

GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION
SPONSORED PROJECT INITIATION

Date: June 12, 1979

Project Title: Mathematical Programming With and Without Differentiability

Co-Project No: E-24-687/G-37-614 *Green card*

Co-Project Directors Dr. Mokhtar S. Bazaraa and Dr. Jonathan E. Spingarn

Sponsor: Air Force Office of Scientific Research; Bolling AFB, DC 20332

Agreement Period: From 6/15/79 Until 6/14/80 (Performance Period)

Type Agreement: Contract No. F49620-79-C-0120

Amount:	\$32,983 E-24-687	Partially funded at \$30,000 through 9/30/79
	<u>10,056 G-37-614</u>	\$19,944 E-24-687
	\$43,039 AFOSR	<u>10,056 G-37-614</u>
	<u>3,080 GIT (E-24-337)</u>	<u>\$30,000</u>

Reports Required:
\$46,119 TOTAL Final Technical Report

Sponsor Contact Person (s):

Technical Matters

Program Manager: Dr. Joseph Bram

Merle M. Andrew
Director of Mathematical & Information Sciences
Air Force Office of Scientific Research
ATTN: NM
Building 410
Bolling AFB, DC 20332

Contractual Matters

(thru OCA)

Lt. Parsons
ATTN: PKZ
Air Force Office of Scientific Res.
Building 410
Bolling AFB, DC 20332
202/767-4959

Marion R. Harrington, Major, USAF
Contracting Officer

Defense Priority Rating: DO-C9 under DMS Reg. 1

Assigned to: Industrial & Systems Engineering/Mathematics (School/Laboratory)

COPIES TO:

Project Director
Division Chief (EES)
School/Laboratory Director
Dean/Director-EES
Accounting Office
Procurement Office
Security Coordinator (OCA)
Reports Coordinator (OCA)

Library, Technical Reports Section
EES Information Office
EES Reports & Procedures
Project File (OCA)
Project Code (GTRI)
Other _____

GEORGIA INSTITUTE OF TECHNOLOGY
OFFICE OF CONTRACT ADMINISTRATION
SPONSORED PROJECT TERMINATION

Date: 10/20/80

Project Title: Mathematical Programming With and Without Differentiability

✓
Co- Project No: E-24-687/G-37-614

Co- Project Director: Dr. M. S. Bazaraa & Dr. J. E. Spingarn

Sponsor: Air Force Office of Scientific Research; Bolling AFB, DC 20332

Effective Termination Date: 6/14/80

Clearance of Accounting Charges: 8/14/80 (for reporting)

Grant/Contract Closeout Actions Remaining:

- ☒ Final Invoice and Closing Documents
- ☐ Final Fiscal Report
- ☐ Final Report of Inventions
- ☐ Govt. Property Inventory & Related Certificate
- ☐ Classified Material Certificate
- ☐ Other _____

Assigned to: ISyE/Mathematics (School/~~Laboratory~~ ~~XXXXXXXX~~)

COPIES TO:

Project Director
Division Chief (EES)
School/Laboratory Director
Dean/Director-EES
Accounting Office
Procurement Office
Security Coordinator (OCA)
✓ Reports Coordinator (OCA)

Library, Technical Reports Section
EES Information Office
Project File (OCA)
Project Code (GTRI)
Other OCA Research Property Coordinator

FINAL REPORT

**MATHEMATICAL PROGRAMMING WITH
AND WITHOUT DIFFERENTIABILITY**

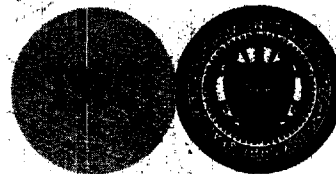
Principal Investigator: M. S. Bazaraa

Co-Principal Investigator: J. Spingarn

**Submitted to
AIR FORCE OFFICE OF SCIENTIFIC RESEARCH**

AUGUST 1980

**GEORGIA INSTITUTE OF TECHNOLOGY
SCHOOL OF INDUSTRIAL & SYSTEMS ENGINEERING
ATLANTA, GEORGIA 30332**



MATHEMATICAL PROGRAMMING WITH AND
WITHOUT DIFFERENTIABILITY

FINAL REPORT

submitted to

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH

Principal Investigator: M. S. Bazaraa

Co-Principal Investigator: J. Spingarn

AUGUST 1980

TABLE OF CONTENTS

<u>TOPIC</u>	<u>PAGE</u>
I. QUASI-NEWTON ALGORITHMS FOR CONSTRAINED NONLINEAR PROGRAMMING (M. S. Bazaraa)	1
I.1 Introduction	1
I.2 Quadratic Approximation Methods	3
I.2-1 General Description of the Algorithm	3
I.2-2 The Quadratic Programming Subproblem	7
I.2-3 Feasible Region for the Quadratic Program	15
I.2-4 Updating the Lagrangian Multipliers	19
I.2-5 Local Convergence	21
I.2-6 Global Convergence	24
I.4 Summary of Completed Research	28
References	31
II. GENERIC OPTIMALITY CONDITIONS AND NONDIFFERENTIABLE OPTIMIZATION (J. Spingarn)	35
II.1 Introduction	35
II.1-1 Generic Optimality Conditions	35
II.1-2 Nondifferentiable Optimization	36
II.2 Research and Publications Summary - Generic Conditions	38
II.3 Research and Publications Summary - Nondifferentiable Optimization	41
References	43
APPENDIX	44

I. QUASI-NEWTON ALGORITHMS FOR CONSTRAINED NONLINEAR PROGRAMMING

(M. S. Bazaraa)

I.1 Introduction

Nonlinear programming has long been of interest to mathematicians, engineers, and management scientists. Recent developments in the field of nonlinear programming, especially those related to computing a search direction and to computing a stepsize, and the advent of the high-speed and large-memory computers have made it possible to numerically solve nonlinear programming problems of great complexity. This capability has not only motivated immense research in the development of nonlinear programming methods, but also expanded its applications to problems in optimal control, optimal design, nonlinear networks, chemical processing, refinery operations and water resources management.

The study of nonlinear programming methods is an area of prime interest. This research concerns itself with the development of nonlinear programming methods based on quadratic approximation of the objective function and linearization of the constraints.

A nonlinear programming problem can be stated as follows:

$$\begin{array}{ll}\text{minimize} & f(x) \\ \text{subject to} & x \in S\end{array}$$

where f is a function defined on E^n , S is a subset of E^n , and x is an n -dimensional vector. The function f and the set S are usually called the objective function and the feasible region, respectively. A decision vector x is called a feasible solution if $x \in S$. The nonlinear program aims at finding a feasible solution \bar{x} such that $f(x) \geq f(\bar{x})$ for each feasible point x . Such a point \bar{x} is called an optimal solution to the problem.

The set S can be defined in terms of inequality and equality restrictions leading to the following general constrained nonlinear program:

$$\begin{aligned} P: \quad & \text{minimize} \quad f(x) \\ & \text{subject to} \quad g_i(x) \leq 0, \quad i = 1, \dots, m \\ & \quad \quad \quad h_i(x) = 0, \quad i = 1, \dots, \ell \end{aligned}$$

Each of the constraints $g_i(x) \leq 0$ for $i = 1, \dots, m$ is called an inequality constraint and each of the constraints $h_i(x) = 0$ for $i = 1, \dots, \ell$ is called an equality constraint. Most practical nonlinear programming problems have the above form, and this research concerns itself with quadratic approximation methods for solving this general constrained problem.

I.2 Quadratic Approximation Methods

In this section, we will briefly discuss the published literature on quadratic approximation methods, commonly known as quasi-Newton or Newton-type methods. The basis of these methods is to successively form a quadratic subprogram by linearizing the original nonlinear constraints around a given point and replacing the objective function with a suitable quadratic form. The optimal solution to the quadratic subprogram is used to update the current solution to the original problem.

This class of methods was originally proposed by Wilson [1963] and further extended by several authors including Garcia and Mangasarian [1976], Han [1976, 1977] and Powell [1978]. Perhaps the most important property which is shared by these algorithms is the fact that they enjoy a super-linear rate of convergence in the vicinity of Kuhn-Tucker points that satisfy second order optimality conditions. In [1977], Han was able to show that the optimal solution to the quadratic problem is indeed a descent direction to a suitable penalty function. Through the use of a line search, he showed convergence of the sequence of iterates even if the starting solution is remote from a Kuhn-Tucker point, thus establishing global convergence.

I.2-1 General Description of the Algorithm

In this section, we will provide a general description of the quadratic approximation algorithm for solving a general constrained nonlinear programming problem of the form

$$\begin{aligned} \text{P: minimize} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \leq 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, \ell \end{aligned}$$

Each iteration consists of two major steps, namely, a direction finding step and a line search step. In the direction finding step, a quadratic programming subproblem is first formed. The solution to this quadratic program yields a search direction. Once the direction is determined, a line search is performed to produce a new point.

Suppose that at iteration k , the vectors $x^k \in E^n$, $u^k \in E^m$, $v^k \in E^\ell$ and an $n \times n$ matrix B_k are given. The following steps are successively performed.

Direction finding step

A quadratic subprogram $Q(x^k, B_k)$ is formulated as follows:

$$\begin{aligned} Q(x^k, B_k): \quad & \text{minimize} \quad \nabla f(x^k)^t d + \frac{1}{2} d^t B_k d \\ & \text{subject to} \quad g_i(x^k) + \nabla g_i(x^k)^t d \leq 0, \quad i = 1, \dots, m \\ & \quad \quad \quad h_i(x^k) + \nabla h_i(x^k)^t d = 0, \quad i = 1, \dots, \ell \end{aligned}$$

Note that the original nonlinear constraints are linearized around the point x^k . Let d^k be a solution of $Q(x^k, B_k)$. This vector will be called a search direction or simply a direction. The dual vectors p^k and q^k are the Lagrangian multipliers associated with the linear inequality and equality constraints respectively, and will be used to update the Lagrangian multipliers of the original problem P . Note that the construction of the constraints forces the direction d^k to point towards the feasible region. Particularly, if $g_i(x^k) > 0$, that is, if the i th inequality constraint is violated, then the i th constraint of the quadratic program will guarantee that $\nabla g_i(x^k)^t d^k \leq -g_i(x^k) < 0$. Therefore, moving along d^k will reduce the infeasibility of the i th constraint of the original problem. Similar interpretation can be given for equality constraints.

Line search step

Using a suitable descent function ϕ , once the direction d^k is determined, a line search along it is performed, resulting in a stepsize λ_k and a new point $x^{k+1} = x^k + \lambda_k d^k$ such that $\phi(x^{k+1}) < \phi(x^k)$. In the vicinity of a Kuhn-Tucker solution, as will be discussed later, superlinear convergence is attained by simply letting $\lambda_k = 1$. For the purpose of the next iteration, u^{k+1} and v^{k+1} are replaced with p^k and q^k respectively. These vectors can also be used to form the matrix B_{k+1} , as will be discussed later.

The algorithm starts with a point x^1 , which is not required to be feasible. Under certain assumptions, the algorithm terminates at a Kuhn-Tucker point in a finite number of iterations or else generates an infinite sequence $\{x^k\}$, any accumulation point of which is a Kuhn-Tucker point. We note that the generated sequence $\{x^k\}$ may not be feasible, thus deviating from conventional feasible direction methods as in the works of Zantendijk [1960] and Topkis-Veinott [1967].

We note that a linearly constrained subprogram can be used in place of the quadratic subprogram. The solution to the linearly constrained problem is used as the next iterate point x^{k+1} . We briefly discuss below the linear constrained programs proposed by Rosen and Kreuser [1972] and Robinson [1972]. Rosen and Kreuser's subprogram is as follows:

$$\text{minimize} \quad f(x) + \sum_{i=1}^m u_i^k g_i(x) + \sum_{i=1}^{\ell} v_i^k h_i(x)$$

$$\text{subject to} \quad g_i(x^k) + \nabla g_i(x^k)^t (x - x^k) \leq 0, \quad i = 1, \dots, m$$

$$g_i(x^k) + \nabla h_i(x^k)^t (x - x^k) = 0, \quad i = 1, \dots, \ell$$

The objective function is the Lagrangian function for problem P, and the con-

straints are linear approximations to the original constraints.

Robinson used a slightly different objective function of the form:

$$f(x) + \sum_{i=1}^m u_i^k [g_i(x) - g_i(x^k) - \nabla g_i(x^k)^t (x - x^k)] \\ + \sum_{i=1}^{\ell} v_i^k [h_i(x) - h_i(x^k) - \nabla h_i(x^k)^t (x - x^k)]$$

The main difference is that linear approximations to the original constraints are subtracted from the Lagrangian objective function. When the original problem is linearly constrained, the objective function proposed by Robinson is equivalent to the original criterion function. This is not the case for the method of Rosen and Kreuser unless, of course, $u^1 = 0$ and $v^1 = 0$.

Line search is usually used to control the convergence of the generated sequence $\{x^k\}$. However, if the point x^k is sufficiently close to a solution point \bar{x} , the new point $x^{k+1} = x^k + d^k$ satisfies $\|x^{k+1} - \bar{x}\| < \|x^k - \bar{x}\|$, so that the distance function from \bar{x} can itself be used as a descent function. Hence the step size rule $\lambda_k = 1$ is useful in the vicinity of a solution point. This rule has been used by Wilson [1963], Rosen and Kreuser [1971], Robinson [1972], Garcia and Mangasarian [1976], Han [1976, 1977], and Powell [1978]. If a starting point is far from a solution, the use of line search is necessary to achieve global convergence.

Han [1977], and Bazaraa and Goode [1979] used line search in the context of quadratic approximation methods in order to maintain the monotonic decrease of an exact penalty function.

We note that the algorithm under study can be thought of as an extension of a certain class of descent algorithms for unconstrained optimization. Particularly in the absence of constraints, and by choosing the descent function to be the objective function itself, various choices of B_k lead to

distinct methods. If $B_k = I$, the algorithm is the method of the steepest descent. When the matrix B_k is taken as the Hessian of the objective function, the algorithm reduces to Newton's method. If updating schemes are used to approximate the Hessian of the objective function, then the algorithm turns out to be a quasi-Newton method.

I.2-2 The Quadratic Programming Subproblem

In this section, we will discuss various methods proposed for forming the quadratic programming direction finding problem. The linearization of all constraints is the common property of these methods. However, various objective functions for the quadratic program have been proposed by several authors. Particularly the quadratic objective function at iteration k is given by $\nabla f(x^k)^T d + \frac{1}{2} d^T B_k d$, where B_k approximates the Hessian of the objective function or the Lagrangian function

$$L(x, u, v) = f(x) + \sum_{i=1}^m u_i g_i(x) + \sum_{i=1}^{\ell} v_i h_i(x)$$

In this section, we will discuss some methods for computing and updating the matrix B_k . These include exact computation, finite difference approximation, and the use of quasi-Newton updates for the Hessian of the Lagrangian function or the original objective function. Other choices of interest are identity and diagonal matrices.

Exact Computation of the Hessian

The matrix B_k is taken as the Hessian of the objective function $\nabla^2 f(x^k)$ or the Hessian of the Lagrangian $\nabla_{xx} L(x^k, u^k, v^k)$ given by:

$$\nabla_{xx} L(x^k, u^k, v^k) = \nabla^2 f(x^k) + \sum_{i=1}^m u_i^k \nabla^2 g_i(x^k) + \sum_{i=1}^{\ell} v_i^k \nabla^2 h_i(x^k).$$

In [1963], Wilson used the Hessian $\nabla_{xx}L$ of the Lagrangian function and was able to show superlinear convergence of the algorithm. One disadvantage caused by this choice, however, is the requirement that the Hessian be determined at each iteration k . This involves the evaluation of $\frac{n^2}{2} (1+m+l)$ scalar functions even if all gradient vectors are given. For most functions this operation is very costly. If the Hessian $\nabla_{xx}L(x,u,v)$ is relatively easy to obtain and is positive definite, then this approach may prove attractive. Keeping in mind the difficulties associated with solving a nonconvex quadratic program, several methods have been proposed to maintain positive definiteness of B_k even if the Hessian $\nabla_{xx}L(x^k, u^k, v^k)$ were not. In [1967], Greenstadt suggested

$$B_k = \sum_{i=1}^n \beta_i b_i b_i^t$$

where $\beta_i = \max \{|a_i|, \delta\}$, δ is a positive scalar, a_i is the i th eigenvalue of $\nabla_{xx}L(x^k, u^k, v^k)$ and b_i is its corresponding eigenvector with $\|b_i\| = 1$. The method of Levenberg-Marquardt is to let

$$B_k = \nabla_{xx}L(x^k, u^k, v^k) + \beta I$$

where β is a positive scalar large enough to assure that B_k is positive definite. One particular implementation of this scheme is to attempt to use Cholesky's factorization of $\nabla_{xx}L(x^k, u^k, v^k)$ into the form LDL^t , where L is a lower triangular matrix with ones on the diagonal and D is a diagonal matrix with positive diagonal elements. If $\nabla_{xx}L(x^k, u^k, v^k)$ is not positive definite, the factorization would fail, but as described in Gill and Murray [1972], a factorization of a modified matrix B_k will be at hand. For other methods,

see Goldfeld, Quandt, Trotter [1966], Fiacco and McCormick [1968], Gill and Murray [1972], Mathews and Davies [1971], Fletcher and Freeman [1977].

Finite Difference Approximation of the Hessian

If obtaining the Hessian $\nabla_{xx}L(x,u,v)$ or $\nabla^2 f(x)$ is relatively difficult, a finite difference approximation to the Hessian can be used. This is done as follows:

$$[B_k]_{ij} = \frac{\nabla_x L(x^k + h e_j, u^k, v^k)_i - \nabla_x L(x^k, u^k, v^k)_i}{h}, i, j = 1, \dots, n$$

where h is a suitably chosen scalar, and e_j denotes a unit vector whose j th entry is one.

There is a significant amount of theoretical and computational support for this approximation. For example, see Goldstein [1965], Stewart [1967] and Goldstein and Price [1967] and Dennis [1972]. The expense of computing $\frac{n^2}{2} (1+m+l)$ scalar functions still remains and positive definiteness of B_k is not guaranteed.

A technique to reduce the overall computational effort is to hold the matrix B_k fixed for a certain number of iterations. This is practically useful when the change of the Hessian is not significant. However, it is difficult to decide how long the matrix should be held fixed. For details of this technique, see Brent [1973].

Quasi-Newton Updates

To avoid calculating second derivatives, quasi-Newton updates have been investigated by several authors. The basic scheme is of the form:

$$B_{k+1} = B_k + D_k$$

Here D_k is called a correction matrix and is chosen to assure that B_{k+1}

satisfies the quasi-Newton equation:

$$B_{k+1} s_k = y_k$$

where $s_k = x^{k+1} - x^k$ and $y_k = \nabla_x L(x^{k+1}, u^{k+1}, v^{k+1}) - \nabla_x L(x^k, u^{k+1}, v^{k+1})$. First, we discuss updates for dense and symmetric Hessian matrices. Later, we will discuss updates for the sparse case.

Garcia and Mangasarian [1976]

Garcia and Mangasarian proposed a suitable update similar to those used in quasi-Newton methods for unconstrained optimization. They used an updating mechanism for an $(n+m+l) \times (n+m+l)$ matrix which approximates the Hessian of the Lagrangian. The upper left $n \times n$ submatrix is used as the quadratic form in the direction finding problem. To be specific, the updating scheme is given below:

$$H_{k+1} = H_k + \frac{\theta}{s_k^t C_k s_k} (y_k s_k^t C_k + C_k s_k y_k^t) - \frac{\theta^2 s_k^t y_k}{(s_k^t C_k s_k)^2} C_k s_k s_k^t C_k$$

where

$$s_k = z^{k+1} - z^k$$

$$z^k = (x^k, u^k, v^k)$$

$$y_k = \nabla_z L(z^{k+1}) - \nabla_z L(z^k) - H_k s_k$$

$$\theta \in (0, 1)$$

$$C_{k+1} = \begin{cases} I & \text{if } k+1 \equiv 0 \pmod{(n+m+l)} \\ C_k - \frac{\theta}{s_k^t C_k s_k} C_k s_k s_k^t C_k & \text{otherwise} \end{cases}$$

The initial matrices H_1 and C_1 are equal to the $(n+m+l) \times (n+m+l)$ identity matrices. Since B_k is the upper left $n \times n$ submatrix of H_k , the scheme seems to be wasteful especially if the number of constraints is very large. Furthermore, it does not guarantee that the matrix B_k is positive semi-definite.

Han [1976]

As opposed to updating the overall Hessian of the Lagrangian, Han proposed updating the Hessian $\nabla_{xx}L(x^k, u^k, v^k)$ only with respect to the vector x . The updates are extensions of some well known double rank updates for unconstrained optimization problems. The general formula is given below:

$$B_{k+1} = B_k + \frac{(y_k - B_k s_k) c_k^t + c_k (y_k - B_k s_k)^t}{c_k^t s_k} - \frac{s_k^t (y_k - B_k s_k) c_k c_k^t}{(c_k^t s_k)^2}$$

where $s_k = x^{k+1} - x^k$, $y_k = \nabla_x L(x^{k+1}, u^{k+1}, v^{k+1}) - \nabla_x L(x^k, u^{k+1}, v^{k+1})$, and c_k is any vector with $c_k^t s_k \neq 0$. Even though the above formula updates the Hessian of the Lagrangian only with respect to the x vector, it has the disadvantage that it does not preserve positive definiteness.

Powell [1978]

Powell presented a quasi-Newton update which preserves positive definiteness of the matrix B_k even if the Hessian $\nabla_{xx}L(x, u, v)$ is itself not positive definite. Powell's update can be thought of as an extension of the well known BFGS formula given below.

$$B_{k+1} = B_k - \frac{B_k s_k s_k^t B_k}{s_k^t B_k s_k} + \frac{y_k y_k^t}{s_k^t y_k}$$

where $s_k = x^{k+1} - x^k$ and $y_k = \nabla_x L(x^{k+1}, u^{k+1}, v^{k+1}) - \nabla_x L(x^k, u^{k+1}, v^{k+1})$. If

the matrix B_k is positive definite, then the matrix B_{k+1} is also positive definite provided that $s_k^t y_k > 0$ holds. However, Powell pointed out that $s_k^t y_k > 0$ may not be satisfied due to the negative curvature of the Lagrangian function. Rather than using H_k in the third term of the BFGS formula, Powell used the vector ξ_k which is a convex combination of y_k and $B_k s_k$. The convex combination is chosen so that $s_k^t \xi_k > 0$ holds in all cases, thus maintaining positive definiteness of B_{k+1} . This update is given below.

$$B_{k+1} = B_k - \frac{B_k s_k s_k^t B_k}{s_k^t B_k s_k} + \frac{\xi_k \xi_k^t}{\xi_k^t \xi_k}$$

where $\xi_k = \theta y_k + (1-\theta) B_k s_k$, and

$$\theta = \begin{cases} 1 & \text{if } s_k^t y_k \geq 0.2 s_k^t B_k s_k \\ \frac{0.2 s_k^t B_k s_k}{s_k^t B_k s_k - s_k^t y_k} & \text{otherwise} \end{cases}$$

Sparse and Symmetric Updates

For sparse problems, the quasi-Newton updates discussed so far have several drawbacks. First, because of symmetry, $\frac{n^2}{2}$ memory locations are needed, which becomes impractical as n increases. Second, zero elements in the Hessian of the Lagrangian will be approximated by generally nonzero elements resulting from the updating formula. Finally, the update formulae may waste a substantial computational effort in carrying out unnecessary matrix and vector multiplications. Here we discuss sparse and symmetric updates where the Hessian $\nabla_{xx} L(x, u, v)$ of the Lagrangian function or the Hessian of the objective function has a known sparsity pattern.

Let J be the set of indices denoting the positions of the known zero entries of the Hessian and let K be the set of all indices not in J .

In [1977, 1978], Toint proposed the sparse and symmetric update given as follows:

First the vector τ_i , $i = 1, \dots, n$ is defined as follows:

$$\tau_{ij} = \begin{cases} s_i^k & \text{if } (i,j) \in K \\ 0 & \text{otherwise} \end{cases}$$

An $n \times n$ matrix Φ is formed using the vectors τ_i 's as follows, where δ_{ij} is the Kronecker delta.

$$\Phi_{ij} = \tau_{ij}\tau_{ji} + \|\tau_i\|^2 \delta_{ij}, \quad i = 1, \dots, n, \quad j = 1, \dots, n$$

Note that Φ satisfies the sparsity conditions, and is symmetric and positive definite provided that none of the vector τ_i , $i = 1, \dots, k$ is identically zero. Then

$$(B_{k+1})_{ij} = \begin{cases} 0 & \text{if } (i,j) \in J \\ \beta_i s_j^k + \beta_j s_i^k + (B_k)_{ij} & \text{otherwise} \end{cases}$$

where the vector β is

$$\beta = \Phi^{-1}(y^k - B_k s^k).$$

Note that the above update satisfies the Quasi-Newton equation. See Schubert [1970] for an update of the Jacobian matrix for nonlinear systems of equations. The interested reader may refer to Goldfarb [1970] for an update based on the Cholesky decomposition, Marwil [1978] and Shanno [1980] for an update based on Greenstadt's [1970] variational method.

Special Choices of B_k

Here we will consider two special choices of B_k . When B_k is chosen to be the identity matrix, the subprogram $Q(x^k, B_k)$ is equivalent to the problem of finding the least distance from the point $-\nabla f(x^k)$ to the feasible region of the direction-finding problem. Several authors have provided efficient methods to handle this special problem. For example, see the survey paper by Cottle and Djang [1979]. Here we may expect that the direction d^k produced by $Q(x^k, I)$ would be inferior to the direction produced by $Q(x^k, B_k)$ around the solution \bar{x} . However, the subprogram $Q(x^k, I)$ has some advantages. One principal advantage is that this program is usually much easier to solve than $Q(x^k, B_k)$. Another factor is the fact the program $Q(x^k, B_k)$ yields super-linear convergence only in the vicinity of a solution point \bar{x} , but actually has no theoretical advantage in early stages of the optimization process. The use of the program $Q(x^k, I)$ can be interpreted as an extension of the steepest descent method for unconstrained optimization.

Another choice is that each B_k is taken as a diagonal matrix whose diagonal entry approximates the Hessian of the Lagrangian function or the objective function by finite difference methods. To be specific, let

$$(B_k)_{ii} = \max \left\{ 1, \frac{\nabla_x L(x^k + h e_i, u^k, v^k)_i - \nabla_x L(x^k, u^k, v^k)_i}{h} \right\}$$

$$i = 1, \dots, m$$

where h is a suitably chosen positive number and e_i denotes an n -dimensional unit vector whose i th entry is one. We note that the $(1+m+l)$ gradient vectors are evaluated to produce the diagonal matrix at each iteration. Note that the matrix B_k is positive definite, and $\|B_k\|$ and $\|B_k^{-1}\|$ are both bounded if the gradient vectors are bounded. Other choices for the diagonal matrix B_k will be investigated.

I.2-3 Feasible Region for the Quadratic Program

Here we let S be the feasible region of problem P . That is,

$$S = \{x \mid g_i(x) \leq 0, i = 1, \dots, m, h_i(x) = 0, i = 1, \dots, \ell\}$$

We assume that S is nonempty and that $g_i(\cdot)$, $i = 1, \dots, m$ and $h_i(\cdot)$, $i = 1, \dots, \ell$ are continuously differentiable. Let $\hat{S}(x^k)$ denote the linearization of the set S at the point x^k so that

$$\begin{aligned} \hat{S}(x^k) = \{x \mid & g_i(x^k) + \nabla g_i(x^k)^t (x - x^k) \leq 0, i = 1, \dots, m, \\ & h_i(x^k) + \nabla h_i(x^k)^t (x - x^k) = 0, i = 1, \dots, \ell\} \end{aligned}$$

Note that the feasible region of the quadratic program $Q(x^k, B_k)$ is nonempty only if $\hat{S}(x^k)$ is nonempty. If the latter is empty, then the quadratic program is inconsistent and the quadratic approximation algorithm will stop prematurely. This point is illustrated by the following example.

Example 1: minimize $x_1 + x_2$
 subject to $h_1(x) = x_1^2 + x_2^2 - 2 = 0$
 $x \in E^2$

Note that the feasible region of the problem is nonempty and that the optimal solution \bar{x} is $(-1, -1)^t$. Let $B_k = I$ and consider the quadratic subprogram at the point $x^k = (0, 0)^t$ given below:

$$\begin{aligned} \text{minimize} \quad & (d_1 + d_2) + \frac{1}{2} (d_1^2 + d_2^2) \\ \text{subject to} \quad & -2 = 0 \end{aligned}$$

Clearly this problem is inconsistent and would result in premature termination of the algorithm.

In the vicinity of a Kuhn-Tucker point satisfying the second order sufficiency optimality conditions, the region $\hat{S}(x^k)$ is nonempty. If the point x^k is feasible, the region $\hat{S}(x^k)$ is indeed nonempty because $d = 0$ is feasible. However, if the point x^k is infeasible and remote from the solution point, we must provide a resolution to the case where the region $\hat{S}(x^k)$ is empty. Han [1977] provided a sufficient condition to assure that the region $\hat{S}(x^k)$ is nonempty. The result is summarized in the following lemma.

Lemma 1

Let g_i , $i = 1, \dots, m$ be continuously differentiable and convex, and h_i , $i = 1, \dots, \ell$ be affine. If the set $\{x \mid g_i(x) < 0, h_i(x) = 0, i = 1, \dots, m, i = 1, \dots, \ell\}$ is nonempty, then $\hat{S}(x^k)$ is nonempty for any $x^k \in E^k$.

Clearly, this sufficient condition is very restrictive. Bazaraa and Goode [1979] introduced artificial variables to prevent the constraint set from being empty. Through the use of a penalty term, these artificial variables will be equal to zero, unless of course the region $\hat{S}(x^k)$ is itself empty. This quadratic program is given below:

$$D(x^k, B_k): \text{ minimize } \nabla f(x^k)^t d + \frac{1}{2} d^t B_k d + r \left[\sum_{i=1}^m y_i + \sum_{i=1}^{\ell} (z_i^- + z_i^+) \right]$$

$$\text{subject to } g_i(x^k) + \nabla g_i(x^k)^t d \leq y_i, \quad i = 1, \dots, m$$

$$h_i(x^k) + \nabla h_i(x^k)^t d = z_i^+ - z_i^-, \quad i = 1, \dots, \ell$$

$$y_i \geq 0, \quad i = 1, \dots, m$$

$$z_i^+ > z_i^- \geq 0, \quad i = 1, \dots, \ell$$

where r is a sufficiently large positive number. The introduction of the artificial variables y_i , z_i^- and z_i^+ assures that the feasible region of $D(x^k, B_k)$ is maintained nonempty. However, we will show through a simple example that quasi-Newton updates of B_k are inadequate in this case unless some additional considerations are taken into account.

Example 2: We will reconsider Example 1.

$$\begin{aligned} &\text{minimize} && x_1 + x_2 \\ &\text{subject to} && x_1^2 + x_2^2 - 2 = 0 \\ &&& x \in X^2 \end{aligned}$$

Let the point $x^k = (0,0)^t$ and $B_k = I$. Then we get the quadratic program $D(x^k, B_k)$ given below:

$$\begin{aligned} &\text{minimize} && d_1 + d_2 + d_1^2 + d_2^2 + r(y^+ + y^-) \\ &\text{subject to} && -2 = y^+ - y^- \\ &&& y^+ \geq 0, y^- \geq 0 \end{aligned}$$

The optimal solution to the above problem is

$$\begin{aligned} d^k &= \left(-\frac{1}{2}, -\frac{1}{2}\right)^t \\ y^+ &= 0, y^- = 2 \end{aligned}$$

The Lagrangian multiplier \bar{q} associated with the linear equality constraint is

$$\bar{q} = r$$

Note that the Lagrangian multiplier $\bar{u} = \frac{1}{2}$ at the optimal solution $\bar{x} = (-1, -1)^t$.

If r is sufficiently large, the estimate \bar{q} of the Lagrangian multiplier is unnecessarily large. The Lagrangian function will thus be

$$L(x, \bar{q}) = x_1 + x_2 + r(x_1^2 + x_2^2 - 2)$$

which means that a big penalty is imposed on the constraint because it was inconsistent at the point $x^k = (0, 0)^t$. The unnecessarily large number \bar{q} may result in ill-conditioning of the next iterate B_{k+1} like penalty function methods. We note here that the choice of B_k in Bazaraa and Goode (1979) does not depend on the estimates of the Lagrangian multipliers. When an update of B_k is applied, one approach is to keep the values of the Lagrangian multipliers corresponding to the inconsistent constraints fixed rather than replacing them with the Lagrangian multipliers produced by the quadratic subprogram $D(x^k, B_k)$. In this study, we will investigate the subprogram $D(x^k, B_k)$ further.

Another approach is to eliminate some inconsistent constraints. Let $I(x^k) = \{i \mid \|\nabla g_i(x^k)\| \neq 0\}$ and $J(x^k) = \{i \mid \|\nabla h_i(x^k)\| \neq 0\}$. Then we have the following linear system to represent the feasible region of the quadratic subprogram $Q(x^k, B_k)$

$$g_i(x^k) + \nabla g_i(x^k)^t d \leq 0, \quad i \in I(x^k)$$

$$h_i(x^k) + \nabla h_i(x^k)^t d = 0, \quad i \in J(x^k)$$

We will investigate some sufficient conditions to guarantee that the above system is not empty.

I.2-4 Updating the Lagrangian Multipliers

In this section, we will discuss updating the Lagrangian multipliers. The estimates u^{k+1} and v^{k+1} of the Lagrangian multipliers may be used to determine the matrix B_{k+1} if B_{k+1} is chosen to approximate the Hessian of the Lagrangian function. Here we will discuss the updating scheme employed by most authors and then discuss some variations to be investigated further. The most popular updating scheme is given below:

$$u^{k+1} = p^k$$

and

$$v^{k+1} = q^k$$

where p^k and q^k are the Lagrangian multipliers obtained from problem $Q(x^k, B_k)$. Note that since $p^k \geq 0$, the nonnegativity of u^{k+1} is automatically maintained. This scheme has a certain advantage that if the sequence $\{x^k\}$ converges to a Kuhn-Tucker point \bar{x} , the estimates u^k and v^k converge to the vectors \bar{u} and \bar{v} of the Lagrangian multipliers, respectively. Under this method, the dual solution (p^k, q^k) may affect the numerical stability of the matrix B_{k+1} . If the length of the vector (p^k, q^k) is unnecessarily large, the next iterate B_{k+1} may suffer from ill-conditioning. This situation may arise if $Q(x^k, B_k)$ is inconsistent and if the search direction is obtained by solving $D(x^k, B_k)$ as explained in Example 2 in Section I.2-3.

Han [1977] presented a sufficient condition that the ∞ -norm of the dual solution (p^k, q^k) is bounded by a certain positive number. The result is summarized in the following theorem.

Theorem 1

Let f and g_i , $i = 1, \dots, m$ be continuously differentiable, g_i , $i = 1, \dots, m$ be convex, and h_i , $i = 1, \dots, \ell$ be affine. Suppose that the feasible region of

the original problem P is nonempty. Further, suppose that the matrix B_k satisfies the following condition:

$$\delta_1 \|d\|^2 \leq d^t B_k d \leq \delta_2 \|d\|^2 \quad \text{for any } d \in E^k, \text{ for all } k$$

Then there exists $\bar{r} > 0$ such that if (p^k, q^k) is a dual solution to $Q(x^k, B_k)$ then the ∞ -norm of the dual solution (p^k, q^k) is bounded by \bar{r} for each k .

The sufficient condition seems restrictive mainly because of convexity of the inequality constraints and linearity of the equality constraints. Since the number \bar{r} is unknown a priori, there still remains the possibility of ill-conditioning of the matrix B_k if \bar{r} is sufficiently large.

Revising the Updating Scheme

Let d^k be a solution to $Q(x^k, B_k)$. Then the dual vector (p^k, q^k) solves the following system:

$$\nabla f(x^k) + B_k d^k + \sum_{i=1}^m p_i \nabla g_i(x^k) + \sum_{i=1}^{\ell} q_i \nabla h_i(x^k) = 0$$

$$p_i (g_i(x^k) + \nabla g_i(x^k)^t d^k) = 0, \quad i = 1, \dots, m$$

$$p_i \geq 0 \quad i = 1, \dots, m$$

Note that the system may not have a unique solution. In particular, we are interested in finding a solution (p^k, q^k) with minimum ∞ -norm to prevent the possibility of ill-conditioning of the matrix B_{k+1} . Furthermore, we will investigate other updating rules. One such rule is:

$$u_i^{k+1} = \max \{0, u_i^k + \delta g_i(x^{k+1})\}$$

$$v_i^{k+1} = v_i^k + \delta h_i(x^{k+1})$$

where δ is a suitably chosen positive number. This method can be interpreted as a subgradient optimization scheme where a fixed step along the subgradient $(g(x^{k+1}), h(x^{k+1}))$ to the Lagrangian function is taken, and then forcing any negative components of the Lagrangian multipliers of the inequality constraints to be equal to zero.

I.2-5 Local Convergence

One of the key advantages of quadratic approximation methods is the fact that they enjoy a superlinear rate of convergence in the vicinity of a Kuhn-Tucker point satisfying second order sufficiency conditions. In this section, we will discuss the major results and assumptions which guarantee superlinear convergence.

First, we review the second order sufficiency condition which was first studied by Fiacco and McCormick [1968].

Definition

A Kuhn-Tucker triple $(\bar{x}, \bar{u}, \bar{v})$ of problem P satisfies the second order sufficiency conditions if the following conditions are simultaneously satisfied:

- (i) $\bar{u}_i = 0$ if $i \in I(\bar{x})$, where $I(\bar{x}) = \{j \mid g_j(\bar{x}) = 0\}$.
- (ii) The set N, the collection of the gradient vectors $\nabla g_i(\bar{x})$, $i \in I(\bar{x})$ and $\nabla h_i(\bar{x})$, $i = 1, \dots, \ell$, is linearly independent.
- (iii) The Hessian $\nabla_{xx} L(\bar{z})$ is positive definite on the tangent subspace $T = \{y \mid y^t d = 0, d \in N\}$.

Local convergence can be established through the use of a contraction mapping defined on a sufficiently small ball $B_\epsilon(\bar{z}) = \{z \mid \|z - \bar{z}\| \leq \epsilon\}$ such that

$$\|z^{k+1} - \bar{z}\| < \|z^k - \bar{z}\|$$

where \bar{z} denotes a Kuhn-Tucker triple satisfying the second order sufficiency conditions. The following theorem summarizes the main local convergence result of the algorithm.

Theorem 2

Let $\bar{z} = (\bar{x}, \bar{u}, \bar{v})$ be a Kuhn-Tucker triple of problem P. Suppose that \bar{z} satisfies the second order sufficiency condition, and that $f, g_i, (i=1, \dots, m), h_i, (i=1, \dots, \ell)$ have a second derivative which is Lipschitz continuous at the point \bar{x} . Then for $r \in (0, 1)$, there exist positive numbers ϵ and δ such that if $\|z^k - \bar{z}\| \leq \epsilon$ and $\|B_k - \nabla_{xx} L(\bar{z})\| \leq \delta$ at the point $z^k = (x^k, u^k, v^k)$, there exists a closest solution (d^k, u^{k+1}, v^{k+1}) of $Q(x^k, B_k)$ to $(0, u^k, v^k)$ such that

$$\|z^{k+1} - \bar{z}\| \leq r \|z^k - \bar{z}\|$$

where $z^{k+1} = (x^k + d^k, u^{k+1}, v^{k+1})$.

Proof

See Han [1976].

We note that the theorem holds only when z^k and B_k are sufficiently close to \bar{z} and $\nabla_{xx} L(\bar{z})$, respectively. Obviously, since $\|z^{k+1} - \bar{z}\| \leq r \|z^k - \bar{z}\|$, the convergence is guaranteed. However, as we will discuss later in the section, a fast rate of convergence characterized by superlinear convergence, is actually realized.

For the discussion of the superlinear convergence, we present the following definitions of linear and superlinear convergence.

Let $\{z^k\}$ converge to \bar{z} . Then the sequence $\{z^k\}$ is said to converge linearly if there exists an $r \in (0,1)$ and $k_0 \geq 0$ such that

$$\|z^{k+1} - \bar{z}\| \leq r \|z^k - \bar{z}\| \quad \text{for all } k \geq k_0$$

If there exists a sequence $\{\gamma_k\}$ convergent to zero such that

$$\|z^{k+1} - \bar{z}\| \leq \gamma_k \|z^k - \bar{z}\|$$

then the sequence $\{z^k\}$ is said to converge superlinearly. If $\{z_k\}$ converges superlinearly to \bar{z} , then

$$\lim_{k \rightarrow \infty} \frac{\|z^{k+1} - z^k\|}{\|z^k - \bar{z}\|} = 1$$

provided that $z^k \neq \bar{z}$. However, the converse is not true. For more details on superlinear convergence properties, refer to Dennis and Moré [1974, 1977] and Ortega and Reinboldt [1970].

To obtain the linear and superlinear rate of convergence, several sufficient conditions have been provided. The conditions are mainly based on the absolute and relative error of approximations to the Hessian, measured by some fixed matrix norms. A sufficient condition for the linear rate of convergence is that $\|B_k - \nabla_{xx} L(\bar{z})\| \leq \delta$. Here $\|\cdot\|$ denote any fixed matrix norm and δ is a sufficiently small positive number. The interested reader may refer to Garcia and Mangasarian [1976], and Han [1976]. A sufficient condition for the superlinear rate of convergence is that

$$\lim_{k \rightarrow \infty} \frac{\|(B_k - \nabla_{xx} L(\bar{z}))(x^{k+1} - x^k)\|}{\|z^{k+1} - z^k\|} = 0$$

This condition is credited to Han [1976]. For similar conditions, refer to Garcia and Mangasarian, and Powell [1978]. We note that if $\|B_k - \nabla_{xx}L(\bar{z})\|$ converges to zero, then the sequence $\{z^k\}$ converges superlinearly to \bar{z} . The reader may easily note that the methods of Wilson [1963], Robinson [1972] and the finite difference procedure are superlinearly convergent because $\lim_{k \rightarrow \infty} \|B_k - \nabla_{xx}L(\bar{z})\| = 0$. However, the condition $\lim_{k \rightarrow \infty} \|B_k - \nabla_{xx}L(\bar{z})\| = 0$ is not necessary for superlinear convergence.

1.2-6. Global Convergence

In this section, we will discuss global convergence of quadratic approximation algorithms employing line search. As mentioned before, in the vicinity of a Kuhn-Tucker point which satisfies the second order sufficiency condition, the distance function from the Kuhn-Tucker point can be used as a descent function, thus establishing convergence. If a starting point is remote from the Kuhn-Tucker point, a line search scheme employing a suitable descent function is needed to achieve convergence. The choice of descent functions and their convergence results will be discussed in this section.

An Exact Penalty Function

A successful descent function is the penalty function $\phi_r(x)$ of the form

$$\phi_r(x) = f(x) + r \left[\sum_{i=1}^m \max\{0, g_i(x)\} + \sum_{i=1}^{\ell} |h_i(x)| \right]$$

The parameter r will be called an exact penalty parameter. The function was first used as a descent function in the context of quadratic approximation

methods by Han [1977]. In [1979], Bazaraa and Goode simplified their minimax algorithm to directly handle the penalty function problem to minimize $\phi_r(x)$. The algorithms of Han and Bazaraa and Goode are discussed below. Both algorithms are globally convergent in the sense that each accumulation point of the sequence $\{x^k\}$ is a Kuhn-Tucker point. Both algorithms have the form $x^{k+1} = x^k + \lambda_k d^k$, where d^k is obtained from solving a quadratic program and λ_k is obtained by a suitable line search scheme. Han [1977] showed that the direction d^k obtained from the quadratic programming problem $Q(x^k, B_k)$ is indeed a descent direction for the exact penalty function. The line search along the direction d^k is performed as follows:

$$\phi_r(x^{k+1}) \leq \min_{0 \leq \lambda \leq \delta} \phi_r(x^k + \lambda d^k) + \epsilon_k$$

where δ is a prescribed positive number and ϵ_k is an error term allowed for the line search such that

$$\sum_{k=1}^{\infty} \epsilon_k < \infty$$

We note that since the function $\phi_r(x)$ is nondifferentiable, derivative-based search methods cannot be applied directly.

Bazaraa and Goode [1979]

Their algorithm was originally designed to solve minimax problems. Hence the algorithm can be specialized to solve the exact penalty function. The corresponding quadratic subprogram $D(x^k, B_k)$ is of the form

$$D(x^k, B_k): \text{ minimize } \nabla f(x^k)^t d + r \left[\sum_{i=1}^m y_i + \sum_{i=1}^{\ell} (z_i^- + z_i^+) \right] + \frac{1}{2} d^t B_k d$$

$$\text{subject to } g_i(x^k) + \nabla g_i(x^k)^t d \leq y_i, \quad i = 1, \dots, m$$

$$h_i(x^k) + \nabla h_i(x^k)^t d = z_i^+ - z_i^-, \quad i = 1, \dots, \ell$$

$$y_i \geq 0, \quad i = 1, \dots, m$$

$$z_i^+, z_i^- \geq 0, \quad i = 1, \dots, \ell$$

Note that each subprogram $D(x^k, B_k)$ has a nonempty feasible region. They specialized Armijo [1964] search rule under the assumption that f , g_i , $i = 1, \dots, m$, and h_i , $i = 1, \dots, \ell$, are upper uniformly differentiable. Each λ_k is determined by:

$$\lambda_k = \left(\frac{1}{2}\right)^{m_k}$$

where m_k is the smallest nonnegative integer such that

$$\phi_r(x^k + \left(\frac{1}{2}\right)^{m_k} d^k) \leq \phi_r(x^k) + \left(\frac{1}{2}\right)^{m_k+1} \nabla^* \phi_r(x^k, d^k)$$

where

$$\begin{aligned} \nabla^* \phi_r(x^k, d^k) &= \nabla f(x^k)^t d^k + r \left(\sum_{i=1}^m y_i + \sum_{i=1}^{\ell} (z_i^+ + z_i^-) \right) \\ &\quad - r \left[\sum_{i=1}^m \max\{0, g_i(x^k)\} + \sum_{i=1}^{\ell} |h_i(x^k)| \right] \end{aligned}$$

The two algorithms can be interpreted as an exact penalty function method which attempts to solve a single unconstrained penalty function $\phi_r(x)$, resulting

in a solution to problem P. This exact penalty function approach was first introduced by Fletcher [1970] who transformed the original problem into a completely unconstrained program. The basic idea is that if \bar{x} is a Kuhn-Tucker point to problem P, there exists a number \bar{r} such that \bar{x} is a local optimal solution to the problem to minimize $\phi_r(x)$ for all $r \geq \bar{r}$. The lower bound \bar{r} is estimated by the Lagrangian multipliers. For a review of exact penalty functions, the reader may refer to Pietrzykowski [1969], Evans, Gould and Tolle [1973], Howe [1973], Conn [1973], Conn and Pietrzykowski [1973], and Fletcher [1975]. For the existence of a globally exact penalty function in the convex case and in the nonconvex case refer to Bertsekas [1975], and Bazaraa and Goode [1979], Han and Mangasarian [1979].

I.4 Summary of Completed Research

In this section, we briefly summarize the major findings of the research completed thus far. A detailed description is given in the Appendix which reproduces the following papers:

1. M. S. Bazaraa and J. J. Goode, "A Globally Exact Penalty Function Without Convexity," submitted to Mathematical Programming.
2. M. S. Bazaraa and J. J. Goode, "An Extension of Armijo's Rule to Minimax and Quasi-Newton Methods for Constrained Optimization," submitted to Journal of Optimization Theory and Applications.
3. M. S. Bazaraa and J. J. Goode, "An Algorithm for Linearly Constrained Nonlinear Programming," Journal of Mathematical Analysis and Applications, to appear.

Globally Exact Penalty Functions

It is well known, under a suitable constraint qualification, that if \bar{x} is an isolated local optimal solution to the problem:

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x) \leq 0 \quad \text{for } i = 1, \dots, m \end{array}$$

then there exists a number λ_0 so that \bar{x} is a local optimal solution to the problem:

$$\text{minimize } f(x) + \lambda \sum_{i=1}^m \max\{0, g_i(x)\}$$

for all $\lambda \geq \lambda_0$. Unfortunately, however, in the absence of convexity, the above result does not hold globally.

In this paper we show, under mild conditions, that if a compact constraint set X is added to the constraints $g_i(x) \leq 0$ for $i = 1, \dots, m$, then the set of global optimal solutions to the original problem and the set of global optimal solutions to the penalty problem, for a sufficiently large penalty parameter λ , are equivalent. In order to prove this result, we use the fact that a family of relatively open sets that cover X must have a finite subcover. An estimate of the size of the penalty parameter is also given.

Minimax and Quasi-Newton Algorithms

An algorithm for solving a minimax problem over a closed convex set is developed. Using a newly developed continuous pseudo-directional derivative, a direction is found by minimizing a positive-semidefinite quadratic program over the feasible region. A step size is then computed using an extension of Armijo's inexact line search.

The algorithm is specialized to both unconstrained and constrained nonlinear programs. For the unconstrained case, various steepest descent and quasi-Newton methods are produced through different choices of the quadratic form. Using an exact penalty function to handle the nonlinear constraints, the direction-finding problem reduces to a convex quadratic programming problem. Unlike other available direction-finding routines that linearize the nonlinear constraints, our program is always feasible. A suitable step size is then found using Armijo's rule. It is shown that accumulation points of the algorithm are indeed Kuhn-Tucker points to the original problem.

Algorithm for Linearly Constrained Nonlinear Programs

Here an algorithm for solving a linearly constrained nonlinear program is developed. Given a feasible solution, to avoid jamming, binding and near

binding constraints are identified. A direction is calculated by solving a least distance programming problem which is defined in terms of the gradients of these constraints.

Once a direction is found, an estimate of the step size, using quadratic approximation of the objective function, is first computed. This estimate is then used in conjunction with Armijo's inexact line search to calculate a new point. It is shown that each accumulation point is a Kuhn-Tucker solution to a slight perturbation of the original problem. Under suitable second order optimality conditions, we show that eventually one functional evaluation is needed to compute the step size.

References

1. Armijo, L., "Minimization of Functions Having Continuous Partial Derivatives," Pacific Journal of Mathematics, Volume 16, pp. 1-3, 1966.
2. Arrow, K. J., F. J. Gould and S. M. Howe, "A General Saddle Point Result for Constrained Optimization," Mathematical Programming, Volume 5, pp. 225-234, 1973.
3. Bazaraa, M. S. and J. J. Goode, "A Globally Exact Penalty Function Without Convexity," Georgia Institute of Technology, 1979.
4. Bazaraa, M. S. and J. J. Goode, "An Extension of Armijo's Rule to Minimax and Quasi-Newton Methods for Constrained Optimization," Georgia Institute of Technology, 1979.
5. Beale, E. M. L., "On Quadratic Programming," Naval Research Logistics Quarterly, Volume 6, pp. 227-244, 1959.
6. Bertsekas, D. P., "On Penalty and Multiplier Methods for Constrained Minimization," in Nonlinear Programming 2, O. L. Mangasarian, R. Meyer, and S. M. Robinson (EDS), Academic Press, New York, 1975.
7. Brent, R. P., Algorithms for Minimization Without Derivatives, Prentice-Hall, Englewood Cliffs, New Jersey, 1973.
8. Broyden, C. G., "The Convergence of a Class of Double Rank Minimization Algorithm 2. The New Algorithm," Journal Institute of Mathematics and its Applications, Volume 6, pp. 222-231, 1970.
9. Conn, A. R., "Constrained Optimization using a Nondifferentiable Penalty Function," SIAM Journal on Numerical Analysis, Volume 10, pp. 760-784, 1973.
10. Conn, A. R. and T. Pietrzykowski, "An Exact Penalty Function Method Converging Directly to a Constrained Minimum," Research Report 73-11, Department of Combinatorics and Optimization, University of Waterloo, May 1973.
11. Cottle, R. W. and A. Djang, "Algorithmic Equivalence in Quadratic Programming, I: A Least Distance Programming Problem," Journal of Optimization Theory and Applications, Volume 28, pp. 275-301, 1979.
12. Dennis, J. E. and J. J. Moré, "A Character of Superlinear Convergence and its Application to Quasi-Newton Methods," Mathematics of Computation, Volume 28, pp. 549-560, 1974.
13. Dennis, J. E. and J. J. Moré, "Quasi-Newton Methods, Motivation and Theory," SIAM Review, Volume 19, pp. 46-89, 1977.
14. Evans, J. P., F. J. Gould and J. W. Tolle, "Exact Penalty Functions in Nonlinear Programming," Mathematical Programming, Volume 4, pp. 72-97, 1973.

15. Fiacco, A. V. and G. P. McCormick, Nonlinear Programming: Sequential Unconstrained Minimization Techniques, John Wiley & Sons, New York, 1968.
16. Fletcher, R., "A General Quadratic Programming Algorithm," SIAM Journal on Applied Mathematics, Volume 7, pp. 76-91, 1971.
17. Fletcher, R., "An Exact Penalty Function for Nonlinear Programming With Inequalities," Mathematical Programming, Volume 5, pp. 129-150, 1973.
18. Fletcher, R., "An Ideal Penalty Function for Constrained Optimization," SIAM Journal on Applied Mathematics, Volume 15, pp. 319-342, 1975.
19. Fletcher, R. and T. L. Freeman, "A Modified Newton Method for Minimization," Journal of Optimization Theory and Applications, Volume 23, pp. 357-372, 1977.
20. Garcia Palomares, U. M. and O. L. Mangasarian, "Superlinearly Convergent Quasi-Newton Algorithms for Nonlinearly Constrained Optimization Problems," Mathematical Programming, Volume 11, pp. 1-13, 1976.
21. Gill, P. E. and W. Murray, Numerical Methods for Unconstrained Optimization, Academic Press, New York, 1972.
22. Gill, P. E. and W. Murray, Numerical Methods for Constrained Optimization, Academic Press, New York, 1974.
23. Goldfarb, D., "A Family of Variable Metric Methods Derived by Variational Means," Mathematics of Computation, Volume 24, pp. 23-26, 1970.
24. Goldfarb, D., "Extensions of Newton's Method and Simplex Methods for Solving Quadratic Programs," Numerical Methods for Nonlinear Optimization, Lootma (Eds.), pp. 121-143, 1972.
25. Goldfeld, S. M., R. E. Quandt, and M. F. Trotter, "Maximization by Improved Quadratic Hill Climbing and Other Methods," Econ. Res. Memo 95, Princeton University Research Program, 1968.
26. Goldstein, A. A., "On Newton's Method," Numerische Mathematik, Volume 7, pp. 391-393, 1965.
27. Goldstein, A. A. and J. F. Price, "An Effective Algorithm for Minimization," Numerische Mathematik, Volume 10, pp. 184-189, 1967.
28. Greenstadt, J., "Variations on Variable - Metric Methods," Mathematics of Computation, Volume 24, pp. 1-18, 1970.
29. Greenstadt, J., "A Quasi-Newton Method with No Derivatives," Mathematics of Computation, Volume 26, pp. 145-166, 1972.
30. Han, S. P., "Superlinearly Convergent Variable Metric Algorithms for General Nonlinear Programming Problems," Mathematical Programming, pp. 263-282, 1967.
31. Han, S.P., "Dual Variable Metric Algorithms for Constrained Optimization," SIAM Journal Control and Optimization, Volume 15, pp. 546-565, 1977.

32. Han, S. P., "A Globally Convergent Method for Nonlinear Programming," Journal of Optimization Theory and Applications, Volume 22, pp. 297-309, 1977.
33. Han, S. P. and O. L. Mangasarian, "Exact Penalty Functions in Nonlinear Programming," Mathematical Programming, Volume 17, pp. 251-269, 1979.
34. Howe, S., "New Conditions for Exactness of a Simple Penalty Function," SIAM Journal on Control, pp. 378-381, 1973.
35. Lemke, C. E., "Bimatrix Equilibrium Points and Mathematical Programming," Management Science, Volume 11, pp. 681-689, 1965.
36. Lemke, C. E., "On Complementary Pivot Theory," in Mathematics of the Decision Science, G. B. Dantzig and A. F. Veinott (Eds.), 1968.
37. Marwil, E. S., "Exploiting Sparsity in Newton-Like Methods," Report TR 78-335, Department of Computer Science, Cornell University, Ithaca, New York, 1978.
38. Mathews, A., and D. Davies, "A Comparison of Modified Newton Methods for Unconstrained Optimization," Computer Journal, Volume 14, pp. 293-294, 1971.
39. Mukai, H., "A Scheme for Determining Step Sizes for Unconstrained Optimization Methods," IEEE Transactions on Automatic Control, Volume AC23, pp. 987-995, 1978.
40. Pietrzykowski, T., "An Exact Potential Method for Constrained Maxima," SIAM Journal on Numerical Analysis, Volume 6, pp. 299-304, 1969.
41. Pietrzykowski, T., "The Potential Method for Conditional Maxima in the Locally Compact Metric Spaces," Numerische Mathematik, Volume 14, pp. 325-329, 1970.
42. Powell, M. J. D., "Algorithms for Nonlinear Constraints that use Lagrangian Functions," presented at the Ninth International Symposium on Mathematical Programming, Budapest, 1976.
43. Powell, M. J. D., "A Last Algorithm for Nonlinearly Constrained Optimization Calculations," presented at the 1977 Dundee Conference on Numerical Analysis, 1977.
44. Powell, M. J. D., "The Convergence of Variable Metric Methods for Nonlinearly Constrained Optimization Calculations," in Nonlinear Programming 3, edited by O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, Academic Press, New York, 1978.
45. Ortega, J. M. and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, 1970.
46. Ritter, K., "A Method for Solving Maximum Problems with a Nonconcave Quadratic Objective Function," Z Wahrscheinlichkeitstheorie und Verwandte Gebiete, Volume 4, pp. 340-351, 1966.

47. Robinson, S. M., "A Quadratically Convergent Algorithm for General Nonlinear Programming Problems," Mathematical Programming, Volume 3, pp. 145-156.
48. Rosen, J. B. and J. Kreuser, "A Gradient Projection Algorithm for Nonlinear Constraints," Numerical Methods for Nonlinear Optimization, Lootsma (Eds.), pp. 297-300, 1972.
49. Schubert, L. K., "Modification of a Quasi-Newton Method for Nonlinear Equations with a Sparse Jacobian," Mathematics of Computation, Volume 24, pp. 27-30, 1970.
50. Shanno, D. F., "On Variable-Metric Methods for Sparse Hessians," Mathematics of Computation, Volume 34, pp. 499-514, 1980.
51. Stewart, G. W., "A Modification of Davidon's Minimization Method to Accept Difference Approximation of Derivatives," Journal Association for Computing Machinery, Volume 14, pp. 72-83, 1967.
52. Toint, P. H. L., "On Sparse and Symmetric Updating Subject to a Linear Equation," Mathematics of Computation, Volume 31, pp. 354-961, 1977.
53. Toint, P. H. L., "Some Numerical Results Using A Sparse Matrix Updating Formula in Unconstrained Optimization," Mathematics of Computation, Volume 32, pp. 839-852, 1978.
54. Topkis, D. M. and A. F. Veinott, "On the Convergence of Some Feasible Direction Algorithms for Nonlinear Programming," SIAM Journal on Control, Volume 5, pp. 268-279, 1967.
55. Wilson, R. B., "A Simplicial Method for Concave Programming," Ph.D. Dissertation, Harvard University, Cambridge, Massachusetts, 1963.

II. GENERIC OPTIMALITY CONDITIONS AND NONDIFFERENTIABLE OPTIMIZATION

(J. Spingarn)

II.1 Introduction

Our research during the period covered by this contract has centered on two themes, both within the compass of mathematical programming: generic optimality conditions and nondifferentiable optimization.

II.1-1 Generic Optimality Conditions

Our work on generic conditions continued the investigation that was begun in Spingarn and Rockafellar [5]. In that paper, it had been shown that for almost all $(v,u) \in \mathbb{R}^{n+m}$, at every local minimizer for the problem

$$\begin{aligned} Q(v,u) \quad & \text{minimize} \quad f(x) - x \cdot v \quad \text{over all } x \in \mathbb{R}^n \\ & \text{satisfying } g_i(x) \leq u_i \quad \text{for all } i = 1, \dots, m \end{aligned}$$

the so called "strong second-order optimality conditions" hold (assuming that the functions f and g_i possess derivatives of sufficiently high order). In this sense, the strong second-order conditions are "generically" necessary for (local) optimality with respect to the class $Q(v,u)$.

When studying questions of genericity, the precise class of problems to which the results apply is crucial. The family $Q(v,u)$ is only one example of a family for which the conditions are generic. So the question naturally arises: For what other families will the strong second-order conditions, or similar conditions, be generically necessary for optimality? This is the question addressed by our recent work on generic conditions. Our principal

accomplishment in this direction has been to obtain an easily verifiable criterion which ensures the genericity of the conditions.

In some circumstances, we found that it is necessary to modify the strong conditions themselves. This situation occurs when the family includes both "fixed" and "variable" constraints. "Fixed" constraints are those that do not vary with the problem parameters, while "variable" constraints do.

The exact manner in which the generic conditions depend on the fixed constraints is also described by our results.

II.1-2 Nondifferentiable Optimization

If $f: R^n \rightarrow R$ is a locally Lipschitz function, the generalized subdifferential of f is the set-valued mapping $\partial f: R^n \rightarrow R^n$ defined by taking $\partial f(x)$ to be the convex hull of the set of all limit points of sequences of the form $(\nabla f(x_n))$, where $x_n \rightarrow x$ and f is differentiable at x_n . (This definition is due to F. Clarke [9]). If f happens to be convex then $\partial f(x)$ is just the set of "subgradients" of f at x , i.e., the set $\{\xi: f(z) - f(x) \geq \langle \xi, z-x \rangle \forall z \in R^n\}$.

When the generalized subdifferential was first studied, the motive was to provide a tool that would be of use in handling optimization problems in which a function which is neither convex nor differentiable is to be minimized. Most algorithms for solving constrained or unconstrained minimization problems make heavy use of derivatives or, in the nondifferentiable but convex case, of subgradients. To generalize such algorithms to a broader class of functions, it is necessary to have a substitute; hence the need for the generalized subdifferential.

Our work in this area has concentrated on the relationship between certain properties of nondifferentiable functions and properties of their generalized subdifferentials. The basic goal has been to identify subclasses of functions

which are both likely to arise in optimization problems and whose subdifferentials possess properties which are likely to facilitate the development of algorithms.

Our principal achievement in this direction has been to characterize the class of "lower- C^1 " functions in terms of their subdifferentials. Lower- C^1 functions are a desirable class of functions to study because of the natural way they arise in optimization problems. Anytime a function is obtained by maximizing in one argument a second function of two arguments (e.g., $f(x) = \max_s g(x,s)$) one obtains a lower- C^1 function, provided the second function has a continuous derivative and the maximum is taken over a compact set. Such functions arise in decomposition schemes for minimizing a function of two arguments.

The most remarkable feature of our characterization of lower- C^1 functions is that the corresponding property of the subdifferential mapping is so closely related to the "monotone" property that characterizes the subdifferential of a convex function. Because of this resemblance, we have coined the word "submonotone" for the related property. The close resemblance is more than a curiosity. There is reason to hope that the similarity will facilitate the transfer to nondifferentiable optimization of algorithms originally intended for convex programming.

II.2 Research and Publications Summary - Generic Conditions

The results of our work in this area form the basis for two articles: "On optimality conditions for structured families of nonlinear programming problems" (submitted to Mathematical Programming) and "Second-order optimality conditions that are necessary with probability one" (to appear in Proceedings, Symposium on Mathematical Programming with Data Perturbations, George Washington University, May 1979). The latter article is a survey without proofs of all our research on this subject to date, while the former contains the main results and their proofs.

We investigated problems of the general form indexed by a parameter $p \in P$, with $P \subset \mathbb{R}^{\ell}$ an open set:

$$\begin{aligned} Q(p) \quad & \text{minimize } f(x,p) \quad \text{over all } x \in C \subset \mathbb{R}^n \\ & \text{satisfying } g_i(x,p) \leq 0 \quad \text{for all } i=1,\dots,m, \text{ and} \\ & \quad h_j(x,p) = 0 \quad \text{for all } j=1,\dots,k \end{aligned}$$

This class is more general than $Q(v,u)$ in two important respects. First, the manner in which f , g_i and h_j depend on the parameter is given more freedom. Rather than requiring perturbation of a special type (e.g., linear perturbations of the objective function and right-hand-side perturbations of the constraints), we only required that the family $Q(p)$ satisfy a general criterion. Second, in addition to the constraints $g_i \leq 0$ and $h_j = 0$, which we refer to as the "variable" constraints, we also investigated the effect of the "structural" or "fixed" constraint $x \in C$ that does not vary with p . The distinction between these two types of constraints is important because the two types play different roles in both the analysis of the conditions

and in the statement of the conditions themselves: the conditions that turn out to be generically necessary for optimality depend on the particular class of problems under consideration.

Our principal accomplishment here was to give appropriate criteria for the family $Q(p)$ which guarantee the genericity of the second-order conditions, and also to describe the form of the second-order conditions and how they depend on the fixed constraint set C .

In order to discuss second-order conditions, we found it necessary to make certain second-order regularity assumptions about the set C . The conditions that we imposed on the set C were incorporated into our definition of "cyrtohedron". Cyrtohedra, which we introduced in [4], are piecewise smooth sets that can be represented locally by a finite number of nonlinear inequality and equality constraints. A cyrtohedron is a union of submanifolds, called the "faces" of C , and each $x \in C$ belongs to a unique such face. In a natural way, with each $x \in C$, we can associate the normal cone $N_C(x)$ to C at x , and the tangent space at x , $L_C(x)$, to the face containing x .

The second-order conditions which we showed to be generically necessary for optimality are the generalized strong second-order conditions discussed previously in Spingarn [4]. A triple $(\bar{x}, \bar{y}, \bar{z}) \in C \times \mathbb{R}_+^m \times \mathbb{R}^k$ is said to satisfy these conditions for the problem $Q(p)$ if

- (SSOC) (i) \bar{x} is feasible for $Q(p)$
- (ii) $-\nabla_x L(\bar{x}, \bar{y}, \bar{z}, p) \text{ relint } N_C(\bar{x})$, where L is the usual Lagrangian, and "relint" denotes relative interior
- (iii) $\bar{y}_i > 0$ iff $g_i(\bar{x}, p) = 0$, for each i
- (iv) The projections onto $L_C(\bar{x})$ of the gradients of the active constraints are linearly independent
- (v) If F is the face of C containing \bar{x} then $\nabla_x^2(LIF)(\bar{x}, \bar{y}, \bar{z}, p)(s, s) > 0$ for all $s \in \mathbb{R}^n$ satisfying $0 \neq s \in L_C(\bar{x})$, s perpendicular to the gradients of the active constraints.

The family $Q(p)$ is full provided the function $p^1 \mapsto \nabla_{x,y,z} L(x,y,z,p^1)$ has Jacobian of full rank at all $(x,y,z,p) \in C \times R_+^m \times R^k \times R^\ell$ (where $L(x,y,z,p) = f(x,p) + \sum y_i g_i(x,p) + \sum z_j h_j(x,p)$ is the usual Lagrangian). Our main result is the following:

Theorem 1

Let $C \subset R^n$ be a d -dimensional cyrtohedron of class C^s , f of class C^2 , and g and h of class C^s on $R^n \times P$ with $s > \max\{1, \alpha-m\}$. If $Q(p)$ is full, there is a subset $P_0 \subset P$ with P/P_0 having measure zero, such that for all $\bar{p} \in P_0$: if $\bar{x} \in C$ is a local minimizer for $Q(\bar{p})$ there exists $(\bar{y}, \bar{z}) \in R_+^m \times R^k$ satisfying SSOC.

II.3 Research and Publications Summary - Nondifferentiable Optimization

We have published our results from this line of work in "Submonotone subdifferentials of Lipschitz functions" (to appear in Trans. Amer. Math. Soc.).

$f: R^n \rightarrow R$ is a lower- C^1 function if every $\bar{x} \in R^n$ has a neighborhood U such that for all $x \in U$, $f(x) = \max_{s \in S} g(x, s)$, where S is some compact set and g and $\nabla_x g$ are continuous jointly in x and s . If f is a locally Lipschitz function $R^n \rightarrow R$, we say that ∂f is strictly submonotone if for all $x \in R^n$,

$$\liminf_{\substack{x_1 \neq x_2 \\ x_i \rightarrow x \\ y_i \in \partial f(x_i) \\ i = 1, 2}} \frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{|x_1 - x_2|} \geq 0$$

Our principal result is the following

Theorem 2

f is lower- C^1 iff ∂f is strictly submonotone.

Notice the close relationship between strict submonotonicity and monotonicity. The latter property clearly implies the former since if ∂f is monotone, the numerator in the "lim inf" above is always nonnegative.

We also investigated the property of "submonotonicity", which is stronger than strict submonotonicity, but weaker than monotonicity. ∂f is submonotone if for all $x \in R^n$,

$$\liminf_{\substack{x^1 \rightarrow x \\ x^1 \neq x \\ y \in \partial f(x) \\ y^1 \in \partial f(x^1)}} \frac{\langle x^1 - x, y^1 - y \rangle}{|x^1 - x|} \geq 0$$

In terms of the function f , we showed that the submonotonicity of ∂f corresponds to a certain "regularity" property of the directional derivative of f . We also proved several results which relate submonotonicity to properties that have been studied by other authors, such as semismoothness (Mifflin [7]), lower semi-differentiability (Rockafellar [6]), quasi-differentiability (pshenichnyi [8]), and Clarke regularity [10]. For instance, we showed that ∂f is submonotone if f is both semismooth and Clarke regular.

References

- *1. J. E. Spingarn, "Submonotone Subdifferentials of Lipschitz Functions," accepted by Transactions of the American Mathematical Society.
- *2. J. E. Spingarn, "On Optimality Conditions for Structured Families of Nonlinear Programming Problems," submitted to Mathematical Programming.
- *3. J. E. Spingarn, "Second-order Optimality Conditions that are Necessary with Probability One," accepted by Proceedings, Symposium on Mathematical Programming with Data Perturbations, George Washington University, May 1979.
- 4. "Fixed and Variable Constraints in Sensitivity Analysis," SIAM Journal of Control and Optimization, May 1980.
- 5. "The Generic Nature of Optimality Conditions in Nonlinear Programming," with R. T. Rockafellar, Mathematics of Operations Research, November, 1979.
- 6. R. T. Rockafellar, "Directionally Lipschitzian functions and subdifferential calculus," Proc. London Math. Soc., 1979, 331-355.
- 7. R. Mifflin, "Semismooth and semiconvex functions in constrained optimization," SIAM Journal on Control and Optimization, 1977, 959-972.
- 8. B. N. Pshenichnyi, Necessary Conditions for an Extremum, Marcel Dekker, New York, 1971.
- 9. F. H. Clarke, "Generalized gradients and applications," Trans. Amer. Math. Soc., 1975, 247-262.
- 10. F. H. Clarke, "Generalized gradients of Lipschitz functionals," to appear in Advances in Math.

* Published under this research contract.

APPENDIX

This appendix contains articles resulting from research conducted under this contract. These articles are:

1. M. S. Bazaraa and J. J. Goode, "A Globally Exact Penalty Function Without Convexity," submitted to Mathematical Programming.
2. M. S. Bazaraa and J. J. Goode, "An Extension of Armijo's Rule to Minimax and Quasi-Newton Methods for Constrained Optimization," submitted to Journal of Optimization Theory and Applications.
3. M. S. Bazaraa and J. J. Goode, "An Algorithm for Linearly Constrained Nonlinear Programming," Journal of Mathematical Analysis and Applications, to appear.
4. J. Spingarn, "On Optimality Conditions for Structured Families of Nonlinear Programming Problems," submitted to Mathematical Programming.
5. J. Spingarn, "Second-order Optimality Conditions that are Necessary with Probability One," to appear in Proceedings, Symposium on Mathematical Programming with Data Perturbations, George Washington University, May 1979.
6. J. Spingarn, "Submonotone Subdifferentials of Lipschitz Functions," to appear in Trans. Amer. Math. Soc.

A GLOBALLY EXACT PENALTY FUNCTION WITHOUT CONVEXITY

Mokhtar S. Bazaraa[†] and Jamie J. Goode^{††}

In this paper, we consider the nonlinear programming problem to minimize $f(x)$ subject to $g_i(x) \leq 0$ for $i = 1, \dots, m$ and $x \in X$. If X is compact, we show under a suitable constraint qualification that a globally exact penalty function exists. Particularly, we show a one-to-one correspondence between global optimal solutions to the original problem and global optimal solutions to the penalty problem for a sufficiently large, but finite, penalty parameter. A lower bound on the penalty parameter is established in terms of the Kuhn-Tucker Lagrangian multipliers and lower bounds on the functions involved.

[†] School of Industrial and Systems Engineering, Georgia Institute of Technology. This author's work is supported under USAFOSR contract F49620-79-C0120

^{††} School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia

1. Introduction

A great deal of attention has been given to the subject of exact penalty functions where a constrained nonlinear programming problem is transformed into a single unconstrained problem or into a finite sequence of unconstrained problems.

Without convexity, the current theory applies only locally. Specifically, if \bar{x} is a strict local minimum to problem P_0 to minimize $f(x)$ subject to $g_i(x) \leq 0$ for $i = 1, \dots, m$, under a suitable constraint qualification, there exists a number λ_0 such that \bar{x} is a local optimal to the problem to minimize $\theta(x, \lambda)$ for all $\lambda \geq \lambda_0$, where $\theta(x, \lambda)$ is an appropriate penalty function. For a review of exact penalty functions, the reader may refer to Evans, Gould, and Tolle [4], Fletcher [5], Han and Mangasarian [8], Howe [9], McCormick [11], and Pietrzykowski [12,13]. For the existence of a globally exact penalty function in the convex case, see Bertsekas [3] and Zangwill [15].

The main result of this paper is to show, under mild assumptions, the existence of a globally exact penalty function in the nonconvex case. Before proceeding, it is worthwhile to briefly review the cases under which an exact penalty does not exist. In this regard, consider problem P_0 and let $g_i(x)_+ = \max \{0, g_i(x)\}$. Given the penalty parameter λ , the penalty problem is to minimize $\theta(x, \lambda)$ where $\theta(x, \lambda) = f(x) + \lambda \sum_{i=1}^m g_i(x)_+$. Figure 1 shows, for $m=1$, the set $\Lambda = \{(g_1(x)_+, f(x)) : x \in R^n\}$. It is clear that if \bar{x} solves problem P_0 , then there exists a λ_0 so that \bar{x} also solves the penalty problem to minimize $\theta(x, \lambda)$ for all $\lambda \geq \lambda_0$, if and only if there is a nonver-

tical supporting hyperplane with slope $-\lambda_0$, to the set Λ at the point $(g_1(\bar{x})_+, f(\bar{x}))$. In Figure 1a, such a supporting hyperplane exists, whereas in Figures 1b and 1c, a globally exact penalty function does not exist. The case illustrated in Figure 1b can be easily overcome by the stipulation of a suitable constraint qualification of the kind that is needed to validate the Kuhn-Tucker conditions.

If we modify problem P_0 so that a compact set X is included in the constraints yielding the compact set $\Lambda' = \{(g_1(x)_+, f(x)) : x \in X\}$, as shown in Figure 1d, a supporting hyperplane can be found.

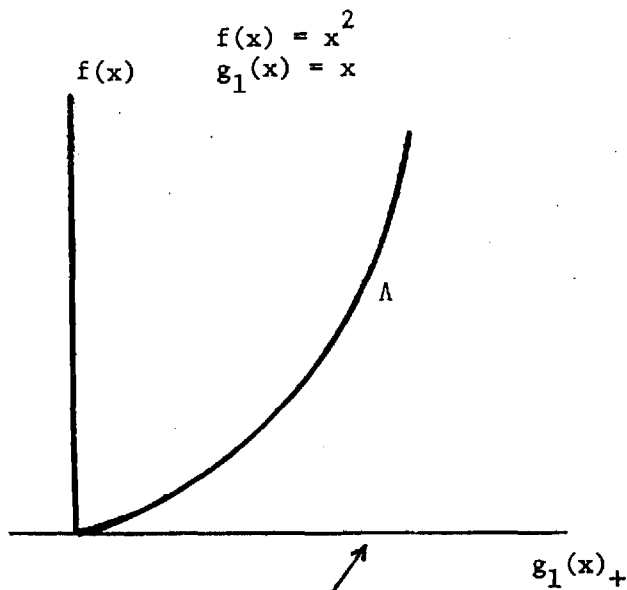
In this paper we consider the following problem:

$$\begin{aligned} \text{Problem P: } & \text{minimize } f(x) \\ & \text{subject to } g_i(x) \leq 0 \quad \text{for } i = 1, \dots, m \\ & \quad \quad \quad x \in X \end{aligned}$$

We think of the constraints defined by X as easy constraints that must be handled explicitly and of the constraints $g_i(x) \leq 0$ for $i = 1, \dots, m$ as those that are treated by a penalty function. Typically, X contains lower and upper bounds on the variables, and possibly linear constraints. As discussed above, we prove that if X is compact and under a suitable constraint qualification, a globally exact penalty exists. The penalty problem under consideration is:

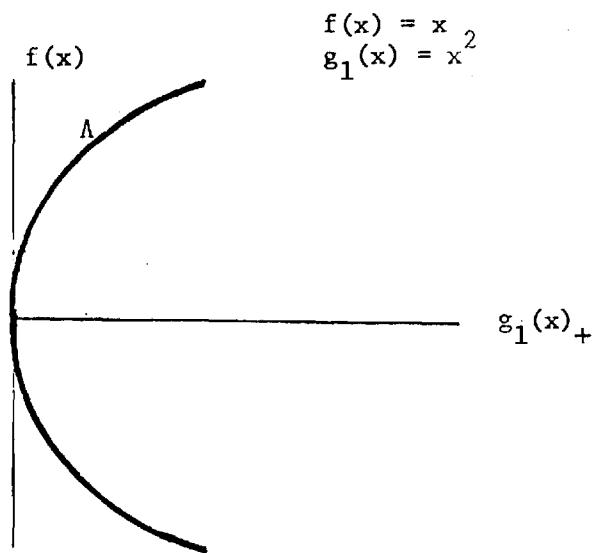
$$\begin{aligned} \text{Problem P}(\lambda): & \text{minimize } \theta(x, \lambda) \\ & \text{subject to } x \in X \end{aligned}$$

In this study, we let $\theta(x, \lambda) = f(x) + \lambda \sum_{i=1}^m g_i(x)_+$. All the qualitative results given in this paper are valid if the expression $\sum_{i=1}^m g_i(x)_+$ is



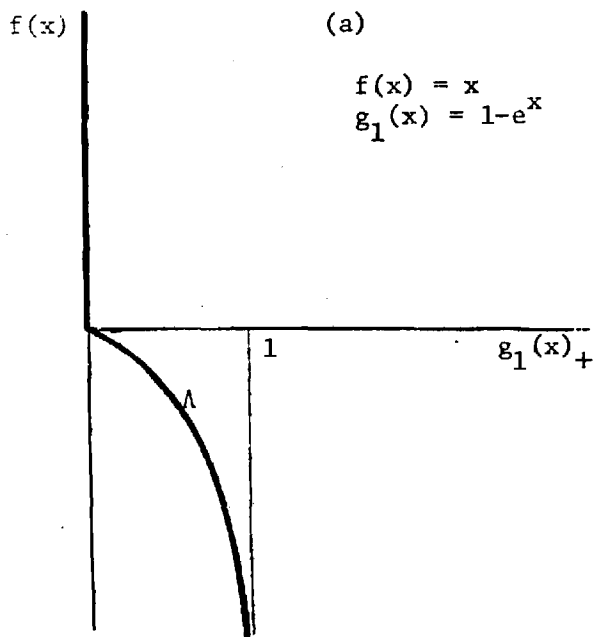
A nonvertical supporting hyperplane exists in the convex case

(a)



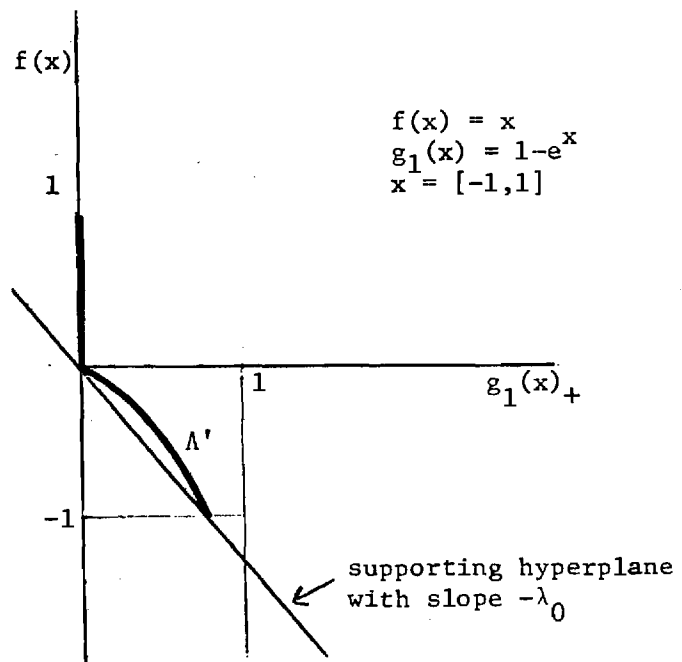
A nonvertical supporting hyperplane does not exist in the convex case due to the lack of a constraint qualification

(b)



A nonvertical supporting hyperplane does not exist in the nonconvex case because of noncompactness of Λ

(c)



A nonvertical supporting hyperplane exists in the nonconvex case in the presence of the compact set X

(d)

Figure 1. Illustration of a Globally Exact Penalty Function in the (g_+, f) plane

replaced by the expression $Q(\|g(x)_+\|)$, where $Q: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies:

$$Q(0) = 0, \quad Q(\delta) > 0 \quad \text{for } \delta > 0, \quad \infty > \lim_{\delta \rightarrow 0^+} Q(\delta)/\delta > 0$$

This assertion follows directly from a Theorem in [8].

Throughout the paper, we assume that f and g are continuously differentiable, and that X is closed. Further, we suppose that problem P is consistent. These assumptions will not be repeated in the statements of the theorems given in the paper. We also note that equality constraints of the form $h_i(x) = 0$ for $i = 1, \dots, \ell$ can be incorporated without any difficulty. In order to keep the notation and development simple, we chose to omit their inclusion.

In Section 2, we give two different sufficient conditions that ensure the existence of an exact penalty strict local minimum. Using compactness of X and the fact that a relatively open cover has a finite subcover, we establish in Section 1, the existence of a globally exact penalty function. Finally, in Section 4, we provide some insight into determining the size of the penalty parameter.

2. Sufficient Conditions for an Exact Penalty

Strict Local Minimum

In this section, we show that an exact penalty strict local minimum exists under two different conditions. These conditions generalize similar conditions which are available in the literature in that they handle the presence of the set X . Particularly, Theorems 2.1 and 2.2 extend similar results of Howe [9] and Han and Mangasarian [8], respectively. They assert that there exists a positive number λ_0 such that if \bar{x} is a strict local minimum for Problem P , then \bar{x} is also a strict local minimum for Problem $P(\lambda)$ for all $\lambda \geq \lambda_0$. These theorems will be used in the next section to prove our main result showing the existence of a globally exact penalty function.

The following notation and definitions will be used throughout the manuscript. Given $x \in X$, let

$$I^+(x) = \{i: g_i(x) > 0\}$$

$$I^-(x) = \{i: g_i(x) < 0\}$$

$$I(x) = \{i: g_i(x) = 0\}$$

\bar{x} is a strict local minimum for Problem $P \leftrightarrow$ there exists $\epsilon > 0$ such that $f(x) > f(\bar{x})$ for each $\bar{x} \neq x \in X$ such that $\|x - \bar{x}\| < \epsilon$ and $g_i(x) \leq 0$ for $i = 1, \dots, m$.

\bar{x} is a strict local minimum for Problem $P(\lambda) \leftrightarrow$ there exists $\epsilon > 0$ such that $\theta(x, \lambda) > \theta(\bar{x}, \lambda)$ for each $\bar{x} \neq x \in X$ such that $\|x - \bar{x}\| < \epsilon$.

Next, we need to provide suitable tangential approximations to the set X at a point $x \in X$. Following Rockafellar [14], consider the contingent cone $K(x)$ and the cone of hypertangents $H(x)$ defined below:

$y \in K(x) \leftrightarrow$ there exist a sequence $\{y_k\}$ converging to y and a positive sequence $\{\lambda_k\}$ converging to 0 such that $x + \lambda_k y_k \in X$ for each k .

$y \in H(x) \leftrightarrow$ for each sequence $\{x_k\}$ in X converging to x , there exists a positive sequence $\{\lambda_k\}$ converging to 0 such that $x_k + \lambda_k y \in X$ for all $\lambda_k \in (0, \lambda_k)$

Note that $H(x)$ is a convex cone which is not necessarily closed and that $K(x)$ is a closed cone, but not necessarily convex. Further, $H(x) \subset K(x)$.

Theorem 2.1 below gives a sufficient condition for the existence of an exact penalty strict local minimum, where the closed convex cone $C(x)$ is defined by:

$$y \in C(x) \leftrightarrow \nabla g_i(x)^T y \leq 0 \quad \text{for each } i \in I(x)$$

Theorem 2.1

Let \bar{x} be feasible for Problem P and suppose that $\nabla f(\bar{x})^T y > 0$ for each $0 \neq y \in C(\bar{x}) \cap K(\bar{x})$. Then:

1. \bar{x} is a strict local minimum for Problem P .
2. there is a number $\lambda_0 > 0$ so that for all $\lambda \geq \lambda_0$, \bar{x} is a strict local minimum for Problem $P(\lambda)$.

Proof

Suppose by contradiction to part (1) that there exists a sequence $\{x_k\}$ converging to \bar{x} such that $x_k \neq \bar{x}$, $x_k \in X$, $g_i(x_k) \leq 0$ for $i = 1, \dots, m$, and $f(x_k) \leq f(\bar{x})$. Let $y_k = (x_k - \bar{x}) / \|x_k - \bar{x}\|$. Then, $\|y_k\| = 1$ and there exist a subsequence $\{y_k\}_K$ and a vector y as that $\|y\| = 1$ and $y_k \rightarrow y$ as $k \rightarrow \infty$ in K .

Then, $y \in K(\bar{x})$. Since $g_i(x_k) \leq 0 = g_i(\bar{x})$ for $i \in I(\bar{x})$, then

$$\nabla g_i(\bar{x})^t y_k + \frac{R_i(\bar{x}, x_k - \bar{x})}{\|x_k - \bar{x}\|} \leq 0 \quad (1)$$

where $R_i(\bar{x}, h)/\|h\| \rightarrow 0$ as $\|h\| \rightarrow 0$. By taking the limit of (1) as $k \rightarrow \infty$ in K , it follows that $\nabla g_i(\bar{x})^t y \leq 0$ for $i \in I(\bar{x})$. Therefore, $y \in C(\bar{x}) \cap K(\bar{x})$. Since $\|y\| = 1$, then by assumption, $\nabla f(\bar{x})^t y > 0$. But

$$\frac{f(x_k) - f(\bar{x})}{\|x_k - \bar{x}\|} = \nabla f(\bar{x})^t y_k + \frac{R(\bar{x}, x_k - \bar{x})}{\|x_k - \bar{x}\|} \quad (2)$$

where $R(\bar{x}, h)/\|h\| \rightarrow 0$ as $\|h\| \rightarrow 0$. Since $f(x_k) \leq f(\bar{x})$, the left hand side of (2) is nonpositive while the right hand side converges to a positive number as $k \rightarrow \infty$ in K . This contradiction implies that \bar{x} is a strict local minimum for Problem P.

To prove part (2), suppose by contradiction that there is a sequence $\{\lambda_k\}$ such that $\lambda_k \rightarrow \infty$ and \bar{x} is not a strict local minimum for Problem $P(\lambda_k)$. Thus, there is a sequence $\{x_k\}$ converging to \bar{x} so that $\bar{x} \neq x_k \in X$ and

$$\theta(x_k, \lambda_k) \leq \theta(\bar{x}, \lambda_k) = f(\bar{x}) \quad (3)$$

Again, let $y_k = (x_k - \bar{x})/\|x_k - \bar{x}\|$. As in the proof of part (1), there is a vector $y \in K(\bar{x})$ with $\|y\| = 1$ and a set K so that $y_k \rightarrow y$ as $k \rightarrow \infty$ in K . Now suppose by contradiction that for some $j \in I(\bar{x})$, $\nabla g_j(\bar{x})^t y > 0$. Since g_j is continuously differentiable, for k in K large enough, $g_j(x_k) > g_j(\bar{x}) = 0$. Hence, by (3)

$$f(x_k) + \lambda_k g_j(x_k) \leq \theta(x_k, \lambda_k) \leq f(\bar{x})$$

so that

$$\frac{f(x_k) - f(\bar{x})}{\|x_k - \bar{x}\|} + \lambda_k \frac{g_j(x_k) - g_j(\bar{x})}{\|x_k - \bar{x}\|} \leq 0 \quad (4)$$

for large k in K . As $k \in K$ goes to ∞ , the first term in (4) converges to $\nabla f(\bar{x})^t y$, $[g_j(x_k) - g_j(\bar{x})]/\|x_k - \bar{x}\|$ converges to $\nabla g_j(\bar{x})^t y > 0$, and $\lambda_k \rightarrow \infty$. Since this is impossible, we conclude that $\nabla g_i(\bar{x})^t y \leq 0$ for each $i \in I(\bar{x})$. Thus $y \in K(\bar{x}) \cap C(\bar{x})$ and so $\nabla f(\bar{x})^t y > 0$. Since

$$\frac{f(x_k) - f(\bar{x})}{\|x_k - \bar{x}\|} = \nabla f(\bar{x})^t y_k + \frac{R(\bar{x}, x_k - \bar{x})}{\|x_k - \bar{x}\|}$$

and since $R(\bar{x}, x_k - \bar{x})/\|x_k - \bar{x}\| \rightarrow 0$ and $\nabla f(\bar{x})^t y_k \rightarrow \nabla f(\bar{x})^t y > 0$, we conclude that $f(x_k) - f(\bar{x}) > 0$ for $k \in K$ large enough. But by (3), $f(x_k) \leq \theta(x_k, \lambda_k) \leq f(\bar{x})$, a contradiction. This completes the proof.

The assumption that $\nabla f(\bar{x})^t y > 0$ for each nonzero vector $y \in C(\bar{x}) \cap K(\bar{x})$ guarantees that \bar{x} is a strict local minimum for Problem P. It also acts as a qualification that ensures an exact penalty strict local minimum. Theorem 2.2 gives a similar result if \bar{x} is a strict local minimum to Problem P and satisfies a suitable constraint qualification that does not involve the objective function. Theorem 2.2 extends similar results of Pietrzykowski [12] and Han and Mangasarian [8]. The following lemma is needed to prove the theorem.

Lemma 2.1

Let \bar{x} be feasible to Problem P and suppose that there is a vector $y \in H(\bar{x})$ such that $\nabla g_i(\bar{x})^t y < 0$ for each $i \in I(\bar{x})$. Let x_λ be a local optimal solution to Problem P(λ). If $x_\lambda \rightarrow \bar{x}$ as $\lambda \rightarrow \infty$, then x_λ is feasible to Problem P for λ sufficiently large.

Proof

Suppose by contradiction that there exist a sequence $\{\lambda_k\}$ and a sequence $\{x_k\}$ so that $\lambda_k \rightarrow \infty$ and $x_k \rightarrow \bar{x}$, where x_k is a local optimal solution to Problem $P(\lambda_k)$ which is not feasible to Problem P . Since $x_k \rightarrow \bar{x}$ and $g_i(\bar{x}) \leq 0$ for all i , then $I_k^+ \cup I_k \subset I(\bar{x})$, where I_k^+ and I_k denote $I^+(x_k)$ and $I(x_k)$, respectively. From [7], the directional derivatives of $\theta(\cdot, \lambda_k)$ at x_k along y is given by:

$$\theta'(x_k, \lambda_k, y) = \nabla f(x_k)^t y + \lambda_k \left[\sum_{i \in I_k^+} \nabla g_i(x_k)^t y + \sum_{i \in I_k} (\nabla g_i(x_k)^t y)_+ \right] \quad (5)$$

Since g_i is continuously differentiable and $\nabla g_i(x)^t y < 0$ for $i \in I(\bar{x})$, then there is an $\varepsilon > 0$ so that $\nabla g_i(x_k)^t y < -\varepsilon$ for $i \in I(\bar{x})$ and for k sufficiently large. Thus, $(\nabla g_i(x_k)^t y)_+ = 0$ for $i \in I_k$ and from (5) we get:

$$\theta'(x_k, \lambda_k, y) = \nabla f(x_k)^t y + \lambda_k \sum_{i \in I_k^+} \nabla g_i(x_k)^t y < \nabla f(x_k)^t y - \varepsilon \lambda_k |I_k^+| \quad (6)$$

where $|I_k^+|$ is the number of elements in the set I_k^+ . Since x_k is not feasible to Problem P , then $|I_k^+| \geq 1$. Since $\lambda_k \rightarrow \infty$ and $\nabla f(x_k)^t y \rightarrow \nabla f(\bar{x})^t y$, then (6) implies that:

$$\theta'(x_k, \lambda_k, y) < 0 \quad (7)$$

for k large enough. But, $y \in H(\bar{x})$ and $x_k \rightarrow \bar{x}$ so that there is a $\mu_k > 0$ so that $x_k + \mu y \in X$ for each $\mu \in (0, \mu_k)$. In view of (7), x_k could not have been a local minimum for Problem $P(\lambda_k)$. This completes the proof.

Theorem 2.2

Let \bar{x} be a strict local minimum for Problem P and suppose there is a vector $y \in H(\bar{x})$ such that $\nabla g_i(\bar{x})^t y < 0$ for each $i \in I(\bar{x})$. Then, there is a $\lambda_0 > 0$ so that \bar{x} is a strict local minimum for Problem $P(\lambda)$ for all $\lambda \geq \lambda_0$.

Proof

By Pietrzykowski's theorem [13], there is a number $\lambda_1 > 0$ so that for $\lambda > \lambda_1$ there exist x_λ and $\varepsilon(\lambda)$ such that:

$$\|x_\lambda - \bar{x}\| < \varepsilon(\lambda) \quad (8)$$

$$\lim_{\lambda \rightarrow \infty} \varepsilon(\lambda) = 0 \quad (9)$$

$$\theta(x_\lambda, \lambda) \leq \theta(x, \lambda) \text{ for all } x \in X \text{ with } \|x - \bar{x}\| < \varepsilon(\lambda) \quad (10)$$

By (8) and (9), $x_\lambda \rightarrow \bar{x}$ as $\lambda \rightarrow \infty$. From (8) and (10), it follows that x_λ is a local minimum for Problem $P(\lambda)$. In view of this and the assumptions of the theorem, it follows that Lemma 2.1 applies, and hence x_λ is feasible to Problem P for λ sufficiently large. Thus, from (10) we get:

$$f(x_\lambda) = \theta(x_\lambda, \lambda) \leq \theta(\bar{x}, \lambda) = f(\bar{x})$$

Since x_λ is feasible for P and $x_\lambda \rightarrow \bar{x}$, then $f(x_\lambda) = f(\bar{x})$ for λ large enough. But, since \bar{x} is a strict local minimum for Problem P , then there is a number $\lambda_0 > 0$ so that $x_\lambda = \bar{x}$ for $\lambda \geq \lambda_0$. Thus, for $\lambda \geq \lambda_0$, \bar{x} is a local minimum for Problem $P(\lambda)$. We wish to show that it is strict. If not, there exist a sequence $\{\lambda_k\}$ and a sequence $\{x_k\}$ so that $\lambda_k \rightarrow \infty$, $\bar{x} \neq x_k \rightarrow \bar{x}$, where x_k is a local minimum for Problem $P(\lambda_k)$. By Lemma 2.1, for k large enough, x_k is feasible to Problem P . However, since \bar{x} is a local minimum for Problem $P(\lambda)$ for λ sufficiently large, then $f(\bar{x}) = \theta(\bar{x}, \lambda_k) = \theta(x_k, \lambda_k) = f(x_k)$. We have thus exhibited a sequence $\{x_k\}$ feasible to Problem P so that $\bar{x} \neq x_k \rightarrow \bar{x}$ and $f(x_k) = f(\bar{x})$. This contradicts the strict local optimality of \bar{x} for Problem P , and the proof is now complete.

3. A Globally Exact Penalty Function

In this section, we present our main theorem which asserts the existence of a globally exact penalty function. This is done by requiring the set X to be compact, in addition to the existence of a suitable qualification that guarantees a strict local exact penalty.

Theorem 3.1

Consider Problem P and suppose that the set X is compact. Denote the set of global optimal solutions $\{x_1, \dots, x_n\}$ to Problem P by Q . Suppose that for each $x_j \in Q$ one of the following two conditions hold:

- a. $\nabla f(x_j)^t y > 0$ for each $0 \neq y \in C(x_j) \cap K(x_j)$
- b. there exists a vector $y \in H(x_j)$ such that $\nabla g_i(x_j)^t y < 0$ for all $i \in I(x_j)$

Then there exists a number $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, x_λ is a global optimal solution to Problem $P(\lambda)$ if and only if $x_\lambda \in Q$.

Proof

Denote the optimal objective value to Problem P by \bar{f} and consider the family of sets $A(\bullet)$ and $B(\bullet)$ defined below:

$$A(\lambda) = \{x: \theta(x, \lambda) - \bar{f} > 0\} \quad (11)$$

$$B(\lambda) = A(\lambda) \cup Q \quad (12)$$

We first show that $B(\lambda)$ is open in the relative topology of X for λ sufficiently large, that is, given $x \in B(\lambda)$ there exists an open neighborhood

$N_\lambda(x)$ around x so that $X \cap N_\lambda(x) \subset B(\lambda)$. Since θ is continuous, then $A(\lambda)$ is open so that the existence of the desired neighborhood is clear for $x \in A(\lambda)$. Now suppose that $x = x_j \in Q$. From Theorems 2.1 and 2.2, it follows under conditions (a) or (b) that x_j is a strict local minimum for Problem $P(\lambda)$ for λ sufficiently large. Thus, there exists a neighborhood $N_\lambda(x_j)$ so that $\bar{f} = \theta(x_j, \lambda) < \theta(y, \lambda)$ for each $x_j \neq y \in N_\lambda(x_j) \cap X$, which shows that $N_\lambda(x_j) \cap X \subset B(\lambda)$.

We have thus proved that there is a number $\lambda_1 > 0$ so that the collection $\{B(\lambda): \lambda \geq \lambda_1\}$ is a family of open sets relative to X . Next, we show that this family covers X . Let $x \in X$ and consider the following three cases:

Case 1: $f(x) > \bar{f}$

Here, $\theta(x, \lambda) \geq f(x) > \bar{f}$ for all $\lambda \geq 0$
so that $x \in A(\lambda) \subset B(\lambda)$ for all $\lambda \geq 0$.

Case 2: $f(x) < \bar{f}$

There must exist an index i such that $g_i(x) > 0$. Thus, for λ large enough, $\theta(x, \lambda) \geq f(x) + \lambda g_i(x) > \bar{f}$ so that $x \in A(\lambda) \subset B(\lambda)$.

Case 3: $f(x) = \bar{f}$

If $g_i(x) > 0$ for some i , as in Case 2, $x \in B(\lambda)$ for $\lambda > 0$.

If $g_i(x) \leq 0$ for $i = 1, \dots, m$ so that x is feasible to Problem P , then $x \in Q$. Thus, $x \in B(\lambda)$ for each λ .

Since X is compact, this relatively open cover has a finite subcover. Let λ_0 be the largest λ in this subcover. Noting that $\lambda' \geq \lambda$ implies that $B(\lambda) \subset B(\lambda')$, then

$$X \subset B(\lambda) = A(\lambda) \cup Q \quad \text{for all } \lambda \geq \lambda_0$$

The above set inclusion can be restated as follows. If $\lambda \geq \lambda_0$ and $x \in X$ then either $\theta(x, \lambda) > \bar{f}$ or else $x \in Q$ in which case $\theta(x, \lambda) = \bar{f}$. This is the desired result and the proof is complete.

The following example shows that in order to validate the conclusion of the above theorem, the qualification given by (a) or (b) in Theorem 3.1 must hold for each global optimal solution to Problem P.

Example 3.1

Problem P: minimize $f(x)$
 subject to $g(x) \leq 0$
 $x \in X$

where,

$$f(x_1, x_2) = -x_1^2 - x_2^2$$

$$g(x_1, x_2) = \begin{cases} x_2 - (x_1 - 1)^2 & \text{if } x_1 \leq 1 \\ x_2 + (x_1 - 1)^2 & \text{if } x_1 \geq 1 \end{cases}$$

$$X = \{(x_1, x_2): x_1 + x_2 \leq 2, x_1, x_2 \geq 0\}$$

Note that the set of global optimal solutions Q to Problem P is given by $\{(0,1), (1,0)\}$. Thus, we have:

At $x_1 = (0,1)^t$

$$C(x_1) = \{(y_1, y_2): 2y_1 + y_2 \leq 0\}$$

$$K(x_1) = \text{cl}H(x_1) = \{(y_1, y_2): y_1 \geq 0\}$$

Note that $0 \neq y \in C(x_1) \cap K(x_1)$ implies that $y_2 < 0$, so that $\nabla f(x_1)^t y > 0$. Also, there exists a vector $y \in H(x_1)$ so that $\nabla g(x_1)^t y < 0$, say $y = (0, -1)^t$. Therefore, both conditions (a) and (b) of Theorem 3.1 hold at x_1 .

At $x_2 = (1,0)^t$

$$C(x_2) = \{(y_1, y_2): y_2 \leq 0\}$$

$$K(x_2) = \text{cl}H(x_2) = \{(y_1, y_2): y_2 \geq 0\}$$

4. Estimating the Size of the Penalty Parameter

Theorem 4.1 gives some insight into determining a lower bound on the penalty parameter in terms of the Kuhn-Tucker multipliers and in terms of suitable lower bounds of the functions f and g_+ . Conclusion (1) asserts the existence of a Kuhn-Tucker multiplier vector at an optimal solution to Problem P. This is assured by assumptions (a) and (b). Here, the former acts as a qualification and the latter enables us to use separation of disjoint convex sets. We note that convexity of $K(x_j)$ is not very restrictive, and indeed holds if X is convex or smooth at x_j . Similar optimality conditions can be found in Bazaraa and Goode [1], Guignard [6], and Mangasarian [10, P. 168]. Conclusion (2) of the Theorem shows the existence of a strict exact local penalty if the penalty parameter exceeds the value of each of the Kuhn-Tucker Lagrangian multipliers. Here, again, assumption (a) is used. This assumption can be replaced by a suitable second order sufficiency condition. A similar result, in the absence of the set X , can be found in Han and Mangasarian [8]. Conclusions (3) and (4) yield the form of the size of the penalty parameter needed for a global exact penalty.

Theorem 4.1

Consider Problem P and suppose that the set X is compact. Denote the set of global optimal solutions $\{x_1, \dots, x_h\}$ to Problem P by Q and denote $f(x_j)$ for $j \in Q$ by \bar{f} . Suppose that for each $x_j \in Q$ the following conditions hold:

$$a. \quad \forall f(x_j)^t y > 0 \quad \text{for each } 0 \neq y \in C(x_j) \cap K(x_j).$$

- b. $K(x_j)$ is convex.

Then:

1. For each $x_j \in Q$ there exist scalars $P_{ij} \geq 0$ for $i \in I(x_j)$ such that:

$$[\nabla f(x_j) + \sum_{i \in I(x_j)} P_{ij} \nabla g_i(x_j)]^t y \geq 0 \quad \text{for } y \in K(x_j)$$

2. For each $x_j \in Q$, there exists a $\delta_j > 0$ such that $x \neq x_j$, $\|x - x_j\| < \delta_j$, and $\lambda \geq \lambda_j$ imply that $\theta(x, \lambda) > \theta(x_j, \lambda) = \bar{f}$, where $\lambda_j > \max \{P_{ij} : i \in I(x_j)\}$.

3. There exists a number $\epsilon > 0$ so that $\sum_{i=1}^m g_i(x)_+ \geq \epsilon$ for each $x \in A \cap B$, where $A = \{x : f(x) \leq \bar{f}\}$, $B = \{x \in X : \|x - x_j\| \geq \delta_0 \text{ for } j = 1, \dots, h\}$, and $\delta_0 = \min \{\delta_j : 1 \leq j \leq h\}$.

4. For $\lambda \geq \lambda_0$, x_λ is a global optimal solution to Problem $P(\lambda)$ if and only if $x_\lambda \in Q$, where $\lambda_0 = \max \{\lambda_1, \dots, \lambda_h, \frac{\bar{f}+b}{\epsilon}\}$ and b is such that $f(x) > -b$ for each $x \in X$.

Proof

Part (1)

This part is equivalent to showing that $-\nabla f(x_j) \in K^*(x_j) + C^*(x_j)$, where $C^*(x_j) = \{ \sum_{i \in I(x_j)} \alpha_{ij} \nabla g_i(x_j) : \alpha_{ij} \geq 0 \text{ for } i \in I(x_j) \}$ and $K^*(x_j)$ is the polar cone of $K(x_j)$, that is, $K^*(x_j) = \{y : y^t z \leq 0 \text{ for each } z \in K(x_j)\}$. If this were not the case, by convexity of $K^*(x_j) + C^*(x_j)$, there exist a nonzero vector c and a scalar α so that:

$$-c^t \nabla f(x_j) \geq \alpha \quad (13)$$

$$c^t y \leq \alpha \quad \text{for each } y \in K^*(x_j) + C^*(x_j) \quad (14)$$

Since $0 \in K^*(x_j) + C^*(x_j)$, then $\alpha \geq 0$ from (14). Thus, by (13), $c^t \nabla f(x_j) \leq 0$.

Letting $y = \sum_{i \in I(x_j)} \alpha_{ij} \nabla g_i(x_j)$ in (14), where $\alpha_{ij} \geq 0$ for $i \in I(x_j)$, it follows that $\sum_{i \in I(x_j)} \alpha_{ij} c^t \nabla g_i(x_j) \leq \alpha$. Since this is true for all $\alpha_{ij} \geq 0$, it

follows that $c^t \nabla g_i(x_j) \leq 0$ for each $i \in I(x_j)$ so that $c \in C(x_j)$. Now, consider $z \in K^*(x_j)$. Then $c^t z \leq 0$ because otherwise (14) would not hold for $y = \lambda z$ for sufficiently large $\lambda > 0$. Since $c^t z \leq 0$ for each $z \in K^*(x_j)$, then, $z \in K^{**}(x_j)$, the polar of $K^*(x_j)$. However, since $K(x_j)$ is a closed convex cone, then $K(x_j) = K^{**}(x_j)$ [2, P. 52].

To summarize, we exhibited a nonzero vector $c \in C(x_j) \cap K(x_j)$ with the property $c^t \nabla f(x_j) \leq 0$. This violates assumption (a). Thus $-\nabla f(x_j) \in K^*(x_j) + C^*(x_j)$, and part (1) follows.

Part (2)

We first show that x_j is a strict local minimum for Problem $P(\lambda_j)$. Suppose, by contradiction, that this is not the case. Then, there is a sequence $\{x_k\}$ in X so that $x_k \rightarrow x_j$, $x_k \neq x_j$, and

$$f(x_k) + \lambda_j \sum_{i=1}^m g_i(x_k)_+ = \theta(x_k, \lambda_j) \leq \theta(x_j, \lambda_j) = \bar{f} \quad (15)$$

Let $y_k = (x_k - x_j) / \|x_k - x_j\|$. Then there is an index set K of positive integers such that $y_k \rightarrow y$ as $k \in K$ approaches ∞ . Note that $\|y\| = 1$ and $y \in K(x_j)$. It can be easily verified from (15) that

$$\nabla f(x_j)^t y + \lambda_j \sum_{i \in I(x_j)} (\nabla g_i(x_j)^t y)_+ \leq 0 \quad (16)$$

From Part (1) and (16) above, we get:

$$0 \geq \lambda_j \sum_{i \in I(x_j)} (\nabla g_i(x_j)^t y)_+ - \sum_{i \in I(x_j)} P_{ij} \nabla g_i(x_j)^t y \geq \sum_{i \in I(x_j)} (\lambda_j - P_{ij}) (\nabla g_i(x_j)^t y)_+$$

Since $\lambda_j > P_{ij}$, the above inequality implies that $(\nabla g_i(x_j)^t y)_+ = 0$, and hence $\nabla g_i(x_j)^t y \leq 0$ for $i \in I(x_j)$. Therefore, $y \in C(x_j) \cap K(x_j)$. By assumption (a), $\nabla f(x_j)^t y > 0$, which is not possible from (16). Thus x_j is a strict local

minimum of Problem $P(\lambda_j)$, and there must exist a number $\delta_j > 0$ so that $x_j \neq x$, $\|x - x_j\| < \delta_j$ implies that $\theta(x, \lambda_j) > \theta(x_j, \lambda_j) = \bar{f}$. Since $\theta(x, \lambda) \geq \theta(x, \lambda_j)$ for $\lambda > \lambda_j$, part (2) follows.

Part (3)

Consider the following sets:

$$\bar{B} = \{x: \|x - x_j\| < \delta_0 \text{ for some } x_j \in Q\}$$

$$E(v) = \{x: \sum_{i=1}^m g_i(x)_+ > v\}, \quad v > 0$$

$$F(v) = E(v) \cup \bar{B}$$

Obviously, \bar{B} , $E(v)$, and $F(v)$ are all open for any $v > 0$. Furthermore, the open family $\bigcup_{v>0} F(v)$ covers $A \cap X$. To show this, let $x \in A \cap X$. If $\sum_{i=1}^m g_i(x)_+ = 0$ then x must belong to Q and hence $x \in \bar{B} \subset F(v)$ for all $v > 0$. If $\sum_{i=1}^m g_i(x)_+ > 0$, then $x \in F(v)$ for any $v < \sum_{i=1}^m g_i(x)_+$. Therefore, there exists a finite subcover, say $A \cap X \subset E(\epsilon) \cup \bar{B}$ for some $\epsilon > 0$. In other words, if $x \in X$ is such that $f(x) \leq \bar{f}$, then either $\sum_{i=1}^m g_i(x)_+ > \epsilon$ or else $\|x - x_j\| < \delta_0$ for some $x_j \in Q$. Thus part (3) follows.

Part (4)

Noting part (2), it suffices to show that $\theta(x, \lambda) > \bar{f}$ for $x \in B$ and $\lambda \geq \lambda_0$. If $f(x) > \bar{f}$, the result is immediate. Now suppose that $f(x) \leq \bar{f}$ so that $x \in A \cap B$. By part (3), $\sum_{i=1}^m g_i(x)_+ \geq \epsilon$. Thus:

$$\theta(x, \lambda) = f(x) + \lambda \sum_{i=1}^m g_i(x)_+ > -b + \lambda \epsilon \geq -b + \left(\frac{\bar{f} + b}{\epsilon}\right) \epsilon = \bar{f}$$

This completes the proof.

REFERENCES

1. Bazaraa, M. S. and J. J. Goode, "Necessary Optimality Criteria in Mathematical Programming in the Presence of Differentiability," Journal of Mathematical Analysis and Applications, Vol. 40, No. 3, pp. 609-621, 1972.
2. Bazaraa, M. S. and C. M. Shetty, Nonlinear Programming: Theory and Algorithms, John Wiley and Sons, New York, 1979.
3. Bertsekas, D. P., "Necessary and Sufficient Conditions for a Penalty Method to be Exact," Mathematical Programming, Vol. 9, pp. 87-99, 1975.
4. Evans, J. P., F. J. Gould, and J. W. Tolle, "Exact Penalty Functions in Nonlinear Programming," Mathematical Programming, Vol. 4, pp. 72-97, 1973.
5. Fletcher, R., "An Exact Penalty Function for Nonlinear Programming With Inequalities," Mathematical Programming, Vol. 5, No. 2, pp. 129-150, 1973.
6. Guignard, M., "Generalized Kuhn-Tucker Conditions for Mathematical Programming Problems in a Banach Space," SIAM Journal on Control, Vol. 7, pp. 232-241, 1969.
7. Han, S. P., "A Globally Convergent Method for Nonlinear Programming," Journal of Optimization Theory and Applications, Vol. 22, No. 3, pp. 297-309, 1977.
8. Han, S. P. and O. L. Mangasarian, "Exact Penalty Functions in Nonlinear Programming," Mathematics Research Center, University of Wisconsin, Madison, Wisconsin (to appear in Mathematical Programming).
9. Howe, S., "New Conditions for Exactness of a Simple Penalty Function," SIAM Journal on Control, Vol. 11, No. 2, pp. 378-381, 1973.
10. Mangasarian, O. L., Nonlinear Programming, McGraw-Hill, New York, 1969.
11. McCormick, G. P., "An Idealized Exact Penalty Function," in Nonlinear Programming 3, edited by O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, Academic Press, New York, 1978.
12. Pietrzykowski, T., "An Exact Potential Method for Constrained Maxima," SIAM Journal on Numerical Analysis, Vol. 6, No. 2, pp. 299-304, 1969.
13. Pietrzykowski, T., "The Potential Method for Conditioned Maxima in Locally Compact Metric Spaces," Numerische Mathematik, Vol. 14, pp. 325-329, 1970.
14. Rockafellar, R. T., "Theory of Subgradients and its Applications to Problems of Optimization," Lecture Notes, University of Montreal, Canada, February - March, 1978.
15. Zangwill, W. I., "Nonlinear Programming via Penalty Functions," Management Science, Vol. 13, pp. 344-358, 1967.

An Extension of Armijo's Rule to Minimax and Quasi-Newton Methods for Constrained Optimization

Mokhtar S. Bazaraa[†] and Jamie J. Goode^{††}

In this study, we propose an algorithm for solving a minimax problem over a closed convex set. At each iteration a direction is found by solving a problem having a quadratic objective function and then a suitable step size along that direction is taken through an extension of Armijo's approximate line search technique. We show that each accumulation point is a Kuhn-Tucker solution and give a condition that guarantees convergence of the whole sequence of iterates. The special cases of unconstrained and constrained nonlinear programming are studied. Through suitable choices of the quadratic form, our procedure retrieves various steepest descent and quasi-Newton algorithms for unconstrained optimization. For the constrained case and using an exact penalty function to handle the nonlinear constraints, our algorithm resembles that of Han, but differs from it both in the direction-finding and the step-determination processes.

Key Words: Minimax Problems, Unconstrained and Constrained Nonlinear Programming, Armijo's Rule, Global Convergence, Quasi-Newton Methods, Steepest Descent Methods

[†] School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia. This author's work is supported and AFOSR grant #F49620-79-C-0120.

^{††} School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia.

The authors wish to thank Professor Jon Spingarn of the School of Mathematics at Georgia Institute of Technology for suggesting the line of proof of Lemma 4.2.

1. INTRODUCTION

In this paper we consider the following problem:

$$\begin{aligned} P: \quad & \text{minimize} \quad \theta(x) \\ & \text{subject to} \quad x \in X \end{aligned}$$

Here X is a closed convex set in R^n and θ is of the form:

$$\begin{aligned} \theta(x) &= f(x) + \sum_{j=1}^{\ell} \alpha_j(x) \\ \alpha_j(x) &= \max_{i \in I_j} \{\beta_{ij}(x)\} \quad j=1, \dots, \ell \end{aligned}$$

We assume that I_j is a finite set of positive integers and that the functions f and β_{ij} are continuously differentiable on an open set S that contains X .

Minimax problems of the above type arise in various contexts and have been studied by many authors. For an excellent exposition of this subject, both from theoretical and algorithmic points of view, the reader is referred to the works of Danskin [5], Dem'yanov [6], and Dem'yanov and Malozemov [7]. The reader is also referred to Chatelon, Hearn and Lowe [4] and Han [11] for the special case of unrestricted minimax problems and to Madsen and Schjaer-Jacobson [15] for the linearly constrained minimax problem.

In addition to Problem P itself, the special case where $\alpha_j(x) = 0$ for each j has been extensively studied. In [10], Goldstein described a gradient projection method for solving the problem to minimize $f(x)$ subject to $x \in X$, and a similar procedure was proposed by Levitin and Polyak [14]. These methods proceed as follows. Given x_k , the next point x_{k+1} is determined by

projecting $x_k - \lambda_k \nabla f(x_k)$ on X , where λ_k is a suitable step size that depends on the Lipschitz constant associated with ∇f . In [16], McCormick proposed an anti-jamming procedure for solving the problem in the special case where X consists of bounds on the variables, and in a joint paper with Topia [17], the procedure was extended to the case of a general closed convex set. In [3], Bertsekas further studied this class of methods with emphasis on the choice of the step size. He also described various ways of achieving super-linear convergence.

We also note the class of subgradient optimization methods for solving the problem to minimize $f(x)$ subject to $x \in X$ in the case where f is convex but not necessarily differentiable. Similar to the methods described above, given a point x_k , x_{k+1} is computed by projecting $x_k - \lambda_k \xi_k$ on X , where ξ_k is any subgradient of f at x_k . For conditions on the step size λ_k that assure convergence, the reader is referred to Polyak [18,19].

In this paper, we propose an algorithm for solving Problem P. We concern ourselves primarily with global convergence properties of the algorithm. Local and superlinear convergence through appropriate choices of the quadratic approximation are only discussed very briefly. At any iteration the algorithm solves a subproblem that finds a search direction and then takes a suitable step along that direction. In the case where X is polyhedral, the direction finding problem reduces to a quadratic program, and in that respect, our method resembles quasi-Newton procedures for solving constrained nonlinear programs. Our direction-finding problem is also similar to the one proposed by Han [11] for solving minimax problems and primarily differs from it in the inclusion of the set X . The step size along the search direction is obtained through an extension of Armijo's [1] rule that handles the nondifferentiability of the objective function θ .

In Section 2, we introduce an approximation to the directional derivative that maintains continuity. This approximation is the key tool in overcoming the difficulties associated with discontinuity of the directional derivative in determining a search direction. In Section 3, we present our algorithm and in Section 4, we prove its convergence to a stationary point. Section 5 is devoted to various specializations of our method. Particularly, we discuss the cases of unconstrained and constrained nonlinear programming. For unconstrained problems, depending on the choice of the direction-finding problem, our algorithm gives rise to different steepest descent and Newton-type algorithms coupled with the efficient Armijo's step size rule. For constrained programs, linear constraints are handled by the set X and nonlinear constraints are treated by an exact penalty function. As a byproduct, a slight modification to the method of finding a search direction for the class of quasi-Newton methods is suggested. This modification overcomes the difficulty of premature termination in case the linearization of the feasible region at the current point is empty.

2. APPROXIMATING THE DIRECTIONAL DERIVATIVE

Note that the objective function θ is not differentiable but has a derivative along any direction d . Particularly, the directional derivative $\theta'(x, d)$ is given by:

$$\theta'(x, d) = \nabla f(x)^t d + \sum_{j=1}^{\ell} \max_{i \in I_j(x)} \{ \nabla \beta_{ij}(x)^t d \} \quad (2.1)$$

where

$$I_j(x) = \{ i : \beta_{ij}(x) = \alpha_j(x) \} \quad (2.2)$$

Since θ' is not continuous in x , a difficulty which could ultimately lead to jamming, we introduce the following approximate directional derivative $\theta^*(x, d)$ which is continuous in both x and d :

$$\theta^*(x, d) = f(x) + \nabla f(x)^t d + \sum_{j=1}^{\ell} \max_{i \in I_j} \{ \beta_{ij}(x) + \nabla \beta_{ij}(x)^t d \} - \theta(x) \quad (2.3)$$

If the functions f and β_{ij} satisfy a strong version of differentiability, which we refer to as upper uniform differentiability, then a one-sided second order approximation of $\theta(x+\lambda d)$ using the pseudo directional derivative $\theta^*(x, d)$ can be devised. As will be seen in the remainder of the paper, this approximation is instrumental in proving convergence of the proposed algorithm.

Definition 2.1

Let S be an open convex set in \mathbb{R}^n and let $f: \mathbb{R}^n \rightarrow \mathbb{R}$. f is said to be upper uniformly differentiable in S if f is continuously differentiable in S and if there is a number $K_f > 0$ so that

$$f(x+d) \leq f(x) + \nabla f(x)^t d + 1/2 K_f \|d\|^2 \quad (2.4)$$

whenever $x, x+d \in S$.

Note that if f has a Lipschitz continuous derivative in S then it is upper uniformly differentiable. That is, if there is a number $1/2 K_f$ so that

$$\|\nabla f(y) - \nabla f(x)\| \leq 1/2 K_f \|x-y\| \quad \text{for } x, y \in S$$

then for x and d such that $x, x+d \in S$, by the mean value theorem, we can write

$$f(x+d) - f(x) = \nabla f(y)^t d$$

for some y between x and $x+d$. But then

$$\begin{aligned} f(x+d) - f(x) - \nabla f(x)^t d &= [\nabla f(y) - \nabla f(x)]^t d \\ &\leq 1/2 K_f \|y-x\| \|d\| \\ &\leq 1/2 K_f \|d\|^2 \end{aligned}$$

and hence f is upper uniformly differentiable in S .

Lemma 2.1

Let S be an open convex set in R^n and suppose that f and β_{ij} for $i \in I_j$ and $j=1, \dots, \ell$ are upper uniformly differentiable in S . Then, there is a number $K > 0$ so that the following hold for all $x, x+d \in S$:

1. $\theta(x+d) \leq \theta(x) + \theta^*(x, d) + 1/2 K \|d\|^2$
2. $\theta^*(x, \lambda d) \leq \lambda \theta^*(x, d)$ for all $\lambda \in [0, 1]$
3. $\theta(x, \lambda d) \leq \theta(x) + \lambda \theta^*(x, d) + 1/2 \lambda^2 K \|d\|^2$ for all $\lambda \in [0, 1]$

Proof

Since f and β_{ij} are upper uniformly differentiable, then there exist scalars K_f and $K_{ij} > 0$ so that:

$$f(x+d) \leq f(x) + \nabla f(x)^t d + 1/2 K_f \|d\|^2 \quad (2.5)$$

$$\beta_{ij}(x+d) \leq \beta_{ij}(x) + \nabla \beta_{ij}(x)^t d + 1/2 K_{ij} \|d\|^2 \quad (2.6)$$

for all $x, x+d \in S$. Let $K_j = \max_{i \in I_j} K_{ij}$ and suppose that $x, x+d \in S$. Then from (2.6) we get:

$$\begin{aligned} \beta_{ij}(x+d) &\leq \beta_{ij}(x) + \nabla \beta_{ij}(x)^t d + 1/2 K_{ij} \|d\|^2 \\ &\leq \max_{r \in I_j} \{\beta_{rj}(x) + \nabla \beta_{rj}(x)^t d\} + 1/2 K_j \|d\|^2 \\ &= \alpha_j(x) + \alpha_j^*(x, d) + 1/2 K_j \|d\|^2 \end{aligned} \quad (2.7)$$

where

$$\alpha_j^*(x, d) = \max_{r \in I_j} \{\beta_{rj}(x) + \nabla \beta_{rj}(x)^t d\} - \alpha_j(x) \quad (2.8)$$

Since (2.7) holds for each $i \in I_j$, then

$$\alpha_j(x+d) \leq \alpha_j(x) + \alpha_j^*(x,d) + 1/2 K_j \|d\|^2 \quad (2.9)$$

Summing (2.5) and (2.9) for $j=1, \dots, \ell$ and noting (2.3) and (2.8) we get:

$$\theta(x+d) \leq \theta(x) + \theta^*(x,d) + 1/2 K \|d\|^2$$

where

$$K = K_f + \sum_{j=1}^{\ell} K_j \quad (2.10)$$

which proves part (1). Now let $\lambda \in [0,1]$ and consider $\alpha_j^*(x, \lambda d)$ below:

$$\begin{aligned} \alpha_j^*(x, \lambda d) &= \max_{i \in I_j} \{ \beta_{ij} + \lambda \nabla \beta_{ij}(x)^t d \} - \alpha_j(x) \\ &= \max_{i \in I_j} \{ \lambda [\beta_{ij} + \nabla \beta_{ij}(x)^t d] + (1-\lambda) \beta_{ij} \} - \alpha_j(x) \\ &\leq \lambda [\alpha_j^*(x) + \alpha_j(x)] + (1-\lambda) \alpha_j(x) - \alpha_j(x) \\ &= \lambda \alpha_j^*(x) \end{aligned} \quad (2.11)$$

Thus, part (2) follows immediately from (2.11). Now part (3) is obvious from parts (1) and (2) and the proof is complete.

It is well known that

$$\theta(x+\lambda d) = \theta(x) + \lambda \theta'(x, d) + o(d, \lambda)$$

where

$$\frac{o(d, \lambda)}{\lambda} \rightarrow 0 \text{ as } \lambda \rightarrow 0^+$$

uniformly in d with $\|d\| = 1$ (see for example Dem'yanov and Molozemov [7, p.53]). However, conclusion (3) of the lemma would be false if $\theta^*(x, d)$ is replaced by $\theta'(x, d)$. This is evident by considering $\theta(x) = |x|$ which corresponds to $f(x) = 0$, $\ell = 1$, $\beta_{11}(x) = x$ and $\beta_{21}(x) = -x$.

3. DESCRIPTION OF THE ALGORITHM

We present below a procedure for solving Problem P.

Initialization Step

Choose $x_1 \in X$ and choose δ_1, δ_2 with $0 < 2\delta_1 < \delta_2$. Let $k=1$ and go to Step 1.

Step 1 (Find a direction)

Given $x_k \in X$, let B_k be a positive semidefinite matrix satisfying

$$d^t B_k d \leq \delta_2 \|d\|^2 \quad \text{for all } d \in \mathbb{R}^n$$

Consider Problem $D(x_k)$ below:

$$\begin{aligned} D(x_k) : \quad & \text{minimize} \quad \theta^*(x_k, d) + 1/2 d^t B_k d \\ & \text{subject to } x_k + d \in X \end{aligned}$$

If Problem $D(x_k)$ has an unbounded optimal solution go to Step 2. Otherwise, let d_k be an optimal solution to Problem $D(x_k)$. If $\theta^*(x_k, d_k) = 0$; stop. If $\theta^*(x_k, d_k) \leq -\delta_1 \|d_k\|^2$, go to Step 3. If $\theta^*(x_k, d_k) > -\delta_1 \|d_k\|^2$, go to Step 2.

Step 2 (Modify the search direction)

Replace B_k by $[1 - (2\delta_1/\delta_2)]B_k + 2\delta_1 I$. Let d_k be an optimal solution to Problem $D(x_k)$ and go to Step 3.

Step 3 (Find Armijo step size)

Given x_k and d_k , let m_k be the smallest nonnegative integer v such that:

$$\theta(x_k + (\frac{1}{2})^v d_k) - \theta(x_k) \leq (\frac{1}{2})^{v+1} \theta^*(x_k, d_k)$$

Let $x_{k+1} = x_k + (\frac{1}{2})^{m_k} d_k$. Replace k by $k+1$ and go to Step 1.

By convexity of X it is clear that the algorithm always generates feasible points to Problem P so that $x_k \in X$ for each k . The direction-finding problem is equivalent to:

$$\begin{aligned} D'(x_k) : \text{minimize} \quad & f(x_k) + \nabla f(x_k)^t d + \sum_{j=1}^{\ell} y_j - \theta(x_k) + \frac{1}{2} d^t B_k d \\ \text{subject to} \quad & y_j \geq \beta_{ij}(x_k) + \nabla \beta_{ij}(x_k)^t d \quad i \in I_j, j=1, \dots, \ell \\ & x_k + d \in X \end{aligned}$$

In the next section, we show that $\theta^*(x_k, d_k) = 0$ if and only if x_k is a Kuhn-Tucker point to Problem P' defined below:

$$\begin{aligned} P' : \text{minimize} \quad & f(x) + \sum_{j=1}^{\ell} y_j \\ \text{subject to} \quad & y_j \geq \beta_{ij}(x) \quad i \in I_j, j=1, \dots, \ell \\ & x \in X \end{aligned}$$

Since this latter problem is equivalent to Problem P , then the algorithm stops only when a Kuhn-Tucker solution is at hand.

If X is polyhedral, then Problem $D(x_k)$ is a convex quadratic program. Note that in Step 1, we do not require B_k to be positive definite. In fact,

the case where $B_k = 0$ is of special interest since it leads to a linear program. If the optimal solution is unbounded, however, B_k is modified slightly in Step 2 in order to guarantee a bounded optimal solution d_k . Note that the identity in Step 2 can be replaced by another sufficiently positive definite matrix if that is deemed more desirable.

Step 2 is also needed for cases where the pseudo directional derivative θ^* goes to zero too fast compared to $\|d_k\|^2$. This would cause the Armijo integers m_k 's to become large. Step 2 recomputes d_k with a positive definite quadratic form to prevent this and to assure the uniform upper bound on m_k given by Lemma 3.1. Note also that if Step 2 is used then the new vector d_k automatically satisfies $\theta^*(x_k, d_k) \leq -\delta_1 \|d_k\|^2$. It is also interesting to note that if $d_k^t B_k d_k \geq 2\delta_1 \|d_k\|^2$ at Step 1 then Step 2 is not needed. This follows directly from the fact that $0 \geq \theta^*(x_k, d_k) + \frac{1}{2} d_k^t B_k d_k$. Therefore, if B_k is chosen to be sufficiently positive definite so that $d^t B_k d \geq 2\delta_1 \|d\|^2$ for all $d \in \mathbb{R}^n$, then Step 2 is never used. As will be demonstrated in Section 5, in some special cases, we can devise schemes for generating a nonpositive definite matrix B_k in such a way that it is a priori guaranteed that $d_k^t B_k d_k \geq 2\delta_1 \|d_k\|^2$ which eliminates the need for Step 2.

Lemma 3.1

The integers m_k 's defined in Step 3 of the algorithm exist and $m_k \leq [y] + 1$, where $[y]$ is the greatest integer in y , and $y = \ln \left(\frac{K}{\delta_1} \right) / \ln 2$, where K is given by (2.10).

Proof

By part (3) of Lemma 2.1 we have:

$$\theta(x_k + \left(\frac{1}{2}\right)^v d_k) - \theta(x_k) \leq \left(\frac{1}{2}\right)^v \theta^*(x_k, d_k) + \left(\frac{1}{2}\right)^{2v+1} K \|d_k\|^2$$

But by the algorithm, we must have $\|d_k\|^2 \leq -\theta^*(x_k, d_k)/\delta_1$ so that

$$\theta(x_k + (\frac{1}{2})^v d_k) - \theta(x_k) \leq (\frac{1}{2})^{v+1} \theta^*(x_k, d_k) [2 - (\frac{1}{2})^v \frac{K}{\delta_1}] \quad (3.1)$$

The right hand side in (3.1) is less than $(\frac{1}{2})^{v+1} \theta^*(x_k, d_k)$ provided that $2 - (\frac{1}{2})^v \frac{K}{\delta_1} > 1$. Therefore,

$$\theta(x_k + (\frac{1}{2})^v d_k) - \theta(x_k) \leq (\frac{1}{2})^{v+1} \theta^*(x_k, d_k)$$

provided that $v \geq y = \frac{\ln(\frac{K}{\delta_1})}{\ln 2}$. Thus m_k exists and is bounded above by $[y] + 1$.

4. GLOBAL CONVERGENCE

In this section, we prove global convergence of the scheme described in Section 3. The following two lemmas are needed. Lemma 4.1 asserts that the algorithm stops only if the point at hand is a Kuhn-Tucker solution to Problem P', which is equivalent to Problem P. The second lemma shows that if \bar{x} , $\bar{x} + \bar{d} \in X$ and if $\{x_k\}$ in X converges to \bar{x} , then there is a direction d sufficiently close to \bar{d} such that $x_k + d \in X$ for large k .

Lemma 4.1

Let $\bar{x} \in X$. Then $(\bar{x}, \alpha(\bar{x}))$ is a Kuhn-Tucker solution to Problem P' if and only if $\theta^*(\bar{x}, \bar{d}) = 0$, where \bar{d} is any optimal solution to Problem $D(\bar{x})$ to minimize $\theta^*(\bar{x}, d) + \frac{1}{2} d^t B d$ subject to $\bar{x} + d \in X$ and where B is positive semidefinite.

Proof

Let \bar{d} be an optimal solution to problem $D(\bar{x})$. Further suppose that $\theta^*(\bar{x}, \bar{d}) = 0$. Since $d = 0$ is feasible to Problem $D(\bar{x})$ and has an objective value equal to 0, and since B is positive semidefinite, then $\bar{d}^t B \bar{d} = 0$. Thus, the optimal objective value is equal to 0 so that $\hat{d} = 0$ is an optimal solution to Problem $D(\bar{x})$. Therefore, $(\hat{d} = 0, \hat{y} = \alpha(\bar{x}))$ is an optimal solution to Problem $D'(\bar{x})$. This further implies that the Fritz John conditions stated in [2] hold at (\hat{d}, \hat{y}) . That is, there exist nonnegative scalars u_0 and v_{ij} , not all equal to 0, such that:

$$[u_0 \nabla f(\bar{x}) + u_0 B d + \sum_{j=1}^{\ell} \sum_{i \in I_j} v_{ij} \nabla \beta_{ij}(\bar{x})]^t (d - \hat{d}) \geq 0 \quad \text{if } \bar{x} + d \in X \quad (4.1)$$

$$u_0 - \sum_{i \in I_j} v_{ij} = 0 \quad j = 1, \dots, \ell \quad (4.2)$$

$$v_{ij}[\hat{y}_j - \beta_{ij}(\bar{x}) - \nabla \beta_{ij}(\bar{x})^t \hat{d}] = 0 \quad i \in I_j, \quad j = 1, \dots, \ell \quad (4.3)$$

Note that $u_0 > 0$ because if $u_0 = 0$ then by (4.2), $v_{ij} = 0$ for all i, j , which is impossible. Noting that $\hat{d} = 0$ and that $u_0 > 0$, (4.1), (4.2), and (4.3) show that $(\bar{x}, \alpha(\bar{x}))$ satisfy the Kuhn-Tucker conditions for Problem P'.

Conversely, suppose that $(\bar{x}, \hat{y} = \alpha(\bar{x}))$ is a Kuhn-Tucker solution to Problem P'. Then there exist scalars $u_{ij} \geq 0$ for $i \in I_j$ and $j = 1, \dots, \ell$ such that:

$$[\nabla f(\bar{x}) + \sum_{j=1}^{\ell} \sum_{i \in I_j} u_{ij} \nabla \beta_{ij}(\bar{x})]^t d \geq 0 \quad \text{if } \bar{x} + d \in X$$

$$\sum_{i \in I_j} u_{ij} = 1 \quad j = 1, \dots, \ell$$

$$u_{ij}[\hat{y}_j - \beta_{ij}(\bar{x})] = 0 \quad i \in I_j, \quad j = 1, \dots, \ell$$

These conditions are precisely (4.1), (4.2), and (4.3) with $\hat{d} = 0$, $u_0 = 1$, $v_{ij} = u_{ij}$. Therefore, $(\hat{d} = 0, \hat{y} = \alpha(\bar{x}))$ is a Kuhn-Tucker solution to Problem $D'(\bar{x})$. Since this problem is convex, then this solution is optimal. Clearly, Problems $D(\bar{x})$ and $D'(\bar{x})$ are equivalent and hence $\hat{d} = 0$ is an optimal solution to Problem $D(\bar{x})$. Thus the optimal objective value is equal to 0, and hence any optimal solution \bar{d} to Problem $D(\bar{x})$ must satisfy $\theta^*(\bar{x}, \bar{d}) = 0$. This follows because if $\theta^*(\bar{x}, \bar{d}) < -z$ for some $z > 0$, then

$$\theta^*(\bar{x}, \lambda \bar{d}) + \frac{1}{2} \lambda^2 \bar{d}^t B \bar{d} \leq -\lambda z + \frac{1}{2} \lambda^2 \bar{d}^t B \bar{d} < 0$$

for $\lambda > 0$ and sufficiently small, violating the fact that the optimal objective value for Problem $D(\bar{x})$ is equal to 0. This completes the proof.

Lemma 4.2

Let X be a convex set in \mathbb{R}^n and let $\bar{x} \in X$. Let $\bar{d} \neq 0$ be such that $\bar{x} + \bar{d} \in X$ and let $\{x_k\}$ be a sequence in X converging to \bar{x} . Then given an $\varepsilon > 0$, there exists a vector d such that $\|d - \bar{d}\| < \varepsilon$ and $x_k + d \in X$ for k sufficiently large.

Proof

Let $\text{ri}(X)$ denote the relative interior of X . Then there exists a point $y \neq \bar{x} + \bar{d}$ such that $y \in \text{ri}(X)$. Now consider d given by

$$d = \bar{d} + \delta \frac{(y - \bar{x} - \bar{d})}{\|y - \bar{x} - \bar{d}\|}, \text{ where}$$

$$\delta = \min \left\{ \frac{\varepsilon}{2}, \frac{\|y - \bar{x} - \bar{d}\|}{2} \right\}$$

Then

$$\begin{aligned} \bar{x} + d &= (\bar{x} + \bar{d}) + \delta \frac{(y - \bar{x} - \bar{d})}{\|y - \bar{x} - \bar{d}\|} \\ &= \frac{\delta}{\|y - \bar{x} - \bar{d}\|} y + \left(1 - \frac{\delta}{\|y - \bar{x} - \bar{d}\|}\right) (\bar{x} + \bar{d}) \end{aligned}$$

Thus, $\bar{x} + d$ is a convex combination of y and $\bar{x} + \bar{d}$ so that $\bar{x} + d \in \text{ri}(X)$. Therefore, there exists a $z > 0$ so that if $\|x + d - h\| < z$ and if h lies in the affine manifold generated by X then $h \in X$. Since x_k , \bar{x} , $\bar{x} + d \in X$, it is clear that $x_k + d$ is in the affine manifold generated by X . Now let $h = x_k + d$. Then $\|\bar{x} + d - h\| = \|\bar{x} - x_k\|$, and since $x_k \rightarrow \bar{x}$, it follows that $\|(\bar{x} + d) - (x_k + d)\| < z$ for k sufficiently large so that $x_k + d \in X$. This completes the proof.

Now we are ready to state our main convergence theorem. The theorem shows that each accumulation point \bar{x} corresponds to a Kuhn-Tucker solution $(\bar{x}, \alpha(\bar{x}))$ to Problem P' . As a corollary, we demonstrate that if \bar{x} is a strong

local minimum then indeed the whole sequence $\{x_k\}$ converges to \bar{x} . Here, \bar{x} is a strong local minimum to Problem P if there exists a number $\gamma > 0$ so that for each $\delta > 0$ there is a number $z(\delta) > 0$ so that

$$x \in X, \|x - \bar{x}\| < \gamma, \text{ and } \theta(x) - \theta(\bar{x}) < z(\delta) \Rightarrow \|x - \bar{x}\| < \delta \quad (4.4)$$

Theorem 4.1

Consider the algorithm described in Section 3 for solving Problem P. If the algorithm stops at iteration k then $(x_k, \alpha(x_k))$ is a Kuhn-Tucker point for Problem P'. Otherwise the algorithm generates an infinite sequence $\{(x_k, d_k)\}$. In this case, if (\bar{x}, \bar{d}) is an accumulation point, then:

1. $\lim_{k \rightarrow \infty} d_k = 0$ and in particular $\bar{d} = 0$.
2. $(\bar{x}, \alpha(\bar{x}))$ is a Kuhn-Tucker point for Problem P'.

Proof

If the algorithm stops at iteration k then $\theta^*(x_k, d_k) = 0$ and by Lemma 4.1 it follows that $(x_k, \alpha(x_k))$ is a Kuhn-Tucker point for Problem P'. Now suppose that the algorithm generates the infinite sequence $\{(x_k, d_k)\}$ and suppose that there is an infinite set K of positive integers such that $(x_k, d_k) \xrightarrow{K} (\bar{x}, \bar{d})$.

First, note that $\theta(x_k)$ is decreasing and that $\theta(x_k) \xrightarrow{K} \theta(\bar{x})$, and hence

$\lim_{k \rightarrow \infty} \theta(x_k) = \theta(\bar{x})$. Also we have

$$\theta(x_{k+1}) - \theta(x_k) \leq \left(\frac{1}{2}\right)^{m_{k+1}} \theta^*(x_k, d_k) \leq - \left(\frac{1}{2}\right)^{m_{k+1}} \delta_1 \|d_k\|^2 \quad \text{for all } k$$

and hence the right hand side must converge to 0. But by Lemma 3.1, m_k is bounded above so that $d_k \rightarrow 0$, and particularly $\bar{d} = 0$. This proves part (1).

Since $d^t B_k d \leq \delta_2 \|d\|^2$ for all $d \in \mathbb{R}^n$ and all k , then there exists an infinite set of positive integers $K' \subset K$ such that $B_k \xrightarrow{K'} B$, and furthermore B is positive semidefinite. Now, suppose by contradiction to Part (2) that $(\bar{x}, \alpha(\bar{x}))$ is not a Kuhn-Tucker point for Problem P' . Then by Lemma 4.1 an optimal solution d' to the problem to minimize $\theta^*(\bar{x}, d) + \frac{1}{2} d^t B d$ subject to $\bar{x} + d \in X$ must satisfy $\theta^*(\bar{x}, d') < -z$ for some $z > 0$. By continuity of θ^* and by Lemma 4.2, there exists a vector d such that $\theta^*(x_k, d) < -z$ and $x_k + d \in X$ for $k \in K'$ sufficiently large. By Lemma 2.1, for $\lambda \in (0, 1)$ we have:

$$\begin{aligned} \theta^*(x_k, \lambda d) + \frac{1}{2} \lambda^2 d^t B_k d &\leq \lambda \theta^*(x_k, d) + \frac{1}{2} \lambda^2 d^t B_k d \\ &\leq -\lambda z + \frac{1}{2} \lambda^2 \delta_2 \|d\|^2 \end{aligned}$$

Let $\hat{\lambda} = \min \{1, \frac{z}{\delta_2 \|d\|^2}\}$. Then $\theta^*(x_k, \hat{\lambda} d) + \frac{1}{2} \hat{\lambda}^2 d^t B_k d \leq -h$, where

$$h = \begin{cases} z - \frac{1}{2} \delta_2 \|d\|^2 & \text{if } z \geq \delta_2 \|d\|^2 \\ \frac{1}{2} \frac{z^2}{\delta_2 \|d\|^2} & \text{if } z < \delta_2 \|d\|^2 \end{cases}$$

We have thus constructed a vector $\hat{d} = \hat{\lambda} d$ so that $x_k + \hat{d} \in X$ for large $k \in K'$ and furthermore $\theta^*(x_k, \hat{d}) + \frac{1}{2} \hat{d}^t B_k \hat{d} \leq -h < 0$. But since d_k solves Problem $D(x_k)$, then $\theta^*(x_k, d_k) + \frac{1}{2} d_k^t B_k d_k \leq -h$. Letting k in K' approach ∞ and noting that $\bar{d} = 0$, it follows that $0 \leq -h$. This contradiction proves part (2).

Corollary

If the accumulation point \bar{x} is a strong local minimum for Problem P , then

$$\lim_{k \rightarrow \infty} x_k = \bar{x}.$$

Proof

Let $\gamma > 0$ be the number given in the definition of a strong local minimum.

Fix $0 < \delta < \frac{\gamma}{2}$. We will show that there exists an ℓ such that $\|x_k - \bar{x}\| < \delta$ for all $k \geq \ell$, which proves the result. Since $d_k \rightarrow 0$ and $x_k \xrightarrow{K} \bar{x}$, then there is an $\ell \in K$ such that

$$\|x_\ell - \bar{x}\| < \delta, \theta(x_\ell) - \theta(\bar{x}) < z(\delta), \|d_k\| < \frac{\gamma}{2} \quad \text{for all } k \geq \ell \quad (4.5)$$

We show the desired result by induction. For $k = \ell$, the result immediately follows from (4.5). Now let $k \geq \ell$ and suppose that $\|x_k - \bar{x}\| < \delta$ and note that:

$$\|x_{k+1} - \bar{x}\| \leq \|x_{k+1} - x_k\| + \|x_k - \bar{x}\| \leq \|d_k\| + \delta < \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma \quad (4.6)$$

Further, since $\theta(x_{k+1}) < \theta(x_\ell)$, from (4.5) it follows that $\theta(x_{k+1}) - \theta(\bar{x}) < z(\delta)$. In view of (4.6) and (4.4) it is then clear that $\|x_{k+1} - \bar{x}\| < \delta$. This completes the induction argument.

It may be noted that if the directions generated by the algorithm do not converge to zero, then $\theta(x_k) \rightarrow -\infty$ so that the problem has an unbounded solution. This follows by noting that θ is decreasing and that if there exists a set of positive integers K so that $\|d_k\| \geq \varepsilon > 0$ for $k \in K$, then

$$\begin{aligned} \theta(x_{k+1}) - \theta(x_k) &\leq - \left(\frac{1}{2}\right)^{m_{k+1}} \theta(x_k, d_k) \leq - \left(\frac{1}{2}\right)^{m_{k+1}} \delta_1 \|d_k\|^2 \\ &\leq - \left(\frac{1}{2}\right)^{[y]+2} \delta_1 \varepsilon^2 \quad \text{for each } k \in K \end{aligned}$$

If $\{x: \theta(x) \leq \theta(x_1), x \in X\}$ is compact then $\{x_k\}$ has an accumulation point. If the functions f and β_{ij} for all i, j are convex, then every accumulation point is an optimal solution to Problem P.

5. SPECIAL CASES

In this section, we discuss various specializations of the algorithm to unconstrained and constrained nonlinear programming problems.

Unconstrained Nonlinear Programming

Here we let $X = \mathbb{R}^n$ and $\alpha_j(x) = 0$ for $j = 1, \dots, \ell$. Under different choices of B_k our algorithm produces various methods for solving the problem to minimize $f(x)$ subject to $x \in \mathbb{R}^n$.

Steepest Descent Methods

At any iteration k the direction-finding Problem $D(x_k)$ is to minimize $\nabla f(x_k)^t d + \frac{1}{2} d^t B_k d$. The following choices of B_k are examined. For each of these choices all entries of B_k are uniformly bounded so that any sequence $\{B_k\}$ has a convergent subsequence as needed in Theorem 4.1.

Steepest Descent Under the Euclidean-Norm

Let $B_k = I$. Here $d_k = -\nabla f(x_k)$ and $d_k^t B_k d_k = -\|\nabla f(x_k)\|^2$, where $\|\cdot\|$ denotes the Euclidean norm. Note that $\theta^*(x_k, d_k) = \nabla f(x_k)^t d_k = -\|d_k\|^2$ so that Step 2 of the algorithm is never used by letting $\delta_1 = 1$. In this case, our algorithm reduces to that of Armijo [1].

Steepest Descent Under the Sup-Norm

Let B_k be a diagonal matrix whose i th diagonal entry b_i is given by

$$b_i = \left| \frac{\partial f(x_k)}{\partial x_i} \right| / \|\nabla f(x_k)\|_1 \quad i = 1, \dots, n$$

where $\|\cdot\|_1$ denotes the ℓ_1 -norm. Note that B_k is positive semidefinite. An optimal solution d_k to Problem $D(x_k)$ is given by

$$d_{ik} = \begin{cases} -\|\nabla f(x_k)\|_1 & \text{if } \partial f(x_k)/\partial x_i > 0 \\ \|\nabla f(x_k)\|_1 & \text{if } \partial f(x_k)/\partial x_i < 0 \\ 0 & \text{if } \partial f(x_k)/\partial x_i = 0 \end{cases}$$

Note that $\theta^*(x_k, d_k) = \nabla f(x_k)^t d_k = -\|\nabla f(x_k)\|_1^2 = -\|d_k\|_s^2$, where $\|\cdot\|_s$ denotes the sup-norm. If we let $\delta_1 = 1$, it is clear that Step 2 of the algorithm is never used.

Steepest Descent Under the ℓ_1 -Norm

Let $\|\cdot\|_s$ denote the sup-norm and let

$$c_i = \frac{\partial f(x_k)/\partial x_i}{\|\nabla f(x_k)\|_s} \quad i = 1, \dots, n$$

Let $I = \{i: |c_i| = 1\}$, and without loss of generality suppose that $I = \{1, \dots, v\}$.

Let

$$d^t = (c_1, \dots, c_v)$$

$$e^t = (c_{v+1}, \dots, c_n)$$

Now consider the matrix B_k given below:

$$B_k = \begin{array}{c} \begin{array}{cc} \begin{array}{|c|c|} \hline v \text{ columns} & n-v \text{ columns} \\ \hline \end{array} & \begin{array}{|c|c|} \hline v \text{ rows} & n-v \text{ rows} \\ \hline \end{array} \\ \begin{array}{|c|c|} \hline \begin{array}{cc} d & d^t \end{array} & \begin{array}{cc} 0 & \end{array} \\ \hline \begin{array}{cc} e & d^t \end{array} & \begin{array}{cc} \frac{n-v}{4} & I \end{array} \\ \hline \end{array} \end{array}$$

We will demonstrate that B_k is positive semidefinite, give the form of an optimal solution d_k which turns out to be a steepest descent direction under the ℓ_1 -norm, and then show that Step 2 of the algorithm is not needed. Let y and z be arbitrary vectors in R^v and R^{n-v} . Then:

$$(y^t, z^t) B_k \begin{pmatrix} y \\ z \end{pmatrix} = y^t d^t y + z^t e^t y + \frac{n-v}{4} z^t z$$

Denote $y^t d$ by a and $z^t e$ by g . Then, the above equation yields:

$$(y^t, z^t) B_k \begin{pmatrix} y \\ z \end{pmatrix} = (a + \frac{1}{2} g)^2 - \frac{1}{4} g^2 + \frac{n-v}{4} z^t z \quad (5.1)$$

By the Schwartz inequality and noting that the absolute value of each component of e is less than 1, we have:

$$g^2 \leq \|e\|^2 \|z\|^2 < (n-v) \|z\|^2$$

From (5.1) it is then clear that B_k is positive semidefinite. Next note that d_k given below is a solution to the system $\nabla f(x_k) + B_k d = 0$, which shows that under this particular choice of B_k , our quadratic program yields a steepest descent direction under the ℓ_1 -norm.

$$d_{ik} = \begin{cases} -\frac{1}{v} \frac{\partial f(x_k)}{\partial x_i} & i = 1, \dots, v \\ 0 & i = v+1, \dots, n \end{cases}$$

Finally, note that

$$\theta^*(x_k, d_k) = \nabla f(x_k)^t d_k = -\frac{1}{v} \sum_{i=1}^v \left| \frac{\partial f(x_k)}{\partial x_i} \right|^2 = -\|\nabla f(x_k)\|_s^2 = -\|d_k\|_1^2$$

where $\|\cdot\|_1$ denotes the ℓ_1 -norm. Therefore, Step 2 of the algorithm is not needed by letting $\delta_1 = 1$.

A Newton-Type Method for Unconstrained Optimization

In [9], Gill and Murray proposed a Newton-type procedure that produces a positive definite matrix B_k through a modified version of Cholesky's factorization of the Hessian H_k . If H_k is sufficiently positive definite then $B_k = H_k$. Otherwise B_k is of the form $H_k + E_k$, where E_k is a diagonal matrix with nonnegative elements.

If during the factorization process of H_k into the form LDL^t , a diagonal element of D is not sufficiently positive, then it is replaced by a suitable positive scalar q . The factorization is stable and can be performed within $\frac{n^3}{6}$ multiplications. At the end, $B_k = L_k D_k L_k^t$ is at hand and the search direction d_k is obtained by solving the system $\nabla f(x_k) + L_k D_k L_k^t d = 0$. One can easily choose the scalar q so that $y^t B_k y \geq 2\delta_1 \|y\|^2$ for any desired δ_1 , thus eliminating the need for Step 2 of the algorithm.

The above scheme of Gill and Murray [9] can thus be used in conjunction of our algorithm. If the Hessian at any accumulation point of the method is sufficiently positive definite, this method reduces to Newton's method, and quadratic convergence is assured.

Constrained Nonlinear Programming

Consider the following nonlinear programming problem:

$$\begin{aligned} \text{NLP: } & \text{minimize } f(x) \\ & \text{subject to } g_j(x) \leq 0 \quad j = 1, \dots, m \end{aligned}$$

Recently, a great deal of attention has been given by many authors to extending quasi-Newton procedures from the unconstrained case so that they can handle

problems of the above type. For a review of these methods, the reader is referred to Garcia-Palomares and Mangasarian [8], Han [13], and Powell [20].

A typical method in the class of quasi-Newton methods proceeds as follows. Given x_k , let d_k be an optimal solution to the following problem:

$$\begin{aligned} \bar{D}(x_k): \quad & \text{minimize} \quad \nabla f(x_k)^t d + \frac{1}{2} d^t B_k d \\ & \text{subject to} \quad g_j(x_k) + \nabla g_j(x_k)^t d \leq 0 \quad j = 1, \dots, m \end{aligned}$$

If x_k is sufficiently close to a Kuhn-Tucker point \bar{x} and if B_k is sufficiently close to the Hessian of the Lagrangian at \bar{x} , then the algorithm $x_{k+1} = x_k + d_k$ converges to \bar{x} at a superlinear rate.

In [12], Han was able to prove convergence of the procedure starting from points remote from \bar{x} . He showed that if μ is sufficiently large so that $\mu > u_j$ for $j = 1, \dots, m$, where u_j is the Lagrangian multiplier associated with the j th constraint in Problem $\bar{D}(x_k)$, then d_k is indeed a descent direction for the penalty function $\phi(x) = f(x) + \mu \sum_{j=1}^m \max\{0, g_j(x)\}$ at x_k . He was able to show global convergence by letting $x_{k+1} = x_k + \lambda_k d_k$, where λ_k essentially solves the problem to minimize $\phi(x_k + \lambda d_k)$ subject to $0 \leq \lambda \leq \delta$, where $\delta > 0$ is a fixed number.

We will now show that our minimax algorithm specializes to Han's method and extends it in two ways. First, rather than performing a line search, our procedure uses the easily implementable Armijo's search. In [12], Han suggested that it is of some practical value to devise such an approximate search procedure for the nondifferentiable function ϕ . Second, a typical quasi-Newton method could stop prematurely if Problem $\bar{D}(x_k)$ has an empty feasible region, that is, if there exists no vector p such that $\nabla g_j(x_k)^t p < 0$ for $j \in I$, where

$I = \{j: g_j(x_k) > 0\}$. As will be seen shortly, our direction-finding problem is always feasible, and furthermore it reduces to Problem $\bar{D}(x_k)$ if the latter is feasible.

Note that Problem NLP can be put in the minimax format as follows.

Let $\ell = m$ and let $\alpha_j(x) = \mu \max \{0, g_j(x)\}$, where μ is an exact penalty parameter. Then Problem P becomes:

$$\text{minimize } f(x) + \mu \sum_{j=1}^{\ell} \max \{0, g_j(x)\}$$

At any particular iteration, our direction-finding problem reduces to:

$$\begin{aligned} D'(x_k): \quad & \text{minimize} \quad \nabla f(x_k)^t d + \mu \sum_{j=1}^m y_j + \frac{1}{2} d^t B_k d \\ & \text{subject to} \quad g_j(x_k) + \nabla g_j(x_k)^t d \leq y_j \quad j = 1, \dots, m \\ & \quad y_j \geq 0 \quad j = 1, \dots, m \end{aligned}$$

The relationship between problems $\bar{D}(x_k)$ and $D'(x_k)$ is given by Lemma 5.1 below.

Lemma 5.1

If Problem $\bar{D}(x_k)$ is not feasible then any feasible point (d, y) to Problem $D'(x_k)$ must have $\sum_{j=1}^m y_j > 0$. Now suppose that B_k is positive semidefinite and symmetric. Further suppose that Problem $\bar{D}(x_k)$ is feasible and that it has (d_k, u) as a Kuhn-Tucker solution. If $\mu > u_j$ for $j = 1, \dots, m$, then $(d_k, y=0)$ is an optimal solution to Problem $D'(x_k)$. Further, if B_k is positive definite, then any optimal solution (\hat{d}, \hat{y}) to Problem $D'(x_k)$ must satisfy $\hat{y} = 0$ and $\hat{d} = d_k$.

Proof

Obviously, if Problem $\bar{D}(x_k)$ is not feasible then any feasible point (d, y) to Problem $D'(x_k)$ must satisfy $\sum_{j=1}^m y_j > 0$. Now suppose that (d_k, u) is a Kuhn-Tucker

solution to problem $\bar{D}(x_k)$. Then:

$$\begin{aligned}
 \nabla f(x_k) + B_k d_k + \sum_{j=1}^m u_j \nabla g_j(x_k) &= 0 \\
 u_j [g_j(x_k) + \nabla g_j(x_k)^t d_k] &= 0 \quad j = 1, \dots, m \\
 g_j(x_k) + \nabla g_j(x_k)^t d_k &\leq 0 \quad j = 1, \dots, m \\
 u_j &\geq 0 \quad j = 1, \dots, m
 \end{aligned} \tag{5.1}$$

But (\hat{d}, \hat{y}) is a Kuhn-Tucker solution to Problem $D'(x_k)$ if there exists a vector v such that

$$\begin{aligned}
 \nabla f(x_k) + B_k \hat{d} + \sum_{j=1}^m v_j \nabla g_j(x_k) &= 0 \\
 \mu - v_j &\geq 0 \quad j = 1, \dots, m \\
 v_j [g_j(x_k) + \nabla g_j(x_k)^t \hat{d} - \hat{y}_j] &= 0 \quad j = 1, \dots, m \\
 g_j(x_k) + \nabla g_j(x_k)^t \hat{d} &\leq \hat{y}_j \quad j = 1, \dots, m \\
 v_j &\geq 0 \quad j = 1, \dots, m \\
 (\mu - v_j) \hat{y}_j &= 0 \quad j = 1, \dots, m
 \end{aligned} \tag{5.2}$$

Noting that $\mu > u_j$, it follows that the system defined by (5.2) holds by letting $\hat{d} = d_k$, $\hat{y} = 0$, and $v = u$. By convexity of Problem $D'(x_k)$ it follows that $(d_k, y=0)$ is indeed an optimal solution.

Now suppose that B_k is positive definite and let (\hat{d}, \hat{y}) be an optional solution to Problem $D'(x_k)$. Therefore $\lambda(\hat{d}, \hat{y}) + (1-\lambda)(d_k, 0)$ is also an optimal

solution for all $\lambda \in (0,1)$. This further implies that $\Psi(\lambda)$ defined below is constant for all $\lambda \in (0,1)$:

$$\begin{aligned}\Psi(\lambda) &= \nabla f(x_k)^t d_k + \lambda \nabla f(x_k)^t (\hat{d} - d_k) + \lambda \mu \sum_{j=1}^m \hat{y}_j \\ &\quad + \frac{1}{2} d_k^t B_k d_k + \frac{1}{2} \lambda^2 (\hat{d} - d_k)^t B_k (\hat{d} - d_k) \\ &\quad + \lambda (\hat{d} - d_k)^t B_k d_k\end{aligned}$$

This implies that $\Psi'(\lambda) = 0$ for $\lambda \in (0,1)$ and hence

$$\nabla f(x_k)^t (\hat{d} - d_k) + (\hat{d} - d_k)^t B_k d_k + \mu \sum_{j=1}^m \hat{y}_j + \quad (5.3)$$

$$\lambda (\hat{d} - d_k)^t B_k (\hat{d} - d_k) = 0 \quad \text{for all } \lambda \in (0,1)$$

But this is possible only if $(\hat{d} - d_k)^t B_k (\hat{d} - d_k) = 0$, and since B_k is positive definite, we must have $\hat{d} = d_k$. From (5.3) we have $\hat{y} = 0$ and the proof is complete.

The above lemma shows that if B_k is positive definite and if μ is sufficiently large, then an optimal solution to Problem $D'(x_k)$ has $\sum_{j=1}^m y_j > 0$ only if Problem $\bar{D}(x_k)$ is not feasible. To illustrate, consider the problem to minimize $f(x)$ subject to $g(x) \leq 0$, where $f(x) = (x-2)^2$ and

$$g(x) = \begin{cases} 1 - (x-1)^2 & x \leq 1 \\ 1 & \text{otherwise} \end{cases}$$

If the starting solution is $x_1 = 1$, then Problem $\bar{D}(x_1)$ is infeasible and the

quasi-Newton method would stop prematurely at the infeasible point x_1 . Our minimax algorithm will not stop at this point and would eventually converge to the optimal solution $\bar{x} = 0$. It is thus proposed that quasi-Newton methods should solve Problem $D'(x_k)$ rather than Problem $\bar{D}(x_k)$ in order to find a search direction d_k .

REFERENCES

1. Armijo, L., "Minimization of Functions Having Lipschitz Continuous First Derivatives," Pacific Journal of Mathematics, Volume 16, pp. 1-3, 1966.
2. Bazaraa, M. S. and J. J. Goode, "Extension of Optimality Conditions via Supporting Functions," Mathematical Programming, Volume 5, pp. 267-285, 1973.
3. Bertsekas, D. P., "On the Goldstein-Levitin-Polyak Gradient Projection Method," IEEE Transactions on Automatic Control, Volume AC-21, pp. 174-183, 1976.
4. Chatelon, J. A., D. W. Hearn, and T. J. Lowe, "A Subgradient Algorithm for Certain Minimax and Minisum Problems," Mathematical Programming, Volume 15, pp. 130-145, 1978.
5. Danskin, J. M., "The Theory of Max-Min With Applications," SIAM Journal on Applied Mathematics, Volume 14, pp. 641-664, 1966.
6. Dem'yanov, V. F., "Algorithms for Some Minimax Problems," Journal of Computer and System Sciences, Volume 2, pp. 342-380, 1968.
7. Dem'yanov, V. F. and V. N. Malozemov, Introduction to Minimax, John Wiley and Sons, New York, 1974.
8. Garcia-Palomares, U. M. and O. L. Mangasarian, "Superlinearly Convergent Quasi-Newton Algorithms for Nonlinearly Constrained Optimization Problems," Mathematical Programming, Volume 11, pp. 1-13, 1976.
9. Gill, P. E. and W. Murray, "Two Methods for the Solution of Linearly Constrained and Unconstrained Optimization Problems," National Physical Laboratory, Teddington, England, Report Number NAC 25, 1972.
10. Goldstein, A. A., "Convex Programming in Hilbert Space," Bulletin American Mathematical Society, Volume 70, pp. 709-710, 1964.
11. Han, S. P., "Variable Metric Methods for Minimizing a Class of Nondifferentiable Functions," University of Illinois, 1979.
12. Han, S. P., "A Globally Convergent Method for Nonlinear Programming," Journal of Optimization Theory and Applications, Volume 22, pp. 297-309, 1977.
13. Han, S. P., "Superlinearly Convergent Variable Metric Algorithms for General Nonlinear Programming Problems," Mathematical Programming, Volume 11, pp. 263-282, 1976.
14. Levitin, E. S. and B. T. Polyak, "Constrained Minimization Problems," USSR Computational Mathematics and Mathematical Physics, Volume 6, pp. 1-50, 1966.

15. Madsen, J. and H. Schjaer-Jacobson, "Linearly Constrained Minimax Optimization, Mathematical Programming, Volume 14, pp. 208-223, 1978.
16. McCormick, G. P., "Anti-Zig-Zagging by Bending," Management Science, Volume 15, pp. 315-319, 1969.
17. McCormick, G. P. and R. A. Topia, "The Gradient Projection Method Under Mild Differentiability Conditions," SIAM Journal on Control, Volume 10, pp. 93-98, 1972.
18. Polyak, B. T., "A General Method of Solving Extremum Problems," Soviet Mathematics, Volume 8, pp. 593-597, 1967.
19. Polyak, B. T., "Minimization of Unsmooth Functionals," USSR Computational and Mathematical Physics, Volume 9, pp. 14-29, 1969.
20. Powell, M. J. D., "The Convergence of Variable Metric Methods for Nonlinearly Constrained Optimization Calculations," in Nonlinear Programming 3, Edited by O. L. Mangasarian, R. R. Meyer, and S. M. Robinson, Academic Press, 1978.

AN ALGORITHM FOR LINEARLY CONSTRAINED
NONLINEAR PROGRAMMING PROBLEMS

Mokhtar S. Bazaraa[†] and Jamie J. Goode^{††}

In this paper an algorithm for solving a linearly constrained nonlinear programming problem is developed. Given a feasible point, a correction vector is computed by solving a least distance programming problem over a polyhedral cone defined in terms of the gradients of the "almost" binding constraints. Mukai's approximate scheme for computing step sizes is generalized to handle the constraints. This scheme provides an estimate for the step size based on a quadratic approximation of the function. This estimate is used in conjunction with Armijo line search to calculate a new point. It is shown that each accumulation point is a Kuhn-Tucker point to a slight perturbation of the original problem. Furthermore, under suitable second order optimality conditions, it is shown that eventually only one trial is needed to compute the step size.

[†] School of Industrial and Systems Engineering, Georgia Institute of Technology, Atlanta, Georgia. This author's work is supported under USAFOSR contract number F49620-79-C-0120.

^{††} School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia.

1. Introduction

This paper addresses the following linearly constrained nonlinear programming problem:

$$\begin{aligned} P: \quad & \text{minimize} \quad f(x) \\ & \text{subject to} \quad Ax \leq b \end{aligned}$$

where f is a twice continuously differentiable function on R^n , and A is an $l \times n$ matrix whose j th row is denoted by a_j^t , and where a superscript t denotes the transpose operation.

There are several approaches for solving this problem. The first one relies on partitioning the variables into basic, nonbasic, and superbasic variables. The values of the superbasic and basic variables are modified while the nonbasic variables are fixed at their current values. Examples of methods in this class are the convex simplex method of Zangwill [18], the reduced gradient method of Wolfe [17], the method of Murtagh and Saunders [12], and the variable reduction method of McCormick [8].

Another class of methods is the extension of quasi-Newton algorithms from unconstrained to constrained optimization. Here, at any iteration, a set of active restrictions is identified, and then a modified Newton procedure is used to minimize the objective function on the manifold defined by these active constraints. See for example Goldfarb [6], and Gill and Murray [5].

Other approaches for solving problems with linear constraints are the gradient projection method and the method of feasible directions. The former computes a direction by projecting the negative gradient on the space orthogonal to the gradients of a subset of the binding constraints while the latter method determines a search direction by solving a linear programming problem.

For a review of these methods the reader may refer to Rosen [14], Zoutendijk [19], Frank and Wolfe [4], and Topkis and Veinott [15].

In this paper, an algorithm for solving problem P is proposed. At each iteration a correction vector is computed by finding the minimum distance from a given point to a polyhedral cone defined in terms of the gradients of the "almost" binding constraints. An approximate line search procedure which extends those of Armijo [1] and Mukai [10, 11] for unconstrained optimization is developed for determining the step size. First, an estimate of the step size based on a quadratic approximation to the objective function is computed, and then adjusted if necessary.

In Section 2, we outline the algorithm. In Section 3, we show that accumulation points of the algorithm are Kuhn-Tucker points to a slight perturbation of the original problem. Finally, in Section 4, assuming that the algorithm converges, and under suitable second order sufficiency optimality conditions, we show that the step size estimates which are based on the quadratic approximation are acceptable so that only one functional evaluation is eventually needed for performing the line search.

2. Statement of the Algorithm

Consider the following algorithm for solving Problem P.

Step 0

Choose values for the parameters c , z , δ , and ϵ . Select a point x_0 such that $Ax_0 \leq b$ and let $\delta_0 = \delta$. Let $i = 0$ and go to Step 1.

Step 1

Let w_1 be the optimal solution to Problem $D(x_1)$ given below:

$$D(x_i): \begin{array}{ll} \text{minimize} & \nabla f(x_i)^t w + \frac{1}{2} z^t w \\ \text{subject to} & a_j^t w \leq 0 \end{array} \quad \text{for } j \in I(x_i)$$

where

$$I(x_i) = \{j: a_j^t x_i > b_j - c\} \quad (2.1)$$

If $w_i = 0$, stop. Else, go to Step 2.

Step 2

Let

$$I^+(w_i) = \{j: a_j^t w_i > 0\} \quad (2.2)$$

and let

$$\beta_i = \min \left\{ 1, \frac{b_j - a_j^t x_i}{a_j^t w_i} \text{ for } j \in I^+(w_i) \right\} \quad (2.3)$$

Let

$$d_i = \beta_i w_i \quad (2.4)$$

and go to Step 3.

Step 3

If

$$f(x_i + \epsilon d_i) + f(x_i - \epsilon d_i) - 2f(x_i) \geq \epsilon^2 \delta_i \|d_i\|^2 \quad (2.5)$$

let

$$\lambda_i = \frac{-\epsilon^2 \nabla f(x_i)^t d_i}{f(x_i + \epsilon d_i) + f(x_i - \epsilon d_i) - 2f(x_i)} \quad (2.6)$$

and let $\delta_{i+1} = \delta_i$, and go to Step 4. Otherwise, let $\lambda_i = 1$, $\delta_{i+1} = \frac{1}{2} \delta_i$, and go to Step 4.

Step 4

Let

$$\alpha_i = \min \{1, \lambda_i\} \quad (2.7)$$

and compute the smallest nonnegative integer k satisfying

$$f(x_i + (\frac{1}{2})^k \alpha_i d_i) - f(x_i) \leq \frac{1}{3} (\frac{1}{2})^k \alpha_i \nabla f(x_i)^t d_i \quad (2.8)$$

Let $k_i = k$, $x_{i+1} = x_i + \alpha_i (\frac{1}{2})^{k_i} d_i$, $i = i + 1$, and go to Step 1.

The following remarks are helpful in interpreting the above algorithm.

1. A direction w_i is determined by solving Problem $D(x_i)$. This problem finds the point in the convex polyhedral cone $\{w: a_j^t w \leq 0 \text{ for } j \in I(x_i)\}$ which is closest to the vector $-\frac{1}{2} \nabla f(x_i)$. Methods of least distance programming, as in the works of Bazaraa and Goode [2], and Wolfe [16] can be used for solving this problem. Special methods that take advantage of the structure of the cone constraints may prove quite useful in this regard.
2. The restrictions enforced in Problem $D(x_i)$ are the c -binding constraints at x_i , that is, those satisfying $b_j - c < a_j^t x_i \leq b_j$. If $w_i = 0$, then the algorithm is terminated with x_i . In this case, from the Kuhn-Tucker conditions for Problem $D(x_i)$, there exist u_j for $j \in I(x_i)$ such that:

$$\nabla f(x_i) + \sum_{j \in I(x_i)} u_j a_j = 0$$

$$u_j \geq 0 \quad \text{for } j \in I(x_i)$$

These conditions imply that x_i is a Kuhn-Tucker point for the following problem:

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && a_j^t x \leq a_j^t x_i \quad \text{for } j \in I(x_i) \\ &&& a_j^t x \leq b_j \quad \text{for } j \notin I(x_i) \end{aligned}$$

Noting that $b_j - c < a_j^t x_i \leq b_j$ for $j \in I(x_i)$, if c is sufficiently small, it is clear that the algorithm is terminated if x_i is a Kuhn-Tucker solution to a slightly perturbed version of Problem P. The following definition will thus be useful.

Definition 2.1

Let x^* be a feasible point to Problem P. If the optimal solution to Problem $D(x^*)$ is equal to zero, then x^* is called a c -KT solution to Problem P.

3. If $x_i + w_i$ is feasible to Problem P, then the search vector d_i is taken as w_i . Otherwise, d_i is taken to be the vector of maximum length along w_i which maintains feasibility of $x_i + d_i$.

4. Steps 3 and 4 of the algorithm compute the step size taken along the vector d_i in order to form x_{i+1} . As proposed by Mukai [10, 11], first an estimate of the step size λ_i is calculated. When appropriate, λ_i is computed by utilizing a quadratic approximation of the function f at x_i , otherwise λ_i is taken equal to 1. In order to ensure feasibility to Problem P, the first trial step size α_i used in conjunction with Armijo line search [1], is the minimum of λ_i and 1. As will be shown in Section 4, under suitable assumptions, for large i , test (2.5) passes, $k_i = 0$, and $\alpha_i = \lambda_i < 1$. This confirms efficiency of the line search scheme where eventually only one trial is needed to compute the step size.

3. Accumulation Points of the Algorithm

Theorem 3.1 shows that each accumulation point of the proposed algorithm is a c-KT point. In order to prove this theorem, lemmas 3.1 and 3.2 are needed. These two lemmas extend similar results of Mukai [10] for unconstrained problems.

In order to facilitate the development in this section, the following notation is used. Let $w(x)$ be the optimal solution to Problem D(x) and let $\beta(x)$ be as given in (2.3) with x_i replaced with x . Finally, let $d(x) = \beta(x)w(x)$.

Lemma 3.1

Suppose that x^* is not a c-KT point for Problem P. Then, there exist scalars μ and $s > 0$ so that $\mu \leq \alpha(x) \leq 1$ for each x with $\|x - x^*\| < s$.

Proof

There exists $s_1 > 0$ so that $I(x) = I(x^*)$ for all $\|x - x^*\| < s_1$. Thus, the feasible region for Problem D(x) is equal to that of Problem D(x^{*}) for all x satisfying $\|x - x^*\| < s_1$. By continuous differentiability of f , it then follows that $w(\cdot)$ is continuous in x at x^* , see for example Daniel [3]. Particularly,

there exists a number $s_2 > 0$ such that $I^+(w(x)) = I^+(w(x^*))$ if $\|x - x^*\| < s_2$. This together with the continuity of $w(\cdot)$ and the formula for computing $\beta(\cdot)$ imply that $\beta(\cdot)$ is continuous in x at x^* . Hence, $d(\cdot)$ is also continuous. Since x^* is not a c-KT point, then $w(x^*) \neq 0$. Furthermore, $b_j - a_j^t x^* \geq c$ if $a_j^t w^* > 0$ which implies that $\beta(x^*) > 0$. Therefore $d(x^*) \neq 0$. By continuity of $\beta(\cdot)$ and $d(\cdot)$ at x^* there exist scalars q and $s > 0$ so that

$$\beta(x) \|d(x)\|^2 \geq \frac{1}{2} \beta(x^*) \|d(x^*)\|^2 \quad \text{if } \|x - x^*\| < s \quad (3.1)$$

$$f(x + \epsilon d(x)) + f(x - \epsilon d(x)) - 2f(x) < q \quad \text{if } \|x - x^*\| < s \quad (3.2)$$

Now, let x be such that $\|x - x^*\| < s$. Since $w(x)$ solves Problem $D(x)$, then $\nabla f(x)^t w(x) \leq -\frac{1}{2} z \|w(x)\|^2$. This, in turn, implies that $-\nabla f(x)^t d(x) \geq \frac{1}{2} z \beta(x) \|d(x)\|^2$ and from (3.1) we get:

$$-\nabla f(x)^t d(x) \geq \frac{1}{4} z \beta(x^*) \|d(x^*)\|^2 = y > 0 \quad (3.3)$$

If test (2.5) passes, then from (3.2) and (3.3) the following lower bound on λ_1 is at hand:

$$\lambda_1 = \frac{-\epsilon^2 \nabla f(x)^t d(x)}{f(x + \epsilon d(x)) + f(x - \epsilon d(x)) - 2f(x)} \geq \frac{\epsilon^2 y}{q}$$

If test (2.5) fails, then $\lambda_1 = 1$ and hence $\lambda_1 \geq \min \left\{ 1, \frac{\epsilon^2 y}{q} \right\} = \mu$. Since $\alpha_1 = \min \{1, \lambda_1\}$, the desired result follows.

Lemma 3.2

If x^* is not a c-KT point for Problem P , then there exist a number $s > 0$ and an integer m so that $k(x) \leq m$ if $\|x - x^*\| < s$, where $k(x)$ is the Armijo integer

given by (2.8) with x_i and α_i replaced with x and $\alpha(x)$ respectively.

Proof

As in the proof of Lemma 3.1 and by continuous differentiability of f , there exist scalars s , h , and $y > 0$ so that for $\|x - x^*\| < s$ the following hold:

$$\nabla f(x)^t d(x) \leq -y \quad (3.4)$$

$$|\nabla f(x + g d(x))^t d(x) - \nabla f(x)^t d(x)| \leq \frac{2}{3} y \quad \text{for each } g \in [0, h] \quad (3.5)$$

Now let m be the smallest nonnegative integer so that $(\frac{1}{2})^m \leq h$ and let x be such that $\|x - x^*\| < s$. Then there exists $\theta \in [0, 1]$ such that:

$$\begin{aligned} f(x + (\frac{1}{2})^m \alpha(x) d(x)) - f(x) - \frac{1}{3} (\frac{1}{2})^m \alpha(x) \nabla f(x)^t d(x) \\ = (\frac{1}{2})^m \alpha(x) \nabla f(x + \theta (\frac{1}{2})^m \alpha(x) d(x))^t d(x) - \frac{1}{3} (\frac{1}{2})^m \alpha(x) \nabla f(x)^t d(x) \\ = (\frac{1}{2})^m \alpha(x) \left[\{ \nabla f(x + \theta (\frac{1}{2})^m \alpha(x) d(x))^t d(x) - \nabla f(x)^t d(x) \} + \frac{2}{3} \nabla f(x)^t d(x) \right] \end{aligned} \quad (3.6)$$

Since $\theta (\frac{1}{2})^m \alpha(x) \leq h$, (3.4) and (3.5) imply that the right hand side of (3.6) is ≤ 0 which in turn shows that $k(x) \leq m$, and the proof is complete.

Theorem 3.1

Either the algorithm terminates with a c-KT point for Problem P or else generates an infinite sequence $\{x_i\}$ of which any accumulation point is a c-KT point for Problem P.

Proof

Clearly the algorithm stops at x_i only if x_i is a c-KT point. Now, suppose that the algorithm generates the infinite sequence $\{x_i\}$. Suppose that x^* is an accumulation point so that $x_i \xrightarrow{K} x^*$ for some infinite set K of positive integers. Since $f(x_i)$ is decreasing monotonically and since $f(x_i) \xrightarrow{K} f(x^*)$ then $f(x_i) \rightarrow f(x^*)$. Suppose by contradiction to the desired conclusion that x^* is not a c-KT point. From Lemmas 3.1 and 3.2, there exist positive numbers μ and y and an integer m so that $\alpha_i \geq \mu$, $\nabla f(x_i)^t d_i \leq -y$, and $k_i \leq m$ for large i in K . Therefore,

$$f(x_{i+1}) - f(x_i) \leq \frac{1}{3} \left(\frac{1}{2}\right)^{k_i} \alpha_i \nabla f(x_i)^t d_i \leq -\frac{1}{3} \mu y \left(\frac{1}{2}\right)^m$$

for large i in K . This implies that $f(x_i) \rightarrow -\infty$, contradicting the fact that $f(x_i) \rightarrow f(x^*)$. This completes the proof.

4. Eventual Acceptance of the Step Size Estimate

In the previous section, we showed that an accumulation point x^* of the sequence $\{x_i\}$ generated by the algorithm is a KT point to the perturbed problem P' given below:

$$\begin{aligned} P': \quad & \text{minimize} \quad f(x) \\ & \text{subject to} \quad a_j^t x \leq a_j^t x^* \quad \text{for } j \in I(x^*) \\ & \quad \quad \quad a_j^t x \leq b_j \quad \text{for } j \notin I(x^*) \end{aligned}$$

Here, we assume that the whole sequence $\{x_i\}$ converges to a point x^* which satisfies suitable second order sufficiency conditions. Under this assumption, we show that test (2.5) is eventually passed. Furthermore, we show

that $\lambda_i < 1$ and that $k_i = 0$ for i large enough.

The second order condition is given in Definition 4.1. It is well-known that x^* satisfying this condition is a strong local minimum for problem P' . That is, there exists a number $\gamma > 0$ so that $f(x^*) < f(x)$ if x is feasible to problem P' and $\|x - x^*\| < \gamma$, see for example McCormick [9] and Han and Mangasarian [7].

Definition 4.1

Let x^* be such that $Ax^* \leq b$ and let $I(x^*) = \{j: a_j^t x^* > b_j - c\}$. x^* is said to satisfy the second order sufficiency optimality conditions for problem P' if there exist scalars $u_j \geq 0$ for $j \in I(x^*)$ and $\gamma > 0$ so that:

$$\nabla f(x^*) + \sum_{j \in I(x^*)} u_j a_j = 0$$

$$f(x^*)^t d \leq 0, a_j^t d \leq 0 \text{ for } j \in I(x^*), \|d\| = 1 \Rightarrow d^t H(x^*) d > \gamma \quad (4.1)$$

Theorem 4.1 shows that test (2.5) will eventually be passed so that λ_i is given by (2.6). The following two intermediate results are needed to prove this theorem.

Lemma 4.1

If $Cd \leq 0$ and $\|d\| = 1$ imply that $d^t H d \geq \gamma > 0$ then there is a number $\theta > 0$ so that $Cd \leq \theta 1$ and $\|d\| = 1$ imply that $d^t H d \geq \gamma/2$.

Proof

Suppose by contradiction that for each integer k there is a vector d_k such that

$$\|d_k\| = 1, Cd_k \leq \frac{1}{k}, \text{ and } d_k^t H d_k < \gamma/2 \quad (4.2)$$

Since the sequence $\{d_k\}$ is bounded, it has an accumulation point d . From (4.2), $\|d\| = 1$, $Cd \leq 0$, and $d^t H d \leq \gamma/2$ which contradicts the assumption of the lemma.

Lemma 4.2

If either $\{x_i\}$ converges or $\{x: Ax \leq b, f(x) \leq f(x_0)\}$ is bounded, then $d_i \rightarrow 0$.

Proof

Since $0 < \beta_i \leq 1$ and $d_i = \beta_i w_i$, it suffices to prove that $w_i \rightarrow 0$. Suppose there exist an infinite set of positive integers K and a number $\epsilon > 0$ so that

$$\|w_i\| \geq \epsilon \quad \text{for } i \in K. \quad (4.3)$$

Clearly, under either of the assumptions of the lemma, there exist an infinite set $K' \subset K$ and a point x^* so that $x_i \xrightarrow{K'} x^*$. By Theorem 3.1 x^* is a c-KT point for Problem P. Thus, $w^* = 0$ is the unique optimal solution to Problem $D(x^*)$. But for large $i \in K'$, $I(x^*) = I(x_i)$, and by continuity of the solutions to $D(\cdot)$ we must have $\|w_i\| < \epsilon/2$ for large i in K' . This contradicts (4.3) and the proof is complete.

Throughout the remainder of this section, the following notation will be used for any scalar γ :

$$H_i^\gamma = 2 \int_0^1 (1-y) H(x_i + y\gamma d_i) dy \quad (4.3)$$

We can integrate by parts to obtain

$$f(x_i + \gamma d_i) - f(x_i) = \gamma \nabla f(x_i)^t d_i + \frac{1}{2} \gamma^2 d_i^t H_i^\gamma d_i \quad (4.4)$$

For further details, the reader may refer to Polak [13, p. 293].

Theorem 4.1

Let $\{x_i\}$ be a sequence generated by the algorithm. Suppose that $x_i \rightarrow x^*$ and x^* satisfies the second order optimality conditions for problem P'. Then there exists an integer m so that test (2.5) passes for all $i \geq m$.

Proof

From (4.3) and (4.4) we get:

$$f(x_i + \epsilon d_i) - f(x_i) = \epsilon \nabla f(x_i)^t d_i + \frac{1}{2} \epsilon^2 d_i^t H_i^\epsilon d_i$$

$$f(x_i - \epsilon d_i) - f(x_i) = -\epsilon \nabla f(x_i)^t d_i + \frac{1}{2} \epsilon^2 d_i^t H_i^{-\epsilon} d_i$$

Adding we obtain:

$$f(x_i + \epsilon d_i) + f(x_i - \epsilon d_i) - 2f(x_i) = \frac{1}{2} \epsilon^2 d_i^t (H_i^\epsilon + H_i^{-\epsilon}) d_i \quad (4.5)$$

Now for $j \in I(x^*)$, $a_j^t x^* > b_j - c$. Since $x_i \rightarrow x^*$ then for i large enough, $a_j^t x_i > b_j - c$ so that $j \in I(x_i)$. By step 1 of the algorithm $a_j^t w_i \leq 0$ and so $a_j^t \frac{d_i}{\|d_i\|} \leq 0$ for i large enough and $j \in I(x^*)$. Likewise, from step 1 of the

algorithm $\nabla f(x_i)^t w_i \leq 0$ and hence $\nabla f(x_i)^t \frac{d_i}{\|d_i\|} \leq 0$. Since $x_i \rightarrow x^*$, then for any number $\theta > 0$, $\nabla f(x^*)^t \frac{d_i}{\|d_i\|} \leq \theta$ for i large enough. Thus, Lemma 4.1 and the second order conditions imply that

$$d_i^t H(x^*) d_i \geq \frac{\gamma}{2} \|d_i\|^2 \quad \text{for large } i \quad (4.6)$$

Now note that

$$\begin{aligned} \|H_i^\varepsilon - H(x^*)\| &= \left\| 2 \int_0^1 (1-y) [H(x_i + y d_i) - H(x^*)] dy \right\| \\ &\leq 2 \int_0^1 (1-y) \|H(x_i + y d_i) - H(x^*)\| dy \end{aligned} \quad (4.7)$$

Since $x_i \rightarrow x^*$, then by Lemma 4.2, $d_i \rightarrow 0$. Particularly, for i large enough, $\|H(x_i + y d_i) - H(x^*)\| < \frac{\gamma}{4}$ for all $y \in [0, 1]$. From (4.7), $\|H_i^\varepsilon - H(x^*)\| < \frac{\gamma}{4}$.

This together with (4.6) yields:

$$\begin{aligned} d_i^t H_i^\varepsilon d_i &= d_i^t H(x^*) d_i + d_i^t (H_i^\varepsilon - H(x^*)) d_i \\ &\geq \frac{\gamma}{2} \|d_i\|^2 - \|d_i\|^2 \|H_i^\varepsilon - H(x^*)\| \\ &\geq \frac{\gamma}{4} \|d_i\|^2 \quad \text{for large } i \end{aligned} \quad (4.8)$$

Similarly,

$$d_i^t H_i^{-\varepsilon} d_i \geq \frac{\gamma}{4} \|d_i\|^2 \quad \text{for large } i \quad (4.9)$$

From (4.5), (4.8), and (4.9) it immediately follows that

$$f(x_i + \varepsilon d_i) + f(x_i - \varepsilon d_i) - 2f(x_i) \geq \varepsilon^2 \frac{\gamma}{4} \|d_i\|^2 \quad \text{for large } i \quad (4.10)$$

From (4.10), if test (2.5) fails for a large i , we must have:

$$\varepsilon^2 \delta_i \|d_i\|^2 > f(x_i + \varepsilon d_i) + f(x_i - \varepsilon d_i) - 2f(x_i) \geq \varepsilon^2 \frac{\gamma}{4} \|d_i\|^2$$

that is, $\delta_i > \frac{\gamma}{4}$. If the conclusion of the lemma does not hold, then test (2.5) fails infinitely often and then $\delta_i \rightarrow 0$. This contradicts $\delta_i > \frac{\gamma}{4}$ for large i , and the proof is complete.

Theorem 4.2

Let $\{x_i\}$ be a sequence generated by the algorithm. Suppose that $x_i \rightarrow x^*$ and that x^* satisfies the second order optimality conditions for Problem P'. Then there exists an integer m so that $f(x_i + \alpha_i d_i) - f(x_i) \leq \frac{1}{3} \alpha_i \nabla f(x_i)^t d_i$ for all $i \geq m$, that is, $k_i = 0$ for all $i \geq m$.

Proof

By Theorem (4.1), test (2.5) passes for large i so that λ_i is given by

$$\lambda_i = \frac{-\varepsilon^2 \nabla f(x_i)^t d_i}{f(x_i + \varepsilon d_i) + f(x_i - \varepsilon d_i) - 2f(x_i)} = \frac{-\nabla f(x_i)^t d_i}{\frac{1}{2} d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i} \quad (4.11)$$

If $\lambda_i \leq 1$ so that $\alpha_i = \lambda_i$, then from (4.4) and (4.11) we get:

$$\begin{aligned} f(x_i + \alpha_i d_i) - f(x_i) - \frac{1}{3} \alpha_i \nabla f(x_i)^t d_i &= \frac{1}{2} \lambda_i^2 d_i^t H_i^{\lambda_i} d_i + \frac{2}{3} \lambda_i \nabla f(x_i)^t d_i \\ &= \frac{1}{2} \lambda_i^2 [d_i^t H_i^{\lambda_i} d_i - \frac{1}{2} d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i] - \frac{1}{12} \lambda_i^2 d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i \end{aligned} \quad (4.12)$$

Since $x_i \rightarrow x^*$, then by Lemma 4.2, $d_i \rightarrow 0$. Thus $H_i^{\lambda_i}$, H_i^ε , and $H_i^{-\varepsilon}$ converge to $H(x^*)$ and the first term in (4.12) will be less than $\frac{\gamma}{24} \lambda_i^2 \|d_i\|^2$ for i large enough. As in the proof of Theorem 4.1, $d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i \geq \frac{\gamma}{2} \|d_i\|^2$ for large i . Substituting in (4.12), the desired result holds.

Now suppose that $\lambda_i > 1$ so that $\alpha_i = 1$. Then

$$f(x_i + \alpha_i d_i) - f(x_i) - \frac{1}{3} \alpha_i \nabla f(x_i)^t d_i = \frac{1}{2} d_i^t H_i^1 d_i + \frac{2}{3} \nabla f(x_i)^t d_i \quad (4.13)$$

Since $\lambda_i > 1$, then from (4.11) we must have

$$\nabla f(x_i)^t d_i < -\frac{1}{2} d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i$$

Substituting in (4.13) we get:

$$\begin{aligned} f(x_i + \alpha_i d_i) - f(x_i) - \frac{1}{3} \alpha_i \nabla f(x_i)^t d_i &< \frac{1}{2} [d_i^t H_i^1 d_i - \frac{1}{2} d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i] \\ &\quad - \frac{1}{12} d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i \end{aligned} \quad (4.14)$$

That the right hand side of (4.14) is ≤ 0 for large i follows exactly in the same manner in which we proved that (4.12) is ≤ 0 . This completes the proof.

Finally, we state certain conditions in Theorem 4.3 below which guarantee that $\lambda_i < 1$ so that $\alpha_i = \lambda_i$ for i large enough.

Theorem 4.3

Let $\{x_i\}$ be a sequence generated by the algorithm. Suppose that $x_i \rightarrow x^*$ and that x^* satisfies the second order optimality conditions for Problem P' . If $z < \frac{\gamma}{4}$, then there is an integer m so that $\lambda_i < 1$ for all $i \geq m$, that is, $\alpha_i = \lambda_i$ for all $i \geq m$.

Proof

By Theorem 4.1 there is an integer m so that for $i \geq m$ we have:

$$\lambda_i = \frac{-\varepsilon^2 \nabla f(x_i)^t d_i}{f(x_i + \varepsilon d_i) + f(x_i - \varepsilon d_i) - 2f(x_i)} = \frac{-\nabla f(x_i)^t d_i}{\frac{1}{2} d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i} \quad (4.15)$$

As in the proof of Theorem 4.1

$$\frac{1}{2} d_i^t (H_i^\varepsilon + H_i^{-\varepsilon}) d_i \geq \frac{\gamma}{4} \|d_i\|^2 \quad \text{for } i \text{ large enough} \quad (4.16)$$

Since w_i solves Problem $D(x_i)$, then there exist scalars $u_{ij} \geq 0$ for $j \in I(x_i)$ such that

$$\nabla f(x_i) + z w_i + \sum_{j \in I(x_i)} u_{ij} a_j = 0 \quad (4.17)$$

$$u_{ij} a_j^t w_i = 0 \quad \text{for } j \in I(x_i) \quad (4.18)$$

From (4.17) and (4.18) it follows that $\nabla f(x_i)^t w_i = -z \|w_i\|^2$. But by Theorem 3.1 x^* is a c-KT point and hence the optimal solution w^* to Problem $D(x^*)$ is $w^* = 0$. Since $x_i \rightarrow x^*$, by continuity of the optimal solution to Problem $D(\cdot)$, and since $b_j - a_j^t x_i > c$ for each $j \in I^+(w_i)$, it follows from (2.3) that $\beta_i = 1$ for large i . Thus $d_i = w_i$ so that

$$\nabla f(x_i)^t d_i = -z \|d_i\|^2 \quad \text{for large } i \quad (4.19)$$

Substituting (4.19) and (4.16) in (4.15), it is clear that $\lambda_i < 1$ for i large enough, and the proof is complete.

REFERENCES

1. L. Armijo, "Minimization of Functions Having Continuous Partial Derivatives," Pacific Journal of Mathematics, Volume 16, pp. 1-3, 1966.
2. M. S. Bazaraa and J. J. Goode, "An Algorithm for Finding the Shortest Element of a Polyhedral Set with Application to Lagrangian Duality," Journal of Mathematical Analysis and Applications, Volume 65, pp. 278-288, 1978.
3. J. W. Daniel, "Stability of the Solution of Definite Quadratic Programs," Mathematical Programming, Volume 5, pp. 41-53, 1973.
4. M. Frank and P. Wolfe, "An Algorithm for Quadratic Programming," Naval Research Logistics Quarterly, Volume 3, pp. 95-110, 1956.
5. P. E. Gill and W. Murray, "Newton-type Methods for Unconstrained and Linearly Constrained Optimization," Mathematical Programming, Volume 7, pp. 311-350, 1974.
6. D. Goldfarb, "Extension of Davidson's Variable Metric Method to Maximization Under Linear Inequality and Equality Constraints," SIAM Journal Applied Mathematics, Volume 17, pp. 739-764, 1969.
7. S. P. Han and O. L. Mangasarian, "Exact Penalty Functions in Nonlinear Programming," Mathematical Programming, Volume 17, pp. 251-269, 1979.
8. G. P. McCormick, "The Variable-Reduction Method for Nonlinear Programming," Management Science, Volume 17, pp. 146-160, 1970.
9. G. P. McCormick, "Second Order Conditions for Constrained Minima," SIAM Journal Applied Mathematics, Volume 15, pp. 641-652, 1967.
10. H. Mukai, "Readily Implementable Conjugate Gradient Methods," Mathematical Programming, Volume 17, pp. 298-319, 1979.
11. H. Mukai, "A Scheme for Determining Step Sizes for Unconstrained Optimization Methods," IEEE Transactions on Automatic Control, Volume AC23, pp. 987-995, 1978.
12. B. A. Murtagh and M. A. Saunders, "Large Scale Linearly Constrained Optimization," Mathematical Programming, Volume 14, pp. 41-72, 1978.
13. E. Polak, Computational Methods in Optimization: A Unified Approach, Academic Press, New York, 1971.
14. J. B. Rosen, "The Gradient Projection Method for Nonlinear Programming, Part I, Linear Constraints," SIAM Journal Applied Mathematics, Volume 9, pp. 514-553, 1961.
15. D. M. Topkis and A. F. Veinott, "On the Convergence of Some Feasible Direction Algorithms for Nonlinear Programming," SIAM Journal Control Volume 5, pp. 263-279, 1967.

16. P. Wolfe, "Algorithm for a Least-Distance Programming Problem," Mathematical Programming Study 1, pp. 190-205, 1974.
17. P. Wolfe, "Methods of Nonlinear Programming," in Nonlinear Programming, edited by J. Abadie, North Holland Publishing Company, Amsterdam, 1967.
18. W. I. Zangwill, "The Convex Simplex Method," Management Science, Volume 14, pp. 221-283, 1967.
19. G. Zontendijk, Methods of Feasible Directions, Elsevier, Amsterdam, 1960.

ON OPTIMALITY CONDITIONS FOR
STRUCTURED FAMILIES OF
NONLINEAR PROGRAMMING PROBLEMS*

J. E. Spingarn

School of Mathematics

Georgia Institute of Technology

Atlanta, Georgia 30332

ABSTRACT

Optimality conditions for families of nonlinear programming problems in R^n are studied from a generic point of view. The objective function and some of the constraints are assumed to depend on a parameter, while others are held fixed. Under suitable conditions, certain strong second-order conditions are shown to be necessary for optimality except possibly for parameter values lying in a negligible set.

*Research sponsored, in part, by the Air Force Office of Scientific Research, under grants number 77-3204 and F49620-79-C-0120.

I. Introduction.

For families of nonlinear programming problems of the type

$$(Q_p) \quad \min f(x,p) \text{ in } x \text{ subject to } g(x,p) \leq 0, \quad h(x,p) = 0, \text{ and} \\ x \in C$$

we derive optimality conditions which are generically necessary in the sense that they hold at all local minimizers for (Q_p) , unless p belongs to a certain first category set of measure zero. Here, P is an open subset of Euclidean space (or more generally a manifold), f , g , and h map $R^n \times P$ into R , R^I , and R^J , respectively, I and J being finite sets, and the inequality $g(x,p) \leq 0$ [resp., the equality $h(x,p) = 0$] is interpreted coordinatewise.

In Spingarn and Rockafellar [7], such conditions for one specific class (Q_p) were derived: right-hand-side perturbations of the constraints and linear perturbations of the objective function. For that class it was demonstrated that, except possibly for problems (Q_p) for p in a set of measure zero, the "strong second-order conditions" (the Kuhn-Tucker conditions with strict complementary slackness, linear independence of the active constraint gradients, and positive definiteness of the Hessian of the Lagrangian on the subspace perpendicular to the gradients of the active constraints) hold at every local minimizer for (Q_p) .

When studying questions of genericity, the class of problems to which the results apply is crucial. The classes of problems

considered in this paper are more general than in [7] in two ways. First, the manner in which f , g , and h depend on p is given more freedom. Rather than requiring perturbations of a special (e.g. right-hand-side) type, we will only require that the family of problems satisfy a general and easily verifiable criterion. Second, in addition to the constraints $g \leq 0$ and $h = 0$, which we refer to as the "variable" constraints, we also investigate the effect of the "structural" or "fixed" constraint $x \in C$ that does not vary with p . The distinction between these two types of constraints is important here because the two types play different roles both in the analysis of the conditions and in the statement of the conditions themselves: the conditions that turn out to be generically necessary for optimality depend on the particular class of problems under consideration.

The regularity conditions that we impose on the set C have been incorporated into our definition of "cyrtohedron". Cyrtohedra, which were introduced in [5], are piecewise smooth sets that can be represented locally by a finite number of nonlinear inequality and equality constraints. They are similar to, but more general than the "manifolds - with - corners" studied by Schecter [4].

The idea to study mathematical programming problems from the generic point of view goes back to the Saigal and Simon study [3] of the complementarity problem. Several others have studied questions which arise in economics concerning the generic properties of equilibrium models and Pareto optima. The dominant notion of

a "generic" property in all of these studies has been the category theoretic one, relative to spaces of differentiable mappings under the Whitney topology, rather than the "measure zero" notion used here, and which we feel is better suited for studying nonlinear programming problems.

II. Preliminaries and notation.

A set $M \subset \mathbb{R}^n$ is a k -dimensional C^s submanifold ($s \geq 1$) if for each $x \in M$ there is an open set $U \subset \mathbb{R}^k$ and a C^s diffeomorphism ϕ mapping U onto a neighborhood of x in M [2]. For any $x = \phi(q) \in M$, $M_x = \text{range } d\phi(q)$ is the tangent space to M at x . If $f : \mathbb{R}^n \rightarrow \mathbb{R}$, then " $f|_M$ " denotes the restriction of f to M . For any $x \in \mathbb{R}^n$, " $\nabla f(x)$ " denotes the ordinary gradient of f at x , while " $\nabla(f|_M)(x)$ " denotes the gradient of $f|_M$ at x , the latter being a linear function on M_x . If $\nabla(f|_M)(x) = 0$ (i.e., if $\nabla f(x)$ is perpendicular to M_x), then x is a critical point for f on M , and in this case the Hessian for $f|_M$ at $x = \phi(q)$ is the bilinear function on M_x defined by

$$(\nabla^2(f|_M)(x))(\bar{u}, \bar{v}) = (\nabla^2(f \circ \phi)(q))(u, v)$$

where $\bar{u} = d\phi(x)u$, $\bar{v} = d\phi(x)v$, and $\nabla^2(f \circ \phi)(q)$ is the ordinary Hessian of $f \circ \phi$. If $\nabla^2(f \circ \phi)(q)$ is nonsingular, then x is a nondegenerate critical point [1].

A subset $S \subset \mathbb{R}^n$ is of measure zero provided for every $\varepsilon > 0$, S can be covered by a countable family of n -rectangles, the sum of whose measures is less than ε [1]. $S \subset \mathbb{R}^n$ is of first category provided S is a countable union of sets whose closures have empty interior. We will call S a negligible set if S is both of measure zero and first category.

If F, N, S are submanifolds, $S \subset N$, $f : F \rightarrow N$, then $f : F \rightarrow N$ is transverse to S if $N_y = S_y + df(x)(F_x)$ whenever $y = f(x) \in S$. For a proof of the following, consult Hirsch [1]:

(2.1) THEOREM (Parametric Transversality) Let F, S, N be C^s submanifolds, P open, with $S \subset N$, $\phi : F \times P \rightarrow N$ of class C^s , $s > \max\{0, \dim F + \dim S - \dim N\}$, and let ϕ be transverse to S . Then there is a subset $P' \subset P$ such that $P \setminus P'$ is negligible and for all $p \in P'$, $\phi(\cdot, p) : F \rightarrow N$ is transverse to S .

(2.2) COROLLARY. Let $f : F \times P \rightarrow R$ be C^2 , P open, $F \subset R^n$ a C^2 submanifold, and assume for each $x \in F$ that the Jacobian of the function $p \mapsto \nabla_x f(x, \cdot)$ is of rank n at all $p \in P$. Then except for p in a negligible set, all critical points of $f(\cdot, p)$ on F are nondegenerate.

Proof: Let $TF = \{(x, \zeta) \in R^n \times R^n : x \in F, \zeta \in F_x\}$, $\phi(x, p) = (x, \nabla_x f(x, p))$. For each $p \in P$, $\phi(\cdot, p)$ is transverse to $F \times \{0\}$ if, and only if, all the critical points of $f(\cdot, p)$ on F are nondegenerate. But the hypothesis implies that $\phi(x, \cdot)$ is transverse to $F \times \{0\}$ for each $x \in F$, and hence that $\phi(\cdot, \cdot)$ is transverse to $F \times \{0\}$. We then apply the theorem with $s = 1$, $N = TF$, and $S = F \times \{0\}$. \square

(2.3) COROLLARY. Let F, S, N be C^1 submanifolds, P open, $S \subset N$, $\phi : F \times P \rightarrow N$ of class C^1 , $\dim F + \dim S - \dim N < 0$, and let ϕ be transverse to S . Then there is a subset $P' \subset P$ such that $P \setminus P'$ is negligible and $\phi(x, p) \notin S$ for all $p \in P'$, $x \in F$.

Proof: It follows from the fact that if $\phi(\cdot, p)$ is transverse to S , then the dimension requirements force $\phi(x, p) \notin S$ for all $x \in F$. \square

For any $S \subset \mathbb{R}^n$, "rank S " denotes the dimension of the linear subspace "span S " spanned by S . "relint S " is the interior of S relative to the affine flat spanned by S .

Let $U \subset \mathbb{R}^n$ be an open set, G_α , $\alpha \in A$ and H_β , $\beta \in B$, finite collections of differentiable functions on U . For any $A_0 \subset A$ and $x \in U$, define

$$\Gamma(x, A_0) = \{\nabla G_\alpha(x) : \alpha \in A_0\} \cup \{\nabla H_\beta(x) : \beta \in B\}$$

$$Z(A_0) = \{y \in U : 0 = G_\alpha(y) = H_\beta(y) \forall \alpha \in A_0, \forall \beta \in B\}.$$

A nonempty connected set $C \subset \mathbb{R}^n$ is a cyrtohedron of class C^s ($s \geq 1$) if for every $\bar{x} \in C$, there are finitely many C^s functions G_α , $\alpha \in A$, and H_β , $\beta \in B$, defined on a neighborhood $U \subset \mathbb{R}^n$ of \bar{x} such that $\bar{x} \in Z(A)$ and

(2.4) (a) For all $x \in U$, $x \in C$ if, and only if,

$$G_\alpha(x) \leq 0 \quad \forall \alpha \in A \text{ and } H_\beta(x) = 0 \quad \forall \beta \in B.$$

(b) If $\sum_A a_\alpha \nabla G_\alpha(\bar{x}) + \sum_B b_\beta \nabla H_\beta(\bar{x}) = 0$ for some $a \in \mathbb{R}_+^A$ and $b \in \mathbb{R}^B$, then $a = 0$ and $b = 0$.

(c) For each $A_0 \subset A$ there is an integer $s(A_0)$ such that $\text{rank } \Gamma(x, A_0) = s(A_0)$ for all $x \in U$.

If C is a cyrtohedron, then U may always be chosen [5] so that

- (b') For all $x \in U$, (b) holds with x in place of \bar{x}
- (c') If $A_0 \subset A_1 \subset A$ and $s(A_0) = s(A_1)$ then $Z(A_0) = Z(A_1)$
- (d) For all $A_0 \subset A$, $Z(A_0)$ is connected $(n-s(A_0))$ - dimensional submanifold

and when this is done, we will say that $(G_\alpha (\alpha \in A), H_\beta (\beta \in B), U)$, or more briefly (G_α, H_β, U) , is a local representation (abbr. l.r.) for C .

Let (G_α, H_β, U) be a l.r., $x \in C \cap U$. Letting $A_+(x) = \{\alpha \in A : G_\alpha(x) = 0\}$, we define

$$T_C(x) = \{\zeta \in \mathbb{R}^n : \zeta \cdot \nabla G_\alpha(x) \leq 0 \ \forall \alpha \in A_+(x), \ \zeta \cdot \nabla H_\beta(x) = 0 \ \forall \beta \in B\}$$

$$L_C(x) = \{\zeta \in \mathbb{R}^n : \zeta \cdot \nabla G_\alpha(x) = 0 \ \forall \alpha \in A_+(x), \ \zeta \cdot \nabla H_\beta(x) = 0 \ \forall \beta \in B\}.$$

The dimension of C is defined to be $\dim C = n - |B|$. It does not depend on x or on the particular local representation.

For $x, y \in C$, define an equivalence relation \sim by specifying $x \sim y$ if, and only if, there is a sequence $x = x_0, x_1, \dots, x_p = y$ in C such that for each pair (x_i, x_{i+1}) ($i = 0, \dots, p-1$), there is a l.r. (G_α, H_β, U) such that $Z(A) \supset \{x_i, x_{i+1}\}$. The equivalence classes under this relation are the faces of C . The proof of the following may be found in [5]:

(2.5) THEOREM. Let $C \subset \mathbb{R}^n$ be a cyrtohedron of class C^s ($s \geq 1$), $x \in C$. Then x lies on a unique face F of C , and F is a connected C^s submanifold of \mathbb{R}^n . The tangent space F_x to F at x is $L_C(x)$. There is a l.r. (G_α, H_β, U) for C such that $x \in Z(A)$, and for any such l.r., $Z(A) = F \cap U$ and $\dim F = \dim L_C(x) = n - s(A)$.

III. First-order conditions.

In this section, certain first-order conditions (3.2) are shown to be generically necessary for optimality. This will be done by showing that a constraint qualification, called the "independence criterion" is generically satisfied at all feasible points. We will then appeal to a result from [5] stating that in the presence of this qualification, these conditions are necessary for optimality.

It is assumed here that f , g , and h are of class C^1 on R^n , and $C \subset R^n$ is a d -dimensional cyrtohedron.

If x is feasible for (Q), the independence criterion (IC) is satisfied for (Q) at x if for any $a \in R^{I_+}$ and $b \in R^J$,

$$(IC) \quad \sum_{I_+} a_i \nabla g_i(x) + \sum_J b_j \nabla h_j(x) \in L_C(x)^\perp \text{ implies } 0 = a = b.$$

It is trivially satisfied if $I_+ = J = \emptyset$. If $C = R^n$, IC says that the gradients of the active constraints at x are linearly independent. More generally, if F is the face of C that contains x , IC says that the gradients of $g_i|_F$, $i \in I_+$ and $h_j|_F$, $j \in J$ at x form a linearly independent set. From [5], we have:

(3.1) THEOREM. If \bar{x} is a local minimizer for (Q) and if the independence criterion is satisfied at \bar{x} , then there exist $\bar{y} \in R_+^I$ and $\bar{z} \in R^J$ such that

- (3.2) (i) $-\nabla_{\bar{x}} L(\bar{x}, \bar{y}, \bar{z}) \in N_C(\bar{x})$
 (ii) $\bar{y}_i > 0$ implies $g_i(\bar{x}) = 0 \quad \forall i \in I$.

Showing that the first-order conditions 3.2 are necessary for optimality in "most" problems reduces, by this theorem, to showing that IC holds for "most" problems.

Let $E : R^n \rightarrow R^I \times R^J$ be given by $E(x) = (g(x), h(x))$. (If $I = J = \emptyset$, then $R^I \times R^J = \{0\}$ and $E(x) = 0$), and for any $I' \subset I$, define $\Omega(I') = \{(x, 0) \in R^I \times R^J : x_i = 0 \quad \forall i \in I'\}$.

(3.3) LEMMA. Let x be feasible for (Q). The independence criterion for (Q) is satisfied at x if, and only if,

$$(3.4) \quad R^I \times R^J = dE(x)(L_C(x)) + \Omega(I_+(x)).$$

Proof: $dE(x)$ is the $(|I| + |J|) \times n$ matrix whose rows are the gradients of f_i , $i \in I$, and g_j , $j \in J$. Let $c = \begin{pmatrix} a \\ b \end{pmatrix}$ represent an arbitrary $(|I| + |J|)$ -dimensional column vector. IC holds at x if, and only if, there exists no $c = \begin{pmatrix} a \\ b \end{pmatrix} \neq 0$ with $a \in R^{I_+}$ such that $c'dE(x)z = 0$ for all $z \in L_C(x)$, an assertion that is easily seen to be equivalent to 3.4. \square

(3.5) LEMMA. Let F be a face of C . If $E|_F : F \rightarrow R^I \times R^J$ is transverse to $\Omega(I')$ for every $I' \subset I$, then IC is satisfied at every $x \in F$ which is feasible for (Q).

Proof: Immediate from the definition of transversality and the preceding lemma. \square

Now suppose that f , g , and h are of class C^1 on $R^n \times P$, and let $E : R^n \times P \rightarrow R^I \times R^J$ be given by $E(x,p) = (g(x,p), h(x,p))$. We say the family (Q_p) is full with respect to constraints if the Jacobian of the function $p' \mapsto E(x,p')$ has rank $|I| + |J|$ at every $(x,p) \in C \times P$. The usual right-hand-side perturbations fit this requirement; here, $P = R^I \times R^J$, and for any $p = (s,t) \in P$, $g(x,p) = u(x) - s$ and $h(x,p) = v(x) - t$ for some C^1 functions u and v .

(3.6) PROPOSITION. Let F be a face of C . Assume that C , g , and h are of class C^s , with $s > \max(0, d - |J|)$ ($d = \dim C$), and that (Q_p) is full with respect to constraints. Then there is a subset $P_F \subset P$ such that $P \setminus P_F$ is negligible, and for all $p \in P_F$, IC holds at all $x \in F$ which are feasible for (Q_p) .

Proof: Since (Q_p) is full with respect to constraints, the Jacobian of the function $p' \mapsto E(x,p')$ has rank $|I| + |J|$ at all $(x,p) \in F \times P$. In particular, $E|(F \times P) : F \times P \rightarrow R^I \times R^J$ is trivially transverse to any submanifold of $R^I \times R^J$.

For each $I' \subset I$, $\Omega(I') \subset R^I \times R^J$ is a subspace of dimension $|I| - |I'| \leq |I|$. Since $E|(F \times P)$ is transverse to $\Omega(I')$, and since

$$\dim F + \dim \Omega(I') - \dim(R^I \times R^J) \leq d + |I| - (|I| + |J|) = d - |J|,$$

and since F and E are of class C^s with $s > \max(0, d - |J|)$, it follows by 2.1 that there is a subset $P_F \subset P$ with negligible complement such that for all $p \in P_F$, the function $E|_{(F \times \{p\})} : F \rightarrow R^I \times R^J$ is transverse to $\Omega(I')$. Clearly, it may be assumed that P_F has this property for all $I' \subset I$. By Lemma 3.5, for all $p \in P_F$, if $x \in F$ is feasible for (Q_p) , then IC is satisfied at x . \square

(3.7) LEMMA. A cyrtohedron has only countably many faces.

Proof: Let (G_α, H_β, U) be a l.r. for C . It is enough to show that U meets only countably many faces of C . For each $x \in U \cap C$, define $A_+(x) = \{\alpha \in A : G_\alpha(x) = 0\}$. Fix $A' \subset A$, and let $T(A') = \{x \in U \cap C : A_+(x) = A'\}$. Clearly it is enough to show that $T(A')$ meets only countably many faces of C . For each $y \in T(A')$ there is an open ball $V_y \subset U$ about y , such that $(G_\alpha(\alpha \in A'), H_\beta(\beta \in B), V_y)$ is a l.r. for C and $G_\alpha < 0$ in V_y for all $\alpha \in A \setminus A'$. By definition of "face", the set $V_y \cap T(A')$ is contained in a single face of C . Thus each $y \in T(A')$ has a neighborhood in $T(A')$ lying in a single face of C , showing $T(A')$ meets only countably many faces of C . \square

(3.8) PROPOSITION. Let C , g , and h be of class C^s with $s > \max(0, d - |J|)$ ($d = \dim C$), and let (Q_p) be full with respect to constraints. Then there is a subset $P_C \subset P$ with negligible complement such that if $p \in P_C$ and x is feasible for (Q_p) , then x satisfies IC for (Q_p) .

Proof: For each face F of C , let P_F be as in Proposition 3.6.

By Lemma 3.7, $P_C = \cap_F P_F$ has the desired property. \square

Combining this with Theorem 3.1, we obtain

(3.9) THEOREM. Let C , g , and h be of class C^s with $s > \max(0, d-|J|)$ ($d = \dim C$), and let (Q_p) be full with respect to constraints. Then there is a subset $P_C \subset P$ with negligible complement such that if $\bar{p} \in P_C$ and $\bar{x} \in C$ is a local minimizer for $(Q_{\bar{p}})$, then there exists $(\bar{y}, \bar{z}) \in R_+^I \times R^J$ such that

- (3.10) (i) $-\nabla_x L(\bar{x}, \bar{y}, \bar{z}, \bar{p}) \in N_C(\bar{x})$
- (ii) $\forall i \in I, \bar{y}_i > 0$ implies $g_i(\bar{x}, \bar{p}) = 0$

The assumption that (Q_p) is full with respect to constraints can be weakened somewhat:

(3.11) COROLLARY. If there is a closed subset $P' \subset P$ of measure zero such that the subfamily $\{(Q_p) : p \in P \setminus P'\}$ is full with respect to constraints, then the conclusion of 3.9 holds.

Proof: Apply Theorem 3.9 to the subfamily. \square

IV. Generic Second-Order Conditions

Henceforth, f , g , h , and C are assumed to be of class C^2 .

Let $R^r = R^n \times R^I \times R^J$, and define $\tau: R^r \rightarrow R^r$ by

$$\tau(w) = (\nabla_x L(w), -\nabla_y L(w), -\nabla_z L(w)) \quad (w = (x, y, z)).$$

If we let $\tilde{C} = C \times R_+^I \times R^J$, then $\tilde{C} \subset R^r$ is also a cyrtohedron of class C^2 .

The second-order conditions which we show here to be generically necessary for optimality are the generalized strong second-order conditions discussed previously in Spingarn [5]. A point $\bar{w} = (\bar{x}, \bar{y}, \bar{z}) \in \tilde{C}$ is said to satisfy these conditions for the problem (Q) if

- (SSOC)
- (i) \bar{x} is feasible for (Q)
 - (ii) $-\nabla_x L(\bar{w}) \in \text{relint } N_C(\bar{x})$
 - (iii) $\forall i \in I, \bar{y}_i > 0$ if, and only if, $g_i(\bar{x}) = 0$
 - (iv) The independence criterion for (Q) holds at \bar{x}
 - (v) If F is the face of C containing \bar{x} , then $(\nabla_x^2 (L|_F)(\bar{w}))(\zeta, \zeta) > 0$ for all $\zeta \in R^n$ satisfying $0 \neq \zeta \in L_C(\bar{x})$, and $\zeta \cdot \nabla g_i(\bar{x}) = \zeta \cdot \nabla h_j(\bar{x}) = 0$ for all $i \in I_+, j \in J$.

For a more detailed discussion of these conditions, and a discussion of their relationship to the classical conditions, we refer to [5].

If a particular representation (G_α, H_β, U) for C near \bar{x} is chosen, these conditions could be rephrased in terms of the functions G_α and H_β , without ever mentioning the set C . We have avoided doing this for several reasons. Most important, the roles played by the two types of constraints, fixed and variable, are not the same, and the above formulation emphasizes the different ways they enter into the conditions. Also, this formulation suggests the possibility of generalizing the conditions to a broader class of sets C . Consider, for example, the set

$$C = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : |x| \leq 1 \text{ and } x_1 + x_2 + x_3 \geq |x|\}.$$

Because no representation of the type 2.4 exists for C near $\bar{x} = 0$, C is not a cyrtohedron. But, like a cyrtohedron, C can be partitioned into "faces" (four in this case) that are submanifolds, and $N_C(x)$ and $L_C(x)$ have obvious meanings, so the conditions SSOC, as stated, are still meaningful. In fact, C has all the properties that are required for our proof of the genericity of SSOC. We do not know if there is a "natural" broader class to which our results apply. It seems that the conditions should be generic for sets C that look (in some sense) locally like the intersection of a cone with a neighborhood of the origin. One possible class would be those sets C such that each $x \in C$ has a neighborhood U such that for some diffeomorphism ϕ , and some closed convex cone

K , $\phi(x) = 0$ and $\phi(C \cap U) = \phi(U) \cap K$. For this class, the proof of the genericity of the above conditions does indeed go through, but since this class does not seem to include cyrtohedra, it is not as broad as one would like.

We observed in [5] that for any $\bar{w} = (\bar{x}, \bar{y}, \bar{z}) \in \tilde{C}$ with \bar{x} feasible for (Q),

$$(4.1) \quad \bar{w} \text{ satisfies 3.2} \iff -\tau \bar{w} \in N_{\tilde{C}}(\bar{w})$$

$$(4.2) \quad \text{if } \bar{x} \text{ is a local minimizer, SSOC holds} \iff$$

$$(a) \quad -\tau \bar{w} \in \text{relint } N_{\tilde{C}}(\bar{w}) \text{ and}$$

$$(b) \quad \bar{w} \text{ is a nondegenerate critical point for } L \text{ on } \tilde{G}.$$

Our proof of the generic necessity of SSOC will proceed as follows. If \bar{x} is a local minimizer, then from the previous section we have the (generic) existence of \bar{y} and \bar{z} satisfying the first-order conditions 3.2. Let $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$. From 4.1, it follows that $-\tau \bar{w} \in N_{\tilde{C}}(\bar{w})$. By 4.3, $-\tau \bar{w} \in N_{\tilde{C}}(\bar{w})$ implies (generically) that $-\tau \bar{w} \in \text{relint } N_{\tilde{C}}(\bar{w})$, so it will follow that 4.2a holds. By 2.2 we know (generically) that all critical points of L on all faces of \tilde{C} are nondegenerate, so that 4.2b also, and hence SSOC holds. \square

(4.3) PROPOSITION. Let $C \subset \mathbb{R}^n$ be a cyrtohedron of class C^2 , P open, and $\tau: \mathbb{R}^n \times P \rightarrow \mathbb{R}^n$ a C^1 function. Suppose that for each $(x, p) \in C \times P$, the map $p' \mapsto \tau(x, p')$ has Jacobian of rank n at (x, p) .

Then there is a subset $P_0 \subset P$ such that $P \setminus P_0$ is negligible and for all $p \in P_0$ and all $x \in C$,

$$(4.4) \quad (x, p) \in N_C(x) \Rightarrow \tau(x, p) \in \text{relint } N_C(x).$$

Proof: Let F be a face of C . For every $x \in F$, there is a l.r. (G_α, H_β, U) for which $x \in Z(A) = F \cap U$. For each such l.r., we will show that there is a subset $\bar{P} \subset P$ with $P \setminus \bar{P}$ negligible such that if $p \in \bar{P}$ and $x \in F \cap U$, then 4.4 holds. F may be covered by sets U corresponding to countably many such l.r. Taking the intersection of the corresponding sets \bar{P} gives a set P_F such that 4.4 is satisfied for all $p \in P_F$ and all $x \in F$. By Lemma 3.7, the set $P_0 = \bigcap_F P_F$ (taking the intersection over all faces F of C) will have the desired property.

So fix a face F , $x \in F$, and (G_α, H_β, U) such that $x \in Z(A) = F \cap U$. For any $\tau \in N_C(x) \setminus \text{relint } N_C(x)$, it follows from the definition of $N_C(x)$ that there exists $A_0 \subset A$ such that $\tau \in \text{span } \Gamma(x, A_0) \not\subset \text{span } \Gamma(x, A)$. Now, for any $A_0 \subset A$, $s(A_0) = \text{rank } \Gamma(x, A_0)$ for all $x \in U$, so it suffices to show for any $A_0 \subset A$ with $s(A_0) < s(A)$, that except for $p \in P$ belonging to a negligible subset, $\tau(x, p) \notin \text{span } \Gamma(x, A_0)$ for all $x \in F \cap U$. Henceforth, we fix $A_0 \subset A$ such that $s(A_0) < s(A)$.

Let $N = (F \cap U) \times \mathbb{R}^n$ and $S = \{(x, w) \in N : w \in \text{span } \Gamma(x, A_0)\}$. Since C is of class C^2 , S is a $(\dim F + s(A_0))$ -dimensional C^1 submanifold, and N is a $(\dim F + n)$ -dimensional C^2 submanifold. Define $\phi(x, p) = (x, \tau(x, p))$, and fix $x \in F \cap U$, $p \in P$ such that $\phi(x, p) \in S$. By hypothesis, $\text{range } d_p \phi(x, p) = \{0\} \times \mathbb{R}^n$. Also,

$N_{\phi(x,p)} = F_x \times \mathbb{R}^n$ and $S_{\phi(x,p)} = F_x \times K$ for some subspace $K \subset \mathbb{R}^n$.

Hence $N_{\phi(x,p)} = S_{\phi(x,p)} + \text{range } d_p \phi(x,p)$, showing that $\phi(x, \cdot) :$

$P \rightarrow N$ is transverse to S , and hence that $\phi : (F \cap U) \times P \rightarrow N$ is transverse to S . By 2.5,

$$\dim(F \cap U) + \dim S - \dim N = \dim F + s(A_0) - n < \dim F + s(A) - n = 0.$$

So, by 2.3, there is a subset $P(A_0) \subset P$ with $P \setminus P(A_0)$ negligible, such that for all $p \in P(A_0)$ and all $x \in F \cap U$, we have $\phi(x,p) \notin S$, or equivalently, $\tau(x,p) \notin \text{span } \Gamma(x, A_0)$. \square

The family (Q_p) will be called full provided the function $p' \mapsto \nabla_w L(w, p') \in \mathbb{R}^r$ has Jacobian of rank r at all $(w, p) \in \tilde{C} \times P$. This notion should not be confused with "full with respect to constraints", which is a weaker property:

(4.5) PROPOSITION. If (Q_p) is full, then it is full with respect to constraints.

Proof: (Q_p) is full with respect to constraints if, and only if, the Jacobian of $p' \mapsto \nabla_{y,z} L(w, p')$ has full rank $|I| + |J|$ at every $(w, p) \in \tilde{C} \times P$. When it does not have full rank, then neither does the Jacobian of $p' \mapsto \nabla_w L(w, p') = \nabla_{x,y,z} L(w, p')$, so (Q_p) is not full. \square

For an example, suppose that $u : \mathbb{R}^n \rightarrow \mathbb{R}^I$, $v : \mathbb{R}^n \rightarrow \mathbb{R}^J$, and $\ell : \mathbb{R}^n \rightarrow \mathbb{R}$ are C^2 functions. Let $P = \mathbb{R}^n \times \mathbb{R}^I \times \mathbb{R}^J$, and for any $p = (q, s, t) \in P$, define $g(x, p) = u(x) - s$, $h(x, p) = v(x) - t$, and $f(x, p) = \ell(x) - x \cdot q$. Then the Jacobian of $p \mapsto \nabla_w L(w, p)$ is minus the identity matrix, and hence of rank r .

Previously, we saw that the first-order conditions 3.2 and 3.10 are necessary for optimality for most $p \in P$ if (Q_p) is full with respect to constraints and sufficient differentiability is assumed. When (Q_p) is full then, for most p , the stronger conditions SSOC are also satisfied provided that the first-order conditions are:

(4.6) THEOREM. Let $C \subset \mathbb{R}^n$ be a cyrtohedron of class C^2 , P open, and let f , g , and h be C^2 functions on $\mathbb{R}^n \times P$. If (Q_p) is full, then there is a subset $P_0 \subset P$ such that $P \setminus P_0$ is negligible and for all $\bar{p} \in P_0$: if $\bar{x} \in C$ is a local minimizer for $(Q_{\bar{p}})$, and if $\bar{y} \in \mathbb{R}_+^I$ and $\bar{z} \in \mathbb{R}^J$ satisfy 3.10, then SSOC holds.

Proof: Since (Q_p) is full, the hypotheses for Proposition 4.3 are satisfied with $\tilde{C} \subset \mathbb{R}^r$ in place of $C \subset \mathbb{R}^n$ and $-\tau$ in place of τ . So, there is a subset $P' \subset P$ with negligible complement such that for any $p \in P'$ and $w \in \tilde{C}$, $-\tau(w, p) \in N_{\tilde{C}}(w)$ implies $-\tau(w, p) \in \text{relint } N_{\tilde{C}}(w)$.

Since (Q_p) is full, the Jacobian of $p' \mapsto \nabla_w L(w, p') \in \mathbb{R}^r$ is of rank r at every $(w, p) \in \tilde{C} \times P$. By 2.2, for every face \tilde{G} of \tilde{C} , there is a set $P(\tilde{G})$ with negligible complement in P such that $L(\cdot, p)$ has only nondegenerate critical points on \tilde{G} for all $p \in P(\tilde{G})$. Let

$P'' = \cap P(\tilde{G})$, taking the intersection over all (countably many by Lemma 3.7) faces of \tilde{C} , and define $P_0 = P' \cap P''$.

Fix $\bar{p} \in P_0$, \bar{x} a local minimizer for $(Q_{\bar{p}})$, and let $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$

satisfy 3.10. Then $-\tau(\bar{w}, \bar{p}) \in N_{\tilde{C}}(\bar{w})$ by 4.1, which implies that \bar{w} is a critical point for $L(\cdot, \bar{p})$ on the face \tilde{G} of \tilde{C} containing \bar{w} by [5, Lemma 3.1c], and that $-\tau(\bar{w}, \bar{p}) \in \text{relint } N_{\tilde{C}}(\bar{w})$ since $\bar{p} \in P'$. Since $\bar{p} \in P''$, \bar{w} is a nondegenerate critical point. Thus both parts of 4.2 are satisfied and SSOC holds. \square

(4.7) THEOREM. Let $C \subset \mathbb{R}^n$ be a d -dimensional cyrtohedron of class C^s , P open, f of class C^2 and g and h of class C^s on $\mathbb{R}^n \times P$ with $s > \max\{1, d - |J|\}$. If (Q_p) is full, there is a subset $P_0 \subset P$ with $P \setminus P_0$ negligible such that for all $\bar{p} \in P_0$: if $\bar{x} \in C$ is a local minimizer for $(Q_{\bar{p}})$ there exists $(\bar{y}, \bar{z}) \in \mathbb{R}_+^I \times \mathbb{R}^J$ satisfying SSOC.

Proof: Combine Theorems 3.9 and 4.6 and Proposition 4.5. \square

In the manner of Corollary 3.11, it follows that the conclusion of Theorems 4.7 is still valid if there is a closed measure zero subset $P' \subset P$ such that the subfamily $\{(Q_p) : p \in P \setminus P'\}$ is full.

Acknowledgement. I wish to thank Professor R. T. Rockafellar for his many helpful suggestions.

References

1. M. Hirsch, Differential Topology, Springer-Verlag, 1976.
2. J. W. Milnor, Topology from the Differentiable Viewpoint, Univ. Press of Virginia (1965).
3. R. Saigal and C. Simon, "Generic properties of the complementarity problem", Math. Programming 4 (1973), 324-335.
4. S. Schecter, "Structure of the demand function and pareto optimal set with natural boundary conditions", J. Math. Econ. 5 (1978) 1-21.
5. J. E. Spingarn, "Fixed and variable constraints in sensitivity analysis", to appear in Siam J. Control and Optimization.
6. _____, "Second-order optimality conditions that are necessary with probability one", Proceedings, Symposium on Mathematical Programming with Data Perturbations, George Washington University, May 1979, forthcoming.
7. _____, and R. T. Rockafellar, "The generic nature of optimality conditions in nonlinear programming", Math of O. R. 4 (1979).

Second-Order Conditions that are
Necessary with Probability One*

Jonathan E. Spingarn**

School of Mathematics

Georgia Institute of Technology

Atlanta, Georgia 30332

* Presented at the Symposium on Mathematical Programming with Data Perturbations, George Washington University, May 1979.

** This research was supported, in part, by the Air Force Office of Scientific Research, under grant number 79-0120 at the Georgia Institute of Technology, Atlanta.

I. Introduction.

In nonlinear programming theory there is a large gap between the weak first-order conditions that are necessary for optimality and the much stronger second-order conditions that have been found useful in the design and analysis of algorithms. It is common practice to assume (without giving any real mathematical justification) that very strong optimality conditions are satisfied at a minimizer, and to base convergence proofs, and thus to justify algorithms, on the basis of such assumptions. Of course, for any given problem, those a priori assumptions cannot be checked, unless the solution is already known.

In this paper, we discuss a "generic" approach to optimality conditions that has been developed in Spingarn and Rockafellar [10] and Spingarn [7,8,9]. Rather than talking about conditions that are necessary for optimality in specific problems, we discuss instead conditions necessary for optimality for most problems in a family of problems. More precisely, for a family $(Q(p))$ of nonlinear programming problems indexed by a parameter $p \in P \subset \mathbb{R}^n$ we study conditions which, unless p belongs to a negligible set, hold at all local minimizers for $(Q(p))$ where by negligible we mean a first category set of measure zero in P .

This approach gives a rigorous mathematical underpinning to the a priori assumption of conditions which are not truly necessary

for optimality, by describing the exact sense and the circumstances in which these conditions can be expected to hold. Another attractive feature of the theory is that "constraint qualifications", which are normally required to prove the necessity of Kuhn-Tucker type first-order conditions, need not be assumed to obtain conditions which are merely generically necessary.

In this paper, no proofs are presented. Instead, we refer the reader to the references [7,8,10].

II. A simple class of perturbations.

Consider the basic problem

$$(Q) \quad \min f(x) \text{ over all } x \in R^n \text{ such that} \\ g(x) \leq 0 \quad \text{and} \quad h(x) = 0 ,$$

where the functions $f : R^n \rightarrow R$, $g : R^n \rightarrow R^m$, and $h : R^n \rightarrow R^k$ are continuously differentiable.

The standard first-order conditions for local optimality of x in (Q) are that x should be feasible and there should exist vectors $y \in R_+^m$ and $z \in R^k$ such that

$$(KT) \quad \nabla f(x) + y' \nabla g(x) + z' \nabla h(x) = 0 \\ \text{and for all } i \notin I_+(x), \quad y_i = 0$$

where

$$I_+(x) = \{i : 1 \leq i \leq m, g_i(x) = 0\}.$$

These conditions are not actually necessary for optimality. They are only necessary under an additional assumption called a "constraint qualification", the simplest such being

$$(CQ) \quad \{\nabla g_i(x) : i \in I_+(x)\} \cup \{\nabla h_j(x) : j=1, \dots, k\}$$

is linearly independent.

When the functions f , g , and h are twice differentiable, a vector x is said to satisfy the strong second-order conditions for local optimality in (Q) if (CQ) holds, and there exists $y \in R_+^m$ and $z \in R^k$ such that (KT) holds with

$$\begin{aligned} &y_i > 0 \quad \text{for all } i \in I_+(x), \quad \text{and} \\ &\text{every nonzero } w \in R^n \text{ for which } w \cdot \nabla g_i(x) = 0 \\ &\text{for all } i \in I_+(x) \quad \text{and } w \cdot \nabla h_j(x) = 0 \text{ for all } j \quad \text{also} \\ &\text{satisfies } w \cdot H(x, y, z)w > 0, \end{aligned}$$

where $H(x, y, z)$ is the Hessian of the Lagrangian function in (Q) :

$$H(x, y, z) = \nabla^2 f(x) + \sum_{i=1}^m y_i \nabla^2 g_i(x) + \sum_{j=1}^k z_j \nabla^2 h_j(x).$$

These conditions are known to guarantee that x is an isolated locally optimal solution to (Q) . They also have other important consequences, for example with respect to the sensitivity of x

to changes in a parameter; cf. Hestenes [3], Fiacco [1].

The strong conditions are useful for proving convergence results; for example, cf. Robinson [5], Rockafellar [6], Powell [4], Fiacco and McCormick [2].

Let us embed (Q) in the following family of nonlinear programming problems

$$(Q(v,u,t)) \quad \min f(x) - x \cdot v \text{ over all } x \in R^n \\ \text{such that } g(x) \leq u, \quad h(x) = t.$$

The original problem (Q) then coincides with $Q(0,0,0)$. Any particular problem in this family may be "bad" in the sense that the strong conditions may fail to hold at some local minimizer for that problem. However, the set of bad problems is small, as the following shows [10]:

THEOREM 1. Suppose f is of class C^2 and g and h are of class C^{n-k} . Then except for (v,u,t) belonging to a set of measure zero in $R^n \times R^m \times R^k$, $(Q(v,u,t))$ is such that every local optimal solution x satisfies the strong second-order conditions.

III. General perturbations.

Next, we examine what happens when more general families of problems are allowed. The families we wish to consider are of the form

$$\begin{aligned} (Q(p)) \quad & \min f(x,p) \text{ over all } x \text{ satisfying} \\ & g(x,p) \leq 0, \quad h(x,p) = 0 \end{aligned}$$

with p ranging over some open subset P of Euclidean space.

The family $Q(v,u,t)$ just considered clearly is a special case.

Obviously, some additional assumption is required in order to guarantee that the strong conditions fail only in a negligible subfamily. After all, we could start with a "bad" problem (Q) for which the strong conditions fail at some local minimum, and then, by introducing trivial perturbations so that $(Q(p)) = (Q)$ for all p , we would obtain a family for which the conditions fail for every problem. The problem here is that the indicated family would not be "rich" enough; it would not contain enough perturbations.

The following definitions specify two different ways a family can be "rich". If g and h are of class C^1 , let us say that the family $(Q(p))$ is full with respect to constraints if the Jacobian of the function $p' \rightarrow (g(x,p), h(x,p)) \in R^{m+k}$ has full rank $m+k$ at every $(x,p) \in R^n \times P$. For any $w = (x,y,z) \in R^r$ ($r=n+m+k$) and $p \in P$, let

$$L(w,p) = f(x,p) + y'g(x,p) + z'h(x,p)$$

be the Lagrangian for $(Q(p))$. If f , g , and h are of class C^2 , the family $(Q(p))$ will be called full provided the function $p' \rightarrow \nabla_w L(w,p') \in R^r$ has full rank r at all $(w,p) \in R^r \times P$. Every full family is automatically full with respect to constraints.

These two properties are sufficient to guarantee the generic necessity of the first-order (KT) and strong second-order conditions, respectively:

THEOREM 2. (a) Let g and h be of class C^s on $R^n \times P$ with $s > \max(0, n-k)$ and let $(Q(p))$ be full with respect to constraints. Then there is a subset $P' \subset P$ with negligible complement such that if $\bar{p} \in P'$ and \bar{x} is a local minimizer for $(Q(\bar{p}))$, then there exists $(\bar{y}, \bar{z}) \in R_+^m \times R^k$ satisfying (KT).

(b) Let f be of class C^2 and g and h of class C^s on $R^n \times P$ with $s > \max(1, n-k)$. If $(Q(p))$ is full, then there is a subset $P' \subset P$ with negligible complement such that for all $\bar{p} \in P'$: if x is a local minimizer for $(Q(\bar{p}))$ there exists $(\bar{y}, \bar{z}) \in R_+^m \times R^k$ satisfying the strong second-order conditions.

To see how Theorem 2 can be applied, consider again the family $(Q(v, u, t))$. We take $p = (v, u, t)$, so for any $w = (x, y, z)$,

$$L(w, p) = f(x) - x \cdot v + y'(g(x) - u) + z'(h(x) - t).$$

We may then compute

$$\nabla_w L(w, p) = \begin{pmatrix} \nabla f(x) - v + \sum y_i \nabla g_i(x) + \sum z_j \nabla h_j(x) \\ \vdots \\ g_i(x) - u_i \\ \vdots \\ h_j(x) - t_j \\ \vdots \end{pmatrix}$$

and hence $\nabla_p \nabla_w L(w, p) = -I$, where I is the $(n+m+k)$ -dimensional identity matrix, which is trivially of rank $n+m+k$.

The full rank criteria given in Theorem 2 are sufficient, but not necessary for the generic necessity of the strong conditions. However, the rank criteria can be weakened (and thus the theorem strengthened) slightly. To illustrate, consider the family

$$(Q(p)) \quad \text{minimize } x^4 + p^2 x \quad \text{over all } x \in \mathbb{R}.$$

The Lagrangian for $(Q(p))$ is $L(x, p) = x^4 + p^2 x$ (since there are no constraints) so $\nabla_p \nabla_x L(x, p) = 2p$. For Theorem 2 to apply, it would have to be true that $2p \neq 0$ for all p . This is not a real obstacle though, since the theorem could be applied to the subfamily $\{Q(p) : p \neq 0\}$. The same reasoning shows in general that the result of the theorem holds whenever the set of p values for which the rank condition fails is contained in a closed measure zero subset of P :

COROLLARY 1. If there is a closed subset $P' \subset P$ of measure zero such that the subfamily $\{(Q(p)) : p \in P \setminus P'\}$ is full [with respect to constraints], then the conclusion of Theorem 2a[resp., of Theorem 2b] holds.

Another minor extension is suggested by the family

$$(Q(p)) \quad \text{minimize } px^2 + (1-p)x \quad \text{over all } x \in R$$

where $p \in R$. In this case, $\nabla_p \nabla_x L(x, p) = 2x - 1$. For Theorem 1 to apply, it would have to be the case that $2x - 1 \neq 0$ for all $x \in R$. Nonetheless, it is possible to conclude in such an instance that except for p in a negligible set, the strong conditions hold for $(Q(p))$ at all local minimizers other than possibly $x = \frac{1}{2}$:

COROLLARY 2. If there is a closed set $K \in R^n$ such that the rank condition of Theorem 2 holds except for $x \in K$, then the conclusion of that theorem holds, except possibly at minimizers which are in K .

IV. Families with selective perturbations.

We are confronted with additional questions when we consider a family like the following one:

$$\begin{aligned} (S(v, u, t)) \quad & \min f(x) - x \cdot v \quad \text{over } x \in R^n \\ & \text{subject to } g(x) \leq u, \quad h(x) = t, \quad \text{and } x \geq 0. \end{aligned}$$

This family is identical to $Q(v,u,t)$, with the important exception that here there is an additional "fixed" constraint $x \geq 0$ that is independent of the parameters. Neither Theorem 1 nor 2 can be applied in this situation.

Those theorems would apply, were we to alter the family by replacing the fixed constraint with a perturbed constraint $x \geq s$. This would yield a family $Q(v,u,t,s)$ for which the strong conditions are necessary except for (v,u,t,s) in a set of measure zero. However, the family of interest, namely $(S(v,u,t)) = (Q(v,u,t,0))$, would be a measure zero subfamily of $(Q(v,u,t,s))$. Thus, although the set of "bad" problems in $(Q(v,u,t,s))$ is negligible, it does not follow that the bad problems in $S(v,u,t)$ are negligible with respect to $S(v,u,t)$.

Rather than concentrate on this particular family, we study the generic behavior of more general families of the form

$$\begin{aligned} (S(p)) \quad & \min f(x,p) \text{ over all } x \in \mathbb{R}^n \\ & \text{subject to } g(x,p) \leq 0, \quad h(x,p) = 0, \quad \text{and } x \in C, \end{aligned}$$

where C is a fixed set. For the family $S(v,u,t)$, we would take $C = \mathbb{R}_+^n$, while the situation in Theorems 1 and 2 requires $C = \mathbb{R}^n$. Concerning the family $(S(p))$, we will address ourselves here to three questions: (1) What reasonable assumptions can we impose on the set C which allow us to develop a theory of generic second-order conditions for $(S(p))$? Intuition suggests that C must be "piecewise C^2 -smooth" in some sense. (2) What are the appropriate

generic second-order conditions? It turns out that these conditions actually depend on the set C , and are not always (but sometimes are) exactly the same as the conditions that would be obtained by replacing the constraint $x \in C$ with inequality or equality constraints and then writing down the usual strong conditions for the problem so obtained. (3) What "rank condition" ensures that these conditions are generic for $(S(p))$?

We begin by stating our assumptions on the set C . These have been incorporated into the definition of "cyrtohedron". The name is taken from the Greek " $\kappa\upsilon\rho\tau\omicron\sigma$ " (= curved, bent) + " $\epsilon\delta\rho\alpha$ " (= side), and is motivated by the fact that these sets look like polyhedra, except that the "faces" instead of being polyhedral, are submanifolds.

Let $U \subset \mathbb{R}^n$ be an open set, G_α , $\alpha \in A$ and H_β , $\beta \in B$, finite collections of differentiable functions on U . For any $A_0 \subset A$ and $x \in U$, define

$$\Gamma(x, A_0) = \{\nabla G_\alpha(x) : \alpha \in A_0\} \cup \{\nabla H_\beta(x) : \beta \in B\}$$

$$Z(A_0) = \{y \in U : 0 = G_\alpha(y) = H_\beta(y) \forall \alpha \in A_0, \forall \beta \in B\}.$$

A nonempty connected set $C \subset \mathbb{R}^n$ is a cyrtohedron of class C^s ($s \geq 1$) if for every $\bar{x} \in C$, there are finitely many C^s functions G_α , $\alpha \in A$, and H_β , $\beta \in B$, defined on a neighborhood $U \subset \mathbb{R}^n$ of \bar{x} such that $\bar{x} \in Z(A)$ and

- (a) For all $x \in U$, $x \in C$ if, and only if,
 $G_\alpha(x) \leq 0 \quad \forall \alpha \in A$ and $H_\beta(x) = 0 \quad \forall \beta \in B$.
- (b) If $\sum_A a_\alpha \nabla G_\alpha(\bar{x}) + \sum_B b_\beta \nabla H_\beta(\bar{x}) = 0$ for some $a \in R_+^A$ and $b \in R^B$, then $a = 0$ and $b = 0$.
- (c) For each $A_0 \subset A$ there is an integer $s(A_0)$ such that
 $\text{rank } \Gamma(x, A_0) = s(A_0)$ for all $x \in U$.

Examples of cyrtohedra. (a) A differentiable submanifold in R^n is a cyrtohedron for which the set A may always be taken to be empty.

(b) Cyrtohedra for which the set A may always be taken either empty or of cardinality one are submanifolds with boundary.

(c) A polyhedral convex set is the intersection of a finite number of closed half-spaces in R^n .

(d) Sets that can be expressed as $C = \{x \in R^n : g_i(x) \leq 0, i = 1, \dots, m, \text{ and } h_j(x) = 0, j = 1, \dots, p\}$, where the functions g_i and h_j are of class C^k and have the property that for every $x \in C$, $\{\nabla g_i(x) : i \in I_+(x)\} \cup \{\nabla h_j(x) : j = 1, \dots, p\}$ is linearly independent, where $I_+(x) = \{i : g_i(x) = 0\}$.

For an example of a simple set that is not a cyrtohedron, consider the set $C \subset R^3$ which consists of all $x = (x_1, x_2, x_3)$ such that $|x| \leq 1$, $x_1 + x_3 \leq 1$, and $-x_1 + x_3 \leq 1$. For this set, there exist no functions G_α, H_β which satisfy the above requirements in a neighborhood of the point $(0, 0, 1)$.

If C is a cyrtohedron, then U may always be chosen so that

- (b') For all $x \in U$, (b) holds with x in place of \bar{x}
- (c') If $A_0 \subset A_1 \subset A$ and $s(A_0) = s(A_1)$ then $Z(A_0) = Z(A_1)$
- (d) For all $A_0 \subset A$, $Z(A_0)$ is connected $(n-s(A_0))$ - dimensional submanifold

and when this is done, we will say that $(G_\alpha(\alpha \in A), H_\beta(\beta \in B), U)$, or more briefly (G_α, H_β, U) , is a local representation (abbr. l.r.) for C .

Let (G_α, H_β, U) be a l.r., $x \in C \cap U$. Letting $A_+(x) = \{\alpha \in A : G_\alpha(x) = 0\}$, we define

$$L_C(x) = \{\zeta \in \mathbb{R}^n : \zeta \cdot \nabla G_\alpha(x) = 0 \ \forall \alpha \in A_+(x), \ \zeta \cdot \nabla H_\beta(x) = 0 \ \forall \beta \in B\}.$$

$$N_C(x) = \left\{ \sum_{\alpha \in A_+(x)} a_\alpha \nabla G_\alpha(x) + \sum_{\beta \in B} b_\beta \nabla H_\beta(x) : a_\alpha \in \mathbb{R}_+^{A_+(x)} \text{ and } b \in \mathbb{R}^B \right\}$$

$N_C(x)$ is the normal cone to C at x , and $L_C(x)$ is the linear approximation to C at x ; the latter is the tangent space at x to the "face" (definition below) of C containing x . The dimension of C is defined to be $\dim C = n - |B|$. It does not depend on x , and none of these definitions depend on the particular local representation chosen.

For $x, y \in C$, define an equivalence relation \sim by specifying $x \sim y$ if, and only if, there is a sequence $x = x_0, x_1, \dots, x_p = y$ in C such that for each pair (x_i, x_{i+1}) ($i=0, \dots, p-1$), there is a l.r. (G_α, H_β, U) such that $Z(A) \supset \{x_i, x_{i+1}\}$. The equivalence classes under this relation are the faces of C .

A few examples help to clarify the latter definition:

- (a) The faces of a polyhedral convex set are the relative interiors of its "faces" in the usual sense (that is, subsets which are the intersection with some supporting hyperplane).
- (b) A submanifold $C \subset \mathbb{R}^n$ has only one face.
- (c) If C is the hemisphere $C = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : |x| \leq 1 \text{ and } x_n \geq 0\}$, then C has four faces, corresponding to the choices of equality or strict inequality in the definition of C :

$$F_1 = \{x : |x| < 1 \text{ and } x_n > 0\}$$

$$F_2 = \{x : |x| = 1 \text{ and } x_n > 0\}$$

$$F_3 = \{x : |x| < 1 \text{ and } x_n = 0\}$$

$$F_4 = \{x : |x| = 1 \text{ and } x_n = 0\}$$

To state the optimality conditions, we need some more definitions. Consider a specific problem

$$(S) \quad \min f(x) \text{ over all } x \in \mathbb{R}^n \text{ such that} \\ g(x) \leq 0, \quad h(x) = 0, \quad \text{and } x \in C.$$

If x is feasible for (S), the independence criterion (IC) is satisfied for (S) at x if for any $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^k$ with $a_i = 0$ for all $i \in I_+$,

$$(IC) \quad \sum_{i=1}^m a_i \nabla g_i(x) + \sum_{j=1}^k b_j \nabla h_j(x) \in L_C(x)^\perp \text{ implies } 0 = a = b.$$

It is trivially satisfied if $m = k = 0$. If $C = \mathbb{R}^n$, IC says that the gradients of the active constraints at x are linearly independent. More generally, IC says that the projections of the gradients of g_i , $i \in I_+$ and h_j at x onto $L_C(x)$ form a linearly independent set.

A set $M \subset \mathbb{R}^n$ is a k -dimensional C^s submanifold ($s \geq 1$) if for each $x \in M$ there is an open set $U \subset \mathbb{R}^k$ and a C^s diffeomorphism ϕ mapping U onto a neighborhood of x in M . For any $x = \phi(q) \in M$, $M_x = \text{range } d\phi(q)$ is the tangent space to M at x . If $f : \mathbb{R}^n \rightarrow \mathbb{R}$, then " $f|_M$ " denotes the restriction of f to M . For any $x \in \mathbb{R}^n$, " $\nabla f(x)$ " denotes the ordinary gradient of f at x , while " $\nabla(f|_M)(x)$ " denotes the gradient of $f|_M$ at x , the latter being a linear function on M_x . If $\nabla(f|_M)(x) = 0$ (i.e., if $\nabla f(x)$ is perpendicular to M_x), then x is a critical point for f on M , and in this case the Hessian for $f|_M$ at $x = \phi(q)$ is the bilinear function on M_x defined by

$$(\nabla^2(f|_M)(x))(\bar{u}, \bar{v}) = (\nabla^2(f \circ \phi)(q))(u, v)$$

where $\bar{u} = d\phi(x)u$, $\bar{v} = d\phi(x)v$, and $\nabla^2(f \circ \phi)(q)$ is the ordinary Hessian of $f \circ \phi$. If $\nabla^2(f \circ \phi)(q)$ is nonsingular, then x is a nongenerate critical point for f on M .

Suppose henceforth that f , g , and h are of class C^2 on \mathbb{R}^n , and that $C \subset \mathbb{R}^n$ is a cyrtohedron of class C^2 . We extend the definition

of the strong second order conditions to the problem (S) by declaring a point $\bar{w} = (\bar{x}, \bar{y}, \bar{z})$ with $\bar{x} \in C$, $\bar{y} \in R_+^m$, and $\bar{z} \in R^k$ to satisfy the conditions whenever

- (SSOC) (i) \bar{x} is feasible for (S)
 (ii) $-\nabla_x L(\bar{w}) \in \text{relint } N_C(\bar{x})$
 (iii) $i \in I$, $\bar{y}_i > 0$ if, and only if, $g_i(\bar{x}) = 0$
 (iv) The independence criterion for (S) holds at \bar{x}
 (v) If F is the face of C containing \bar{x} , then
 $(\nabla_x^2(L|F)(\bar{w}))(\zeta, \zeta) > 0$ for all $\zeta \in R^n$ satisfying $0 \neq \zeta \in L_C(\bar{x})$, and $\zeta \cdot \nabla g_i(\bar{x}) = \zeta \cdot \nabla h_j(\bar{x}) = 0$ for all $i \in I_+$, and all j .

As before, we say the family $(S(p))$ is full provided the map $p' \rightarrow \nabla_w L(w, p') \in R^r$ has full rank r at all $(w, p) \in R^r \times P$. We now have covered all the preliminaries needed to state the final result.

THEOREM 3. Let $C \subset R^n$ be a d-dimensional cyrtohedron of class C^S , P open, f of class C^2 and g and h of class C^S on $R^n \times P$ with $s > \max\{1, d-k\}$. If $(S(p))$ is full, there is a subset $P_0 \subset P$ with $P \setminus P_0$ negligible such that for all $\bar{p} \in P_0$: if $\bar{x} \in C$ is a local minimizer for $(S(\bar{p}))$ there exists $(\bar{y}, \bar{z}) \in R_+^m \times R^k$ satisfying SSOC.

Of course, this result can be slightly improved in the manner of Corollaries 1 and 2.

V. Comparison with the classical conditions.

For problems of the form (Q) we have seen that under mild assumptions, the classical strong conditions

- (SC)
- i) \bar{x} is feasible for (Q).
 - ii) $\nabla f(\bar{x}) + \sum \bar{y}_i \nabla g_i(\bar{x}) + \sum \bar{z}_j \nabla h_j(\bar{x}) = 0$.
 - iii) Strict complementary slackness: $\bar{y}_i > 0 \Leftrightarrow g_i(\bar{x}) = 0$.
 - iv) The gradients of the active constraints, i.e. $\{\nabla g_i(\bar{x}) : i \in I_+\} \cup \{\nabla h_j(\bar{x}) : j=1, \dots, k\}$ form a linearly independent set.
 - v) For any $\zeta \in \mathbb{R}^n$ satisfying $\zeta \neq 0$,
 $\zeta \cdot \nabla g_i(\bar{x}) = 0 \quad \forall i \in I_+$, and $\zeta \cdot \nabla h_j(\bar{x}) = 0, j=1, \dots, k$,
 we have $\zeta' [\nabla^2 f(\bar{x}) + \sum \bar{y}_i \nabla^2 g_i(\bar{x}) + \sum \bar{z}_j \nabla^2 h_j(\bar{x})] \zeta > 0$

are generically necessary for optimality in families of problems containing (Q) (cf. Theorems 1 and 2), and that for problems of the form (S) (i.e., families with fixed cyrtohedron constraints), the more general conditions SSOC are generically necessary for optimality.

Locally, the fixed set C can be represented by inequality and equality constraints; if (G_α, H_β, U) is a local representation for C, then $C \cap U = \{x \in U : G_\alpha(x) \leq 0, \alpha \in A, H_\beta(x) = 0, \beta \in B\}$. So, at least locally, (S) is equivalent to a problem (Q') of the type (Q) (i.e., without "fixed" constraints):

$$\begin{aligned}
 (Q') \quad & \min f(x) \text{ subject to } g_i(x) \leq 0, \quad i = 1, \dots, m, \\
 & h_j(x) = 0, \quad j = 1, \dots, k, \quad G_\alpha(x) \leq 0, \quad \alpha \in A, \\
 & H_\beta(x) = 0, \quad \beta \in B.
 \end{aligned}$$

It is natural to ask what the relationship is between the conditions SSOC for (S) and SC for (Q').

In most cases, the two sets of conditions are essentially equivalent in the following sense. If $(\bar{x}, \bar{y}, \bar{a}, \bar{z}, \bar{b}) \in R^n \times R_+^m \times R_+^A \times R^k \times R^B$ satisfies SC for (Q'), then $(\bar{x}, \bar{y}, \bar{z})$ satisfies SSOC for (S). If $(\bar{x}, \bar{y}, \bar{z}) \in R^n \times R_+^m \times R^k$ satisfies SSOC for (S), then it is possible to find $\bar{a} \in R_+^A$ and $\bar{b} \in R^B$ such that $(\bar{x}, \bar{y}, \bar{a}, \bar{z}, \bar{b})$ satisfies SCi, ii, iii, and for any such \bar{a} and \bar{b} , SCv will automatically hold for (Q'). However, SCiv may fail. For example, if C is a four-sided pyramid in R^3 with apex \bar{x} , SCiv can never be satisfied for (Q') because no set of four vectors in R^3 can be linearly independent. However, SSOCiv can (and usually will) be satisfied at \bar{x} . In fact, $(\bar{x}, \bar{y}, \bar{z})$ will satisfy SSOCiv if and only if the projections onto $L_C(\bar{x})$ of the gradients of the (nonfixed) constraints active at \bar{x} are linearly independent. But $L_C(\bar{x}) = \{0\}$ in this case, so SSOCiv merely says that there are no active constraints at \bar{x} . Of course, one would expect the generic conditions to assert this. If $k > 0$, one would expect the apex of the pyramid to be a minimizer with probability zero. If $k = 0$, it is not unusual that the apex should be a minimizer, but one would expect one or more of the inequality constraints to be active there only with probability zero.

In the most common cases, such as $C = R_+^n$, the set C will be expressible as the set of points which satisfy a finite number of equality and inequality constraints with linearly independent gradients (cf. section III, example (d) under "examples of cyrtohedra"). Then, the two sets of conditions are essentially the same. The main difference is that in the SSOC formulation, no multipliers are associated with the constraints defining the cyrtohedron.

We also remark that the SSOC formulation suggests what the generic conditions should look like if we generalize them to a wider class of fixed sets C . Consider, for example, the set

$$C = \{x = (x_1, x_2, x_3) \in R^3 : |x| \leq 1 \text{ and } x_1 + x_2 + x_3 \geq |x|\}.$$

Because no local representation exists for C near $\bar{x} = 0$, C is not a cyrtohedron. But, like a cyrtohedron, C can be partitioned into "faces" (four in this case) that are submanifolds, and $N_C(x)$ and $L_C(x)$ have obvious meanings, so the conditions SSOC, as stated above, are still meaningful. In fact, C has all the properties that are required for our proof of the genericity of SSOC. For such a set C , it would be impossible to reformulate the problem (S) as a problem in the form of (Q'), so the old conditions SC have no bearing here, although the new conditions SSOC would apply and can be shown to be generically necessary for optimality. We do not know if there is a "natural" broader class to which our results apply. The above example suggests conditions should be generic for sets C that look

(in some sense) locally like the intersection of a cone with a neighborhood of the origin. One possible class would be those sets C such that each $x \in C$ has a neighborhood U such that for some diffeomorphism ϕ , and some closed convex cone K , $\phi(x) = 0$ and $\phi(C \cap U) = \phi(U) \cap K$. For this class, the proof of the genericity of the above conditions does indeed go through. However, this is not as broad a class as we would like; it does not seem even to include the class of cyrtohedra.

REFERENCES

1. A. V. Fiacco, Sensitivity analysis for nonlinear programming using penalty methods, *Math. Programming* 10, pp. 287-311 (1976).
2. A. V. Fiacco and G. P. McCormick, Nonlinear Programming: Sequential Unconstrained Minimization Techniques, Wiley, New York (1968).
3. M. R. Hestenes, Optimization Theory, the Finite Dimensional Case, Wiley, New York (1975).
4. M. J. D. Powell, A method for nonlinear constraints in minimization problems, in Optimization (R. Fletcher, ed.), Academic Press, New York, N. Y. (1972).
5. S. M. Robinson, Perturbed Kuhn-Tucker points and rates of convergence for a class of nonlinear programming algorithms, *Mathematical Programming* 7, 1-16 (1974).
6. R. T. Rockafellar, Augmented Lagrange multiplier functions and duality in nonconvex programming, *SIAM J. Control* 12, 2 268-285 (1974).
7. J. E. Spingarn, Fixed and variable constraints in sensitivity analysis, *SIAM J. Control* 18, 3 (1980).
8. J. E. Spingarn, On optimality conditions for structured families of nonlinear programming problems, forthcoming.
9. J. E. Spingarn, Generic conditions for optimality in constrained minimization problems, Dissertation, Dept. of Mathematics, Univ. of Washington (1977).
10. J. E. Spingarn and R. T. Rockafellar, The generic nature of optimality conditions in nonlinear programming, *Math. of O. R.* 4 (1979).

Jonathan E. Spingarn

SUBMONOTONE SUBDIFFERENTIALS OF
LIPSCHITZ FUNCTIONS

Jonathan E. Spingarn

School of Mathematics
Georgia Institute of Technology
Atlanta, Georgia 30332

ABSTRACT

The class of "lower- C^1 " functions, that is functions which arise by taking the maximum of a compact family of C^1 functions, is characterized in terms of properties of the Clarke subdifferential. A locally Lipschitz function is shown to be lower- C^1 if, and only if, its subdifferential is "strictly submonotone". Other properties of functions with "submonotone" subdifferentials are investigated.

Key words: submonotone mapping, Clarke generalized gradient, lower- C^1 function, nondifferentiable optimization.

Presented at the Tenth International Symposium on Mathematical Programming in Montreal, August 27-31, 1979.

This research was supported by the Air Force Office of Scientific Research under grant number F49620-79-C-0120.

0. Introduction

One of the nice features of convex optimization is the link with "monotone" mappings. Due to this, convex problems can be rephrased as "variational problems", often resulting in considerable simplification. This can be useful for theoretical reasons, by emphasizing when the central justification for a proof or procedure is the monotonicity of the subdifferential. For example, Rockafellar [7,8] has exploited the link between monotone mappings and saddle functions to unify and simplify the existing theory of multiplier methods in convex programming.

It is the aim of this paper to show that a concept closely related to monotonicity, e.g. "submonotonicity", also plays a natural role in the analysis of nondifferentiable, nonconvex problems. We will do this by demonstrating how properties of nondifferentiable functions can be related to monotone-type properties of their Clarke subdifferentials.

Our most important result appears in section IV, where a complete characterization is obtained, in terms of properties of the Clarke subdifferential, for the class of "lower- C^1 " functions, that is functions that arise by taking the maximum of a compact family of C^1 functions. It is shown that these functions are precisely those locally Lipschitz functions whose Clarke subdifferentials are "strictly submonotone".

In section III, some implications of the submonotonicity property are developed, and several equivalent characterizations

are given. This concept is then contrasted with properties that have been discussed by other authors. Among these are regularity in the sense of Clarke [2], quasi-differentiability in the sense of Pshenichnyi [5], lower semi-differentiability in the sense of Rockafellar [9], and semismoothness in the sense of Mifflin [4].

We wish to thank Professor Rockafellar for sharing many valuable insights with us.

I. Notation

R^n denotes Euclidean space with the usual inner product $x \cdot y = \langle x, y \rangle = \sum x_i y_i$. The closed unit ball in R^n is denoted by $B = \{x \in R^n : |x| \leq 1\}$.

If $K \subset R^n$ is a compact convex set, then ψ_K^* is the support function of K , defined by $\psi_K^*(u) = \sup\{\langle u, x \rangle : x \in K\}$. For any $u \in R^n$, we let $K_u = \{x \in K : \langle u, x \rangle = \psi_K^*(u)\}$.

The notation $T : R^n \rightrightarrows R^n$ indicates that T is a set-valued mapping. T is closed provided the set $\{(x, y) : y \in T(x)\}$ is closed. T is locally bounded if for every $\bar{x} \in R^n$ there is $\epsilon > 0$ and $R > 0$ such that $y \in T(x)$, $|x - \bar{x}| < \epsilon$ implies $|y| < R$.

We will say the sequence (x_n) converges to x in the direction $u \in R^n$, written $x_n \xrightarrow{u} x$, provided either $x_n \rightarrow x$ and $u = 0$, or $u \neq 0$, $\frac{x_n - x}{|x_n - x|} \rightarrow \frac{u}{|u|}$, and $x_n \neq x$ for all n .

If $f : R^n \rightarrow R$, the directional derivative of f at x (when it exists) is

$$f'(x; u) = \lim_{t \downarrow 0} \frac{f(x + tu) - f(x)}{t}.$$

II. Submonotonicity

In this section, $T : R^n \rightrightarrows R^n$ denotes a convex-valued closed multifunction. T will be called submonotone at $x \in R^n$ provided

$$\liminf_{\substack{x' \rightarrow x, x' \neq x \\ y \in T(x), y' \in T(x')}} \frac{\langle y' - y, x' - x \rangle}{|x' - x|} \geq 0 .$$

(T is trivially submonotone at x if $T(x) = \emptyset$). T is directionally upper semicontinuous (d.u.s.c.) at x provided that for all $u \in R^n$, whenever $x_k \xrightarrow{u} x$ and $y_k \in T(x_k)$ for all k , then for every $\varepsilon > 0$ there exists k_0 such that

$$T(x_k) \subset T(x)_u + \varepsilon B \quad \forall k \geq k_0 .$$

For $u = 0$, this is automatically satisfied since T is assumed to be closed. If T is locally bounded near x then T is d.u.s.c. at x if, and only if, for all $u \neq 0$, whenever $x_k \xrightarrow{u} x$ and $T(x_k) \ni y_k \rightarrow y$, then $y \in T(x)_u$. If T is submonotone [respectively, d.u.s.c.] at all $x \in R^n$, then T is submonotone [resp., d.u.s.c.].

(2.1) THEOREM. Let $T : R^n \rightrightarrows R^n$ be locally bounded near x (as is the case if $T = \partial f$ with f locally Lipschitz). Then T is d.u.s.c. at x if, and only if, T is submonotone at x .

Proof. If T is not submonotone at x , there is $\varepsilon > 0$ and there are sequences $x_n \rightarrow x$, $x_n \neq x$, $y_n \in T(x_n)$, $y_n' \in T(x)$, such that $\frac{\langle x_n - x, y_n - y_n' \rangle}{|x_n - x|} \leq -\varepsilon < 0$, $\forall n$. We may clearly assume $x_n \xrightarrow{u} x$ for some $u \neq 0$, and since T is closed and locally bounded, that $y_n \rightarrow y \in T(x)$ and $y_n' \rightarrow y' \in T(x)$. Then $\Psi_{T(x)}^*(u) > \langle u, y' \rangle - \varepsilon \geq \langle u, y \rangle$, so T is not d.u.s.c.

Suppose that T is submonotone at x . Let $x_n \xrightarrow{u} x$, $u \neq 0$, $y_n \in T(x_n)$, $y_n \rightarrow y$. Since T is closed and locally bounded, $y \in T(x)$ and we will be done if we can show $y \in T(x)_u$. If $z \in T(x)$,

$$(y - z) \cdot u = \lim \frac{\langle y_n - z, x_n - x \rangle}{|x_n - x|} \geq 0$$

since T is submonotone at x . Since this holds for all $z \in T(x)$, $y \cdot u \geq \Psi_{T(x)}^*(u)$, showing that T is d.u.s.c. at x . ■

Of course if $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is convex, ∂f is monotone, and hence submonotone. The fact that ∂f is directionally upper semicontinuous is proved by Rockafellar [6, Theorem 24.6].

The multifunction $T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ will be called strictly submonotone at x provided

$$\liminf_{\substack{x_1 \neq x_2 \\ x_i \rightarrow x, i=1,2 \\ y_i \in T(x_i), i=1,2}} \frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{|x_1 - x_2|} \geq 0.$$

Strict submonotonicity clearly implies submonotonicity.

Next, we state a characterization of strict submonotonicity similar to the one provided in Theorem 2.1 for submonotonicity. The proof is similar, so it has been omitted.

(2.2) THEOREM. Let $T : R^n \rightarrow R^n$ be locally bounded near x .

Then T is strictly submonotone at x if, and only if, whenever

$x_n \rightarrow x, x'_n \rightarrow x, x_n \neq x'_n, y_n \in T(x_n), y'_n \in T(x'_n), y_n \rightarrow y, y'_n \rightarrow y',$

$x_n - x'_n \xrightarrow{v} 0$, one also has $v \cdot y' \leq v \cdot y$.

III. Lipschitzian functions

Next, we turn our attention to a particular class of multi-functions, namely those that are the Clarke generalized gradient mapping [1] for a locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Thus, if $T = \partial f$, we ask what the submonotonicity of ∂f implies about f .

If f is locally Lipschitz, the Clarke derivative of f is the function

$$f^\circ(x, u) = \limsup_{\substack{t \downarrow 0 \\ h \rightarrow 0}} \frac{f(x+h+tu) - f(x+h)}{t}.$$

$f^\circ(x, \cdot)$ is a continuous sublinear function which is the support function of the compact convex set $\partial f(x)$ called the Clarke generalized gradient of f at x . For every $u, v \in \mathbb{R}^n$, $f^\circ(x; \cdot)$, being a finite convex function, possesses a finite directional derivative at u in the direction v which we denote by $f^\circ(x; u; v)$. Alternatively, we could define $f^\circ(x; u; \cdot)$ to be the support function of $\partial f(x)_u$. Clearly $f^\circ(x; 0; \cdot) = f^\circ(x; \cdot)$. Let us also define

$$f^\rightarrow(x; u; v) = \begin{cases} \limsup_{\substack{h \rightarrow 0 \\ u \rightarrow u \\ t/|h| \downarrow 0}} \frac{f(x+h+tv) - f(x+h)}{t} & \text{if } u \neq 0 \\ f^\circ(x; v) & \text{if } u = 0 \end{cases}$$

Clearly $f^\rightarrow(x; u; v) \leq f^\circ(x; v)$. Also, $f^\rightarrow(x; u; \cdot)$ is sublinear, so $f^\rightarrow(x; u; \cdot)$ is the support function of some subset of $\partial f(x)$. As

we shall see, the case where that subset is $\partial f(x)_u$ corresponds to the case where ∂f is submonotone or, equivalently, d.u.s.c. To see that $f^\rightarrow(x;u;\cdot)$ is sublinear, note that

$$\begin{aligned} f^\rightarrow(x;u;v_1+v_2) &= \limsup_t \frac{f(x+h+t(v_1+v_2)) - f(x+h)}{t} \\ &\leq \limsup_t \frac{f(x+(h+tv_1)+tv_2) - f(x+(h+tv_1))}{t} \\ &\quad + \limsup_t \frac{f(x+(h+tv_1)) - f(x+h)}{t} \\ &= f^\rightarrow(x;u;v_2) + f^\rightarrow(x;u;v_1) . \end{aligned}$$

(3.1) THEOREM. Let $f : R^n \rightarrow R$ be locally Lipschitz. ∂f is d.u.s.c. at x if, and only if, $f^\circ(x;u;v) = f^\rightarrow(x;u;v)$ for all $u, v \in R^n$.

Proof: (\Leftarrow) Let $u \neq 0$ (if $u = 0$, the assertion is trivial), $x_k \xrightarrow{u} x$, $\partial f(x_k) \ni y_k \rightarrow y$. To show ∂f is d.u.s.c., it must be demonstrated that $y \in \partial f(x)_u$. Fix an arbitrary $v \in R^n$. Then

$$v \cdot y_k \leq f^\circ(x_k;v) = \limsup_{\substack{h \rightarrow 0 \\ t \downarrow 0}} \frac{f(x_k+h+tv) - f(x_k+h)}{t}$$

so $h_k, t_k > 0$ can be found with

$$v \cdot y_k - \frac{1}{k} \leq \frac{f(x_k+h_k+t_k v) - f(x_k+h_k)}{t_k}$$

$$\text{and} \quad \max \{t_k, |h_k|\} \leq |x_k - x|/k .$$

Hence,

$$\begin{aligned} v \cdot y &= \lim_k v \cdot y_k \\ &\leq \limsup_k \frac{f(x_k + h_k + t_k v) - f(x_k + h_k)}{t_k} \\ &\leq f^+(x; u; v), \end{aligned}$$

where the last inequality follows from the fact that $x_k - x + h_k \xrightarrow{u} 0$ and $t_k/|x_k - x + h_k| \downarrow 0$. But $f^+(x; u; v) = f^\circ(x; u; v)$ by assumption, so $v \cdot y \leq f^\circ(x; u; v) = \Psi_{\partial f(x)_u}^*(v)$ for all v , which implies that $y \in \partial f(x)_u$.

(\Rightarrow) Fix $u \neq 0$, $v \in \mathbb{R}^n$. First we show that $f^\circ(x; u; v) \geq f^+(x; u; v)$.

Pick sequences $h_n \xrightarrow{u} 0$, $t_n/|h_n| \downarrow 0$ such that

$$f^+(x; u; v) = \lim_{n \rightarrow \infty} \frac{f(x + h_n + t_n v) - f(x + h_n)}{t_n}.$$

By the mean-value property [Lebourg, 3], there is, for each n ,

$y_n \in \partial f(x + h_n + c_n t_n v)$ with $0 < c_n < 1$ such that

$$v \cdot y_n = \frac{f(x + h_n + t_n v) - f(x + h_n)}{t_n}.$$

Without loss of generality, we can assume that $y_n \rightarrow y$ for some

$y \in \partial f(x)$. Since ∂f is assumed to be d.u.s.c. at x , we have

$y \in \partial f(x)_u$. Hence $f^+(x; u; v) = \lim v \cdot y_n = v \cdot y \leq \Psi_{\partial f(x)_u}^*(v) =$

$f^\circ(x; u; v)$, as desired.

To prove the opposite inequality, fix $u \neq 0$, $v \in \mathbb{R}^n$, $w \in \partial f(x)_u$, and we will show $w \cdot v \leq f^+(x; u; v)$. From this, the desired inequality follows by taking the supremum in w .

By d.u.s.c., we may find $\delta_n > 0$ ($n=1, 2, \dots$) such that $0 < \delta \leq \delta_n$ implies

$$\partial f(x + \delta(u + \frac{1}{n}v)) \subset \partial f(x)_{u + \frac{1}{n}v} + \frac{1}{n^2} B.$$

Clearly we may assume $\delta_n \rightarrow 0$. Let $x_n = x + \delta_n(u + \frac{1}{n}v)$ and choose $y_n \in \partial f(x_n)$. Then $x_n \xrightarrow{u} x$ and $y_n \in \partial f(x)_{u + \frac{1}{n}v} + \frac{1}{n^2} B$. Since $y_n \in \partial f(x_n)$, we may find $t_n > 0$ and $h_n \in \mathbb{R}^n$ such that

$$v \cdot y_n - \frac{1}{n} \leq \frac{f(x_n + h_n + t_n v) - f(x_n + h_n)}{t_n}$$

$$\max\{|h_n|, t_n\} < |x_n - x|/n.$$

Next, we will show that $\liminf_n y_n \cdot v \geq w \cdot v$. Since $x_n + h_n \xrightarrow{u} x$ and $t_n/|x_n - x + h_n| \rightarrow 0$, this will imply

$$\begin{aligned} w \cdot v &\leq \liminf_n \frac{f(x_n + h_n + t_n v) - f(x_n + h_n)}{t_n} \\ &\leq f^+(x; u; v) \end{aligned}$$

which is the desired result.

For each n , choose $y'_n \in \partial f(x)_{u + \frac{1}{n}v}$ such that $|y_n - y'_n| \leq \frac{1}{n^2}$.

Then

$$\begin{aligned}
y_n \cdot (u + \frac{1}{n}v) &= y'_n \cdot (u + \frac{1}{n}v) + (y_n - y'_n) \cdot (u + \frac{1}{n}v) \\
&\geq w \cdot (u + \frac{1}{n}v) - \frac{1}{n^2} |u + \frac{1}{n}v|
\end{aligned}$$

(because $w \in \partial f(x)$, $y'_n \in \partial f(x)_{u + \frac{1}{n}v}$)

$$\geq y'_n \cdot u + \frac{1}{n} w \cdot v - \frac{1}{n^2} |u + \frac{1}{n}v|$$

(because $w \in \partial f(x)_u$, $y'_n \in \partial f(x)$)

$$\geq y_n \cdot u + \frac{1}{n} w \cdot v - \frac{1}{n^2} (|u| + |u + \frac{1}{n}v|)$$

(because $|y_n - y'_n| \leq \frac{1}{n^2}$). So

$$y_n \cdot v \geq w \cdot v - \frac{1}{n} (|u| + |u + \frac{1}{n}v|)$$

and hence $\liminf y_n \cdot v \geq w \cdot v$, as desired. \square

Combining our results so far, we obtain the following:

(3.2) COROLLARY. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz, then the following are equivalent

- i. ∂f is submonotone at x
- ii. ∂f is d.u.s.c. at x
- iii. $f^\rightarrow(x; \cdot; \cdot) = f^\circ(x; \cdot; \cdot)$

Now that we have acquired a better understanding of the submonotonicity property of ∂f and what it implies about f , a logical question to ask next is: Just how strong is this property? In other words, if we take a look at "regularity" or "subdifferentiability" properties that have been studied for nondifferentiable functions by other authors, then which of these imply or are implied by the submonotonicity of ∂f ?

A locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be semismooth at $x \in \mathbb{R}^n$ [Mifflin, 4] provided that $x_k \xrightarrow{u} x$ and $y_k \in \partial f(x_k)$ imply that $\langle u, y_k \rangle \rightarrow f'(x; u)$.

(3.3) PROPOSITION. If ∂f is submonotone at x then f is semismooth at x .

Proof. If $x_k \xrightarrow{u} x$ and $y_k \in \partial f(x_k)$ then every subsequence of (y_k) has a subsubsequence converging to some point in $\partial f(x)_u$ by directional upper semicontinuity. Hence $\langle u, y_k \rangle \rightarrow \psi_{\partial f(x)}^*(u)$. By Proposition 3.5, $\psi_{\partial f(x)}^*(u) = f'(x; u)$. ■

The function $f(x) = -|x|$ is semismooth, but ∂f is not submonotone at $x = 0$, so the converse of 3.3 is false.

Following Pshenichnyi [5], let us say that f is quasi-differentiable at x if there is a closed convex set K such that $f'(x; \cdot) = \psi_K^*(\cdot)$. The function $f(x) = -|x|$ is not quasi-differentiable, so it is natural to ask whether every locally Lipschitz function which is both semismooth and quasi-differentiable has

a submonotone subgradient mapping. The answer is negative.

Consider the function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined as follows:

$$f(a,b) = \begin{cases} 0 & \text{if } a \leq 0 \\ a^2/4 & \text{if } a > 0, |b| \geq a^2/2 \\ |b| - b^2/a^2 & \text{if } a > 0, |b| < a^2/2 \end{cases}$$

Then f is differentiable at all points where either $b \neq 0$ or $a \leq 0$. At all points $x = (a, 0)$ with $a > 0$, f is quasi-differentiable since $f'(x; \cdot) = \Psi_K^*(\cdot)$ with $K = \{(0, -1), (0, 1)\}$. f is also locally Lipschitz, and it is not hard to check that f is everywhere semismooth. However, ∂f is not d.u.s.c. since $\partial f(0) = K$ but $(0, 0) \in \partial f(0, b)$ for all $b \neq 0$.

A locally Lipschitz function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ will be called regular at x [Clarke, 2] provided that $f'(x; \cdot) = \Psi_{\partial f(x)}^*(\cdot)$. Clearly this is a stronger property than quasi-differentiability. The function f of the previous paragraph is not regular at 0 , so it is natural to ask whether semismoothness plus regularity implies the submonotonicity of ∂f . This time the answer is affirmative:

(3.4) PROPOSITION. ∂f is submonotone at x if, and only if,
 f is semismooth and regular at x .

Proof. Suppose f is semismooth and regular at x . If $x_n \xrightarrow{u} x$ ($u \neq 0$), $y_n \in \partial f(x_n)$, and $y_n \rightarrow y$ then $y \in \partial f(x)$ and

$$\begin{aligned}
\langle y, u \rangle &= \lim \langle y_n, u \rangle \\
&= f'(x; u) \quad (\text{by semismoothness}) \\
&= \psi_{\partial f(x)}^*(u) \quad (\text{by regularity})
\end{aligned}$$

so $y \in \partial f(x)_u$. Hence ∂f is d.u.s.c., hence submonotone at x .

The other direction follows by Propositions 3.3 and 3.5. ■

Rockafellar [9] has defined $z \in \mathbb{R}^n$ to be a lower semigradient for f at x if

$$\liminf_{\substack{v \rightarrow u \\ t \downarrow 0}} \frac{f(x+tv) - f(x)}{t} \geq \langle u, z \rangle \quad \forall u \in \mathbb{R}^n.$$

If such a z exists, f is lower semidifferentiable.

(3.5) PROPOSITION. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz, ∂f submonotone at x . Then

$$\lim_{\substack{t \downarrow 0 \\ v \rightarrow u}} \frac{f(x+tv) - f(x)}{t} = \psi_{\partial f(x)}^*(u) \quad \forall u \in \mathbb{R}^n.$$

In particular, f is lower semidifferentiable at x and $\partial f(x)$ is the set of lower semigradients. Also, f is regular at x .

Proof. If $u = 0$, this follows easily from the fact that f is locally Lipschitz, so suppose $u \neq 0$. Let $t_n \downarrow 0$, $v_n \rightarrow u$. For each n , there is $c_n \in (0, 1)$ and $y_n \in \partial f(x + c_n t_n v_n)$ such that

$$\frac{f(x+t_n v_n) - f(x)}{t_n} = y_n \cdot v_n.$$

Since $x + c_n t_n v_n \xrightarrow{u} x$, we must have $y_n \cdot u \rightarrow \Psi_{\partial f(x)}^*(u)$. Thus

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(x+t_n v_n) - f(x)}{t_n} &= \lim y_n \cdot v_n \\ &= \lim y_n \cdot u = \Psi_{\partial f(x)}^*(u). \end{aligned}$$

Hence f is lower semidifferentiable and $\partial f(x)$ is the set of lower semigradients. It is then obvious that f is regular at x . ■

The converse of 3.5 is false: $f(x) = x^2 \sin \frac{1}{x}$ is locally Lipschitz and differentiable but ∂f is not submonotone at $x = 0$.

and lower semidifferentiable

It is also possible for a function to be regular but for ∂f not to be submonotone. Consider, for example, any function $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfying the following properties:

- (i) $f(x) = x - \frac{1}{x^2}$ for $x = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$
- (ii) f' exists and is decreasing on $\left(\frac{1}{n+1}, \frac{1}{n}\right)$,
 $f'_+\left(\frac{1}{n+1}\right) = 1$, and $f'_-\left(\frac{1}{n}\right) = 0$, $n=2,3,4,\dots$
- (iii) $f(x) = \frac{1}{4}$ for $x \geq \frac{1}{2}$ and $f(0) = 0$
- (iv) $f(-x) = f(x)$ for all x .

Since $|x| - x^2 \leq f(x) \leq |x|$ for all x , $f'(0;u) = |u|$ for all u . Also, $\partial f(0) = [-1,1]$ so f is regular at 0 . But ∂f is clearly not submonotone at 0 . Note that the behavior of f is nice at all points $x \neq 0$.

Since the property of strict submonotonicity is central to this paper, it is useful to mention an example of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that ∂f is submonotone everywhere, but is not strictly submonotone. The function is

$$f(x,y) = \begin{cases} |y| & \text{if } x \leq 0 \\ |y| - x^2 & \text{if } x \geq 0, |y| \geq x^2 \\ \frac{x^4 - y^2}{2x^2} & \text{if } x \geq 0, |y| \leq x^2 \end{cases}$$

It is easily checked that f is locally Lipschitz, that ∂f is everywhere submonotone, and $\partial f(0,0) = [(0,-1), (0,1)]$. If we let $x_n = \left(\frac{1}{n}, \frac{1}{n^2}\right)$, $x'_n = \left(\frac{1}{n}, -\frac{1}{n^2}\right)$, $y_n = \left(\frac{2}{n}, -1\right)$, $y'_n = \left(\frac{2}{n}, 1\right)$, $n=1,2,\dots$, and $u = (1,0)$, then $x_n \xrightarrow{u} 0$, $x'_n \xrightarrow{u} 0$, $y_n \in \partial f(x_n)$, and $y'_n \in \partial f(x'_n)$ for all n . However,

$$\frac{\langle x_n - x'_n, y_n - y'_n \rangle}{|x_n - x'_n|} = -2 \quad \text{for all } n$$

so ∂f is not strictly submonotone.

IV. Lower- C^1 functions

In this section, we characterize the class of "lower- C^1 functions" in terms of their Clarke gradients. $f : R^n \rightarrow R$ is lower- C^1 provided f can be represented locally as $f(x) = \max_{s \in S} g(x, s)$, where S is compact and g and $\nabla_x g$ are continuous jointly in x and s . In Theorem 4.9, it is demonstrated that a locally Lipschitz f is lower- C^1 if, and only if, ∂f is strictly submonotone. The term "lower- C^1 function" was suggested to us by Professor R. T. Rockafellar.

(4.1) LEMMA. Let $f : R^n \rightarrow R$ be locally Lipschitz, $x, y \in R^n$. For every $\epsilon > 0$, there are neighborhoods U of x and V of y such that if $x' \in U$ and $y' \in V$, then $|\psi_{\partial f(x')}^*(y) - \psi_{\partial f(x')}^*(y')| \leq \epsilon$.

Proof. Let κ be a Lipschitz constant for f on a neighborhood U of x . Then $\partial f(x') \subset \kappa B$ for all $x' \in U$, and it follows that κ is a (global) Lipschitz constant for $\psi_{\partial f(x')}^*(\cdot)$. Take V to be the open ball of radius ϵ/κ centered at y . ■

(4.2) LEMMA. Let $f : R^n \rightarrow R$ be locally Lipschitz. Then

$$(4.3) \quad \liminf_{\substack{x' \rightarrow x \\ t \downarrow 0}} \frac{f(x' + ty) - f(x')}{t} - \psi_{\partial f(x')}^*(y) \geq 0, \quad \forall y \in R^n$$

if, and only if, for any compact $K \subset \mathbb{R}^n$, and any $\epsilon > 0$, there is a neighborhood U of x and $\lambda > 0$ such that

$$(4.4) \quad \frac{f(x'+ty') - f(x')}{t} - \psi_{\partial f(x')}^*(y') \geq -\epsilon$$

whenever $x' \in U$, $y' \in K$, $0 < t < \lambda$.

Proof. Assume 4.3 holds, and fix $K \subset \mathbb{R}^n$ and $\epsilon > 0$. Since f is locally Lipschitz, 4.3 implies

$$\liminf_{\substack{x' \rightarrow x \\ y' \rightarrow y \\ t \downarrow 0}} \frac{f(x'+ty') - f(x')}{t} - \psi_{\partial f(x')}^*(y) \geq 0, \quad \forall y \in \mathbb{R}^n.$$

This, and Lemma 4.1, imply that for each $y \in K$ we may find neighborhoods U_y of x , V_y of y , and $\lambda_y > 0$ such that

$$\psi_{\partial f(x')}^*(y) - \psi_{\partial f(x')}^*(y') \geq -\epsilon/2$$

and

$$\frac{f(x'+ty') - f(x')}{t} - \psi_{\partial f(x')}^*(y) \geq -\epsilon/2$$

whenever $x' \in U_y$, $y' \in V_y$, and $0 < t < \lambda_y$. Pick a finite subcover V_{y_1}, \dots, V_{y_m} for K , and let $U = U_{y_1} \cap \dots \cap U_{y_m}$ and $\lambda = \min\{\lambda_{y_1}, \dots, \lambda_{y_m}\}$.

For any $x' \in U$, $y' \in K$, and $t \in (0, \lambda)$, let i be such that $y' \in V_{y_i}$, and we get

$$\begin{aligned}
& \frac{f(x'+ty') - f(x')}{t} - \psi_{\partial f(x')}^*(y') \\
&= \left(\frac{f(x'+ty') - f(x')}{t} - \psi_{\partial f(x')}^*(y_1) \right) + \left(\psi_{\partial f(x')}^*(y_1) - \psi_{\partial f(x')}^*(y') \right) \\
&\leq -\varepsilon/2 - \varepsilon/2 = -\varepsilon,
\end{aligned}$$

as desired. The opposite direction of the lemma is obvious. ■

(4.5) PROPOSITION. If $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is locally Lipschitz, then ∂f is strictly submonotone at x if, and only if, 4.3 holds.

Proof. (\Rightarrow) If $y = 0$, the assertion is trivial. Without any loss of generality, we may assume that $|y| = 1$. Fix $\varepsilon > 0$. Since ∂f is strictly submonotone at x , there is $r > 0$ such that

$$\frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{|x_1 - x_2|} \geq -\varepsilon$$

whenever $|x_i - x| < 2r$, $y_i \in \partial f(x_i)$ for $i = 1, 2$, and $x_1 \neq x_2$.

Let x' and t be chosen so that $|x' - x| < r$ and $0 < t < r$. We will complete the proof by showing that

$$\frac{f(x'+ty) - f(x')}{t} - \psi_{\partial f(x')}^*(y) \geq -\varepsilon.$$

Choose any $y_1 \in \partial f(x')_y$. By the mean-value theorem of Lebourg [3], we may find $s \in (0, t)$ and $y_2 \in \partial f(x'+sy)$ such that $f(x'+ty) - f(x') = t\langle y, y_2 \rangle$. Letting $x_1 = x'$ and $x_2 = x' + sy$, we have

$$\begin{aligned} \frac{f(x'+ty) - f(x')}{t} - \Psi_{\partial f(x')}^*(y) &= \langle y, y_2 - y_1 \rangle \\ &= \frac{\langle x_2 - x_1, y_2 - y_1 \rangle}{|x_2 - x_1|} \geq -\varepsilon. \end{aligned}$$

(\Leftarrow) Next, suppose 4.3 holds, and let $\varepsilon > 0$ be given. By Lemma 4.2, there is a neighborhood U of x and $\lambda > 0$ such that

$$\frac{f(x'+tu) - f(x')}{t} - \Psi_{\partial f(x')}^*(u) \geq -\varepsilon/2$$

whenever $x' \in U$, $|u| \leq 1$, and $0 < t < \lambda$. We may also assume that U is small enough so that $|z - z'| < \lambda$ for all $z, z' \in U$. Fix $x_i \in U$, $y_i \in \partial f(x_i)$ for $i=1,2$, with $x_1 \neq x_2$. Let $t = |x_2 - x_1|$ and $u = (x_2 - x_1)/t$. Then

$$\begin{aligned} \frac{\langle x_1 - x_2, y_1 - y_2 \rangle}{|x_1 - x_2|} &= -\langle u, y_1 \rangle - \langle -u, y_2 \rangle \\ &\geq -\Psi_{\partial f(x_1)}^*(u) - \Psi_{\partial f(x_2)}^*(-u) \\ &= \frac{f(x_1+tu) - f(x_1)}{t} - \Psi_{\partial f(x_1)}^*(u) \\ &\quad + \frac{f(x_2-tu) - f(x_2)}{t} - \Psi_{\partial f(x_2)}^*(-u) \\ &\geq -\frac{\varepsilon}{2} - \frac{\varepsilon}{2} = -\varepsilon, \end{aligned}$$

which shows that ∂f is strictly submonotone at x . ■

(4.6) LEMMA. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz, let C and K be compact sets in \mathbb{R}^n , and suppose that ∂f is strictly submonotone on C . Then

$$\liminf_{\substack{x \in C \\ y \in K \\ t \downarrow 0}} \frac{f(x+ty) - f(x)}{t} - \Psi_{\partial f(x)}^*(y) \geq 0.$$

Proof. Let $\varepsilon > 0$ be given. By Proposition 4.5 and Lemma 4.2, for each $x \in C$, there is $\lambda_x > 0$ such that

$$\frac{f(x'+ty) - f(x')}{t} - \Psi_{\partial f(x')}^*(y) \geq -\varepsilon$$

whenever $|x' - x| < \lambda_x$, $y \in K$, and $0 < t < \lambda_x$. Let $x_1, \dots, x_r \in C$ be such that for every $x \in C$ we have $|x - x_i| < \lambda_{x_i}$ for some i .

Let $\lambda = \min(\lambda_{x_1}, \dots, \lambda_{x_r})$. Then for any $x \in C$, $y \in K$, we have

$$\frac{f(x+ty) - f(x)}{t} - \Psi_{\partial f(x)}^*(y) \geq -\varepsilon$$

whenever $0 < t < \lambda$. ■

(4.7) LEMMA. Let $\phi(t)$ be real-valued, defined for $t > 0$ sufficiently small, such that $\lim_{t \downarrow 0} \phi(t) = 0$. Then there is a continuously differentiable function $\alpha(t)$ defined on $[0, a]$ for some $a > 0$ such that

$$\alpha(0) = \alpha'(0) = 0$$

$$\alpha(t) \geq t\phi(t), \quad \forall t \in (0, a] .$$

Proof. Let $a > 0$ be such that ϕ is bounded above on $(0, 2a]$, and let $a_k = a/2^k$, $k=0, 1, \dots$. If β is the infimum of all affine functions $\ell : \mathbb{R} \rightarrow \mathbb{R}$ which satisfy $\ell(a_k) \geq \phi(t)$ for all $t \in (0, 2a_k]$ and all $k=0, 1, 2, \dots$ then the following properties are easily checked:

β is continuous, concave, nondecreasing on $[0, a]$

$$\beta(0) = 0$$

$$\beta \geq \phi \text{ on } (0, a]$$

β is affine on $[a_{k+1}, a_k]$, $k=0, 1, 2, \dots$.

Also, β'_+ , the right derivative of β has these properties:

β'_+ is finite, nonnegative, and nonincreasing on $(0, a)$

β'_+ is constant on $[a_{k+1}, a_k)$, $k=0, 1, 2, \dots$

β'_+ is integrable on $[0, a]$.

This last assertion is proven as follows. Whenever $0 < u < v < a$,

$$\beta(v) - \beta(u) = \int_u^v \beta'_+(s) \, ds$$

(cf. Rockafellar [6, 24.2.1]). Since $\beta'_+ \geq 0$ and β is continuous,

$$\int_0^a \beta'_+(s) \, ds = \lim_{\substack{u \rightarrow 0 \\ v \rightarrow a}} \int_u^v \beta'_+(s) \, ds = \beta(a) - \beta(0) < \infty,$$

so β is integrable. Note that since $\beta(0) = 0$, $\beta(t) = \int_0^t \beta'_+(s) \, ds$ for all $t \in [0, a]$.

For each $k=1, 2, \dots$, pick c_k such that

$$\frac{1}{2}(a_k + a_{k+1}) < c_k < a_k$$

$$(a_k - c_k)(\beta'_+(a_{k+1}) - \beta'_+(a_k)) < a_{k+1}.$$

Define $\mu : (0, a) \rightarrow \mathbb{R}$ to be the function that agrees with $1 + \beta'_+$ on the intervals $[a_{k+1}, c_k]$ ($k=1, 2, \dots$) and on $[a_1, a_0)$, and is affine on the intervals $[c_k, a_k]$ ($k=1, 2, \dots$). Then μ is continuous, nonnegative, and nonincreasing on $(0, a)$ and

$$\int_{a_{k+1}}^t \mu(s) - \beta'_+(s) \, ds \geq 0 \quad \text{for all } k=0, 1, 2, \dots, \\ t \in [a_{k+1}, a_k]$$

Since $0 \leq \mu \leq \beta'_+ + 1$ and β'_+ is integrable, it follows that μ is integrable. Then for all $t \in [0, a]$,

$$\int_0^t \mu(s) \, ds \geq \int_0^t \beta'_+(s) \, ds = \beta(t).$$

Define $\alpha(t) = t \int_0^t \mu(s) \, ds$ for all $t \in [0, a]$. Clearly,

α is continuously differentiable on $(0, a]$.

$$\alpha(0) = 0$$

$$\alpha(t) \geq t\phi(t) \text{ for } t \in (0, a] .$$

It remains only to show that α is continuously differentiable at

0. We have $\alpha'(0) = \lim_{t \rightarrow 0^+} \frac{\alpha(t)}{t} = \lim_{t \rightarrow 0} \int_0^t \mu(s) \, ds = 0$. Also, for $t > 0$,

$$\begin{aligned} \alpha'(t) &= \int_0^t \mu(s) \, ds + t\mu(t) \\ &= \int_0^t (\mu(s) + \mu(t)) \, ds \\ &\leq 2 \int_0^t \mu(s) \, ds \quad (\text{since } \mu \text{ is nondecreasing}) \end{aligned}$$

so $\lim_{t \rightarrow 0} \alpha'(t) = 0$. ■

(4.8) PROPOSITION. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz. If ∂f is strictly submonotone then for every compact $C \subset \mathbb{R}^n$, there is a continuously differentiable $\alpha : [0, a] \rightarrow \mathbb{R}$ such that $\alpha(0) = \alpha'(0) = 0$ and

$$f(x + ty) \geq f(x) + t\psi_{\partial f(x)}^*(y) - \alpha(t)$$

whenever $x \in C$, $|y| = 1$, and $0 < t \leq a$.

Proof. For $t > 0$, define

$$\phi(t) = - \inf_{\substack{t' \leq t \\ x \in C \\ |y|=1}} \min \left(\frac{f(x+t'y) - f(x)}{t'} - \psi_{\partial f(x)}^*(y), 0 \right).$$

Then $\phi \geq 0$ and by Lemma 4.6, $\lim_{t \rightarrow 0} \phi(t) = 0$. By Lemma 4.7, there is a real-valued function $\alpha(t)$ which is continuously differentiable on $[0, a]$ for some $a > 0$ such that $\alpha(0) = \alpha'(0) = 0$ and $\alpha(t) \geq t\phi(t)$ for all $t \in (0, a]$. It follows that $f(x + ty) \geq f(x) + t\psi_{\partial f(x)}^*(y) - \alpha(t)$ whenever $x \in C$, $|y| = 1$ and $0 < t \leq a$. ■

(4.9) THEOREM. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be locally Lipschitz. ∂f is strictly submonotone if, and only if, for every $\bar{x} \in \mathbb{R}^n$ there is a neighborhood U of \bar{x} , a compact set S and a continuous function $g : U \times S \rightarrow \mathbb{R}$ such that $\nabla_x g(x, s)$ exists and is continuous in (x, s) and such that

$$f(x) = \max_{s \in S} g(x, s) \quad \forall x \in U.$$

Proof. (\Rightarrow) Suppose ∂f is strictly submonotone, and fix $\bar{x} \in \mathbb{R}^n$. By Proposition 4.8, there is a $a > 0$, and a C^1 function $\alpha : [0, a] \rightarrow \mathbb{R}$ such that $\alpha(0) = \alpha'(0) = 0$ and

$$f(x+y) \geq f(x) + \langle \zeta, y \rangle - \alpha(|y|)$$

whenever $|x - \bar{x}| \leq 1$, $|y| \leq a$, and $\zeta \in \partial f(x)$. Let $b = \min\{1, a/2\}$.

Then

$$f(x) \geq f(x') + \langle x - x', \zeta \rangle - \alpha(|x - x'|)$$

whenever $|x - \bar{x}| \leq b$, $|x' - \bar{x}| \leq b$, and $\zeta \in \partial f(x')$. Let $U = \{x : |x - \bar{x}| < b\}$ and $S = \{(x', \zeta) : |x' - \bar{x}| \leq b, \zeta \in \partial f(x')\}$. If we define

$$g(x, x', \zeta) = f(x') + \langle x - x', \zeta \rangle - \alpha(|x - x'|),$$

then g has the desired properties.

(\Rightarrow) Fix $\bar{x} \in \mathbb{R}^n$, let U , S , and g be as indicated, and let $K \subset U$ be a compact convex neighborhood of \bar{x} . By compactness, $\nabla_x g(x, s)$ is uniformly continuous on $K \times S$. So, defining for $t > 0$

$$\eta(t) = \sup_{\substack{z, z' \in K \\ s \in S \\ |z - z'| \leq t}} |\nabla_x g(z, s) - \nabla_x g(z', s)|$$

we have $\lim_{t \downarrow 0} \eta(t) = 0$. By Lemma 4.7 there is, for some $a > 0$,

a C^1 function $\alpha : [0, a] \rightarrow \mathbb{R}$ such that $\alpha(0) = \alpha'(0) = 0$ and $\alpha(t) \geq t\eta(t)$ for all $t \in (0, a]$.

Fix $x, x' \in K$ such that $x \neq x'$. For each $s \in S$, by the mean-value theorem, there is $x'' \in K$ on the line segment (x, x') such that $g(x', s) - g(x, s) = (x' - x) \cdot \nabla_x g(x'', s)$. Then

$$\begin{aligned}
& [g(x', s) - g(x, s) - (x' - x) \cdot \nabla_x g(x, s)] / |x' - x| \\
&= (\nabla_x g(x'', s) - \nabla_x g(x, s)) \cdot \frac{x' - x}{|x' - x|} \\
&\geq -\eta(|x'' - x|) \geq -\eta(|x' - x|) \geq -\frac{\alpha(|x' - x|)}{|x' - x|} .
\end{aligned}$$

Hence, for all $s \in S$,

$$g(x', s) \geq g(x, s) + (x' - x) \cdot \nabla_x g(x, s) - \alpha(|x' - x|) .$$

Let $\zeta \in \partial f(x)$ be arbitrary. By Clarke [1, Theorem 2.1], we may find $s_1, \dots, s_k \in S$ and numbers $\lambda_1, \dots, \lambda_k$ such that

$$\begin{aligned}
\zeta &= \sum \lambda_i \nabla_x g(x, s_i) \\
\lambda_i &\geq 0, \quad \sum \lambda_i = 1, \quad g(x, s_i) = f(x) .
\end{aligned}$$

Then

$$\begin{aligned}
f(x') &\geq \sum \lambda_i g(x', s_i) \\
&\geq \sum \lambda_i (g(x, s_i) + (x' - x) \cdot \nabla_x g(x, s_i) - \\
&\quad \alpha(|x' - x|)) \\
&= f(x) + (x' - x) \cdot \zeta - \alpha(|x' - x|) .
\end{aligned}$$

Since this holds for all $\zeta \in f(x)$, we have shown that for all $x, x' \in K$ with $x \neq x'$, we have

$$f(x') \geq f(x) + \psi_{\partial f(x)}^*(x' - x) - \alpha(|x' - x|) .$$

It then follows easily by Lemma 4.5 that ∂f is strictly submonotone at every interior point of K , and hence in particular at \bar{x} . ■

References

1. Clarke, F. H., "Generalized gradients and applications", Trans. Amer. Math. Soc., Vol. 205, pp. 247-262, 1975.
2. _____, "Generalized gradients of Lipschitz functionals", Advances in Math.
3. Lebourg, G., "Valeur moyenne pour gradient généralisé", C. R. Acad. Sc. Paris, t. 281 (10 Novembre 1975) Serie A, 795-7.
4. Mifflin, R., "Semismooth and semiconvex functions in constrained optimization", SIAM J. Control and Optimization, Vol. 15, 6, 1977.
5. Pshenichnyi, B. N., Necessary Conditions for an Extremum, Marcel Dekker, New York, 1971.
6. Rockafellar, R. T., Convex Analysis, Princeton U. Press, 1972.
7. _____, "The multiplier method of Hestenes and Powell applied to convex programming", J. Opt. Theory and Appl., 12, 6, 1973.
8. _____, "Augmented Lagrangians and applications of the proximal point algorithm in convex programming", Math. of O. R. 1, 1976.
9. _____, "The theory of subgradients and its applications to problems of optimization", lecture notes, U. of Montreal, Feb.-March, 1978.