STATISTICS FOR GEOMETRIC GRAPHS

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IUUK - ITI
CHARLES UNIVERSITY
PRAGUE

ATLANTA, ROBIN FEST
MAY 2012

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&

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PARIS

ATLANTA, ROBIN FEST
MAY 2012

$\odot_1$: EVERY GRAPH IS A MINOR OF A CUBIC GRAPH

(NOT TRUE FOR SUBDIVISIONS
IMERSIONS)

RADIUS OF CONTRACTED PARTS $\to \infty$

$\odot_2$: EVERY GRAPH IS AN IMERSION OF A PLANAR GRAPH

(NOT TRUE FOR SUBDIVISIONS
FOR MINORS)

LENGTH OF PATHS $\longrightarrow \infty$

WHAT IF WE RESTRICT

RADIUS $\leq i$

PATH LENGTH $\leq i$

$\Downarrow$

HIERARCHY OF CLASSES

$\boxed{\text{CLASS RESOLUTION IN TIME}}$

$$\mathcal{C} \subseteq \mathcal{C}\nabla 0 \subseteq \mathcal{C}\nabla 1 \subseteq \mathcal{C}\nabla 2 \subseteq \ldots$$

$$\xrightarrow[\text{TIME}]{}$$

$\boxed{\text{DEF}}$ (PLOTKIN, RAO, SMITH)

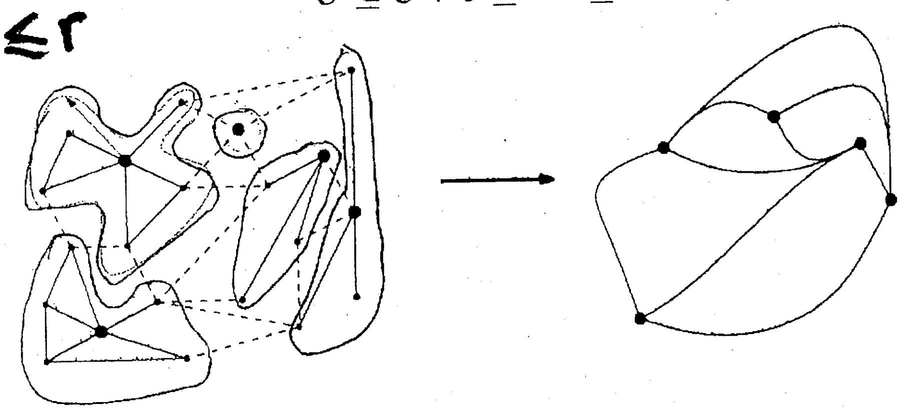
$\mathcal{C}\nabla i$ IS THE CLASS OF ALL i- SHALLOW MINORS (SHALLOW MINORS AT DEPTH i) OF GRAPHS FROM $\mathcal{C}$

DEPTH OF A MINOR $\equiv$ MAX RADIUS OF CONTRACTED SUBGRAPH

(ALL GRAPHS ARE SIMPLE)

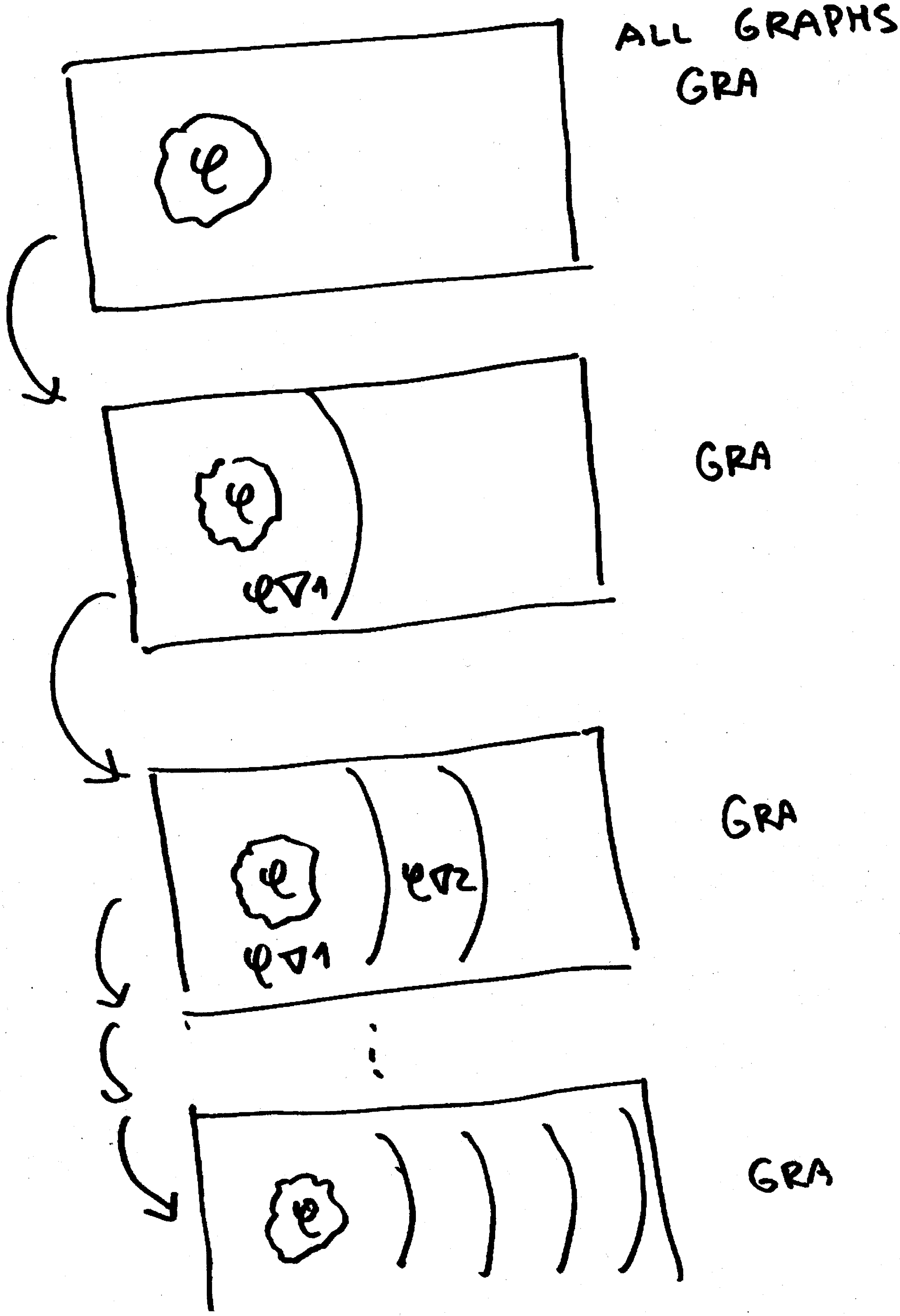
Class Resolution $(r=1)$

$$\mathcal{C} \subseteq \mathcal{C} \nabla 0 \subseteq \cdots \subseteq \mathcal{C} \nabla r \subseteq \ldots$$



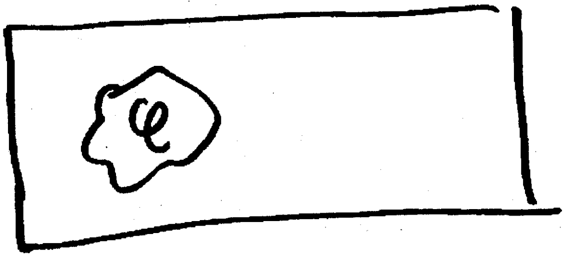
- If there exists r such that $\mathcal{C} \nabla r$ contains all graphs, then $\mathcal{C}$ is *somewhere dense*,
- Otherwise $\mathcal{C}$ is *nowhere dense*

PARAMETRIZATION OF GRAPHS



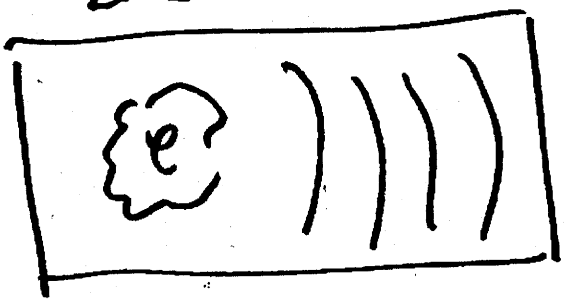
EXAMPLES

PLANAR

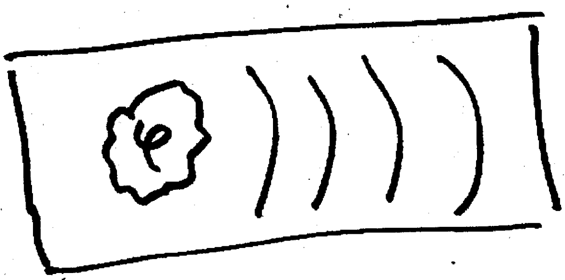


$\varphi \nabla i = \varphi$

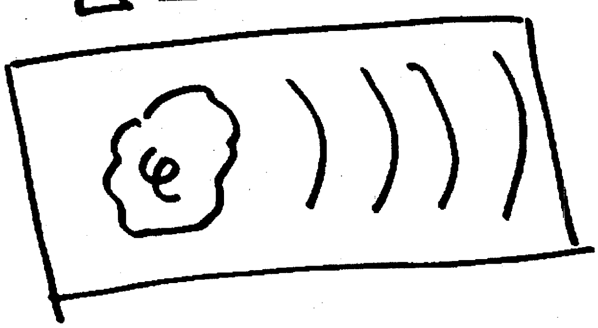
$\Delta \leq d$ BOUNDED DEGREE



FORBIDDEN TOPOLOGICAL MINOR



$\Delta \leq$ GIRTH



SCALLING OF ALL GRAPHS

- CLASS RESOLUTION CAPTURES MANY COMBINATORIAL PROPERTIES

- TIME IS A GOOD PARAMETRIZATION
  LEADS TO EFFECTIVE ALGORITHMS

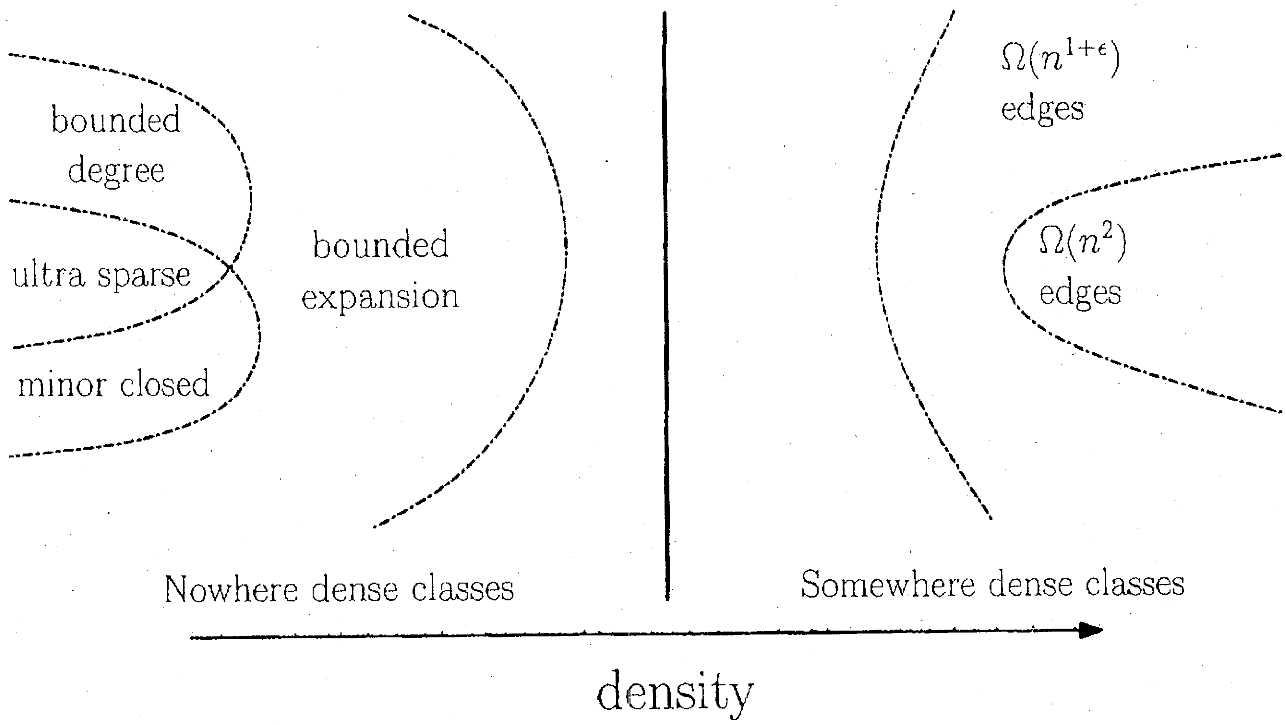
- LEADS TO DICHOTOMY

DEF

$\mathcal{C}$ is SOMEWHERE DENSE
IF THERE EXISTS i_0 SUCH THAT
$\mathcal{C} \nabla i_0$ IS THE CLASS OF ALL GRAPHS

$\mathcal{C}$ is NOWHERE DENSE OTHERWISE

ND VS SD
DICHOTOMY


bounded
degree
ultra sparse
minor closed
bounded
expansion
$\Omega(n^{1+\epsilon})$
edges
$\Omega(n^2)$
edges
Nowhere dense classes
Somewhere dense classes
density

NOT ARBITRARY DEF

POSSIBLE TO DESCRIBE BY
MANY (VIRTUALLY ALL)
GRAPH PARAMETERS :

$\alpha, \omega, \chi,$ WCOL

ALSO BY
EDGE DENSITIES OF

$\mathcal{C} \nabla i$ .... SHALLOW MINORS

$\mathcal{C} \tilde{\nabla} i$ .... SHALLOW TOPOLOGICAL MINORS

$\mathcal{C} \overset{\propto}{\nabla} i$ .... SHALLOW IMMERSIONS

ALSO BY COUNTING
(THUS THE TITLE)

CHARACTERISATION OF ND

Let $\mathcal{C}$ be an unbounded size infinite class of graphs, let F be a graph with at least one edge and let q be a positive integer. Then the following conditions are equivalent:

(1) *$\mathcal{C}$ is a class of nowhere dense graphs,*

(2) *for every integer r, $\mathcal{C} \triangledown r$ is not the class of all finite graphs,*

(3) *for every integer r, $\mathcal{C} \,\widetilde{\triangledown}\, r$ is not the class of all finite graphs,*

(4) *$\mathcal{C}$ is a uniformly quasi-wide class,*

(5) *$H(\mathcal{C})$ is a quasi-wide class,*

(6) $\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\triangledown r} \frac{\log\|G\|}{\log|G|} = 1,$

(7) $\lim_{r\to\infty} \limsup_{G\in\mathcal{C}\widetilde{\triangledown} r} \frac{\log\|G\|}{\log|G|} = 1,$

(8) $\lim_{r\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \nabla_r(G)}{\log|G|} = 0,$

(9) $\lim_{r\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \widetilde{\nabla}_r(G)}{\log|G|} = 0,$

(10) $\lim_{p\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \chi_p(G)}{\log|G|} = 0,$

(11) $\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\triangledown i} \frac{\log \chi_{\cdot}(G)}{\log|G|} = 0,$

(12) $\lim_{p\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \mathrm{col}_p(G)}{\log|G|} = 0,$

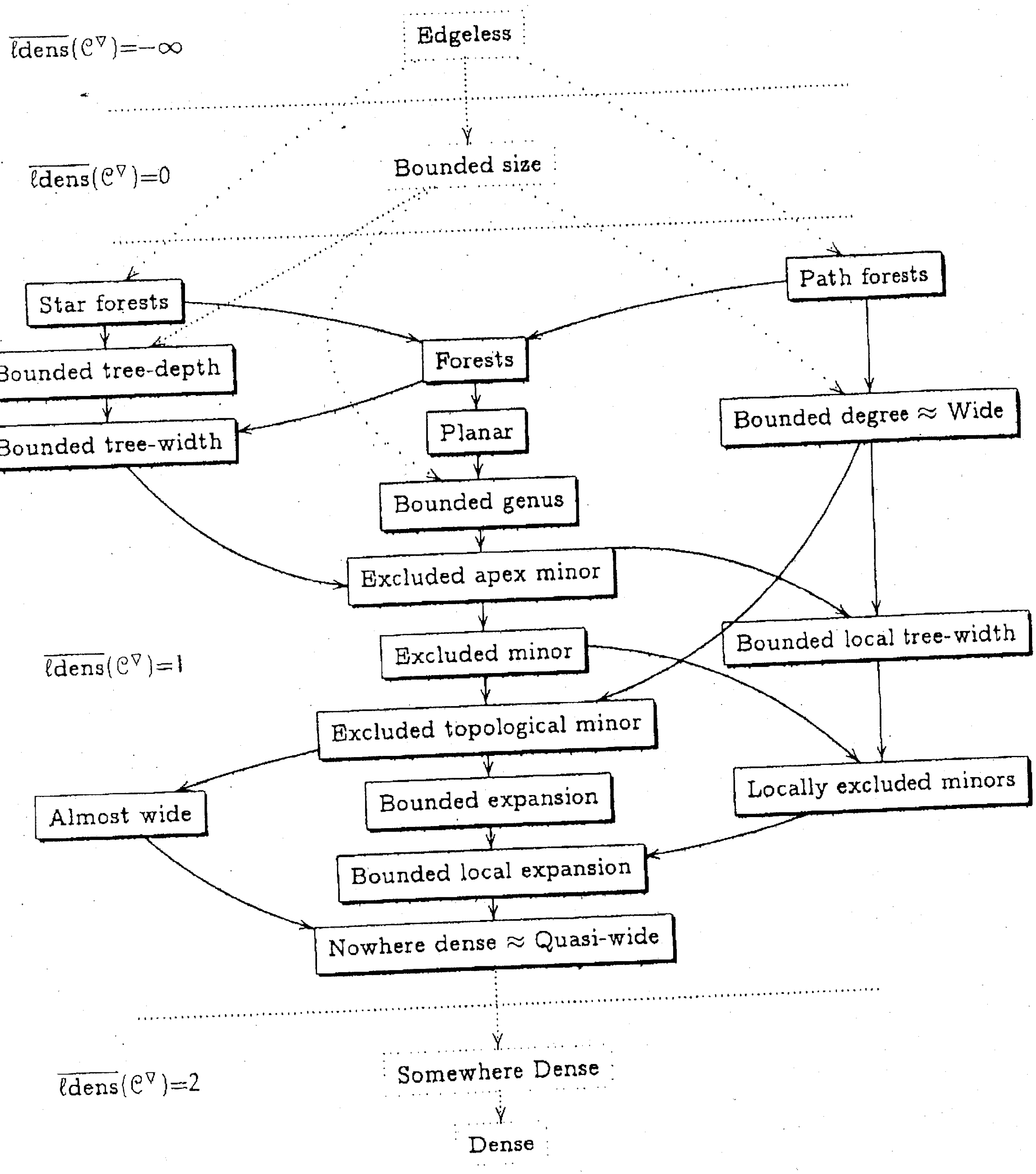
(13) $\lim_{p\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \mathrm{wcol}_p(G)}{\log|G|} = 0,$

(14) *for every integer c, the class $\mathcal{C} \bullet K_c = \{G \bullet K_c : G \in \mathcal{C}\}$ is a class of nowhere dense graphs,*

(15) $\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\triangledown i} \frac{\log(\#F \subseteq G)}{\log|G|} < |F|,$

(16) *for every polynomial P, the class $\mathcal{C}'$ of the 1-transitive fraternal augmentations of directed graphs $\vec{G}$ with $\Delta^-(\vec{G}) \le P(\nabla_0(G))$ and $G \in \mathcal{C}$ form a class of nowhere dense graphs,*

5''



Inclusion map of some hereditary classes.

$\boxed{\text{EXPLANATION}}$ I.

$\boxed{\log\text{-DENSITY}}$:

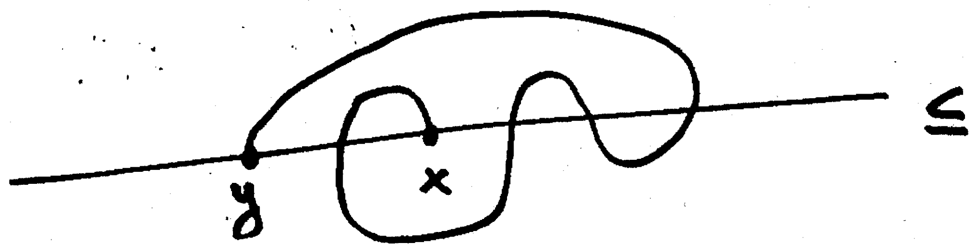
$$\frac{\log \|G\|}{\log |G|} < 1+\varepsilon \quad \dots \quad \|G\| < |G|^{1+\varepsilon}$$

$\boxed{\text{GRAD}}$

$$\nabla_r(G) = \max_{H \in G\nabla r} \frac{\|H\|}{|H|}$$

"r-DEGENERACY"

$\boxed{\text{WCOL}_p}$

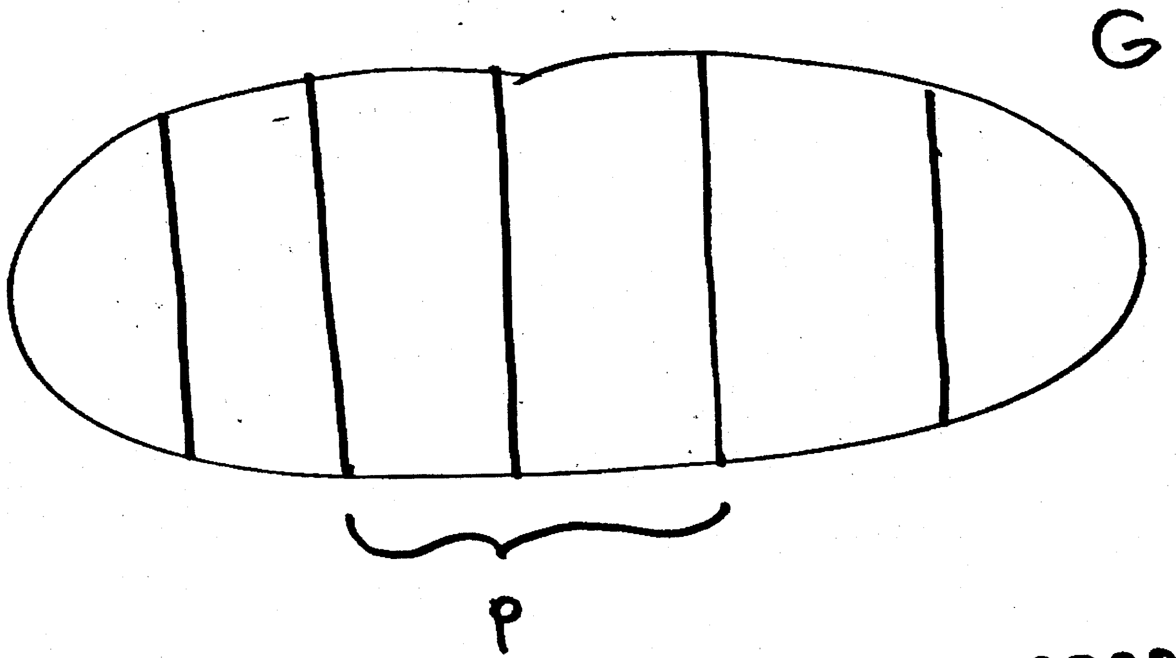


y TO THE LEFT WHICH ARE p-REACHABLE

(KIERSTED, YANG)

EXPLANATION II

χ_p ≡ LOW TREE DEPT DECOMPOSITION



ANY p CLASSES INDUCE SUBGRAPH WITH TREE DEPTH $\leq p$

$td(G) =$ MIN HIGHT OF A ROOTED TREE SUCH THAT $G \subseteq CLOS(T)$

"ANCESTOR RELATION"

$\boxed{\text{EXPLANATION}}$ III.

$\boxed{\text{r- INDEPENDENT}}$

$A \subseteq V(G)$ $\quad \mathrm{dist}_G(x,y) > r.$

$\boxed{\text{QUASIWIDE } \mathcal{C}}$

$\forall r \; \exists s(r) \; \forall n \; \exists N :$

$\forall G \in \mathcal{C}$ HOLDS

$|G| > N \Rightarrow \exists S \; (|S| \leq s(r))$

$G \setminus S$ HAS r- INDEPENDE[NT] SET A

$|A| \geq n$

$\sim$ SCATTERED SETS

$$\text{ND} \iff \lim_{r\to\infty} \limsup_{G\in\mathcal{C}} \frac{\log \nabla_r(G)}{\log|G|} = 0$$

$$\nabla_r(G) = \max_{H\in G\nabla r} \frac{\|H\|}{|H|}$$

$$\Downarrow$$

$$\|H\| = O\left(|H|^{\varepsilon+1}\right)$$

FOR EVERY $\varepsilon > 0$

ALMOST LINEAR ALGORITHMS

$\boxed{\text{SPECIAL CASE}}$

$$f(r) \stackrel{\text{DEF}}{=} \limsup_{G\in\mathcal{C}} \nabla_r(G) < \infty$$

$\boxed{\text{BOUNDED EXPANSION}}$

EXPANSION FUNCTION

- CONSTANT (PROPER MINOR CLOSED)

- EXPONENTIAL (REGULAR α)

- POLYNOMIAL (GEOMETRIC INTERSECTION)

$\vdots$

- ~~EVERY~~ (GROWTH POSSIBLE ARBITRARY FAST)

SMALL EXPANSION $\Rightarrow$ GOOD PROPERTIES

— SUBEXPONENTIAL GROWTH

$\Downarrow$

SUBLINEAR SEPARATORS

$\Downarrow$ (DVOŘÁK, NORINE)

— EXPANSION $c^{r^{1/3-\varepsilon}}$
YIELDS SMALL CLASS

(# OF LABELLED GRAPHS)
$\leq n!\,\alpha^n$

- SUBEXPONENTIAL CLIQUE-GROWTH

$$\Downarrow$$

$\mathcal{C}$ IS HYPERFINITE
AND THUS

EVERY MONOTONNE PROPERTY
IS TESTABLE IN THE BOUNDED
DEGREE MODEL

(BENJAMINI, SCHRAMM, SHAPIRA)

EXTENSION OF

ω - EXPANSION $\qquad i \longmapsto \sup_{G \in \mathcal{C} \nabla i} \omega(G)$

SUB-EXPONENTIAL
ω - EXPANSION

$$\limsup_{i \to \infty} \sup_{G \in \mathcal{C} \nabla i} \frac{\log \omega(G)}{i} = 0$$

(VERY) SMALL EXPANSION

SUBLINEAR SEPARATORS

ALL MONOTONNE PROPERTIES TESTABLE IN THE BOUNDED DEGREE MODEL

SMALL CLASS

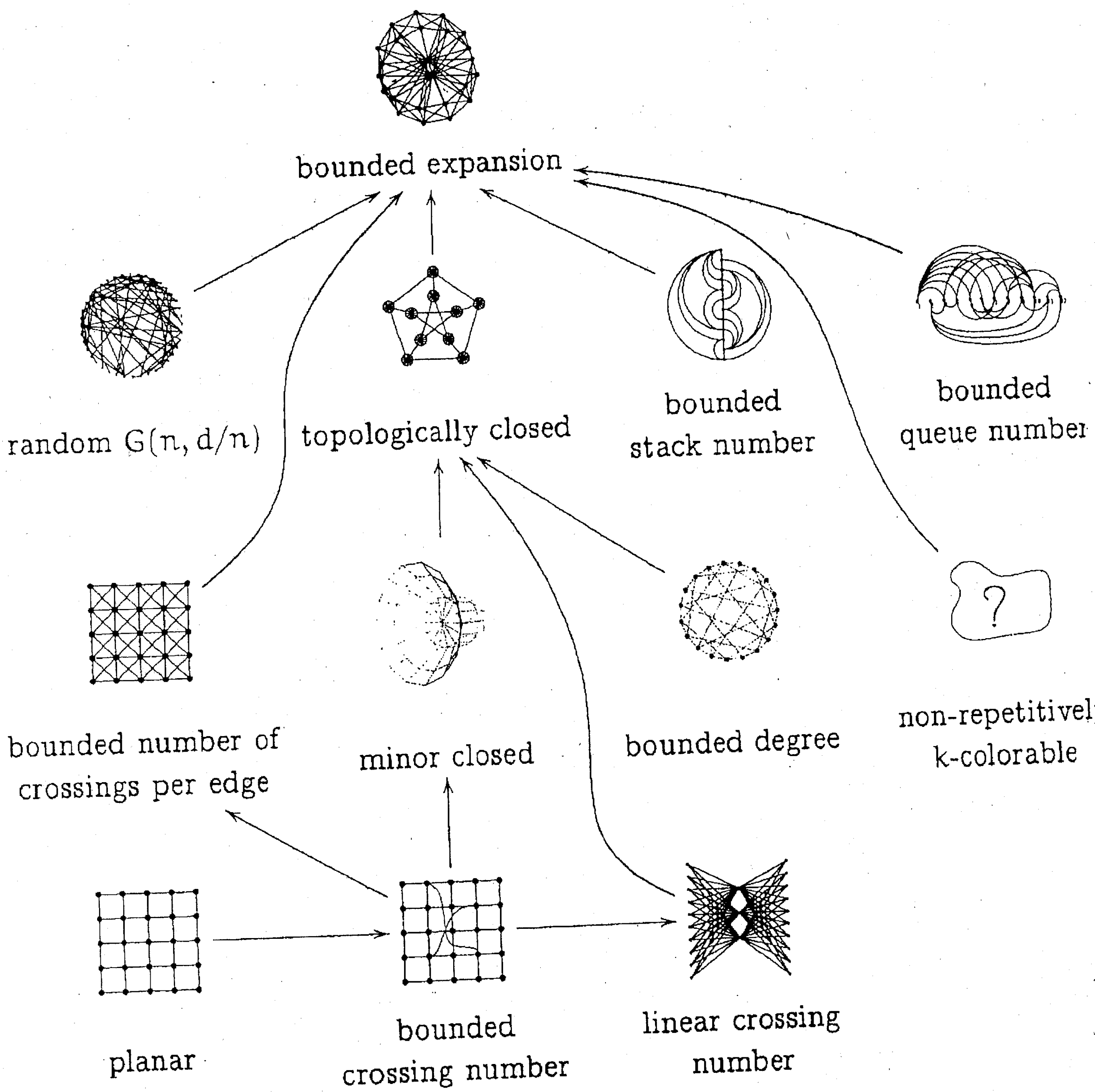
INCLUDES ALL PROPER MINOR CLOSED CLASSES

CHARACTERISATION OF BOUNDED EXPANSION CLASSES

Let $\mathcal{C}$ be a class of graphs. Then the following conditions are equivalent:

(1) *$\mathcal{C}$ has bounded expansion,*
(2) *for every integer r, $\sup_{G\in\mathcal{C}} \nabla_r(G) < \infty$,*
(3) *for every integer r, $\sup_{G\in\mathcal{C}} \widetilde{\nabla}_r(G) < \infty$,*
(4) *for every integer p, $\sup_{G\in\mathcal{C}} \chi_p(G) < \infty$,*
(5) *for every integer p, $\sup_{G\in\mathcal{C}} \mathrm{col}_p(G) < \infty$,*
(6) *for every integer p, $\sup_{G\in\mathcal{C}} \mathrm{wcol}_p(G) < \infty$,*
(7) *for every integer c, the class $\mathcal{C} \bullet K_c = \{G \bullet K_c : G \in \mathcal{C}\}$ has bounded expansion,*
(8) *$\mathcal{C}$ has low tree-width colorings,*
(9) *$\mathcal{C}$ has low tree-depth colorings,*
(10) *for every integer p, there exists an integer $X(p)$ such that every graph $G \in \mathcal{C}$ has a p-centered colorings using at most $X(p)$ colors,*
(11) *for every integer k, the class $\mathcal{C}'$ of the 1-transitive fraternal augmentations of directed graphs $\vec{G}$ with $\Delta^-(\vec{G}) \le k$ and $G \in \mathcal{C}$ form a class with bounded expansion,*
(12) *the class $\mathcal{C}$ is a degenerate class of graphs (that is: $\nabla_0(G)$ is bounded on $\mathcal{C}$) and there exists a function F such that every orientation $\vec{G}$ of a graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \le Q(\Delta^-(\vec{G}), i)$,*
(13) *there exists a function f such that every graph $G \in \mathcal{C}$ has a transitive fraternal augmentation $\vec{G} = \vec{G}_1 \subseteq \vec{G}_2 \subseteq \cdots \subseteq \vec{G}_i \subseteq \cdots$ where $\Delta^-(\vec{G}_i) \le f(i)$.*

EXAMPLES OF BOUNDED EXPANSION



(+ D. WOOD)

BOUNDED EXPANSION

$\Downarrow$

LINEAR ALGORITHM FOR SUBGRAPH PROBLEM

LINEAR ALGORITHM FOR COLORING χ_p

$$\chi = \chi_1 \leq \chi_2 \leq \chi_3 \leq \dots \leq \chi_\infty = td$$

$\uparrow$ STAR COLORING (χ_2)

$\uparrow$ CENTERED COLORING (χ_∞)

HOMOMORPHISM ORDER	$\times$	COUNTING
DECISION PROBLEM		STATISTICS TESTING

DENSE GRAPHS (LOVÁSZ & ....)

$$t(F,G) = \frac{\hom(F,G)}{|G|^{|F|}}$$

VERY SPARSE GRAPHS (BENJAMINI, SCHRAMM ...

$$\mathrm{dens}(F,G) = \frac{\#\, F \subseteq G}{|G|}$$

$$F_1 \to G \qquad F_2 \to G \qquad F_3 \to G$$

$$F_1 \hookrightarrow G \qquad F_2 \hookrightarrow G \qquad F_\infty \hookrightarrow G$$

LOOKING FOR THE EXPONENT (DEGREE OF FREEDOM)

– HOW IS $\dfrac{\log(\# F \subseteq G)}{\log |G|}$ BOUNDED WHEN G IS RESTRICTED TO A CLASS $\mathcal{C}$?

– HOW MANY VERTICES CAN BE CHOSEN INDEPENDENTLY WHEN LOOKING FOR A COPY OF F?

$\boxed{\text{THM}}$

FOR EVERY INFINITE CLASS OF GRAPHS $\mathcal{C}$

AND EVERY GRAPH F

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \nabla i} \frac{\log(\# F \subseteq G)}{\log |G|} \in \{-\infty, 0, 1, \dots, \alpha(F), |F|\}$$

WHERE $\alpha(F)$ IS INDEPENDENCE NUMBER

- DEGREES OF FREEDOM $\longrightarrow$ RANDOMNES
- $\varrho_F(G) = \dfrac{\hom(F, G)}{|G|^{\alpha(F)}}$ $\longrightarrow$ LIMITS
- TREE REDUCTIONS $\longrightarrow$ REGULARI

$\boxed{\text{COROLLARY}}$

$\mathcal{C}$ NOWHERE DENSE

$$\Updownarrow$$

$$\lim_{i\to\infty} \limsup_{G\in\mathcal{C}\nabla i} \frac{\log(\# F \subseteq G)}{\log|G|}$$

$$\in$$

$$\{-\infty, 0, 1, \ldots, \alpha(F)\}$$

FOR EVERY F. WITH AT LEAST ONE EDGE SO THAT $\alpha(F) < |F|$.

$\boxed{\text{PROOF}}$

IT SUFFICES TO PROVE $\Downarrow$.

PROOF COMBINES ALL CHARACTERIZATION

- TREE DEPTH td
- LOW TREE DEPTH DECOMPOSITIONS χ_p
- STEPPING UP LEMMA FOR "(k, F) - SUNFLOWERS"
- COUNTING OF COLORED TREES

$\boxed{\text{PROOF II}}$

FIRST CASE: $F = K_2$

$\boxed{\text{THM}}$ (TRICHOTOMY THM)

FOR EVERY INFINITE CLASS $\mathcal{C}$
THE LIMIT

$$\lim_{i \to \infty} \limsup_{G \in \mathcal{C} \nabla i} \frac{\log \|G\|}{\log |G|}$$

$$\in$$

$$\{-\infty, 0, 1, 2\}$$

≤ 1 IFF $\mathcal{C}$ IS NOWHERE DENSE

2 IFF $\mathcal{C}$ IS SOMEWHERE DENSE

GENERAL FRAMEWORK

X A CLASS OF FO FORMULAS

- ALL
- EXISTENTIAL POSITIVE

  $\exists \quad \wedge \quad \vee$

- PRIMITIVE POSITIVE

  $\exists \quad \wedge$

$\phi(x_1, \dots, x_p)$ FO FORMULA WITH p FREE VARIABLES

$$\langle \phi, G \rangle = \frac{|\{(v_1, \dots, v_p) : G \models \phi(v_1, \dots, v_p)\}|}{|G|^p}$$

EX: $p = 2$ $\phi(x_1, x_2) = \exists x \left(x_1 x \in E \;\&\; x x_2 \in E \right)$

$\Updownarrow$

$\exists$ PATH OF LENGTH 2 FROM x_1 TO x_2

$\langle \phi, G \rangle =$ PROB THAT A PAIR JOINED BY PATH OF L. 2

$G_1, G_2, G_3, \ldots$

IS X - CONVERGENT

IFF

FOR EVERY $\phi \in X$

$\langle \phi, G_i \rangle$ CONVERGES.

$\boxed{\text{EXAMPLES}}$

ϕ PP $\leftrightsquigarrow$ EXISTENCE OF HOMOMORPHISM

$$(F, x_1, \dots, x_p) \to (G, y_1, \dots, y_p)$$

$p = |F|$

$$\langle \phi, G \rangle = t(F, G)$$

$$= \frac{\hom(F, G)}{|G|^{|F|}}$$

LOVÁSZ DENSE CASE

$p = 1$

$$\langle \phi, G \rangle = \frac{|\{y : (F, x) \longrightarrow (G, y)\}|}{|G|}$$

BENJAMINI – SCHRAMM

$p = 0$

$$\langle \phi, G \rangle = \frac{F \xrightarrow{?} G}{|G|^{0}} \begin{matrix} \nearrow 1 \\ \searrow 0 \end{matrix}$$

LEFT LIMITS

?

IDEALS IN HOMOMORPHISM ORDER

THE CASE $p=0$

$$X = FO$$

$$\langle \phi, G \rangle = \begin{cases} 1 & G \models \phi \\ 0 & \end{cases}$$

ELEMENTARY CONVERGENCE

$G_1, G_2, \ldots$ ELEMENTARY CONVERGENT

IFF

FOR EVERY SENTENCE ϕ

$\exists\, n_\phi$ SUCH THAT

EITHER FOR ALL $i > n_\phi$

$$G_i \models \phi$$

OR FOR ALL $i > n_\phi$

$$G_i \not\models \phi$$

$\boxed{\text{THM}}$

$G_1, G_2, \ldots$ OF GRAPHS WITH DEGREES $\leq d$ IS FO-CONVERGENT

$$\Updownarrow$$

(G_i) IS ELEMENTARY CONVERGENT

&

(G_i) IS BENJAMINI-SCHRAMM CONVERGENT

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