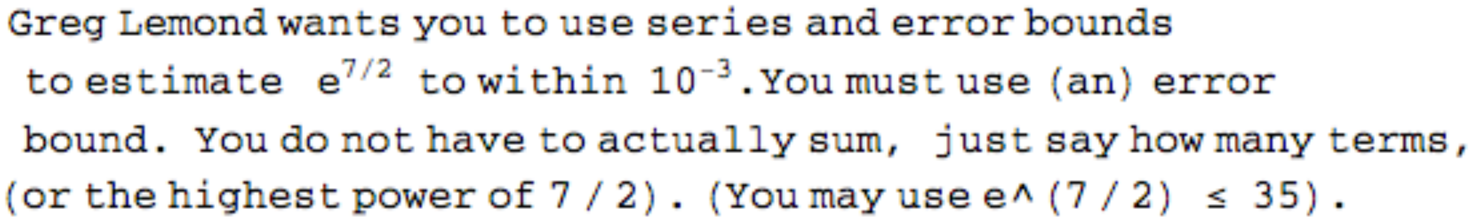
5) *From 2012 Quiz 1*



Tyler Hamilton would like you to find

a) A series for $\int_0^x \sin(\pi t^2/2)\, dt$

6) *From 2012 Quiz 1*



Greg Lemond wants you to use series and error bounds to estimate $e^{7/2}$ to within 10^{-3}. You must use (an) error bound. You do not have to actually sum, just say how many terms, (or the highest power of 7 / 2). (You may use e^ (7 / 2) ≤ 35).

7) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

P: **i** + **j** – **k**

Q: 2**i** – **j**

R: 3**i** – **j** – **k**

8) Find a parametrization of the line that is the intersection of the planes

P: $x + y + z + 1 = 0$
Q: $x - y + z + 2 = 0$

Note: this is a modified version of a question from a 2012 Math 1502 quiz.

9) a) $\vec{v} \cdot \vec{v} = \left\|\vec{v}\right\|^2$, is the ________________ of vector **v**.

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

Recitation 15

Today: Solving linear systems of equations

The following was a 2013 pop quiz question.
For what values of **a** does the following system have a solution?

$$\begin{aligned} 7x + 2y - 3z &= 25 \\ y + 3z &= 5 \\ 3y + \mathbf{a}z &= 3 \end{aligned}$$

Please take a few minutes to fill out the technical issues survey.
There are 30 recitations in the semester: we're ~50% through Math 1502!

A Few Definitions

a) A system of linear equations is **consistent** if it has _______________________ .

b) A system of linear equations is **inconsistent** if it has _____________________ .

c) A system of linear equations that is overdetermined has _____________________ .

d) Can a system of linear equations be overdetermined and consistent? If yes, provide an example with at least 3 equations.

3) Find *h* and *k* such that the system has

   a) no sol’n
   b) a unique sol’n
   c) infinitely many solutions

$$x_1 + hx_2 = 2$$

$$4x_1 + 8x_2 = k$$

4) A 3 x 4 coefficient matrix has three pivot columns. Is the system consistent? Why/why not?

5) Find the general solution to the system whose augmented matrix is given below.

6) True or false.

a) The reduced echelon form of a matrix is unique.

b) If a system has free variables, the solution set has many solutions.

c) If a row in an echelon form of an augmented matrix is [0 0 0 3 0], then the linear system is inconsistent.

Recitation 16

Today: Span, Linear Dependence

Definitions

Assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are arbitrary vectors.

a) The sum $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is a ______________ combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) The set of all possible ________ combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the _________ of $\mathbf{v}_1$ and $\mathbf{v}_2$.

c) Any vector in the __________ of $\mathbf{v}_1$ and $\mathbf{v}_2$ can be written as a ________________________ of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Question 1

If you haven't already, please take a few minutes to fill out the technical issues survey.
No lectures/recitations next Monday/Tuesday.

1) This is similar to the **first** question on your next HW.

Linear Dependence

Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 ..., \vec{v}_N$ are linearly dependent (LD) if ∃ $c_1, c_2, c_3, ..., c_N$ *not* all ____________ , such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + ...c_N\vec{v}_N = \vec{0}$$

If the vectors are not LD, they are ______________________.

Example:

To determine whether a set of vectors are _____, we solve:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + ... + c_N\vec{v}_N = \vec{0}$$

which has the same solution as the linear system whose augmented matrix is $\left[\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3 \; ... \; \vec{v}_N \; \vec{0}\right]$.

2) Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

Group Work Suggestions

- before moving to the next question:
    - ask if everyone agrees with answer
    - ask if everyone understands how to get the answer

- everyone pick a color to write in, and match it with their text

- use mics if you have them

- message me if your group gets stuck

3) This is similar to the second question on your next HW.

Determine whether vector **b** is in the set spanned by the columns of matrix A.

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

Hint: let the columns of A be $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. *If* **b** *is in the set spanned by these 3 vectors, there is a linear combination of* $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ *that equals* **b**.

4) This is similar to the third question on your next HW.

Determine whether vector **b** can be written as a linear combination of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. In other words, determine whether x_1 and x_2 exist such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$. If possible, find x_1 and x_2.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

5) Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}$$

6) Find values of h so that the following vectors are LD.

$$\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$$

Recitation 18

Today: Span, Linear Dependence, Linear Transforms

Quiz 3 Next Thursday

Make sure you can solve these questions from old quizzes:

- 2012: Quiz 2 #2 and #3
- 2012: Quiz 3 #1 and #2
- 2013: Quiz 3 #1 and #3

We'll solve some of these in Tuesday's recitation.

QH8 Office Hours Next Week

Tuesday and Wednesday 7:30 to 8:30 pm

At the same place as last time:

Online Drop-in Tutoring

Wednesdays, 5:30 to 7:00 pm

For all ~450 distance calculus students

Facilitated by Greg, who will answer questions and review problems from QH8 recitations

If you haven't already, please take a few minutes to fill out the technical issues survey.

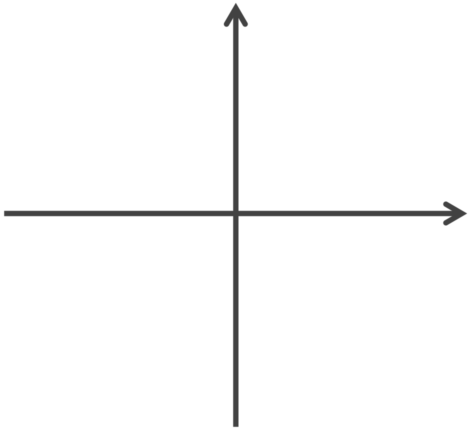
1) This is similar to the second and third questions on the transforms HW.

Let $\vec{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and T be a linear transformation that maps $\vec{u}_1$ onto $\vec{v}_1$, and $\vec{u}_2$ onto $\vec{v}_2$. Find T and the image of $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ under T.

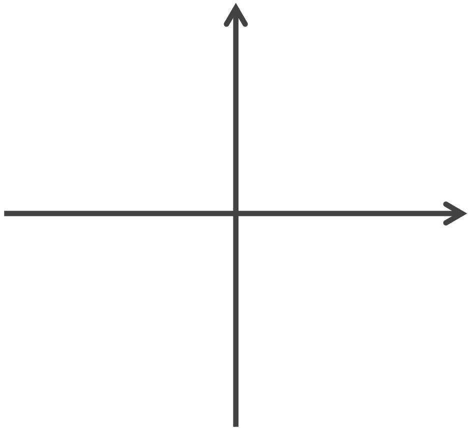
2) Plot u and v, their images under T, and provide a geometric interpretation of what T does to vectors in R^2.

$$\mathrm{u} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, \mathrm{v} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

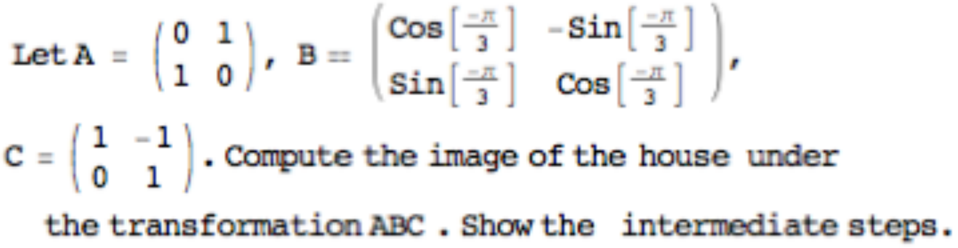
$$a) \quad T(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



$$b) \quad T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

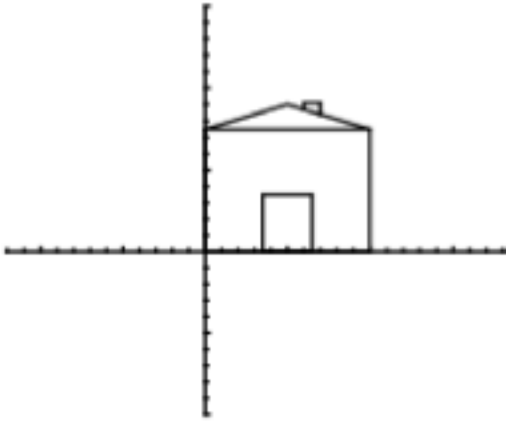


3) *From 2012 Quiz 2*



Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \text{Cos}\left[\frac{-\pi}{3}\right] & -\text{Sin}\left[\frac{-\pi}{3}\right] \\ \text{Sin}\left[\frac{-\pi}{3}\right] & \text{Cos}\left[\frac{-\pi}{3}\right] \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute the image of the house under the transformation ABC . Show the intermediate steps.



4) Fill in the elements of the 3x3 matrix.

Hint: the elements can be identified by inspection.

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 6x_3 \\ x_3 - 7x_1 \\ -x_2 - 5x_3 \end{bmatrix}$$

5) Find values of h so that the vectors are LD, and values of h so that the vectors are LI.

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$

If ___________ , then the vectors are LD.

If ___________ , then the vectors are LI.

Recitation 19

Today: Quiz 3 Review

Adobe 9.3: What's New?

- new Adobe Connect Add-in (not needed, I hope)
- fewer technical issues?
- new drawing tools
- participants can be given drawing powers
- writing on board is still anonymous

Quiz 3 Next Thursday

Make sure you can solve these questions from old quizzes:

2012: Quiz 2 #2 and #3
2012: Quiz 3 #2
2013: Quiz 3 #1 and #3

Review recent HWs on Span, Lin Transforms, Gauss Jordan
Review sections 1.1, 1.2, 1.3, 1.7, 1.8, 1.9

QH8 Office Hours Next Week

Tuesday and Wednesday 7:30 to 8:30 pm
We'll solve practice quiz problems & go over specific areas you'd like to review.
At the same place as last time:

Online Drop-in Tutoring

Wednesdays, 5:30 to 7:00 pm
For all ~450 distance calculus students
Facilitated by Greg, who will answer questions and review problems from QH8 recitations
https://georgiatech.adobeconnect.com/dcp-online-drop-in-tutor-center-2014-fall

1) From 2013, Quiz 3, #3

If T is a linear trasformation consisting of rotating counterclockwise by π /3 radians followed by a reflection about the line x = y, find the matrix such that T(x) = Ax.

2) From 2012, Quiz 2, #2

For what values of b is **y** a linear combination of **u** and **v**?

$$\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}$$

If ____________ , then **y** is a linear combination of **u** and **v**.

If ____________ , then **y** is *not* a linear combination of **u** and **v**.

3) From 2013, Quiz 3, #1

If T is a linear trasformation, and:

$$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)=\begin{bmatrix} 1 \\ 0 \end{bmatrix},\ T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)=\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

find the matrix, A, such that T(x) = Ax.

4) From 2012, Quiz 3, #2

For what values of *a* are the following vectors LI?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ a \end{bmatrix}$$

If ___________ , then the vectors are LD.

If ___________ , then the vectors are LI.

5) State whether the following statements are true or false and explain your reasoning.

a) The columns of matrix A are LI if the equation $A\mathbf{x} = \mathbf{0}$ has the trivial solution.

The statement is ____________ because:

b) If S is a set of LI vectors, then each vector in S is a linear combination of the other vectors in S.

The statement is ____________ because:

c) If a set of vectors contains fewer vectors than there are entries in the vectors, the set is LI.

The statement is ____________ because:

Recitation 21

Today: Matrix Inverses, LU Decomposition

1a) State the formula for the inverse of the matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \ M^{-1} =$$

1b) Use the formula in (1a) to find a 2×2 matrix P such that: $P\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \ P\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

2) Solve the equation Ax = b, using the LU decomposition of A, where

$$A = \begin{bmatrix} 3 & -7 & -2 \\ -3 & 5 & 1 \\ 6 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 \\ 0 & -2 & -1 \\ 0 & 0 & -1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -7 \\ 5 \\ 2 \end{bmatrix}$$

3) Find the LU decomposition of A.

$$A = \begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix}$$

4) State whether the following statements are true or false and explain your reasoning.

a) A can be row reduced to the identity matrix iff A is invertible.

The statement is ____________ because:

b) If matrix B is the inverse of matrix A, the equations AB = I and BA = I must both be true.

The statement is ____________ because:

c) If A and B are both N×N and invertible, then the product $A^{-1}B^{-1}$ is the inverse of AB.

The statement is ____________ because:

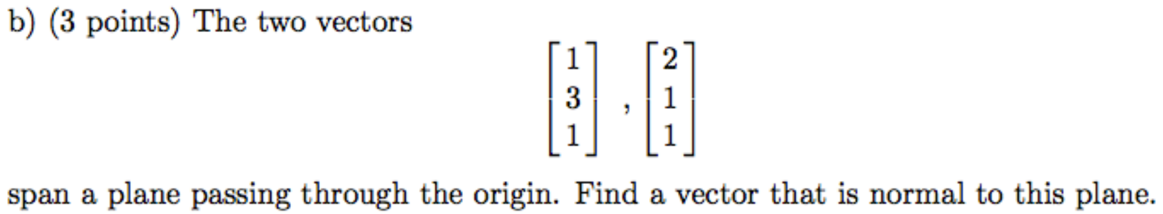
d) If an N×N matrix A is invertible, then the columns of A^T are LI.

The statement is ____________ because:

Recitation 22

Today: Column Space and Null Space, LU Decomposition

From quiz 3:



b) (3 points) The two vectors

$$\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

span a plane passing through the origin. Find a vector that is normal to this plane.

POP QUIZ #2

- Start time: 8:10?
- Ends at: start time + 15 minutes.
- Pop quiz grading
  - 5 points: correct
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, choose any of the following:

  A. **work on whiteboard in breakout room**
     - type A in text chat so I know you want to work in breakout room
     - submit work by letting me know when done, and/or email me a screen capture of your work

  B. **work on paper and give work to facilitator**
     - type B in text chat so I know you're doing this
     - leave 2 inch margin on paper
     - write your name and QH8 at the top
     - facilitator is receiving instructions today on how to submit your work

  C. **work on paper and email a photo of your work to me**
     - type C in text chat so I know you are emailing your work to me
     - email your photo to me before: end time

Find a basis for the column space of:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 & 6 & 9 \end{pmatrix}$$

2) Fill in the blanks:

a) Coll A is the set of all linear combinations of the __________ of matrix A.

b) Nul A is the set of all solutions to ______________ .

c) The __________ columns of matrix A form a basis for the column space of A.

d) The rank of matrix A is the ______________________ .

e) The nullity of matrix A is the ______________________ .

3) Find i) a basis for Col A, and ii) a basis for Nul A.

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recitation 23

Today: Column and Null Space

From 2012 Quiz 3: Find a basis for the nullspace of A. Also find its rank and nullity.

$$A = \begin{bmatrix} 3 & -1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 1 & 3 & 2 & 4 \end{bmatrix}$$

Starting This Week:

Evidence of inappropriate behavior will be forwarded to the course instructors, and possibly also to the chair of the School of Mathematics and High school facilitators. Evidence will be reviewed to determine if further action is required. Such action could either result in

1) the Georgia Tech's Office of Undergraduate Admissions being made aware of student behavior, and/or
2) all students from a particular school moved to another section where interactions between students from different schools is not possible.

Behavior is inappropriate if it can interpreted as hurtful or disrespectful.

Students can request to be moved to another section at any time.

Questions can be directed to the students teaching assistant and/or the course instructors at any time.

2) A 2013 pop quiz question: find the coordinates of **b** with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\mathbf{b} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

3) Is λ = 2 an eigenvalue of matrix A? Why/why not?

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$$

4) Find a basis for the eigenspace of A, for the eigenvalue λ = -5.

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$$

5) Find the characteristic polynomial and eigenvalues of:

a) $X = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

b) $Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

6) Find the LU decomposition of the following matrix (*likely not enough time for this*).

Hints: start by finding U with row operations, eliminating one element at a time.

$$B = \begin{bmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{bmatrix}$$

Recitation 24

Today: Determinants (3.1, 3.2), Diagonalization (5.3)

Suppose A and B are square matrices.

1) If a multiple of one row of A is added to another row to produce B, then det(B) = ________ .
2) If two rows of A are interchanged to produce B, then det(B) = ___________ .
3) If one row of A is multiplied by K to produce B, then det(B) = ____________ .
4) If A is a trianglular matrix, then det(A) = ___________________ .

Compute det(A)

$$A = \begin{bmatrix} 1 & -4 & 2 \\ -2 & 8 & -9 \\ -1 & 7 & 0 \end{bmatrix}$$

Theorems from Section 3.2

Suppose A and B are square matrices.

1) A is not invertible iff det(A) = ____________ .
2) det(AB) = _______________ .
3) det(A + B) = ________________ .
4) det(A^T) = _______________ .

Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

Section 5.3: Diagonalization

A matrix A is diagonalizable if it can be written in the form:

A = ___________________

where

P is ___________________

D is ___________________

Suppose A is N×N. To diagonalize A:

1. find all _________________ of A to construct D
2. find N ______ eigenvectors of A to construct P
3. find P^{-1} (we don't yet have a method for finding inverse of 3×3 matrix)
4. write A = ____________ .

Group Work

Writing on board disappears when I enter room (sometimes). So lets try this:

- None of the breakout rooms have the questions.
- Every breakout room has a whiteboard.
- Write the following down, and write it on the whiteboard when you get into the breakout room.

Diagonalize the following matricies, if possible.

$$A=\begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix};\ B=\begin{bmatrix} 3 & 2 \\ 0 & 3 \end{bmatrix};\ C=\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \text{where } \lambda\text{'s of } C \text{ are } 2,2,5$$

Recitation 25

Today: Quiz Review

Quiz 4 on Thursday

Review recent HWs on span, determinants, and eigenvalues

Review sections 2.8, 2.9, 3.1, 3.2, 5.1, 5.2

QH8 Office Hours

Tuesday and Wednesday 7:30 to 8:30 pm

We'll solve practice quiz problems & go over specific areas you'd like to review.

At the same place

Online Drop-in Tutoring

Wednesdays, 5:30 to 7:00 pm

For all ~450 distance calculus students

Facilitated by Greg, who will answer questions and review problems from QH8 recitations

Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

b) Find as many LI eigenvectors for this eigenvalue as possible.

$$A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix}$$

Group Work

Writing on board disappears when I enter room (sometimes). So lets try this:

- None of the breakout rooms have the questions.
- Every breakout room has a whiteboard.
- Write Question on the whiteboard when you get into the breakout room, solve it, then move to Question 2.
- You've got about 15 minutes.

Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

b) Find as many LI eigenvectors for this eigenvalue as possible.

$$A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix}$$

Question 2

Compute det(B) by using row reduction.

$$B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}$$

Question 1 (from 2013 Quiz 4)

$$A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix}$$

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

b) Find as many LI eigenvectors for this eigenvalue as possible.

Question 2

Compute det(B) by using row reduction.

$$B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}$$

3) State whether the following statements are true or false and explain your reasoning.

a) The dimension of Col A is the number of pivot columns of A.

The statement is ____________ *because:*

b) The dimension of Nul A is the number of variables in equation $A\mathbf{x} = 0$.

The statement is ____________ *because:*

c) If A is a square matrix, and $\det(A^4) = 0$, then A is not invertible.

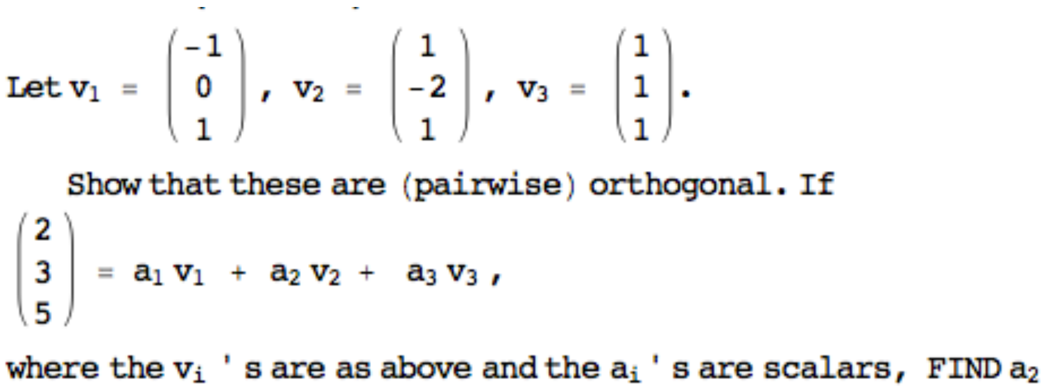
The statement is ____________ *because:*

4) Find all values of h so that the eigenspace for D, for λ = 4, is two dimensional.

$$D = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Recitation 27

Today: Diagonalization (5.3), Orthogonality (6.1, 6.2)



Let $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Show that these are (pairwise) orthogonal. If

$$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3,$$

where the v_i ' s are as above and the a_i ' s are scalars, FIND a_2

Question 2: Group Work

Writing on board disappears when I enter room (sometimes). So lets try this:

- None of the breakout rooms have the questions.
- Every breakout room has a whiteboard.
- Write Question on the whiteboard when you get into the breakout room
- You've got about 10 minutes.

Question 2 (parts a and b are from 2014 Quiz 4)

a) Find all eigenvalues

b) Find a eigenbasis for each eigenvalue.

c) Is it possible to diagonalize A? Why/why not?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

Question I (parts a and b are from 2014 Quiz 4, Question 3)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

a) Find all eigenvalues

b) Find a eigenbasis for each eigenvalue.

c) Is it possible to diagonalize A? Why/why not?

Recitation 28

Today: Orthogonality (6.1, 6.2, 6.3)

True or False:

a) Eigenvalues must be nonzero scalars.

This is _________ , because

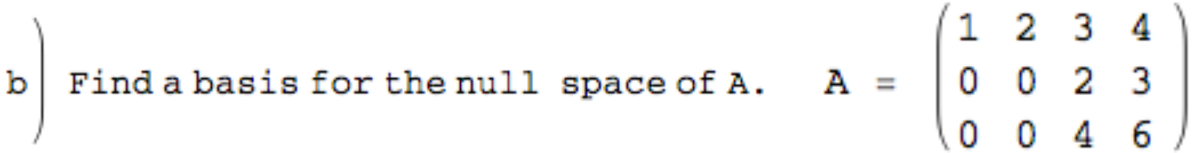
b) Eigenvectors must be nonzero vectors.

This is _________ , because

Quiz 4, Question 1b

Solutions were emailed to students yesterday.

Question



b) Find a basis for the null space of A. $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 6 \end{pmatrix}$

An Answer

A basis for Nul(A) is the set: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \right\}$

Is this correct? How can we check to see if this answer is correct?

The basis vectors must ______________ , and ________________

Orthogonality (6.2)

a) Compute the orthogonal projection (OP) of y onto the line, L, that passes through the origin and is parallel to u.

b) Sketch y, u, L, and the OP.

$$y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \quad u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Orthogonality (6.2)

c) Calculate the distance between y and its OP.

d) Write y as a sum of a vector in Span(u) and a vector orthogonal to u.

$y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}, \quad u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

Orthogonality and Linear Independence

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

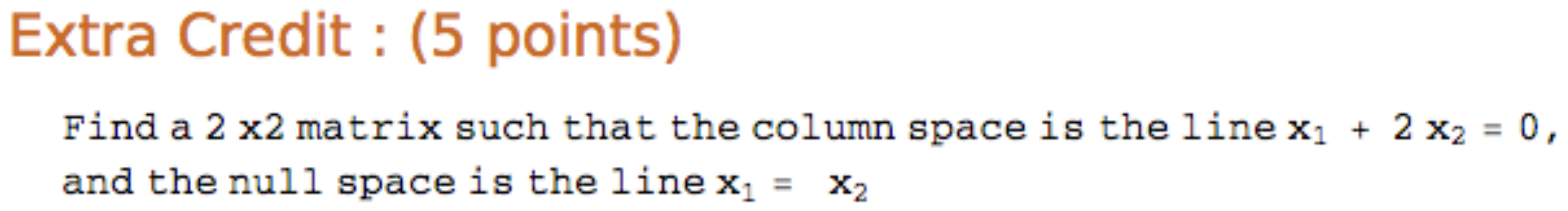
a) Are the columns of A LI?
b) Do the columns of A form a basis for R^4?
c) Are the columns of A mutually orthogonal?

Orthogonality and Linear Independence

Find an orthogonal basis for the column space of A.

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

If time permits *(from 2014 Quiz 4)*:



Extra Credit : (5 points)

Find a 2 x2 matrix such that the column space is the line $x_1 + 2x_2 = 0$, and the null space is the line $x_1 = x_2$

If time permits *(from 2012 Quiz 4):*

Find the eigenvalues and eigenvectors of A and use them to find a formula for A^k.

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$$

Recitation 29

Today: Orthogonality (6.1 to 6.5)

Orthogonality and Linear Independence

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

a) Are the columns of A LI?
b) Do the columns of A form a basis for R^4?
c) Are the columns of A mutually orthogonal?

Orthogonality and Linear Independence

Find an orthogonal basis for the column space of A.

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Orthogonality and Linear Independence

Find an orthogonal basis for the column space of A.

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

2) Least Squares (slide 1/3)

Consider the system Ax = b, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \; b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

i. Does A have LI columns?

ii. Do the columns of A form a basis for R^3?

iii. Is b in Col(A)?

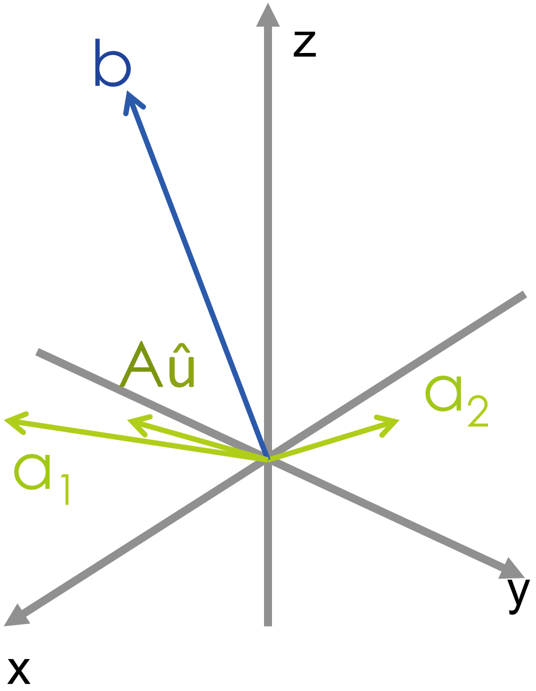
iv. Is there a solution to Ax = b?

2) Least Squares (slide 2/3)

Consider the system Ax = b, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Our best solution to this system: the vector û, such that ||b – Aû|| ≤ ||b – A__|| .
What does this mean?



Is b in Col(A)?

Is Aû in Col(A)?

Is (b – Aû) perpendicular to all vectors in Col(A)?

Thus, $A^T \bullet (b - A\hat{u}) =$

Solving for û gives us:

2) Least Squares (slide 3/3)

Consider the system Ax = b, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Find the least squares solution to this system.

$A^TA =$

$A^Tb =$

$(A^TA)^{-1} =$

û =

The vector we found, û, has the property that:

Least Squares (slide 3/3)

Consider the system Ax = b, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \ b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

Find the least squares solution to this system.

$A^T A =$

$A^T b =$

$(A^T A)^{-1} =$

x =

The vector we found, x, has the property that:

Least Squares: A Special Case

Find a least squares sol’n to Ax=b, where: $A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \; b = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}$

Recitation 31

Today: QR Decomposition, Orthogonality Review (6.1 to 6.5)

QR Factorization

A=QR, and R is an upper triangular matrix.

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -6 \\ 1 & 5 \end{bmatrix}, Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

a) If we weren't given Q, we could find it by using _______________ .

b) The columns of Q are _________________ .

c) The columns of Q form a __________________ basis for ___________

QR Factorization

A=QR, and R is an upper triangular matrix.

$$A=\begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -6 \\ 1 & 5 \end{bmatrix}, Q=\begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

d) The dimensions of R have to be ___________ , because ____________________ .

e) Find R.

f) Show that your answer for part (e) is correct.

Least Squares (LS) Formulas

For each case, state a formula for the LS solution of Ax = b.

1) A is a square matrix and A^TA is invertible.

2) A is square and has **orthogonal** columns.

3) A is square and has **orthonormal** columns.

Least Squares Solutions

If A is square, and $A^T A$ is not invertible, can we find a LS solution to Ax = b? Why/why not?

Least Squares Solutions

Describe all LS solutions to the system:

$x_1 + x_2 = 2$
$x_1 + x_2 = 4$

Recitation 32

Today: Final Exam Review

Quiz Grades

- Quiz grades "locked" today.
- Please check your graded quizzes to see if they were graded correctly.

If You are Writing Math 1502 Final Exam

- Part I: Mon, Dec 8
- Part II: Tue, Dec 9
- Work with your facilitator to find a time/place to write.

QH8 Office Hours

Sat Dec 6, Sun Dec 7

Please use text chat when you are free with "I'm free Sat ___ and Sun __" so that we can try to find times that work, for most of you.

Today

Group work: 3 groups, we may return to main room if/when groups are getting stuck

1) Section 4.6: Row and Col Space of A^T (Slide 1 of 2)

Row(A) is the set of all possible linear combinations of the rows of A.

Theorem (from Section 4.6)

If two matrices A and B are row equivalent, then their row spaces are __________.

If B is in echelon form, the nonzero rows of B form a basis for ________________ ,

as well as ___________________ .

A proof of this theorem uses the fact: if B is obtained from row operations on A, the rows of B are ________________________ of the rows of A.

1) Section 4.6: Row and Col Space of A^T (Slide 2 of 2)

Matrix A and its row echelon form are given. Find a basis for

1) Col(A)
2) Row(A)
3) Row(A^T)
4) Col(A^T)

Hint: the answers for all of the above do not require any calculation.

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

2) F*rom 2012 Quiz 4:*

Find the eigenvalues and eigenvectors of A and use them to find a formula for A^k.

Hint: one of the eigenvalues is 3.

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$$

3) QR Factorization

A=QR, and R is an upper triangular matrix.

$$A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}, \quad Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$$

a) The dimensions of R have to be ___________ .

b) Calculate matrix R. *Hint: save time by factoring 1/7 out of matrix Q first*.

c) Compute a few elements of the product QR to check your answer for part (b)

4) Fill in the Blanks

Matrix A is invertible and has dimensions NxN.

a) The columns of A form a basis for _____

b) rank(A) = _____

c) Nul(A) = _____

d) dim(Nul(A)) = ______

e) dim(Nul(A^T)) = _____

f) dim(Row(A^T)) = ______

g) dim(Col(A^T)) = ______

h) dim(Col(A)) + dim(Nul(A)) = ______ + ______ = ________

i) dim(Row(A)) + dim(Nul(A)) = ______ + ______ = _______

5) A modified version of a 2014 Quiz 2 Question

Compute the integral, using a Taylor polynomial, to an accuracy of at least 0.01.

Hint: N is bigger than 30.

$$\int_0^1 e^{-x^4}\,dx$$

6) True or False

A) The LS solution of Ax = b is the point in Col(A) closest to b.

This statement is _______ because

B) If x is in subspace W, then $x - \text{proj}_W x \neq 0$

This statement is _______ because

7) True or False

A) If $\{v_1,v_2,v_3\}$ is an orthogonal basis and c is a constant, then $\{v_1,v_2,cv_3\}$ is another, different orthogonal basis.

This statement is _______ because

B) If $\hat{\mathbf{u}}$ is a LS solution to $A\mathbf{u} = \mathbf{b}$, then $\hat{\mathbf{u}} = (A^TA)^{-1}A^T\mathbf{b}$.

This statement is _______ because

8) Eigenvalues and Orthogonality

1) What can we say about the eigenvalues of an orthonormal matrix?
Hint: look at $||Av||^2$*, where A is orthonormal, and v is an e-vector of A.*

---

2) What can we say about the eigenvalues of a matrix that has LD columns?

3) What can we say about the eigenvalues of a matrix that is upper triangular?

Welcome to Your Distance Calculus Recitation!

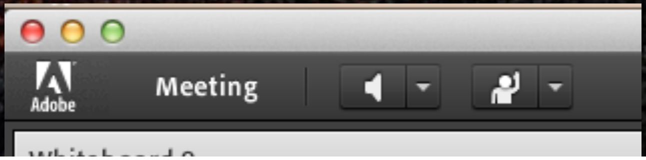
We'll get started at 8:05.

While we are waiting, see if you can use the chat window (bottom right) to join the discussion.

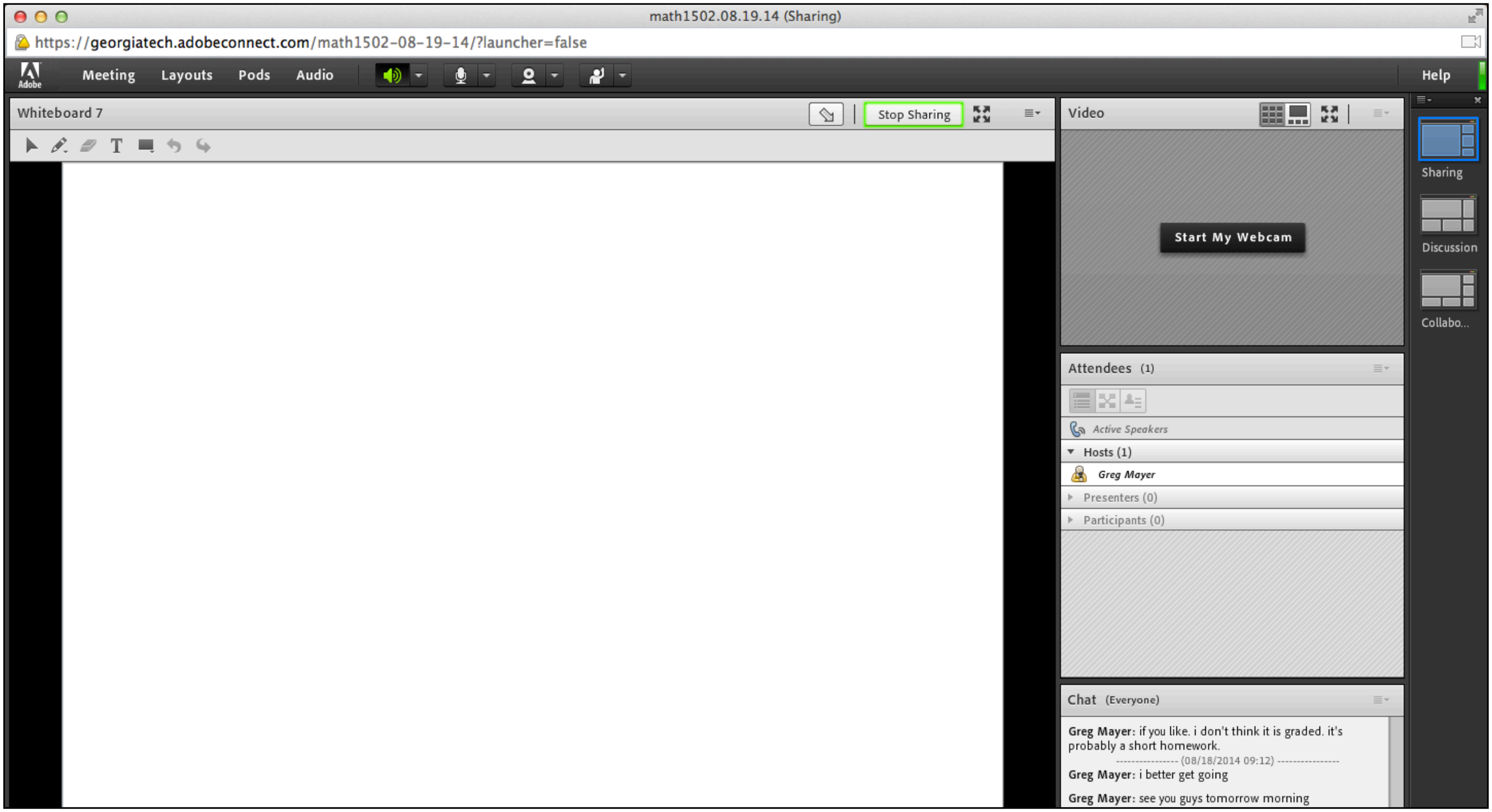
Today:

- Introduction to Adobe Connect
- What are Recitations?
- Icebreaker
- Numerical integration (8.7)

If you can't hear the TA, click the speaker icon



Adobe Connect



Greg Mayer: if you like. i don't think it is graded. it's probably a short homework.

---------------- (08/18/2014 09:12) ----------------

Greg Mayer: i better get going

Greg Mayer: see you guys tomorrow morning

Microphones, Webcams, Tablets

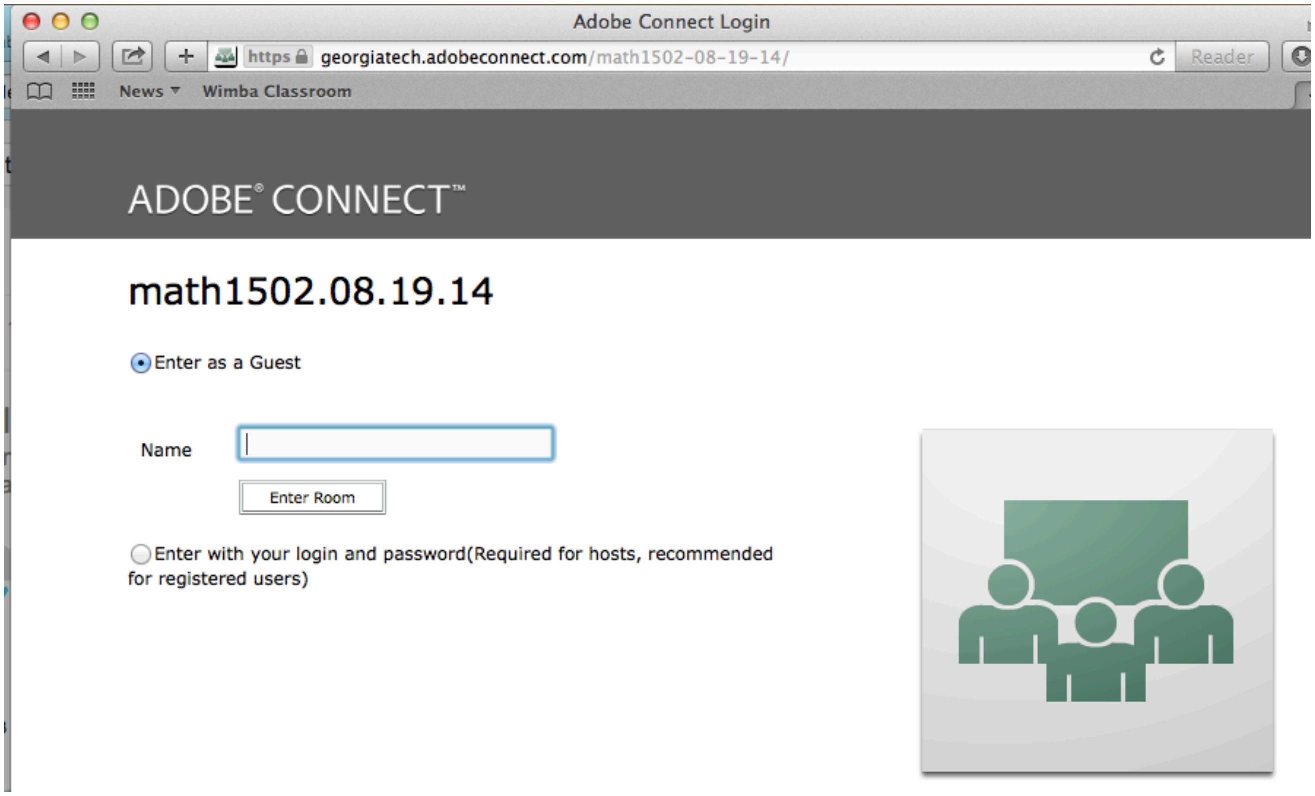
We can loan you a wacom bamboo tablet, if you'd like to borrow one please send me an email.

If you have a mic or a webcam, you are welcome to use them.

The Whiteboard

- You need to be a presenter to write on the board
- Only a host can chanage permission levels
- All writing on the board is anonymous
- Please respect other students taking this course (and your TA): you are responsible for your learning

Logging Into Adobe Connect for Recitations



math1502.08.19.14

● Enter as a Guest

Name

○ Enter with your login and password(Required for hosts, recommended for registered users)

Thursday's recitation is at
https://georgiatech.adobeconnect.com/math1502-08-19-14

Adobe Connect Technical Problems?

You can:

- reload your browser
- log in/out
- use a different web browser
- reboot
- get help from another student and/or your TA

I strongly recommend that, if possible, you use a wired connection.

What are Recitations?

- Our goal: help students understand course material so that they can complete assignments and prepare for quizzes and exams.
- please bring questions about the homework or lectures

Our Section in a Nutshell

- students in Math 1502 are divided into many sections
- ours is the only section that
  - doesn’t have on-campus students
  - uses Connect for recitations
- Why Adobe Connect?
  - it’s cheaper
  - you can interact with students at other schools

Tablets

- Students in our section can borrow tablets.
- If you already have a tablet you want to use, that's ok
- Equipment need to be returned to your facilitator
- If you don’t have a tablet and want to borrow one, email me
- Tablets (should) come with a CD, use it to configure tablet settings

Course Websites

- Recordings of recitations and lectures: tegrity.gatech.edu
- Discussion forum: piazza.com
- Live lectures: gtcourses.gatech.edu
- Textbook and homework: www.mymathlab.com

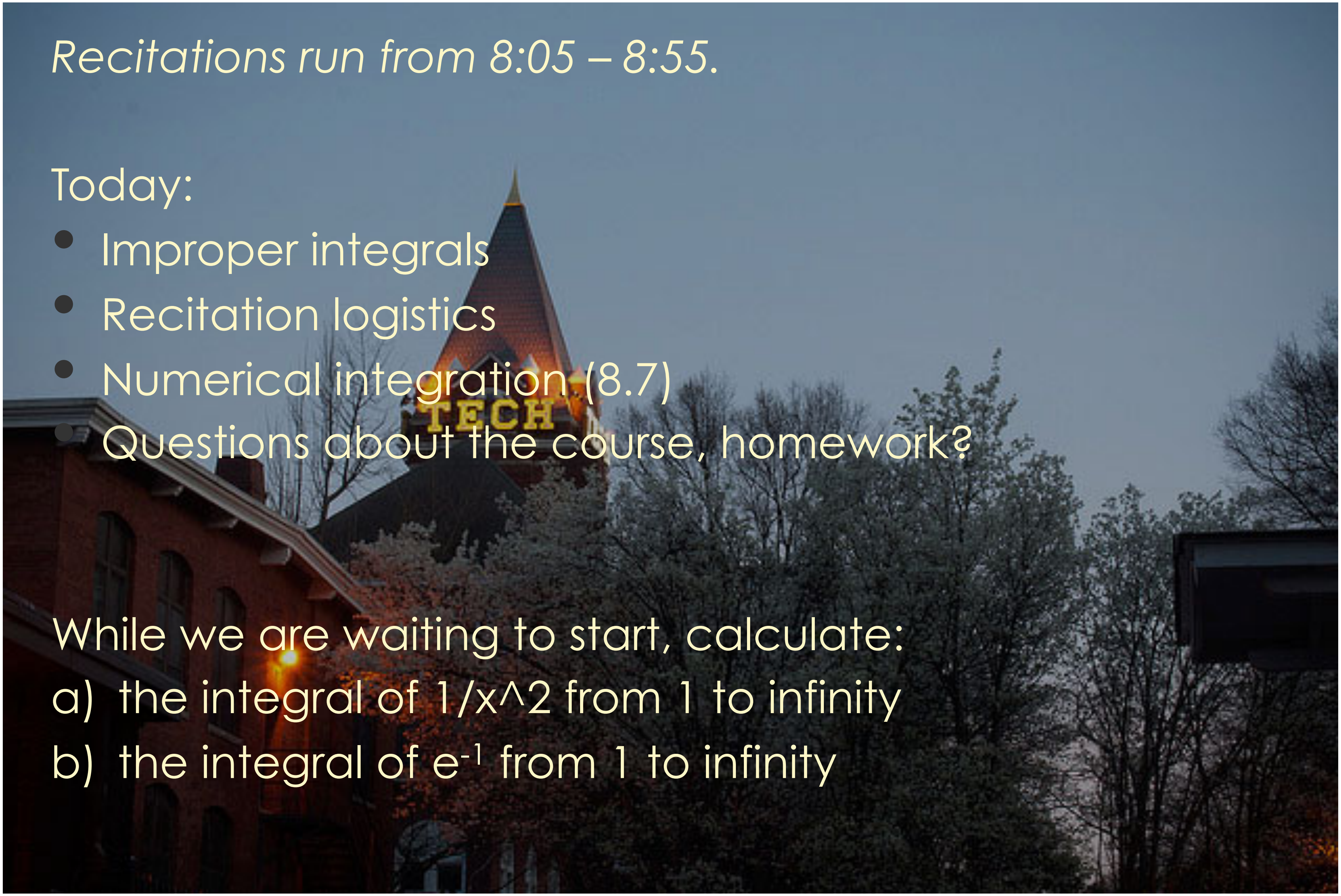
- First homework due ________________

Grading Weights

	Weight (%)
Homework	10
Final	25
Quizzes	60
Pop Quizzes	5
Total	100

Grades will be made available through T-Square

Recitation 02



Recitations run from 8:05 – 8:55.

Today:

- Improper integrals
- Recitation logistics
- Numerical integration (8.7)
- Questions about the course, homework?

While we are waiting to start, calculate:

a) the integral of 1/x^2 from 1 to infinity

b) the integral of e^{-1} from 1 to infinity

Improper Integrals

$$\int_1^\infty \frac{1}{x^2}dx = \lim_{b\to\infty} x^{-1}\Big|_1^b = \lim_{b\to\infty}\left(\frac{-1}{b} - \frac{-1}{1}\right) = +1$$

$$\int_1^\infty e^{-x}\,dx = \lim_{b\to\infty}\int_1^b e^{-x}dx = \lim_{b\to\infty}\left(-e^{-x}\right)\Big|_1^b$$

$$= \lim\left(-e^{-b}\right)-\left(-e^{-1}\right)$$

$$= 0 + 1/e$$

$$= 1/e$$

Example: Integrate 1/x from 1 to 2

a) What is the exact answer?

$$\boxed{\int_1^2 \frac{1}{x}dx = \ln x \Big|_1^2 = \ln 2} \approx 0.69314718\ldots$$

b) Set up but don't evaluate an expression for the area using Simpson's Rule.

Let n=4:

$$S = \frac{1/4}{3}\left(\frac{1}{1} + 4\frac{1}{1+1/4} + 2\frac{1}{1+2/4} + 4\frac{1}{1+3/4} + \frac{1}{1+4/4}\right), \quad \Delta x = \frac{b-a}{N} = \frac{2-1}{4} = \frac{1}{4}$$

$$S = \frac{1}{3}\left(\frac{1}{4} + \frac{4}{4}\frac{4}{5} + \frac{2}{4}\cdot\frac{4}{6} + \frac{4}{4}\cdot\frac{4}{7} + \frac{1}{4}\cdot\frac{4}{8}\right)$$ (NOT NECESSARY)

$$= \frac{1}{3}\left(\frac{1}{4} + \frac{4}{5} + \frac{2}{6} + \frac{4}{7} + \frac{1}{8}\right) = 1747/2520$$

$$\approx 0.693253968\ldots$$

NOTE: N=4 implies 5 function evaluations

Example: Integrate 1/x from 1 to 2

c) Find the number of subintervals required for four digit accuracy using Simpson's rule.

$$E_S \leq \frac{b-a}{180} \max|f''''| (\Delta x)^4,$$

$$= \frac{(b-a)^5}{180N^4} 24$$

$$= \frac{1}{180N^4} 24$$

for 4 digit accuracy, we need

$$E_s < 0.00005$$

$$\Rightarrow \frac{24}{180N^4} < 0.00005$$

$$\Rightarrow N^4 > \frac{8000}{3} \approx 7.2 \Rightarrow N = 8$$

$$f'''' = x^{-5}(-4)(-3)(-2)(-1)$$

max value of x^{-5} occurs at $x=1$.

$\Rightarrow$ let $\max|f''''|$ be $4!(1)^{-5} = 24$

NOTE: E_T: $\frac{24}{12N^2} < 0.00005$, $N > 14.2 \Rightarrow N = 15$

Recitation 03

Today:

1. Improper integrals: comparison test
2. A few announcements
3. Improper integrals: techniques of integration

Use the comparison test to determine whether the following integral converges.

$$\int_1^\infty \frac{1}{\sqrt{1+x^2}}\,dx$$

NOTE THAT: $\sqrt{1+x^2} < 1+x$ on $x \in [1, \infty)$

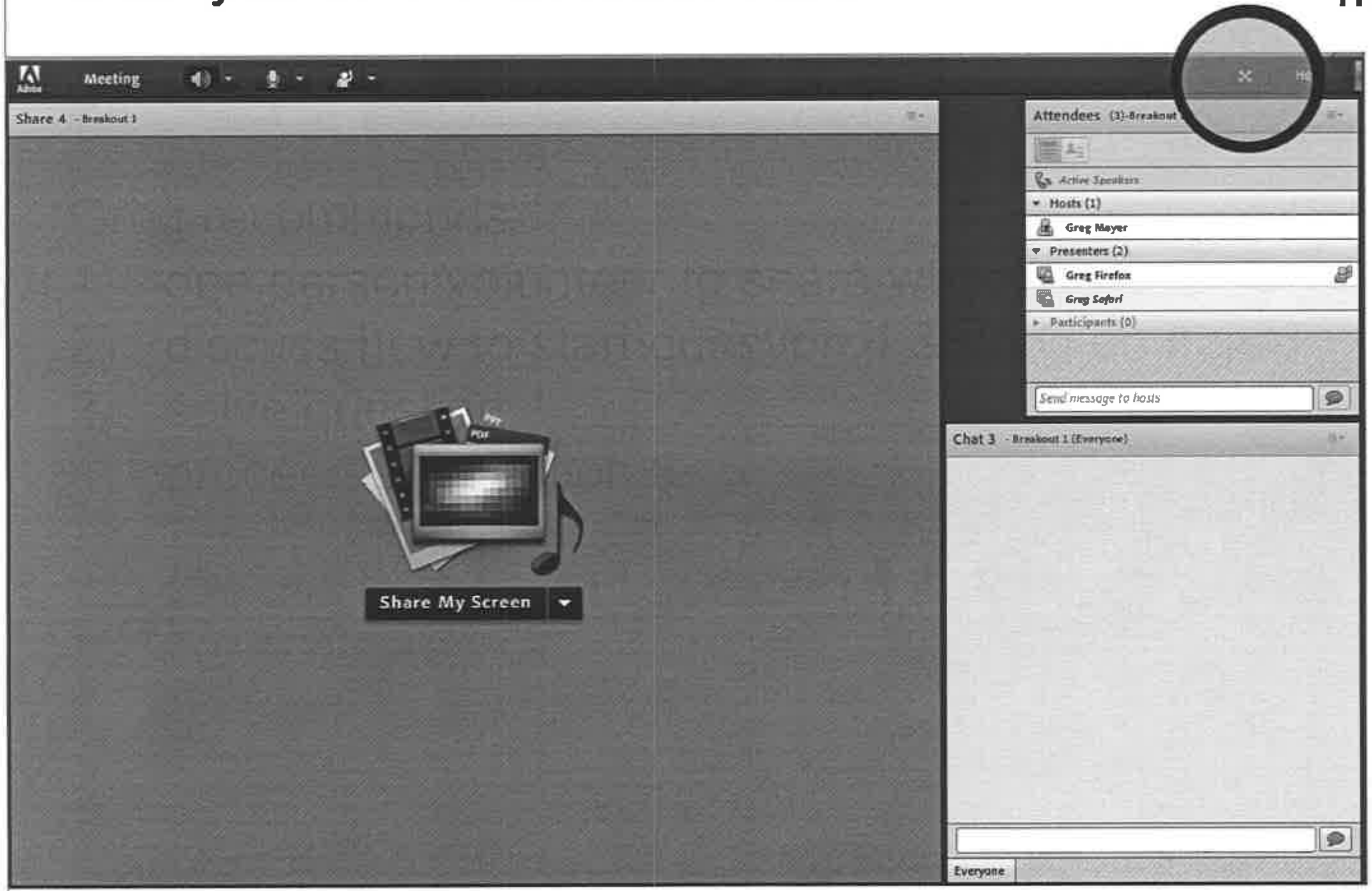
So: $\frac{1}{\sqrt{1+x^2}} > \frac{1}{1+x}$ on $x \in [1, \infty)$

$$\Rightarrow \int_1^\infty \frac{1}{\sqrt{1+x^2}}\,dx > \int_1^\infty \frac{1}{1+x}\,dx = \ln|1+x|\Big|_1^\infty$$

$\Rightarrow$ by comparison test, integral diverges

Group Work

when you are in a breakout room:



message TA

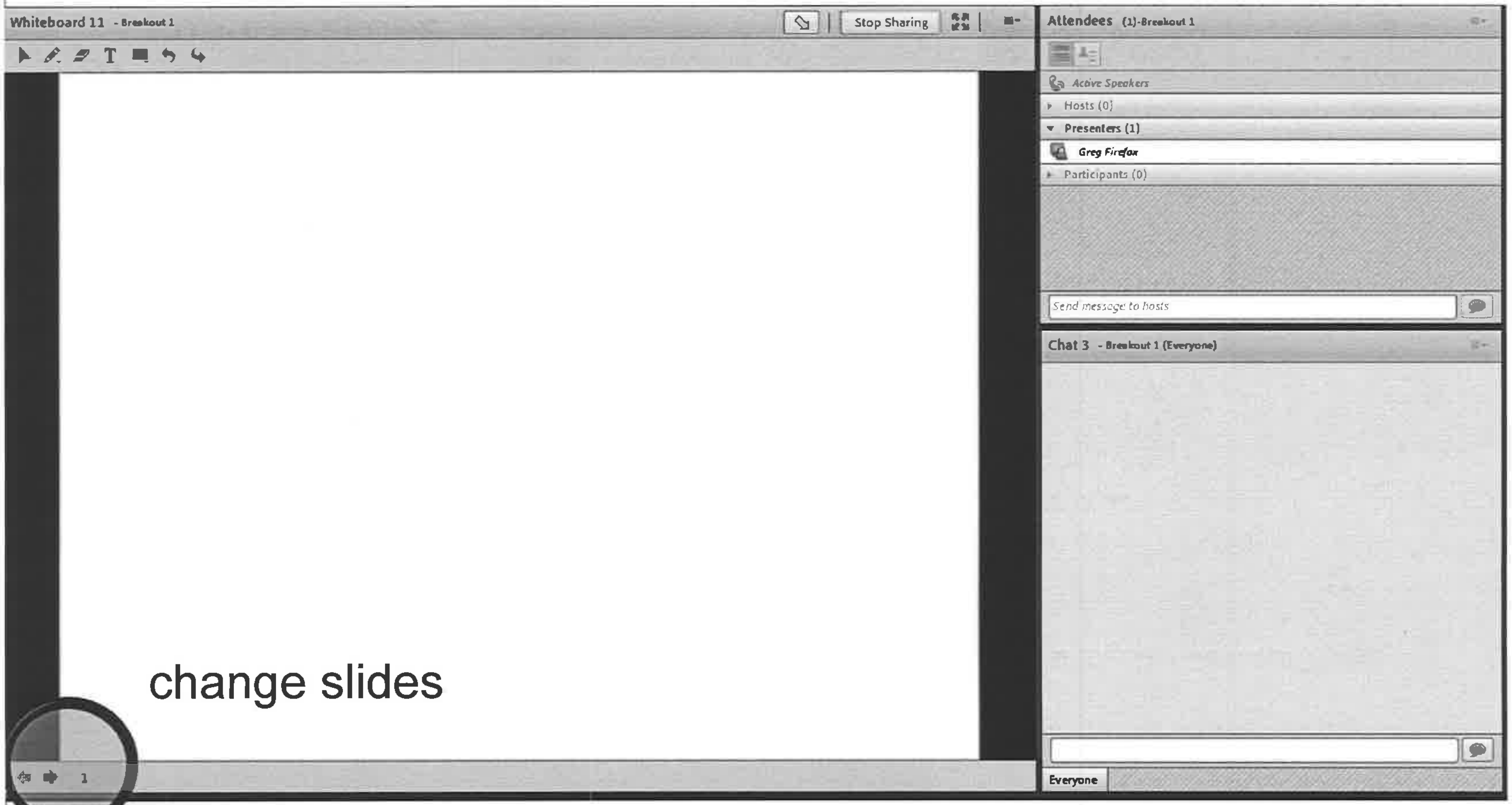
Greg recommends:

1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5

28

Group Work

when you are in a breakout room:



change slides

Greg recommends:

1) one person volunteer to share whiteboard
2) discuss how to start question 4 & draw on board
3) solve question 4
4) proceed to question 5

Group Work Questions

Evaluate the following integrals.

4) $\displaystyle\int_{5}^{\infty} \frac{1}{x^2+25}\,dx$

5) $\displaystyle\int_{0}^{144} \frac{1}{\sqrt{144-x}}\,dx$

6) $\displaystyle\int_{0}^{144} \frac{1}{\sqrt{144-x^2}}\,dx$

7) $\displaystyle\int_{0}^{\infty} \frac{1}{x^2+7x+6}\,dx$

8) $\displaystyle\int_{-\infty}^{\infty} \frac{Ax}{\left(x^2+B\right)^{12}}\,dx$

9) $\displaystyle\int x^3 \ln x\,dx$

4 [illegible]

4) $\int_5^\infty \frac{1}{x^2+25}dx = \frac{1}{25}\int_5^\infty \frac{1}{(\frac{x}{5})^2+1}dx$, $u = x/5$, $dx = 5du$

$= \frac{1}{25}\int_1^\infty \frac{1}{u^2+1}(5du)$

$= \frac{1}{5} \arctan u \Big|_1^\infty$

$= \frac{1}{5}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{20}$

5) $\int_0^{144} \frac{1}{\sqrt{144-x}}dx = \int_{144}^0 \frac{1}{\sqrt{u}}(-du)$, $u = 144-x$, $du = -dx$

$= \int_0^{144} \frac{1}{\sqrt{u}}du$

$= 2 \cdot u^{1/2} \Big|_0^{144}$

$= 24$

6) $\int_0^{12} \frac{1}{\sqrt{144-x^2}}dx \Rightarrow \int_0^{\pi/2} \frac{1}{12\sqrt{1-s^2}}(12c\, d\theta)$ let $\frac{x}{12} = \sin\theta$, so $dx = 12\cos\theta d\theta$

$= \theta\Big|_0^{\pi/2} = \pi/2$

NOTE: $\arcsin 1 = \pi/2$
$\arcsin 0 = 0$

5 8

7) $\int_0^\infty \frac{1}{x^2+7x+6}dx = \int_{\cdot}^{\cdot} \frac{1}{(x+6)(x+1)}dx$,

$\frac{1}{x+6}\,\frac{1}{x+1} = \frac{A}{x+6}+\frac{B}{x+1}$

$1 = A(x+1) + B(x+6)$

$\begin{bmatrix} 1 & 6 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ....

or 'just: $A+6B=1$
$A+B=0$

$\Rightarrow 5B=1$, so $B=1/5$, $A=-1/5$

$= \int_{\cdot}^{\cdot} \frac{-1/5}{x+6} + \frac{1/5}{x+1}\,dx$

$= +\frac{1}{5}\ln\frac{x+1}{x+6}\Big|_0^\infty$

$= \frac{1}{5}\left(\ln 1 - \ln\frac{1}{6}\right)$

$= -\frac{1}{5}\ln\left(\frac{1}{6}\right)$

$= \frac{1}{5}\ln(6)$

8) $\int_{-\infty}^{+\infty} \frac{Ax}{(x^2+B)^{12}}dx$

$= 0$

ODD FUNCTION OVER
SYMMETRIC INTERVAL

9) $\int x^3 \ln x\,dx$

let $u = \ln x$ $\quad dv = x^3 dx$

$du = \frac{1}{x}dx$ $\quad v = x^4/4$

$\int x^3 \ln x\,dx = \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\,\frac{1}{x}dx + C$

$= \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C$

Recitation 04

Today:

1. 1st order linear DE
2. A few announcements
3. Group Work: Seperable & Linear DEs

Consider:

STANDARD FORM: $y' - \frac{1}{x} y = 2x \ln x$

$$xy' - y = 2x \ln x$$

a) Is the DE seperable? NO

b) Is the DE 1st order linear in y(t)? YES

c) What is the integrating factor?

I.F. is $e^{\int \frac{-1}{x} dx} = e^{-\ln x} = e^{\ln 1/x} = 1/x$

$xy' - y = 2x\ln x$

VERY PROCEDURAL ☺

STEP 1 Standard form:

$$y' - \frac{1}{x}y = 2\ln x$$

STEP 2: I.F: $1/x$

STEP 3 multiply by I.F.

$$\underbrace{x^{-1}y' - x^{-2}y}_{\text{this is a derivative}} = 2\frac{\ln x}{x}$$

STEP 4

$$\frac{d}{dx}\left(x^{-1}y\right) = 2\frac{\ln x}{x}$$

by — rule

STEP 5

$$x^{-1}y = 2\int \frac{\ln x}{x}\,dx$$

$$= 2(\ln x)^2 + c$$

$$y = 2x(\ln x)^2 + Cx$$

Announcements

Invitation to Participate in a Study

- you're receiving snail mail soon!
- please review forms with your parents and send them back

Online Survey

- anonymous!
- available at:
- please provide feedback on web cams, collaboration, the chat pod, etc

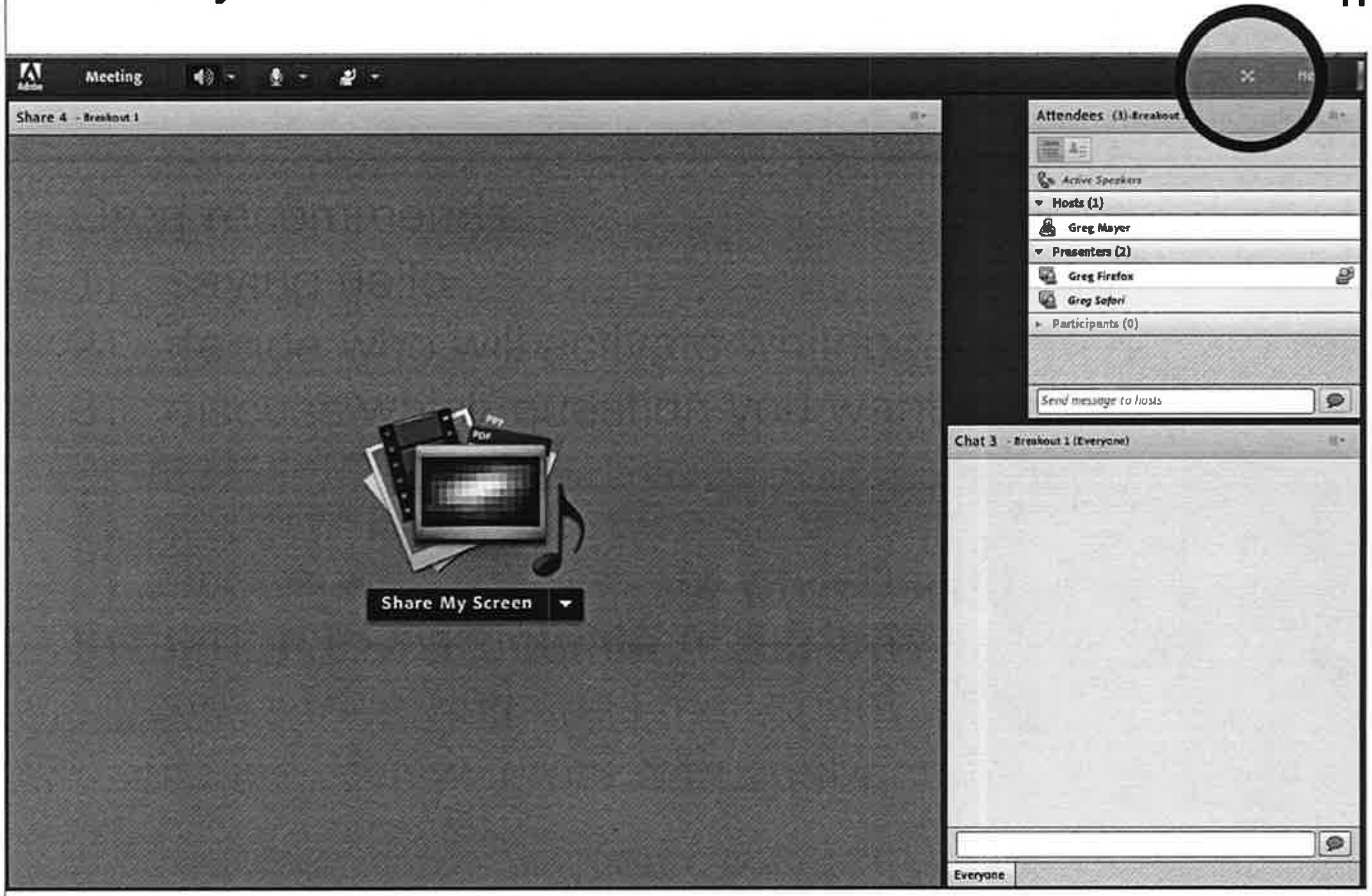
Quiz 1

- Sep 11 (two weeks from today)

Group Work

when you are in a breakout room:

message TA



Greg recommends:

1) saying hello
2) decide who will activate whiteboard
3) discuss which question you want to start on

Group Work Questions

For each of the following DE's,

a) State whether the DE is seperable.

b) State whether the DE is 1st order linear in y.

c) Solve the DE.

1) $xy' + 3y = 3 - \dfrac{4}{x}$

2) $\dfrac{1}{x} y' = ye^{x^2} + 2\sqrt{ye^{x^2}}$

3) $xy' + y = (1 + x)e^x$

4) $(\sin t)y' + (\cos t)y = \tan t$

5) $\cos(y) + (1 + e^{-x})(\sin y)y' = 0$

1) $xy' + 3y = 3 - 4/x$ not sep is lin. STANDARD FORM: $y' + \frac{3}{x}y = \frac{3}{x} - 4/x^2$

I.F. is $e^{\int \frac{3}{x} dx} = e^{\ln x^3} = x^3$

MULTIPLY BY I.F.:

$$x^3 y' + 3x^2 y = 3x^2 - 4x$$

$$x^3 y = \int 3x^2 - 4x \, dx + C$$

$$= x^3 - 2x^2 + C$$

$$y = 1 - 2/x + C/x^3$$

2) $xy' + y = (1+x)e^x$

S.F.: $y' + \frac{1}{x}y = \frac{1+x}{x}e^x$

I.F. is $e^{\int 1/x \, dx} = x$

$\Rightarrow \quad xy' + y = (1+x)e^x$

$$xy = \int (1+x)e^x \, dx$$

$$= e^x + \int xe^x \, dx + C, \qquad \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = e^x \end{matrix}$$

$$= e^x + xe^x - \int e^x \, dx + C$$

$$= xe^x + C$$

$$y = e^x + C/x$$

3) $\frac{1}{x}y' = ye^{x^2} + 2\sqrt{y}\,e^{x^2}$

not linear, but seperable:

$$\frac{1}{x}y' = e^{x^2}(y + 2\sqrt{y})$$

$$\frac{1}{y+2\sqrt{y}}\,dy = xe^{x^2}\,dx$$

let $\sqrt{y} = u$, $dy\left(\frac{1}{2}\frac{1}{\sqrt{y}}\right) = du$

$dy = 2u\,du$

$$\frac{1}{u^2+2u}\,2u\,du = \int xe^{x^2}\,dx$$

, $u = x^2$, $du = 2x\,dx$

$$2\,\frac{1}{u+2}\,du = \frac{1}{2}\int e^u\,du$$

$$\boxed{2\ln(\sqrt{y}+2) = \frac{1}{2}e^{x^2} + C}$$

$$\ln(\sqrt{y}+2| = \frac{e^{x^2}}{4} + c$$

$$y = \left(e^{c + e^{x^2}/4} - 2\right)^2$$

4) $\sin t \; y' + \cos t \; y = \tan t$

SF: $y' + \frac{c}{s} y = \frac{t}{s}$

IF: $e^{\int c/s \, dt}$. Let $u = \sin t$, $du = \cos t \, dt$

$= e^{\int \frac{1}{u} du} = e^{\ln u} = u = s$

$\Rightarrow s y' + c y = \tan t$

$s y = \int \tan t \, dt = -\ln \cos t + c$ etc

5) $\cos y + (1+e^{-x})(\sin y) \, y' = 0$

$\frac{s}{c} dy = -\frac{1}{1+e^{-x}}$, $u = 1+e^{-x}$

$du = -e^{-x} dx = (1-u) dx$

$\ln|\sec y| = -\int \frac{e^x}{e^x+1} dx$, $u = e^x$, $du = u \, dx$

$= -\int \frac{u}{u+1} \; \frac{1}{u} du$

$= -\int \frac{1}{u+1} du$

$\ln|\sec y| = -(1+e^{-x}) + c$

Recitation 05

Today:

1. Geometric Series
2. Group Work: Infinite Series

The sum of a geometric series is: $\sum_{k=1}^{\infty} ar^{k-1} = \frac{a}{1-r}$

The geometric series is convergent iff $|r| < 1$.

Express 1.7979797979 … as a rational number.

$$1.\overline{79} = 1 + 79 \cdot 10^{-2} + 79 \cdot 10^{-4} + \ldots$$

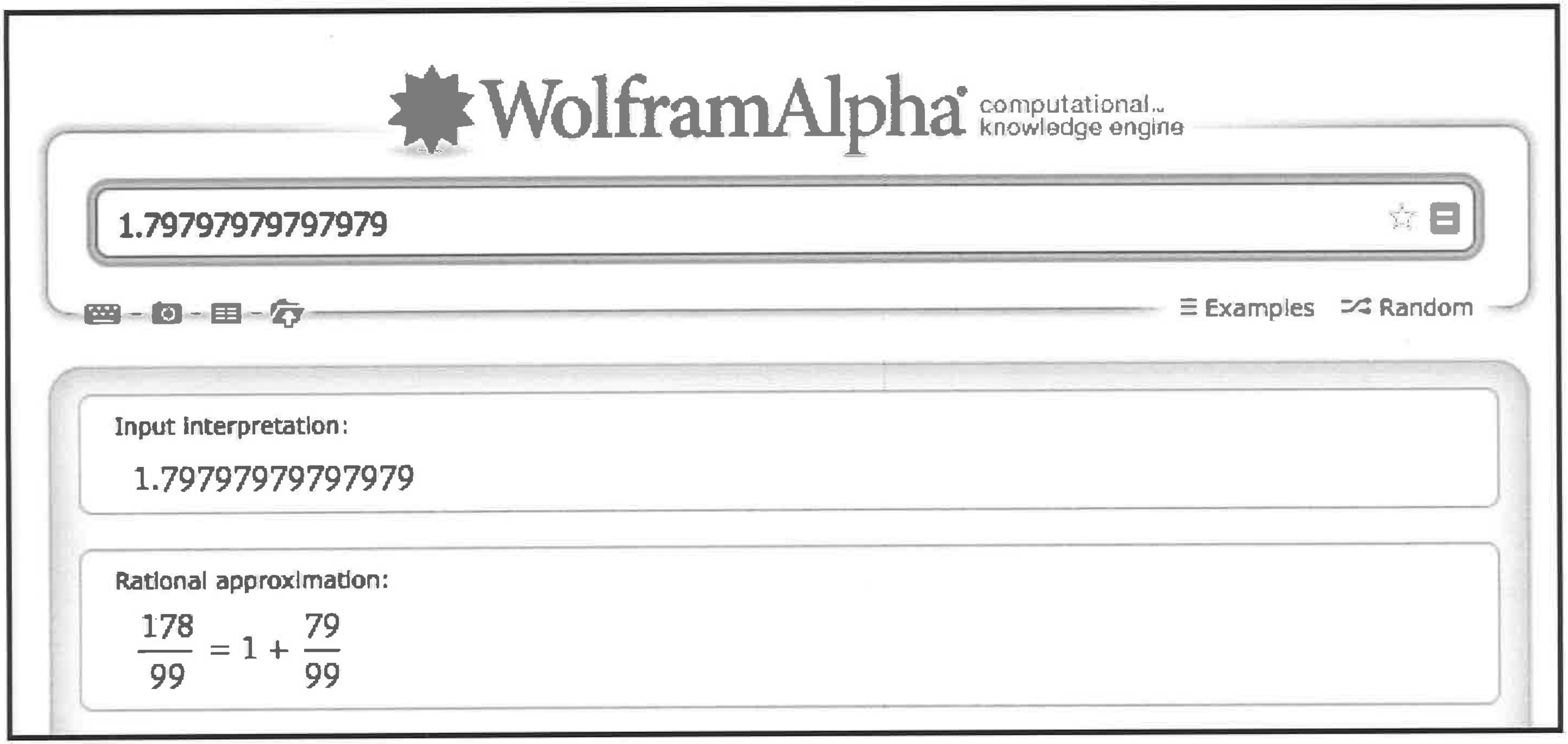
$$= 1 + 79\left(10^{-2} + 10^{-4} + 10^{-6} + \ldots\right)$$

$$= 1 + 79 \sum_{k=1}^{\infty} 10^{-k \cdot 2} = 1 + 79 \sum_{k=1}^{\infty} \left(\frac{1}{100}\right)^{k} = 1 + 79 \frac{1}{100} \sum_{k=1}^{\infty} \left(\frac{1}{100}\right)^{k-1}$$

$$= 1 + \frac{79}{100} \frac{1}{1 - 1/100}$$

$$= 1 + 79/99$$

Please complete Start-of-Term Survey at:
https://www.surveymonkey.com/s/Math1501StartOfTerm2014



1.79797979797979

Input interpretation:

1.79797979797979

Rational approximation:

$$\frac{178}{99} = 1 + \frac{79}{99}$$

Determine if the following series convergent or divergent. If it is convergent, find its sum.

$$\sum_{k=1}^{\infty}\frac{4^{k+1}}{5^k} = \sum \frac{4\cdot 4\cdot 4^{k-1}}{5\cdot 5^{k-1}} = \frac{16}{5}\sum\left(\frac{4}{5}\right)^{k-1}$$

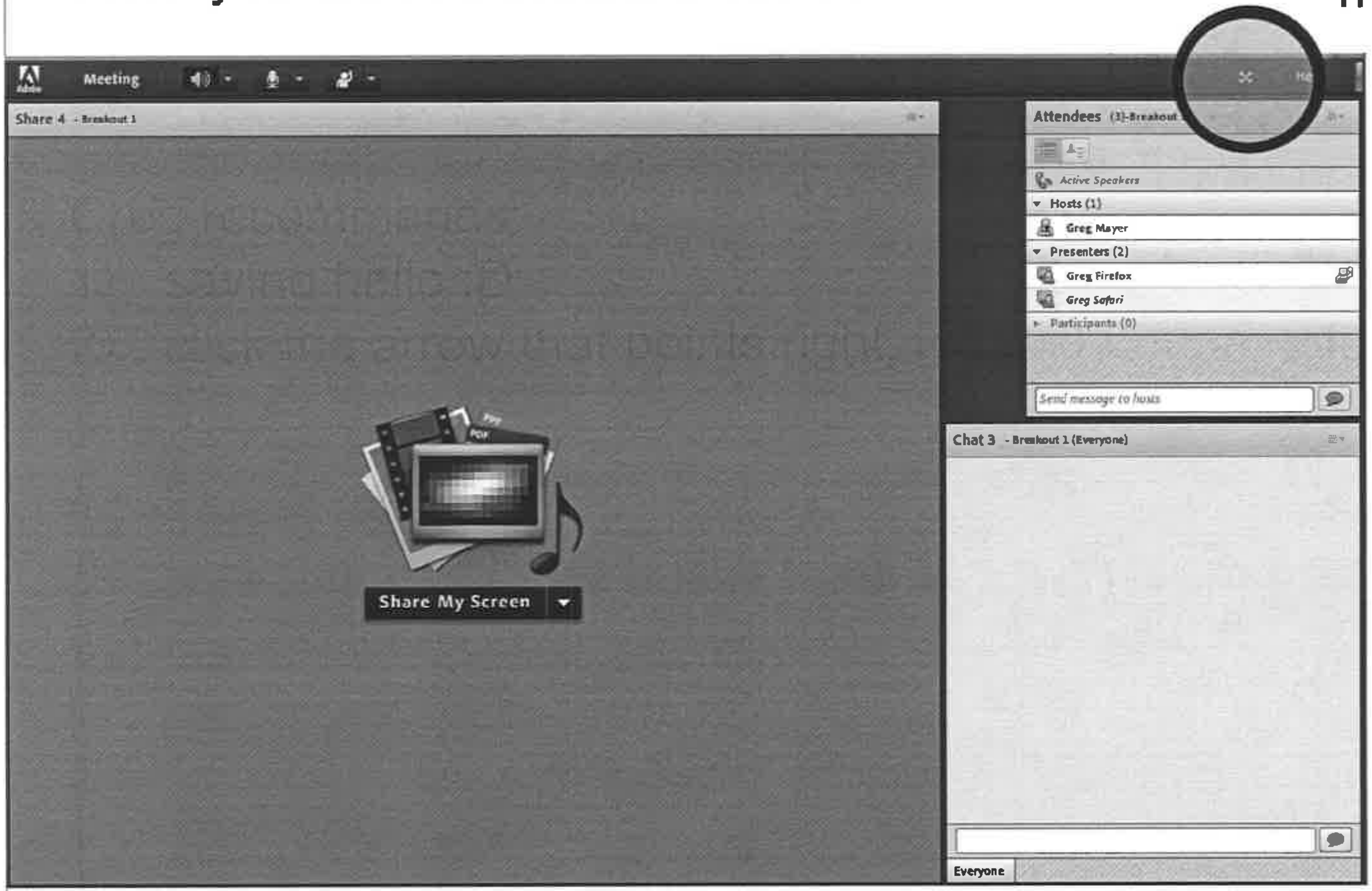
$$= \frac{16}{5}\frac{1}{1-4/5}$$

$$=16$$

CONVERGENT BECAUSE (ABS VAL) COMMON RATIO IS $\left|4/5\right| < 1$

Group Work

when you are in a breakout room:



message TA

Greg recommends:

1) saying hello :D
2) click the arrow that points right, located bottom left of screen

Group Work Questions

0.301301....

1) Express ~~1.1232323~~.... as a ratio of integers.

2) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k}$$

$$B)\ \sum_{k=1}^{\infty} 2^{2k} 3^{1-k}$$

$$C)\ \sum_{k=1}^{\infty} \frac{(-5)^k}{4^{k+1}}$$

$$D)\ \sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)}$$

1) 1.1232323 -----

1) $1.1\overline{23} = 1.1 + 23\left(10^{-3} + 10^{-5} + \dots\right)$

$$= 1.1 + \frac{23}{10}\left(10^{-2} + 10^{-4} + \dots\right)$$

$$= 1.1 + \frac{23}{100}\sum_{k=1} 10^{-2k}$$

$$= \frac{11}{10} + \frac{23}{10}\sum_{k=1}^{\infty}\left(\frac{1}{100}\right)^{k}$$

$$= \frac{11}{10} + \frac{23}{10}\,\frac{1}{100}\sum_{k=1}^{\infty}\left(\frac{1}{100}\right)^{k-1}$$

$$= \frac{11}{10} + \frac{23}{1000}\,\frac{1}{1-1/100}$$

$$= \frac{11}{10} + \frac{23}{1000}\,\frac{100}{99}$$

$$= \frac{11}{10} + \frac{23}{990}$$

$$= \frac{556}{495} = 1\frac{61}{495}$$

*)

$$\left.\begin{aligned}
0.\overline{301} &= 301\left(10^{-3} + 10^{-6} + \dots\right) \\
&= 301\left(\sum_{k=1}^{\infty} 10^{-3k}\right) \\
&= 301 \sum_{k=1}^{\infty} \left(\frac{1}{1000}\right)^{k} \\
&= \frac{301}{1000} \sum_{k=1}^{\infty} \left(\frac{1}{1000}\right)^{k-1} \\
&= \frac{301}{1000} \frac{1}{1 - \frac{1}{1000}} \\
&= \frac{301}{999} \quad \checkmark
\end{aligned}\right\}$$

WE DIDN'T DO THIS ONE

2A)

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{3^k + 2^k}{6^k} &= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{k} + \left(\frac{1}{3}\right)^{k} \\
&= \frac{1}{2} \sum 0.5^{k-1} + \frac{1}{3} \sum \left(\frac{1}{3}\right)^{k-1} \\
&= \frac{1}{2} \frac{1}{1-1/2} + \frac{1}{3} \frac{1}{1-1/3} \\
&= 1 + 1/2
\end{aligned}$$

2B) $\sum_{k=1}^{\infty} 2^{2k} 3^{1-k} = \sum 4^k \cdot 3 \cdot 3^{-k} = 3 \sum \left(\frac{4}{3}\right)^k \Rightarrow$ diverges

2C) $\sum_{k=1}^{\infty} \frac{(-1)^k}{4^{k+1}} = \sum \frac{(-1)(-1)^{+k-1}}{4 \cdot 4 \cdot 4^{k-1}} = \frac{-1}{16} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{4^{k-1}}$

$= \frac{-1}{16} \frac{1}{1-(-1/4)}$

$= \frac{-1}{16} \frac{4}{5} = \frac{-1}{20}$ ✓

(we did a slightly different question, ($\sum\left(\frac{-5}{4}\right)^k$ diverges))

2D) $\sum_{k=1}^{\infty} \frac{1}{(3k-2)(3k+1)} = \sum_{k=1}^{\infty} \frac{1}{3} \frac{1}{3x-2} - \frac{1}{3} \frac{1}{3x+1}$

$= \left(\frac{1}{3} - \cancel{\frac{1}{12}}\right) + \left(\cancel{\frac{1}{12}} - \cancel{\frac{1}{21}}\right) + \left(\cancel{\frac{1}{21}} - \cancel{\frac{1}{30}}\right) + \left(\cancel{\frac{1}{30}} - \ldots\right.$

$= 1/3$

Recitation 06

Today:

1. Test for Divergence
2. Group Work: Infinite Series, Differential Equations

The **Test for Divergence (Theorem 7 from 10.2)**

$$\text{If } \lim_{k\to\infty} a_k \text{ is not zero or does not exist, then } \sum_{k=1}^{\infty} a_k \text{ diverges.}$$

Note that:

If $\lim_{k\to\infty} a_k$ is equal to zero, then the divergence test is inconclusive

Use the test for divergence to determine whether these series converge.

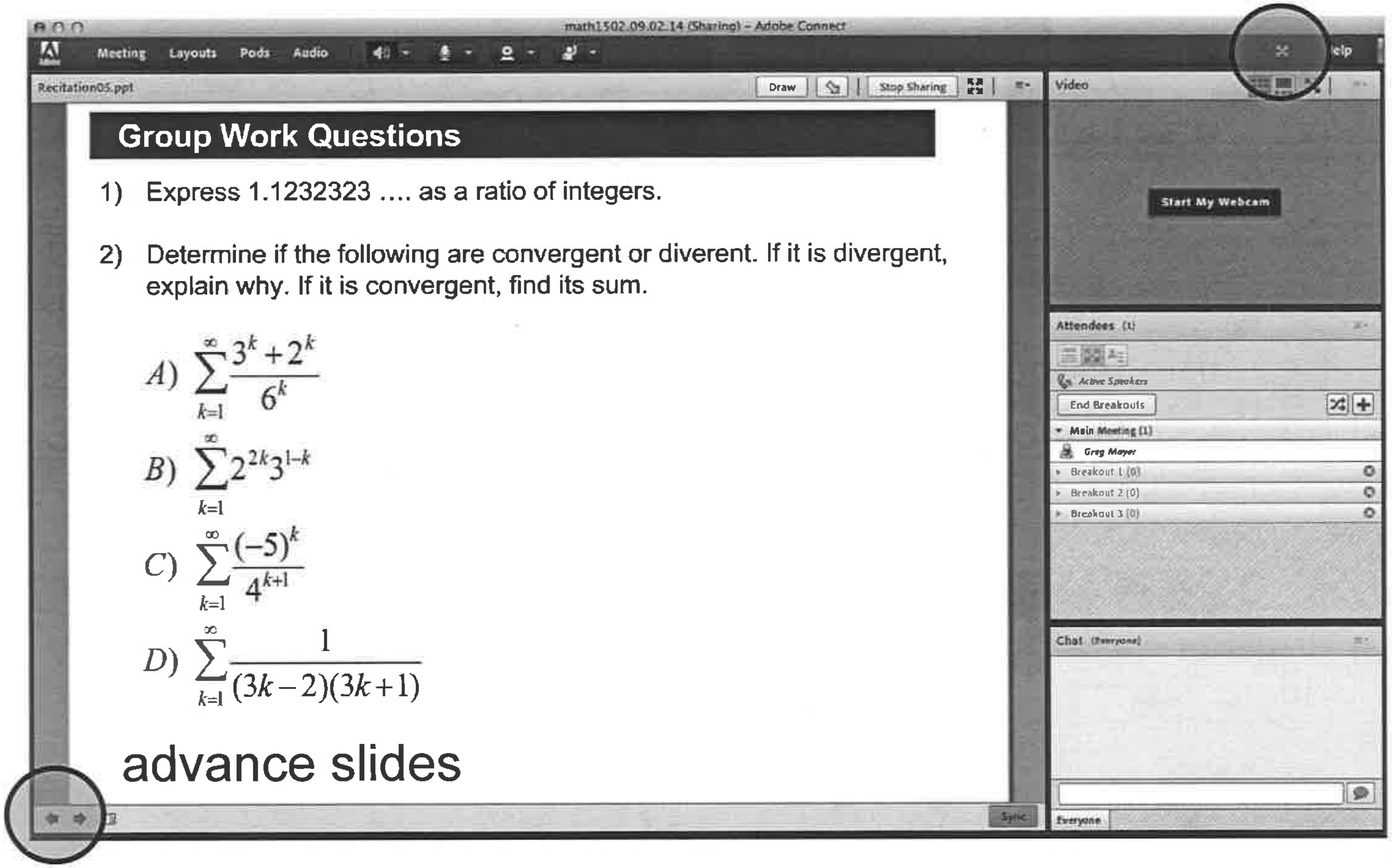
A) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k+2}$ the test for divergence is inconclusive, because $\lim_{k\to\infty} \frac{1}{k+2} = 0$

B) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{k+2}$ diverges, because $\lim_{k\to\infty} \frac{k}{k+2} = \lim_{k\to\infty} \frac{1}{1+2/k} = 1 \neq 0$

Group Work

When you are in a breakout room:

message TA



Group Work Questions

1) Express 1.1232323 …. as a ratio of integers.

2) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty}\frac{3^k+2^k}{6^k}$$

$$B)\ \sum_{k=1}^{\infty}2^{2k}3^{1-k}$$

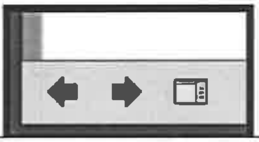
$$C)\ \sum_{k=1}^{\infty}\frac{(-5)^k}{4^{k+1}}$$

$$D)\ \sum_{k=1}^{\infty}\frac{1}{(3k-2)(3k+1)}$$

advance slides

Greg recommends:

1) Saying hello :D
2) Each student write & chat in their favourite color
3) Advance to the next slide with arrows located bottom left



Group Work Questions

1) Express 0.002100210021 …. as a ratio of integers.

2) Solve the following DE for x > 0: $x^3 y' + (2 - 3x^2)y = x^3$

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$A)\ \sum_{k=1}^{\infty} \frac{k^2}{5k^2+5}$

$B)\ \sum_{k=1}^{\infty} \frac{1}{k(k+3)}$

$C)\ \sum_{k=1}^{\infty} 3(0.4)^k - 2(-0.1)^{k+1}$

$D)\ \sum_{k=1}^{\infty} \left(\sin\left(\frac{1}{k}\right) - \sin\left(\frac{1}{k+1}\right)\right)$

$E)\ \sum_{k=1}^{\infty} \arctan(k)$

4) Consider the DE $y' + y = y^2 e^x$
   a) Determine if the DE is seperable, and/or 1st order linear in y.
   b) Solve the DE for x > 0. *Hint: let z = 1/y.*

Group Work Questions

1) Express 0.002100210021 …. as a ratio of integers.

$$= 21\left(10^{-4} + 10^{-8} + 10^{-12} + , \dots\right)$$

$$= 21\left(\sum_{k=1}^{\infty} 10^{-4k}\right)$$

$$= 21\left(\sum_{k=1}^{\infty} \left(\frac{1}{10000}\right)^{k}\right.$$

$$= \frac{21}{10000} \sum_{k=1}^{\infty} \left(\frac{1}{10000}\right)^{k-1}$$

$$= \frac{21}{10000} \frac{1}{1 - \frac{1}{10000}}$$

$$= \frac{21 \cdot 10000}{10000 \cdot 9999}$$

$$= \frac{21}{9999} = 7/3333$$

Group Work Questions

2) Solve the following DE for x > 0: $x^3 y' + (2 - 3x^2)y = x^3$

STANDARD FORM:

$$y' + \frac{2-3x^2}{x^3} y = 1$$

I.F. is $\exp\left(\int \frac{2-3x^2}{x^3}\,dx\right) = \exp\left(x^{-2} - \ln x^3\right) = F(x)$

DE BECOMES:

$$(F)y' + (F)\left(\frac{2-3x^2}{x^3}\right)y = F$$

$$Fy = \int F\,dx = \int \frac{e^{x^{-2}}}{x^3}\,dx = \frac{1}{2}e^{-1/x^2} + C$$

$$y = \frac{x^3}{2} + Cx^3 e^{1/x^2}$$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

A) $\sum_{k=1}^{\infty} \frac{k^2}{5k^2+5}$

$$\lim_{k \to \infty} \frac{k^2}{5k^2+2} = \lim_{k \to \infty} \frac{1}{5+2/k^2} = 1/3$$

$\Rightarrow$ diverges by the divergence test

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$B)\ \sum_{k=1}^{\infty}\frac{1}{k(k+3)} = \sum_{k=1}^{\infty} \frac{1/3}{k} - \frac{1/3}{k+3}$$

$1 = A(k+3) + B(k)$

$A + B = 0$

$3A = 1$

$$= \frac{1}{3}\left[\left(1 - \cancel{\tfrac{1}{4}}\right) + \left(\tfrac{1}{2} - \cancel{\tfrac{1}{5}}\right) + \left(\tfrac{1}{3} - \cancel{\tfrac{1}{6}}\right) + \left(\cancel{\tfrac{1}{4}} - \tfrac{1}{7}\right) + \dots\right]$$

$$= \frac{1}{3}\left(1 + \tfrac{1}{2} + \tfrac{1}{3}\right]$$

$$= \frac{1}{3}\left(\frac{6}{6} + \frac{3}{6} + \frac{2}{6}\right)$$

$$= \frac{11}{18}$$

$$= 0.61111 \dots$$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

C) $\sum_{k=1}^{\infty} 3(0.4)^{k} - 2(-0.1)^{k+1}$

$$= 3\left(\frac{0.4}{1-0.4}\right) - 2\sum_{k=1}^{\infty}(-0.1)^{k+1}$$

$$= 2 - 2(0.1)(0.1)\sum_{k=1}^{\infty}(-0.1)^{k-1}$$

$$= 2 - 0.02\,\frac{1}{1-(-0.1)}$$

$$= 2 - \frac{0.02}{1.1}, \quad \left(\text{NOTE: } \frac{0.02}{1.1} = 0.0181818\text{------}\right)$$

$$= 2 - \frac{2}{110} = \frac{218}{110} = 1.981818\ldots$$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

D) $\sum_{k=1}^{\infty}\left(\sin\left(\frac{1}{k}\right)-\sin\left(\frac{1}{k+1}\right)\right)$

$= \left(\sin 1 - \sin\frac{1}{2}\right) + \left(\sin\frac{1}{2} - \sin\frac{1}{3}\right) + \cdots\cdots$

$= \sin 1$

Group Work Questions

3) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$E)\ \sum_{k=1}^{\infty} \arctan(k)$

$\arctan k \to \frac{\pi}{2}$ as $k \to \infty$

so $\sum_{k=1}^{\infty} \arctan k$ D.N.E. by the divergence test.

$y' + y = y^2 e^x$. Not seperable. Not linear in y.

Let $z = \frac{1}{y}$. Then $\overbrace{\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}}^{\text{chain rule}} = \frac{\frac{dy}{dx}}{-y^2}$, because $\frac{dz}{dy} = \frac{-1}{y^2}$.

so $\frac{dy}{dx} = -y^2 \frac{dz}{dx}$

and $y = 1/z$.

The DE becomes:

$$y' + y = y^2 e^x$$

$$-y^2 \frac{dz}{dx} + \frac{1}{z} = \frac{1}{z^2} e^x$$

$$-\frac{1}{z^2}\frac{dz}{dx} + \frac{1}{z} = \frac{1}{z^2} e^x$$

$$\frac{dz}{dx} - z = e^x$$

This is a linear DE in z. I.F. is e^{-x}

$$e^{-x}\frac{dz}{dx} - e^{-x} z = 1$$

$$e^{-x} z = x + c \Rightarrow z = \frac{1}{y} = xe^x + ce^x$$

or: $y = 1/(xe^x + ce^x)$

Recitation 07

Today:

1. Announcements
2. Quiz Review

For what values of x does the following converge? Why?

$$\sum_{k=1}^{\infty} x^{k-1} = 1 + x + x^2 + x^3 + \ldots$$

$|x| < 1$, geometric series

When the series converges, what is the series equal to?

$$= \frac{1}{1-x}$$

Express the following as an infinite series.

$$\frac{1}{1-x^4} = \frac{1}{1-u}, \quad u = x^4$$

$$= \sum_{k=1}^{\infty} u^{k-1} = \sum_{k=1}^{\infty} (x^4)^{k-1} = \sum_{k=1}^{\infty} x^{4k-4}$$

How Quizzes Work

How Quizzes Work

1. Facilitator gets a copy of quiz
2. You write quiz on Thursday, during recitation (8:05 – 8:55)
3. Quiz is proctored (perhaps by your facilitator)
4. Give your completed quiz to your proctor.

Format of Quiz

- Are calculators allowed?
- Are textbooks and notes allowed?
- Quiz length: probably three pages, 1 question per page, questions can have multiple parts
- Don't write on back of pages
- Leave a 1 inch margin around edges of page
- If run out of space, you can use extra pages

How Quizzes Work

If You Have Questions During the Quiz

1. If your proctor agrees, you can use to Adobe Connect to ask your TA questions (you'd need to be closely proctored)
2. Otherwise: call/text me at [redacted]

Before The Quiz

Find your facilitator, and discuss

- where you are writing the quiz, and
- whether you are connecting via Adobe Connect.

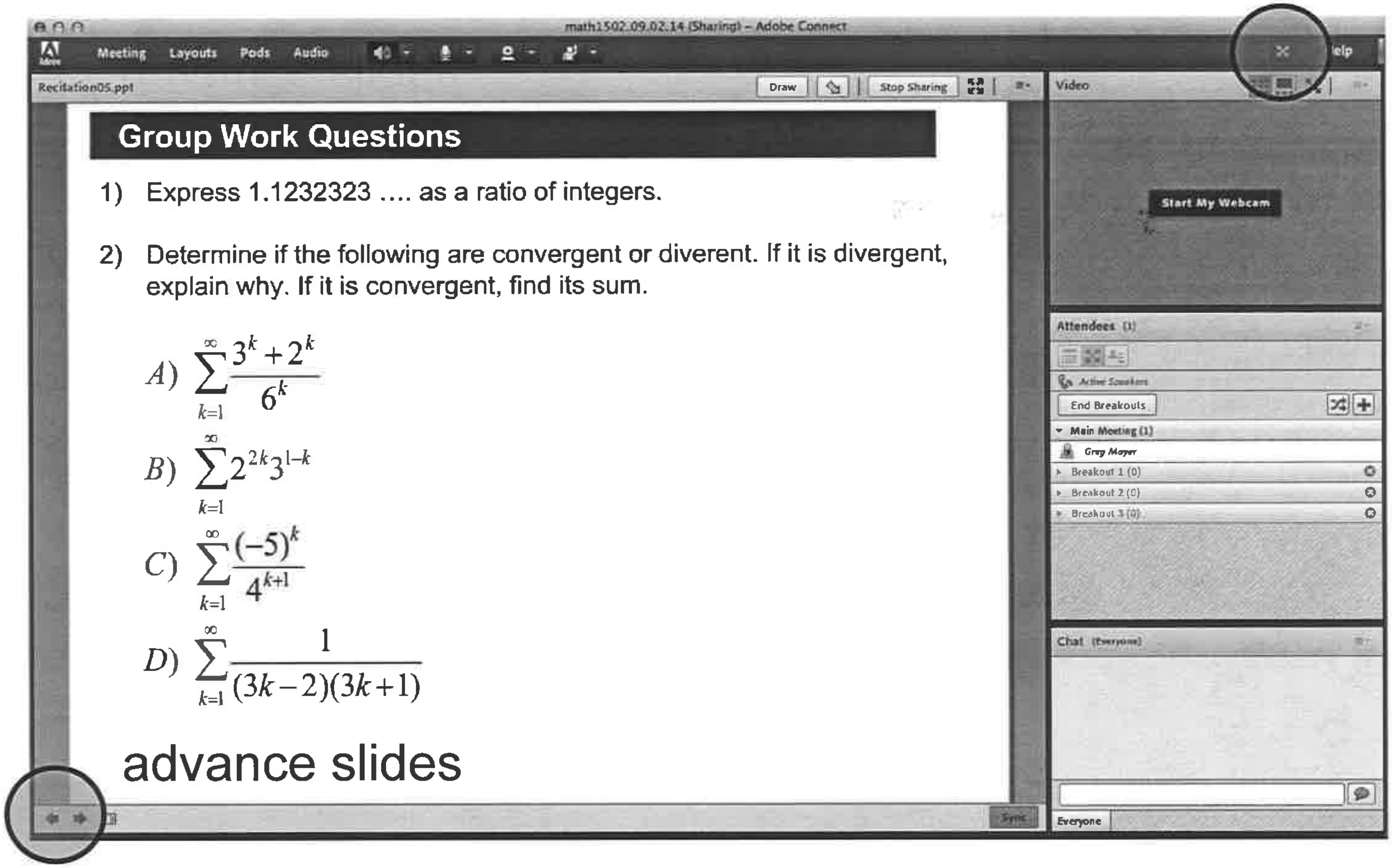
What Happens After The Quiz

1. Your facilitator scans/emails your quiz to GT,
2. they get graded in about a week or two,
3. your TA enters grades in T-Square,
4. your graded quiz is returned via email to you.

Group Work

When you are in a breakout room:

message TA



Group Work Questions

1) Express 1.1232323 …. as a ratio of integers.

2) Determine if the following are convergent or diverent. If it is divergent, explain why. If it is convergent, find its sum.

$$A)\ \sum_{k=1}^{\infty}\frac{3^k+2^k}{6^k}$$

$$B)\ \sum_{k=1}^{\infty}2^{2k}3^{1-k}$$

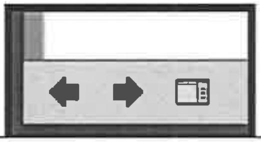
$$C)\ \sum_{k=1}^{\infty}\frac{(-5)^k}{4^{k+1}}$$

$$D)\ \sum_{k=1}^{\infty}\frac{1}{(3k-2)(3k+1)}$$

advance slides

Greg recommends:

1) Saying hello :D
2) Each student write & chat in their favourite color
3) Advance to the next slide with arrows located bottom left



1) Solve xy’ + 2y = cos(x) / x, with y(π) = 1.

STANDARD FORM:

$$y' + \frac{2}{x} y = \frac{\cos x}{x^2}$$

I.F.: $e^{\int 2/x\,dx} = e^{2\ln x} = x^2$

DE. BECOMES:

$$x^2 y' + 2xy = \cos x$$

$$x^2 y = \int \cos x\, dx + C$$

$$= +\sin x + C$$

$$y = \frac{\sin x}{x^2} + C/x^2$$

$$C = \pi^2$$

$$\Rightarrow y = \frac{\sin x}{x^2} + \pi^2/x^2$$

2) Find the sum of the following series.

$$\sum_{k=3}^{\infty}\frac{(-2)^{k+2}}{5^{k-1}} = \sum_{k=1}^{\infty}\frac{(-2)^{k+4}}{5^{k+1}}$$

$$= \sum_{k=1}^{\infty}\frac{(-2)^5}{5^2}\frac{(-2)^{k-1}}{5^{k-1}}$$

$$= \frac{-32}{25}\sum_{k=1}^{\infty}\left(\frac{-2}{5}\right)^{k-1}$$

$$= \frac{-32}{25}\frac{1}{1+2/5}$$

$$= \frac{-32}{25}\frac{5}{7}$$

$$= \frac{-32}{35}$$

3) Find the number of intervals, N, needed to ensure an accuracy of at least 0.01 using Simpson's Rule for the following integral.

$$\int_1^e \ln x \, dx$$

$$E_s \le \frac{(b-a)^5}{180 N^4} M, \quad \begin{cases} M = \max_{x \in [1,e]} \left| \frac{d^4}{dx^4} \ln x \right| \\ \frac{d^4}{dx^4} \ln x = \frac{-6}{X^4} \\ \Rightarrow \text{use } M = 6 \end{cases}$$

$$\frac{(e-1)^5}{180 N^4} 6 \le 0.01$$

$$\frac{(e-1)^5 \cdot 6}{180 \cdot 0.01} \le N^4$$

$$N \ge 2.658 \ldots$$

$\Rightarrow$ N must be at least 3

4) Determine whether the following converge. Do not evaluate these integrals.

A) $\displaystyle\int_e^\infty \frac{1}{x\ln x}\,dx$ $\displaystyle = \lim_{b\to\infty}\int_e^b \frac{1}{x\ln x}\,dx = \lim_{b\to\infty} \frac{1}{2}\left(\ln(\ln x)\right)\Big|_e^b$ D.N.E.

(can use u substitution with $u = \ln x$)

(DNE = DOES NOT EXIST)

B) $\displaystyle\int_0^\infty \cos(x)\sin(x)e^{-2x}\,dx$

$|\cos x \sin x\, e^{-2x}| \le e^{-2x}$ for $x \in [0,\infty]$

and $\displaystyle\int_0^\infty e^{-2x}\,dx$ exists, so the given integral

converges.

5~~6~~) Express as an infinite series.

$$\frac{x}{1+x^2} = \frac{x}{1-(-x^2)} = \frac{x}{1-u}, \quad u = -x^2$$

$$= x\sum_{k=1}^{\infty} u^k$$

$$= x\sum_{k=1}^{\infty} (-x^2)^k$$

$$= \sum_{k=1}^{\infty} (-1)^k x^{2k+1}$$

6) Find the sum of the following series.

$$\sum_{k=0}^{\infty}\frac{5}{(k+1)(k+5)} = 5\sum_{k=0}^{\infty}\frac{1/4}{k+1}-\frac{1/4}{k+5}$$

(partial fractions)

$$= \frac{5}{4}\left(\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+0+0+0+\dots\right)$$

(TELESCOPIC SERIES)

$$= \frac{125}{48}$$

<u>WORK FOR PARTIAL FRACTION DECOMPOSITION</u>

$$\frac{1}{(k+1)(k+5)} = \frac{A}{k+1}+\frac{B}{k+5}$$

$$1 = A(k+5) + B(k+1)$$

$$1 = k(A+B) + (5A+B)$$

$$\underline{0\cdot k} + 1 = \underline{k(A+B)} + (5A+B)$$

$$\Rightarrow \left.\begin{matrix} A+B=0 \\ 5A+B=1 \end{matrix}\right\} \begin{matrix} A=-B,\ 5A+(-A)=1 \\ A=\frac{1}{4},\ B=-\frac{1}{4} \end{matrix}$$

7) Determine whether the following series converges.

A) $\sum_{k=1}^{\infty} \frac{k^2+3}{k3^k}$

USE RATIO TEST

$$\lim_{k\to\infty} \frac{(k+1)^2+3}{k^2+3} \cdot \frac{k3^k}{(k+1)(3^{k+1})}$$

$$= \lim_{k\to\infty} \underbrace{\frac{k^2+2k+4}{k^2+3}}_{\text{GOES TO 1}} \cdot \underbrace{\left(\frac{k}{k+1}\right)}_{\text{GOES TO 1}} \cdot \frac{1}{3}$$

$$= 1/3$$

$\Rightarrow$ converges by ratio test.

Recitation 09

Today: 10.7 (Power Series, Radius/Interval of Convergence and Absolute Convergence)

A) The interval of convergence is set of all values of x s.t. the series converges

B) If an infinite series Σa_n is called **absolutely convergent** if $\Sigma |a_n|$ converges

C) If a series is absolutely convergent, then it is convergent

D) A series is called **conditionally convergent** if it is convergent but not abs. convergent.

Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

15

Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k 2^k}$$

1) Try ratio test. Let $a_k = \frac{(-1)^k x^k}{k\,2^k}$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim\left|\frac{x^{k+1}}{x^k}\cdot\frac{k}{k+1}\cdot\frac{2^k}{2^{k+1}}\right|$$

$$= \lim\left|\frac{x}{2}\right|$$

$$= \left|\frac{x}{2}\right|$$

we need $\left|\frac{x}{2}\right| < 1 \Rightarrow \boxed{\text{radius is } 2}$

At $x = \pm 2$, ratio test is inconclusive

At $x = +2$

$\sum \frac{(-1)^k 2^k}{k\,2^k}$ is convergent

At $x = -2$

$\sum \frac{(-1)^k (-2)^k}{k\,2^k} = \sum \frac{1}{k}$

is divergent.

$$\boxed{\begin{array}{l}\text{INTERVAL IS } x \in (-2,+2] \\ \text{OR } -2 < x \le +2\end{array}}$$

Given the series below, determine

1) the radius and interval of convergence
2) values of x where series is absolutely convergent
3) values of x where the series is conditionally convergent

$$\sum_{k=1}^{\infty} \frac{(-1)^k x^k}{k2^k}$$

2) series is abs. convergent where $\Sigma \frac{x^k}{k2^k}$ converges

- by ratio test: abs. convergent for $(-2, +2)$
- at $x = \pm 2$: divergent

$\Rightarrow$ A.C. for $x \in (-2, +2)$

3) C.C. at $x = +2$

(because at $x=2$, series is convergent but not A.convergent)

1) Find the radius and interval of convergence.

$$\sum_{k=1}^{\infty} \frac{x^k}{k}$$

TRY: Ratio test with $a_k = x^k / k$

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim \left| \frac{X^{k+1}}{X^k} \cdot \frac{k}{k+1} \right| = \left| \frac{X}{1} \right| = |x| \Rightarrow \boxed{\text{RADIUS is } 1}$$

AT $x = +1$, DIVERGENT P-SERIES, $P = 1$

AT $x = -1$, CONVERGENT BY ALTERNATING SERIES TEST

$$\boxed{\text{INTERVAL IS } -1 \le X < +1}$$

2) Find the radius and interval of convergence.

$$\sum_{k=1}^{\infty}(-k)^{4k}x^{4k}$$

TRY RATIO TEST:

$$\lim_{k\to\infty}\left|\frac{(-(k+1))^{4(k+1)}}{(-k)^{4k}}\cdot\frac{x^{4(k+1)}}{x^{4k}}\right| = \lim_{k\to\infty}\left|\frac{(k+1)^4(k+1)^{4k}}{k^{4k}}x^4\right|$$

$$=\lim_{k\to\infty}\left|(k+1)^4x^4\right|$$

The limit DNE. unless $x=0$.

$\Rightarrow$ RADIUS IS ZERO

INTERVAL IS THE POINT $x=0$

3) Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

a) $\sum_{k=1}^{\infty}(-1)^k \frac{k}{k^2+1}$

Not abs. convergent: $\frac{k}{k^2+1} > \frac{k}{2k^2} = \frac{1}{2k^2}$ (divergent p-series)

But $\lim_{k\to\infty} \frac{k}{k^2+1} = 0$, so the series is conditionally convergent by the alt. series test.

b) $\sum_{k=1}^{\infty} \frac{\cos(\pi k)}{k} = \sum \frac{(-1)^k}{k}$, because $\cos(\pi k) = (\pm 1)^k$

$\Rightarrow$ not abs. convergent

But is conditionally convergent by alternating series test.

4) Find the radius and interval of convergence. Then find the values of x for which series is

a) absolutely convergent
b) conditionally convergent
c) divergent.

$$1-\frac{x}{2}+\frac{2x^2}{4}-\frac{3x^3}{8}+\frac{4x^4}{16}-\ldots$$

$$=\left(\sum_{k=1}^{\infty}\frac{k x^k (-1)^k}{2^k}\right)+1$$

TRY RATIO TEST: $\left|\frac{k+1}{k}\cdot\frac{2^{k+1}}{2^k}\cdot\frac{x^{k+1}}{x^k}\right| = |2x| < 1$ ⇒ RADIUS IS $\frac{1}{2}$

AT $x = \pm 2$, THE SERIES IS DIVERGENT BY THE TEST FOR DIVERGENCE.

⇒ INTERVAL IS JUST $x \in (-0.5, +0.5)$.

a) ABS. CONVERGENT ONLY ON $-\frac{1}{2} < x < \frac{1}{2}$

b) C. CONVERGENT NO WHERE

c) DIVERGENT FOR $|x| \geq \frac{1}{2}$

Recitation 10

Today: 10.8, 10.9 (Taylor Polynomials and Series)

Complete the Following Forumlas

The Nth order Taylor Polynomial about x = a is:

$$P_N(x)=\sum_{k=0}^{N}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

The Taylor Series about x = a is:

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k = P_N(x) - R_N(x)$$

The phrase "about x = a" means that the expansion is "centered" at $x=a$

Remainder formula for expansion about x = a: $|R_n(x)|\le\left(\max_{t\in[0,x]}\left|f^{(n+1)}(t)\right|\right)\frac{|x-a|^{n+1}}{(n+1)!}$

Quiz 1 has been graded, grades are in T-Square, you'll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.

Example

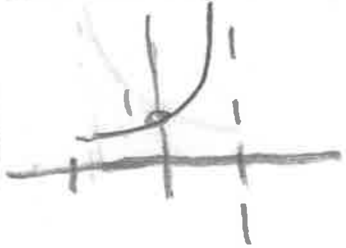
a) Plot a rough sketch of f(x) = $1/(1 - x)^2$ for x between -1 and + 1.

b) Use the Taylor expansion for 1/(1 – x) to find $P_0(x)$, $P_1(x)$, and $P_2(x)$ of f(x) about x = 0.

c) Sketch P_0, P_1, P_2

Hint: the Taylor expansion of 1 / (1 – x) about x = 0 is $\frac{1}{(1-x)} = 1 + x + x^2 + x^3 + \dots$

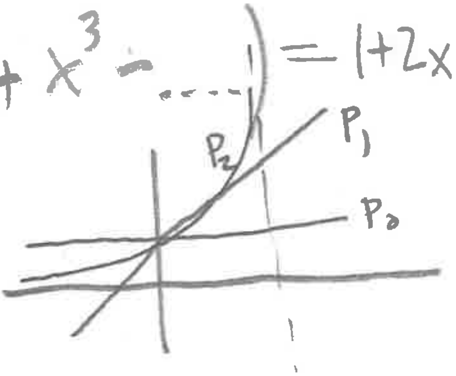
a)



x	f(x)
-1	1/4
0	1
1	+∞

b) $$\frac{1}{(1-x)^2} = \frac{d}{dx}\left(\frac{-1}{1-x}\right)\cdot(-1) = \frac{d}{dx}\left(+1+x+x^2+x^3+\dots\right) = 1+2x+3x^2+\dots$$

$$\Rightarrow P_0 = 1,\quad P_1 = 1+2x,\quad P_2 = 1+2x+3x^2$$



P_2 is best approximation

ALTERATE APPROACHES:

1) EXPAND & COLECT $$\frac{1}{(1-x)^2} = \frac{1}{1-x}\,\frac{1}{1-x} = (1+x+x^2+\dots)(1+x+x^2+\dots) = 1+2x+3x^2+\dots$$

2) USE FORMULA: $P_0 = \frac{f^{(0)}(0)}{0!}(x-0)^0 = 1,\ P_1 = \frac{f^{(1)}(0)}{1!}(x-0)^1 + P_0 = 1+2x$, etc

Group Work Suggestions

- everyone pick a color to write in, and match it with their text
- use mics if you have them
- before moving to the next question:
    - ask if everyone agrees with answer
    - ask if everyone understands how to get the answer
- message me if your group gets stuck

Make sure you can solve Question #4 before the next quiz.

1) Find the Taylor expansion about x = 0 of $f(x) = \dfrac{2x}{1+x^2}$

Hint: write the Taylor expansion for 1/(1-x) at x = 0, and then apply suitable modifications to the expansion.

$$\frac{1}{1-x} = 1 + x + x^2 + \ldots$$

$$\frac{1}{1+x^2} = 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \ldots$$

$$\frac{2x}{1+x^2} = 2x\left(1 - x^2 + x^4 - x^6 + \ldots\right) = f(x)$$

DONE.

IF YOU PREFER: $f(x) = \sum_{k=0}^{\infty} 2x(-1)^k (x^2)^k$

2) Fill in the blank: the Maclaurin series is just the Taylor series about the point $x=0$.

The Maclaurin series for e^x is

$$e^x = \sum_{k=0}^{\infty} \frac{(e^x)^{(k)}(0)}{k!} x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

Use this expansion to find the Maclaurin series for f(x), where:

$$f(x) = e^{-x^2} = \exp(-x^2) .$$

ALSO FIND $P_2(x)$.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \left(\text{because } (e^x)^{(k)}(0) = 1 \quad \forall k\right)$$

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!}$$

$$P_2(x) = 1 + (-x^2) + \frac{(-x^2)^2}{2!} = 1 - x^2 + \frac{x^4}{2}$$

3) Estimate the following integral to within 0.01 using series.
Hint: use the Alternating Series Remainder Theorem.

$$\int_0^1 x^4 e^{-x^2}\,dx$$

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-x^2)^k}{k!} = 1 - x^2 + x^4/2! - \frac{x^6}{3!} + \dots$$

$$x^4 e^{-x^2} = x^4 - x^6 + x^8/2! - \frac{x^{10}}{3!} + \dots$$

$$\int_0^1 x^4 e^{-x^2}\,dx = \left(\frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9\cdot 2} - \frac{x^{11}}{14\cdot 3\cdot 2} + \dots\right)\Big|_0^1$$

$$= \frac{1}{5} - \frac{1}{7} + \frac{1}{18} - \frac{1}{66} + \frac{1}{288} - \dots$$

THE ALT. SERIES REMAINDER THEOREM IS $|R_N| < |a_{N+1}|$

$a_4 = 1/66 > 0.01$, $a_5 = \frac{1}{288} < 0.01$. THUS $\int_0^1 x^4 e^{-x^2}\,dx \approx \frac{1}{5} - \frac{1}{7} + \frac{1}{18} - 1/66.$

ie: first four terms needed.

4) Estimate $e^{3/2}$ to within 10^{-4}.

Note: this is a question from a Math 1502 quiz from 2012.

Hint: use the Taylor Series Remainder Theorem, and approximate $e^{3/2}$ with 5.

$$|R_N(x)| = |f(x) - P_{N-1}(x)| = \max_c f^{(N+1)}(c)\,|x|^{N+1} / (N+1)!,\quad c \in [0, x].$$

WHAT DO WE USE FOR $\max_c f^{(N+1)}(c)$? Use 5, because $e^{3/2} \approx 4.48$.

N	$5 \cdot (3/2)^{N+1} / (N+1)!$
1	5.625
2	2.8125
⋮	⋮
8	0.0005296 ....
9	$0.00008 < 10^{-4} = 0.0001$

$$\text{THUS} \quad e^{3/2} \approx P_9\left(\tfrac{3}{2}\right) = \left.\left(1 + x + \frac{x^2}{2!} + \dots + \frac{x^9}{9!}\right)\right|_{x=3/2}$$

DONE. (NOTE IF YOU EVALUATE THIS, YOU GET 4.48167.....)

5) Let f(x) = (exp(2x²) – 1 – x²) / x⁴. Note: exp(x) = e^x = e^x

a) Find $P_2(x)$ about x = 0
b) Use the result from part a) to evaluate the limit of f(x) as x goes to zero.

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + x^4/4! + \ldots$$

Substitute x with $2x^2$:

$$e^{2x^2} = 1 + \frac{(2x^2)}{1!} + \frac{(2x^2)^2}{2!} + \frac{(2x^2)^3}{3!} + \ldots$$

Therefore:

$$f(x) = \left(e^{2x^2} - 1 - x^2\right)/x^4 = \left[\left(1 + \frac{2x^2}{1!} + \frac{4x^4}{2!} + \frac{8x^6}{3!} + \ldots\right) - 1 - x^2\right]/x^4$$

$$= \left[0 + x^2 + 2x^4 + \frac{8}{3!}x^6 + \ldots\right]/x^4$$

$$= \underbrace{0 + x^{-2} + 2}_{P_2(x)} + \frac{8}{3!}x^2 + \ldots$$

a) $\Rightarrow P_2(x) = 2 + 1/x^2$

b) $\lim_{x \to 0} f(x) = \lim_{x \to 0}\left(0 + \frac{1}{x^2} + 2 + \frac{8}{3!}x^2 + \ldots\right)$ which does not exist.

$\Rightarrow f(x) \to \infty$ as $x \to 0$.

(THANK YOU ANDREW DURDEN FOR CORRECTING MY ERROR ON THIS QUESTION!)

Recitation 11

Today: 4.5 (l'Hospital's Rule)

Describe how you would evaluate the following limits.

$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ is of the form 0/0 or ∞/∞, then ~~we can:~~

we evaluate $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)}$ ⊛

$\lim\limits_{x\to a} f(x)-g(x)$ is of the form ∞ − ∞, then we can:

find common denominator, use ⊛

$\lim\limits_{x\to a} f(x)^{g(x)}$ is of the form 0^0, 1^∞, or ∞^0, then we can:

use $\lim\limits_{x\to a} e^{\ln f^g} = \lim\limits_{x\to a} e^{g\ln f}$, use ⊛

~~*Quiz 1 has been graded, grades are in T-Square, you'll receive your graded quizzes this week via email, please make sure they were graded correctly and let me know if you have questions.*~~

POP QUIZ #1

- Start time: 8:10?
- Ends at: 8:25
- Pop quiz grading
  - 5 points: correct
  - 4 points: something correct
  - 3 points: name on the page
  - 0 points: did not take pop quiz

- To submit your work, choose any of the following:

A. **work on whiteboard in breakout room**
   - type A in text chat so I know you want to work in breakout room
   - submit work by letting me know when done, and/or email me a screen capture of your work

B. **work on paper and give work to facilitator**
   - type B in text chat so I know you're doing this
   - leave 2 inch margin on paper
   - write your name and QH8 at the top
   - facilitator is receiving instructions today on how to submit your work

C. **work on paper and email a photo of your work to me**
   - type C in text chat so I know you are emailing your work to me
   - email your photo to me before 8:40

Find the Taylor series, centered at a = 0, of sin(5 x^2).

Taylor series at a=0 of sin x is

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k \, x^{2k+1}}{(1+2k)!}$$

$$\Rightarrow \sin(5x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k \, (5x^2)^{2k+1}}{(1+2k)!}$$

DONE.

Evaluate: $\lim\limits_{x\to\infty}\left(\cos\frac{1}{x}\right)^{x} = 1^{\infty}$,

$$= \lim_{x\to\infty} e^{\ln \cos(1/x)^{x}}$$

$$= \lim e^{x \ln \cos 1/x}$$

$$= \lim e^{(\ln \cos 1/x) / 1/x}$$

$$\overset{\textcircled{H}}{=} \lim e^{\left(\frac{d}{dx} \ln \cos(\frac{1}{x})\right) / \left(\frac{d}{dx}(\frac{1}{x})\right)}$$

$$= \lim e^{\left[\sec(\frac{1}{x}) \cdot \sin(1/x)(-1/x^2)\right] / \left[-1/x^2\right]}$$

$$= e^{0} = 1$$

1) Evaluate the following limit by using l'Hospital's rule, if possible.

Note: this limit is a well-known definition for an important number.

$$\lim_{x\to 1} x^{1/(x-1)}$$

$$= e^{\ln x^{1/x-1}}$$

$$= \lim_{x\to 1} e^{\frac{1}{x-1}\ln x}$$

Ⓗ

$$= \lim_{x\to 1} e^{\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}(x-1)}}$$

$$= \lim_{x\to 1} e^{1/x} = e$$

2) Evaluate the following limit by using l'Hospital's rule, if possible.

$\lim\limits_{x\to\infty} x\sin\left(\frac{\pi}{x}\right)$, I.F. type $0\cdot\infty$ (I.F. = indeterminate form)

$$= \lim_{x\to\infty} \frac{\sin(\pi/x)}{1/x} \text{ , I.F. type } 0/0$$

Ⓗ
$$= \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\cdot \frac{d}{dx}\left(\frac{\pi}{x}\right)}{-x^{-2}}$$

$$= \lim_{x\to\infty} \frac{\cos\left(\frac{\pi}{x}\right)\cdot \pi \cdot \left(-x^{-2}\right)}{-x^{-2}}$$

$$= \pi$$

3) Evaluate the following limit by using l'Hospital's rule, if possible.

$$\lim_{x\to 0^+} x(\ln|x|)^2$$

$$= \lim_{X\to 0} \frac{(\ln x)^2}{1/X} \quad , \quad \text{I.F. type } \infty/\infty$$

$$= \lim_{X\to 0} \frac{2\ln(x)\cdot \frac{1}{(x)}}{-1/x^2}$$

$$= 2\lim_{X\to 0} \frac{\ln(x)}{1/(x)} \quad , \quad \text{I.F. type } \infty/\infty$$

Ⓗ

$$= 2\lim_{X\to 0} \frac{1/x}{-1/x^2}$$

$$= 2\lim_{X\to 0} (-x) = 0$$

4) Evaluate the following limit by using l'Hospital's rule, if possible.

a) $$\lim_{x\to\infty}\frac{1}{x}\int_0^x e^{t^2}\,dt \overset{(H)}{=} \lim_{x\to 0}\frac{e^{x^2}}{1} = \infty$$

b) $$\lim_{x\to 0}\left(\frac{1}{\ln(1+x)}-\frac{1}{x}\right) = \lim_{x\to 0}\frac{x-\ln(1+x)}{x\ln(1+x)}, \quad \text{I.F. } 0/0$$

$$\overset{(H)}{=} \lim_{x\to 0}\frac{1-(1+x)^{-1}}{\ln(1+x)+\frac{x}{1+x}}$$

$$= \lim_{x\to 0}\frac{(1+x)-1}{(1+x)\ln(1+x)+x}, \quad \text{I.F. } 0/0$$

$$\overset{(H)}{=} \lim_{x\to 0}\frac{1}{\ln(1+x)+1+1}$$

$$= 1/2$$

5) a) $\vec{v} \cdot \vec{v} = \left\|\vec{v}\right\|^2$, is the length of vector **v**.

b) A rhombus is a parallelogram with four sides of equal length. Show that the diagonals of a rhombus are perpendicular.

Solution next recitation

Recitation 12

Today: Lines, Planes, Dot Products

Suppose we have the points P(1,2,3), Q(1,3,4), R(2,2,2).

a) Find a vector that is normal to the plane that contains the three points.

b) Find an equation of the plane.

a) $\overrightarrow{PQ} = \begin{pmatrix}1-1\\3-2\\4-3\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix}1\\0\\-1\end{pmatrix}$. $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are vectors parallel to the plane.

Thus, $\vec{N} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k\\ 0 & 1 & 1\\ 1 & 0 & -1\end{vmatrix} = \boxed{-\hat{i}+\hat{j}-\hat{k}}$ is ⊥ to the plane, **DONE**

b) Let $\vec{r} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ be a point in the plane, So $\vec{r} - P$ is a vector in the plane.

$$\text{THUS: } 0 = \vec{N} \cdot \begin{pmatrix}x-1\\y-2\\z-3\end{pmatrix} = \begin{pmatrix}-1\\1\\-1\end{pmatrix} \cdot \begin{pmatrix}x-1\\y-2\\z-3\end{pmatrix} = \overbrace{-x+y-z}^{(Ax+By+Cz)}\overbrace{+1-2+3}^{(-D)}$$

$$\begin{pmatrix}A\\B\\C\end{pmatrix} \nearrow$$

$$\Rightarrow \boxed{-x+y-z=-2}$$

2) The equation Ax + By + Cz = D is a plane. A vector perpendicular to the plane is:

$\begin{pmatrix}A\\B\\C\end{pmatrix}$ or $A\hat{i}+B\hat{j}+C\hat{k}$

Find a parametrization of the line that is the intersection of the planes

$$\left.\begin{array}{ll} P: & x - 2y + z = 3 \\ Q: & 2x + y + z = 1 \end{array}\right\} \text{ we need a line of form } \vec{r} = \vec{r}_0 + \vec{v}\,t$$

Note: this is a question from a 2012 Math 1502 quiz.

- vectors ⊥ to P & Q are $\underbrace{\begin{pmatrix}1\\-2\\1\end{pmatrix} \text{ and } \begin{pmatrix}2\\1\\1\end{pmatrix}}_{\text{normals to plane}}$
- vector ⊥ to normals is $\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -3\hat{i} + \hat{j} + 5\hat{k} = \vec{v}$
- a point in intersection can be found by finding any point that satisfies both equations:

  let $z=0$, then $\left.\begin{array}{l} x - 2y = 3 \\ 2x + y = 1 \end{array}\right\}$ $\begin{array}{l} x = -1 \\ y = 1 \end{array}$

  $\Rightarrow (-1, 1, 0)$ is a point in intersection.

$\Rightarrow$ line is $\vec{r} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}-3\\1\\5\end{pmatrix} t$

1) Find the equation for the line that is perpendicular to the yz-plane, and also passes through P(1,4,3).

a vector $\perp$ to yz-plane is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ OR $\hat{i}+0\hat{j}+0\hat{k}$.

$\Rightarrow$ line is $\boxed{\vec{r}=\begin{pmatrix}1\\4\\3\end{pmatrix}+\begin{pmatrix}1\\0\\0\end{pmatrix}t}$. DONE.

TO SEE WHY THE VECTOR $\begin{pmatrix}1\\0\\0\end{pmatrix}$ IS $\perp$ TO THE YZ-PLANE, CONSIDER THAT:

- THE YZ-PLANE HAS THE EQUATION $\boxed{x=0}$, OR $\underset{\sim}{1}x+\underset{\sim}{0}y+\underset{\sim}{0}z=0$.
- THE VECTOR, OR $\underset{\sim}{A}$ VECTOR $\perp$ TO THE PLANE IS THE COEFFICIENTS OF x,y,z. THE COEFFICIENTS ARE 1, 0, 0, SO A VECTOR $\perp$ IS $\begin{pmatrix}1\\0\\0\end{pmatrix}$.

2) Do these lines intersect each other? Why/why not?

$x_1 = 1 + t$　　$x_2 = 1 - u$

$y_1 = -1 - t$　　$y_2 = 1 + 3u$

$z_1 = -4 + 2t$　　$z_2 = -2u$

If they intersect, $\exists$ t, u st.

① $1 + t = 1 - u \Rightarrow t = -u$

② $-1 - t = 1 + 3u \Rightarrow t + 3u = 2$, or $t + 3(-t) = 2$

$t = -1$

③ $-4 + 2t = -2u$

so $u = +1$ by ①.

Equation ③ must be satisfied by $t = 1$ and $u = +1$. Is it?

$-4 + 2t = -2u$

$-4 + 2(-1) = -2(+1)$

$-6 = -2$ NOT SATISFIED.

$\Rightarrow$ LINES DON'T INTERSECT (ie - they are "skew")

3) a) If **a×b** = $\vec{0}$ and **a•b** = 0, what can we conclude about vectors **a** and **b**? Explain your reasoning.

$\vec{a}\times\vec{b}=\vec{0}$ means $\vec{a} \parallel \vec{b}$, or at least one of $\vec{a}, \vec{b}$ is $\vec{0}$

$\vec{a}\cdot\vec{b}=0$ means $\vec{a}\perp\vec{b}$, or at least one of $\vec{a}, \vec{b}$ is $\vec{0}$

$\Rightarrow$ at least one of $\vec{a}, \vec{b}$ is zero vector

b) Which of the following make sense? Explain why/why not.

i. **a × (b • c)** NONSENSE: $a \cdot b$ is a scalar, $\vec{a}\times(\text{scalar})$ is illegal

ii. **a • (b • c)** NONSENSE $a\cdot(\text{scalar})$ is illegal

iii. **a × (b×c)** OK: $\vec{a}\times$ vector is legal

iv. **a • (b×c)** OK: $\vec{a}\cdot$ vector is legal

4) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

$\vec{P}$: **j** – **k**

$\vec{Q}$: 3**i** – **j** + 2**k**

$\vec{R}$: 3**i** – 2**j** + 3**k**

if $(\vec{P}\times\vec{Q})\cdot\vec{R} = \vec{0}$, then co-planar.

if $(\vec{P}\times\vec{Q})\cdot\vec{R} \neq \vec{0}$, not co-planar

$$\vec{P}\times\vec{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 3 & -1 & 2 \end{vmatrix} = \hat{i} + 3\hat{j} - 3\hat{k}$$

$$\vec{P}\times\vec{Q}\cdot\vec{R} = \begin{pmatrix} 1 \\ 3 \\ -3 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} = 3-6-9 \neq 0$$

$\Rightarrow$ NOT CO-PLANAR

Recitation 13

Today: Quiz 2 Review

Estimate to within 0.0001 by using series.

$$I = \int_0^{1/2} \frac{\ln(1+x)}{x}dx$$

$$= \int_0^{.5} \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right)dx$$

$$= \int\left(1 - \frac{x}{2} + \frac{x^2}{3} - \dots\right)dx$$

$$= x - \frac{x^2}{2\cdot 2} + \frac{x^3}{3\cdot 3} - \frac{x^4}{4\cdot 4} + \dots \Big|_0^{1/2}$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2\cdot 2^k}$$

By the alternating series remainder theorem, the first term that is less than 0.0001 is $\underline{k=8}$.

guess and check.

$$\Rightarrow \sum^{7} < I < \sum^{8}$$

$$I \simeq \sum_{k=1}^{7} \frac{(-1)^{k-1}}{k^2\cdot 2^k}$$

Office hours: Tuesday and Wednesday 7:30 to 8:30 pm
https://georgiatech.adobeconnect.com/distancecalculusofficehours

Group Work Suggestions

- before moving to the next question:
    - ask if everyone agrees with answer
    - ask if everyone understands how to get the answer
- everyone pick a color to write in, and match it with their text
- use mics if you have them
- message me if your group gets stuck

1) Evaluate by a) using power series, and b) l'Hopital's rule.

$$\lim_{x\to 0}\frac{\cos(x)-1}{x\sin x}$$

a) $\lim_{x\to 0}\frac{\left(1-x^2/2+x^4/4-\ldots\right)-1}{x\left(x/1!-x^3/3!+x^5/5!-\ldots\right)} = \lim_{x\to 0}\frac{-1/2+x^2/4-\ldots}{1-x^2/3!+\ldots} = -1/2$

(divide numer. and denom. by x^2)

b) $\lim_{x\to 0}\frac{-\sin x}{\sin x + x\cos x} = \lim_{x\to 0}\frac{-\cos x}{\cos x+\cos x - x\sin x}$

$$= \frac{-1}{1+1-0}$$

$$= -1/2$$

(PART b) USES L'HOP. TWICE)

2) Find the interval of convergence.

$$\sum_{k=1}^{\infty}\frac{\ln k}{k}(x+1)^k$$

$$\left(\begin{array}{l} -1 < x+1 < +1 \\ -2 < x < 0 \end{array}\right)$$

ratio test: $\lim_{k\to\infty}\left|\frac{\ln(k+1)}{\ln(k)}\cdot\frac{k}{k+1}\cdot\frac{(x+1)^{k+1}}{(x+1)^k}\right| = \overbrace{|x+1| < 1}$

$\Rightarrow$ converges for $x \in (-2,0)$, and at endpoints:

$x=0$, the series is $\sum_{k=1}^{\infty}\frac{\ln k}{k}$ diverges by comparison with harmonic series.

$x=-2$, the series is $\sum_{k=1}^{\infty}\frac{\ln k}{k}(-1)^k$, converges by A.S.T.

$\Rightarrow$ interval is $[-2,0)$.

3) Find the distance between planes P1 and P2.

P1: $x + 2y + z = 3$
P2: $x + 2y + z = 9$

A point on P_1 is $(0, 0, 3)$. ← WE CAN FIND THIS POINT BY SELECTING VALUES FOR X AND Y AND THEN USING $x + 2y + z = 3$ TO FIND z.

The normal to P_1 is $\begin{bmatrix}1\\2\\1\end{bmatrix}$.

A line connecting P_1 to P_1 is $\vec{L} = \begin{bmatrix}0\\0\\3\end{bmatrix} + t\begin{bmatrix}1\\2\\1\end{bmatrix}$

The point on P_2 closest to $(0,0,3)$ is the point where:

$$x + 2y + z = 9$$
$$(0+t) + 2(0+2t) + (3+t) = 9$$

SOLVING: $6t = 6$, $t = 1$. Thus, line intersects P_2 when $t = 1$.

$\Rightarrow$ this point is $(0+1, 0+2, 3+1)$ or $(1,2,4)$

Distance between $(1,2,4)$ and $(0,0,3)$ is $\sqrt{1^2+2^2+1^2}$
$= \sqrt{6}$

4) The line L is determined from P_1 and P_2. The plane Q is determined by Q_1, Q_2, Q_3.

Does L intersect Q? If so, where?

$P_1(1,-1,2)$
$P_2(-2,3,1)$

$$\left.\begin{matrix} P_1(1,-1,2) \\ P_2(-2,3,1) \end{matrix}\right\} \text{ L is } \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + t\begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$$

$Q_1(2,0,-4)$
$Q_2(1,2,3)$
$Q_3(-1,2,1)$

$$\text{Q has normal } \begin{vmatrix} -1 & 2 & 7 \\ -3 & 2 & 5 \end{vmatrix} = \begin{bmatrix} -4 \\ -16 \\ 4 \end{bmatrix}, \text{ or } \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

$$\Rightarrow \text{Q has equation } \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \cdot \begin{pmatrix} x-2 \\ y-0 \\ z+4 \end{pmatrix}, \quad x+4y-z=6$$

If L intersects Q, $\exists$ t s.t.

$$(1-3t)+4(-1+4t) - (2-t) = 6 \quad \Rightarrow t = 11/14$$

$$\Rightarrow \quad \begin{aligned} x &= 1 - 3 \cdot 11/14 = -19/14 \\ y &= -1 + 4 \cdot 11/14 = 15/7 \\ z &= 2 - 11/14 = 17/14 \end{aligned}$$

5) From 2012 Quiz 1

Tyler Hamilton would like you to find

a) Aseries for $\int_0^x \sin(\pi t^2/2)\,dt$

$$= \int_0^x \left(1 - \frac{\left(\frac{\pi t^2}{2}\right)^3}{3!} + \frac{\left(\frac{\pi t^2}{2}\right)^5}{5!} - \ldots\right) dt$$

$$= X - \frac{\pi^3 t^7}{2^3 \cdot 3! \cdot 7} + \frac{\pi^5 t^{11}}{2^5 \cdot 5! \cdot 11} - \ldots$$

6) *From 2012 Quiz 1*

Greg Lemond wants you to use series and error bounds to estimate $e^{7/2}$ to within 10^{-3}. You must use (an) error bound. You do not have to actually sum, just say how many terms, (or the highest power of 7 / 2). (You may use e^ (7 / 2) ≤ 35).

LAGRANGE FORM OF ERROR: $35 \cdot \frac{\left(\frac{7}{2}\right)^{N+1}}{(N+1)!}$

FIND A VALUE OF N s.t. this is less than 10^{-3}

15 works.

$\Rightarrow$ 15 terms.

7) Vectors that are co-planar are in the same plane. Determine whether the vectors are co-planar:

$\vec{P}$: i + j − k

$\vec{Q}$: 2i − j

$\vec{R}$: 3i − j − k

METHOD 1

$$\vec{P} \times \vec{Q} = \begin{vmatrix} i & j & k \\ 1 & 1 & -1 \\ 2 & 1 & 0 \end{vmatrix} = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \cdot \vec{R} = 3 + 2 + 1 = 6$$

if co-planar, $\vec{P} \times \vec{Q} \cdot \vec{R} = 0$, $\Rightarrow$ not coplanar.

ALTERNATE METHOD: co-planar iff $\vec{P}\vec{Q} \parallel \vec{P} \times \vec{R}$, NOTATION: $\parallel$ = PARALLEL

$$\left.\begin{aligned} \vec{P} \times \vec{Q} &= \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \\ \vec{P} \times \vec{R} &= \begin{bmatrix} -2 \\ -2 \\ -4 \end{bmatrix} \end{aligned}\right\}$$

THESE VECTORS ARE NOT SCALAR MULTIPLES OF EACH OTHER, AND SO ARE NOT PARALLEL. $\Rightarrow$ NOT CO-PLANAR

8) Find a parametrization of the line that is the intersection of the planes

P: ~~$-x + 2y + z = 2$~~ $x+y+z+1=0$
Q: ~~$x + y + 2z = 11$~~ $x-y+z+2=0$

Note: this is a modified version of a question from a 2012 Math 1502 quiz.

normals to P & Q are $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\\1\end{bmatrix}$

a vector ∥ to desired line is

$$\begin{vmatrix} i & j & k \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix}, \text{ or } \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

a point in intersection is $x=0$, $\left.\begin{matrix} y+z=-1 \\ -y+z=-2 \end{matrix}\right\}$ $\begin{matrix} y=1/2 \\ z=-3/2 \end{matrix}$

$$\Rightarrow \vec{l} = \begin{bmatrix}0\\1/2\\-3/2\end{bmatrix} + t\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Recitation 15

Today: Solving linear systems of equations

The following was a 2013 pop quiz question.

For what values of **a** does the following system have a solution?

R_1 7x + 2y - 3z = 25

R_2 y + 3z = 5

R_3 3y + **a**z = 3

$R_3 - 3R_2$ is: $0x + 0y + (a-9)z = -12$. So if $a = 9$, then we have "$0 = -12$" $\Rightarrow$ no sol'n. DONE. $\Rightarrow a \neq 9$

IF YOU PREFER, WE CAN WRITE OUT AUGMENTED MATRICES:

$$\left(\begin{array}{ccc|c} 7 & 2 & -3 & 25 \\ 0 & 1 & 3 & 5 \\ 0 & 3 & a & 3 \end{array}\right) \underset{R_3 - 3R_2}{\overset{R_1 - 2R_2}{\sim}} \begin{pmatrix} 7 & 0 & -9 & 15 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & a-9 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -9/7 & 15/7 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & -12/(a-9) \end{pmatrix}$$

$\Rightarrow x_3 = \frac{-12}{a-9} \Rightarrow$ as $a \to 9$, then $x_3 \to +\infty$. $\boxed{\text{Thus } a \neq 9.}$

But if $a \neq 9$, x_3 has a solution, as does the system.

$\left(x_1 = \frac{1}{7}(15 + 9x_3), \; x_2 = 5 - 3x_3, \; x_3 = \frac{12}{9-a} \right)$

There are 30 recitations in the semester (including quizzes). We're ~50% through the course.

A Few Definitions

a) A system of linear equations is **consistent** if it has at least one solution.

b) A system of linear equations is **inconsistent** if it has no solutions.

c) A system of linear equations that is overdetermined has more equations than unknowns

d) Can a system of linear equations be overdetermined and consistent? If yes, provide an example with at least 3 equations.

YES: $\left(\begin{array}{cc|c} 1 & 2 & 5 \\ 1 & 3 & 7 \\ 2 & 4 & 10 \end{array}\right)$, ie $x_1 = 1$, $x_2 = 2$

3) Find h and k such that the system has
   - a) no sol'n
   - b) a unique sol'n
   - c) infinitely many solutions

$$x_1 + hx_2 = 2$$
$$4x_1 + 8x_2 = k$$

$$\left(\begin{array}{cc|c} 1 & h & 2 \\ 4 & 8 & k \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & h & 2 \\ 0 & 8-4h & k-8 \end{array}\right), \text{ or } \sim \boxed{\begin{array}{l} x_1 + hx_2 = 2 \\ 0\cdot x_1 + (8-4h)x_2 = k-8 \end{array}} \begin{array}{l} ① \\ ② \end{array}$$

PART a)

a) ⇒ if $\boxed{h=2, \text{ and } k \neq 8}$ then ② is inconsistant, so the system is inconsistent (because LHS$=0$, RHS $\neq 0$)

c) If $\boxed{k=8, h=2}$ we have $0x_1 + 0x_2 = 0$ and $x_1 = 2 - hx_2$ which has ∞ many solutions. (because x_2 is free)

b) If $k=$ anything and $h \neq 2$, unique sol'n.

4) A 3 x 4 coefficient matrix has three pivot columns. Is the system consistent? Why/why not?

CONSIDER: $\begin{bmatrix} 1 & 6 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ IS A COEFF. MATRIX, WHICH YIELDS AN INF. # OF SOLUTIONS SO YES.

(ie - the aug. matrix can't have row of form $[0\,0\,0\,0\,c]$ where $c \in \mathbb{R}$)

5) Find the general solution to the system whose augmented matrix is given below.

$$\begin{bmatrix} 3 & -2 & 4 & 0 \\ 9 & -6 & 12 & 0 \\ 6 & -4 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 3 & -2 & 4 & 0 \\ & 0 & & \\ & & & \end{bmatrix}$$

$$\Rightarrow \quad x_1 = \frac{1}{3}(2x_2 - 4x_3)$$

x_2, x_3 are free

6) True or false.

① each leading element in a row is 1

② " " 1 is the only non-zero element in column.

a) The reduced echelon form of a matrix is unique.

b) If a system has free variables, the solution set has many solutions.

c) If a row in an echelon form of an augmented matrix is [0 0 0 3 0], then the linear system is inconsistent.

a) TRUE

Ⓐ needs to be 0 for REF

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & 1 & * \\ 0 & 0 & 0 & 0 \end{array}\right)$$

NOT REF

Ⓑ

$$\left(\begin{array}{cccc} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

IS REF (can't change mtx in a way s.t. ①&② don't hold)

b) FALSE! eg AUG MTX $\left.\begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}\right\}$ INCONSISTENT 3 F.V.

c) FALSE! row just says that $3x_4 = 0$. By itself does not mean system is inconsistent.

Recitation 16

Today: Span, Linear Dependence

Definitions

Assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are arbitrary vectors.

SIMPLE EXAMPLE: THE XY-PLANE IN $\mathbb{R}^3$. THE SET OF ALL $c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}$ SPAN THE XY PLANE.

EG - $\begin{bmatrix}2\\1\\0\end{bmatrix}$ IS IN THE SPAN BECAUSE $c_1=2$, $c_2=1$.

$\begin{bmatrix}1\\2\\3\end{bmatrix}$ IS <u>NOT</u> IN THE SPAN. WE CAN'T FIND c_1, c_2.

a) The sum $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ is a linear combination of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) The set of all possible linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ is the SPAN of $\mathbf{v}_1$ and $\mathbf{v}_2$.

c) Any vector in the SPAN of $\mathbf{v}_1$ and $\mathbf{v}_2$ can be written as a LINEAR COMBINATION of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$.

Question 1

Let $\mathbf{a}_1 = \begin{bmatrix}1\\4\\-1\end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix}-6\\-17\\2\end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix}4\\2\\h\end{bmatrix}$. For what value(s) of h is b in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

If you haven't already, please take a few minutes to fill out the technical issues survey.

1) This is similar to the **first** question on your next HW.

Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -6 \\ -17 \\ 2 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ h \end{bmatrix}$. For what value(s) of h is b in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?

IF $\vec{b}$ IS IN THE PLANE, $\exists$ c_1 & c_2 s.t.

$$c_1\vec{a}_1 + c_2\vec{a}_2 = \vec{b}$$

(or $A\vec{c} = \vec{b}$, A is the mtx formed from $\vec{a}$.)

Aug. matrix: $\left[\begin{array}{cc|c} 1 & -6 & 4 \\ 4 & -17 & 2 \\ -1 & 2 & h \end{array}\right] \sim \begin{bmatrix} 1 & -6 & 4 \\ 0 & \boxed{7 \quad -14} \\ 0 & -4 & h+4 \end{bmatrix} \rightarrow 0c_1 + 7c_2 = -14 \rightarrow c_2 = -2$

$0c_1 - 4c_2 = h+4,$

$-4(-2) = h+4 \Rightarrow \boxed{h=4}$

if $h=4$, $\vec{b}$ is co-planar with $\vec{a}_1, \vec{a}_2$

Linear Dependence

Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \ldots, \vec{v}_N$ are linearly dependent (LD) if $\exists\ c_1, c_2, c_3, \ldots, c_N$ *not* all ZERO , such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \ldots . c_N\vec{v}_N = \vec{0}$$

If the vectors are not LD, they are LINEARLY INDEPENDENT

Example: vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\2\end{bmatrix}$ are LD, because $1\begin{bmatrix}1\\1\end{bmatrix} + \left(\frac{-1}{2}\right)\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

- vectors $\begin{bmatrix}1\\1\end{bmatrix}$ & $\begin{bmatrix}2\\3\end{bmatrix}$ are LI, $c_1\begin{bmatrix}1\\1\end{bmatrix} + c_2\begin{bmatrix}2\\3\end{bmatrix} = \vec{0}$ only when $c_1 = c_2 = 0$.

To determine whether a set of vectors are LD, we solve:

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 + \ldots + c_N\vec{v}_N = \vec{0}$$

which has the same solution as the linear system whose augmented matrix is $\left[\vec{v}_1\ \ \vec{v}_2\ \ \vec{v}_3\ \ldots\ \vec{v}_N\ \vec{0}\right]$.

if c_i not all zero, LD

if $c_i = 0\ \forall i$, LI.

2) Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}$$

If LI, $\exists$ c_1, c_2, c_3, c_4 s.t.

$$c_1\vec{V_1} + c_2\vec{V_2} + c_3\vec{V_3} + c_4\vec{V_4} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2 EQUATIONS, 4 unknowns.

$\Rightarrow$ at least 2 f.v.

$\Rightarrow$ not all c_1, c_2, c_3, c_4 are zero

$\Rightarrow$ LD.

3) This is similar to the second question on your next HW.

Determine whether vector **b** is in the set spanned by the columns of matrix A.

$$A = \begin{pmatrix} 1 & 0 & 5 \\ -2 & 1 & -6 \\ 0 & 2 & 8 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix}$$

if $\vec{b}$ spanned by $\vec{a}_1, \vec{a}_2, \vec{a}_3$, $\exists\ \overbrace{c_1, c_2, c_3}^{\text{not all zero}}$ s.t.

$$c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 = \vec{b} \quad ⊛$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ -2 & 1 & -6 & -1 \\ 0 & 2 & 8 & 6 \end{array}\right) \sim \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & +4 & 3 \\ 0 & 2 & 8 & 6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow c_1 = 3 - 4c_3,\ c_2 = 2 - 5c_3,\ c_3$ free $\Rightarrow \vec{b}$ is in the set because ⊛ is satisfied

(ALSO: $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b}$ are LD, because c_1, c_2, c_3 exist and are not all zero)

4) This is similar to the third question on your next HW.

Determine whether vector **b** can be written as a linear combination of vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. In other words, determine whether x_1 and x_2 exist such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$. If possible, find x_1 and x_2.

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix},\ \mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix},\ \mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}.$$

If $\vec{b}$ is a lin. comb. of $\vec{a}_1$ & $\vec{a}_2$, $\exists$ x_1 & x_2 s.t. $x_1\vec{a}_1 + x_2\vec{a}_2 = \vec{b}$.

Form aug. mtx.:

$$\left[\begin{array}{cc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array}\right] \sim \begin{bmatrix} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $x_1 + 0x_2 = 3$

$0x_1 + x_2 = 2$

$0x_1 + 0x_2 = 0$

$\Rightarrow x_1 = +3$, $x_2 = +2$

$$\boxed{\Rightarrow \vec{b} \text{ is a lin. comb. of } \vec{a}_1, \vec{a}_2 \text{ and } x_1 = +3,\ x_2 = +2}$$

5) Determine whether the following vectors are LI.

$$\overset{\vec{v}_1}{\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix} -7 \\ 2 \\ 4 \end{bmatrix}}$$

If L.I., then $\exists$ c_1, c_2, c_3 not all zero s.t.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

make augmented matrix:

$$\begin{bmatrix} 5 & 0 & -7 & 0 \\ -3 & 0 & 2 & 0 \\ -1 & 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 0 & -10 & 0 \\ 0 & 0 & -13 & 0 \end{bmatrix}$$

$c_1 = 4c_3$

c_2, c_3 free

$\Rightarrow$ c's exist, not all zero

s.t. $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$

$\Rightarrow$ LD, not LI.

6) Find values of h so that the following vectors are LD.

$$\begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \\ h \end{bmatrix}$$

$$\begin{pmatrix} 2 & 4 & -2 & 0 \\ -2 & -6 & 2 & 0 \\ 4 & 7 & h & 0 \end{pmatrix} \underset{R_3-2R_1}{\overset{\substack{R_1/2 \\ R_2+R_1}}{\sim}} \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & h+4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -h-4 & 0 \end{pmatrix}$$

If $h=-4$, $C_3=0$. But $C_2=0$ and $C_1=C_3$.
So if $h=-4$, $C_1=C_2=C_3=0$, vectors are LI.

If $h\neq-4$, vectors are LD.

Recitation 19

Today: Span, Linear Dependence, Linear Transforms

Quiz 3 Next Thursday

Make sure you can solve these questions from old quizzes:

- 2012: Quiz 2 #2 and #3
- 2012: Quiz 3 #1 and #2
- 2013: Quiz 3 #1 and #3

We'll solve some of these in Tuesday's recitation.

QH8 Office Hours Next Week

Tuesday and Wednesday 7:30 to 8:30 pm
At the same place as last time:
https://georgiatech.adobeconnect.com/distancecalculusofficehours

Online Drop-in Tutoring

Wednesdays, 5:30 to 7:00 pm
For all ~450 distance calculus students
Facilitated by Greg, who will answer questions and review problems from QH8 recitations
https://georgiatech.adobeconnect.com/dcp-online-drop-in-tutor-center-2014-fall

If you haven't already, please take a few minutes to fill out the technical issues survey.

1) This is similar to the second and third questions on the transforms HW.

Let $\vec{u}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \vec{v}_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$, and T be a linear transformation that maps $\vec{u}_1$ onto $\vec{v}_1$, and $\vec{u}_2$ onto $\vec{v}_2$. Find T and the image of $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$ under T.

Let $T(\vec{x}) = A\vec{x} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, so $3a_{11} + 4a_{12} = 4$ ①

$3a_{21} + 4a_{22} = 1$ ②

Similarily: $3a_{11} + 3a_{12} = -1$ ③

$3a_{21} + 3a_{22} = 3$ ④

Combining ① and ③ gives $a_{12} = 5$, $a_{11} = -16/3$

Equations ② and ④ gives $a_{21} = -2$, $a_{22} = 3$

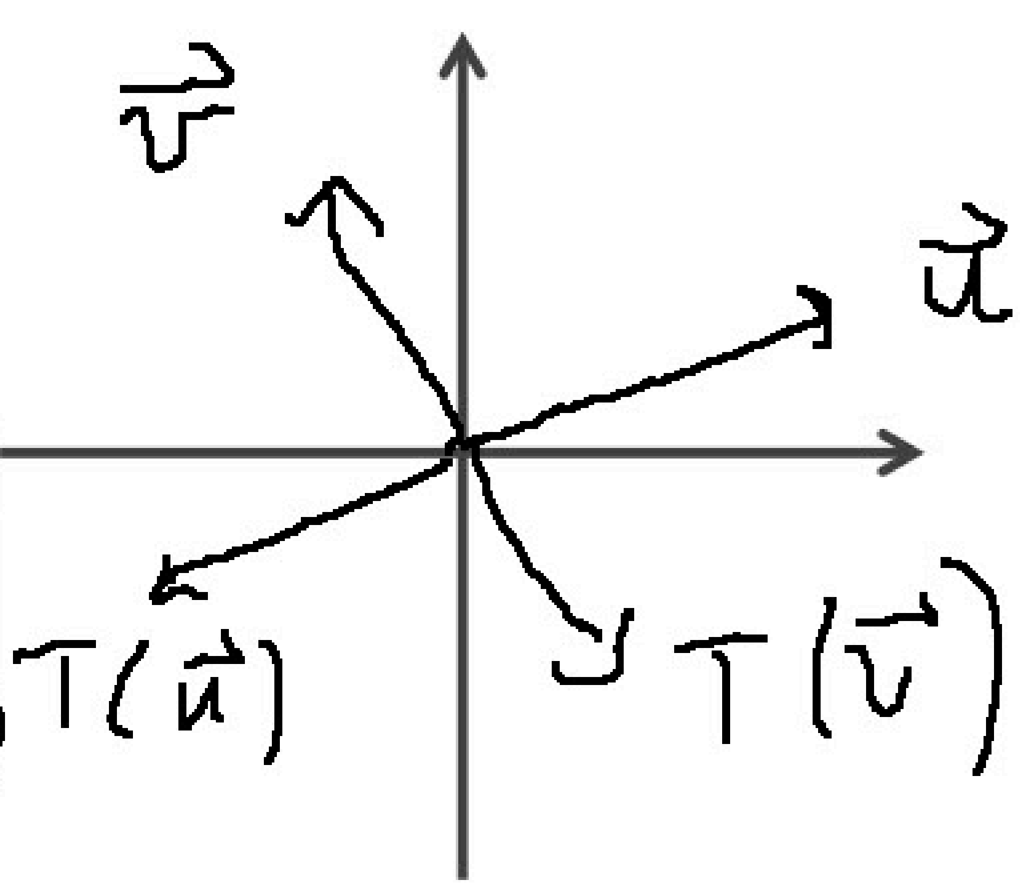
$$\left.\right\} \quad T(\vec{x}) = \begin{pmatrix} -16/3 & 5 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad T\begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} -16 \\ -6 \end{pmatrix}$$

2) Plot u and v, their images under T, and provide a geometric interpretation of what T does to vectors in R^2.

$$u = \begin{bmatrix} 5 \\ 2 \end{bmatrix}, v = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$a) \quad T(x) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

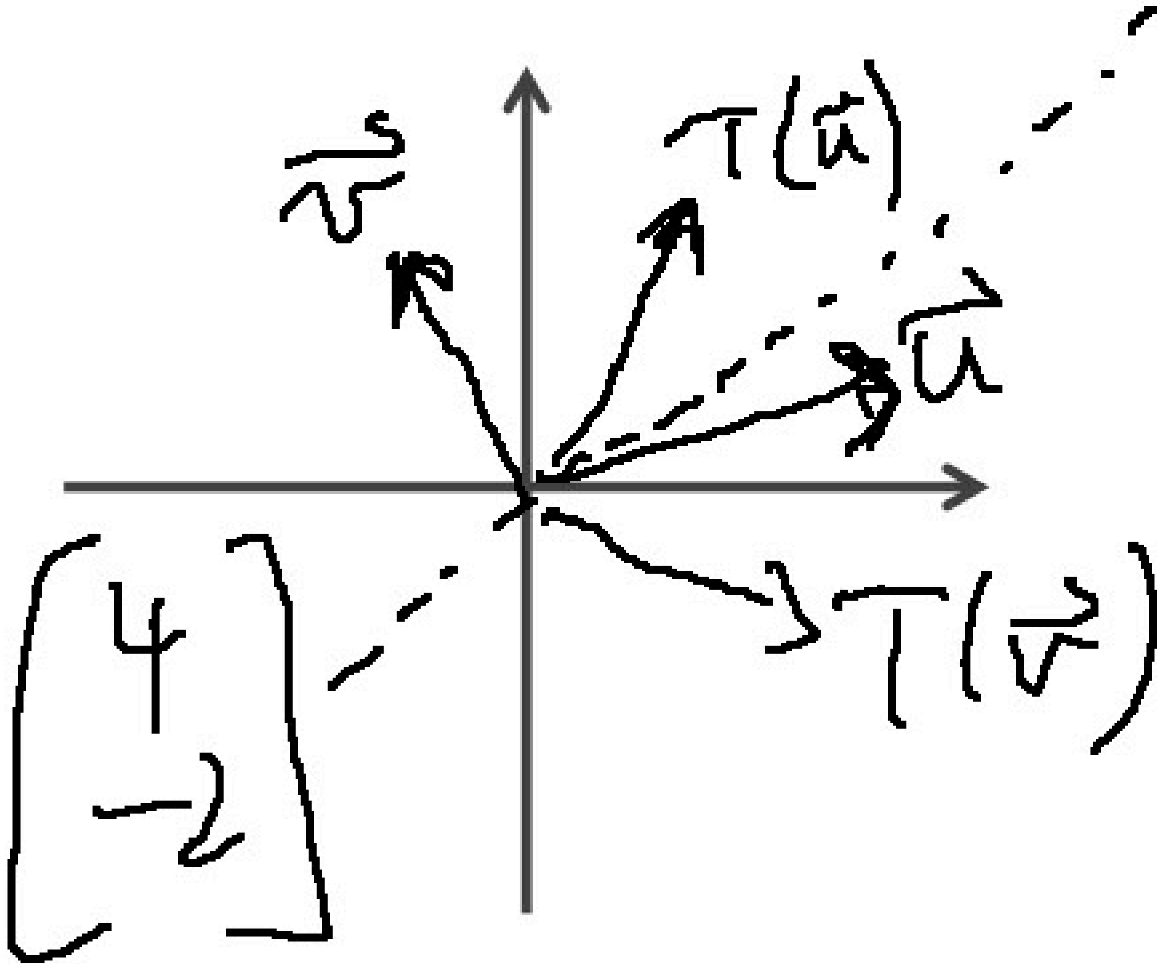
$$T(\vec{u}) = \begin{bmatrix} -5 \\ -2 \end{bmatrix}, T(\vec{v}) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$



reflection through origin

$$b) \quad T(x) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{u}) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, T(\vec{v}) = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$



reflection through line y=x

3) *From 2012 Quiz 2*

Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} \mathrm{Cos}\left[\frac{-\pi}{3}\right] & -\mathrm{Sin}\left[\frac{-\pi}{3}\right] \\ \mathrm{Sin}\left[\frac{-\pi}{3}\right] & \mathrm{Cos}\left[\frac{-\pi}{3}\right] \end{pmatrix}$,

$C = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Compute the image of the house under the transformation ABC. Show the intermediate steps.

$$\cos\left(\frac{-\pi}{3}\right) = 1/2$$

$$\sin(-\pi/3) = -\sqrt{3}/2 \approx -0.866$$

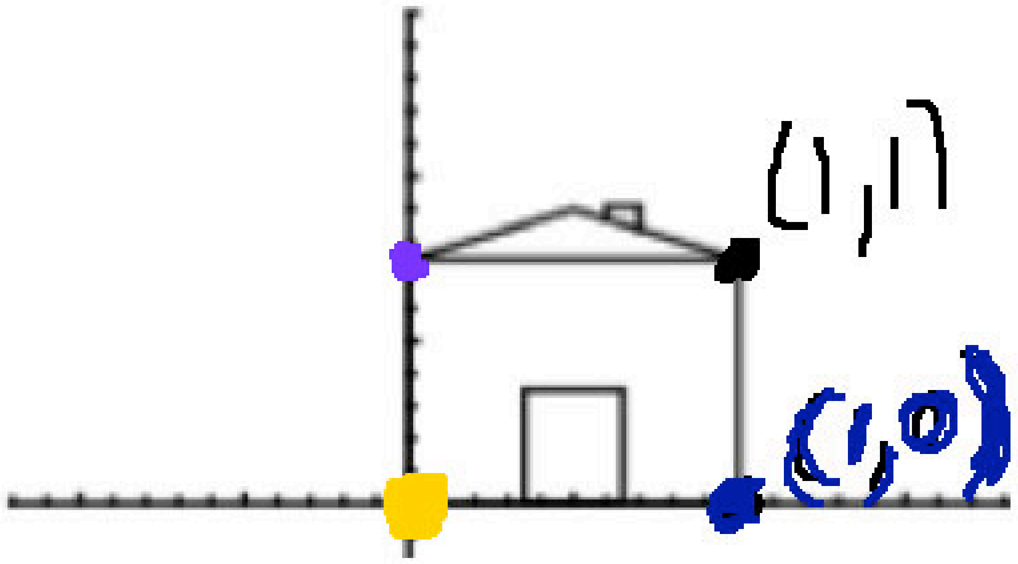


image of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ under C: $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

image of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ under C: $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

transformation under C

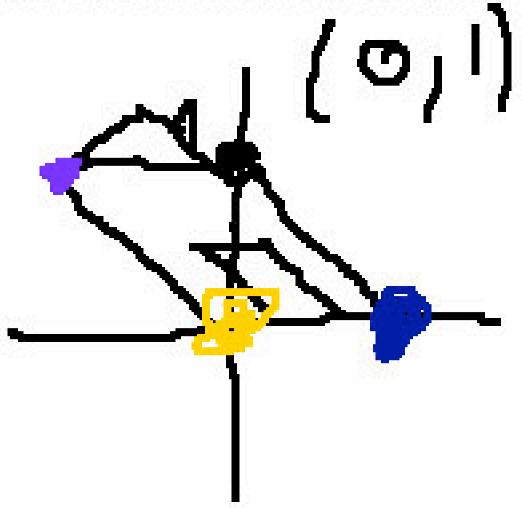
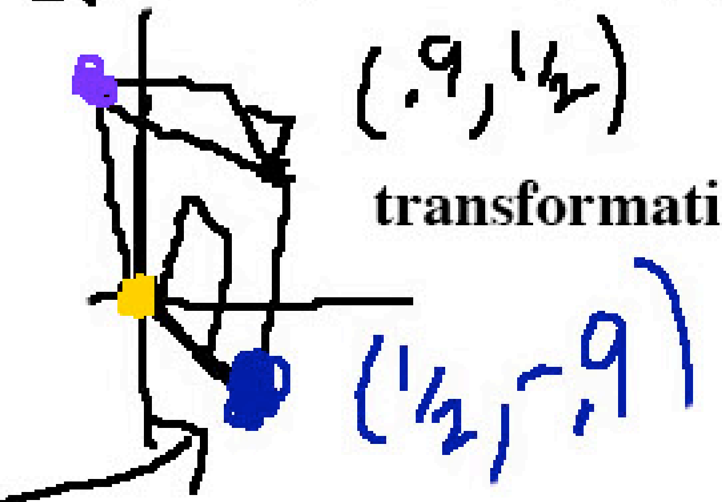


Image under BC is a shear, then clockwise rotation by $\pi/3$:

$$BC\begin{pmatrix} 1 \\ 1 \end{pmatrix} = B\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 & +0.9 \\ -0.9 & 1/2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{bmatrix} .9 \\ .5 \end{bmatrix}$$

$$BC\begin{pmatrix} 0 \\ 1 \end{pmatrix} = B\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .5 & .9 \\ -.9 & .5 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} .5 \\ -.9 \end{pmatrix}$$



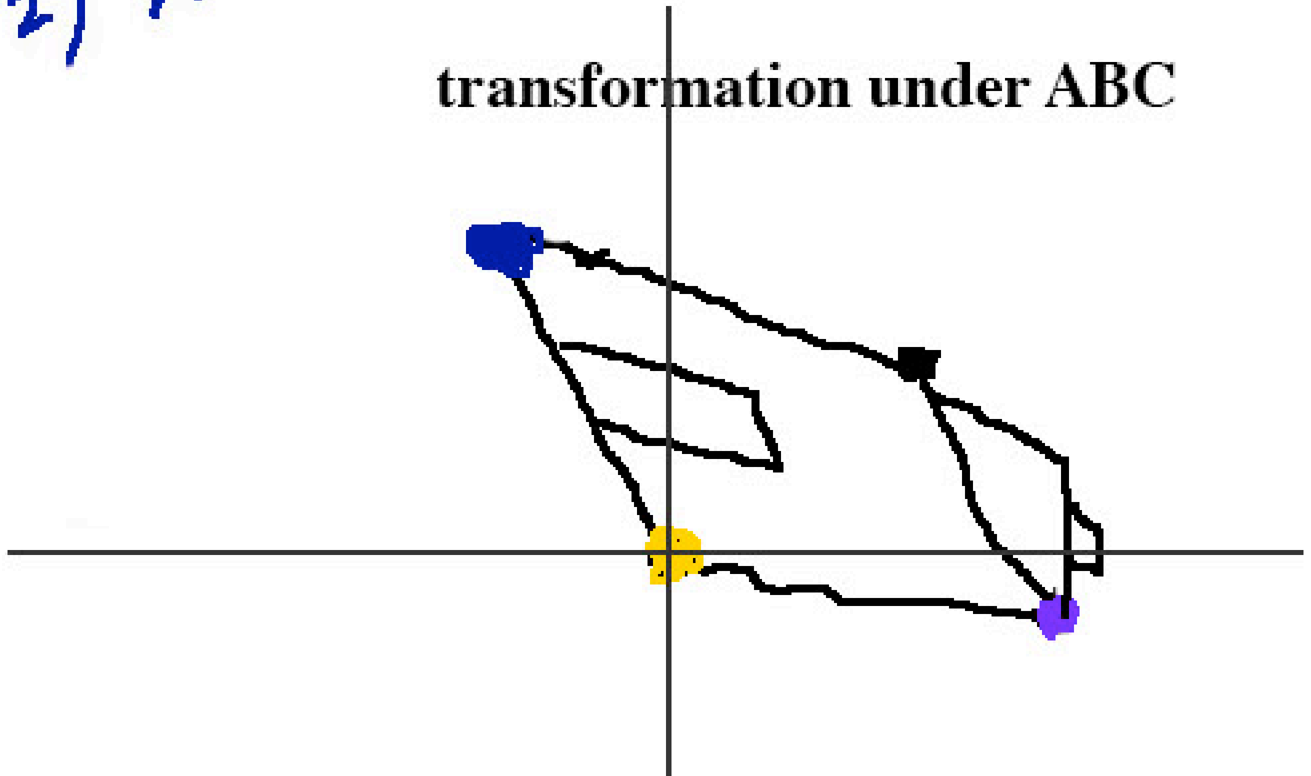
transformation under BC

Image under ABC:

$$ABC\begin{pmatrix} 1 \\ 1 \end{pmatrix} = A\begin{pmatrix} .9 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} .9 \\ 1/2 \end{pmatrix} = \begin{pmatrix} .4 \\ .5 \end{pmatrix}$$

$$ABC\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} .5 \\ -.9 \end{pmatrix} = \begin{pmatrix} -.9 \\ .5 \end{pmatrix}$$

transformation under ABC



notice how the yellow dot never moved.

4) Fill in the elements of the 3x3 matrix.

Hint: the elements can be identified by inspection.

$$\begin{bmatrix} 2 & 0 & -6 \\ -7 & 0 & 1 \\ 0 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 - 6x_3 \\ x_3 - 7x_1 \\ -x_2 - 5x_3 \end{bmatrix}$$

5) Find values of h so that the vectors are LD, and values of h so that the vectors are LI.

$$\begin{bmatrix} 3 \\ -6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 9 \\ h \\ 3 \end{bmatrix}$$

If $h = -18$, then the vectors are LD.

If $h \neq -18$, then the vectors are LI.

$$\begin{pmatrix} 3 & -6 & 9 & 0 \\ -6 & 4 & h & 0 \\ 1 & -3 & +3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -2 & 3 & 0 \\ 0 & -8 & h+18 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$\Rightarrow x_2 = 0$

second row gives:

$$0x_1 - 8x_2 + (h+18)x_3 = 0$$

But $x_2 = 0$, so $(h+18)x_3 = 0$

If $h \neq -18$, $x_3 = 0$, and vectors are LI because $x_1 = x_2 = x_3 = 0$

Recitation 20

Today: Quiz 3 Review

Quiz 3 Next Thursday

Make sure you can solve these questions from old quizzes:

- 2012: Quiz 2 #2 and #3
- 2012: Quiz 3 #2
- 2013: Quiz 3 #1 and #3

Review recent HWs on Span, Lin Transforms, Gauss Jordan

Review sections 1.1, 1.2, 1.3, 1.7, 1.8, 1.9

QH8 Office Hours Next Week

Tuesday and Wednesday 7:30 to 8:30 pm

At the same place as last time:

https://georgiatech.adobeconnect.com/distancecalculusofficehours

Online Drop-in Tutoring

Wednesdays, 5:30 to 7:00 pm

For all ~450 distance calculus students

Facilitated by Greg, who will answer questions and review problems from QH8 recitations

https://georgiatech.adobeconnect.com/dcp-online-drop-in-tutor-center-2014-fall

If you haven't already, please take a few minutes to fill out the technical issues survey.

1) From 2013, Quiz 3, #3

If T is a linear trasformation consisting of rotating counterclockwise by $\pi/3$ radians followed by a reflection about the line x = y, find the matrix such that T(x) = Ax.

Let $T(\vec{x}) = A\vec{x}$. Then matrix A is a 2x2 matrix, and $\vec{x}$ is a column vector with two elements.

$$T(\vec{x}) = A\vec{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \sin\pi/3 & \cos\pi/3 \\ \cos\pi/3 & -\sin\pi/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

✓ DONE

If you prefer:

$$T(x) = A\vec{x} = \begin{pmatrix} \sqrt{3}/2 & 1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

A common mistake would be to write $T = \begin{pmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

but that is wrong: the rotation should be applied 1st.

2) From 2012, Quiz 2, #2

For what values of b is **y** a linear combination of **u** and **v**?

$$\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\mathbf{u} = \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ b \end{bmatrix}$$

If $\vec{y}$ is a L.C. of $\vec{u}$ and $\vec{v}$, then $\exists\, c_1, c_2$ such that $c_1\vec{u} + c_2\vec{v} = \vec{y}$. Form augmented matrix:

$$\left(\begin{array}{cc|c} 4 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & b & 3 \end{array}\right) \sim \begin{pmatrix} 0 & -1 & -3 \\ 2 & 1 & 1 \\ 1 & b/3 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 3 \\ 2 & 0 & -2 \\ 1 & b/3 & 1 \end{pmatrix} \sim \left(\begin{array}{cc|c} 0 & 1 & 3 \\ 1 & 0 & -1 \\ 1 & b/3 & 1 \end{array}\right)$$

First and 2^{ND} rows give us $c_1 = -1$, $c_2 = +3$.

The 3^{RD} row is: $c_1 + b/3\, c_2 = 1$

$$-1 + b/3\,(3) = 1$$

$$b = 2.$$

If $b \neq 2$, then the system is not consistent, ie $\vec{y}$ is not a L.C. of $\vec{u}$ and $\vec{v}$.

If $b=2$, then **y** is a linear combination of **u** and **v**.

If $b \neq 2$, then **y** is *not* a linear combination of **u** and **v**.

3) From 2013, Quiz 3, #1

If T is a linear trasformation, and:

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix},\quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix}$$

find the matrix, A, such that T(x) = Ax.

Let $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$. Then:

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=A\begin{bmatrix}1\\1\end{bmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \quad ①$$

$$T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)=A\begin{bmatrix}1\\2\end{bmatrix}=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\3\end{bmatrix} \quad ②$$

First rows of equations ① and ②:

$$\left.\begin{matrix}a_{11}+a_{12}=1\\ a_{11}+2a_{12}=2\end{matrix}\right\} \begin{matrix}a_{11}=0,\ a_{12}=1\\ \text{(by inspection)}\end{matrix}$$

Second rows of equations ① and ②:

$$\left.\begin{matrix}a_{21}+a_{22}=0\\ a_{21}+2a_{22}=3\end{matrix}\right\} \begin{matrix}a_{22}=3,\ a_{12}=-3\\ \text{(also by inspection)}\end{matrix}$$

$$T(\vec{x})=A\vec{x}=\begin{pmatrix}0 & 1\\ -3 & 3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}$$

4) From 2012, Quiz 3, #2

For what values of *a* are the following vectors LI?

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ a \end{bmatrix}$$

If LI, then the equation $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$ can only be satisfied when $c_1 = c_2 = c_3 = 0$.

Form augmented matrix: $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 4 & 2 & 0 \\ 2 & 3 & a & 0 \end{array}\right) \sim \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -1 & a-4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & 0 \\ 0 & 0 & 2a-7 & 0 \\ 0 & 1 & 4-a & 0 \end{pmatrix}$

The second row is $0c_1 + 0c_2 + (2a-7)c_3 = 0$, or $(2a-7)c_3 = 0$.

If $2a-7 \neq 0$, then c_3 must be zero. But if $c_3 = 0$, then from the other rows, $c_2 = c_1 = 0$. Thus, if $2a-7 \neq 0$, $c_1 = c_2 = c_3 = 0$ and vectors must be LI.

If vectors are not LI they are LD, and vice versa.

If $a = 7/2$, then the vectors are LD.

If $a \neq 7/2$, then the vectors are LI.

Recitation 21

Today: Matrix Inverses, LU Decomposition

1a) State the formula for the inverse of the matrix

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad M^{-1} = \frac{1}{\det M}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

1b) Use the formula in (1a) to find a 2×2 matrix P such that: $P\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $P\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

By construction:

$$P\underbrace{\begin{bmatrix} 3 & 3 \\ 4 & 3 \end{bmatrix}}_{B} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}.$$

The inverse of B is $\frac{1}{3\cdot 3 - 3\cdot 4}\begin{bmatrix} 3 & -3 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & +1 \\ 4/3 & -1 \end{bmatrix}$

$$P\underbrace{B\,B^{-1}}_{=I} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix} B^{-1} = \begin{bmatrix} 4 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 4/3 & -1 \end{bmatrix} = \begin{bmatrix} -16/3 & 5 \\ 3 & -2 \end{bmatrix}$$

$$\Rightarrow \boxed{P = \begin{bmatrix} -16/3 & 5 \\ 3 & -2 \end{bmatrix}}.$$

(CHECK: $P\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} -16/3 & 5 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ ✓)

2) Solve the equation Ax = b, using the LU decomposition of A, where

$$A=\begin{bmatrix}3 & -7 & -2\\ -3 & 5 & 1\\ 6 & -4 & 0\end{bmatrix}=\underbrace{\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -5 & 1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}3 & -7 & -2\\ 0 & -2 & -1\\ 0 & 0 & -1\end{bmatrix}}_{U},\mathbf{b}=\begin{bmatrix}-7\\ 5\\ 2\end{bmatrix}$$

L = lower triangular mtx

U = upper tri. mtx

LET $\underline{y} = Ux$. Then $Ly = b$:

$$\begin{bmatrix}1 & 0 & 0\\ -1 & 1 & 0\\ 2 & -5 & 1\end{bmatrix}\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}-7\\ 5\\ 2\end{bmatrix} \Rightarrow \left.\begin{array}{l} y_1 = -7\\ -y_1 + y_2 + 0y_3 = 5,\ y_2 = -2\\ 2y_1 - 5y_2 + y_3 = 2,\ y_3 = 6\end{array}\right\}\text{"FORWARD SUBS"}$$

But $Ux = y$:

$$\begin{bmatrix}3 & -7 & -2\\ 0 & -2 & -1\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}-7\\ -2\\ 6\end{bmatrix} \Rightarrow \left.\begin{array}{l} x_3 = -6\\ -2x_2 - x_3 = -2,\ x_2 = 4\\ 3x_1 - 7x_2 - 2x_3 = -7,\ x_1 = 3\end{array}\right\}\text{"BACKWARD SUBS"}$$

$$\Rightarrow \boxed{x = \begin{bmatrix}3\\ 4\\ -6\end{bmatrix}}$$

E_{21} = "elimination matrix" that eliminates a_{21}.

3) Find the LU decomposition of A.

$$A = \begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix}$$

$$\underset{E_{21}=\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}{\sim} \begin{pmatrix} 5 & 0 & 4 \\ 0 & 2 & 3 \\ 10 & 10 & 16 \end{pmatrix} = E_{21}A$$

$$\underset{E_{31}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}}{\sim} \begin{pmatrix} 5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 10 & 24 \end{pmatrix} = E_{31}E_{21}A$$

$$\underset{E_{32}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{pmatrix}}{\sim} \begin{pmatrix} 5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 9 \end{pmatrix} = E_{32}E_{31}E_{21}A = \mathcal{U}$$

$$A = \overbrace{\left(E_{32}E_{31}E_{21}\right)^{-1}}^{=L}\mathcal{U}, \quad L = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix}$$

3) Find the LU decomposition of A.

$$A = \begin{bmatrix} -5 & 0 & 4 \\ 10 & 2 & -5 \\ 10 & 10 & 16 \end{bmatrix}$$

ALTERNATE METHOD

THIS IS THE METHOD THE TEXTBOOK USES

FIND u: (sequence of row ops to obtain REF)

$$\begin{matrix} R_2+2R_1 \\ \sim \\ R_3+2R_1 \end{matrix} \begin{pmatrix} -5 & 0 & 4 \\ 0 & \boxed{2} & +3 \\ 0 & \boxed{10} & 24 \end{pmatrix} \underset{\sim}{R_3-5R_2} \begin{pmatrix} -5 & 0 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & \boxed{+9} \end{pmatrix} = u$$

FIND L

$$\begin{bmatrix} -5 \\ 10 \\ 10 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \end{bmatrix} \begin{bmatrix} 9 \end{bmatrix}$$

$$\div(-5) \quad \div 2 \quad \div 9$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 5 & 1 \end{pmatrix} = L$$

ACTUALLY: THIS METHOD IS THE SAME AS THE OTHER, WE JUST WROTE OUT FEWER STEPS

4 ~~3)~~ State whether the following statements are true or false and explain your reasoning.

a) A can be row reduced to the identity matrix iff A is invertible.

The statement is T *because:* "this is how we find inverse" (THEOREM 7, 2.2)

b) If matrix B is the inverse of matrix A, the equations AB = I and BA = I must both be true.

The statement is T *because:* by def'n of invertible

c) If A and B are both NxN and invertible, then the product $A^{-1}B^{-1}$ is the inverse of AB.

The statement is F *because:* $(AB)^{-1} = B^{-1}A^{-1}$

d) If an NxN matrix A is invertible, then the columns of A^T are LI.

The statement is T *because:* if A is invert., then A^T is also invert., and any invertable matrix (square) has LI cols. See invertable matrix theorem, section 2.3

Recitation 22

Today: Column Space and Null Space, LU Decomposition

From quiz 3:

b) (3 points) The two vectors

$$\begin{bmatrix}1\\3\\1\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}$$

span a plane passing through the origin. Find a vector that is normal to this plane.

The given vectors are both in the plane.

A vector perpendicular to them is given by:

$$\begin{bmatrix}1\\3\\1\end{bmatrix} \times \begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{vmatrix} i & j & k\\ 1 & 3 & 1\\ 2 & 1 & 1\end{vmatrix} = \begin{bmatrix}3-1\\-(1-2)\\1-6\end{bmatrix} = \begin{bmatrix}2\\1\\-5\end{bmatrix}$$

DONE.

WHY DOES THAT WORK?

$\begin{bmatrix}2\\1\\-5\end{bmatrix}$ is normal to all vectors in this plane:

$$\begin{bmatrix}2\\1\\5\end{bmatrix} \cdot \left(c_1\begin{bmatrix}1\\3\\1\end{bmatrix} + c_2\begin{bmatrix}2\\1\\1\end{bmatrix}\right) = c_1 \cdot 0 + c_2 0 = 0$$

POP QUIZ #2

- Start time: 8:10?
- Ends at: 8:25
- Pop quiz grading
    - 5 points: correct
    - 4 points: something correct
    - 3 points: name on the page
    - 0 points: did not take pop quiz

- To submit your work, choose any of the following:

A. work on whiteboard in breakout room

- type A in text chat so I know you want to work in breakout room
- submit work by letting me know when done, and/or email me a screen capture of your work

B. work on paper and give work to facilitator

- type B in text chat so I know you're doing this
- leave 2 inch margin on paper
- write your name and QH8 at the top
- facilitator is receiving instructions today on how to submit your work

C. work on paper and email a photo of your work to me

- type C in text chat so I know you are emailing your work to me
- email your photo to me before 8:40

Find a basis for the column space of:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 3 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Pivot columns are in columns 1 and 4, so $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$ form a basis for the column space of the given matrix.

Check: column 6 is:

$$\begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = 3\begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix} - 6\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \checkmark$$

(similar calculations for other columns)

2) Fill in the blanks:

a) Coll A is the set of all linear combinations of the columns (or pivot columns) of matrix A.

b) Nul A is the set of all solutions to $A\vec{x} = \vec{0}$.

c) The LI columns of matrix A form a basis for the column space of A. ↖ or the pivot columns

d) The rank of matrix A is the number of LI cols.

e) The nullity of matrix A is the (number of cols) − rank.

$\Rightarrow$ num of cols = rank + nullity

3) Find i) a basis for Col A, and ii) a basis for Nul A.

$$A = \begin{bmatrix} 3 & -1 & -3 & -1 & 8 \\ 3 & 1 & 3 & 0 & 2 \\ 0 & 3 & 9 & -1 & -4 \\ 6 & 3 & 9 & -2 & 6 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -3 & 0 & 6 \\ 0 & 2 & 6 & 0 & -4 \\ 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

i) a basis for Col A is just $\begin{bmatrix} 3 \\ 3 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$.
ie - the pivot columns.

ii) x_3 & x_5 are free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_5 - x_3 - x_2/3 \\ 2x_5 - 3x_3 \\ x_3 \\ 2x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4/3 \\ 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

(these vectors form basis for Nul A)

(first row gives: $3x_1 - x_2 - 3x_3 + 0x_4 + 6x_5 = 0$

$3x_1 - (2x_5 - 3x_3) - 3x_3 + 6x_5 = 0 \Rightarrow 3x_1 = -4x_5 + 0x_3$

$\Rightarrow x_1 = (-4/3)\, x_5$)

Recitation 23

Today: Column and Null Space, Eigenvalues, LU Decomposition

From 2012 Quiz 3: Find a basis for the nullspace of A. Also find its rank and nullity.

$$A = \begin{bmatrix} 3 & -1 & 1 & -1 & 2 \\ 0 & 1 & 3 & 2 & 4 \\ 0 & 1 & 3 & 2 & 4 \end{bmatrix} \begin{matrix} R_1+R_2 \\ \sim \\ R_3-R_2 \end{matrix} \left(\begin{array}{ccccc|c} 3 & 0 & 4 & 1 & 6 & 0 \\ 0 & 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow \underline{2}$ pivot col's $\Rightarrow$ rank $= \underline{2}$

And nullity $= (\#\text{ of cols}) - (\text{rank}) = \underline{5-2} = \underline{3}$ ("rank theorem" from Section 2.9)

Find basis for nullspace. Three free variables: x_3, x_4, x_5.

Row 1: $\underline{3}x_1 + \underline{0}x_2 + \underline{4}x_3 + \underline{\ }x_4 + \underline{6}x_5 = 0 \Rightarrow x_1 = -\frac{4}{3}x_3 - \frac{1}{3}x_4 - 2x_5$

Row 2: $0x_1 + x_2 + 2x_3 + 3x_4 + 4x_5 = 0 \Rightarrow x_2 = -2x_3 - 3x_4 - 4x_5$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -4/3 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{V}_1} + x_4 \underbrace{\begin{bmatrix} -1/3 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{\vec{V}_2} + x_5 \underbrace{\begin{bmatrix} -2 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\vec{V}_3}.$$

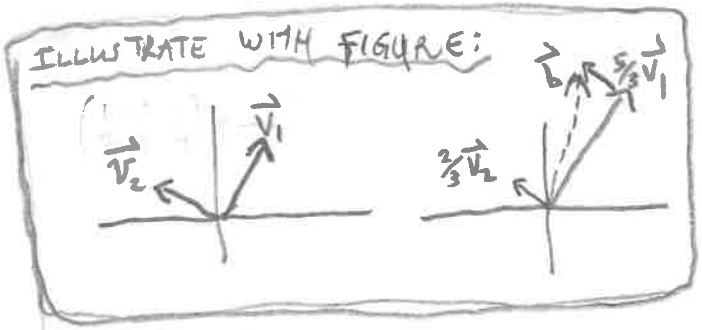
Vectors $\vec{V}_1, \vec{V}_2, \vec{V}_3$ are a basis for nullspace of A.

2) A 2013 pop quiz question: find the coordinates of **b** with respect to $\mathbf{v}_1$ and $\mathbf{v}_2$.

$$\vec{\mathbf{b}} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \vec{\mathbf{v}}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{\mathbf{v}}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

if $\vec{b}$ is in the basis $\{\vec{v}_1, \vec{v}_2\}$, then $\exists$ c_1, c_2 s.t.

$$\boxed{c_1\vec{V}_1 + c_2\vec{V}_2 = \vec{b}}$$



or: $$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

If c_1 & c_2 exist, then $\vec{b}$ in basis, & c_1 & c_2 are its "coordinates".

Solving sytem (with aug. matrix) yields

$$\left.\begin{aligned} c_1 &= 5/3 \\ c_2 &= 2/3 \end{aligned}\right\}$$ thus, coordinates are $\left(\frac{5}{3}, \frac{2}{3}\right)$

3) Is λ = 2 an eigenvalue of matrix A? Why/why not? (from Section 5.1)

$$A = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix}$$

If $\lambda=2$ is an e-vector, then it must satisfy:

$$\underline{A\vec{v} = \lambda\vec{v}}, \quad \text{or} \quad \underline{(A - \lambda I)\vec{v} = \vec{0}}. \quad (\text{by def'n})$$

Substitute in our known quantities:

$$\vec{0} = (A - \lambda I)\vec{v}$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \left(\begin{bmatrix}3&2\\3&8\end{bmatrix} - 2\begin{bmatrix}1&0\\0&1\end{bmatrix}\right)\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Are columns of $A-\lambda I$ LI? NO.

$\Rightarrow \exists$ many sol'ns (infinitely many)

$\Rightarrow \lambda = 2$ is an eigenvalue.

COMPREHENSION CHECK:

Q/ IF the columns of $A-\lambda I$ were LI, what would $\vec{v}$ have to be?

A/ $\vec{v}$ would have to be $\vec{0}$.

λ would not be an e-value, because e-vectors must be non-zero.

4) Find a basis for the eigenspace of A, for the eigenvalue λ = -5.

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & 2 \\ 3 & 3 & -2 \end{bmatrix}$$

The corresponding e-vectors to $\lambda = -5$ are solutions to:

$$(A - \lambda I)\vec{v} = \vec{0}, \text{ or}$$

$$\begin{bmatrix} -4-(-5) & 1 & 1 \\ 2 & -3-(-5) & 2 \\ 3 & 3 & -2+5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}$$

or: (aug. matrix)
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 0 \\ 3 & 3 & 3 & 0 \end{array}\right)$$

$\Rightarrow x_2 \,\&\, x_3$ free!

And $x_1 = -x_2 - x_3$.

If $x_2 = 1, x_3 = 0$, then $x_1 = -1$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

If $x_2 = 0, x_3 = 1$, then $x_1 = -1$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\vec{v}_1, \vec{v}_2$ are LI e-vectors that form basis for e-space of A for $\lambda = -5$.

5) Find the characteristic polynomial and eigenvalues of:

a) $X = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

$0 = \det(A - \lambda I)$

$= \begin{vmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{vmatrix}$

$= \lambda^2 - 4\lambda - 45$

$= (\lambda - 9)(\lambda + 5) \Rightarrow \lambda = 9, -5$

b) $Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & -2 \end{bmatrix}$

$0 = \det(A - \lambda I)$

$= (\lambda - 3)(\lambda + 2)\lambda$

$\Rightarrow$ e-values are $0, +3, -2$

6) ~~#~~ Find the LU decomposition of the following matrix *(if its before 8:40 am, in group work)*.

$$B = \begin{bmatrix} 3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{bmatrix}$$

$$\sim \quad E_{21} = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 9 & 9 & 14 \end{pmatrix} = E_{21} A$$

$$\sim \quad E_{31} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 8 \end{pmatrix} = E_{31} E_{21} A$$

$$\sim \quad E_{32} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \quad \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix} = E_{32} E_{31} E_{21} A = U$$

$$L = E_{21}^{-1} E_{31}^{-1} E_{32}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ +3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & +2 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ +3 & +2 & 1 \end{pmatrix}}$$

~~4)~~ 6) Find the LU decomposition of the following matrix ~~(if its before 8:40 am~~, *in group work)*.

$$B = \begin{bmatrix} -3 & 1 & 2 \\ -9 & 0 & -4 \\ 9 & 9 & 14 \end{bmatrix}$$

ALTERNATE METHOD

FIND U:

$$\underset{R_3-3R_1}{\overset{R_2+3R_1}{\sim}} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 6 & 8 \end{pmatrix} \overset{R_3-2R_2}{\sim} \begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 4 \end{pmatrix} = U$$

FIND L:

$$\left.\begin{array}{ll} \text{COLUMN 1} & \begin{bmatrix} 3 \\ -9 \\ 9 \end{bmatrix} \div 3 \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} \\ \text{COL 2} & \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix} \div 3 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \\ \text{COL 3} & \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \div 4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{array}\right\} L = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

Recitation 24

Today: Determinants (3.1, 3.2), Diagonalization (5.3)

Suppose A and B are square matrices.

1) If a multiple of one row of A is added to another row to produce B, then det(B) = det A .
2) If two rows of A are interchanged to produce B, then det(B) = −det(A) .
3) If one row of A is multiplied by K to produce B, then det(B) = K det A .
4) If A is a trianglular matrix, then det(A) = product of diagonal elements,

there ops do not change determinant

Compute det(A).

$$A = \begin{bmatrix}1 & -4 & 2\\ -2 & 8 & -9\\ -1 & 7 & 0\end{bmatrix} \underset{\boxed{R_3+R_1}}{\overset{\boxed{R_2+2R_1}}{\sim}} \begin{pmatrix}1 & -4 & 2\\ 0 & 0 & -5\\ 0 & 3 & 2\end{pmatrix} \overset{\text{SWAP } R_2 \,\&\, R_3}{\sim} \begin{pmatrix}1 & -4 & 2\\ 0 & 3 & 2\\ 0 & 0 & -5\end{pmatrix}$$

$$\Rightarrow \det A = -\big((1)(3)(-5)\big) = +15.$$

Theorems from Section 3.2

Suppose A and B are square matrices.

1) If A is not invertible, det(A) = 0.
2) det(AB) = det A · det B.
3) det(A + B) = det(A+B). ie- $\det(A+B) \neq \det A + \det B$
4) det(A^T) = det A.

Determine whether the following vectors are LI.

$$\begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$$

$$\begin{pmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{pmatrix} \overset{R_1-5R_2}{\sim} \begin{pmatrix} 0 & 15 & 9 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & -2 \\ 0 & 5 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

⇒ det is zero.

⇒ vectors are LD.

If LI, then $\det\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix}$ would be non-zero.

COFACTOR METHOD

$$\det\begin{vmatrix} 5 & 0 & -1 \\ 1 & -3 & -2 \\ 0 & 5 & 3 \end{vmatrix} = 5\left((-3)(3)-(-2)(5)\right) - 0(3-0) + (-1)(5-0)$$

$$= 5(-9+10) - 5$$

$$= 5-5 = 0$$ ⇒ vectors are LD

Section 5.3: Diagonalization

A matrix A is diagonalizable if it can be written in the form:

$$A = \underline{\; P D P^{-1} \;}$$

where

P is a matrix of e-vectors (columns are e-vectors)

D is a matrix of e-values (diagonal elements are e-values, off diagonal are 0)

Suppose A is N×N. To diagonalize A:

1. find all e-values of A
2. find N LI eigenvectors of A
3. construct P from eigenvectors
4. construct D from eigenvalues
5. find P^{-1}.

2) Diagonalize the following matricies, if possible.

$$A=\begin{bmatrix}1&3\\4&2\end{bmatrix};\ B=\begin{bmatrix}3&2\\0&3\end{bmatrix};\ C=\begin{bmatrix}3&1&1\\1&3&1\\1&1&3\end{bmatrix}\text{ where }\lambda\text{'s of }C\text{ are }2,2,5$$

A)

$$0=\det(A-\lambda I)$$
$$=\det\begin{pmatrix}1-\lambda&3\\4&2-\lambda\end{pmatrix}$$
$$=(1-\lambda)(2-\lambda)-12$$
$$=\lambda^2-3\lambda-10$$
$$\Rightarrow \lambda=-2,5$$

$\underline{\text{if } \lambda_1=-2}$, $\vec{v}=\begin{bmatrix}1\\-1\end{bmatrix}$

$\underline{\text{if } \lambda_2=+5}$, $\vec{v}=\begin{bmatrix}3/4\\1\end{bmatrix}$

$$P=\begin{bmatrix}1&3/4\\-1&1\end{bmatrix},\ P^{-1}=\frac{1}{1+3/4}\begin{pmatrix}1&-3/4\\+1&1\end{pmatrix}$$
$$=\frac{4}{7}\begin{pmatrix}1&-3/4\\1&1\end{pmatrix}$$

$$\Rightarrow A=\begin{pmatrix}1&3/4\\-1&1\end{pmatrix}\begin{pmatrix}-2&0\\0&5\end{pmatrix}\begin{pmatrix}4/7&-3/7\\4/7&4/7\end{pmatrix}$$

B)

$$0=(3-\lambda)(3-\lambda)-0$$
$$\Rightarrow \lambda_{1,2}=+3.$$

$\underline{\text{When } \lambda_1=3:}$

$$(A-\lambda I)\vec{v}=\begin{pmatrix}0&2\\0&0\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

$\Rightarrow$ one eigenvector is $\begin{bmatrix}1\\0\end{bmatrix}$, but we can't find another LI eigenvector. So P does not exist, and B is not diagonalizable.

C)

$\underline{\text{if } \lambda_{1,2}=2:}$ $A-2I=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}$

$$\Rightarrow \vec{v}_1=\begin{bmatrix}1\\0\\-1\end{bmatrix},\ \vec{v}_2=\begin{bmatrix}1\\-1\\0\end{bmatrix}.$$

$\underline{\text{if } \lambda_3=5:}$ $A-5I=\begin{pmatrix}-2&1&1\\1&-2&1\\1&1&-2\end{pmatrix}$

$$\sim\begin{pmatrix}2&-1&-1\\0&-3&3\\0&3&-3\end{pmatrix}\sim\begin{pmatrix}2&-1&-1\\0&1&-1\\0&0&0\end{pmatrix}$$

$$\Rightarrow \vec{v}_3=\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

$$\Rightarrow C=\begin{pmatrix}1&1&1\\0&-1&1\\-1&0&1\end{pmatrix}\begin{pmatrix}2&&\\&2&\\&&5\end{pmatrix}P^{-1}$$

To find P^{-1}: (use Section 2.2)

$$\left(\begin{array}{ccc|ccc}1&1&1&1&0&0\\0&-1&1&0&1&0\\-1&0&1&0&0&1\end{array}\right)\sim\begin{pmatrix}1&0&2&1&1&0\\0&1&-1&0&-1&0\\0&1&2&1&0&1\end{pmatrix}\sim\begin{pmatrix}1&0&2&1&1&0\\0&1&-1&0&-1&0\\0&0&3&1&1&1\end{pmatrix}$$

$$\sim\begin{pmatrix}1&0&0&2/3&2/3&-1/3\\0&1&0&1/3&-2/3&1/3\\0&0&1&1/3&1/3&1/3\end{pmatrix},\ P^{-1}=\frac{1}{3}\begin{pmatrix}2&2&-1\\1&-2&1\\1&1&1\end{pmatrix}$$

(the last three columns of the final matrix are braced and labelled P^{-1})

(*FINDING INVERSE OF 3×3 HASN'T BEEN COVERED IN MATH 1502)

Recitation 25

Today: Quiz Review

Quiz 4 Next Thursday

Review recent HWs on span, determinants, and eigenvalues

Review sections 2.8, 2.9, 3.1, 3.2, 5.1, 5.2

QH8 Office Hours

Tuesday and Wednesday 7:30 to 8:30 pm

We'll solve practice quiz problems & go over specific areas you'd like to review.

At the same place as last time:

https://georgiatech.adobeconnect.com/distancecalculusofficehours

Online Drop-in Tutoring

Wednesdays, 5:30 to 7:00 pm

For all ~450 distance calculus students

Facilitated by Greg, who will answer questions and review problems from QH8 recitations

https://georgiatech.adobeconnect.com/dcp-online-drop-in-tutor-center-2014-fall

Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

b) Find as many LI eigenvectors for this eigenvalue as possible.

$$A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix}$$

Group Work

Writing on board disappears when I enter room (sometimes). So lets try this:

- None of the breakout rooms have the questions.
- Every breakout room has a whiteboard.
- Write Question on the whiteboard when you get into the breakout room, solve it, then move to Question 2.
- You've got about 15 minutes.

Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

b) Find as many LI eigenvectors for this eigenvalue as possible.

$$A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix}$$

Question 2

Compute det(B) by using row reduction.

$$B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}$$

Question 1 (from 2013 Quiz 4)

a) Determine whether 12 is an eigenvalue (hint: there is a faster method than finding the characteristic polynomial)

b) Find as many LI eigenvectors for this eigenvalue as possible.

$$A = \begin{bmatrix} 10 & 3 & -1 \\ 2 & 9 & 1 \\ -2 & 3 & 11 \end{bmatrix}$$

a) EASY WAY: IF 12 IS AN E-VALUE, THERE ARE NON-TRIVIAL SOL'NS TO $(A-12I)\vec{v}=0$.
SINCE $A-12I = \begin{bmatrix} -2 & 3 & -1 \\ 2 & -3 & 1 \\ -2 & 3 & -1 \end{bmatrix}$ HAS LD COLUMNS, NON-TRIVIAL SOLUTIONS EXIST, SO $\lambda=12$ MUST BE AN E-VALUE.

ANOTHER WAY: $\lambda=12$ IS AN E-VALUE IF ITS A ROOT OF THE CHAR. POLY:

$$\det(A-12I) = \begin{vmatrix} -2 & 3 & -1 \\ 2 & -3 & 1 \\ -2 & 3 & -1 \end{vmatrix} = -2(3-3) - 3(-2+2) - 1(6-6) = 0$$

$\Rightarrow$ 12 IS AN E-VALUE.

YET ANOTHER APPROACH: FIND CHARACTERISTIC POLYNOMIAL, FIND ITS ROOTS, AND SEE IF 12 IS ONE OF THEM.

b) $$\begin{pmatrix} -2 & 3 & -1 & 0 \\ 2 & -3 & 1 & 0 \\ -2 & 3 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 2 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$2x_1 = 3x_2 - x_3$

x_2, x_3 free.

$\Rightarrow$ basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow$ only 2 LI e-vectors.

Question 2

Compute det(B) by using row reduction.

$$B = \begin{bmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{bmatrix}$$

$$\begin{vmatrix} 1 & 5 & -3 \\ 3 & -3 & 3 \\ 2 & 13 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -3 \\ 0 & -18 & +12 \\ 0 & -3 & -1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 3 & 1 \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 6 \cdot 1(-3) \cdot 1$$

$$= -18$$

CHECK ANSWER:

$$|B| = 1(21-39) - 5(-21-6) - 3(39+6)$$

$$= -18 + 5 \cdot 27 - 3 \cdot 45$$

$$= -18 + 135 - 135$$

$$= -18$$

3) State whether the following statements are true or false and explain your reasoning.

a) The dimension of Col A is the number of pivot columns of A.

The statement is TRUE *because:* BY DEFINITION, SEE SECTION 2.9.
\# PIVOT COLS = rank A = dim(Col A)
(dim(colA) is also the rank A)

b) The dimension of Nul A is the number of variables in equation A**x** = 0.

The statement is FALSE *because:* dim(Nul A) = # of free variables

c) If A is a square matrix, and det(A^4) = 0, then A is not invertible.

The statement is TRUE *because:* $\det(A^4) = \det(A)\det(A)\det|A|\det|A| = 0$

$\Rightarrow \det A = 0$

$\Rightarrow$ A not invertible.

4) Find all values of h so that the eigenspace for D, for λ = 4, is two dimensional.

$$D = \begin{bmatrix} 4 & 2 & 3 & 3 \\ 0 & 2 & h & 3 \\ 0 & 0 & 4 & 14 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

THE BASIS FOR E-SPACE SPANS SOLUTIONS TO:

$$(D - 4I)\vec{v} = \vec{0}.$$

$$D - 4I = \begin{pmatrix} 0 & 2 & 3 & 3 \\ 0 & -2 & h & 3 \\ 0 & 0 & 0 & 14 \\ 0 & 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & h+3 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(for a two-dimen. eigenspace, we need two free variables)

IF $h=-3$: THERE ARE __2__ PIVOT COLUMNS,
⇒ THERE ARE __2__ FREE VARIABLES } dim (eigenspace) = 2

IF $h \neq -3$ dim (eigenspace) = 1

⇒ h must be −3 for there to be a 2D eigenspace for $\lambda = 4$.

STILL CONFUSED?
If $h=-3$, then our aug. matrix is $\begin{pmatrix} 0 & 2 & 3 & 3 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,
so x_1 & x_3 free, $x_2 = -3x_3/2$, $x_4 = 0$.
Basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 2 \\ 0 \end{bmatrix} \right\}$, which has two LI vectors.

Recitation 27

Today: Diagonalization (5.3), Orthogonality (6.1, 6.2, 6.3, 6.4)

Let $v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Show that these are (pairwise) orthogonal. If

$$\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = a_1 v_1 + a_2 v_2 + a_3 v_3,$$

where the v_i 's are as above and the a_i 's are scalars, FIND a_2

$$\left.\begin{aligned} \vec{v}_1 \cdot \vec{v}_2 &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = -1+0+1=0 \\ \vec{v}_1 \cdot \vec{v}_3 &= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = -1+0+1=0 \\ \vec{v}_2 \cdot \vec{v}_3 &= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1-2+1=0 \end{aligned}\right\} \text{vectors must be pairwise orthogonal}$$

SECOND PART: FIND a_2.

THE TEDIOUS METHOD: solve $\begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

THE FAST METHOD: $v_2 \cdot \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = v_2 \cdot (a_1 v_1 + a_2 v_2 + a_3 v_3) = a_1 \underbrace{v_2 \cdot v_1}_{=0} + a_2 v_2 \cdot v_2 + a_3 \underbrace{v_2 v_3}_{=0}$

$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} = 2-6+5 = 1$, and $a_2 v_2 \cdot v_2 = 6$,

So $1 = a_2 \cdot 6$, so $a_2 = 1/6$.

Question I (parts a and b are from 2014 Quiz 4, Question 3)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

a) Find all eigenvalues

b) Find a eigenbasis for each eigenvalue.

c) Is it possible to diagonalize A? Why/why not?

a) $0 = \det(A - \lambda I) = (1-\lambda)(\lambda-2)^2 \Rightarrow \lambda_1 = 1, \ \lambda_{2,3} = 2.$

b) $\lambda_1 = 1$: $A - (1)I = \begin{pmatrix} 0 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

first row: $0 + 2x_2 + 0 = 0 \Rightarrow x_2 = 0$

second row: $0 + 0 + x_3 = 0 \Rightarrow x_3 = 0$

$\Rightarrow x_1$ is free. choose $x_1 = 1$.

$\Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$

$\lambda_{2,3} = 2$: $A - 2I = \begin{pmatrix} -1 & 2 & 3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$

x_3 free

$x_2 = x_3$

$x_1 = 5x_3$

$\Rightarrow \vec{v}_{2,3} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$

$\Rightarrow$ Basis for λ_1 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\Rightarrow$ Basis for $\lambda_{2,3}$ is $\left\{ \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix} \right\}$

c) Not possible to diagonalize, because we don't have three LI eigenvectors

Recitation 28

Today: Orthogonality (6.1, 6.2, 6.3)

True or False:

a) Eigenvalues must be nonzero scalars.

This is FALSE *, because* eigenvalues can be zero. When they are we know that the given matrix has LD columns (because $A\vec{v}=\vec{0}=0\vec{v}$ will have non-trivial solutions)

b) Eigenvectors must be nonzero vectors.

This is TRUE *, because* we want to find non-trivial solutions to the equation $A\vec{v}=\lambda\vec{v}$.

The zero vector $\vec{v}=\vec{0}$ will <u>always</u> solve the equation $A\vec{v}=\lambda\vec{v}$, so we're only interested in the non-zero solutions.

Quiz 4, Question 1b

Solutions were emailed to students yesterday.

Question

b) Find a basis for the null space of A. $A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 6 \end{pmatrix}$

An Answer

A basis for Nul(A) is the set: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \right\}$

Is this correct? How can we check to see if this answer is correct?

The basis vectors must be LI, and be in the Null space of A.

$A\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, but $A\begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$ is not in Nul(A), and so cannot be a basis vector.

Why is the orthogonal projection $\frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$? It comes from finding the α s.t. $\|\vec{y}-\alpha\vec{u}\|$ is minimized, which occurs when $\vec{u} \perp (\vec{y}-\alpha\vec{u})$

Orthogonality

a) Compute the orthogonal projection of $\vec{y}$ onto the line, L, that passes through the origin and is parallel to $\vec{u}$.

b) Sketch $\vec{y}$, $\vec{u}$, and the orthogonal projection, and L.

c) Calculate the distance between $\vec{y}$ and L.

d) Write $\vec{y}$ as a sum of a vector in Span($\vec{u}$) and a vector orthogonal to u.

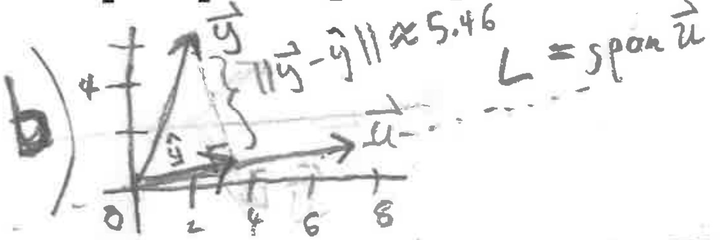
$\vec{y} = \begin{bmatrix} 2 \\ \cancel{5}\,6 \end{bmatrix}$, $\vec{u} = \begin{bmatrix} \cancel{4}\,7 \\ -\cancel{7}\,1 \end{bmatrix}$

a) orthogonal projection $= \frac{\vec{y}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}$ (definition)

$= \frac{14+6}{49+1}\begin{bmatrix} 7 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0.4 \end{bmatrix} = \hat{y}$

b)



c) $\|\vec{y}-\hat{y}\| = \sqrt{(2-2.8)^2+(6-0.4)^2} = \sqrt{0.64+29.16} \approx 5.46$

d) $\vec{y} = \hat{y} + \overbrace{(\vec{y}-\hat{y})}^{\text{component of } \vec{y} \perp \text{ to } \vec{u}}$ (DONE)

$= \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} + \begin{bmatrix} 2-14/5 \\ 6-2/5 \end{bmatrix}$

Question 3

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

a) Are the columns of A LI?
b) Do the columns of A form a basis for R⁴?
c) Are the columns of A mutually orthogonal?

a) $$\underset{R_4-R_1}{\overset{R_3+R_2}{\sim}} \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & -2 & 7 \end{pmatrix} \sim \begin{pmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$ YES, COLUMNS ARE LI.

b) NO. BASIS FOR A SUBSPACE OF $\mathbb{R}^4$.

c)
$$\left.\begin{array}{l} \vec{a}_1 \cdot \vec{a}_2 = -15+1-1-21 \neq 0 \\ \vec{a}_1 \cdot \vec{a}_3 = \text{doesn't matter} \\ \vec{a}_2 \cdot \vec{a}_3 = \text{doesn't matter} \end{array}\right\}$$ not mutually orthogonal.

Recitation 29

Today: Orthogonality (6.1 to 6.5)

Orthogonality and Linear Independence

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

a) Are the columns of A LI?
b) Do the columns of A form a basis for R^4?
c) Are the columns of A mutually orthogonal?

a) YES (we did this last recitation)

b) NO, we would need 4 columns.
The columns of A are a basis for a subspace of $\mathbb{R}^4$.

c) NO: $\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 1 \\ 1 \\ -7 \end{bmatrix} = -36 \neq 0$
(no need to check the other 2 dot products)

~~Question A~~

Find an orthogonal basis for the column space of A.

$$A = \begin{bmatrix} 3 & -5 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -2 \\ 3 & -7 & 8 \end{bmatrix}$$

Let columns of A be $\vec{a}_1, \vec{a}_2, \vec{a}_3$.

Let basis for Col(A) be $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

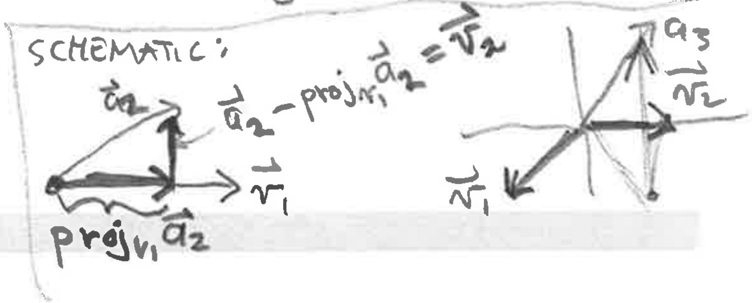
Set $\vec{v}_1 = \vec{a}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$.

$$\vec{v}_2 = \vec{a}_2 - \overbrace{\frac{\vec{a}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1}^{\text{projection of } \vec{a}_2 \text{ onto } \vec{v}_1 = \text{proj}_{\vec{v}_1}\vec{a}_2} = \begin{bmatrix} -5 \\ 1 \\ 1 \\ -7 \end{bmatrix} - \frac{-40}{20}\begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -3 \\ 9 \end{bmatrix}$$

$$\vec{v}_3 = \underbrace{\vec{a}_3 - \frac{\vec{a}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1}_{\text{a vector } \perp \text{ to } \vec{v}_1} - \frac{\vec{a}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix} - \frac{3}{2}\vec{v}_1 - \left(\frac{-1}{2}\right)\vec{v}_2$$

$$= \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

orthogonal basis is $\vec{v}_1, \vec{v}_2, \vec{v}_3$.



Q) WHAT WOULD'VE HAPPENED IF dim(col(A)) = 2?

A) $\vec{v}_3 = 0$ or $\vec{v}_2 = 0$.

Least Squares (slide 1/3)

Consider the system Ax = b, where $A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$

1) Are the columns of matrix A linearly independent?

YES! (They are not co-linear.)

2) Do the columns of A form a basis for R³?

NO! (we would need 3 columns.)

3) Is b in Col(A)?

NOPE. If it were, then there would exist x_1, x_2,

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = b : \quad \left(\begin{array}{cc|c} 4 & 0 & 2 \\ 0 & 2 & 0 \\ 1 & 1 & 11 \end{array}\right) \sim \begin{pmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 1 & 10.5 \end{pmatrix} \Rightarrow \text{INCONSISTENT. AHHH!}$$

4) Is there a solution to Ax = b? NOPE.

5) Therefore, we will:

FIND $\hat{x}$, that is the "closest" solution to

$Ax=b$.

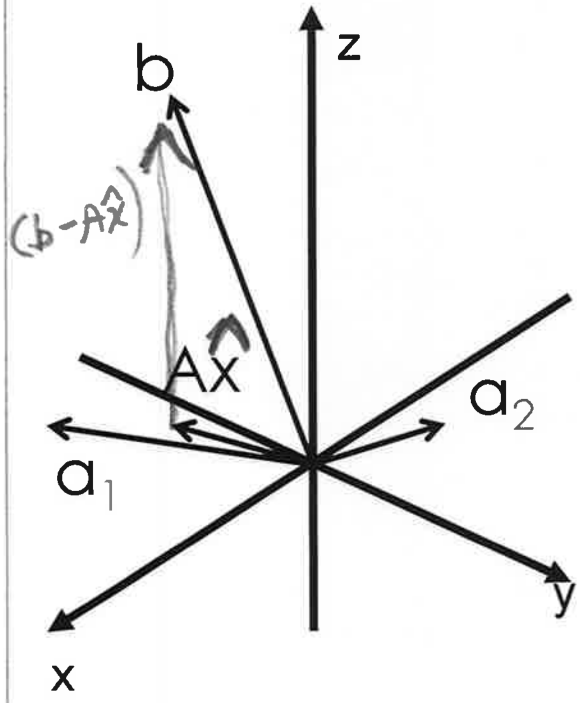
Least Squares (slide 2/3)

Consider the system Ax = b, where

$$A = \begin{array}{cc} \vec{a}_1 & \vec{a}_2 \\ \end{array} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}.$$

Our best solution to this system: the vector $\hat{x}$, such that $|| b - A\hat{x} || \leq \underline{||b - Ax||}$.

What does this mean?



1) Is b in Col A? NO.

2) Is $A\hat{x}$ in Col A? YES. $A\hat{x}$ is co-planar with a_1 and a_2.

3) Is $(b - A\hat{x})$ perpendicular to all vectors in Col(A)?

YES!

$\Rightarrow A^T(b - A\hat{x}) = \vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \boxed{\hat{x} = (A^TA)^{-1}A^Tb}$

Least Squares (slide 3/3)

Consider the system Ax = b, where

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix}$$

Find the least squares solution to this system.

$$A^TA = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 17 & 1 \\ 1 & 5 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 4 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 11 \end{bmatrix} = \begin{bmatrix} 19 \\ 11 \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix}$$

$$\hat{x} = \frac{1}{84} \begin{bmatrix} 5 & -1 \\ -1 & 17 \end{bmatrix} \begin{bmatrix} 19 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

DONE

NOTE

$$A\hat{x} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 3 \end{bmatrix}$$

is the "closest" vector in col(A) to b.

RECITATION 28: LEAST-SQUARES

EXAMPLE; FIND A L.S. SOL'N OF $Ax=b$: $A=\begin{bmatrix}1&5\\3&1\\-2&4\end{bmatrix}$ $b=\begin{bmatrix}4\\-2\\-3\end{bmatrix}$ $\boxed{9, 6.5}$

Q Yes or no? Are columns of A orthogonal?

A Yes.

SAY If they ~~were~~ not orthogonal, we would need to approach this problem differently.

Q What is the orthog proj of b onto column space of A?

A $\hat{b} = \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \frac{b \cdot a_2}{a_2 \cdot a_2} a_2 = \dots = \underbrace{\tfrac{2}{7} a_1 + \tfrac{1}{7} a_2}= \begin{bmatrix}1\\1\\0\end{bmatrix}$

Q what is $\hat{x}$?

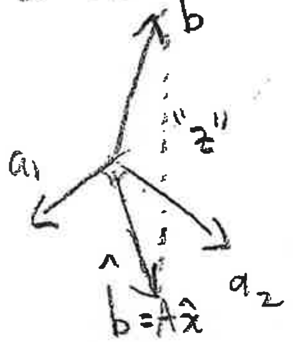
A $\hat{x} = \begin{bmatrix}2/7\\1/7\end{bmatrix}$, because $A\hat{x} = \hat{b}$ $\quad \begin{bmatrix}a_1 & a_2\end{bmatrix}\begin{bmatrix}2/7\\1/7\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$

we solve $\begin{bmatrix}1&5\\3&1\\-2&4\end{bmatrix} \hat{x} = \hat{b} = \begin{bmatrix}1\\1\\0\end{bmatrix}$

TA DA!

WRITE

SCHEMATIC



$\hat{b} \in \text{span}\{a_1, a_2\}$

$\boxed{\hat{x} = \text{L.S. sol'n to } Ax=b}$

inconsistent system

$\boxed{\text{ie: } A\hat{x} = \hat{b}}$

or: $\hat{x}$ is the vector that minimizes

$$\|b - \hat{b}\| = \|b - A\hat{x}\|$$

(why does it minimize? Hopefully covered in class!)

Recitation 31

Today: QR Decomposition, Orthogonality Review (6.1 to 6.5)

QR Factorization

A=QR, and R is an upper triangular matrix.

$$A = \begin{array}{cc} a_1 & a_2 \\ \hline 5 & 9 \\ 1 & 7 \\ -3 & -6 \\ 1 & 5 \end{array}, Q = \begin{array}{cc} q_1 & q_2 \\ \hline 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{array}$$

a) If we weren't given Q, we could find it by using GRAM SCHMIDT.

THE GRAM SCHMIDT ALGORITHM PRODUCES AN <u>ORTHOGONAL</u> BASIS FOR COL(A).

b) The columns of Q are ORTHONORMAL : SO $q_1 \cdot q_1 = q_2 \cdot q_2 = 1, q_1 \cdot q_2 = 0$.

c) The columns of Q form an orthonormal basis for col(A)

NOTE:
- q_1 is a multiple of a_1
- q_2 is a multiple of a_1 & a_2 : $\boxed{q_2 = a_2 - \frac{a_2 \cdot q_1}{q_1 \cdot q_1} q_1}$
- $q_1 \cdot q_1 = q_2 \cdot q_2 = 1, q_1 \cdot q_2 = 0.$

has the weightings for final exam exemption policy been announced?

QR Factorization

A=QR, and R is an upper triangular matrix.

$$A = \begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -6 \\ 1 & 5 \end{bmatrix}, Q = \begin{bmatrix} 5/6 & -1/6 \\ 1/6 & 5/6 \\ -3/6 & 1/6 \\ 1/6 & 3/6 \end{bmatrix}$$

d) The dimensions of R have to be 2x2, because A=QR.

Q R

4x2 2x2

same

dim of A

e) Find R.

f) Show that your answer for part (e) is correct.

e) $A = QR$ (given)

$Q^T A = Q^T Q R$ (multiply by Q^T)

$Q^T A = I R$, because Q is an orthonormal matrix

$$\Rightarrow R = Q^T A = \frac{1}{6}\begin{bmatrix} 5 & 1 & -3 & 1 \\ -1 & 5 & 1 & 3 \end{bmatrix}\begin{bmatrix} 5 & 9 \\ 1 & 7 \\ -3 & -6 \\ 1 & 5 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 36 & 72 \\ 0 & 36 \end{bmatrix} = \begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix}$$

f) QR should equal A. Check: $\frac{1}{6}\begin{bmatrix} 5 & -1 \\ 1 & 5 \\ -3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 6 & 12 \\ 0 & 6 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 30 & 54 \\ \text{YOU CAN STOP CALCULATING HERE} \end{bmatrix} = \begin{bmatrix} 5 & 9 \\ & \end{bmatrix}$

Least Squares (LS) Formulas

For each case, state a formula for the LS solution of Ax = b.

1) A is a square matrix and A^TA is invertible.

$$\hat{x} = (A^TA)^{-1}A^Tb$$

2) A is square, A^TA is invertible, and A has **orthogonal** columns.

for 2x2 or 3x3 this is faster than $(A^TA)^{-1}A^Tb$

Othogonal projection of b onto Col(A) is $\hat{b} = \underbrace{\boxed{\frac{b\cdot a_1}{a_1\cdot a_1}}}_{x_1}\vec{a}_1 + \underbrace{\boxed{\frac{b\cdot a_2}{a_2\cdot a_2}}}_{x_2}\vec{a}_2 + \ldots$

$= \sum \frac{b\cdot a_i}{a_i\cdot a_i}\vec{a}_i$ ← x_i

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$, and in general, $x_i = \frac{b\cdot a_i}{a_i\cdot a_i}$

3) A is square, A^TA is invertible, and A has **orthonormal** columns.

METHOD 1

Since A has orthonormal columns, $A^TA = I$, so

$$\hat{x} = (I)^{-1}A^Tb = A^Tb.$$

METHOD 2

Can also use: $\hat{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, where $x_i = \frac{b\cdot a_i}{a_i\cdot a_i} = \frac{b\cdot a_i}{1} = b\cdot a_i$

Least Squares Solutions

If A is square, and A^TA is not invertible, can we find a LS solution to Ax = b? Why/why not?

the LS solution to $A\vec{x}=\vec{b}$ is the sol'n to $\underline{A^TA}\vec{x}=A^T\vec{b}$.

- A^TA is some matrix that has LD columns
- $A^T\vec{b}$ is some vector

$\Rightarrow$ we are solving a system with an infinite number of solutions

$\Rightarrow$ YES, we can find a LS solution, and there are infinitely many ~~of~~ them.

Describe all LS solutions to the system:

$$x_1 + x_2 = 2$$
$$x_1 + x_2 = 4$$

if $A\vec{x} = \vec{b}$, then $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

The LS solutions of $A\vec{x} = \vec{b}$ solve:

$$A^T A \hat{x} = A^T b$$

But $A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

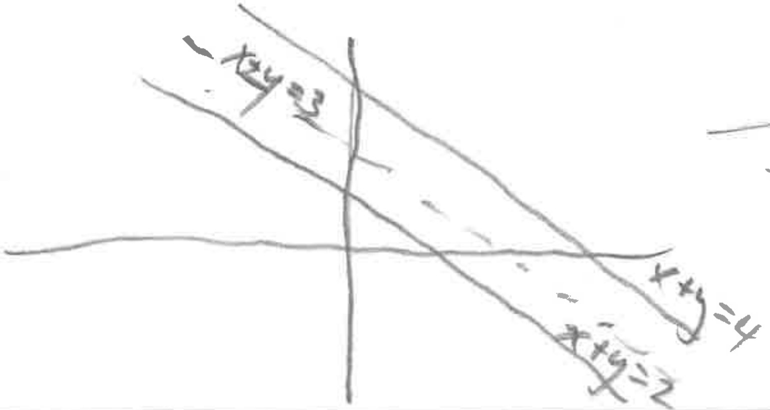
$A^T b = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$

$$\left.\begin{matrix} A^T A = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \\ A^T b = \begin{bmatrix} 6 \\ 6 \end{bmatrix} \end{matrix}\right\} \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

is a system with an infinite # of solutions:

$2x_1 + 2x_2 = 6$

or $\boxed{x_1 + x_2 = 3}$



THE LS SOLUTION, $\hat{x}$ IS THE VECTOR SUCH THAT

$$\|A\hat{x} - \vec{b}\| \le \|A\vec{x} - \vec{b}\|$$

Recitation 32

Today: Final Exam Review

Quiz Grades

- Quiz grades “locked” today.
- Please check your graded quizzes to see if they were graded correctly.

If You are Writing Math 1502 Final Exam

- Part I: Mon, Dec 8
- Part II: Tue, Dec 9
- Work with your facilitator to find a time/place to write.

QH8 Office Hours

Sat Dec 6, Sun Dec 7

Please use text chat when you are free with “I am free on Sat ___ and Sun __” so that we can try to find times that work for most of you.

Today

Group work: 3 groups, we may return to main room if/when groups are getting stuck

1) Section 4.6: Row and Col Space of A^T (Slide 1 of 2)

Row(A) is the set of all possible linear combinations of the rows of A.

Theorem (from Section 4.6)

If two matrices A and B are row equivalent, then their row spaces are equal.

If B is in echelon form, the nonzero rows of B form a basis for Row(A),

as well as Row(B).

A proof of this theorem uses the fact: if B is obtained from row operations on A, the rows of B are a linear combination of the rows of A.

1) Section 4.6: Row and Col Space of A^T (Slide 2 of 2)

Matrix A and its row echelon form are given. Find a basis for

1) Col(A)
2) Row(A)
3) Row(A^T)
4) Col(A^T)

Hint: the answers for all of the above do not require any calculation.

$$A = \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ \hline \end{array}$$

$$A = \left[\begin{array}{rrrr} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{array}\right] \sim \left[\begin{array}{rrrr} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{c} \vec{r}_1 \\ \vec{r}_2 \\ \vec{r}_3 \end{array}$$

1) $\{\vec{a}_1, \vec{a}_2\}$

2) $\{\vec{r}_1, \vec{r}_2\}$

3) $A^T = \begin{bmatrix} 1 & -1 & 5 \\ -4 & 2 & -6 \\ 9 & -4 & 10 \\ -7 & 1 & 7 \end{bmatrix}$, and we already know the third and 4th ~~column~~ rows of A^T are a linear comb of the other two, so basis is $\{\vec{a}_1^T, \vec{a}_2^T\}$

4) Similar to previous question, basis is $\{\vec{r}_1^T, \vec{r}_2^T\}$

QUESTION 2) ~~If time permits~~ *(from 2012 Quiz 4):*

Find the eigenvalues and eigenvectors of A and use them to find a formula for A^k.

$$A = \begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$$

$$0 = A - \lambda I = (5-\lambda)(7-\lambda) - 8 = \lambda^2 - 12\lambda + 27 = (\lambda-3)(\lambda-9), \quad \lambda_1 = 3, \ \lambda_2 = 9$$

$$\lambda_1 = 3 : \quad A - 3I = \begin{pmatrix} 2 & 2 \\ 4 & 4 \end{pmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 = 9 \quad A - 9I = \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Rightarrow D = \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}, \ P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \ P^{-1} = \frac{1}{\det P}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -2 \\ 1 & 1 \end{pmatrix}$$

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & 9^k \end{bmatrix}\begin{bmatrix} 1/3 & -2/3 \\ 1/3 & 1/3 \end{bmatrix}$$

3) QR Factorization

A=QR, and R is an upper triangular matrix.

$$A = \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix}, \quad Q = \begin{bmatrix} -2/7 & 5/7 \\ 5/7 & 2/7 \\ 2/7 & -4/7 \\ 4/7 & 2/7 \end{bmatrix}$$

a) The dimensions of R have to be 2×2 .

b) Calculate matrix R. *Hint: save time by factoring 1/7 out of matrix Q first.*

c) Compute a few elements of the product QR to check your answer for part (b)

b) $R = Q^T A$

$$= \frac{1}{7} \begin{bmatrix} -2 & 5 & 2 & 4 \\ 5 & 2 & -4 & 2 \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 5 & 7 \\ 2 & -2 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix}$$

c) $$QR = \frac{1}{7} \begin{bmatrix} -2 & 5 \\ 5 & 2 \\ 2 & -4 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 7 & 7 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} \overset{\checkmark}{-2} & \overset{\checkmark}{+3} \end{bmatrix}$$

sufficient to stop after two elements

4) Fill in the Blanks

Matrix A has dimensions NxN and is invertible.

a) The columns of A form a basis for $\mathbb{R}^N$

b) rank(A) = N, because A is invertible, it has N LI columns

c) Nul(A) = $\{0\}$

d) dim(Nul(A)) = 0

e) dim(Nul(A^T)) = 0, because A is invertible, it has N LI rows, so A^T has N LI columns

f) dim(Row(A^T)) = dim(Col(A)) = N

g) dim(Col(A^T)) = dim(Row(A)) = N

h) dim(Col(A)) + dim(Nul(A)) = N + 0 = N

i) dim(Row(A)) + dim(Nul(A)) = N + 0 = N

5) A modified version of a 2014 Quiz 2 Question

Compute the integral, using a Taylor polynomial, to an accuracy of at least 0.01.

$$\int_0^2 e^{-x^4}\,dx$$

$$= \int_0^2 \sum_{k=0}^{\infty} \frac{(-x^4)^k}{k!}\,dx$$

$$= \sum \left(\frac{(-1)^k}{k!}\right) \int_0^2 x^{4k}\,dx$$

$$= \sum \left(\frac{(-1)^k}{k!}\right) \frac{x^{4k+1}}{4k+1}\Bigg|_0^2$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}\left(\frac{2^{4k+1}}{4k+1} - 0\right)$$

$$\leq$$

ALT. SERIES REM. THM.:

$$\left|\int_0^2 e^{-x^4}dx - \sum_{k=0}^{N} \frac{(-1)^k}{k!}\,\frac{2^{4k+1}}{4k+1}\right|$$

$$< \frac{2^{4N+1}}{N!(4N+1)}$$

need $N=39$ .... AHH!

True or False

A) If $\{v_1,v_2,v_3\}$ is an orthogonal basis and c is a constant, then $\{v_1,v_2,cv_3\}$ is another, different orthogonal basis.

This statement is F *because* a basis is a set of all possible linear combinations of vectors.

B) The LS solution of Ax = b is the point in Col(A) closest to b.

$\hat{x}$ is the LS solution, and

This statement is F *because* $A\hat{x}$ is the closest point to b, not $\hat{x}$.

C) If x is in subspace W, then $x - \text{proj}_W x \neq 0$

This statement is F *because* if x is in W, then $x = \text{proj}_W x$

ie- the projection of a vector onto that space is itself.

True or False

D) If $\{v_1,v_2,v_3\}$ is an orthogonal basis and c is a constant, then $\{v_1,v_2,cv_3\}$ is another, different orthogonal basis.

This statement is ______ *because*

E) If $\hat{\mathbf{u}}$ is a LS solution to $A\mathbf{u} = \mathbf{b}$, then $\hat{\mathbf{u}} = (A^TA)^{-1}A^T\mathbf{b}$.

This statement is F *because* A^TA may not be invertible

Eigenvalues and Orthogonality

① What can we say about the eigenvalues of an ortho~~gonal~~normal matrix?

Let $A\vec{v} = \lambda\vec{v}$, A is orthogonal

Then: $\|A\vec{v}\| = \|\lambda\vec{v}\|$, but $\|A\vec{v}\| = \|\vec{v}\|$, because A is orthonormal

$$\|\vec{v}\| = |\lambda|\,\|\vec{v}\|$$

$$\Rightarrow \boxed{\lambda = \pm 1}$$

BECAUSE:

$$\|Ax\|^2 = (Ax)^T Ax$$

$$= x^T A^T A x$$

$$= x^T x$$

$$\Rightarrow \|Ax\|^2 = \|x\|^2$$

so $\|Ax\| = \|x\|$

(note: norm is always nonnegative)

② IF A HAS LD COLUMNS, AT LEAST ONE E-VAL IS ZERO.

③ IF A IS UPPER TRIANGULAR, E-VALS ARE THE DIAGONAL ELEMENTS OF A.