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THE DEVELOPMENT OF A QUANTITATIVE METHOD FOR THE OPTIMIZATION OF THE FACILITIES LOCATION PROBLEM

A THESIS

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CHAPTER I

INTRODUCTION

Objective

The objective of this study was to develop a method to obtain a mathematically optimum solution to the problem of the location of facilities within a given area.

Purpose

The purpose of this study is to provide some objective basis by which a layout may be evaluated and which will also generate a layout that will be optimum with respect to some criterion.

Definition of Facilities Planning

Facilities planning is more often defined in the framework of one of its subdivisions, known generally as plant layout, than as an entity unto itself. There are as many definitions of plant layout as there are writers on the subject, but they are generally similar in most respects. One of the better ones is that given by Reed (1):

> Plant layout is the most effective arrangement and coordination of the physical plant facilities to allow greatest efficiency in the combination of men, materials, and machines necessary for operation of any unit of a plant or business.

Selection of Criterion

If one chooses to use the term facilities planning, then the limitation of this area to manufacturing enterprises alone is lost, as it should be. Plant layout is only a subtopic of the larger area, which includes the planning of any structure in which facilities must be provided for some functional purpose. The design of stores, churches, airports, schools, and in fact every building that must satisfy some criterion of function, rather than primarily aesthetic considerations, rightly belongs in the field of facilities planning.

There is only one criterion that can be used to judge the effectiveness of any functional facility; and that is that it must provide the maximum usefulness for the least amount of human expenditure. Normally, a facility is planned with a pre-defined purpose: it must accommodate a certain number of people, or it must produce so many units, or in other words, it must fulfill its purpose, and this can be said to be its maximum usefulness.

Given that the facility accomplishes this task (for if it does not, its reason for existence disappears), then the human expenditure must be kept as low as possible. Human expenditure is difficult to measure; the only objective method is to equate it to money, through the medium of time. Thus the primary consideration in minimizing human expenditure is to minimize cost, in money. This is not meant to imply that human considerations that cannot be equated to money must be ignored; however, objectivity is lost in the process of taking these considerations into account. There are some means of avoiding this pitfall, and they will be mentioned later.

Therefore, the primary objective in designing any functional enterprise is to insure that it accomplishes its intended purpose, and the secondary objective is to accomplish it at minimum cost. The first phase of plant layout concerns itself with the primary objective; in this phase the number and types of equipment that can best fulfill the primary function are selected. The second phase is concerned with the location of this equipment in order to minimize cost.

This study concerns itself with the second phase only. The assumption is made that the first phase has been satisfactorily accomplished and no further mention will be made of it.

It is now necessary to select criteria by which a layout may be evaluated. Given that monetary cost must be minimized, it is necessary to find a measure of a layout that can be expressed as a monetary cost. Fortunately, Freeman (2) has shown that one of the most important costs which is a function of the layout is that of material handling, and the units of weight- or volume-distance per unit time are a reliable measure of this cost (Moore) (3). This is the sole criterion that is used in this study. However, a method will be introduced later by which criteria that cannot be expressed directly in weight- or volume-distance relationships can be introduced into the problem.

In more complete terms, the objective of this study will be to find a method by which facilities may be located in order to minimize this weight- or volume-distance criterion.

CHAPTER II

BACKGROUND AND SURVEY OF LITERATURE

In this chapter a background of the field of facilities planning will be presented in order to acquaint the reader with the state of the technique at the present time. A survey of the literature will then be presented in order to acquaint him with the work of a quantitative nature that is being done in this field.

Background

The problem of facilities location is one that has existed for a long time, but until the last century it has not been recognized as a separate problem area. The historical method of facilities location has been the trial-and-error method, and indeed, even today this method is by far the most prevalent. One might say that most plants designed prior to 1940 were laid out by trial and in many of these cases the error has not yet been discovered.

However, since about 1940, the cost of error has been drastically reduced by the use of scale layouts and scale models. By using these tools, trial layouts may be made, the errors discovered and corrected, and a new layout designed. Unfortunately, there is no way to know whether the resulting design is the best one.

Many "rules of thumb" have been devised, forms have been designed, and elaborate procedures have been developed to take as much guesswork as possible out of the layout problem, but none of them suggests any way to truly "optimize" a layout design. For discussions of these techniques, the reader's attention is invited to Reed (1), Apple (4), and Muther (5). All three of these books give very excellent presentations of the field of plant layout.

Survey of Literature

The first work done on quantitative techniques of facilities planning reviewed by Huffman (6), and a quantitative flow chart was presented by de Villeneuve (7). In both of these works the only techniques presented are those of deciding between alternative solutions. However, the advantage is gained that the decision may be made on a quantitative basis rather than by judgment.

A large gap then exists before any further work is done in this area. In 1958 Wimmert (8) presented a method by which an optimum solution could be obtained to the equipment location problem, but with a very limited application.

In essence, Wimmert first formulates the problem in the same manner as is presented in this study. In his problem, he inserts new machines into already existing layouts, and limits the problem to assigning the new machines to a limited number of available locations. The criterion used is that of weight- or volume-distance. He devises a special square matrix in which the row headings represent the amount of material that moves between each pair of machines to be located, and the column headings represent the distance between each pair of available locations.

The values in each cell are the product of the row heading and the column headings, and represent weight- or volume-distance that would be moved between the two machines <u>if they were located in that pair of locations</u>. In this problem there are n machines; taken two at a time there would result a square matrix with $n!/2!(n-2)!$ rows and columns.

Without going into further detail, he solves this matrix by eliminating the nonoptimum solutions successively until only the optimum one is left. The matrix he uses allows him to do this by virtue of the <u>location</u> of the cells, rather than by the values contained therein.

As the number of machines, n, increases, the $n!/2!(n-2)!$ square matrix increases rapidly in size, until its practicality is much in doubt. A necessary condition is that it be square, which implies the same number of machines and available locations. If there are more machines than available locations, then of course there are no feasible solutions. However, if there are more available locations than machines, then the model cannot handle the situation without modifications.

A serious limitation is that the model recognizes no relationships between the new machines and the existing ones. This is roughly equivalent to establishing a new and independent production line in the middle of an already existing layout. This writer believes this difficulty can be overcome, and more will be said of this later.

Summed up, this method is merely that of choosing among alternatives. However, by virtue of using discrete locations, all the alternatives may be tabulated, and the technique simply gives a feasible method of picking out the best one.

Wimmert is the first writer on the subject to specifically formulate the problem mathematically as

$$Z = \sum_{i=1}^{n} f_i d_i = \text{minimum}$$

where

f_i = amount of material flow between machines

d_i = distance between machines.

This equation is the basis for this thesis, and its development will be taken up in detail in the next chapter.

The next work of major importance that has been done in this field is that of Moore (9). His method is similar in appearance to Wimmert's, but its application is the reverse. In this case, discrete "candidate areas" are specified for the locations of n new machines. A matrix is formulated with the existing machines as column headings and the new machines as row headings, with the cells representing the flow of materials, in weight or volume per unit time, between the rows and columns. Another matrix is then made up with the existing machines as column headings and the candidate areas as row headings. The cells represent distance between the column headings and the row headings. By matrix multiplication, these two matrices are combined into a single one, with the new machines as column headings and the candidate areas as row headings. Each cell contains a value, in weight- or volume-distance units, that represents the effectiveness of assigning a particular machines to a given area.

The following is a short example to clarify this procedure. Assume that the flow between the new machines and the existing machines is known and can be tabulated as follows:

Existing Machines

		1	2	3
New Machines	1	5	2	1
	2	4	6	3

The distance between each candidate area and the existing machines can be tabulated as

To Existing Machines

		1	2	3
From Candidate Areas	1	10	7	10
	2	8	6	6

Multiplying these two matrices results in: *

$$\begin{pmatrix} 10 & 7 & 10 \\ 8 & 6 & 6 \end{pmatrix} \begin{pmatrix} 5 & 4 \\ 2 & 6 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 74 & 112 \\ 58 & 86 \end{pmatrix}$$

---

*This operation is performed as follows:

$$AB = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = C$$

where

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j}.$$

The resulting matrix may easily be solved by a technique of linear programming known as the "assignment method," a good explanation of which appears in Churchman (10). This is again merely a method used to choose between alternatives, although all the alternatives are considered. The matrix used must be square, but this presents no great problem.

The beauty of this method is that it yields an optimum solution very easily. Unfortunately, it has two rather serious limitations: first, specific "candidate areas" must be selected, and second, no relationships can exist between the machines to be located.

Moore suggests that the second limitation may be overcome by the use of general simplex formulation of the problem, but this writer, after considerable research, does not agree. In the general formulation of this problem the relationships are not linear, and therefore linear programming, at its present state of development, is incapable of dealing with it.

However, as a suggestion for further research, it appears that some combination of Wimmert's and Moore's methods might yield promising results.

A third method, by Bindschedler and Moore (11) describes a method of formulating "iso-cost" lines to determine locations for new machines in existing layouts. This method assumes that there is a material handling cost associated with every point in a layout with respect to an existing machine or machines. By connecting all points with equal cost, a series of lines representing fixed material handling costs are generated. These lines are similar to contour lines on maps. Then by simply locating the new machine as closely as possible to the line with the smallest value, the new layout is optimized.

Unfortunately, when more than one new machine is to be located, or new machines have relationships among themselves, or the material moves between the machines in unequal amounts (all of these being the usual situation), the solution becomes so complex as to be utterly unmanageable by this method.

Miehle (12), while not working on the specific facilities planning problem, has presented some valuable information on the solution of the plant layout problem. His specific problem was that of locating variable centers with respect to already fixed centers so as to minimize the distances between them. The work was done on a geographic basis rather than at any local area. He obtains his solution by an iteration procedure, but the actual method is not presented in the article. An iteration procedure, probably the same type, is to be used in this thesis.

Probably the most advanced and promising work done in this field is that of McHose (13). This work is deserving of close examination here.

In this paper, the work of Yaseen (14) is cited to show a common fallacy in location of economic activities. This stems from the use of the physical principle of the center of moments to optimize facility location. This is perhaps best illustrated by a simple example. Consider in Figure 1 that 30 weight units of material must flow from point

A to an unknown point B. From point B 20 weight units must flow to point C. The problem is to locate point B so as to minimize the total flow in units of weight-distance.

A	B	C
$x = 0$	$x = 4$	$x = 10$

Figure 1. Illustration of Center of Moments Solution

The principle of the center of moments states that the distance from A to B times its associated weight must equal the distance from B to C times its associated weight. The solution of the problem in Figure 1 by this method gives a value for point B of $x = 4$, and the total flow for this system is

$$4(30) + 6(20) = 120 + 120 = 240 \text{ weight-distance units.}$$

However, assume that B is located at $x = 2$. Then

$$2(30) + 8(20) = 60 + 160 = 220 \text{ weight-distance units}$$

which is clearly less than the first case. The actual optimum solution to this problem is to locate point B at $x = 0$, which gives the value

$$0(30) + 10(20) = 0 + 200 = 200 \text{ weight-distance units,}$$

but this is obviously an absurd solution.

However, this clearly illustrates the fallacy inherent in using the system of the center of moments as a solution to the problem.

As a general statement of the problem that he solves, McHose derives the equation

$$Z_m = \sum_{i=1}^{n} V_i^m D_i^m$$

where

m = power of the equation

n = number of fixed centers

V_i = weighted factor associated with the i^{th} variable center and each fixed center

D_i = distance between variable center i and each fixed center.

The location of both the fixed and variable centers are expressed in Cartesian coordinates, and this leads to D_i, which is expressed as the square root of the hypotenuse of a right triangle. This expression exists in implicit form only, and no direct solution is possible in this form, except by an iteration procedure.

McHose then removes the radical by letting $m = 2$ in the above equation, and this makes it possible to solve it for explicit solutions. This is only an approximate solution, but the author shows that <u>in his examples</u> the error is very small. Unfortunately, he gives no method by which the error may be determined, other than by comparison with the exact solution. This, of course, defeats the purpose of the entire method.

McHose's work deals almost entirely with the location of only one variable center. A method is suggested for locating more than one center, but it is merely the repeated application of his method for one center, and it becomes very tedious for as few as three variable centers. This difficulty will be removed in this thesis.

Another limitation made in his work is that no restrictions are placed on the variables. It is entirely possible that his procedure will cause points to coincide, or to be located so close to each other that while a mathematical optimum is attained, its application to a practical situation would be impossible. There is not even a guarantee that the points will fall within any given area.

One interesting conclusion he reaches is that the more symmetrical the fixed centers are, and the more symmetrical the associated weight distributions are, the more closely the second-order solution will approach the first-order solution. This is a very interesting point, and more research should be done on it to see whether a quantitative measure of this error could be determined. If this could be done, then one could predict how close the second-order solution would be to the true optimum, and the validity of this solution would be known.

In presenting his final method, McHose states that the second-order solution should be taken as a first approximation, and then a search conducted in the area of the solution for better solutions. While this method works well for only one variable center, the problem is complicated enormously for more than one center. It is apparent that for even as few as three or four variable centers, the method is not feasible.

As a last reference to work in this field, the work of Reis and Andersen (15) is cited. While their work does not contribute to the actual process of attaining an optimum, it can be incorporated into the process to make it more meaningful.

A term introduced in this work is called the "importance factor" and is defined as

> ... any factor other than volume of product or distance to be moved that is to be considered in determining a good plant layout from a materials handling point of view ... .

This factor is used by assuming that all materials handling moves are assigned a factor that represents its importance. As an example, a move that is made with a highly delicate piece of equipment, or with corrosive acids, should be made as short as possible, even though some moves handling a much greater volume of material might be longer. The wisdom of such a procedure should be obvious, although this does introduce a subjective quantity into an otherwise entirely objective approach. The actual approach used in this article was to assume an importance factor of 1.0 to represent no adjustment, and a scale upwards from this point to represent increasing importance. The factor so determined is then used to multiply the amount of material being moved, which has the mathematical effect of increasing the weighting factor.

Some work on this point is necessary to establish bounds on the magnitude of this factor, but this is beyond the scope of this thesis.

In order to appreciate the usefulness of a technique such as this, the reader is referred to Muther (16).

CHAPTER III

AN OPTIMUM SOLUTION TO THE FACILITIES LOCATION PROBLEM

The following assumptions are made in order to limit the practical considerations of the facilities location problem to dimensions that can conveniently be expressed mathematically.

Assumptions and Limitations

1. All distances between points are measured in a straight line. This actually is a very unrealistic situation, and one that eventually <u>must</u> be removed. However, this is beyond the scope of this study.

2. All facilities are symmetrical. This assumption is made so that a point may be located and considered to be the center of the facility. For the partly graphical solution that will be presented, this assumption is not necessary, but is necessary at this point for the mathematical formulation.

3. All information pertaining to the amount of flow between facilities is known and deterministic in nature.

4. Waste, or scrap material, is not to be considered in the solution. This restriction is not necessary, but is made for the sake of simplicity.

5. The problem of location will be considered in two dimensions only. The extension to three dimensions is obvious and is omitted for simplicity.

Incorporation of Points with No Material Handling Contact

This section will indicate a method to be used to incorporate into the solution facilities that have no material handling contact with other facilities. Such would be the case for all office spaces, tool rooms, rest rooms, and other such nonproductive facilities.

The actual process that can be used is to assign "dummy" flows between these points, and simply consider them as real flows for the purposes of the solution. A certain amount of subjectivity will necesarily enter at this point, but several methods are available to reduce it as much as possible. See Muther (18) for an example of such a method.

Definition of Importance Units

In order to clarify the concept of dummy flows, a unit of measurement is introduced that is referred to as an "importance" unit. An importance unit is a unit that measures the relative importance of the degree of closeness of two facilities. Such a unit has no absolute value, but varies infinitely from one situation to another. In order to give it magnitude, one importance unit will be defined as equal to the minimum flow, in units of weight or volume, between two facilities in the layout under consideration.

For example, if the flow from point one to point two is 100 pounds per hour, and no two other points in the layout have less flow per unit time, then one importance unit would equal 100 pounds per hour.

If the flow between points three and four is 150 pounds per hour, then it would be assigned 1.5 importance units.

The purpose of the introduction of importance units is twofold: (1) it eliminates the necessity for dealing with large numbers in the mathematical solution of the problem, and (2) it makes the incorporation of points with no material handling contact more meaningful.

Consider, for example, the following simple example. A layout is proposed that contains five points with material handling contact. A tabulation is made as shown in Table 1.

Table 1. Assignment of Importance Units

flow		pounds per hour	importance units
from	to		
P_1	P_2	300	3.00
P_2	P_3	250	2.50
P_3	P_4	175	1.75
P_4	P_5	100	1.00

In addition to these five points, there are two other points, P_6 and P_7, that are to be located, and one point, P_8, that is already fixed. These points have no material handling with each other, or with any of the other points.

Some type of chart (a good example is that previously cited by Muther (18)) is then constructed in which the relative importance of the closeness is determined in nonquantitative terms. Assume that point P_6 is a support activity to point P_4, has no relationship to any other point, and it is considered very important to have it as close as possible to P_4.

Here some subjectivity enters into the problem, but it cannot be avoided. Suppose the decision is made that the importance of locating P_6 close to P_4 is roughly equal to the importance of having P_1 close to P_2. Then 3.00 importance units would be assigned to the relationship of P_6 and P_4. Since the location of P_6 with respect to any other point is of no importance, these relationships are assigned a value of zero importance units.

In a similar manner, importance values are assigned to the relationship of point P_7 and the other points. Although point P_8 is fixed, its relative importance to the other points can be determined and assigned, and in this manner the location of the variable points will be influenced by the location and relative importance of P_8.

Personnel Flow Considerations

In addition to the foregoing considerations, it is often necessary to make allowances for the number of people that move between facilities. Such cases can be visualized as drinking fountains and rest rooms, where traffic will be considerable, if not strictly productive.

In such cases, a method could be developed to equate the movement of a person with the movement of a certain amount of material. In this way, one person going to a drinking fountain could be said to be equivalent to 50 pounds of material moving through the same distance.

This opens up an entire new area for research, but it will not be pursued further in this study. Suffice it to say that the problem is a formidable one, as it must take into account variables such as the "value" (as manifested by the wage rate) of each employee that is considered.

Rather than assume any measure of equivalence, which would necessarily be a poor guess, this factor will not be included in the example that follows. In a real problem it must, of course, be included, and the reader can see (conceptually, at least) how this may be done.

The Use of Relative Importance Factors

At this point it is appropriate to reexamine Table 1. The flow between facilities is given in units of weight per unit time, but no measure of their relative importance is given. It is at this point that the method of Reis and Andersen becomes useful. Assume for illustrative purposes that the flows between all facilities except P_3 and P_4 are of a simple, easy-to-handle, inexpensive, noninflammable, nontoxic materials, but that of the 175 pounds that flow between P_3 and P_4, 75 pounds of it are a highly corrosive acid. It would then be appropriate to reduce the distance this material flows in order to decrease the probability of an accident occurring during transit. Obviously, the move is of greater importance than the other moves. Since the importance units are assigned proportionally to the amount of material being moved, increasing this value of 75 pounds will increase its value in importance units. In order to do this, it is multiplied by some factor.

No justification is given for the choice of this factor, as this is beyond the scope of this study. Research should be done on this point to establish the magnitude of this factor. One approach to the problem that seems promising to this author is to use the expected loss resulting from an accident as some sort of index for calculating the magnitude of this factor.

However, it is assumed that through some procedure a factor is arrived at for this particular move, and it has a value of 1.5. Then the importance units associated with the move from P_3 to P_4 would be calculated as shown in Table 2.

Table 2. Calculation of Importance Units

flow		pounds per hour	importance factor	adjusted pounds/hour	importance units
from	to				
P_3	P_4	100	1.00	100.0	
		75	1.50	112.5	
				212.5	2.125

Combination of Relative Importance Factor and Importance Units Concepts

The two preceding concepts can now be consolidated. From an operations process chart, or its equivalent, the number and types of facilities necessary for an operation are found. In addition, the amounts and types of materials moving between successive operations are tabulated.

Assume that such a chart has been made, the following information has been obtained from it, and it is now desired to design a layout for this process.

It is found that from P_1, which is already fixed, 100 pounds of material flow to point P_2, which is unknown. This move is assigned a relative importance factor of 3.5. From P_3, which is fixed, 125 pounds flow to P_2. A relative importance factor of 1.0 is assigned to three-fifths of the flow, and a relative importance factor of 1.5 is assigned to the other two-fifths. From P_2, 100 pounds of material flow to P_4, which is fixed, and 100 pounds go to P_5, which is a variable center. Both are assigned a relative importance factor of 1.0. From P_3, 100 pounds of material move to P_5 and a relative importance factor of 1.0 is assigned to this move; and 100 pounds of material come to P_5 from P_4, this with a relative importance factor of 2.0. This is combined with the 100 pounds that came from P_2 and is moved to P_6. A relative importance factor of 1.0 is assigned to this last move.

A point, P_7, has no material handling contact with any other points, and is unknown. It is decided that P_7 must be close to P_6 and P_4. Its relationship to P_6 is about as important as the relationship between P_2 and P_3, and its relationship to P_4 is a little more important than the relationship between P_3 and P_2, but less than that between P_5 and P_6. This is tabulated as shown in Table 3.

It will be noted from this table that the combination of P_7 and P_5, as well as some others, is not included. While it could be included as a possible combination, the flow is zero, and therefore the importance

Table 3. Assignment of Importance Units

flow from	flow to	pounds per hour	importance factor	adjusted pounds per hour	importance units
P_1	P_2	100	3.5	350	3.5
P_3	P_2	75 50	1.0 1.5	75 75 150	1.5
P_2	P_4	100	1.0	100	1.0
P_2	P_5	100	1.0	100	1.0
P_5	P_6	300	1.0	300	3.0
P_3	P_5	100	1.0	100	1.0
P_4	P_5	100	2.0	200	2.0
P_4	P_7				2.0
P_6	P_7				1.5

units assigned will be zero. Likewise, the relationship of P_7 and all points except P_6 and P_4 is considered unimportant and was assigned the value zero, therefore resulting in the assignment of zero importance units.

Development of a Model

Consider a rectangular area, of dimensions x_{max} by y_{max} containing fixed facilities A, B, C, ..., ϕ. The centers of these fixed facilities are denoted by the letters. Superimpose this area on a coordinate axis, with the origin at the lower left corner, as in Figure 2. The coordinates of the fixed points A, B, C, ... φ are known.

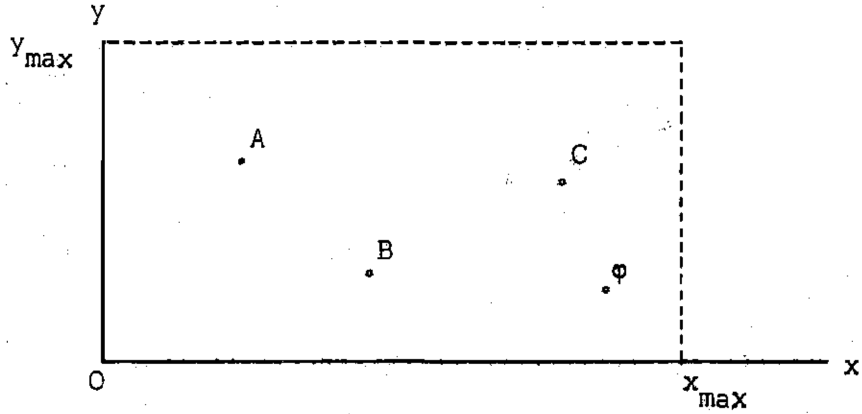

Figure 2. General Area for Consideration

Now suppose n new facilities must be assigned to locations within this area. Each new facility will be assumed to have materials handling contact with each of the fixed centers and with each of the other $(n - 1)$ new facilities.

Each of these n new facilities will be assigned at points P_i $(i = 1, 2, 3, \ldots, n)$, each of which has the coordinates x_i, y_i. The distance from the i^{th} facility to the $i + 1^{th}$ facility will be the distance from P_i to P_{i+1}. Represented algebraically, this distance is

$$d_{i,i+1} = [(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2]^{\frac{1}{2}}. \qquad (1)$$

The distance from facility i to any other facility j is the distance from P_i to P_j and is represented as

$$d_{i,j} = [(x_j - x_i)^2 + (y_j - y_i)^2]^{\frac{1}{2}}. \qquad (2)$$

This formulation allows the distance between any two facilities, in any order, to be expressed solely as a function of its coordinates. Intuitively, the optimum solution is to make all of these distances as short as possible. However, these facilities obviously cannot occupy the same space at the same time, nor for practical reasons, can they even approach this. Therefore, constraints must be put on the distances, in order to provide a lower limit for the value that any $d_{i,j}$ may assume. This distance represents the area the facility itself will occupy, space for the operator, a share of aisle space, a place for stock containers, and so forth. This constraint is written as

$$d_{i,j} \geq K_{i,j} . \tag{3}$$

This, with two other restrictions, makes up a complete set of constraints for the problem. These are simply that the facilities must be located within the original area, and are expressed as

$$0 \leq x_i \leq x_{max} \tag{4}$$

$$0 \leq y_i \leq y_{max} . \tag{5}$$

This confines the solution to nonnegative values of the variables within the predetermined area of solution.

In order to formulate an objective function, consider the following. The total distance (d_t) that material must travel through the system is equal to the sum of its component distances. Expressed mathematically, this is

$$d_T = \sum_{i,j=1}^{n} d_{i,j} \tag{6}$$

The amount of material, in units of weight or volume, that flows between any two points P_i and P_j is represented by $F_{i,j}$. Therefore the product $f_{i,j}\, d_{i,j}$ represents the flow between points P_i and P_j in units of amount-distance (i.e., foot-pounds). For the complete system, the symbol that represents the total flow from P_1 to P_n will be Z, which is equal to the sum of the flows between each pair of P_i's and is expressed as

$$Z_1 = \sum_{i,j=1}^{n} F_{i,j}\, d_{i,j} = \text{minimum}.^{*} \tag{7}$$

Rewriting this equation and the three equations

$$d_{i,j} \geq K_{i,j} \tag{3}$$

$$0 \leq x_i \leq x_{max} \tag{4}$$

$$0 \leq y_i \leq y_{max} \tag{5}$$

in terms of the variables x and y yields the following set of equations, which make up the general statement of the problem.

---

*The subscript on the Z indicates the power to which each separate factor of the equation is raised, in this case, the first power. This corresponds to McHose's notation

$$Z_m = \sum_{i=1}^{n} V_i^m D_i^m .$$

Minimize

$$\sum_{i,j=1}^{n} F_{i,j}[(x_j - x_i)^2 + (y_j - y_i)^2]^{\frac{1}{2}} \tag{8}$$

subject to

$$[(x_j - x_i)^2 + (y_j - y_i)^2]^{\frac{1}{2}} \geq K_{i,j} \tag{9}$$

$$0 \leq x_i \leq x_{max} \tag{10}$$

$$0 \leq y_i \leq y_{max} . \tag{11}$$

A few comments on this system of equations are now in order. As McHose points out, these equations do not present an explicit solution for the variables x and y. Although the distances are linear, the distance expressed as a function of the variables x and y is neither a linear nor a quadratic function, and the methods of linear and quadratic programming cannot cope with this problem.

Lagrange's method of undetermined multipliers may be used to solve this system, but unfortunately the problem rapidly becomes so complex as to make this procedure totally impractical.

The gradient method as presented by Hansell (17) appears to offer a solution to this problem, but because of the implicit nature of the function, fails to do so.

The general category into which this problem falls is known in mathematical literature as maximization-minimization theory, and indeed there is no general method of solution known at this time.

It is therefore necessary to approach the problem by a rather indirect method. However, this procedure is simpler than any other type of approach and can be shown to give results very close to the true optimum.

Illustration of a Method of Solution

A hypothetical problem will now be formulated in order to show the difficulties encountered in the Lagrangian solution, and to illustrate an alternative method of solution. The data given in Table 3 will be used. It is assumed that an area 100 by 120 feet is available for location of the facilities. Points P_1, P_3, P_4, and P_6 will be assumed fixed as shown in Figure 3.

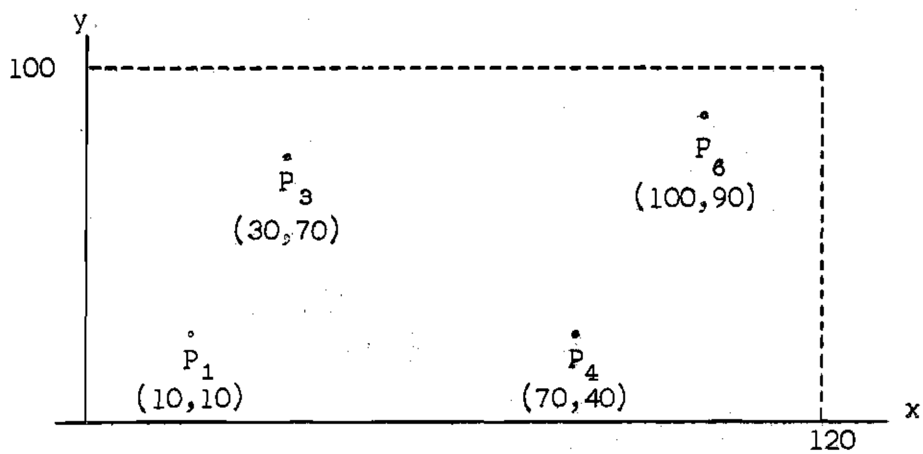

Figure 3. Layout for a Hypothetical Problem

Since these points actually represent centers of areas, it is necessary to represent the areas surrounding these points that are occupied by the facility itself. It is assumed in order to simplify the mathematical formulation that these areas are circles. The areas occupied by each facility are given in Table 4, in terms of the radius of the circle that represents the area occupied by the facility.

Table 4. Areas Occupied by Facilities, Expressed as the Radii of Circles

Facility	Radius (feet)
P_1	2
P_2	3
P_3	3
P_4	2
P_5	4
P_6	2
P_7	6

Therefore, the closest that any point can come to another point will be the sum of the radii of their corresponding circles. This allows the development of a set of constraints for this problem. These are arrived at as follows:

$$d_{i,j} \geq \text{radius } P_i + \text{radius } P_j . \qquad (12)$$

Since P_1, P_3, P_4, and P_6 are fixed, it is necessary to limit the distance between each of these points and each of the variable points to a minimum as in Equation (12). It is also necessary to limit the distance between any two of the three variable points. These are expressed as follows:

$$d_{1,2} \geq 2 + 3 = 5$$

$$d_{1,5} \geq 2 + 4 = 6$$

$$d_{1,7} \geq 2 + 6 = 8$$

$$d_{3,2} \geq 3 + 3 = 6$$

$$d_{3,5} \geq 3 + 4 = 7$$

$$d_{3,7} \geq 3 + 6 = 9$$

$$d_{4,2} \geq 3 + 2 = 5$$

$$d_{4,5} \geq 4 + 2 = 6$$

$$d_{4,7} \geq 2 + 6 = 8$$

$$d_{6,2} \geq 2 + 3 = 5$$

$$d_{6,5} \geq 2 + 4 = 6$$

$$d_{6,7} \geq 2 + 6 = 8$$

and

$$d_{2,5} \geq 3 + 4 = 7$$

$$d_{2,7} \geq 3 + 6 = 9$$

$$d_{5,7} \geq 4 + 6 = 10 .$$

Since each P_i may be expressed in terms of its coordinates, the distance can be written as shown in Equation (9).

The formulation of the objective function is as follows: Equation (7) is used as the objective function and is

$$Z_1 = \sum_{i,j=1}^{n} F_{i,j}\, d_{i,j} . \tag{7}$$

Since distance may be measured between any two points, the number of $d_{i,j}$'s is

$$\frac{n!}{2!(n-2)!}$$

which yields

$$\frac{7!}{2!(7-2)!}$$

or 21 terms in Equation (7). However, each term that has a weighting factor of zero will drop out, and since flows between fixed centers need not be considered, this leaves only the distances between the variable centers and the fixed centers, and between variable centers. These are found from Table 3. This allows the expansion of Equation (7) as follows:

$$Z_1 = F_{1,2}\, d_{1,2} + F_{3,2}\, d_{3,2} + F_{2,5}\, d_{2,5} + F_{5,6}\, d_{5,6} + F_{7,1}\, d_{7,1} \tag{13}$$

$$+ F_{4,2}\, d_{4,2} + F_{3,5}\, d_{3,5} + F_{5,4}\, d_{5,4} + F_{7,4}\, d_{7,4} .$$

Taking the values of $F_{i,j}$ from Table 3, writing each $d_{i,j}$ in terms of x_i, y_i, and substituting each known value of x_i, y_i results in the following equation:

$$Z_1 = 3.5[(x_2 - 10)^2 + (y_2 - 10)^2]^{\frac{1}{2}} + 1.5[(x_2 - 30)^2 + (y_2 - 70)^2]^{\frac{1}{2}} \quad (14)$$

$$+ 1[(x_2 - 70)^2 + (y_2 - 40)^2]^{\frac{1}{2}} + 1\,[(x_2 - x_5)^2 + (y_2 - y_5)^2]^{\frac{1}{2}}$$

$$+ 1[(x_5 - 30)^2 + (y_5 - 70)^2]^{\frac{1}{2}} + 2\,[(x_5 - 70)^2 + (y_5 - 40)^2]^{\frac{1}{2}}$$

$$+ 3[(x_5 - 100)^2 + (y_5 - 90)^2]^{\frac{1}{2}} + 2\,[(x_7 - 70)^2 + (y_7 - 40)^2]^{\frac{1}{2}}$$

$$+ 1.5[(x_7 - 100)^2 + (y_7 - 90)^2]^{\frac{1}{2}},$$

which must be minimized subject to:

$$[(x_2 - 10)^2 + (y_2 - 10)^2]^{\frac{1}{2}} \geq 5 \quad (15)$$

$$[(x_5 - 10)^2 + (y_5 - 10)^2]^{\frac{1}{2}} \geq 6 \quad (16)$$

$$[(x_7 - 10)^2 + (y_7 - 10)^2]^{\frac{1}{2}} \geq 8 \quad (17)$$

$$[(x_2 - 30)^2 + (y_2 - 70)^2]^{\frac{1}{2}} \geq 6 \quad (18)$$

$$[(x_5 - 30)^2 + (y_5 - 70)^2]^{\frac{1}{2}} \geq 7 \quad (19)$$

$$[(x_7 - 30)^2 + (y_7 - 70)^2]^{\frac{1}{2}} \geq 9 \quad (20)$$

$$[(x_2 - 70)^2 + (y_2 - 40)^2]^{\frac{1}{2}} \geq 5 \quad (21)$$

$$[(x_5 - 70)^2 + (y_5 - 40)^2]^{\frac{1}{2}} \geq 6 \quad (22)$$

$$[(x_7 - 70)^2 + (y_7 - 40)^2]^{\frac{1}{2}} \geq 8 \tag{23}$$

$$[(x_2 - 100)^2 + (y_2 - 90)^2]^{\frac{1}{2}} \geq 5 \tag{24}$$

$$[(x_5 - 100)^2 + (y_5 - 90)^2]^{\frac{1}{2}} \geq 6 \tag{25}$$

$$[(x_7 - 100)^2 + (y_7 - 90)^2]^{\frac{1}{2}} \geq 8 \tag{26}$$

$$[(x_2 - x_5)^2 + (y_2 - y_5)^2]^{\frac{1}{2}} \geq 7 \tag{27}$$

$$[(x_2 - x_7)^2 + (y_2 - y_7)^2]^{\frac{1}{2}} \geq 9 \tag{28}$$

$$[(x_5 - x_7)^2 + (y_5 - y_7)^2]^{\frac{1}{2}} \geq 10 . \tag{29}$$

This system of equations has six unknowns, x_2, y_2, x_5, y_5, x_7, and y_7.

In order to set this up for a Lagrangian multiplier solution, one proceeds as follows: Take the partial derivatives of Z_1 with respect to each of the variables. This yields

$$\frac{\partial Z_1}{\partial x_2} = F(x_i, y_i) \tag{30}$$

$$\frac{\partial Z_1}{\partial y_2} = F(x_i, y_i) \tag{31}$$

$$\frac{\partial Z_1}{\partial x_5} = F(x_i, y_i) \tag{32}$$

$$\frac{\partial Z_1}{\partial y_5} = F(x_i, y_i) \tag{33}$$

$$\frac{\partial Z_1}{\partial x_7} = F(x_i, y_i) \tag{34}$$

$$\frac{\partial Z_1}{\partial y_7} = F(x_i, y_i) \tag{35}$$

Representing Equations 15 through 29 by G_1, G_2, G_3, ..., G_{15}, the partial derivatives are taken of each equation with respect to each of its variables. This yields

$$\frac{\partial G_1}{\partial x_2} = F(x_i, y_i) \tag{36}$$

$$\frac{\partial G_1}{\partial y_2} = F(x_i, y_i) \tag{37}$$

$$\frac{\partial G_2}{\partial x_5} = F(x_i, y_i) \tag{38}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{\partial G_{15}}{\partial y_7} = F(x_i, y_i) \ . \tag{71}$$

Each partial derivative 36 through 71 is multiplied by λ_i with the subscript of λ corresponding to the subscript of G. Each partial derivative of G_i with respect to x_i or y_i is then added to the corresponding partial of Z_1 and equated to zero. In other words,

$$\frac{\partial Z_1}{\partial x_2} + \lambda_1 \frac{\partial G_1}{\partial x_2} + \lambda_4 \frac{\partial G_4}{\partial x_2} + \cdots + \lambda_{14} \frac{\partial G_{14}}{\partial x_2} = 0 \tag{72}$$

$$\vdots \qquad \vdots$$

$$\frac{\partial Z_1}{\partial x_5} + \lambda_2 \frac{\partial G_2}{\partial x_5} + \lambda_5 \frac{\partial G_5}{\partial x_5} + \cdots + \lambda_{15} \frac{\partial G_{15}}{\partial x_5} = 0 \tag{74}$$

$$\vdots \qquad \vdots$$

$$\frac{\partial Z_1}{\partial y_7} + \lambda_3 \frac{\partial G_3}{\partial y_7} + \lambda_6 \frac{\partial G_6}{\partial y_7} + \cdots + \lambda_{15} \frac{\partial G_{15}}{\partial y_7} = 0\ . \tag{77}$$

The above set of six equations with 21 unknowns is then solved simultaneously with the 15 equations (15 through 29) to produce an optimum solution for x_2, y_2, x_5, y_5, and y_7, x_7. Unfortunately, because of the implicit nature of these equations, a direct solution is not feasible. Therefore, it is necessary to find some way in which the task presented above can be circumnavigated. Again consider Equation (7)

$$Z_1 = \sum_{i,j=1}^{n} F_{i,j}\ d_{i,j} \tag{7}$$

The most difficult part of the solution to this equation occurs because of the square-root expression of $d_{i,j}$. The differentiation of this one term yields a radical in the denominator that is very difficult to handle. For example, Equation (30) expanded is

$$\frac{\partial Z_1}{\partial x_2} = \frac{x_2 - 10}{[(x_2 - 10)^2 + (y_2 - 10)^2]^{\frac{1}{2}}} + \frac{2.5x_2 - 75}{[(x_2 - 30)^2 + (y_2 - 70)^2]^{\frac{1}{2}}}$$

$$+ \frac{2x_2 - 2x_5}{[(x_2 - x_5)^2 + (y_2 - y_5)^2]^{\frac{1}{2}}}$$

for this problem.

In order to overcome this difficulty, McHose suggests changing Equation (7) to

$$Z_2 = \sum_{i,j=1}^{n} F_{i,j}^2 \, d_{i,j}^2 \, . \tag{78}$$

Indeed, this does simplify the differentiation process and the subsequent operations drastically. Unfortunately, the solution of the Z_2 function is not the solution of the Z_1 function. There exists, however, a method by which this approximate solution may be successively improved until it approaches the exact solution. This method is called the gradient method, and an outline of it can be found in Hansell (17).

Briefly, this method consists of finding an approximate solution and then by an iteration procedure finding better solutions until some termination criterion is satisfied. Although Hansell works with a constrained problem, the method works equally well with an unconstrained problem.

It can be stated as follows: given a function $Z = f(x_i, y_i)$, find the values of x_i and y_i that minimize Z. Assume that an initial approximation is known to the values of x_i and y_i, and use the notation x_i^n, y_i^n to denote successive approximations to x_i, y_i. In this case the superscripts denote the successive number of each approximation, the first approximation being denoted by x_i^1, y_i^1. A theorem then states that a better solution may be determined by the equations

$$x_i^{n+1} = x_i^n - u \left|\frac{\partial F(x_i,y_i)}{\partial x_i}\right|^n \tag{79}$$

and

$$y_i^{n+1} = y_i^n - u \left|\frac{\partial F(x_i,y_i)}{\partial x_i}\right|^n , \tag{80}$$

the superscript on the partial derivatives denoting that it is evaluated at that point. The partial derivatives, when evaluated, give the direction the values of x_i and y_i must move in order to approach the minimum of Z_1, while the factor u gives the distance in this direction that it will move. The determination of u is not a simple process, as too small a value of u will make the convergence too slow, while too large a value of u makes the process miss the optimum value altogether. Numerous ways are available for calculation of u, none of which are easy; one method is presented by Crockett and Chernoff (19).

However, in this particular presentation, a rather pragmatic approach can be taken to this problem; the determination of u will be covered in detail during the solution of the problem.

Equation (14) is now changed to the form of Equation (78) by squaring each term in the equation, and it becomes

$$Z_2 = 12.25[(x_2 - 10)^2 + (y_2 - 10)^2] + 2.25[(x_2 - 30)^2 + (y_2-70)^2]$$
$$+ 1[(x_2 - 70)^2 + (y_2 - 40)^2] + 1[(x_2 - x_5)^2 + (y_2 - y_5)^2]$$
$$+ 1[(x_5 - 30)^2 + (y_5 - 70)^2] + 4[(x_5 - 70)^2 + (y_5 - 40)^2]$$
$$+ 9[(x_5 - 100)^2 + (y_5 - 90)^2] + 4[(x_7 - 70)^2 + (y_7 - 40)^2]$$
$$+ 2.25[(x_7 - 100)^2 + (y_7 - 90)^2] \,. \tag{81}$$

Taking the partial derivatives with respect to each of the variables and equating to zero yields

$$\frac{\partial Z_2}{\partial x_2} = 24.5x_2 - 245 + 4.5x_2 - 135 + 2x_2 - 140 + 2x_2 - 2x_5 = 0 \tag{82}$$

$$\frac{\partial Z_2}{\partial y_2} = 24.5y_2 - 245 + 4.5y_2 - 315 + 2y_2 - 80 + 2y_2 - 2y_5 = 0 \tag{83}$$

$$\frac{\partial Z_2}{\partial x_5} = 2x_5 - 2x_2 + 2x_5 - 60 + 8x_5 - 560 + 18x_5 - 1800 = 0 \tag{84}$$

$$\frac{\partial Z_2}{\partial y_5} = 2y_5 - 2y_2 + 2y_5 - 140 + 8y_5 - 320 + 18\, y_5 - 1620 = 0 \tag{85}$$

$$\frac{\partial Z_2}{\partial x_7} = 8x_7 - 560 + 4.5x_7 - 450 = 0 \tag{86}$$

$$\frac{\partial z_2}{\partial y_7} = 8y_7 - 320 + 4.5y_7 - 405 = 0 \,. \tag{87}$$

The solution of these equations yields

$$x_2^1 = 20.75 \qquad x_5^1 = 82.50 \qquad x_7^1 = 81.00$$

$$y_2^1 = 23.70 \qquad y_5^1 = 71.00 \qquad y_7^1 = 58.00$$

and these points are plotted in Figure 4.

Assuming that these points are a good approximation to the minimum, the iterative procedure described above will be used to converge to the true optimum.

Expanding Equations 30 through 36 and incorporating them in Equations 79 and 80 yields the following set of equations:

$$x_2^{n+1} = x_2^n - u\left[\frac{3.5x_2 - 35}{[(x_2-10)^2+(y_2-10)^2]^{\frac{1}{2}}} + \frac{1.5x_2 - 45}{[(x_2-30)^2+(y_2-70)^2]^{\frac{1}{2}}} + \frac{x_2 - 70}{[(x_2-70)^2+(y_2-40)^2]^{\frac{1}{2}}} + \frac{x_2 - x_5}{[(x_2-x_5)^2+(y_2-y_5)^2]^{\frac{1}{2}}}\right] \tag{88}$$

$$y_2^{n+1} = y_2^n - u\left[\frac{3.5y_2 - 35}{[(x_2-10)^2+(y_2-10)^2]^{\frac{1}{2}}} + \frac{1.5y_2 - 105}{[(x_2-30)^2+(y_2-70)^2]^{\frac{1}{2}}} + \frac{y_2 - 40}{[(x_2-70)^2+(y_2-40)^2]^{\frac{1}{2}}} + \frac{y_2 - y_5}{[(x_2-x_5)^2+(y_2-y_5)^2]^{\frac{1}{2}}}\right] \tag{89}$$

$$x_5^{n+1} = x_5^n - u\left[\frac{x_5 - x_2}{[(x_2-x_5)^2+(y_2-y_5)^2]^{\frac{1}{2}}} + \frac{x_5 - 30}{[(x_5-30)^2+(y_5-70)^2]^{\frac{1}{2}}} + \frac{2x_5 - 140}{[(x_5-70)^2+(y_5-40)^2]^{\frac{1}{2}}} + \frac{3x_5 - 300}{[(x_5-100)^2+(y_5-90)^2]^{\frac{1}{2}}}\right] \tag{90}$$

$Z_1 = 458.24$

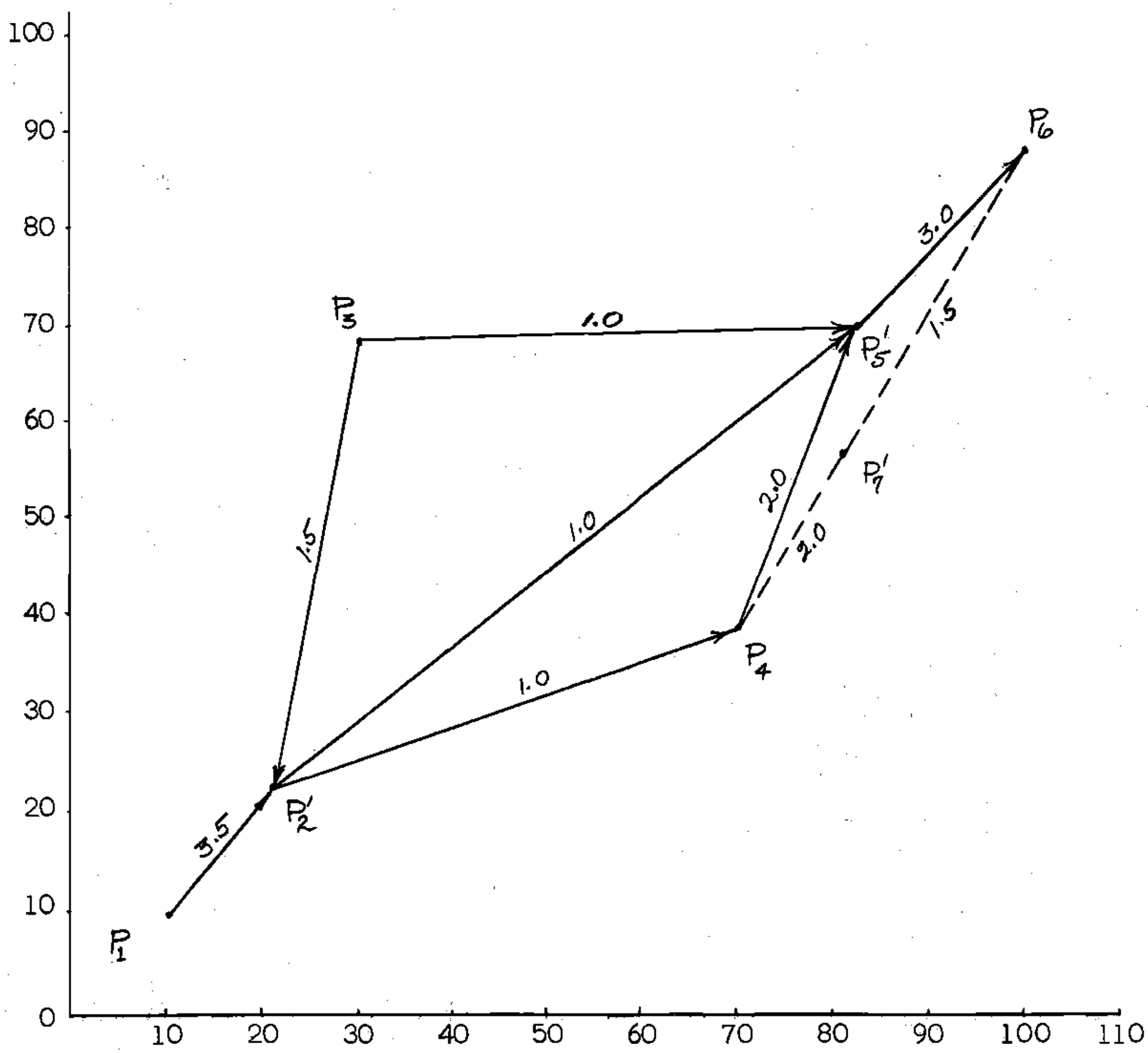

Figure 4. Initial Solution with Weighted Flows Indicated

$$y_5^{n+1} = y_5^n - u\left[\frac{y_5 - y_2}{[(x_2 - x_5)^2 + (y_2 - y_5)^2]^{\frac{1}{2}}} + \frac{y_5 - 70}{[(x_5 - 30)^2 + (y_5 - 70)^2]^{\frac{1}{2}}}\right. \tag{91}$$

$$\left. + \frac{2y_5 - 80}{[(x_5 - 70)^2 + (y_5 - 40)^2]^{\frac{1}{2}}} + \frac{3y_5 - 270}{[(x_5 - 100)^2 + (y_5 - 90)^2]^{\frac{1}{2}}}\right]$$

$$x_7^{n+1} = x_7^n - u\left[\frac{2x_7 - 140}{[(x_7 - 70)^2 + (y_7 - 40)^2]^{\frac{1}{2}}} + \frac{1.5x_7 - 150}{[(x_7 - 100)^2 + (y_7 - 90)^2]^{\frac{1}{2}}}\right] \tag{92}$$

$$y_7^{n+1} = y_7^n - u\left[\frac{2y_7 - 80}{[(x_7 - 70)^2 + (y_7 - 40)^2]^{\frac{1}{2}}} + \frac{1.5y_7 - 135}{[(x_7 - 100)^2 + (y_7 - 90)^2]^{\frac{1}{2}}}\right] \tag{93}$$

Inspection of Equations 88 through 93 shows that the variables x_7 and y_7 are independent of the other variables, and the location of x_2, y_2, x_5, and y_5 will not influence the location of P_7 in any way. Applying the following iteration procedure to P_7 will find the optimum location for it, but the process is long and tedious.

It will be noted that P_7 has contact with two points only, and therefore it can be shown that it will be located on the line joining the two points, as close as possible to the point having the greater associated weight. This is represented by the intersection of the line joining the two points and the constraint boundary of the point with the larger weight.

This location may be found by solving

$$[(x_7 - 70)^2 + (y_7 - 40)^2]^{\frac{1}{2}} = 8$$

simultaneously with

$$3y = 5x - 230$$

which gives the results: $x_7 = 74.00$ and $y_7 = 46.67$. A diagrammatical view of this operation is shown in Figure 5.

It is now possible to eliminate point P_7 from further consideration and work only with the remaining points P_2 and P_5.

Substituting the values previously calculated for the Z_2 function in Equations 88 through 91 results in:

$$x_2^2 = 20.75 - u\left[\frac{37.625}{(115.56 + 187.69)^{\frac{1}{2}}} \quad \frac{-13.87}{(85.56 + 2143.69)^{\frac{1}{2}}} + \frac{-61.75}{(3813.06 + 2237.29)^{\frac{1}{2}}}\right]$$

$$= 20.75 - u\left[\frac{37.63}{17.41} - \frac{-13.87}{47.22} - \frac{49.25}{51.88} - \frac{61.75}{77.79}\right]$$

$$= 20.75 - u[2.1614 - 0.2937 - 0.9493 - 0.7938]$$

$$= 20.75 - u(0.1246)$$

$$y_2^2 = 23.7 - u\left[\frac{47.95}{17.41} - \frac{69.45}{47.22} - \frac{16.3}{51.88} - \frac{47.3}{77.79}\right]$$

$$= 23.7 - u[2.754 - 1.4708 - 0.3142 - 0.608]$$

$$= 23.7 - u(0.361)$$

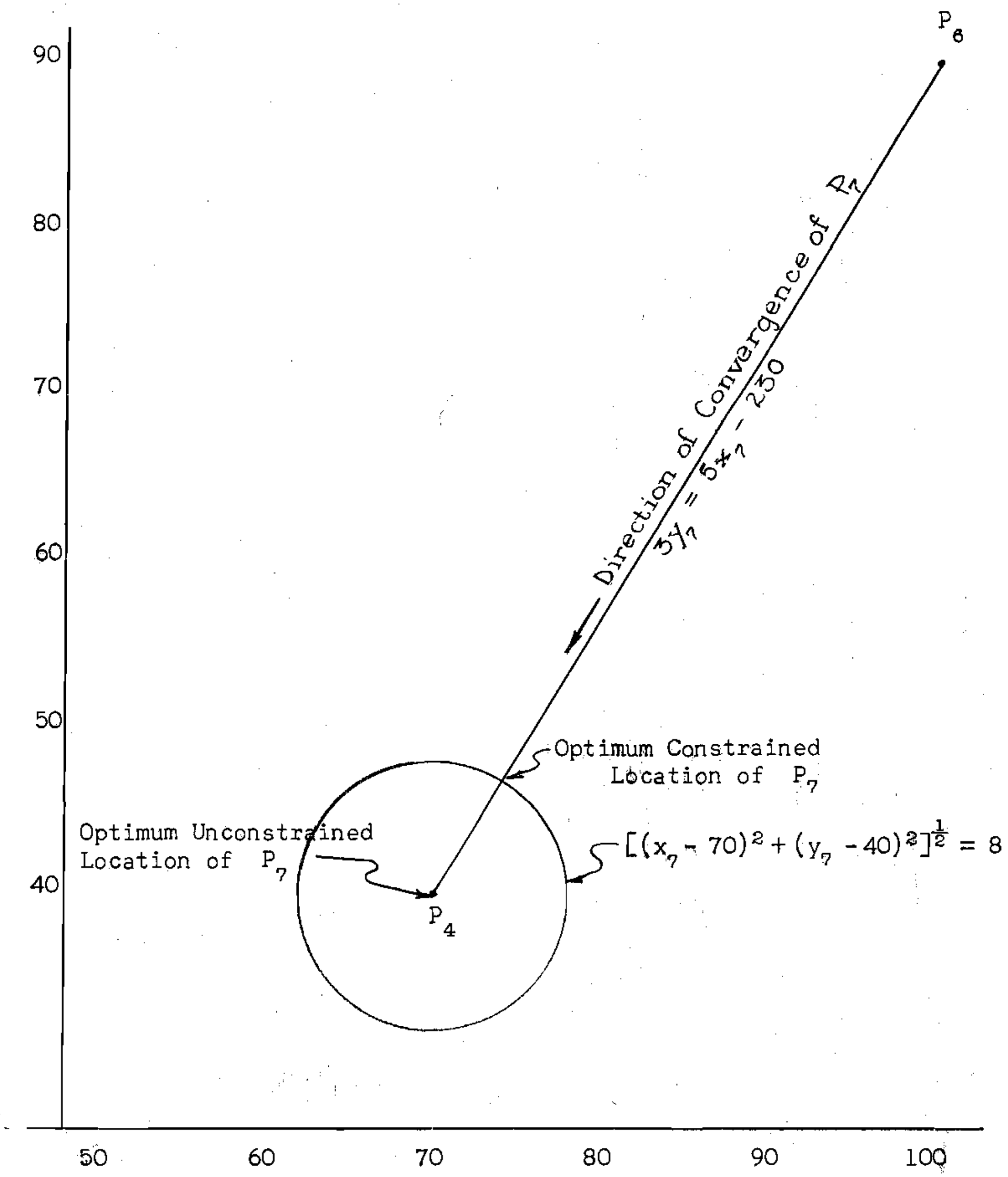

Figure 5. Location of Point P_7

$$x_5^2 = 82.5 - u\left[0.7938 + \frac{52.5}{(2756.25+1)} + \frac{25}{(156.25+961)} + \frac{-52.5}{(306.25+361)}\right]$$

$$= 82.5 - u\left[0.7938 + \frac{52.5}{52.5} + \frac{25}{33.42} - \frac{52.5}{25.82}\right]$$

$$= 82.5 - u[0.7938 + 1 + 0.7481 - 2.0333]$$

$$= 82.5 - u(0.5086) \ .$$

$$y_5^2 = 71 - u\left[0.608 + \frac{1}{52.5} + \frac{62}{33.42} - \frac{57}{25.82}\right]$$

$$= 71 - u[0.608 + 0.019 + 1.8552 - 2.2076]$$

$$= 71 - u(0.2746) \ .$$

The question now arises as to how to determine a magnitude for u. The same value of u must be used in all calculations for any particular set of values of x_i, y_i, but may be changed between iterations. Returning to the original concept of the problem, it can be seen that for values of u less than 2.0, the change in the variables will be less than one foot. Any change this small, in a layout measuring 120 feet by 100 feet, can be seen to be very small. Indeed, actual measuring difficulties may arise. If u is taken to be some value up to 10, then the resulting changes in u will be measured in feet, with a maximum of five feet resulting for x_5. For values of u greater than 10, the resulting change will be even greater. In order to avoid the possibility of overstepping the optimum value, a small value should be selected. A good trial value for u in this case appears to be two.

It must now be decided in which direction the variables x_i and y_i must move in order to optimize the solution, as this is determined by the sign of u. The original equation,

$$Z_1 = \sum_{i,j=1}^{n} F_{i,j}\, d_{i,j} \tag{7}$$

can now be evaluated at x_i^1, y_i^1 to determine a value of Z_1^1; values for x_i^2, y_i^2 can also be used to calculate a value for Z_1^2. Two trial values of u are used: +2 and -2. The one that causes Z_1 to decrease will be the one to be used in the succeeding iterations. Therefore Equation (7) is evaluated at x_i^1, y_i^1 as follows:

$$\begin{aligned} Z_1^1 &= 3.5(17.41) + 1.5(47.22) + 1(51.88) + 1(77.79) \\ &\quad + 1(52.5) + 2(33.42) + 3(25.82) \\ &= 60.94 + 70.83 + 51.88 + 77.79 + 52.50 + 66.84 + 77.46 \\ &= 458.24 \ . \end{aligned}$$

The new values for the variables x_i and y_i are next calculated and tabulated as follows:

Table 5. Tabulation of Trial Values of the Variable

	$x_i^1 y_i^1$	$x_i^2\ y_i^2$	
		$u = +2$	$u = -2$
x_2	20.75	20.5008	20.9992
y_2	23.70	22.9780	24.4220
x_5	82.50	81.4828	83.5172
y_5	71.00	70.4508	71.5492

Using the values in Table 5, Z_1 is evaluated again for the variable determined by $u = +2$ and $u = -2$. The actual calculations are found in the Appendix, the results of which are that for $u = +2$, the corresponding value of Z_1^{2*} is 457.36, while for $u = -2$, the value of Z_1^2 is 459.37. Since using the values obtained by setting $u = +2$ causes Z_1 to decrease, these values are used as the second approximation to the solution. These are plotted in Figure 4.

The iteration procedure is then continued until some termination criterion is satisfied.

It is convenient at this time to reexamine the original statement of the problem. The criterion used to judge the effectiveness of a layout is that of cost of handling material; the use of the criterion of weight- or volume-distance per unit time implies a direct relationship to cost, and while such a relationship can be determined in the case of "real" layouts by empirical means, in the case of this example, a simple relationship must be assumed.

Suppose that the cost of moving one pound of material through a distance of one foot is $0.01. Assume that competent authority has specified that the material handling cost must be as low as possible, and that it must be accurate to <u>less</u> than two dollars per year. In other words, the layout must be repeatedly improved until further improvements cannot reduce the total material handling cost more than two dollars per year. This is admittedly an absurd requirement, but it serves to show the accuracy obtainable with this method.

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*The superscript on the Z indicates that it is evaluated at x_i^2, y_i^2.

Assume that the unit time associated with Z_1 is one hour. Now if the change in Z_1 is 0.09 foot-pounds per hour, then ΔZ_1 for one day (8 hours) is

$$0.09 \text{ foot-pounds/hour} \times 8 \text{ hours/day} = 0.72 \text{ foot-pounds/day}$$

and assuming 300 working days per year, ΔZ_1 for one year is

$$0.72 \text{ foot-pounds/day} \times 300 \text{ days/year} = 216 \text{ foot-pounds/year.}$$

Since the cost of moving one pound one foot is \$0.01, then a ΔZ_1 of 216 foot-pounds per year represents a change in total cost of

$$216 \text{ foot-pounds/year} \times \$0.01\text{/foot-pound} = \$2.16\text{/year}$$

which exceeds the limit of \$2.00 per year.

Using the same procedure with a ΔZ_1 of 0.084 pounds per hour produces a change in the total yearly cost of \$1.992, which is very close to the specification which must be met.

It may then be concluded that any change in the layout that produces a reduction in Z_1 less than 0.084 is unnecessary, and no further improvements are needed.

The assignment of importance units has previously reduced the value of Z_1 by a factor of 10^{-2}, and it is therefore necessary to reduce the value for ΔZ_1 accordingly. This gives the final termination criterion, and it is

$$\Delta Z_1 = 0.084 \times 10^{-2} = 0.00084 \text{ .}$$

The continuation of the iteration procedure outlined above gives a final solution in 15 steps. The results for each iteration are tabulated in Table 6.

Table 6. Results of Iteration Procedure

n	u	x_2	y_2	x_5	y_5	Z_1	ΔZ_1	T.C.
1	-	20.75	23.70	82.50	71.00	458.24	-	0.00084
2	2	20.50	22.98	81.48	70.45	457.36	-0.88	
3	2	20.09	22.28	80.62	69.82	456.50	-0.86	
4	2	19.716	21.715	79.867	69.18	455.73	-0.77	
5	5	18.723	20.379	78.172	67.48	454.13	-1.60	
6	5	17.708	19.177	76.851	65.827	452.58	-1.55	
7	5	16.619	17.548	75.608	64.125	451.606	-0.974	
8	5	15.468	16.831	74.750	62.840	450.822	-0.784	
9	5	15.061	15.424	74.040	61.760	450.558	-0.264	
10	5	13.578	14.977	73.540	60.818	449.995	-0.563	
11	5	14.050	13.018	73.075	60.067	449.625	-0.370	
12	5			72.810	59.341	449.486	-0.139	
13	5			72.547	59.032	449.367	-0.119	
14	5			72.394	58.500	449.362	-0.005	0.00084
15	5			72.225	58.096	449.3615	-0.0005	0.00084

Inspection of Table 6 shows that the initial choice of $u = +2$ is too small, and while this will eventually cause convergence, it is increased to +5 in the fifth iteration in order to speed the convergence. At the end of the eleventh iteration the distance from point P_2 to point P_1 is found to be 5.05 feet, which very closely approaches the limit of five feet previously set. Therefore, it is held constant at this point and the procedure is continued for point P_5. At the end of the fifteenth iteration, the value of ΔZ_1 is 0.0005, which is less than the termination criterion of 0.00084, and the process is stopped.

The final values of the variables are as shown in Table 7.

Table 7. Final Solution

Variable	Value
x_2	14.050
y_2	13.018
x_5	72.225
y_5	58.096
x_7	74.000
y_7	46.670

The value of Z_1 at these points is 449.3615, and represents a minimum for this problem.

A diagrammatical view of the iteration process is shown in Figure 6. It can be seen how each point moves toward its optimum location, point P_2 stopping on the boundary of point P_1, and point P_5 being terminated by convergence.

Figure 7 shows the final layout, with the weighted flows indicated by lines connecting the points.

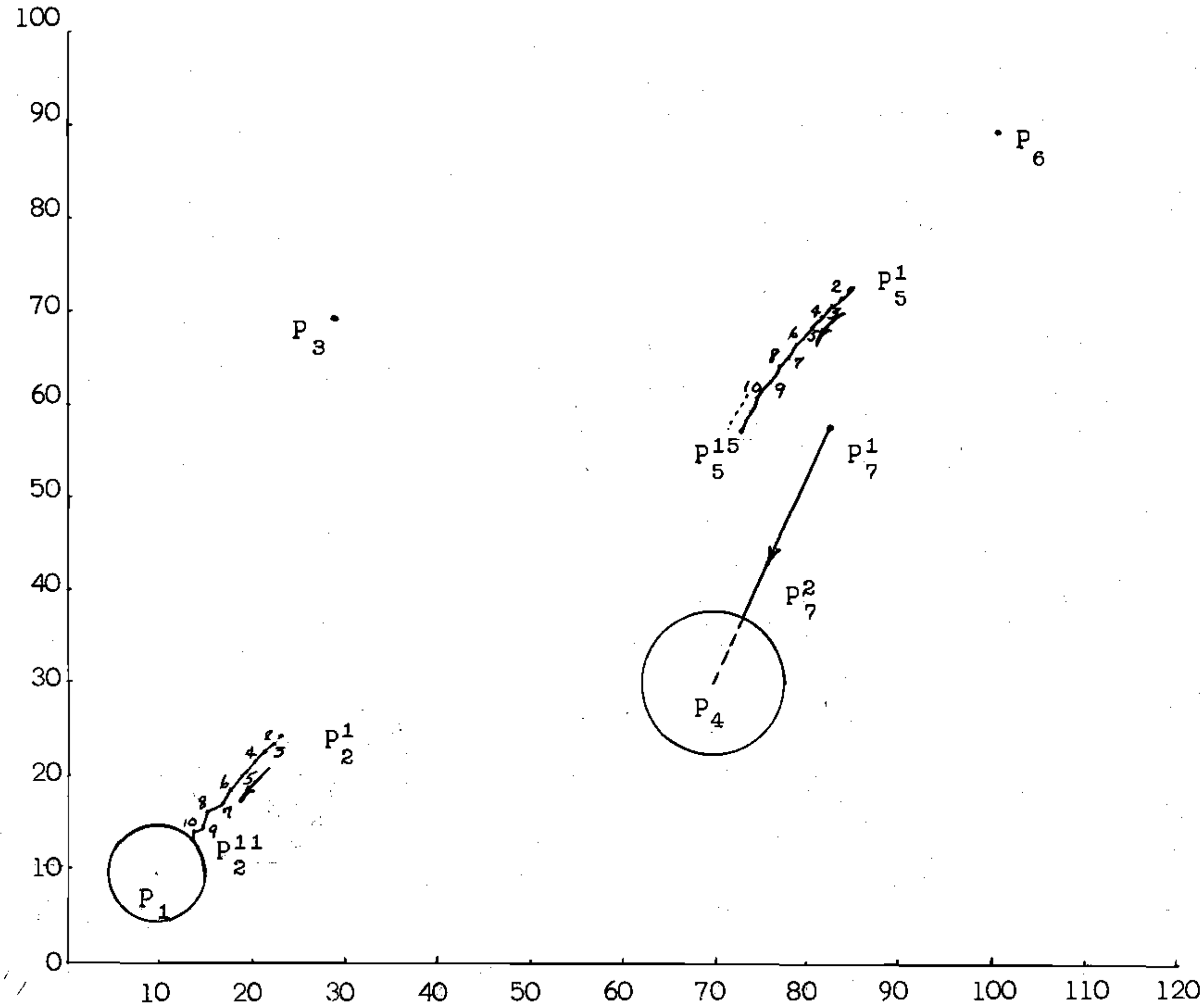

Figure 6. Illustration of General Problem Solution

$Z_1 = 449.3615$

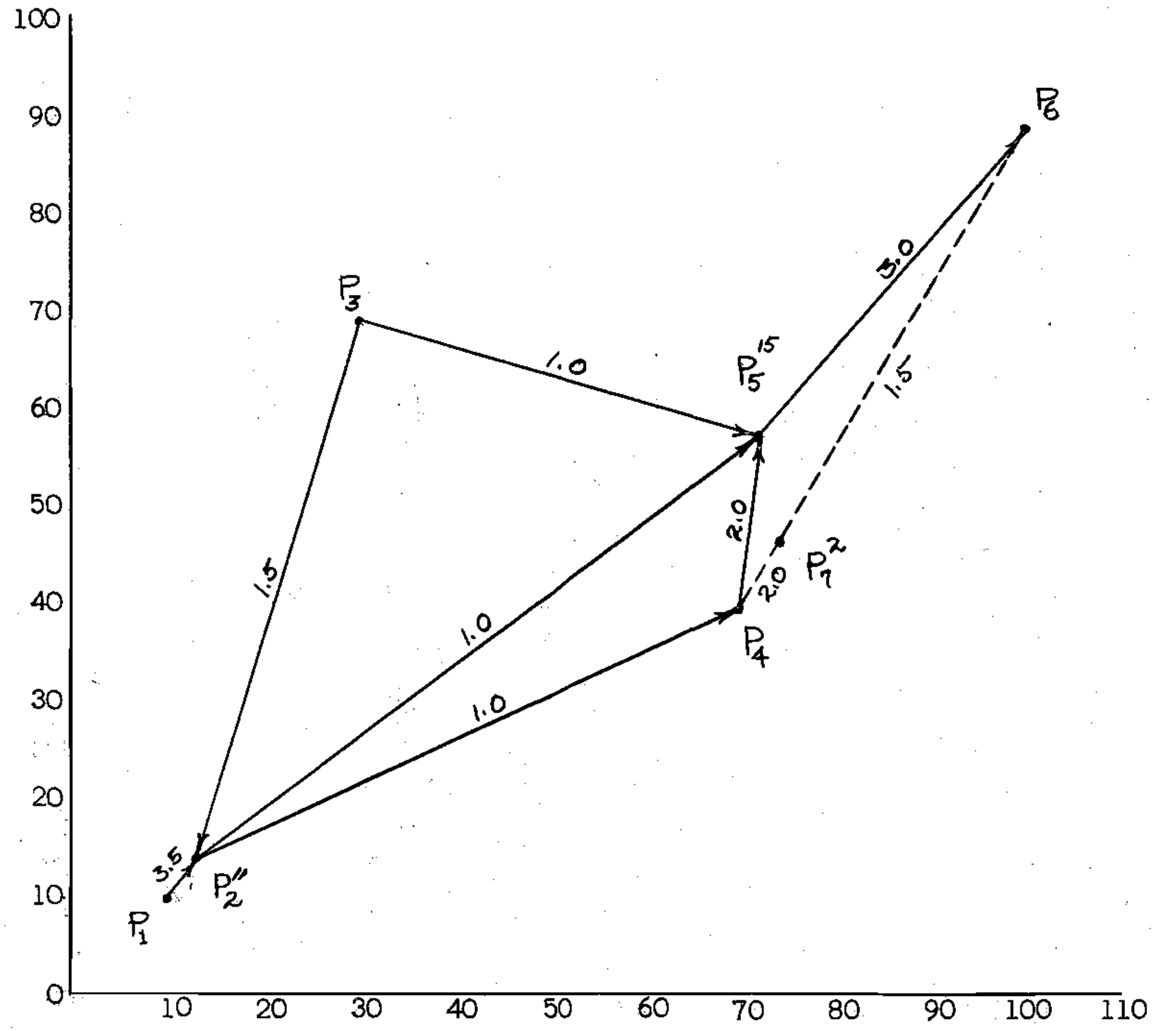

Figure 7. Final Solution with Weighted Flows Indicated

CHAPTER IV

SUMMARY AND CONCLUSIONS

The problem defined in this study is to develop a method to obtain a mathematically optimum solution to the problem of the location of facilities within a given area.

Summary

It has been shown that this problem may be formulated as a non-linear mathematical program of the form

minimize

$$Z_1 = \sum_{i,j=1}^{n} F_{i,j}[(x_i - x_j)^2 + (y_i - y_j)^2]^{\frac{1}{2}}$$

subject to

$$[(x_i - x_j)^2 + (y_i - y_j)^2]^{\frac{1}{2}} \geq K_{i,j}$$

$$0 \leq x_i \leq x_{max}$$

$$0 \leq y_i \leq y_{max}$$

and that at the present state of the art no satisfactory method of solution for this model exists.

In addition, a method has been developed to obtain values for the weighting factor $F_{i,j}$ in order to incorporate material handling flow and the relative importance of that flow. Also, a semi-objective method is established for determining $F_{i,j}$ when no material handling flow exists.

For an actual solution to be problem, the equation

$$Z_2 = \sum_{i,j=1}^{n} F_{i,j}[(x_i - x_j)^2 + (y_i - y_i)^2]$$

is used to determine a first approximation to the unconstrained problem. An iteration technique is then used to obtain successive approximations by the equations

$$x_i^{n+1} = x_i^n - u \left|\frac{\partial Z_1}{\partial x_i}\right|^n$$

$$y_i^{n+1} = y_i^n - u \left|\frac{\partial Z_1}{\partial y_i}\right|^n .$$

It is not necessary to continue this iteration process to its conclusion in all cases. When an unknown point has relationships with only two other points, for instance, it can be shown that it will be on the line connecting the points and will be as close as the constraints permit to the point that has the larger associated weighting factor.

By successively plotting each iteration on a layout of the area under consideration, it can easily be seen when any point violates a constraint. When this occurs, the intersection of the constraint boundary with the line connecting the last point outside the constraint area with the first point inside the area can be found and used as the location of that point, which is held fixed in further iterations. The justification for this is that since the gradient is the direction that causes the <u>most rapid</u> decrease in the objective, each point must move along the gradient until it is halted either by a boundary or by convergence. On the preceding example, one point was located so close to the constraint boundary that this procedure was not followed. It was simply allowed to remain where it fell.

The actual monetary cost represented by the solution to the example, using the relationship of $0.01 per foot-pound, is $107,846,760 per year. This is, of course, totally unrealistic, except perhaps for the United States Post Office. However, since it is known that this is within $2.00 per year of the lowest possible cost, it represents a maximum error of only 0.0000019 per cent. This effectively demonstrates the extreme accuracy that is obtainable with this method.

Conclusions

Due to the lack of an explicit solution for the model presented here, an iteration procedure has been presented. While in theory it is not difficult to use, in practice it is rather tedious. The iteration process becomes sensitive to rounding-off errors near the optimum solution, making close approximation very difficult. However, due to the nature of the variables being considered, the process can be terminated before this happens.

As the number of variables and fixed points increases, the length of the resulting equations increases rapidly, making each iteration longer. However, as is pointed out in McHose, as the number of points increases, and as they become more symmetrical, the closer the Z_2 approximation is to the true optimum. This has the effect of increasing the length of each individual iteration, but of decreasing the number of iterations necessary.

In the example in the preceding section, an interesting situation arises by allowing an exchange of material between points P_7 and P_5. In this case the iteration must be performed on equations containing 10 terms rather than 7, but convergence occurs on the first iteration. This obviously would eliminate a great deal of work, but unfortunately would eliminate the example also, and for this reason this relationship was omitted in order to give a clear picture of the entire procedure.

One limitation not previously discussed occurs when one considers the effect of placing a facility directly between two others. Naturally, this obstructs flow and then the mathematically shortest path can no longer be used. This matter is deserving of further attention, but it is not attempted in this thesis.

It is not meant that this method be used immediately to design a real layout; rather, this is but one more step in approaching a feasible all-inclusive method of quantifying the area of facilities planning. However, it is hoped that this method could find some application on a small scale.

A P P E N D I X

I. Solutions to Equations 82 Through 87

$$12.5x_7 = 1010$$

$$x_7 = \frac{1010}{12.5} = 81$$

$$33x_2 = 520 + 2x_5$$

$$x_2 = \frac{520 + 2x_5}{33}$$

$$30x_5 = 2420 + 2x_2$$

$$30x_5 = 2420 + 2\left(\frac{520 + 2x_5}{33}\right)$$

$$30x_5 = 2420 + \frac{1040}{33} + \frac{4x_5}{33}$$

$$30x_5 = 2420 + 31.515 + 0.1212x_5$$

$$29.8788x_5 = 2451.515$$

$$x_5 = 82.5$$

$$x_2 = \frac{520 + 2(82.5)}{33}$$

$$x_2 = 20.75$$

$$12.5y_7 = 725$$

$$y_7 = \frac{725}{12.5}$$

$$y_7 = 58$$

$$y_2 = \frac{640 + 2y_5}{33}$$

$$30y_5 = 2080 + 2y_2$$

$$30y_5 = 2080 + 2\left(\frac{640 + 2y_5}{33}\right)$$

$$30y_5 = 2080 + \frac{1280}{33} + \frac{4y_5}{33}$$

$$29.88\ y_5 = 2116$$

$$y_5 = 71$$

$$y_2 = \frac{640 + 2(71)}{33}$$

$$y_2 = 23.7$$

$x_2 = 20.75$	$x_5 = 82.5$	$x_7 = 81$
$y_2 = 23.7$	$y_5 = 71$	$y_7 = 58$

II. Evaluation of Z_1 to Determine u

Case I: u - +2

$$Z_1^2 = 3.5(110.25 + 168.43)^{\frac{1}{2}} + 1.5(90.23 + 2211.07)^{\frac{1}{2}} + 1(2450.17 + 289.75)^{\frac{1}{2}}$$

$$+ 1(2718.8 + 2253.67)^{\frac{1}{2}} + 1(2650.48 + .20)^{\frac{1}{2}} + 2(131.85 + 927.25)^{\frac{1}{2}}$$

$$+ 3(342.89 + 382.17)^{\frac{1}{2}}$$

$$= 3.5(16.70) + 1.5(47.95) + 1(52.34) + 1(77.28) + 1(51.47)$$

$$+ 2(32.55) + 3(26.93)$$

$= 58.45 + 71.93 + 52.34 + 77.28 + 51.47 + 65.1 + 80.79$

$= 457.36$

<u>Case II: u = -2</u>

$$Z_1^2 = 3.5(121 + 208)^{\frac{1}{2}} + 1.5(81 + 2077.35)^{\frac{1}{2}} + 1(2401.08 + 242.67)^{\frac{1}{2}}$$

$$+ 1(3908.5 + 2220.97)^{\frac{1}{2}} + 1(2864.09 + 2.4)^{\frac{1}{2}} + 2(182.71 + 995.35)^{\frac{1}{2}}$$

$$+ 3(271.68 + 340.43)^{\frac{1}{2}}$$

$$= 3.5(18.5) + 1.5(46.46) + 1(51.42) + 1(78.3) + 1(53.52)$$

$$+ 2(34.33) + 3(24.75)$$

$$= 63.53 + 69.69 + 51.42 + 78.3 + 53.52 + 68.66 + 74.25$$

$$= 459.37$$

III. Calculations for Remainder of Iteration Procedure

$$x_2^3 = 20.50 - 2\left[\frac{36.75}{16.45} - \frac{14.25}{47.95} - \frac{49.5}{52.35} - \frac{60.98}{77.28}\right]$$

$$= 20.50 - 2[2.234 - 0.2972 - 0.9456 - 0.7891]$$

$$= 20.50 - 2[0.2021] = 20.09$$

$$y_2^3 = 22.98 - 2\left[\frac{45.43}{16.45} - \frac{70.53}{47.95} - \frac{17.02}{52.35} - \frac{47.47}{77.28}\right]$$

$$= 22.98 - 2[2.7617 - 1.4709 - 0.3251 - 0.6143]$$

$$= 22.98 - 2[.3514] = 22.2772$$

$$x_5^3 = 81.48 - 2\left[0.7891 + \frac{51.48}{51.48} + \frac{22.96}{32.55} - \frac{55.56}{26.93}\right]$$

$$= 81.48 - 2[0.7891 + 1 + 0.7054 - 2.0631]$$

$$= 81.48 - 2[0.4314] = 80.6172$$

$$y_5^3 = 70.45 - 2\left[0.6143 + \frac{.45}{51.48} + \frac{60.9}{32.55} - \frac{58.65}{26.93}\right]$$

$$= 70.45 - 2[0.6143 + 0.0027 + 1.871 - 2.1779]$$

$$= 70.45 - 2(0.3161) = 69.8178$$

$$Z_1^3 = 3.5(101.81 + 150.8)^{\frac{1}{2}} + 1.5(98.21 + 2277.2)^{\frac{1}{2}} + 1(2491.01 + 314)^{\frac{1}{2}}$$

$$+ 1(3663.88 + 2260.05)^{\frac{1}{2}} + 1(50.62) + 2(112.78 + 889.23)^{\frac{1}{2}}$$

$$+ 3(375.58 + 407.23)^{\frac{1}{2}}$$

$$= 3.5(15.9) + 1.5(48.75) + 1(52.95) + 1(76.97) + 1(50.62)$$

$$+ 2(31.69) + 3(27.95)$$

$$= 55.64 + 73.12 + 52.94 + 76.96 + 50.62 + 63.37 + 83.85$$

$$= 456.50$$

$$x_2^4 = 20.09 - 2\left[\frac{35.315}{15.9} - \frac{14.865}{48.75} - \frac{49.91}{52.95} - \frac{60.53}{76.97}\right]$$

$$= 20.09 - 2[2.22106 - 0.30492 - 0.94258 - 0.78641]$$

$$= 20.09 - 2[0.18715] = 19.7157$$

$$y_2^4 = 22.28 - 2\left[\frac{42.98}{15.9} - \frac{71.58}{48.75} - \frac{17.72}{52.45} - \frac{47.54}{76.94}\right]$$

$$= 22.28 - 2[2.70314 - 1.46830 - 0.33465 - 0.61764]$$

$$= 22.28 - 2[0.28255] = 21.7149$$

$$x_5^4 = 80.62 - 2\left[0.78641 + \frac{50.62}{50.62} + \frac{21.24}{31.69} - \frac{58.14}{27.95}\right]$$

$$= 80.62 - 2[0.78641 + 1 + 0.67024 - 2.08014]$$

$$= 80.62 - 2[0.37651] = 79.86698$$

$$y_5^4 = 69.82 - 2\left[0.61764 - \frac{.18}{50.62} + \frac{59.64}{31.69} - \frac{60.54}{27.95}\right]$$

$$= 69.82 - 2[0.61764 - .00355 + 1.88198 - 2.16601]$$

$$= 69.82 - 2[0.33] = 69.16$$

$$Z_1^4 = 3.5(94.39 + 137.24)^{\frac{1}{2}} + 1.5(105.77 + 2328.16)^{\frac{1}{2}} + 1(2528.51 + 334.34)^{\frac{1}{2}}$$

$$+ 1(3618.18 + 2252.94)^{\frac{1}{2}} + 1(2486.72 + 0.67)^{\frac{1}{2}} + 2(97.36 + 851.47)^{\frac{1}{2}}$$

$$+ 3(405.34 + 433.47)^{\frac{1}{2}}$$

$$= 3.5(15.21) + 1.5(49.35) + 1(53.5) + 1(76.65) + 1(49.87)$$

$$+ 2(30.8) + 3(28.95$$

$$= 53.235 + 74.025 + 53.500 + 76.650 + 49.870 + 61.600 + 86.85$$

$$= 455.73$$

$$x_2^5 = 19.7157 - 5\left[\frac{34.005}{15.21} - \frac{15.4265}{49.35} - \frac{50.28}{53.5} - \frac{60.151}{76.65}\right]$$

$$= 19.7157 - 5[2.2357 - 0.31259 - 0.93981 - 0.78474]$$

$$= 19.7157 - 5[.19856] = 18.7229$$

$$y_2^5 = 21.7149 - 5\left[\frac{41.002}{15.21} - \frac{72.428}{49.35} - \frac{18.2851}{53.5} - \frac{47.465}{76.65}\right]$$

$$= 21.7149 - 5[2.69572 - 1.46763 - 0.34177 - 0.61924]$$

$$= 21.7149 - 5[0.26708] = 20.3795$$

$$x_5^5 = 79.867 - 5\left[0.78474 + \frac{49.867}{49.87} + \frac{19.734}{30.8} - \frac{60.399}{28.95}\right]$$

$$= 79.867 - 5[0.78474 - 0.99993 + 0.64071 - 2.08632]$$

$$= 79.867 - 5[.33906] = 78.1717$$

$$y_5^5 = 69.18 - 5\left[0.61924 - \frac{0.82}{49.87} + \frac{58.36}{30.8} - \frac{62.46}{28.95}\right]$$

$$= 69.18 - 5[0.61924 - 0.01644 - 1.8948 - 2.15751]$$

$$= 69.18 - 5[0.34] = 67.48$$

$$Z_1^5 = 3.5(76.09 + 107.73)^{\frac{1}{2}} + 1.5(127.17 + 2462.19)^{\frac{1}{2}} + 1(2629.34 + 384.96)^{\frac{1}{2}}$$

$$+ 1(3534.16 + 2218.46)^{\frac{1}{2}} + 1(2320.51 + 6.35)^{\frac{1}{2}} + 2(66.78 + 755.15)^{\frac{1}{2}}$$

$$+ 3(476.47 + 507.15)^{\frac{1}{2}}$$

$$= 3.5(13.55) + 1.5(50.9) + 1(54.9) + 1(75.85 + 1(48.21)$$

$$+ 2(28.67) + 3(31.35)$$

$$= 47.425 + 76.35 + 54.9 + 75.85 + 48.21 + 57.34 + 94.05$$

$$= 454.125$$

$$x_2^6 = 18.7229 - 5\left[\frac{30.53}{13.55} - \frac{16.916}{50.9} - \frac{51.277}{54.9} - \frac{59.449}{75.85}\right]$$

$$= 18.7229 - 5[2.25313 - 0.33233 - 0.934 - 0.78377]$$

$$= 18.7229 - 5[0.20303] = 17.7078$$

$$y_2^6 = 20.3795 - 5\left[\frac{36.328}{13.55} - \frac{74.43}{50.9} - \frac{19.62}{54.9} - \frac{47.1}{75.85}\right]$$

$$= 20.3795 - 5[2.68103 - 1.46227 - 0.35737 - 0.62096]$$

$$= 20.3795 - 5[0.24043] = 19.1772$$

$$x_5^6 = 78.1717 - 5\left[0.78377 + \frac{48.1717}{48.21} + \frac{16.3434}{28.67} - \frac{65.485}{31.35}\right]$$

$$= 78.1717 - 5[0.78377 + 0.99921 + 0.57003 - 2.08883]$$

$$= 78.1717 - 5[0.26418] = 76.8508$$

$$y_5^6 = 67.48 - 5\left[0.62096 - \frac{2.52}{48.21} + \frac{54.96}{28.67} - \frac{67.56}{31.35}\right]$$

$$= 67.48 - 5[0.67096 - 0.05227 + 1.91698 - 2.15502]$$

$$= 67.48 - 5[0.33065] = 65.8267$$

$$Z_1^6 = 3.5(59.41 + 84.22)^{\frac{1}{2}} + 1.5(151.098 + 2582.96)^{\frac{1}{2}} + 1(2734.47 + 433.59)^{\frac{1}{2}}$$

$$+ 1\,(3497.89 + 2176.18)^{\frac{1}{2}} + 1(2195 + 17.42)^{\frac{1}{2}} + 2(46.93 + 755.15)^{\frac{1}{2}}$$

$$+ 3\,(535.89 + 584.35)^{\frac{1}{2}}$$

$= 3.5(11.985) + 1.5(52.288) + 1(56.286) + 1(74.326) + 1(46.037)$

$+ 2(28.271) + 3(33.47)$

$= 41.9475 + 78.432 + 56.286 + 74.326 + 47.037 + 56.542 + 100.41$

$= 452.58$

$$x_2^7 = 17.7078 - 5\left[\frac{26.42535}{11.718} - \frac{18.67485}{52.534} - \frac{52.4499}{56.511} - \frac{59.1341}{75.296}\right]$$

$$= 17.7078 - 5[2.26532 - 0.36559 - 0.93835 - 0.78137)$$

$$= 17.7078 - 5[0.21764] = 16.6196$$

$$y_2^7 = 19.1772 - 5\left[\frac{31.367}{11.718} - \frac{76.557}{52.534} - \frac{21.038}{56.511} - \frac{46.6112}{75.296}\right]$$

$$= 19.1772 - 5[2.67682 - 1.45729 - 0.37228 - 0.61904]$$

$$= 19.1772 - 5[0.32584] = 17.54795$$

$$x_5^7 = 76.8508 - 5\left[0.78536 + \frac{46.6842}{46.894} + \frac{13.3684}{26.432} - \frac{69.9474}{33.768}\right]$$

$$= 76.8508 - 5[0.78536 + 0.99553 + 0.50577 - 2.07141]$$

$$= 76.8508 - 5[0.24857] = 75.60795$$

$$y_5^7 = 65.8267 - 5\left[0.61904 - \frac{4.4268}{46.894} + \frac{51.1464}{26.432} - \frac{73.2804}{33.768}\right]$$

$$= 65.8267 - 5[0.61904 - 0.0944 + 1.93502 - 2.17012$$

$$= 65.8267 - 5[0.3403] = 64.1252$$

$$Z_1^7 = 3.5(43.819+56.972)^{\frac{1}{2}} + 1.5(179.035+2751.212)^{\frac{1}{2}} + 1(2849.467+504.092)^{\frac{1}{2}}$$

$$+ 1(3479.631+2169.436)^{\frac{1}{2}} + 1(2080.089+34.513)^{\frac{1}{2}} + 2(31.45+582.025)^{\frac{1}{2}}$$

$$+ 3(594.97 + 669.505)^{\frac{1}{2}}$$

$$= 3.5(10.04) + 1.5(54.132) + 1(57.91) + 1(75.16) + 1(45.985)$$

$$+ 2(24.768) + 3(35.559)$$

$$= 35.14 + 81.198 + 57.91 + 75.16 + 45.985 + 49.536 + 106.677$$

$$= 451.606$$

$$x_2^8 = 16.6196 - 5\left[\frac{23.1686}{10.04} - \frac{20.0706}{54.132} - \frac{53.3804}{57.91} - \frac{58.9884}{75.16}\right]$$

$$= 16.6196 - 5[2.30763 - 0.37077 - 0.92178 - 0.78484]$$

$$= 16.6196 - 5[0.23024] = 15.4684$$

$$y_2^8 = 17.548 - 5\left[\frac{26.418}{10.04} - \frac{78.678}{54.132} - \frac{22.452}{57.91} - \frac{46.5772}{75.16}\right]$$

$$= 17.548 - 5[2.63127 - 1.45344 - 0.41476 - 0.61971]$$

$$= 17.548 - 5[0.14336] = 16.8312$$

$$x_5^8 = 75.608 - 5\left[0.78484 + \frac{45.608}{45.985} + \frac{11.216}{24.768} - \frac{73.176}{35.559}\right]$$

$$= 75.608 - 5[0.78484 + 0.9918 + 0.45284 - 2.05786]$$

$$= 75.608 - 5[0.17162] = 74.7499$$

$$y_5^8 = 64.1252 - 5\left[0.61971 - \frac{5.8748}{45.985} + \frac{48.2504}{24.768} - \frac{77.6244}{35.559}\right]$$

$$= 64.1252 - 5[0.61971 - 0.12775 + 1.94809 - 2.18297]$$

$$= 64.1252 - 5[0.25708] = 62.8398$$

$$Z_1^8 = 3.5(29.903 + 46.665)^{\frac{1}{2}} + 1.5(211.167 + 2826.921)^{\frac{1}{2}} + 1(2973.695 + 536.793)^{\frac{1}{2}}$$

$$+ 1(3514.296 + 2116.791)^{\frac{1}{2}} + 1(2002.554 + 51.268)^{\frac{1}{2}}$$

$$+ 2(22.562 + 521.656)^{\frac{1}{2}} + 3(637.568 + 737.676)^{\frac{1}{2}}$$

$$= 3.5(8.75) + 1.5(55.119) + 1(59.249) + 1(75.04) + 1(45.319)$$

$$+ 2(23.329 + 3(37.084)$$

$$= 30.625 + 82.6785 + 59.249 + 75.04 + 45.319 + 46.658 + 111.252$$

$$= 450.822$$

$$x_2^9 = 15.4684 - 5\left[\frac{19.1394}{8.75} - \frac{21.7974}{55.119} - \frac{54.5316}{59.249} - \frac{59.2815}{75.04}\right]$$

$$= 15.4684 - 5[2.18736 - 0.39546 - 0.92038 - 0.79]$$

$$= 15.4684 - 5[.08152] = 15.0608$$

$$y_2^9 = 16.8312 - 5\left[\frac{23.9092}{8.75} - \frac{79.7532}{55.119} - \frac{23.1688}{59.249} - \frac{46.0086}{75.04}\right]$$

$$= 16.8312 - 5[2.73248 - 1.44693 - 0.39104 - 0.61312]$$

$$= 16.8312 - 5[0.28139] = 15.4243$$

$$x_5^9 = 74.7499 - 5\left[0.79 + \frac{44.7499}{45.319} + \frac{9.4998}{23.329} + \frac{75.7503}{37.084}\right]$$

$$= 74.7499 - 5[0.79 + 0.98744 + 0.40721 - 2.04267]$$

$$= 74.7499 - 5[0.14196] = 74.04$$

$$y_5^9 = 62.8398 - 5\left[0.61312 - \frac{7.1602}{45.319} + \frac{45.6796}{23.329} - \frac{81.4806}{37.084}\right]$$

$$= 62.8398 - 5[0.61312 - 0.158 + 1.95806 - 2.19719]$$

$$= 62.8398 - 5[0.21599] = 61.7599$$

$$Z_1^9 = 3.5(25.612 + 29.423)^{\frac{1}{2}} + 1.5(223.18 + 2978.507)^{\frac{1}{2}} + 1(3017.525 + 603.965)^{\frac{1}{2}}$$

$$+ 1(3537.525 + 2146.988)^{\frac{1}{2}} + 1(1939.522 + 67.982)^{\frac{1}{2}}$$

$$+ 2(16.322 + 473.493)^{\frac{1}{2}} + 3(673.922 + 797.503)^{\frac{1}{2}}$$

$$= 3.5(7.418) + 1.5(56.583) + 1(60.178) + 1(75.396) + 1(44.805)$$

$$+ 2(22.132) + 3(38.359)$$

$$= 25.963 + 84.8745 + 60.178 + 75.396 + 44.805 + 44.264 + 115.077$$

$$= 450.5575$$

$$x_2^{10} = 15.0608 - 5\left[\frac{17.7128}{7.418} - \frac{22.4088}{56.583} - \frac{54.9392}{60.178} - \frac{58.9792}{75.396}\right]$$

$$= 15.0608 - 5[2.38781 - 0.39603 - 0.91294 - 0.78226]$$

$$= 15.0608 - 5[0.29658] = 13.5779$$

$$y_2^{10} = 15.4243 - 5\left[\frac{18.98505}{7.418} - \frac{81.86355}{56.583} - \frac{24.5757}{60.178} - \frac{46.3356}{75.396}\right]$$

$$= 15.4243 - 5[2.55932 - 1.44679 - 0.40838 - 0.61450]$$

$$= 15.4243 - 5[0.08959] = 14.9766$$

$$x_5^{10} = 74.04 - 5\left[0.78226 + \frac{44.04}{44.805} + \frac{8.08}{22.132} - \frac{77.88}{38.359}\right]$$

$$= 74.04 - 5[0.78226 + 0.98293 + 0.36508 - 2.03029]$$

$$= 74.04 - 5[.09998] = 73.5401$$

$$y_5^{10} = 61.7599 - 5\left[0.61456 - \frac{8.2401}{44.805} + \frac{43.5198}{22.132} - \frac{84.7203}{38.359}\right]$$

$$= 61.7599 - 5[0.61456 - 0.18391 + 1.96637 - 2.20862]$$

$$= 61.7599 - 5[0.1884] = 60.8179$$

$$Z_1^{10} = 3.5(12.801 + 24.767)^{\frac{1}{2}} + 1.5(269.685 + 3027.575)^{\frac{1}{2}} + 1(3183.453 + 626.171)^{\frac{1}{2}}$$

$$+ 1(3595.465 + 2101.425)^{\frac{1}{2}} + 1(1895.74 + 84.311)^{\frac{1}{2}}$$

$$+ 2(12.532 + 433.385)^{\frac{1}{2}} + 3(700.126 + 851.595)^{\frac{1}{2}}$$

$$= 3.5(6.129) + 1.5(57.522) + 1(61.722) + 1(75.478) + 1(44.5)$$

$$+ 2(21.157 + 3(39.442)$$

$$= 21.4515 + 86.283 + 61.722 + 75.478 + 44.5 + 42.334 + 118.326$$

$$= 449.9945$$

$$x_2^{11} = 13.5779 - 5\left[\frac{12.52265}{6.129} - \frac{24.63315}{57.422} - \frac{56.4221}{61.722} - \frac{59.9622}{75.478}\right]$$

$$= 13.5779 - 5[2.04317 - 0.42898 - 0.91413 - 0.79444]$$

$$= 13.5779 + 5[.09438] = 14.0498$$

$$y_2^{11} = 14.9766 - 5\left[\frac{17.4181}{6.129} - \frac{82.5351}{57.422} - \frac{25.0234}{61.722} - \frac{45.8413}{75.478}\right]$$

$$= 14.9766 - 5[2.84192 - 1.43734 - 0.40542 - 0.60735]$$

$$= 14.9766 - 5[.39181] = 13.0176$$

$$x_5^{11} = 73.5401 - 5\left[0.79444 + \frac{43.5401}{44.5} + \frac{7.0802}{21.117} - \frac{79.3797}{39.392}\right]$$

$$= 73.5401 - 5[0.79444 + 0.97843 + 0.33528 - 2.01512]$$

$$= 73.5401 - 5[.09303] = 73.0749$$

$$y_5^{11} = 60.8179 - 5\left[0.60735 - \frac{9.1821}{44.5} + \frac{41.6358}{21.117} - \frac{87.5463}{39.392}\right]$$

$$= 60.8179 - 5[0.60735 - 0.20634 + 1.97167 - 2.22244]$$

$$= 60.8179 - 5[0.15024] = 60.0667$$

$$Z_1^{11} = 3.5(16.401 + 9.106)^{\frac{1}{2}} + 1.5(254.409 + 3246.994)^{\frac{1}{2}} + 1(3130.425 + 728.05)^{\frac{1}{2}}$$

$$+ 1(3483.962 + 2213.618)^{\frac{1}{2}} + 1(1855.447 + 98.67)^{\frac{1}{2}}$$

$$+ 2(9.455 + 402.672)^{\frac{1}{2}} + 3(724.961 + 896.002)^{\frac{1}{2}}$$

$$= 3.5(5.05) + 1.5(59.173 + 1(62.117) + 1(75.482) + 1(44.206)$$

$$+ 2(20.301) + 3(40.261)$$

$$= 17.675 + 88.7595 + 62.117 + 75.482 + 44.206 + 40.602 + 120.783$$

$$= 449.6245$$

$$x_2^{12} = 14.0498 - 5\left[\frac{14.1743}{5.05} - \frac{23.9253}{59.173} - \frac{55.5902}{62.117} - \frac{59.0251}{75.482}\right]$$

$$= 14.0498 - 5[2.80679 - 0.40433 - 0.895 - 0.78198]$$

$$= 14.0498 - 5[0.72548] = 10.4706$$

$$y_2^{12} = 13.0176 - 5\left[\frac{10.5616}{5.05} - \frac{85.4736}{59.173} - \frac{26.9824}{62.117} - \frac{47.0491}{75.482}\right]$$

$$= 13.0176 - 5[2.09141 - 1.44447 - 0.43438 - 0.62332]$$

$$= 13.0176 + 5[0.41076] = 15.0714$$

$$x_5^{12} = 73.0749 - 5\left[0.78198 + \frac{43.0749}{44.206} + \frac{6.1498}{20.301} - \frac{80.7753}{40.261}\right]$$

$$= 73.0749 - 5[0.78198 + 0.97441 + 0.30293 - 2.00629]$$

$$= 73.0749 - 5[.05303] = 72.8098$$

$$y_5^{12} = 60.0667 - 5\left[0.62332 - \frac{9.9333}{44.206} + \frac{40.1334}{20.301} - \frac{89.7999}{40.261}\right]$$

$$= 60.0667 - 5[0.62332 - 0.22471 + 1.97692 - 2.23044]$$

$$= 60.0667 - 5[0.14509] = 59.34125$$

$$Z_1^{12} = 168.5515 + 1(3452.738 + 2145.885)^{\frac{1}{2}} + 1(1832.679 + 113.608)^{\frac{1}{2}}$$

$$+ 2(7.895 + 374.086)^{\frac{1}{2}} + 3(739.307 + 939.956)^{\frac{1}{2}}$$

$$= 168.5515 + 1(74.824) + 1(44.117) + 2(19.514) + 3(40.979)$$

$$= 449.486$$

$$x_5^{13} = 72.8098 - 5\left[\frac{58.76}{74.824} + \frac{42.8098}{44.117} + \frac{5.6196}{19.544} - \frac{81.5706}{40.979}\right]$$

$$= 72.8098 - 5[0.785309 + 0.970369 + 0.287535 - 1.990546]$$

$$= 72.8098 - 5[.052667] = 72.5465$$

$$y_5^{13} = 59.3413 - 5\left[\frac{46.3237}{74.824} - \frac{10.6587}{44.117} + \frac{38.6826}{19.544} - \frac{91.9761}{40.979}\right]$$

$$= 59.3413 - 5[0.619102 - 0.2416 + 1.979257 - 2.29487]$$

$$= 59.3413 - 5[0.061889] = 59.0319$$

$$Z_1^{13} = 168.5515 + 1(3421.864 + 2117.316)^{\frac{1}{2}} + 1(1810.205 + 120.299)^{\frac{1}{2}}$$

$$+ 2(6.485 + 362.213)^{\frac{1}{2}} + 3(753.695 + 959.023)^{\frac{1}{2}}$$

$$= 168.5515 + 1(74.426) + 1(43.936) + 2(19.149) + 3(41.385)$$

$$= 449.3665$$

$$x_5^{14} = 62.5465 - 5\left[\frac{58.4967}{74.426} + \frac{42.5465}{43.936} + \frac{5.093}{19.199} - \frac{82.3605}{41.385}\right]$$

$$= 72.5465 - 5[0.786971 + 0.968374 + 0.265274 - 1.990105]$$

$$= 72.5465 - 5[0.030514] = 72.3939$$

$$y_5^{14} = 59.0319 - 5\left[\frac{46.0143}{74.426} - \frac{10.9681}{43.936} + \frac{38.0638}{19.199} - \frac{92.9043}{41.385}\right]$$

$$= \;\; 59.0319 - 5[0.618255 - 0.249638 + 1.982592 - 2.244878]$$

$$= \;\; 59.0319 - 5[0.106331] \;\; = \;\; 58.5002$$

$$Z_1^{14} \;\; = \;\; 168.5515 + 1(3404.034 + 2068.667)^{\frac{1}{2}} + 1(1797.243 + 132.245)^{\frac{1}{2}}$$

$$+ 2(5.731 + 342.257)^{\frac{1}{2}} + 3(762.097 + 992.237)^{\frac{1}{2}}$$

$$= \;\; 168.515 + 73.924 + 43.926 + 2(18.653 + 3(41.885)$$

$$= \;\; 449.362$$

$$x_5^{15} = \;\; 72.3939 - 5\left[\frac{58.3441}{73.924} + \frac{42.3939}{43.926} + \frac{4.7878}{18.653} - \frac{82.8183}{41.885}\right]$$

$$= \;\; 72.3939 - 5[0.789244 + 0.96512 + 0.256677 - 1.977278]$$

$$= \;\; 72.3939 - 5[0.033763] \;\; = \;\; 72.2251$$

$$y_5^{15} = \;\; 58.5002 - 5\left[\frac{45.4826}{73.924} - \frac{11.4998}{43.926} + \frac{37.0004}{18.653} - \frac{94.4994}{41.885}\right]$$

$$= \;\; 58.5002 - 5[0.615261 - 0.261799 + 1.983616 - 2.256163]$$

$$= \;\; 58.5002 - 5[0.080915] \;\; = \;\; 58.0956$$

$$Z_1^{15} \;\; = \;\; 168.5515 + 1(3384.366 + 2032.026)^{\frac{1}{2}} + 1(1782.959 + 141.715)^{\frac{1}{2}}$$

$$+ 2(4.951 + 327.451)^{\frac{1}{2}} + 3(771.445 + 1017.89)^{\frac{1}{2}}$$

$$= \;\; 168.5515 + 1(73.595) + 1(43.871) + 2(18.222) + 3(42.30)$$

$$= \;\; 449.3615$$

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