Approximate Feedback Linearization: A Homotopy Operator Approach*

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Abstract

In this paper, we present an approach for finding feedback linearizable systems that approximate a given single-input nonlinear system on a given compact region of the state space. First, we show that if the system is close to being involutive then it is also close to being linearizable. Rather than working directly with the characteristic distribution of the system, we work with characteristic one-forms, i.e., with the one-forms annihilating the characteristic distribution. We show that homotopy operators can be used to decompose a given characteristic one-form into an exact and antiexact part. The exact part is used to define a change of coordinates to a normal form that looks like a linearizable part plus nonlinear perturbation terms. The nonlinear terms in this normal form depend continuously on the antiexact part and they vanish whenever the antiexact part does. Thus, the antiexact part of a given characteristic one-form is a measure of nonlinearizability of the system. If the nonlinear terms are small, by neglecting them we obtain a linearizable system approximating the original system. One can design control for the original system by designing it for the approximating linearizable system and applying it to the original one. We apply this approach for design of locally stabilizing feedback laws for nonlinear systems that are close to being linearizable.

1 Introduction

Consider a single-input system

$$\dot{x} = f(x) + g(x)u \tag{1}$$

where f, g are smooth vector fields defined on a compact contractible region $\mathcal{M}$ of R^n containing the origin. (Typically, $\mathcal{M}$ is a closed ball in R^n). We assume that $f(0) = 0$, i.e, that the origin is an equilibrium for $\dot{x} = f(x)$. The classical problem of feedback linearization

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can be stated as follows: find in a neighborhood of the origin a smooth change of coordinates $z = \Phi(x)$ (a local diffeomorphism) and a smooth feedback law $u = k(x) + l(x)u_{new}$ such that the closed loop system in the new coordinates with new control is linear:

$$\dot{z} = Az + Bu_{new}, \tag{2}$$

and controllable. We usually require that $\Phi(0) = 0$.

We assume that the system (1) has *the linear controllability* property

$$\dim \operatorname{span}\, \{g, ad_f g, \ldots, ad_f^{n-1} g\} = n, \ \forall x \in \mathcal{M}, \tag{3}$$

(where $ad_f^i g$ are iterated Lie brackets of f and g). We define the *characteristic distribution* for (1)

$$D := \operatorname{span}\, \{g, ad_f g, \ldots, ad_f^{n-2} g\} \tag{4}$$

(it is an n-1-dimensional smooth distribution by assumption of linear controllability (3)). We shall call any nowhere vanishing one-form ω annihilating D a *characteristic one-form for* (1). All the characteristic one-forms for (1) can be represented as multiples of some fixed characteristic one-form ω_0 by a smooth nowhere vanishing function (zero-form) β. Suppose that there is a nonvanishing β so that $\beta\omega_0$ is exact, i.e., $\beta\omega_0 = d\alpha$ for some smooth function α (d denotes the exterior derivative). Then ω_0 is called *integrable* and β is called an *integrating factor for* ω_0. The following result is standard (Hunt *et al.*, Jakubczyk and Respondek).

Theorem 1.1 *Suppose that the system (1) has the linear controllability property (3) on $\mathcal{M}$. Let D be the characteristic distribution and ω_0 be a characteristic one-from for (1). The following statements are equivalent:*

1. *(1) is feedback linearizable in a neighborhood of the origin in $\mathcal{M}$.*
2. *D is involutive in a neighborhood of the origin in $\mathcal{M}$.*
3. *ω_0 is integrable in a neighborhood of the origin in $\mathcal{M}$.*

As it is well know, a generic nonlinear system is not feedback linearizable for $n > 2$. However, in some cases, it may make sense to consider *approximate* feedback linearization. Namely, if one can find a feedback linearizable system *close* to (1), there is hope that a control designed for the feedback linearizable system and applied to (1) will give satisfactory performance, if the feedback linearizable system is *close* enough to (1). The first attempt in this direction goes back to Krener (1984). He considered applying to (1) a change of variables and feedback that yield system of the form:

$$\dot{z} = Az + Bu_{new} + \mathcal{O}(z, u_{new}) \tag{5}$$

where the term $\mathcal{O}(z, u_{new})$ contains higher order terms. The aim was too make $\mathcal{O}(z, u_{new})$ of as high order as possible. Then we can say that the system (1) is approximately feedback linearized in a small neighborhood of the origin. Recently, Hunt and Turi (1993) have proposed a new algorithm to achieve the same goal with fewer number of steps.

Another idea has been investigated by Hauser *et al.* (1992). Roughly speaking the idea was to neglect nonlinearities in (1) responsible for the failure of the involutivity condition in Theorem 1.1. This approach happened to be successful in the *ball and beam* system, when neglecting of centrifugal force acting on ball yielded a feedback linearizable system. Application of a control scheme designed for the system with centrifugal force neglected to the original system gave much better results than applying a control scheme based on classical Jacobian linearization. This approach has been further investigated by Hauser (1992) for the purpose of approximate feedback linearization about the manifold of constant operating points. However, a general approach to deciding which nonlinearities should be neglected to get the best approximation has not been set forth.

Krener and Maag (1991) design a best L^2 feedback approximation for the design of control law for the systems with cubic nonlinearities using a different approach. They try to come up with a change of variables that directly minimizes the terms P and Q .

All of the above-mentioned work (except Krener and Maag (1991)) dealt with applying a change of coordinates and a preliminary feedback so that the resulting system looks like linearizable part plus nonlinear terms of highest possible order around an equilibrium point or an equilibrium manifold. However, in many applications one requires a large region of operation for the nonlinearizable system. In such a case, demanding the nonlinear terms to be neglected to be of highest possible order may, in fact, be quite undesirable. One might prefer that the nonlinear terms to be neglected be small in a uniform sense over the region of operation. In the present paper we propose an approach to approximate feedback linearization that uses a change of coordinates and a preliminary feedback to put a system (1) in *a perturbed Brunovsky form*

$$\dot{z} = Az + Bu_{new} + P(z) + Q(z)u_{new} \tag{6}$$

where $P(z)$ and $Q(z)$ vanish at $z = 0$ and are "small" on $\mathcal{M}$. We obtain upper bounds on uniform norms of P and Q (depending on some measures of noninvolutivity of D) on any compact, contractible $\mathcal{M}$.

Our approach is an indirect one. We begin with approximating characteristic one-forms by exact forms using *homotopy operators*. Namely, on any contractible region $\mathcal{M}$ one can define a linear operator H that satisfies

$$\omega = d(H\omega) + Hd\omega, \tag{7}$$

for any form ω.

Note that the homotopy identity (7) allows to decompose any given one-form into the *exact part* $d(H\omega)$ and an "error" part $\epsilon := Hd\omega$, that we will call the *antiexact part of* ω. For given ω_0 annihilating D and a scaling factor β we define $\alpha_\beta := H\beta\omega_0$ and $\epsilon_\beta := Hd\beta\omega_0$. Note that the one-form ϵ_β measures how exact $\omega_\beta := \beta\omega_0$ is. If it is zero then ω_β is exact and the system (1) is linearizable, the zero-form α_β and its first $n-1$ Lie derivatives along f are the new coordinates. In the case that ω_0 is not exactly integrable, i.e., when no exact integrating factor β exist, we choose β so that $d\beta\omega_0$ is *smallest* in some sense (for this makes also ϵ_β small). We will call this β an *approximate integrating factor for* ω_0. We will use the zero-form α_β and its first $n-1$ Lie derivatives along f as the new coordinates, as in the linearizable case. In those new coordinates the system (1) is in the form

$$\dot{z} = Az + Bru + Bp + Eu \tag{8}$$

where r, p are smooth functions, $r \neq 0$ around the origin, and the term E (the obstruction to linearizablity) depends *linearly* on ϵ_β and some of its derivatives (in particular, E vanishes whenever ϵ_β does). We choose $u = r^{-1}(u_{new} - p)$, where u_{new} is a new control variable. After this change of coordinates and control variable the system is of the form (6) with $Q := r^{-1}E, P := -r^{-1}pE$. We obtain estimates on the uniform norm of Q and P (via estimates on r, p, and E) in terms of the error one form ϵ_β for any fixed β, on any compact, contractible region $\mathcal{M}$. Most important fact is that Q and P depend in a continuous way on ϵ_β and some of its derivatives, and they vanish whenever ϵ does.

From another point of view our approach can be viewed as a robustness analysis of exact feedback linearization. It is of obvious interest to analyse what happens to an exactly linearizable system subject to a small perturbation that destroys the property of being linearizable. One can expect that if linearization was used as an intermediate tool to achieve stabilization, tracking, disturbance rejection, etc., a small perturbation, yielding a system "close" to being linearizable, still allows one to apply the control designed for the original linearizable system, guaranteeing satisfactory performance. In the present paper we propose some tools to measure a distance of a nonlinearizable perturbed system from a linearizable one, thus allowing to measure how small the small perturbation is. In particular, we provide analysis of robustness of stabilizing feedback design based on feedback linearization.

We anticipate many applications of transforming (1) into (6). The idea behind making Q and P small is to neglect them in design. Intuitively speaking, we can neglect them if they are "small enough". What does it mean "small enough" will depend on particular application.

We should warn that one cannot expect that an exact or approximate feedback linearization will always help to improve performance. The point is that the idea of linearization is to get rid of nonlinearities because we don't know how to deal with them. It may happen though that removing of some nonlinear terms may negatively affect the performance of the system. For instance, consider the problem of stabilization of the feedback linearizable system $\dot{x} = -x^3 + u$. One can remove the nonlinear term $-x^3$ using feedback, but this doesn't help stabilization at all. The term $-x^3$ actually helps to stabilize the system, especially for large initial conditions. Still, there are enough examples of systems in which nonlinear terms cause problems to justify the present study.

The paper is organized as follows.

In Section 2 we introduce notation and some auxiliary results. We also explain construction of characteristic one-forms.

In Section 3 we discuss the problem of optimal scaling of the characteristic one forms. We review the construction of exact integrating factors and introduce and study best approximate L^p integrating factors.

In Section 4 we show how homotopy operators can be use to decompose characteristic one forms into exact and antiexact parts.

In Section 5 we prove that a change of coordinates based on the exact part of any characteristic one-form obtained with a homotopy operator having center at the origin defines a local diffeomorphism that takes the system (1) to a normal form that looks like a linearizable part perturbed by some nonlinear terms. The nonlinear perturbation terms depend linearly on the antiexact part of the characteristic one-form.

In Section 6 we obtain some upper bounds on nonlinear perturbation terms using the antiexact part of a characteristic one form and thus eastablish a continuity relationships between some measures of noninvolutivity and nonlinearizability.

In Section 7 we apply the results of the paper to study locally linearizing feedback laws for the system (1).

2 Notation and auxiliary results

In the present paper we apply the theory of differential alternating forms. We refer to standard texts such as Abraham *et al.*, Bryant *et al.*, Edelen, Flanders, or Hicks, for all the notions not defined here.

We denote by $T\mathcal{M}$ the tangent bundle to $\mathcal{M}$ and by $\Omega^k(\mathcal{M})$ the set of all alternating k-forms on $\mathcal{M}$, i.e., space of all k-linear anti-symmetric functionals on $T\mathcal{M}$. $\Omega(\mathcal{M})$ will denote the direct sum of all $\Omega^k(\mathcal{M})$. Let $\zeta \in \Omega(\mathcal{M})$ and $v \in T\mathcal{M}$. Then $d\zeta$ will denote the exterior derivative of ζ and $L_v\zeta$ will denote the Lie derivative of ζ along v.

By $i_v(\xi)$ we will mean the interior product (contraction) of a vector field v with a k-form ξ, which is a $k-1$ form defined by

$$i_v(\xi)(v_1, v_2, \ldots, v_{k-1}) := \xi(v, v_1, v_2, \ldots, v_{k-1}).$$

Note that if ξ is a one-form, then $i_v(\xi) = \xi(v)$. Below we summarize some properties of interior and exterior (wedge) products and exterior and Lie derivatives.

Proposition 2.1 *Let $\xi_1 \in \Omega^k(\mathcal{M}), \xi_2 \in \Omega(\mathcal{M})$ and $v \in T\mathcal{M}$ be arbitrary. Then*

1. $i_v(\xi_1 \wedge \xi_2) = (i_v\xi_1) \wedge \xi_2 + (-1)^k \xi_1 \wedge (i_v\xi_2)$.
2. $i_v i_v \xi_2 = 0$.
3. $d(\xi_1 \wedge \xi_2) = (d\xi_1) \wedge \xi_2 + (-1)^k \xi_1 \wedge (d\xi_2)$.
4. $d(L_v\xi_2) = L_v(d\xi_2)$.

While we did not need any additional structure except the differential one to study the problem of finding exact integrating factors, in the case of approximate integrating factors we need some means of measuring the distance between k-forms (for instance ω_0 from an exact form $d\alpha$, or $d\beta\omega_0$ from 0), both at a point and globally (on $\mathcal{M}$). For this, we use a Riemannian metric, i.e., a positive definite (pointwise) inner product $\langle , \rangle$ on the tangent space to $\mathcal{M}$. This inner product induces an inner product on p-forms (see Abraham *et al.*, Section 6.2), that we will denote by the same symbol. Namely, let $\{e^i\}$, $i = 1, \ldots, n$ be an orthonormal basis for $\Omega^1(\mathcal{M})$. Then the inner product on p-forms is uniquely defined by requiring $\{e^{i_1} \wedge \ldots \wedge e^{i_p} | i_1 < \ldots < i_p\}$ to be an orthonormal basis for $\Omega^p(\mathcal{M})$. The corresponding pointwise norm will be denoted by $|\cdot|$. We obtain a global inner product $\langle\langle , \rangle\rangle$ of p-forms on $\mathcal{M}$ by integrating the pointwise one over $\mathcal{M}$. A standard metric associated with coordinate system $x_1, x_2, \ldots, x_n$ is the one in which the vector fields $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}$, and thus the one-forms $dx_1, dx_2, \ldots, dx_n$, are orthonormal. The standard (in coordinates x_i) volume element on $\mathcal{M}$ is

$$\mu := dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n.$$

For any one-form η we will denote by $\eta^\#$ the dual vector field to η, i.e., the unique smooth vector field satisfying $\langle \xi, \eta \rangle = i_{\eta^\#}(\xi) = \xi(\eta^\#)$ for any one-form ξ. For instance, if we use the standard metric, we have $(\sum_{i=1}^n \eta_i dx_i)^\# = \sum_{i=1}^n \eta_i \frac{\partial}{\partial x_i}$. We have the following elementary result.

Proposition 2.2 *Let $\xi_1, \xi_2 \in \Omega^1(\mathcal{M})$ and $\zeta \in \Omega^2(\mathcal{M})$. Then*

$$\langle \xi_1 \wedge \xi_2, \zeta \rangle = \langle \xi_2, i_{\xi_1^\#} \zeta \rangle = \zeta(\xi_1^\#, \xi_2^\#).$$

Let $\zeta \in \Omega^p(\mathcal{M})$ and $\xi \in \Omega^{m-p}(\mathcal{M})$. In the sequel we will deal with the operator $W_\xi : \Omega^p(\mathcal{M}) \mapsto \Omega^m(\mathcal{M})$ defined by $W_\xi \zeta := \zeta \wedge \xi$.

To obtain a one-form annihilating an $n-1$-dimensional distribution D in $\mathcal{M}$, we contract any volume element of $\mathcal{M}$ by any basis of D. For instance, we may choose

$$\omega_0 := i_g i_{ad_f g} \ldots i_{ad_f^{n-2} g} \mu, \tag{9}$$

where μ is the standard volume element in coordinates x_i.

3 Approximate Integrating Factors

Before we discuss the approximate integrating factors for nonintegrable characteristic one-form ω_0, let us remind the reader how, given an integrable characteristic one-form ω_0, one constructs an exact integrating factor for ω_0. The construction will suggest what can be done in the case of nonintegrable characteristic one-form ω_0. Let us begin with following standard result (see, e.g., Abraham *et al.*, Section 6.4, Edelen, Section 4.2):

Proposition 3.1 *Let ω_0 be a nonvanishing one-form on $\mathcal{M}$. The following statements are equivalent:*

1. *ω_0 is integrable,*

2. There is a one-form γ such that

$$d\omega_0 = \gamma \wedge \omega_0. \tag{10}$$

3. There is a zero-form θ suth that

$$d\omega_0 = d\theta \wedge \omega_0 \tag{11}$$

4. ω_0 satisfies

$$d\omega_0 \wedge \omega_0 = 0. \tag{12}$$

Note that the statement (4) provides a test for integrability of ω_0, and thus for linearizability of the system (1). Let us present one possible way of proving (4) $\Rightarrow$ (2).

Let X be any smooth vector field on $\mathcal{M}$ satisfying $i_X(\omega_0) = \omega_0(X) = 1$. Then (see Proposition 2.1 (1))

$$0 = i_X(d\omega_0 \wedge \omega_0) = i_X(d\omega_0) \wedge \omega_0 + d\omega_0 \wedge i_X(\omega_0) = i_X(d\omega_0) \wedge \omega_0 + d\omega_0.$$

Choosing $\gamma := -i_X(d\omega_0)$, we see that (4) $\Rightarrow$ (2). Even though it is not immediately seen, one can choose X so that $i_X(\omega_0) = 1$ and $-i_X(d\omega_0) = d\theta$ for some zero-form θ, thus proving (3). The condition (3) is most important in construction of the integrating factor for ω_0. Namely, once we know the zero-form θ, choosing $\beta := e^{-\theta}$, we obtain $d\beta\omega_0 = 0$ and $\beta > 0$ as required. Note that an integrating factor β obtained in that way is not unique. Namely, if $\beta\omega_0 = d\alpha$ for some zero-form α, one can replace β by $h(\alpha)\beta$, where $h(\alpha)$ is smooth and positive. It is now easily checked that $dh(\alpha)\beta\omega_0 = 0$, so that $h(\alpha)\beta$ is an integrating factor for ω_0, whenever β is.

Given a nonintegrable characteristic one-form ω_0, we try to find a *best possible* integrating factor for it. Let us recall that the goal is to make $d\beta\omega_0$ "as small as possible". Below we define some precise meaning for making $d\beta\omega_0$ "small" by an appropriate choice of β. We want to avoid the trivial solution $\beta = 0$. To contrary, we want to formulate the problem of construction of best approximate integrating factor so that the solution yields $\beta > 0$ for all $x \in \mathcal{M}$. In coordinates, minimization of $d\beta\omega_0$ can be understood as making the differences of the mixed partial derivatives $\frac{\partial \beta\omega_{0i}}{\partial x_j} - \frac{\partial \beta\omega_{0j}}{\partial x_i}$ as small as possible.

We will first establish a pointwise measure of exactness for $\beta\omega_0$ and then construct a global one from the pointwise one. Let

$$\kappa(\omega_0)(x) := \frac{|(d\omega_0)(x)|}{|(\omega_0)(x)|}, \tag{13}$$

where $|\cdot|$ is the pointwise norm of a form given by the Riemannian metric. Note that such a measure of exactness of ω_0 is invariant under scaling of the one-form ω_0 by a constant, non-zero function. Now we define global measures of exactness of ω_0. A uniform measure can be obtained by taking the supremum of $\kappa(\omega_0)(x)$ over $\mathcal{M}$

$$\chi_\infty(\omega_0) := \sup\{\kappa(\omega_0)(x), x \in \mathcal{M}\}. \tag{14}$$

Note that the supremum exists since $\mathcal{M}$ is compact (by assumption).

An average measure of integrability is obtained by integrating $\kappa(\omega_0)(x)$ over $\mathcal{M}$. Let $p > 0$, then

$$\chi_p(\omega_0) := (\int_{\mathcal{M}} (\kappa(\omega_0)(x))^p \mu)^{1/p}, \tag{15}$$

where μ is the volume element associated with the Riemannian metric. Now, for $1 \leq p \leq \infty$, we can define the best approximate L^p integrating factor β for ω_0 as the zero-form that minimizes $\chi_p(\beta\omega_0)$.

The best situation one might hope for when facing the problem of construction of the best approximate integrating factor is that there is a single function β that is, in fact, the best L^p approximate integrating factor for every $1 \leq p \leq \infty$. This will be the case if we can find a function β that minimizes $\kappa(\beta\omega_0)(x)$ at every point $x \in \mathcal{M}$. In certain special cases there is, in fact, an easy solution to this problem.

Let ω_0 be a given one-form on $\mathcal{M}$ and consider the decomposition

$$d\omega_0 = \gamma \wedge \omega_0 + \tau. \tag{16}$$

This equation should be interpreted as an "approximation" to (10) with the two-form τ playing the role of an error term. There are infinitely many ways of decomposing $d\omega_0$ as above, because for any γ, one can simply choose $\tau := d\omega_0 - \gamma \wedge \omega_0$. We know from Proposition 3.1 that we can choose $\tau = 0$ in the case of integrable ω_0. If ω_0 fails to be integrable we will try to choose γ and τ in (16) so that the two-form τ is smallest possible in a least square sense, i.e., w.r.t. to the (global) L^2 norm of forms on $\mathcal{M}$

$$||\xi|| := (\int_{\mathcal{M}} |\xi|^2 \mu)^{1/2} \quad \text{for} \quad \xi \in \Omega(\mathcal{M}) \tag{17}$$

Note that this smallest possible τ measures how far ω_0 is from being closed (and thus exact).

It happens that the problem of finding γ and τ satisfying (16) with $||\tau||$ minimal can be solved pointwise.

As in the case of integrable ω_0, we will use an interior product of a vector field X satisfying $i_X(\omega_0) = \omega_0(X) = 1$ with $d\omega_0 \wedge \omega_0$ to obtain a decomposition of type (16). We have (see Proposition 2.1 (1))

$$i_X(d\omega_0 \wedge \omega_0) = i_X(d\omega_0) \wedge \omega_0 + d\omega_0 \wedge i_X(\omega_0) = i_X(d\omega_0) \wedge \omega_0 + d\omega_0$$

so that

$$d\omega_0 = (-i_X(d\omega_0)) \wedge \omega_0 + i_X(d\omega_0 \wedge \omega_0).$$

This relation has the required form (16) with $\gamma := -i_X(d\omega_0)$, $\tau := i_X(d\omega_0 \wedge \omega_0)$.

We will denote by τ_{min} the two form τ satisfying (16) for some γ with a minimum pointwise norm $|\tau(x)|$ at every $x \in \mathcal{M}$. (It is clear that this will also be the two form with minimal global norm $||\cdot||$ among all two forms τ satisfying (16)). Belowe we give an explicit formula for τ_{min}.

Proposition 3.2 *Put* $X_0 := |\omega_0|^{-2}\omega_0^{\#}$. *Then* $i_{X_0}(\omega_0) = 1$ *and*

$$\tau_{min} = i_{X_0}(d\omega_0 \wedge \omega_0). \tag{18}$$

Among all γ *satisfying (16) for* $\tau = \tau_{min}$ *the one with a minimal norm is*

$$\gamma_{min} := -i_{X_0}(d\omega_0). \tag{19}$$

γ_{min} *is pointwise orthogonal to* ω_0. *All other one-forms* γ *satisfying (16) can be represented as* $\gamma_\eta = \gamma_{min} + \eta\omega_0$, *for some zero-form* η.

Proof: First, note that $\tau = \tau_{min}$ if and only if $\langle \xi \wedge \omega_0, \tau \rangle = 0$ pointwise on $\mathcal{M}$ for any one-form ξ. (This follows from the fact that by a standard least-squares argument τ_{min} must be orthogonal to the space $R(W_{\omega_0}) = \{\xi \wedge \omega_0 | \xi \in \Omega^1(\mathcal{M})\}$.) Note also that $i_{X_0}(\omega_0) = |\omega_0|^{-2}\omega_0(\omega_0^{\#}) = |\omega_0|^{-2}\langle \omega_0, \omega_0 \rangle = 1$. Thus, it is immediately seen that (16) is satisfied for τ, γ given by (18) and (19). To see that τ defined by (18) is actually τ_{min} we will show that $\langle \xi \wedge \omega_0, \tau \rangle = 0$ pointwise on $\mathcal{M}$ for any one-form ξ. Using Proposition 2.2 and 2.1 (2), we have

$$\langle \xi \wedge \omega_0, \tau \rangle = -\langle \omega_0 \wedge \xi, i_{X_0}(d\omega_0 \wedge \omega_0) \rangle = -|\omega_0|^{-2}\langle \xi, i_{\omega_0^{\#}} i_{\omega_0^{\#}}(d\omega_0 \wedge \omega_0) \rangle = 0.$$

We have shown that $\tau = \tau_{min}$.

To prove that γ_{min} is pointwise orthogonal to ω_0 note that $\langle \omega_0, \gamma_{min} \rangle = i_{\omega_0^{\#}}\gamma_{min} = -|\omega_0|^{-2} i_{\omega_0^{\#}} i_{\omega_0^{\#}}(d\omega_0) = 0$. □

Note that $\tau_{\min} = 0$ is zero on $\mathcal{M}$ if and only if ω_0 is integrable on $\mathcal{M}$, $\tau_{\min} = 0$ and $\gamma_{min} = 0$ on $\mathcal{M}$ if and only if ω_0 is exact on $\mathcal{M}$. Let $\beta \in \Omega^0(\mathcal{M})$. We have

$$d\beta\omega_0 = (d\beta + \beta\gamma_{min}) \wedge \omega_0 + \beta\tau_{\min}. \tag{20}$$

Now, we can obtain the following pointwise lower bound for $\kappa(\beta\omega_0)$:

Proposition 3.3 *For any* $x \in \mathcal{M}$ *and* $\beta \in \Omega^0(\mathcal{M})$ *we have*

$$\kappa(\beta\omega_0)(x) \geq \frac{|\tau_{\min}(x)|}{|\omega_0(x)|}. \tag{21}$$

Proof: Since the two-forms $(d\ln\beta + \gamma_{min}) \wedge \omega_0$ and $\tau_{\min}$ are pointwise orthogonal for every β (see the proof of Proposition 3.2), we have $\kappa(\beta\omega_0)(x) = \frac{|d\beta \wedge \omega_0 + \beta d\omega_0|}{|\beta\omega_0|} = \frac{|d\beta \wedge \omega_0 + \beta(\gamma_{min} \wedge \omega_0) + \beta\tau_{min}|}{|\beta\omega_0|} = \frac{|(d\beta + \beta\gamma_{min}) \wedge \omega_0 + \beta\tau_{min}|}{|\beta\omega_0|} = \frac{|(d\ln\beta + \gamma_{min}) \wedge \omega_0 + \tau_{min}|}{|\omega_0|} = ((\frac{|(d\ln(\beta) + \gamma_{min}) \wedge \omega_0|}{|\omega_0|})^2 + (\frac{|\tau_{\min}|}{|\omega_0|})^2)^{1/2} \geq \frac{|\tau_{\min}|}{|\omega_0|}$. □

The best we can hope for is that the lower bound for $\kappa(\beta\omega_0)(x)$ obtained above is sharp, i.e., there is a zero-form β such that $\kappa(\beta\omega_0)(x) = \frac{|\tau_{\min}(x)|}{|\omega_0(x)|}$ for every $x \in \mathcal{M}$. This will be the case if $(d\ln(\beta) + \gamma_{min}) \wedge \omega_0 = 0$ for some choice of β. A necessary and suficient condition for this is $\gamma_\eta := \gamma_{min} + \eta\omega_0 = d\theta$, for some zero-forms η and θ. Then we choose $\beta := e^{-\theta}$ and obtain $(d\ln\beta + \gamma_\eta) \wedge \omega_0 = 0$. Note that the zero-form β is everywhere strictly positive, as required.

Example 3.0 Consider

$$\begin{array}{rcl}\dot{x}_1 & = & x_2 + h_1(x_3) + h_2(x_1, x_2) \\ \dot{x}_2 & = & x_3 + h_3(x_3) + h_4(x_1, x_2) \\ \dot{x}_3 & = & u,\end{array} \tag{22}$$

where $h_i(\cdot)$ are any smooth functions with $h_i(0) = \frac{\partial h_i}{\partial x_j}(0) = 0$. We have $g = \frac{\partial}{\partial x_3}$, $f = (x_2 + h_1(x_3) + h_2(x_1, x_2))\frac{\partial}{\partial x_1} + (x_3 + h_3(x_3) + h_4(x_1, x_2))\frac{\partial}{\partial x_2}$, $ad_f g := [f, g] = (-h_1'(x_3))\frac{\partial}{\partial x_1} - (1 + h_3'(x_3))\frac{\partial}{\partial x_2}$, $\omega_0 = (1 + h_3'(x_3))dx_1 - h_1'(x_3)dx_2$, $d\omega_0 = h_1''(x_3)dx_2 \wedge dx_3 - h_3''(x_3)dx_1 \wedge dx_3$, and $d\omega_0 \wedge \omega_0 = (h_1''(x_3)dx_2 \wedge dx_3 - h_3''(x_3)dx_1 \wedge dx_3) \wedge ((1 + h_3'(x_3))dx_1 - h_1'(x_3)dx_2) = ((1 + h_3'(x_3))h_1''(x_3) - h_1'(x_3)h_3''(x_3))dx_1 \wedge dx_2 \wedge dx_3$. We see, that the system is exactly feedback linearizable in a neighborhood of the origin if and only if $(1 + h_3'(x_3))h_1''(x_3) - h_1'(x_3)h_3''(x_3) = 0$, which is the case if $h_1(\cdot) = 0$.

It happens that for this system we can actually construct the best approximate L^p integrating factor β in above-mentioned sense, i.e., the one that works for every $1 \leq p \leq \infty$. Suppose that we use the standard metric in coordinates x_1, x_2, x_3. We have

$$\gamma_{min} = \frac{h_1'(x_3)h_1''(x_3) + (1 + h_3'(x_3))h_3''(x_3)}{h_1'(x_3)^2 + (1 + h_3'(x_3))^2}dx_3,$$

$$\tau_{\min} = \frac{(h_1''(x_3) + h_3'(x_3)h_1''(x_3) - h_1'(x_3)h_3''(x_3))}{h_1'(x_3)^2 + (1 + h_3'(x_3))^2}((1 + h_3'(x_3))dx_2 \wedge dx_3 + h_1'(x_3)dx_1 \wedge dx_3).$$

Note that γ_{min} depends only on x_3, and thus it is exact. One can check that $\gamma_{min} = d\theta$ for $\theta = \ln(h_1'(x_3)^2 + (1 + h_3'(x_3))^2)^{1/2}$. The best approximate integrating factor in above-mentioned sense is $\beta_0 = e^{-\theta} = (h_1'(x_3)^2 + (1 + h_3'(x_3))^2)^{-1/2} = |\omega_0|^{-1}$ (note that this choice makes the pointwise length of $\beta_0\omega_0$ equal to 1 everywhere). □

In Banaszuk *et al.*, 1994a, we show that the lower bound for $\kappa(\beta\omega_0)(x)$ is always sharp if the metric is the standard metric in some special coordinates. There are however examples of systems for which the lower bound for $\kappa(\beta\omega_0)(x)$ is not sharp. In this case a more sophisticated analysis is required (cf. Banaszuk *et al.*, 1994b,c), which leads to some variational problems whose solutions for β are given as solutions of elliptic eigenvalue problems. A simple alternative would be an approximation of γ_η by an exact form $dH\gamma_\eta$ using a homotopy operator H (see the next section).

Note that even though the minimal $\chi_p(\beta\omega_0)$ seems to be a natural measure of integrability of ω_0 and thus also a measure of noninvolutivity of the characteristic distribution D, it may not be a sufficient measure for the problem of approximate linearization. There are some indications that one should actually minimize $d(\beta\omega_0)$ *together* with its first n-1 Lie derivatives along f. This problem is currently being studied.

4 Homotopy operator

On any contractible region $\mathcal{M}$ one can define a linear operator $H : \Omega^k(\mathcal{M}) \mapsto \Omega^{k-1}(\mathcal{M})$ that partially inverts the exterior derivative, i.e.,

$$\omega = (dH + Hd)\omega, \quad \forall \omega \in \Omega^k(\mathcal{M}) \tag{23}$$

or, in other words,

$$\omega = d(H\omega) + H d\omega \tag{24}$$

Any operator with such property will be called a *homotopy* operator. Following Flanders, we present a construction of such an operator. Consider the *cylinder* $I \times \mathcal{M}$ where $I := [0,1]$ and define a family of maps $j_\lambda : \mathcal{M} \mapsto I \times \mathcal{M}$ by $j_\lambda(x) = (\lambda, x)$, for $\lambda \in I$. Note that k-forms on the cylinder can be represented in coordinates $\lambda, x_1, \ldots, x_n$ as sums of monomials of two types: $a(\lambda, x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$ and $a(\lambda, x) d\lambda \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}}$. We now define a linear operator $K : \Omega^k(I \times \mathcal{M}) \mapsto \Omega^{k-1}(\mathcal{M})$ such that its action on these two types of monomials is given by

$$K(a(\lambda, x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}) = 0, \tag{25}$$

$$K(a(\lambda, x) d\lambda \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}}) = (\int_0^1 a(\lambda, x) d\lambda) dx_{i_1} \wedge \ldots \wedge dx_{i_{k-1}}. \tag{26}$$

The operator K satisfies (Flanders, Section 3.6)

$$K(d\omega) + d(K\omega) = j_1^* \omega - j_0^* \omega, \tag{27}$$

where $j_\lambda^* : \Omega^k(I \times \mathcal{M}) \mapsto \Omega^k(\mathcal{M})$ is the pullback induced by j_λ. Note that the above result doesn't require $\mathcal{M}$ to be contractible. Now, by definition, $\mathcal{M}$ is contractible iff there is a smooth mapping $\phi : I \times \mathcal{M} \mapsto \mathcal{M}$ such that $\phi(1, x) = x, \phi(0, x) = x^0$, where x^0 is a distingushed point in $\mathcal{M}$. Such a mapping ϕ is called a *homotopy* or *contraction* (of $\mathcal{M}$ to x^0). The point x^0 is called the homotopy center. Since we have $(\phi \circ j_1)(x) = x$ and $(\phi \circ j_0)(x) = x^0 \ \forall x \in \mathcal{M}$, the pullback $\phi^* : \Omega^k(\mathcal{M}) \mapsto \Omega^k(I \times \mathcal{M})$ induced by the mapping ϕ satisfies

$$j_1^*(\phi^* \omega) = \omega, \ j_0^*(\phi^* \omega) = 0. \tag{28}$$

Therefore, (27), (28), and the fact that the exterior derivative commutes with a pullback together imply

$$K\phi^*(d\omega) + d(K\phi^* \omega) = \omega. \tag{29}$$

Thus, the operator $H := K \circ \phi^*$ satisfies (24) so that it is a homotopy operator.

Note that different choices of homotopy centers x^0 and homotopies ϕ yield different homotopy operators. The one we will use is probably the simplest one: it will act on one-forms by integrating them along straight lines (in coordinates x_i) from a distingushed point x^0 in $\mathcal{M}$ (usually the origin). Such a homotopy operator will be called *radial* (see, e.g., Edelen, Section 5.3 and Flanders, Section 3.7). If $\mathcal{M}$ is star-shaped in coordinates x_i with respect to x^0 (i.e., $\mathcal{M}$ can be contracted to x^0 by straight lines lying entirely in $\mathcal{M}$), a simple choice for a homotopy ϕ is $\phi(\lambda, x) = x^0 + \lambda(x - x^0)$. Let $\omega = \sum_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k}(x) dx_{i_1} \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$. Now, one can explicitly calculate the pullback ϕ^*:

$$\begin{aligned} \phi^* \omega &= \textstyle\sum_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k}(x^0 + \lambda(x - x^0)) d(x^0_{i_1} + \lambda(x_{i_1} - x^0_{i_1})) \\ &\qquad \wedge d(x^0_{i_2} + \lambda(x_{i_2} - x^0_{i_2})) \wedge \ldots \wedge d(x^0_{i_k} + \lambda(x_{i_k} - x^0_{i_k})) \\ &= \textstyle\sum_{i_1 \ldots i_k} \omega^{i_1 \ldots i_k}(x^0 + \lambda(x - x^0))((x_{i_1} - x^0_{i_1}) d\lambda + \lambda dx_{i_1}) \\ &\qquad \wedge((x_{i_2} - x^0_{i_2}) d\lambda + \lambda dx_{i_2}) \wedge \ldots \wedge ((x_{i_k} - x^0_{i_k}) d\lambda + \lambda dx_{i_k}), \end{aligned} \tag{30}$$

and we can express the action of the radial homotopy operator as

$$\begin{array}{rcl}(H\omega)(x) & = & \sum_{i_1\ldots i_k}(\sum_{i_j}((-1)^{j+1}(x_{i_j}-x^0_{i_j}) \\ & & \int_0^1 \lambda^{k-1}\omega^{i_1\ldots i_k}(x^0+\lambda(x-x^0))d\lambda))dx_{i_1}\wedge\ldots\wedge\widehat{dx_{i_j}}\ldots\wedge dx_{i_k}\end{array} \tag{31}$$

where the symbol $\widehat{}$ over dx_{i_j} indicates that it is omitted.

Once again consider the system of Example 3 Choose $\omega_0=(1+h_3'(x_3))dx_1-h_1'(x_3)dx_2$ and $\beta=1$. Then, for $x_0=0$, we see that

$$\alpha=\int_0^1(x_1(1+h_3'(\lambda x_3))-x_2h_1'(\lambda x_3))d\lambda=x_1-\frac{x_2h_1(x_3)}{x_3}+\frac{x_1h_3(x_3)}{x_3}$$

and the error one-form ϵ is given by

$$\begin{array}{rcl}\epsilon & = & -(h_3(x_3)-x_3h_3'(x_3))/x_3^2dx_1+(h_1(x_3)-x_3h_1'(x_3))/x_3^2dx_2 \\ & & +(x_2h_1(x_3)-x_1h_3(x_3)-x_2x_3h_1'(x_3)+x_1x_3h_3'(x_3))/x_3^2dx_3\end{array}$$

(Alternatively, once α is known, one can use the formula $\epsilon=\beta\omega_0-d\alpha$ instead of $\epsilon=Hd\beta\omega_0$ to obtain the error one-form ϵ.)

Let $|\cdot|$ denote the pointwise norm of a form induced by the standard metric in x_i coordinates. We have shown in the previous section that the best approximate integrating factor for the above system in the sense of minimizing $\frac{|(d\beta\omega_0)(x)|}{|(\beta\omega_0)(x)|}$ pointwise everywhere is $\beta_0=|\omega_0|^{-1}=(h_1'(x_3)^2+(1+h_3'(x_3))^2)^{-1/2}$. Contrary to the case $\beta=1$ it is now impossible to evaluate $H\beta_0\omega_0$ and $Hd\beta_0\omega_0$ explicitly. The integration must be performed case by case for specific functions $h_1(\cdot)$ and $h_3(\cdot)$. We don't expect to be always able to perform the integration symbolically, for the result might not be an elementary function.

The homotopy operator we will use in the sequel uses the origin in coordinates x_i as the homotopy center. One can obtain other homotopy operators choosing different homotopy centers x_0. Moreover, one doesn't have to integrate over straight lines from the center. Note that the notion of a straight line is associated with specific choice of coordinates. Hence, if we change coordinates, we immediately obtain a homotopy operator, namely the radial homotopy operator in the new coordinates. Moreover, an arithmetic mean of homotopy operators is again a homotopy operator. For exactly linearizable systems, once we have found an exact integrating factor for a characteristic one-form ω, any homotopy operator will give the same exact part and zero antiexact part. However, for nonlinearizable systems the choice of a homotopy operator will make a difference. Apparently, the choice of a particular homotopy operator will influence the exact and antiexact parts of a characteristic one-form ω. It is not clear to us yet, what should be the best choice for approximate feedback linearization. This issue is currently under investigation. One may expect that the optimal homotopy operator might be rather complicated. Hence, even though there is no reason to believe that the radial homotopy operator in the original coordinates will be the best one (i.e., yielding the smallest antiexact part of ω), there is a good chance that it will be the simplest one to apply. Moreover, any homotopy operator with the center at the origin that satisfies $\phi(\lambda,0)\equiv 0$ (in particular the radial one) will always yield $\epsilon(0)=0$. Since ϵ is a smooth one-form vanishing at the origin, it will be small in a neighborhood of the origin.

The following result shows some that there are some limitations to what can be achieved in approximating non-exact characteristic forms by exact ones.

Theorem 4.1 *Let H be any homotopy operator on $\mathcal{M}$ and ω be any characteristic one-form for the system (1). Let $\alpha_H := H\omega$ and $\epsilon_H := H d\omega$. Then for any closed curve c in $\mathcal{M}$ we have*

$$\int_c L_f^i \epsilon_H = \int_c L_f^i \omega$$

for $i = 0, 1, \ldots$.

Proof: Note that $L_f^i\omega = L_f^i(d\alpha_H + \epsilon_H) = dL_f^i\alpha_H + L_f^i\epsilon_H$. Now, the proof follows from the fact that the integral of an exact form over any closed curve is zero. $\square$

The above result is a law of preservation of hassle. No matter how one chooses a homotopy operator H, the average value of ("a component along c" of) $\epsilon_H := H d\omega$ on any closed curve c is constant. Different homotopy operators may only distribute ϵ along c in a different way. A similar result holds true for any Lie derivative (of any order) of ϵ along any vector field.

This result can be used to obtain lower bounds for the uniform norm of the error one-form ϵ_H and its Lie derivatives along f on $\mathcal{M}$, independent of the choice of homotopy.

Let us conclude the section by an example of a homotopy operator that is optimal in some precise sense. Let ω be a one-form. The so-called Hodge decomposition of ω is a decomposition of the form $\omega = d\alpha + \epsilon$, where $d\alpha$ is the best L^2 approximation of ω among exact one-forms (cf. Abraham *et al.*, Section 7.5, Banaszuk *et al.*, 1992, 1994b,c). Let δ denote the formal adjoint operator to the exterior derivative d and $\Delta := \delta d + d\delta$ denote the Laplace - De Rham operator (Abraham *et al.*, Section 7.5). One can show that $\Delta\alpha = \delta\omega$ and $\Delta\epsilon = \delta d\omega$. These equations (together with some boundary conditions) allow to find α and ϵ appearing in the Hodge decomposition of ω. Therefore, the operator $H_\Delta = \Delta^{-1}\delta$ (formally) satisfies $\omega = d(H_\Delta\omega) + H_\Delta d\omega$, so that it is a homotopy operator. Note that one has to solve a boundary value problem to obtain the zero-form α such that $d\alpha$ best approximates ω in L^2. This should be contrasted with the radial homotopy operator, which requires only simple integration with respect to a parameter. For instance, if the system (1) has polynomial nonlinearities in the original coordinates, the characteristic one-form ω_0 given by (9) will also have polynomial coefficients and thus the integration in (31) can be easily performed, yielding polynomial expressions for α and ϵ. The situation is usually much more difficult after applying an optimal approximate integrating factor β_0. The optimal characteristic one-form $\beta_0\omega$ will rarely be polynomial, and the result of integration in (31) might not be expressed in terms of elementary functions. This is one reason why we might not always be able to apply the optimal approximate integrating factor in practice, even if we find one.

5 Change of coordinates

In this section we prove that a change of coordinates based on the exact part of any characteristic one-form ω obtained with a homotopy operator H having center at the origin defines a local diffeomorphism that takes the system (1) to a normal form that looks like a linearizable part perturbed by some nonlinear terms, that depend linearly on the error one-form $\epsilon := H d\omega$. This approach can be applied to both linearizable and nonlinearizable systems.

For exactly linearizable systems (1), we proceed as follows. First, we construct a characteristic one-form ω_0. Then we choose an exact integrating factor β and obtain a new characteristic form $\omega := \beta\omega_0$ such that $d\omega = 0$. We apply a homotopy operator H to get the zero form $\alpha := H\omega$. Then we use change of variables

$$\begin{aligned} z_1 &= \alpha, \\ z_2 &= L_f\alpha, \\ &\vdots \\ z_n &= L_f^{n-1}\alpha. \end{aligned} \tag{32}$$

The system (8) in new coordinates is

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\vdots \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= p + ru \end{aligned} \tag{33}$$

where

$$r = L_g L_f^{n-1}\alpha, p = L_f^n\alpha,$$

and the feedback $u = r^{-1}(u_{new} - p)$ makes it linear.

For a nonlinearizable system we proceed as follows. First, we construct a characteristic one-form ω_0. Then we choose an integrating factor β, either optimal or not, and obtain a new characteristic one-form $\omega := \beta\omega_0$. We apply a homotopy operator H to get the zero form $\alpha := H\omega$ and the corresponding error one-form $\epsilon := H d\omega$. Then we use change of variables (32) (as for exactly linearizable systems) to get a normal form

$$\begin{aligned} \dot{z}_1 &= z_2 + e_1 u \\ \dot{z}_2 &= z_3 + e_2 u \\ &\vdots \\ \dot{z}_{n-1} &= z_n + e_{n-1} u \\ \dot{z}_n &= p + ru + e_n u \end{aligned} \tag{34}$$

where

$$
\begin{array}{rcl}
e_1 & = & L_g \alpha, \\
e_2 & = & L_g L_f \alpha, \\
 & \vdots & \\
e_n & = & L_g L_f^{n-1} \alpha - (-1)^{n-1} \omega(ad_f^{n-1} g), \\
r & = & (-1)^{n-1} \omega(ad_f^{n-1} g), \\
p & = & L_f^n \alpha.
\end{array} \tag{35}
$$

In the sequel we will need the following result.

Lemma 5.1 *Let ω be any characteristic one-form for the system (1). Let i, j be nonnegative integers. Then*

1. $(L_f^i \omega)(ad_f^j g) = 0$, *for* $i + j < n - 1$,
2. $(L_f^i \omega)(ad_f^{n-1-i} g) = (-1)^i \omega(ad_f^{n-1} g)$, *for* $i = 0, 1, \ldots, n-1$.

Proof: (1) One can proof by induction the formula:

$$
(L_X^i \eta)(Y) = \sum_{l=0}^{l=i} (-1)^l \binom{i}{l} L_X^{i-l}(\eta(ad_X^l Y)). \tag{36}
$$

In particular,

$$
\begin{array}{rcl}
(L_f^i \omega)(ad_f^j g) & = & \sum_{l=0}^{l=i} (-1)^l \binom{i}{l} L_X^{i-l}(\omega(ad_f^l(ad_f^j g))) \\
 & = & \sum_{l=0}^{l=i} (-1)^l \binom{i}{l} L_X^{i-l}(\omega(ad_f^{l+j} g)).
\end{array} \tag{37}
$$

Notice that $\omega(ad_f^{l+j} g) = 0$ for $l = 0, \ldots, i$ if $i + j \leq n - 1$.

(2) Apply the formula (37) for $j = n - 1 - i$ and notice that all the terms $\omega(ad_f^{l+n-1-j} g)$ vanish except when $l = i$. □

To establish continuity relationships between noninvolutivity and nonlinearizability we express the nonlinear perturbation terms e_i in terms of the error one-form ϵ.

Proposition 5.1 *Let ω be a characteristic form for (1), H be a homotopy operator on $\mathcal{M}$, $\alpha := H\omega$, and $\epsilon := H d\omega$. Let $e_i, i = 1, \ldots, n$, and p be given by (35). Then*

$$
\begin{array}{rcl}
e_1 & = & -\epsilon(g), \\
e_2 & = & -(L_f \epsilon)(g), \\
 & \vdots & \\
e_{n-1} & = & -(L_f^{n-2} \epsilon)(g), \\
e_n & = & -(L_f^{n-1} \epsilon)(g).
\end{array} \tag{38}
$$

Proof: It is a straightforward calculation using Lemma 5.1. □

Note that the above choice for e_n and r is not the only possible. Actually, any choice that guarantees $r + e_n = L_g L_f^{n-1}\alpha$ with $e_n(0) = 0$ could be considered, for instance $e_n = 0$ and $r = L_g L_f^{n-1}\alpha$. Our choice is dictated by the fact that it guarantees $r \neq 0$ on the whole $\mathcal{M}$ and $e_n = -(L_f^{n-1}\epsilon)(g)$.

One can also express the function $p := L_f^n \alpha$ using the error one-form ϵ as $p = (L_f^{n-1}\omega)(f) - (L_f^{n-1}\epsilon)(f)$.

A natural question to ask is whether the zero-form α together with its $n-1$ Lie derivatives along f is a well defined change of coordinates. The main result of this section says that in a neighborhood of the origin, (32) indeed defines a local diffeomorphism. Before we prove it, we need some preliminary results.

Lemma 5.2 *Let η be any smooth one-form and X, Y be any smooth vector fields on $\mathcal{M}$. Then*

1. $(L_X\eta)(Y) = L_X(\eta(Y)) - \eta([X,Y])$,
2. *If $\eta(0) = 0$ and $X(0) = 0$, then $(L_X^i\eta)(Y)(0) = 0$, for $i = 0, 1, 2, \ldots$.*

Proof: (1) See Hicks, Section 7.3.

(2) For $i = 0$ the formula is true as $(L_X^0\eta)(Y)(0) = \eta(Y)(0) = 0$. Assume that the formula is true for $i = 0, \ldots, m$. Using the statement (1) one easily shows that $(L_X^{m+1}\eta)(Y)(0) = d((L_X^m\eta)(Y))(X)(0) - (L_X^m\eta)([X,Y])(0)$. The first part of this expression is zero because $X(0) = 0$, the second by assumption. By induction, the formula holds for all nonnegative i. □

Proposition 5.2 *Assume that* $\dim \operatorname{span}\{g, ad_f g, \ldots, ad_f^{n-1} g\} = n$, *$\forall x$ in a neighborhood of 0 in $\mathcal{M}$ (linear controllability). Let ω be any characteristic one-form for the system (1). Then the one-forms ω, $L_f\omega$, $\ldots$, $L_f^{n-1}\omega$ are linearly independent in a neighborhood of the origin.*

Proof: It is sufficient to show that $(L_f^{n-1}\omega \wedge L_f^{n-2}\omega \wedge \ldots \wedge \omega)(0) \neq 0$. Since this form is smooth, it is enough to check that $(L_f^{n-1}\omega \wedge L_f^{n-2}\omega \wedge \ldots \wedge \omega)(g, ad_f g, \ldots, ad_f^{n-1} g)(0) \neq 0$. For this, note that $(L_f^{n-1}\omega \wedge L_f^{n-2}\omega \wedge \ldots \wedge \omega)(g, ad_f g, \ldots, ad_f^{n-1}) = det S$, where S is an $n \times n$ matrix whose (i,j) entry is $(L_f^{i-1}\omega)(ad_f^{j-1} g)$. Now, by Lemma 5.1, S is an upper triangular matrix, whose i-th diagonal element is $(-1)^{n-i}\omega(ad_f^{n-1} g)$. Therefore, $det S = (-1)^n(\omega(ad_f^{n-1} g))^n \neq 0$. □

Now we are ready to prove the main result of this section.

Theorem 5.1 *Assume that* $\dim \operatorname{span}\{g, ad_f g, \ldots, ad_f^{n-1} g\} = n$, *$\forall x$ in a neighborhood of 0 in $\mathcal{M}$ (linear controllability). Let H be any homotopy operator on $\mathcal{M}$ with the center at the origin such that $\phi(\lambda, 0) \equiv 0$ and let ω be any characteristic one-form for the system (1). Set $\alpha := H\omega$. Then (32) defines a local diffeomorphism in a neighborhood of the origin.*

Proof: We will show that the differentials of the zero-forms α, $L_f\alpha$,..., $L_f^{n-1}\alpha$ are linearly independent at the origin (and thus, in a neighborhood of the origin). Let $\epsilon := Hd\omega$. Since $\omega = d\alpha + \epsilon$ and the Lie and exterior derivatives commute, we have $dL_f^i\alpha = L_f^i d\alpha = L_f^i(\omega - \epsilon) = L_f^i\omega - L_f^i\epsilon$. Hence, $dL_f^i\alpha(0) = L_f^i\omega(0) - L_f^i\epsilon(0)$. Note that $\epsilon(0) := (Hd\omega)(0) = 0$, as H is a homotopy operator with the center at 0 such that $\phi(\lambda, 0) \equiv 0$. Now, it follows from Lemma 5.2(2) that ϵ, $L_f\epsilon$, ..., $L_f^{n-1}\epsilon$ all vanish at the origin and hence $dL_f^i\alpha(0) = L_f^i\omega(0)$. Now the result follows from Proposition 5.2. $\square$

We usually cannot guarantee a priori that the change of coordinates (32) will be valid in the whole $\mathcal{M}$. Some conditions for a map to be a global diffeomorphism are quoted in Hunt *et al.*and Zampieri. Below, we show an example of a system that admits a global transformation in R^3 to the normal form (6).

Example 5.0 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + h_1(x_3) \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= u \end{aligned} \tag{39}$$

where $h_1(\cdot)$ is any smooth function with $h_1'(0) = 0$. We have $\omega_0 = dx_1 - h_1'(x_3)dx_2$, $\alpha = H\beta\omega_0 = x_1 - \frac{x_2 h_1(x_3)}{x_3}$, $\epsilon = (\frac{h_1(x_3) - x_3 h_1'(x_3)}{x_3^2})(x_3 dx_2 - x_2 dx_3)$. $L_f\epsilon = L_f^2\epsilon = 0$. The system can be transformed by a global diffeomorphism

$$\begin{aligned} z_1 &= \alpha = x_1 - \tfrac{x_2 h_1(x_3)}{x_3}, \\ z_2 &= L_f\alpha = x_2, \\ z_3 &= L_f^2\alpha = x_3 \end{aligned} \tag{40}$$

to the form

$$\begin{aligned} \dot{z}_1 &= z_2 + \tfrac{z_2(h_1(z_3) - z_3 h_1'(z_3))}{z_3^2} u \\ \dot{z}_2 &= z_3 \\ \dot{z}_3 &= u. \end{aligned} \tag{41}$$

The inverse transformation given by

$$\begin{aligned} x_1 &= z_1 + \tfrac{z_2 h_1(z_3)}{z_3}, \\ x_2 &= z_2, \\ x_3 &= z_3. \end{aligned} \tag{42}$$

$\square$

In the case when the change of coordinates is not valid on the whole region $\mathcal{M}$, we have to restrict to a region on which the change of coordinates is valid. In the sequel we assume that this has been done and the restricted region is also called $\mathcal{M}$.

6 Estimates of the nonlinear part

In this section we estimate the nonlinear perturbation terms $e_1, \ldots, e_n$ using the error one-form ϵ. First, let us rewrite the equations (34) (in the usual matrix-vector notation) as

$$\dot{z} = Az + Bru + Bp + Eu \tag{43}$$

where A, B are in the Brunovsky form, that is,

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ \vdots & & & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $E = (e_1, e_2, \ldots, e_n)^T$ (e_i's, r, and p are defined by (35)). We see that $r \neq 0$ on $\mathcal{M}$ (under the assumption that linear controllability holds on $\mathcal{M}$) and E depends *linearly* on ϵ and vanishes, whenever ϵ does. We will choose $u = r^{-1}(u_{new} - p)$, where u_{new} is a new control variable. After this change of coordinates and control variable the system is of the form (6) with $Q := r^{-1}E, P := -r^{-1}pE$. In this section we obtain estimates on the uniform norm of Q and P (via estimates on r, p, and E) in terms of the error one form ϵ_β for any fixed β, on any compact, contractible region $\mathcal{M}$.

Let h be a smooth vector field on $\mathcal{M}$ and l be a nonnegative integer. Let ζ be a k-form on $\mathcal{M}$. We define C^0 norm $||\zeta||^0$ of ζ as $||\zeta||^0 := \sup|\zeta(x)|$, for $x \in \mathcal{M}$ (uniform norm on $\mathcal{M}$), and C_h^l norm $||\zeta||_h^l$ as $||\zeta||_h^l := \sup(|\zeta(x)|^2 + |L_h\zeta(x)|^2 + \ldots + |L_h^l\zeta(x)|^2)^{1/2}$, for $x \in \mathcal{M}$ (uniform norm on $\mathcal{M}$, together with the first l Lie derivatives along h). It is immediately seen from Proposition 5.1 that whenever the one-forms ϵ, $L_f\epsilon$, $\ldots$, $L_f^{n-1}\epsilon$ are small on $\mathcal{M}$, so is the term E on $\mathcal{M}$.

Theorem 6.1 *Let ω be any characteristic one-form for the system (1) and let ϵ be the error one-form corresponding to a given homotopy operator. Then the mapping $\epsilon \mapsto E$ is a continuous mapping from the space of smooth one forms equipped with the C_f^{n-1} norm on $\mathcal{M}$ into the space of smooth vector fields on $\mathcal{M}$ equipped with the C^0 norm (uniform norm on $\mathcal{M}$). In particular,*

$$||E||^0 \leq ||\epsilon||_f^{n-1}||g||^0.$$

Proof: Immediate in view of Proposition 5.1. □

Note that in the above result we could substitute for ϵ, $L_f\epsilon$, $\ldots$, $L_f^{n-1}\epsilon$ their evaluations at (contractions by) the vector field g. Let h, v be smooth vector fields on $\mathcal{M}$ and l be a nonnegative integer. Let us define $C_{h,v}^l$ seminorm $||\zeta||_{h,v}^l$ of a one-form ζ on $\mathcal{M}$ as $||\zeta||_{h,v}^l := \sup(|\zeta(v)(x)|^2 + |L_h\zeta(v)(x)|^2 + \ldots + |L_h^l\zeta(v)(x)|^2)^{1/2}$, over $x \in \mathcal{M}$. Note that $C_{h,v}^l$ is not quite a norm, for it may happen that $||\zeta||_{h,v}^l = 0$ even though $\zeta \neq 0$ (example: $||\omega_0||_{f,g}^0$ for ω_0

being a characteristic one-form for (1)). However, it happens that for the one-forms on $\mathcal{M}$, the $C^{n-1}_{f,g}$ seminorm becomes a norm if the vector fields f, g satisfy the linear controllability condition of Theorem 1.1. This follows from the following result.

Proposition 6.1 *Let ζ be a one-form on $\mathcal{M}$, and the vector fields f, g satisfy the linear controllability condition of Theorem 1.1. Then $\zeta = 0$ if and only if $||\zeta||^{n-1}_{f,g} = 0$.*

Proof: ($\Rightarrow$) Obvious.

($\Leftarrow$) Note that $L^i_f\zeta(g) = L_f(L^{i-1}_f\zeta(g)) - L^{i-1}_f\zeta(ad_f g)$. We have $L^i_f\zeta(g) = 0$ for $i = 0, 1, \ldots, n-1$. In particular, $\zeta(g) = 0$. Thus, using Lemma 5.2(1), we get $0 = (L_f\zeta)(g) = L_f(\zeta(g)) - \zeta(ad_f g) = -\zeta(ad_f g)$. Continuing in the same fashion, we obtain $\zeta(ad^i_f g) = 0$, for $i = 0, 1, \ldots, n-1$. By the linear controllability assumption, the vector fields $ad^i_f g$ for $i = 0, 1, \ldots, n-1$ are linearly independent. The one-form ζ annihilates n linearly independent fields on an n-dimensional manifold. Thus $\zeta = 0$. $\square$

The above result, when applied to the error one-form ϵ yields an obvious fact that the $\epsilon = 0$ is equivalent with (32) being the linearizing change of coordinates for (1). The fact that we wanted to emphasize here is that, because of (38), the nonlinear perturbation terms e_i can be used to define a norm for the error one-form ϵ, thus making the relationship between a measure of noninvolutivity of the characteristic distribution D and a direct measure of nonlinearity of the system (1) in new coordinates explicit. Namely, we have

Proposition 6.2 $||E||^0 = ||\epsilon||^{n-1}_{f,g}$.

We conclude this section with establishing some upper bounds on the uniform norms $||P||^0$ and $||Q||^0$ of the nonlinear terms $Q := r^{-1}E$, $P := -r^{-1}pE$ in the system (6) after change of coordinates and preliminary feedback.

Proposition 6.3 *Let ω be any characteristic form for the system (1), $\alpha := H\omega$, $\epsilon := H d\omega$. Let $\rho := \inf |\omega(ad^{n-1}_f g)(x)|$ over $x \in \mathcal{M}$ and $\varrho := \sup |L^n_f\alpha(x)|$ over $x \in \mathcal{M}$. Then*

1.
$$||P||^0 \leq \frac{\varrho||\epsilon||^{n-1}_{f,g}}{\rho}. \tag{44}$$

2.
$$||Q||^0 \leq \frac{||\epsilon||^{n-1}_{f,g}}{\rho}. \tag{45}$$

Proof: Immediate, in view of Proposition 6.2 and (35). $\square$

7 Application to stabilization

In this section we will use the results of the previous section to study various locally stabilizing feedback laws for the system (1). The laws that we have in mind will be linear in new coordinates (32), with the gains chosen so that the linear part of the system (6) is asymptotically stable. We will then study robustness of such control laws when applied to the system (6). We will accomplish that studying Lyapunov functions that are quadratic in new coordinates. We shall examine how the nonlinear part of (6) affects the time derivative of the Lyapunov function. The continuity result of Theorem 6.1 will allow us to formulate some robustness criteria for stabilization.

The idea behind transforming a linearizable system (1) to an equivalent form (2) is to design control schemes for (2), which is much easier to analyze and control, and apply them to (1). For example, if Φ happens to be a global diffeomorphism from R^n into R^n, one can globally asymptotically stabilize the system (1). For this, one can choose new control variable $u_{new} = Kz$ (linear feedback in new variables) so that the closed-loop system

$$\dot{z} = (A + BK)z, \tag{46}$$

is globally asymptotically stable (controllability of (2) is equivalent to possibility of arbitrary assignment of the eigenvalues of $(A + BK)$ by an appropriate choice of the feedback gain K). Then $u = k(x) + l(x)u_{new} = k(x) + l(x)K\Phi(x)$ makes the closed-loop system

$$\dot{x} = f(x) + g(x)(k(x) + l(x)K\Phi(x)) \tag{47}$$

globally asymptotically stable, since Φ^{-1} is a diffeomorphism preserving the equilibrium point at the origin.

For nonlinearizable systems the best we can hope for using our approach is to transform (1) to (6) with P and Q small. Then we will try to use the new form (6) to design a locally stabilizing feedback—in this case we expect to improve the basin of attraction of the origin of the closed loop system. We will choose $u_{new} = Kz$ (a feedback law linear in new variables) so that the mapping $A+BK$ is stable (has all eigenvalues with negative real parts) and analyze its robustness as a stabilizing law for (6)—bounds on uniform norms for Q and P should help us to do so. Let us stress that we will actually use new coordinates z and new control u_{new} only as intermediate tools, the control law $u = r^{-1}(u_{new}-p) = r^{-1}(Kz-p)$ will be expressed in the old coordinates x as $u = k_1(x)$ (where $k_1(x) := r(\Phi(x))^{-1}(K\Phi(x) - p(\Phi(x))))$ and applied to (1). Since Φ^{-1} is a diffeomorphism preserving the equilibrium point at the origin, it maps the basin of attraction of the equilibrium for

$$\dot{z} = (A + BK)z + P(z) + Q(z)Kz \tag{48}$$

to the basin of attraction of the equilibrium for

$$\dot{x} = f(x) + g(x)k_1(x) \tag{49}$$

Observe that, to express the feedback laws computed in new coordinates z in the original coordinates x, we don't even need to find the form (6) explicitly. It would be actually very difficult, if not impossible, to do so in general, for we would have to know the inverse transformation $x = \Phi^{-1}(z)$ in order to obtain the form (6).

Of course, we might not always be able to find the best integrating factor β_0 for ω_0 annihilating $D := \text{span}\ \{g, ad_f g, \ldots, ad_f^{n-2} g\}$ to begin with. Still, for any scaling factor β we can choose the corresponding zero-form α_β and its Lie derivatives along f as new coordinates. We can also find the corresponding error one-form ϵ_β and verify the bounds on the corresponding terms Q and P in (6), and decide if they are sufficiently small for our purpose.

Theorem 7.1 *Assume that $z = \Phi(x)$ be a (global) diffeomorphism of $\mathcal{M}$ onto its image given by (32). Let $u_{new} = Kz$ be any linear feedback in new variables so that the linear part*

$$\dot{z} = (A + BK)z, \tag{50}$$

of the system (6) obtained from (1) after change of coordinates and preliminary feedback is asymptotically stable. Let N be a positive definite n by n matrix and let M be the unique positive semidefinite solution of the Lyapunov equation

$$(A + BK)^T M + M(A + BK) + N = 0. \tag{51}$$

Let

$$E(z) := P(z) + Q(z)Kz \tag{52}$$

and $\Omega_r = \{0\} \cup \{z \in \Phi(\mathcal{M}) : \langle z, Mz\rangle < r$ and $\langle z, Nz\rangle - 2\langle z, ME(z)\rangle > 0\}$. Define $r_{max} := \sup\{r \geq 0 : \Omega_r \subseteq \Phi(\mathcal{M})\}$. Then $\Phi^{-1}(\Omega_{r_{max}})$ is an invariant set contained in the basin of attraction of the origin of the system

$$\dot{x} = f(x) + g(x)k_1(x), \tag{53}$$

where $k_1(x) := r(\Phi(x))^{-1}(K\Phi(x) - p(\Phi(x)))$ (p and q are defined by (35)).

Proof: The linearizable part of the system in new coordinates z can be made asymptotically stable by feedback $u_{new} = Kz$ linear in new coordinates. One can define a quadratic Lyapunov function $V(z) := \langle z, Mz\rangle$ with a negative time derivative $\frac{\partial V}{\partial z}(z) = -\langle z, Nz\rangle$ solving the Lyapunov equation (51). The sets Ω_r are invariant sets for the closed loop linear part (50). Now, the time derivative of Lyapunov function for the true system in new coordinates is $\frac{\partial V}{\partial z}(z) = -(\langle z, Nz\rangle - 2\langle z, ME(z)\rangle)$. If this is negative, the sets $\Phi^{-1}(\Omega_{r_{max}})$ are invariant sets for the closed-loop system (53). $\square$

The above result simply states a sufficient condition for a region of $\mathcal{M}$ to be an invariant set contained in the basin of attraction of the origin of the system $\dot{x} = f(x) + g(x)k_1(x)$, and is well known. What is nice about the above result is that we can actually estimate the set $\Phi^{-1}(\Omega_{r_{max}})$ in our approach. Namely, since we have estimates on the uniform norms $\|P(z)\|$ and $\|Q(z)\|$ of the nonlinear terms in the system (6), we obtain an upper bound on the uniform norm $E(z) = P(z) + Q(z)Kz$. Thus, we can check if the time derivative

$-(\langle z, Nz\rangle) + 2\langle z, ME(z)\rangle)$ of the Lyapunov function is negative on the region of interest. Moreover, since we expect $P(z)$ and $Q(z)$ to be small, so will be $E(z)$. Since the first term in $-(\langle z, Nz\rangle) + 2\langle z, ME(z)\rangle)$ is negative and the second term is small, the whole expression is negative in some neighborhood of the origin.

Let us define yet another measure of nonlinearity in new coordinates that is particularly suited for studying stabilization:

$$\eta_{afl}(z) := \frac{2\langle z, ME(z)\rangle}{\langle z, Nz\rangle},\ \mathit{for}\ z \neq 0,\ \eta_{afl}(0) := 0. \tag{54}$$

Now we can replace the condition

$$2\langle z, ME(z)\rangle < \langle z, Nz\rangle,\ \mathit{for}\ z \neq 0 \tag{55}$$

with

$$\eta_{afl}(z) < 1. \tag{56}$$

Note that the quantity η_{afl} actually depends on the choice of characteristic one form, the particular homotopy operator, the stabilizing feedback gain matrix K, and the matrix N. Observe that $|\eta_{afl}(z)| < 1$ means that the linear term dominates the nonlinear one in the time derivative $\frac{\partial V}{\partial z}(z) = -(\langle z, Nz\rangle) + 2\langle z, ME(z)\rangle)$ of the Lyapunov function $V(z) := \langle z, Mz\rangle$ at the particular point z, guaranteeing its negative sign. On the other hand $0 < \eta_{afl}(z)$ means that the nonlinearities contribute to making $\frac{\partial V}{\partial z}(z)$ more positive, and thus have a destibilizing effect, while $\eta_{afl}(z) < 0$ means that the nonlinearities try to make $\frac{\partial V}{\partial z}(z)$ more negative, and hence help to stabilize the system. Therefore, the following terminology is justified: we will say that the nonlinearities are *weak* (respectively, *strong*) at z if $|\eta_{afl}(z)| < 1$ (respectively, $|\eta_{afl}(z)| > 1$) and *friendly* (respectively, *unfriendly*) if $\eta_{afl}(z) < 0$ (respectively, $0 < \eta_{afl}(z)$).

Let us express this condition in terms of system (6) and (8). We have $E(z) = P(z) + Q(z)Kz = (Kz - p(z))(r^{-1}(z)E(z))$. Thus $2\langle z, ME(z)\rangle = 2\langle z, M(Kz - p(z))(r^{-1}(z)E(z))\rangle$, and (56) is equivalent with

$$\frac{2\langle z, M(Kz - p(z))(r^{-1}(z)E(z))\rangle}{\langle z, Nz\rangle} < 1,\ \mathit{for}\ z \neq 0. \tag{57}$$

Using bounds on $||P(z)||$ and $||Q(z)||$ obtained in the previous section, one can formulate the following inequality that implies the previous ones

$$2\frac{(\varrho + |K||z|)\|\epsilon\|_{f,g}^{n-1}}{\rho} < \frac{\inf\ \sigma(N)}{\sup \sigma(M)}|z|,\ \mathit{for}\ z \neq 0 \tag{58}$$

where $\sigma(\cdot)$ denotes a spectrum of a matrix.

It is possible to combine the problems of designing a stabilizing feedback for the linear part of of the system (6) obtained from (1) after change of coordinates and preliminary feedback and construction of a Lyapunov function in a linear quadratic optimal control design: find u_{new} minimizing

$$\int_0^\infty (\langle z(t), Nz(t)\rangle + \langle u_{new}(t), Ru_{new}(t)\rangle)dt,$$

for strictly positive definite R and a positive definite N. (To make life easier, we will assume that N is also strictly positive definite). It is well-known that the optimal control u_{new} has the form of linear feedback $u_{new} = Kz$ for $K = -R^{-1}B^TM$, where M is the unique positive definite solution of the Riccati equation

$$A^TM + MA - MBR^{-1}B^TM + N = 0. \tag{59}$$

Example 7.0 Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 + ax_3^3 + bx_1^3 \\ \dot{x}_2 &= x_3 + cx_1^2x_2 \\ \dot{x}_3 &= u. \end{aligned} \tag{60}$$

□

Note this is a particular case of the system considered in Example 3. We have $\omega_0 = dx_1 - 3ax_3^2dx_2$, and $d\omega_0 = 6ax_3dx_2 \wedge dx_3$. For scaling factor $\beta = 1$ we get $\alpha := H\omega = x_1 - ax_2x_3^2, \epsilon := Hd\omega = (2ax_3)(x_2dx_3 - x_3dx_2)$. New coordinates $z = \Phi(x)$ are given by $z_1 := \alpha = x_1 - ax_2x_3^2, z_2 := L_f\alpha = x_2 + bx_1^3 - acx_1^2x_2x_3^2, z_3 := L_f^2\alpha = x_3 + 3b^2{x_1}^5 + 3b{x_1}^2x_2 + c{x_1}^2x_2 - 2abc{x_1}^4x_2{x_3}^2 - ac^2{x_1}^4x_2{x_3}^2 - 2acx_1{x_2}^2{x_3}^2 + 3ab{x_1}^2{x_3}^3 - ac{x_1}^2{x_3}^3 - 2a^2cx_1x_2{x_3}^5$. Note that $\Phi(x)$ is only a local diffeomorphism around the origin and it is impossible to find an inverse transformation. Thus, in the sequel we express the nonlinear terms $E(z)$, $r(z)$, and $p(z)$ in old coordinates: $E(\Phi(x)) = [-2ax_2x_3, -2ac{x_1}^2x_2x_3, -2acx_1x_2x_3\,(2b{x_1}^3 + c{x_1}^3 + 2x_2 - 4a{x_3}^3)]^T$,

$$\begin{aligned} r(\Phi(x)) &:= 1 + 9ab{x_1}^2{x_3}^2 - 3ac{x_1}^2{x_3}^2 - 18a^2cx_1x_2{x_3}^4 \\ p(\Phi(x)) &:= 15b^3{x_1}^7 + 21b^2{x_1}^4x_2 + 5bc{x_1}^4x_2 + c^2{x_1}^4x_2 + 6bx_1{x_2}^2 + \\ &\quad 2cx_1{x_2}^2 + 3b{x_1}^2x_3 + c{x_1}^2x_3 - 8ab^2c{x_1}^6x_2{x_3}^2 - 6abc^2{x_1}^6x_2{x_3}^2 - \\ &\quad ac^3{x_1}^6x_2{x_3}^2 - 10abc{x_1}^3{x_2}^2{x_3}^2 - 8ac^2{x_1}^3{x_2}^2{x_3}^2 - 2ac{x_2}^3{x_3}^2 + \\ &\quad 21ab^2{x_1}^4{x_3}^3 - 4abc{x_1}^4{x_3}^3 - ac^2{x_1}^4{x_3}^3 + 12abx_1x_2{x_3}^3 - \\ &\quad 4acx_1x_2{x_3}^3 - 10a^2bc{x_1}^3x_2{x_3}^5 - 6a^2c^2{x_1}^3x_2{x_3}^5 - 4a^2c{x_2}^2{x_3}^5 + \\ &\quad 6a^2bx_1{x_3}^6 - 4a^2cx_1{x_3}^6 - 2a^3cx_2{x_3}^8 \end{aligned}$$

(all computationd were done using Mathematica). To design a locally stabilizing feedback, we have solved the LQ regulator problem for the linear part of the system as mentioned above for N being the 3 by 3 identity matrix and $R = 1$. The optimal feedback gain matrix was $K = [-1, -2.41421, -2.41421]$, and eigenvalues of $A+BK$ were $-1, -0.707107 + i0.707107, -0.707107 - i0.707107$. The feedback law applied to the original system (1) was $u_{afl} := r(\Phi(x))^{-1}(K\Phi(x) - p(\Phi(x)))$. We choose the values of parameters $a = 0.01, b = 1, c = 5$ and $\mathcal{M} := \{|x_i| < 0.36, i = 1, 2, 3\}$. We checked the condition (57) was satisfied on

$\mathcal{M}$, with $\sup \eta_{afl}(\Phi(x)) \approx 0.45$. Thus, by Theorem 7.1, the corresponding set $\Phi^{-1}(\Omega_{r_{max}})$ (defined in the formulation of Theorem 7.1) is in the basin of attraction of the origin. As we have checked, the whole $\mathcal{M}$ was in the basin of attraction of the origin. The basin of attraction was actually much larger than $\mathcal{M}$, even though the condition (57) was not satisfied (note that Theorem 7.1 gives only an underestimate of the actual stability region). For comparison, we considered the control based on Jacobian linearization $u_{jac} := Kx$, for the same gain matrix K. Note that x and z coordinates agree up to 1-st order, and that both control schemes u_{afl} and u_{jac} yield the same linear part of the closed loop system with eigenvalues $-1, -0.707107 + i0.707107, -0.707107 - i0.707107$. We checked that for u_{jac} condition (57) failed to hold on $\mathcal{M}$, with $\sup \eta_{jac}(x) \approx 5.6$ (11 times more than for u_{afl}), where $\eta_{jac}(x) := \frac{2\langle x, M(Kx)(E_{jac}(x))\rangle}{\langle x, Nx\rangle}$, $E_{jac} := [ax_3^3 + bx_1^3, cx_1^2x_2, 0]^T$. Not whole $\mathcal{M}$ was in the region of stability for u_{jac}, and the region of stability for u_{jac} was strictly contained in the the region of stability for u_{afl}. We present (in figures 1 through 3) typical plots of the state variables as functions of time (the darker lines represents the time responses for u_{afl}, the lighter lines for u_{jac}). Comparing those responses of our system for both control schemes, we see that u_{afl} offered faster convergence to the origin and less oscillatory responses than u_{jac}. We also plot (in figures 4 and 5) the terms $\eta_{afl}(\Phi(x))$ and $\eta_{jac}(x)$ along trajectories, because they in some sense measure nonlinearity of the corresponding closed loop systems. Observe that the strong and unfriendly nonlinearities prevail in the closed-loop system with u_{jac} control when compared to weak nonlinearities in the closed-loop system with u_{afl} control.

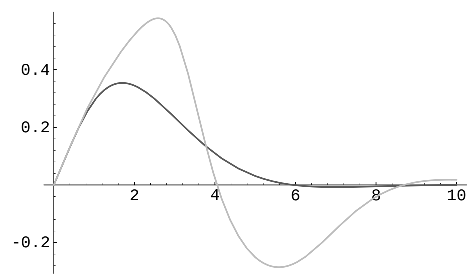

Figure 1: $x_1(t)$ for $x_1(0) = 0, x_2(0) = 0.3, x_3(0) = 0.3$.

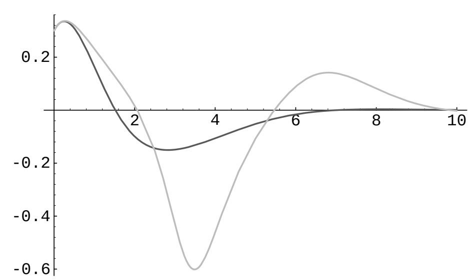

Figure 2: $x_2(t)$ for $x_1(0) = 0, x_2(0) = 0.3, x_3(0) = 0.3$.

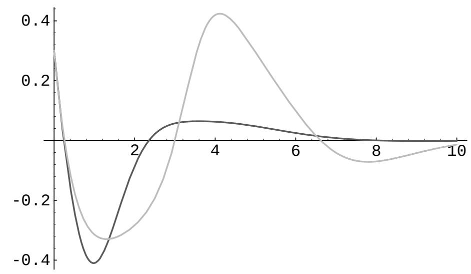

Figure 3: $x_3(t)$ for $x_1(0) = 0, x_2(0) = 0.3, x_3(0) = 0.3$.

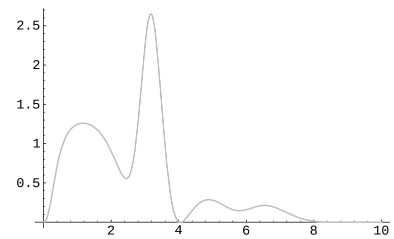

Figure 4: η_{jac} for $x_1(0) = 0, x_2(0) = 0.3, x_3(0) = 0.3$.

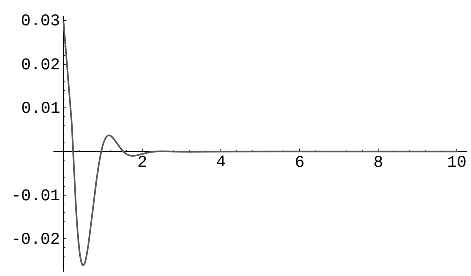

Figure 5: η_{afl} for $x_1(0) = 0, x_2(0) = 0.3, x_3(0) = 0.3$.

Note that from Section 3 we actually know that the optimal integrating factor for $\omega_0 = dx_1 - ax_3^2 dx_2$ is $\beta = |\omega_0|^{-1} = (1+a^2x_3^4)^{-1/2}$. Observe, that with $a = 0.01$ and $|x_3| < 0.36$, we have $\beta \approx 1$ up to six decimal places. We have found the coresponding change of coordinates and performed simulations, but the results were indistinguishable from the case $\beta = 1$.

8 Conclusion

In this paper, we presented an approach for finding feedback linearizable systems that approximate a given single-input nonlinear system on a given compact region of the state space. We have shown that if the system is close to being involutive then it is also close to being linearizable. We have applied this approach for design of locally stabilizing feedback laws for nonlinear systems that are close to being linearizable. The main idea was to study the characteristic one forms rather than deal with the characteristic distribution directly. In this approach two issues have occured. First, how to scale characteristic forms, second, how to approximate them by exact forms. We have presented some ideas on that subject and indicated some open problems.

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References

1. Abraham, R., J.E. Marsden, T. Ratiu: *Manifolds, Tensor Analysis, and Applications*, Addison-Wesley, 1983.

2. Banaszuk, A., J. Hauser, D. Taylor: "L^2-Approximate feedback linearization using the Hodge decomposition theorem," *Proceedings of International Symposium on Implicit and Nonlinear Systems*, Fort Worth, December 1992, 329-333.

3. Banaszuk, A., J. Hauser: "Least squares approximate feedback linearization: a variational approach," *Proceedings of the 32nd Conference on Decision and Control*, San Antonio, December 1993, 2760-2765.

4. Banaszuk, A., J. Hauser: "Approximate feedback linearization: approximate integrating factors," *Proceedings of the 1994 American Control Conference*, Baltimore, MD, June 1994, 1690-1694.

5. Banaszuk, A., A. Święch, J. Hauser: "Least squares approximate feedback linearization," 1994, submitted.

6. Banaszuk, A.,A. Święch, J. Hauser: "Approximate feedback linearization: least squares approximate integrating factors," *Proceedings of the 33rd Conference on Decision and Control*, Lake Buena Vista, FL, December 1994, 1621-1626.

7. Bryant, R.L., S.S. Chern, R.B. Gardner, H.L. Goldschmidt, P.A. Griffiths: *Exterior Differential Systems*, Springer-Verlag, 1991.

8. Edelen, D.G.B.: *Applied Exterior Calculus*, Wiley, 1985.

9. Flanders, H.: *Differential Forms with Applications to the Physical Sciences*, Dover Publications, 1989.

10. Hauser, J., S. Sastry, P. Kokotovic: "Nonlinear control via approximate input-output linearization: the ball and beam example," *IEEE Transactions on Automatic Control*, AC-37, 1992, 392-398.

11. Hauser, J.: "Nonlinear control via uniform system approximation," *Systems and Control Letters*, 17, 1991, 145-154.

12. Hauser, J., Z. Xu, "An approximate Frobenius Theorem," *1993 IFAC World Congress*, Sydney, 8, 43–46.

13. Hicks, N.J.: *Notes on Differential Geometry*, Van Nostrand, 1965.

14. Hunt, L.R., R. Su, G. Meyer, "Global transformations of nonlinear systems", *IEEE Transactions on Automatic Control*, AC-28, 1983, 24-30.

15. Hunt, L.R., J. Turi, "A new algorithm for constructing approximate transformations for nonlinear systems", *IEEE Transactions on Automatic Control*, AC-38, 1993, 1553-1556.

16. Jakubczyk, B., W. Respondek: "On linearization of control systems", *Bull. Acad. Pol. Sci. Ser. Sci. Math.*, 28, 1980, 517-522.

17. Kokotovic, P., "The joy of feedback: nonlinear and adaptive," *IEEE Control Systems*, June 1992, 7-17.

18. Krener, A.: "Approximate linearization by state feedback and coordinate change," *Systems and Control Letters*, 5, 1984, 181-185.

19. Krener, A., Maag: "Controller and observer design for cubic systems," submitted, 1991.

20. Xu, Z., J. Hauser, "Higher order approximate feedback linearization about a manifold," *Journal of Mathematical Systems, Estimation, and Control*, 4, 1994, 451–465.

21. Xu, Z., J. Hauser, "Higher order approximate feedback linearization about a manifold for multi-input systems," *IEEE Transactions on Automatic Control*, 1995, to appear.

22. Zampieri, G.: "Finding domains of invertibility for smooth functions by means of attraction basins", *Journal of Differential Equations*, 1, 1993, 11-19.