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Control of Slowly-Varying Linear Systems

E. W. KAMEN, P. P. KHARGONEKAR, AND A. TANNENBAUM

Abstract—State feedback control of slowly-varying linear continuous-time and discrete-time systems with bounded coefficient matrices is studied in terms of the frozen-time approach. This note centers on pointwise stabilizable systems; that is, systems for which there exists a state feedback gain matrix placing the frozen-time closed-loop eigenvalues to the left of a line $\text{Re } s = -\gamma < 0$ in the complex plane (or within a disk of radius $\rho < 1$ in the discrete-time case). It is shown that if the entries of a pointwise stabilizing feedback gain matrix are continuously differentiable functions of the entries of the system coefficient matrices, then the closed-loop system is uniformly asymptotically stable if the rate of time variation of the system coefficient matrices is sufficiently small. It is also shown that for pointwise stabilizable systems with a sufficiently slow rate of time variation in the system coefficients, a stabilizing feedback gain matrix can be computed from the positive definite solution of a frozen-time algebraic Riccati equation.

I. INTRODUCTION

Linear time-varying systems are sometimes studied using the frozen-time method in which the time variable in the system coefficients is viewed as a parameter. An example of the power of this approach is Rosenbrock's result [1] that a linear continuous-time system is asymptotically stable if the frozen-time eigenvalues of the system matrix are to the left of a line $\text{Re } s = -\gamma < 0$ in the complex plane and if the rate of time variation of the system matrix is sufficiently small. Desoer [2] proved that uniform asymptotic stability can be deduced under the same conditions on the system matrix and gave an explicit bound on the rate of time variation (for results in the nonlinear case, see [3, pp. 218–223]). A corresponding result for linear time-varying discrete-time systems was also proved by Desoer [4]. Recent results on the linear time-varying continuous-time case are given in [5] and [6].

Although the frozen-time method appears to be often utilized in practice in the control of linear time-varying systems, not much is currently known regarding analytical conditions on the given system which guarantee asymptotic stability of the closed-loop system. However, we should note that in [7] sufficient conditions (with the correction given in [8]) are given for the existence of a stabilizing state feedback gain matrix computed from the solution to a frozen-time algebraic Riccati equation. In this note we also consider the application of the frozen-time approach to the construction of a stabilizing state feedback gain matrix. We consider pointwise stabilizable systems for which there exists a state feedback gain matrix placing the frozen-time closed-loop eigenvalues to the left of a line

$\text{Re } s = -\gamma < 0$ in the complex plane, or within a disk of radius $\rho < 1$ in the discrete-time case. Such a feedback is said to be pointwise stabilizing.

In the next section we begin with the continuous-time case. We first consider the question as to when a pointwise stabilizing feedback gain matrix results in a uniformly asymptotically stable closed-loop system, assuming that the rate of variation of the system coefficient matrices is sufficiently small. If the system coefficient matrices are bounded with bounded derivatives, an answer (a sufficient condition) is that the entries of the feedback gain matrix be continuously differentiable functions of the entries of the system coefficient matrices. It follows from Delchamps's lemma [10] that such a feedback can be constructed from the positive definite solution of a frozen-time algebraic Riccati equation. A discrete-time version of the results is presented in Section III, and in Section IV some concluding remarks are given.

II. CONTINUOUS-TIME CASE

With R equal to the field of real numbers, for any positive integer q let R^q denote the space of q -element column vectors with entries in R . The norm $\|x\|$ of an element $x \in R^q$ is defined by

$$\|x\| = (x^T x)^{1/2} \quad (1)$$

where x^T is the transpose of x . Given positive integers p, q , a $p \times q$ matrix M over R will be viewed as an element of R^{pq} . The Frobenius norm $\|M\|$ is defined by

$$\|M\| = \left(\sum_{i=1}^p \sum_{j=1}^q m_{ij}^2 \right)^{1/2} \quad (2)$$

where m_{ij} is the i, j entry of M . For any $x \in R^q$, it is easy to verify that $\|Mx\| \leq \|M\| \|x\|$.

Given positive integers m and n , consider the m -input n -dimensional linear time-varying continuous-time system given by the state equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \quad (3)$$

where $u(t)$ is the m -vector input or control applied at time t and $x(t)$ is the n -dimensional state vector at time t . The $n \times n$ coefficient matrix $A(t)$ and the $n \times m$ coefficient matrix $B(t)$ in (3) are assumed to be bounded differentiable functions of t for $t \geq 0$. Thus,

$$A(t): R_+ \rightarrow \Omega \subset R^{n^2}$$

$$B(t): R_+ \rightarrow \Gamma \subset R^{nm}$$

where $R_+ = \{t \in R: t \geq 0\}$ and Ω, Γ are compact subsets. Finally, it is assumed that the derivatives $\dot{A}(t)$ and $\dot{B}(t)$ are bounded for $t \geq 0$, so that

$$\sup_{t \geq 0} \|\dot{A}(t)\| = \dot{a}_M < \infty \text{ and } \sup_{t \geq 0} \|\dot{B}(t)\| = \dot{b}_M < \infty. \quad (4)$$

The rate of time variation of the coefficient matrices $A(t)$ and $B(t)$ is measured by the magnitudes of \dot{a}_M and \dot{b}_M defined by (4).

In this note we consider state feedback control of the system (3) with the control $u(t)$ given by $u(t) = -K(t)x(t)$, where $K(t)$ is an $m \times n$ time-varying gain matrix. With this control, the resulting closed-loop system is

$$\dot{x}(t) = (A(t) - B(t)K(t))x(t). \quad (5)$$

The particular problem of interest is constructing a feedback gain matrix $K(t)$ (assuming one exists) which results in uniform asymptotic stability of the closed-loop system (5). We shall approach this problem using the frozen-time method in which the time variable t in $A(t)$ and $B(t)$ is viewed as a parameter p with p ranging over all positive numbers. This results in an infinite collection of linear time-invariant systems

$$\dot{x}(t) = A(p)x(t) + B(p)u(t), \quad p \geq 0. \quad (6)$$

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Now given $\gamma > 0$, we say that the system (3) is pointwise stabilizable with order γ if each of the time-invariant systems in the collection (6) can be stabilized with order γ . That is, for each $p \geq 0$ there is a constant gain matrix K_p such that

$$\operatorname{Re} \lambda_i(A(p) - B(p)K_p) \leq -\gamma, \quad i = 1, 2, \dots, n \quad (7)$$

where $\lambda_i(A(p) - B(p)K_p)$ is the i th eigenvalue of $A(p) - B(p)K_p$. Hence, the control $u(t) = -K_p x(t)$ places the eigenvalues of the time-invariant system $\dot{x}(t) = A(p)x(t) + B(p)u(t)$ to the left of $\operatorname{Re} s = -\gamma$ in the complex plane. From known results (Hautus [11]), the system (3) is pointwise stabilizable with order γ if and only if

$$\operatorname{rank} [sI - A(p) : B(p)] = n, \quad \operatorname{Re} s \geq -\gamma, p \geq 0. \quad (8)$$

Now suppose that the system (3) is pointwise stabilizable with order γ , so that there exists a collection K_p , $p \geq 0$ of matrices for which (7) is satisfied. Defining $K(t) := K_p$, we have that the control $u(t) = -K(t)x(t)$ is pointwise stabilizing. We would like to know when the resulting closed-loop system (5) is uniformly asymptotically stable. As seen from the following result, for a certain class of pointwise stabilizing feedback gain matrices $K(t)$, the closed-loop system (5) is uniformly asymptotically stable if the rates of time variation \dot{a}_M and \dot{b}_M are sufficiently small.

Theorem 1: Suppose that the system (3) is pointwise stabilizable with order γ for some $\gamma > 0$, i.e., there exists a time-varying feedback gain matrix $K(t)$ such that

$$\operatorname{Re} \lambda_i(A(t) - B(t)K(t)) < -\gamma, \quad \text{all } i, t \geq 0. \quad (9)$$

Suppose also that $K(t)$ is constructed so that its entries are continuously differentiable (class $C^{(1)}$ on $\Omega \times \Gamma$) functions of the entries of the matrices $A(t)$ and $B(t)$. Then $K(t)$ is bounded with bounded derivative, and if \dot{a}_M and \dot{b}_M are sufficiently small, the closed-loop system $\dot{x}(t) = (A(t) - B(t)K(t))x(t)$ is uniformly asymptotically stable.

Proof: Let $K(t)$ satisfy the hypothesis of the theorem so that the i, j entry k_{ij} of $K(t)$ is a class $C^{(1)}$ function f_{ij} of the entries of $A(t)$ and $B(t)$; that is,

$$k_{ij} = f_{ij}(a_{11}, a_{12}, \dots, a_{nn}, b_{11}, b_{12}, \dots, b_{nm}) \quad (10)$$

where

$$f_{ij} : \Omega \times \Gamma \rightarrow \mathbb{R}$$

and

$$A(t) = (a_{ij}(t)), B(t) = (b_{ij}(t)).$$

Since $A(t)$ and $B(t)$ are bounded and differentiable for $t \geq 0$ and the f_{ij} are continuously differentiable, it follows that $k_{ij}(t)$ is bounded and differentiable for all i, j and $t \geq 0$ with

$$\sup_{t \geq 0} |k_{ij}(t)| \leq \sup_{(\omega, \gamma) \in \Omega \times \Gamma} |f_{ij}(\omega, \gamma)|. \quad (11)$$

Thus, $K(t) = (k_{ij}(t))$ is bounded. In addition, taking the derivative of both sides of (10) gives

$$\dot{k}_{ij}(t) = \sum_{r=1}^n \sum_{s=1}^n \frac{\partial f_{ij}}{\partial a_{rs}} \dot{a}_{rs}(t) + \sum_{r=1}^n \sum_{s=1}^m \frac{\partial f_{ij}}{\partial b_{rs}} \dot{b}_{rs}(t).$$

Thus,

$$\begin{aligned} \sup_{t \geq 0} |\dot{k}_{ij}(t)| &\leq \sum_{r=1}^n \sum_{s=1}^n \left[\sup_{(\omega, \gamma) \in \Omega \times \Gamma} \left| \frac{\partial f_{ij}(\omega, \gamma)}{\partial a_{rs}} \right| \right] [\sup_{t \geq 0} |\dot{a}_{rs}(t)|] \\ &\quad + \sum_{r=1}^n \sum_{s=1}^m \left[\sup_{(\omega, \gamma) \in \Omega \times \Gamma} \left| \frac{\partial f_{ij}(\omega, \gamma)}{\partial b_{rs}} \right| \right] [\sup_{t \geq 0} |\dot{b}_{rs}(t)|]. \end{aligned} \quad (12)$$

Since the partial derivatives of the f_{ij} are continuous, $\Omega \times \Gamma$ is compact, and $A(t)$ and $B(t)$ are bounded for $t \geq 0$ by (12), $\dot{k}_{ij}(t)$ is bounded for all i, j and $t \geq 0$, and therefore $\dot{K}(t)$ is bounded for $t \geq 0$. It also follows

from (12) and the definition (2) of the matrix norm that

$$\sup_{t \geq 0} \|\dot{K}(t)\| = \dot{k}_M \rightarrow 0 \quad \text{as } \dot{a}_M \rightarrow 0 \text{ and } \dot{b}_M \rightarrow 0. \quad (13)$$

Now

$$\frac{d}{dt} (A(t) - B(t)K(t)) = \dot{A}(t) - B(t)\dot{K}(t) - \dot{B}(t)K(t)$$

and thus

$$\sup_{t \geq 0} \left\| \frac{d}{dt} (A(t) - B(t)K(t)) \right\| \leq \dot{a}_M + [\sup_{t \geq 0} \|B(t)\|] \dot{k}_M + \dot{b}_M [\sup_{t \geq 0} \|K(t)\|]. \quad (14)$$

By (11) the bound on $\|K(t)\|$ is independent of \dot{a}_M and \dot{b}_M , and since $B(t)$ is bounded, from (13) and (14) we have

$$\sup_{t \geq 0} \left\| \frac{d}{dt} (A(t) - B(t)K(t)) \right\| \rightarrow 0 \quad \text{as } \dot{a}_M \rightarrow 0 \text{ and } \dot{b}_M \rightarrow 0. \quad (15)$$

By the results in [2], (9) and (15) imply that $\dot{x}(t) = (A(t) - B(t)K(t))x(t)$ is uniformly asymptotically stable if \dot{a}_M and \dot{b}_M are sufficiently small.

A feedback gain matrix $K(t)$ satisfying the hypothesis of Theorem 1 can be constructed as follows. Given $\gamma > 0$ and an $n \times n$ positive definite symmetric matrix Q over \mathbb{R} , consider the algebraic Riccati equation (ARE)

$$(A^T(t) + \gamma I)P(t) + P(t)(A(t) + \gamma I) - P(t)B^T(t)B(t)P(t) + Q = 0. \quad (16)$$

If the given system (3) is pointwise stabilizable with order γ , it follows from known results [9, pp. 237–238] that the ARE (16) has a unique positive definite solution $P(t)$ for all $t \geq 0$, and if we define

$$K(t) = B^T(t)P(t) \quad (17)$$

the frozen-time eigenvalues of the closed-loop system $\dot{x}(t) = (A(t) - B(t)K(t))x(t)$ are to the left of the line $\operatorname{Re} s = -\gamma$. By Delchamps' lemma [10], the entries of $P(t)$ are real analytic functions of the entries of $A(t)$ and $B(t)$, and thus the entries of the feedback matrix (17) are continuously differentiable functions of $A(t)$ and $B(t)$. Hence, we have the following result.

Theorem 2: Suppose that the system (3) is pointwise stabilizable with order γ for some $\gamma > 0$. Given a positive definite $n \times n$ matrix Q over \mathbb{R} , let $P(t)$ denote the positive definite solution to the ARE (16). Then if \dot{a}_M and \dot{b}_M are sufficiently small, the control $u(t) = -B^T(t)P(t)x(t)$ results in a uniformly asymptotically stable closed-loop system.

III. DISCRETE-TIME CASE

All of the results given in the previous section have a counterpart in the discrete-time case. A brief sketch of this case is given below. We continue to assume that the vector and matrix norms are given by (1) and (2), respectively.

Consider the m -input n -dimensional linear time-varying discrete-time system given by the state equation

$$x(k+1) = A(k)x(k) + B(k)u(k) \quad (18)$$

where $k = 0, 1, 2, \dots$ is the discrete-time index. It is assumed that $A(k)$ and $B(k)$ are bounded for $k \geq 0$, so that

$$A(k): N \rightarrow \Omega \subset \mathbb{R}^{n^2}$$

$$B(k): N \rightarrow \Gamma \subset \mathbb{R}^{nm}$$

where N = set of natural numbers and Ω, Γ are compact subsets.

Defining

$$(\Delta a)_M = \sup_{k \geq 0} \|A(k+1) - A(k)\|, (\Delta b)_M = \sup_{k \geq 0} \|B(k+1) - B(k)\| \quad (19)$$

we assume that $(\Delta a)_M < \infty$ and $(\Delta b)_M < \infty$. The rate of time variation of the coefficient matrices $A(k)$ and $B(k)$ is measured by $(\Delta a)_M$ and $(\Delta b)_M$.

Given a positive real number ρ with $0 \leq \rho < 1$, the system (18) is said to be pointwise stabilizable with order ρ if there exists a time-varying feedback gain matrix $G(k)$ such that

$$|\lambda_i(A(k) - B(k)G(k))| \leq \rho, \quad \text{all } i, k \geq 0. \quad (20)$$

From known results (Hautus [11]), the system (18) is pointwise stabilizable with order ρ if and only if

$$\text{rank } [zI - A(k) : B(k)] = n, \quad |z| \geq \rho, k \geq 0.$$

We then have the discrete-time counterpart to Theorem 1. The proof of the following result resembles the proof of Theorem 1, except that instead of using the chain rule, one must use the mean value theorem and the stability theorem in [4]. The straightforward details are omitted.

Theorem 3: Suppose that the system (18) is pointwise stabilizable with order ρ for some ρ with $0 \leq \rho < 1$, so that there exists a feedback gain matrix $G(k)$ satisfying (20). Suppose also that $G(k)$ is constructed so that its entries are continuously differentiable (class $C^{(1)}$) functions of the entries of $A(k)$ and $B(k)$. Then $G(k)$ is bounded, $\sup_{k \geq 0} \|G(k+1) - G(k)\|$ is bounded, and if $(\Delta a)_M$ and $(\Delta b)_M$ are sufficiently small, the closed-loop system $x(k+1) = (A(k) - B(k)G(k))x(k)$ is uniformly asymptotically stable.

Now given ρ with $0 < \rho < 1$ and an $n \times n$ positive definite symmetric matrix Q over R , consider the ARE

$$\rho^{-1}A^T(k)P(k)\rho^{-1}A(k) - P(k) - \rho^{-1}A^T(k)P(k)B(k) \cdot [B^T(k)P(k)B(k) + I]^{-1}B^T(k)P(k)A(k)\rho^{-1} + Q = 0. \quad (21)$$

If the system (18) is pointwise stabilizable with order ρ , it follows from known results [9, pp. 497–498] that the ARE (21) has a unique positive definite solution $P(k)$ for all $k \geq 0$, and if we define the feedback gain matrix

$$G(k) = (B^T(k)P(k)B(k) + I)^{-1}B^T(k)P(k)A(k) \quad (22)$$

the frozen-time eigenvalues of the closed-loop system

$$x(k+1) = (A(k) - B(k)G(k))x(k) \quad (23)$$

are within the disk of radius ρ . By Delchamp's lemma [10], the entries of $P(k)$ are real analytic functions of the entries of $A(k)$ and $B(k)$. Thus, the entries of the feedback matrix (22) are continuously differentiable functions of the entries of $A(k)$ and $B(k)$. We therefore have the following discrete-time counterpart to Theorem 2.

Theorem 4: Suppose that the system (18) is pointwise stabilizable with order ρ for some ρ with $0 < \rho < 1$. Given a positive definite $n \times n$ matrix Q over R , let $P(k)$ denote the positive definite solution to the ARE (21). Then if $(\Delta a)_M$ and $(\Delta b)_M$ are sufficiently small, the control $u(k) = -G(k)x(k)$, where $G(k)$ is given by (22), results in a uniformly asymptotically stable closed-loop system.

IV. CONCLUDING REMARKS

We have shown that if a pointwise stabilizing state feedback gain matrix is constructed so that it is a continuously differentiable function of the system coefficient matrices, the resulting closed-loop system will be uniformly asymptotically stable if the rate of time variation of the system coefficient matrices is sufficiently small. It was also shown that such feedback gain matrices can be computed from the solution to a frozen-time algebraic Riccati equation. It is obvious that this framework can be dualized in order to yield results on observers. An application of these

results is in adaptive control problems where the steady-state rate of time variation of the estimated system parameters is sufficiently small.

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A Two-Sided Interpolation Approach to H_∞ Optimization Problems

U. SHAKED

Abstract—A solution to the two-sided interpolation problem which arises in H_∞ -optimization theory is obtained. This solution is found in closed form, explicitly in terms of the required interpolation directions. It is simple to obtain and it does not require the application of the relatively complicated matrix Pick–Nevanlinna theory. The solution obtained is of minimum order; due to its simplicity, the order reduction, which occurs at the minimum value of the H_∞ -norm, is clearly explained.

I. INTRODUCTION

One of the fundamental problems in H_∞ -optimization theory is the problem of interpolation via inners [1], [2]. This problem, which arises in many areas of linear system theory, has witnessed a significant revival in the last few years. Two main approaches have been proposed for solving the interpolation problem. The first one is a state-space approach, which is based on the technique of Hankel-norm approximation [3], [4], and the second is the classical Pick–Nevanlinna interpolation approach [5], [6], which is generalized to the matrix case in [7]. In spite of its apparent simplicity and the fact that it provides better physical insight and it looks more straightforward, the second approach has not yet come up with closed-form solutions that are simple enough in the sense that they are expressed explicitly in terms of the problem interpolation requirements.

A major achievement in the application of the Pick–Nevanlinna approach for the matrix case has been recently obtained in [8] where it has been recognized that low order and more computationally effective results can be obtained by solving the interpolation problem along prespecified directions. Using the directional interpolation approach, a very simple

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