ESSAYS ON DIGITAL GOODS AND ONLINE MARKETS

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SUMMARY

Information technology has revolutionized the way in which sellers engage with potential customers and distribute their products through online channels. However, they also face increasing challenges to remain competitive. For example, in the software industry, the plethora of available applications leads to a highly competitive landscape, making it difficult for new entrants to gain visibility and attract consumer interest. For online platforms, the platform owner not only serves as an intermediary for sellers and buyers but also introduces its own private-label products, further intensifying competition with third-party sellers.

This dissertation investigates the strategic actions sellers undertake to tackle these challenges. In the first essay, we build a game-theoretical model to examine two prevalent strategies, seeding and time-limited freemium, that developers can employ to spur adoption by helping consumers directly or indirectly learn the value of their products. We offer managerial recommendations on the optimal circumstances for implementing each strategy, considering factors such as social and self-learning dynamics, adoption costs, and product value depreciation.

In the second essay, we study the impacts of Amazon launching its private-label products and engaging in self-preferencing for these products on third-party sellers. Our findings show that although Amazon favors its own products in search results, the average sales of third-party products in the affected categories increase more than those in unaffected categories. We then investigate several mechanisms that could contribute to this change. We find that Amazon's private-label products displace lower-quality sellers, foster variety in

product designs, and serve as valuable references for third-party sellers to improve their searchability. These factors potentially lead to higher sales and ultimately an increase in consumer welfare, with little impact on prices.

CHAPTER 1

INTRODUCTION

Information technology has revolutionized the way sellers connect with potential consumers and distribute their products. However, this evolution also presents new challenges for sellers to thrive in competitive markets. For instance, in the software industry, the vast array of available applications makes it increasingly difficult for new offerings to stand out and capture consumer attention. Additionally, in online platforms, the platform owner not only serves as a gatekeeper connecting sellers and consumers but also introduces private-label products, competing with third-party sellers under a self-preferencing recommendation system. My dissertation delves into these challenges and explores the strategic actions sellers undertake to adapt and respond effectively to these evolving market dynamics.

In the first essay, along with coauthors Marius Florin Niculescu, D.J. Wu, and Yifan Dou, I use a game-theoretical model to explore two popular strategies through which developers can catalyze adoption by helping consumers directly or indirectly learn the value of their products - seeding and time-limited freemium. In software markets, the sheer number of available applications make it rather challenging for any given new one to stand out and be noticed by consumers. While word-of-mouth (WOM) effects may help developers gradually gain visibility for their products, efficiently jumpstarting and propagating adoption prior to product obsolescence are by no means trivial. Often, relying solely on paid adoption may result in sub-optimal outcomes. We explore two popular strategies through which developers can catalyze adoption by helping consumers directly or indirectly learn the value of their products - *seeding* (free full-feature product giveaways to a subset of the consumer base) and *time-limited freemium* (*TLF*). Seeding, as a business strategy, existed for a long time. On the other hand, the feasibility to offer market-wide TLF became mainstream more recently, with the advent of digital goods and services. Thus, a natural question emerges - if TLF represents nowadays a feasible and easily implementable strategy for software applications, has seeding approach been rendered irrelevant in these markets? In this study, we provide managerial recommendations on when each of these strategies with a free full-feature-consumption component is optimal, based on social and self-learning dynamics, consumer priors, adoption costs, and individual product value depreciation. While we see TLF showing up as optimal in some parameter range for each scenario explored, the same cannot be said about seeding. We identify two particular conditions under which the latter can still emerge as a dominant strategy - the presence of (i) user adoption costs and/or (ii) individual depreciation of value by usage. While WOM effects alone are not enough for seeding to dominate other strategies, we do see that in the presence of any of the aforementioned additional market conditions, the parameter range where seeding is dominant expands as social learning is more efficient. We further show that our results are robust under diverse assumptions regarding seeding and the distribution of consumer priors.

In the second essay, we document evidence of Amazon's engagement in self-preferencing and examine the consequences of Amazon launching its private-label products while employing such a strategy. We first present two pieces of evidence for self-referencing. In the direct evidence, we find that Amazon private-label products are ranked higher than third-party products even when accounting for other observables. To control for unobserved product qualities, we further leverage a scenario where Amazon becomes the seller of an existing third-party product, i.e., the product itself remains unchanged. We find that product sales immediately increase after Amazon becomes the seller, indirectly showing that Amazon may actively promote its own products more than third-party counterparts. We then examine the effects of Amazon launching private-label products on third parties in the same category. We find that although Amazon favors its own products in search, the average sales of third-party products in affected categories increase more than those in the unaffected categories. We further explore the mechanisms that may explain the changes. We find that private-labels displace lower-rated sellers, stimulate innovation and variety in product designs, and serve as valuable guidance for third-party sellers to enhance their searchability by improving product descriptions. These factors potentially lead to higher sales and ultimately an increase in consumer welfare, with prices being largely unchanged.

CHAPTER 2

SCORE HIGH WITH A FREE KICK: SEEDING VS. TIME-LIMITED FREEMIUM AS CATALYSTS FOR THE ADOPTION OF SOFTWARE APPLICATIONS

2.1 Introduction

The software app markets have been growing exponentially during the last decade thanks to the advances in Internet technologies, the widespread use of desktop and mobile devices, and a lower entry barrier for developers. Microsoft, the developer of Windows, the leading desktop operating system by market share, has facilitated compatibility with "over 35 million application titles with greater than 175 million application versions, and 16 million unique hardware/driver combinations" (Fortin 2018). On the mobile front, with over a decade of growth, the top two app stores, Google Play and Apple App Store, boast a combined app count above 5.70 million (Statista 2021b).

However, in this supply-flooded app market, significant profits (and therefore market success) can be elusive for developers. According to Slashdata (2018), in the second quarter of 2018, almost three-quarters of the developers made less than USD 1,000 per month in terms of app revenue. Failure to monetize apps can be attributed to a number of factors, including product value, competition, limited consumer attention, high adoption costs, inefficient dissemination of product and brand awareness, and limited product life with consumption-based depreciation. Some app categories exhibit strong competition (e.g., generic games with easily cloneable interfaces and gameplay), while others might be dominated by a few leaders (e.g., productivity software, electronic medical records, and niche products). In this paper, we focus on the latter categories of products, in which the greatest challenge is to get the consumers to discover and adopt the product rather than fending off competition.

It is well established that customers adopt software applications based on their perceptions of the product's value – often known as "priors" – which may be in line with the real value or considerably off (Lambrecht et al. 2007, Weathers et al. 2007, Shulman et al. 2015, Chen et al. 2021, Zhang et al. 2021). Customers can increase their knowledge of the value of the product via several additional learning mechanisms. On one hand, consumers can engage in *social learning* via word-of-mouth (WOM), thus allowing their perceptions to be shaped to a certain degree by the opinions of other consumers or experts. On the other hand, if consumers are able to access the product directly, they can engage in *self-learning*, whereby they update their priors on the value of the product upon using it for a period of time.

Understanding the potential misalignment between consumer valuation perceptions and reality, as well as the dynamics of consumer learning, software producers have been increasingly involved in strategically managing the latter through various forms of *free*-consumption offers ultimately intended to catalyze revenue-generating adoption. Two popular strategies employing the free-consumption approach are *seeding* (*S*) and *time-limited freemium* (*TLF*, otherwise referred to as time-locked *free trials*). Through *S*, developers provide a full-functionality product for free to a subset of the market, counting on these seeded consumers to not only use the product but also help spread awareness and knowledge about it within their respective communities and beyond. For example, many providers such as IBM, Microsoft, and SAS offer a bundle of their developer-grade products for free to students and educators (IBM 2022, Microsoft 2022, SAS 2022). Via its Technology Impact Program, Autodesk (2022b) is giving away free licenses for many of its products to nonprofits, startups, and entrepreneurs using design for environmental or social good. Seeding is also a popular strategy within mobile app markets via free app giveaway opportunities (distributed through portals such as AppAdvice Daily, AppsFree, and https://www.giveawayoftheday.com/). Moreover, seeding and price discounts have been used as popular, albeit frowned-upon incentives by market entrants without established brands to harvest online reviews in order to jumpstart WOM effects (Hautala 2021).

Under *TLF*, all consumers are able to try the full-functionality product at no charge during a limited trial period, after which they are required to pay for continuing the adoption. Free trial windows typically span from a few days to a few months. TLF strategies have been employed for many categories of apps and services in domains including engineering and design (e.g., AutoCAD, VeSys, Adobe Creative Cloud), productivity (e.g., Salesforce CRM products, Microsoft Office 365), IT security (e.g., Crowdstrike, Kaspersky, Norton, Bitdefender), content provision (e.g., Hulu, Apple TV+, Newspapers.com), health and wellness (e.g., Peleton, Calm, Nutrium). And this is just a small subset of applications and services taking advantage of TLF.

Both TLF and S are appealing strategies in the context of *experience goods* - a broad category which encompasses many digital goods whose value and fit are better understood by consumers once they are directly exposed to the product/service. It is important to highlight that a major difference between TLF and S lies in how they cannibalize some demand in order to optimally drive paid consumption. TLF aims to cannibalize demand from *every* potential customer for a limited period of time, leaving open the possibility to charge later each of these customers (whose priors have been updated after the free trial) for the residual value of the product after the completion of the trial. On the other hand, S cannibalizes demand from a subset of the market, albeit for the *entire* product life. What the firm attempts through S is to get seeded customers to influence the purchase decisions of *other* customers. In that sense, it can be argued that TLF shapes paid consumption mostly via self-learning, whereas S takes advantage of self-learning to fuel WOM (which in turn, drives social learning).

Embracing a market strategy that involves some form of free access to the full product involves a delicate balance act, due to intrinsic adoption costs and the potential of value depreciation with use. Adoption costs (associated with installation, setup, hardware and storage, testing, learning to use, etc.) can undermine the effectiveness of both TLF and S as some consumers may shy away from exploring the product in the first place even when there is some free access to it. Furthermore, as software applications are getting more complex in terms of features and functionality the size of their installation footprint has increased considerably, which is particularly challenging for mobile users whose phones have limited storage. If a mobile user does not have enough space to install a new app on her device, in order to explore a free-trial app she must either upgrade her cloud storage or delete other applications both of which are costly actions. Value depreciation through use, otherwise referred to as *individual depreciation* (Dou et al. 2017),[1] can also dilute the

[1]Not to be confused with *obsolescence*, which captures time-based depreciation in value, regardless of usage.

benefit of free trials should consumers be able to utilize the software for a significant portion of their specific related needs prior to the expiration of the trial period. This applies to cases in which the user need is limited in scope and scale (e.g., installing Adobe Auditions for a small project to remove background noise from a handful of audio tracks, or installing Adobe Photoshop Lightroom to remove dust spots from digital photos from a vacation trip, when the consumer realizes ex-post that the camera sensor was not clean of debris when pictures were taken). In the mobile space, it has been documented that, users tend to lose interest in many installed apps relatively quickly and the retention rates drop to single-digit percentages for the majority of app categories after only one month (Statista 2022). Individual depreciation is also present when consumption is more hedonic (e.g., video games, music, movies) rather than utilitarian, switching costs are negligible, and consumers constantly search for "the next" great experience.[2] On the other hand, enterprise applications that are used for daily operations (e.g., ERP systems, electronic medical record systems) would likely exhibit low individual depreciation as their values are not expected to decline through use.

Seeding as a business strategy existed for a long time, being applicable to both physical and digital goods. On the other hand, the feasibility to implement a consistent, market-wide *TLF* strategy has really been ushered in by the advent of digital goods and services for it relies on encapsulating a limited free-for-all consumption component with *encoded automatic expiration* at the end of the trial period. Hence, an important question for soft-

[2]For example, in the context of video games, it has been documented that, on average, players tire quickly of a particular game. According to Shiller (2013), consumers reduce their valuation from $80 in the first month of use to just a couple of dollars after six months. In fact, after only the first week of ownership, the consumption value that owners place on the games they own already deteriorates between 22% and 49% (Ishihara and Ching 2019).

ware vendors is whether or not seeding is still relevant nowadays, given that now they can embrace the *TLF* approach. While various studies explored seeding and free trials separately, little research exists on how they fare against each other. This work aims to fill this gap. Extant research looked at how seeding fares against other freemium models such as feature limited freemium as well as strategies that do not involve a free consumption component (Niculescu and Wu 2014) and the findings indicate that seeding is the dominating model when consumers significantly underestimate the value of the product prior to adoption. However, if *TLF* is among viable strategies, we show that the dominance of seeding strategy is no longer guaranteed in such scenarios. For seeding to dominate *TLF* (and other more traditional models without free consumption), in addition to customers significantly underestimating the product, other market factors have to be present. In particular, in this study, we identify two such factors - adoption costs and individual depreciation. While our study does not completely dispute the continued viability of seeding as a strategy in today's markets for digital goods, we find that more stars need to align to warrant its use. Interestingly, stronger WOM effects alone do not help the seeding strategy dominate the other strategies. Nevertheless, if any of the aforementioned additional market conditions occur, stronger WOM effects do lead to a larger region of the parameter space where seeding dominates. We further show that our results are robust in nature under diverse assumptions regarding seeding (uniform vs. targeted) and the distribution of consumer priors. Moreover, we uncover non-trivial properties of the optimal individual depreciation rate when the firm chooses to endogenize it. Our work expands the research agenda on economics of free in markets for digital goods and services, further refining the theory behind optimality of strategies that involve some form of free offering on behalf of the vendors.

2.2 Literature

Our novel theoretical contribution lies predominantly within the space of *economics of free*, advancing the research agenda on the impact and optimality of (i) time-locked free trials and (ii) seeding strategies.[3] For expositional brevity, we keep the discussion in this section focused predominantly around this extant literature. At the same time, we do acknowledge that our modeling framework also draws on and combines various modeling elements from several other supporting literature streams (including multi-period adoption of digital goods, role of WOM effects on adoption, and individual use-based value depreciation). Relevant works in these ancillary research streams are referenced throughout the main body of the paper, as we introduce various go-to-market models.

The ability to influence consumer perceptions, purchase behavior, and dissemination of awareness through sampling campaigns has been recognized for a long time (Hamm et al. 1969, Holmes and Lett 1977, Goering 1985). Time-locked free trials and free demonstrations represent a special case of sampling where consumers get exposure to the full-feature product for a limited period of time. Heiman and Muller (1996) explore the optimal length of free trials and demonstrations in the context of physical goods, focusing in particular on cars and printers. In general, as physical goods and some digital services have marginal costs, it may not be optimal to cover the entire market through free trial campaigns. Accounting for such unit costs incurred by the vendor, Schlereth et al. (2013) and Tian and Xueying (2018) explore the optimal market coverage of free trial campaigns. While market-wide free trial strategies are not that common in the markets for physical

[3] There exist other forms of freemium (including quantity-limited and feature/quality-limited). This study does not focus on these, and, as such, we do not discuss associated extant literature.

goods, we do see widespread implementation of a seemingly similar, albeit quite different strategy - *free returns* (or full *money-back guarantees, MBGs* within a certain time frame) - occasionally paired with free shipping as well. Similar to free trials, MBG policies also help resolve consumer uncertainty and the risk of a mismatch, and may also positively impact consumer adoption decisions and willingness to pay a higher price (Che 1996, Suwelack et al. 2011, Bower and Maxham III 2012). Unlike with free trials and demonstrations, under $MBGs$ consumers gain experience with the product after the purchase. At the same time, from the perspective of both consumers and providers/retailers, such strategies may add considerable costs. For consumers, there are inconvenience costs associated with the return process (repackaging the item, taking it to the retailer or a collection point, etc) and consumers must incur these costs in order to receive the refund (because, unlike in the case of free trials, consumers are charged upfront in the case of free return policies). Heiman et al. (2001) explore consumer preference for free demonstrations vs. $MBGs$ and analyze scenarios when the two risk-reducing strategies complement or substitute each other. For providers/retailers, free returns add considerable logistical costs as well. Part of it is in terms of labor costs to process returns, which, alone, can in some cases cancel out the increase in revenue if we myopically consider short-term profits (Patel et al. 2021). In addition, goods used and returned during the free returns window in many cases exhibit wear and tear and cannot be re-commercialized as new items. The salvage value of returns represents an important factor in the implementation of *MBGs* for physical goods (Davis et al. 1995, Akçay et al. 2013) - some returns are unopened or in like-new condition and can be put back on the shelf right away, others can be refurbished/recertified and sold at a discount, and some necessitate retiring altogether, with the retailer (along with other entities upstream in the supply chain) absorbing the overall cost associated with the retired item. Moorthy and Srinivasan (1995) explore how offering *MBGs* can also be used to signal product quality. Furthermore, cross-channel full-refund or partial-refund returns (e.g., buy online, return in person) have been considered as a feature to influence consumer adoption in omnichannel operations and, potentially, help fight competition (He et al. 2020, Jin et al. 2020, Nageswaran et al. 2020).

In the context of digital goods and services, and more specifically software applications as well as online services, the provider costs associated with offering time-locked free trials become negligible. Once the product is built, inserting code to lock the product or service access upon the expiration of the free trial can be done with very few resources. As such, it is feasible to offer market wide free trials (TLF). Also, similarly, for this specific category of products and services, the costs of offering $MBGs$ are negligible - once a user requests money back within an acceptable window after purchase, it is very easy for the provider to reverse the online transaction. There are no actual physical or digital returns for the products in this space - the licence gets deactivated or the online access is revoked. In software application and services markets, both TLF and MBG strategies are employed.[4] One difference between TLF and MBG is that with MBG the customer pays upfront, whereas with TLF, in many cases consumers can download the free trial without initiating payment or providing details on how the payment will be processed (e.g., providing a credit card account). Arguably, with TLF, more consumers can try the product even if they cannot afford the paid version. However, with financial instruments that offer short-term

[4] Intuit Quickbooks and TaxAct desktop versions, as well as Autodesk online services all come with risk-free $MBGs$. We already discussed several TLF examples in the Introduction.

access to capital (e.g., a credit card), even such customers can try products offered with $MBGs$. Another difference is that TLF is usually implemented by the developer and can benefit all consumers equally. On the other hand, MBG strategies are traditionally point-of-sale (retailer) strategies and can differ in extent across developer and various resellers of the same digital product.[5] Since in this paper we focus on a single decision-making vendor for the product, this difference is irrelevant for our analysis. As such, in the context of this study, in contrast to physical goods markets, in digital goods markets the aforementioned two risk-reducing strategies - MBG and TLF - are more or less equivalent. Keeping that in mind, for the rest of this study we stick to TLF terminology.

The literature on properties and performance of TLF go-to-market strategies in the digital space has also progressed substantially in the last decade. Cheng and Liu (2012) and Dey et al. (2013) explore when it is optimal to offer TLF in software markets and how the length of the trial period should be calibrated. Cheng et al. (2015) compare TLF against other free sampling strategies (feature limited trials and hybrid feature/time limited trials). Wang and Özkan-Seely (2018) show that price can serve as a quality signal that complements direct experiential learning when TLF is the dominant strategy (the optimal trial length is positive). Datta et al. (2015) and Foubert and Gijsbrechts (2016) explore the impact of free trials on customer acquisition (conversion), churn, and overall customer lifetime value in the long run. Lee and Tan (2013) and Chen et al. (2017) investigate the interaction between WOM effects and free sampling strategies (including TLF) when exploring their market outcome. Sunada (2018) explores optimal free trial length in the presence of

[5]For example, while offering a 30-day MBG for direct purchases, as of January 2022, Autodesk (2022a) also stated that "Return policies for purchases and renewal charges from third-party sellers such as retailers or authorized Autodesk resellers can vary by seller."

demand depreciation. Mehra and Saha (2018) study whether public betas and free trials should be used in tandem or not. Yoganarasimhan et al. (2020) investigate personalized-length vs. uniform-length TLF strategies, and the impact of optimally-personalized free trials on short-term conversions and long-run customer loyalty and overall revenues. Reza et al. (2021) explore how promotion redemption and subsequent usage are impacted when targeting existing users with hybrid time- and quantity-limited free trials.

Unlike TLF, seeding involves handing out free perpetual licenses but only to a fraction of the market. Seeded customers learn about the value of the product through direct use (same learning mechanism as in the case of TLF) and then they may propagate information about the product through the network via WOM effects, or alter the value of the product via network effects. A segment of this literature focuses on how to optimize (or nearly optimize) targeted or stochastic seeding strategies contingent on the topology of the network and the optimization objective (Galeotti and Goyal 2009, Libai et al. 2013, Schlereth et al. 2013, Kim et al. 2015, Chen et al. 2017, Cui et al. 2018, Wilder et al. 2018, Akbarpour et al. 2020, Chin et al. 2021). Aral et al. (2013) and Nejad et al. (2015) explore the role of consumer homophily on the effectiveness of seeding campaigns. At a market level, abstracting from the network structure, several studies employed adaptations of the Bass (1969) model to explain how firms can employ seeding to jumpstart and accelerate the product diffusion process (Jain et al. 1995, Lehmann and Esteban-Bravo 2006, Jiang and Sarkar 2010). Dou et al. (2013) explore how seeding and social media features can be used in tandem to engineer optimal network effects in markets for digital goods and services. Niculescu and Wu (2014) find that uniform seeding dominates feature-limited freemium and no-promotion strategies when consumers significantly underestimate apriori the value of the product. Lin et al. (2019) find that free sampling promotions (including seeding[6]) can have positive effects on product ratings - in other words, seeding can be an effective tool to harvest positive reviews early on in the diffusion process. Han et al. (2021) explore scenarios in which seeding is a desirable strategy for either manufacturer or retailer in a supply chain.

While extensive separate research investigations of each of TLF and seeding strategies have been conducted, the direct comparison of optimal TLF against seeding has gone largely unexplored. Schlereth et al. (2013) explore a numerical optimization of the market coverage of TLF and seeding under exogenous pricing that is kept constant across the sampling methods. They do not draw conclusions as to which strategy dominates in any given parameter range and they do not benchmark these two business models with free consumption against other no-promotion models in terms of profits. To the best of our knowledge, this study is the first to compare and contrast TLF, (random and targeted) seeding, and business models with no promotion (under perpetual-licensing and subscription-based licensing) within a unified framework accounting for WOM effects, endogenous pricing, adoption costs, and individual use-based value depreciation. Complementing this modeling contribution, our main theoretical contribution lies in identifying market and product factors that allow seeding to emerge as a dominant strategy at least in some range of the parameter space when TLF is also in the vendor's feasible go-to-market strategy set. Additional theoretical contributions include explorations of how the optimality regions fluctuate

[6]Lin et al. (2019) classify the products into nine categories. Not all free sampling campaigns fall under our description of seeding. For example, free samples of health food items do not correspond to our definition of seeding because food is a repeated consumption non-durable item and the sample provides only a small portion. On the other hand, free sampling promotions of apparel and home appliances do correspond to our definition of seeding because these goods are semi-durable or durable, with the same item not being purchased very often.

for various strategies contingent on model parameters and also interesting patterns in optimal depreciation rate levels when the latter are endogenized.

2.3 Baseline Setup

This section presents the baseline model setup and related results. Section 2.3.1 introduces the product characteristics and candidate business models. Section 2.3.2 provides the parameterization of the demand structure. Lastly, Section 2.3.3 presents the benchmark results associated with the baseline model.

2.3.1 Supply Structure and Candidate Business Models

We consider a scenario in which a firm has already developed a software product and is exploring the most profitable way to commercialize it. At this pre-release stage, all the development costs are sunk. The product has a life span of two periods, after which it becomes obsolete. The marginal production cost and the time discount factor of future earnings are considered negligible. The firm aims to maximize the undiscounted profit over two periods. Consistent with established literature (Choudhary 2007, Zhang and Seidmann 2010, Niculescu and Wu 2014, Li and Jain 2016, Chen and Jiang 2021), we focus on scenarios where the firm can offer a credible price commitment. In our setup, the firm considers among the following four models:

(a) **Charge for Everything - Perpetual Licensing** (*CE-PL*): Consumers pay a *one-time* fee at the beginning of the adoption period, which in turn grants them the right to use the product throughout its entire lifecycle (i.e., until obsolescence horizon) without any additional charges;

(b) **Charge for Everything - Subscription** (*CE-SUB*): Consumers purchase a single-period license at the beginning of period 1 and/or 2, which expires after that period. Consumers who subscribed in period 1 have the option to renew the subscription at the beginning of period 2 (but are not required to);

(c) **Time-Limited Freemium** (*TLF*): All consumers have the access to the product at no charge in period 1 (i.e., the free trial period). When the free trial expires, consumers are required to purchase a license in period 2 to continue using the product;[7]

(d) **Seeding** (*S*) - paired with **perpetual licensing**: In the baseline model setup, we focus on *uniform* seeding, whereby the firm samples seeded customers uniformly across all tiers of the addressable market (Libai et al. 2005, Li et al. 2019, Akbarpour et al. 2020, Chin et al. 2021). Later, in Section 2.6 and in the Appendix, we also explore *targeted* seeding, which involves firms choosing the seeded consumers selectively.

2.3.2 Demand Structure

Consider a unit mass of consumers with their types θ uniformly distributed on $[0, 1]$. A type-θ consumer derives per-period benefits $a\theta$ from using the product. Coefficient $a > 0$, which we refer hereafter as *quality factor*, quantifies in an aggregate form core quality dimensions of the product such as reliability, versatility, efficiency, ease of use, etc. Type θ captures heterogeneity in the consumers' willingness to pay (WTP) for quality per period as a reflection of diverse needs for the product and individual fit. Consumers cannot observe the product quality a before the product is released. Instead, they rely on the prior on

[7] In practice there also exist subscription-based models with a free trial period in the beginning. However, in this simplified framework of two periods, a subscription model with a free trial in first period is equivalent to *TLF*.

product quality, $a_0 = \alpha a$. In the base model, we assume for simplicity a homogeneous value $\alpha > 0$ across consumers.[8] We relax this assumption later in the paper in Section 2.6.2 and show numerically how heterogeneity of consumer priors on quality (whereby some customers initially overestimate the quality of the product while others underestimate it) impacts the main results.

Consumers adjust their priors over time. Let a_t to denote a consumer's perceived valuation factor before release ($t = 0$), at the beginning of period 1 ($t = 1$), and at the beginning of period 2 ($t = 2$). We employ the same parameterization of the valuation learning process as in Niculescu and Wu (2014), capturing in a unified framework how the value of a_t (for $t \in \{1, 2\}$) is shaped up by *self-learning via use* and *social learning through WOM*. The two dimensions of this learning model are reproduced below for readers' convenience:

- **Self learning.** We assume that *adopting* consumers (whether paying, seeded, or trying the product) can perfectly learn the product quality through use. As most software products are experience goods, adopting consumers can *directly* update their priors through own hands-on experience, which is not necessarily affected by the opinions of others.

- **Social learning via WOM**. Non-adopters in period 1 (for all models except *TLF*), while deprived from direct, own experience with the product, *indirectly* adjust their priors on quality by learning from the "buzz" (WOM) spread by the period 1 adopters. Formally, we assume that $a_2 = a_1 + N_{1,\text{total}}^{\frac{1}{w}}(a - a_1) = a_1(1 - N_{1,\text{total}}^{\frac{1}{w}}) + aN_{1,\text{total}}^{\frac{1}{w}}$, where $N_{1,\text{total}}$ is the total number of period 1 adopters (which includes both paying

[8] if $0 < \alpha < 1$ then all customers initially underestimate the quality of the product, wheres if $\alpha > 1$ then all customers initially overestimate the quality of the product.

and non-paying adopters, if any)[9] and w is the strength of the WOM effects (i.e., the degree of persuasiveness of reviews). We refer readers to Niculescu and Wu (2014) for an elaborate discussion of how this WOM-based learning model is anchored to and motivated by the rich research streams on (i) factors that affect the magnitude of the impact of (online) reviews and (ii) the stickiness/inertia of own beliefs and strategies in the presence of additional information suggesting a potential need for course correction.

In a nutshell, the updated prior a_2 at the beginning of period 2 is a weighted average between the older prior a_1 at the beginning of period 1 and the signal a sent by period 1 adopters after they experience the product. The weight of the new signal ($N_{1,\text{total}}^{\frac{1}{w}}$) captures its overall degree of persuasiveness in convincing non-adopters to deviate from their prior beliefs. Part of the story has to do with how many reviews are out there - the more reviews the higher the higher the likelihood that the non-adopting potential customers will pay attention to the message of the reviews.[10] The degree of persuasiveness, w, captures how the content of these reviews has the ability to influence non-adopters in changing their valuation. If there are any non-adopters in period 1, then $N_{1,total} < 1$. As such, $N_{1,\text{total}}^{\frac{1}{w}}$ is increasing in w, spanning the interval $(0, 1)$ as w spans $(0, \infty)$. A very low w means that reviews, even in large numbers, have limited power in convincing non-adopters to deviate from their priors.

[9] Under *CE-PL* and *CE-SUB*, $N_{1,\text{total}}$ represents the total amount of *paying* customers in period 1. Under *S*, $N_{1,\text{total}}$ includes both *paying* and *seeded* period 1 consumers. Under *TLF*, in the absence of adoption costs, $N_{1,\text{total}} = 1$.

[10] In a more general context of combining priors with outside signals, Bates and Granger (1969) show that the minimum variance unbiased estimator for the updated forecast is a weighted average of the prior and the outside signals. Building on that, Zhang et al. (2018) further explain how the resulting weight of the prior in the updated forecast is decreasing in the number of available outside reviews. Our social learning model, albeit in a more reduced heuristical form, remains true to the essence of the aforementioned theories.

A high w, on the other hand, means that it takes only a handful of reviews for the non-adopters to adjust from a_1 to a value very close to the real quality factor a. In scenarios in which all consumers either paid for or got free access to the product in period 1 (which is the case under TLF model), social learning becomes redundant but remains mathematically consistent with self learning.[11]

Table 2.1: Consumer perceived quality factor across business models

	Before release	Beginning of period 1	Beginning of period 2
CE-PL ***CE-SUB***	All consumers: $a_0 = \alpha a$	All consumers: $a_1 = \alpha a$	Installed base at the end of period 1: $a_2 = a$ All other consumers: $a_2 = a_1 + N_{1,total}^{\frac{1}{w}}(a - a_1)$
S	All consumers: $a_0 = \alpha a$	Seeded consumers: $a_1 = a$ All other consumers: $a_1 = \alpha a$	Installed base at the end of period 1: $a_2 = a$ All other consumers: $a_2 = a_1 + N_{1,total}^{\frac{1}{w}}(a - a_1)$
TLF	All consumers: $a_0 = \alpha a$	All consumers: $a_1 = a$	All consumers: $a_2 = a$

We summarize in Table 2.1 how the two learning mechanisms impact the updating of consumer priors under each of the candidate models for both adopters and non-adopters. Consistent with Niculescu and Wu (2014), we make several additional assumptions. First, we further assume that, while each customer knows her own type, the distribution of θ is not publicly known among consumers, such that they cannot infer the true quality a based on the firm's optimal pricing p. On the other hand, we assume the firm knows

[11] In such a case, $N_{1,total} = 1$ and $a_1 = a$ (via self-learning). As such, regardless of the value of $w > 0$, $a_2 = a + 1 \times (a - a) = a = a_1$. Once customers learn the true quality of the product through self learning, any subsequent WOM effects have no further impact on their perception of the product quality.

the consumer type distribution but does not have information on the precise type of each individual customer and cannot price discriminate. Next, we assume a form of bounded rationality in that consumers in period 1 do not anticipate a change in their priors at a later time (they operate under the belief that their prior is the correct value of quality, especially since they do not know the distribution of θ and they cannot anticipate various scenarios of how demand will be realized). Last but not least, we assume WOM effects take longer to manifest compared to self-learning effects - the former subsumes the latter as an initial stage in that adopters in period one first have to learn themselves the quality of the product, then disseminate information about it, and then, non-adopters need time to internalize that WOM information. In the context of a two-period horizon, for simplicity, we assume WOM effects take a period to apply whereas self-learning via direct experience is instantaneous. We summarize our key notation in Table A.1 in the Appendix.

Without any loss of generality, throughout this manuscript we normalize the true quality factor to $a = 1$. Moreover, the main results are derived under moderate strength of WOM effects ($w = 1$). We relax this assumption and explore numerically in Sections 2.6.3 and 2.6.4 how the results hold under varying strengths of WOM effects.

2.3.3 Baseline Setup - Dominant Strategy

The individual solutions for each of the four strategies under the baseline setup are presented in Appendix B, in Propositions 5-8. We point out that the optimal pricing solutions under *CE-PL* and *S* (reproduced in Propositions 5 and 8) are carried over directly from Niculescu and Wu (2014). Note that *CE-PL* model is essentially a special case of model *S* with the seeding ratio k set to zero. For clarity of exposition, for a given parameter set, we will consider *CE-PL* dominating *S* if the profit optimization under model *S* yields $k^* = 0$. In the result below, we present the dominant strategy, when comparing among the considered four models, for each region of the parameter space.

Proposition 1. *Under the baseline setup, there exists* $\bar{\alpha} \in \left(0, \frac{1}{2}\right)$ [12] *such that the firm's dominant strategy is:*

(i) TLF, if $\alpha \in (0, \bar{\alpha})$*;*

(ii) CE-SUB, if $\alpha \in [\bar{\alpha}, 1)$*;*

(iii) CE-PL, if $\alpha \geq 1$.

In addition, TLF yields the largest social welfare among all models.

Figure 2.1 illustrates firm's optimal price and ensuing adoption pattern and profit under each of the four strategies. An immediate implication from Proposition 1 is that seeding a non-negligible mass of consumers upfront (model *S* with $k^* > 0$[13]) is always a dominated strategy in any region of the parameter space. Niculescu and Wu (2014) discuss in detail the mechanics of how S dominates *CE-PL* under low priors and a similar argument applies to the dominance of S over *CE-SUB* in the same region. In essence, when consumer priors on quality are low, in the absence of a free offering that would facilitate self and social learning, the firm would have to rely on a very low price in order to jumpstart adoption, thus taking in only a small profit. In contrast, in that same region, under S, WOM effects do not need to be triggered by paid adoption. Instead, the firm can forfeit period 1 paid

[12] $\bar{\alpha} \approx 0.3968$ is defined in implicit form the proof of Proposition 1.

[13] When optimizing firm strategy under S *in isolation*, $k^* > 0$ is optimal only when consumers initially severely underestimate the quality of the product ($\alpha \in (0, \alpha_s)$ where $\alpha_s \approx 0.065$ is defined in Proposition 8). As can be seen from Propositions 5, 6, and 8, we have $\lim_{\alpha \downarrow 0} \pi^*_{CE-PL} = \lim_{\alpha \downarrow 0} \pi^*_{CE-SUB} = 0 < \frac{1}{16} = \lim_{\alpha \downarrow 0} \pi^*_S$.

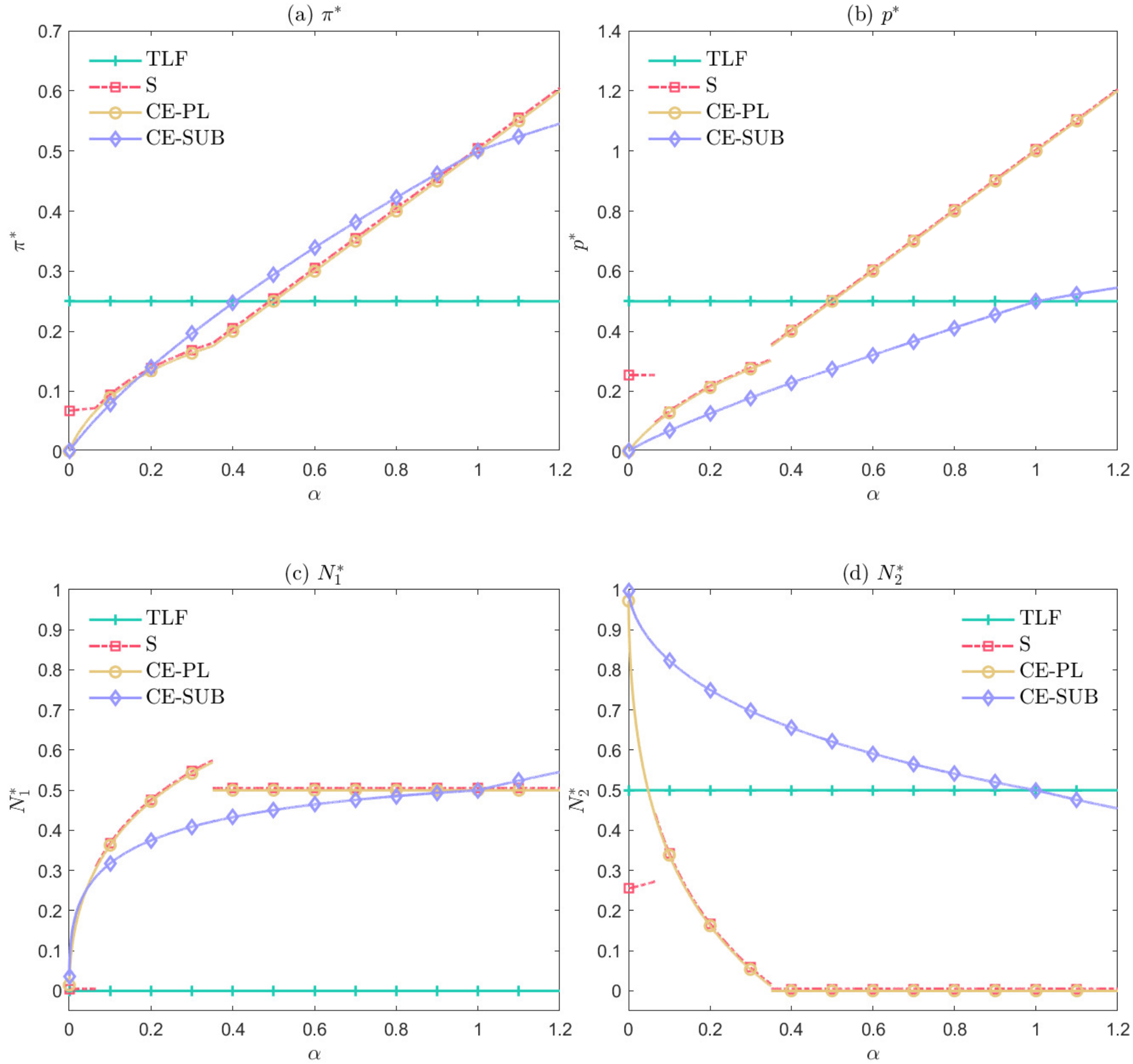


Figure 2.1: Baseline scenario - optimal price, profit, and associated adoption under each model.

adoption altogether, and instead use seeded customers to induce social learning that can achieve a considerable update of the priors of unseeded customers (to a point where a considerable number of the latter are willing to pay a significant price in period 2 for a *single* remaining period of use even though they balked at paying the very same price for two periods of use at the beginning of period 1).

However, in contrast to Niculescu and Wu (2014), what we find is that when TLF

enters the picture, that region of optimality for S evaporates. Under optimal implementation of S and, by definition, under TLF, for low enough priors, the firm does not get any paid adoption in period 1. What ultimately decides the winner between these two strategies is the revenue in period 2. Under S, the seeded customers steer other customers in the direction of the right value of the quality factor. Nevertheless, as seeded customers get perpetual licenses and seeding is uniform, the firm cannot afford loosing too many of the high type customers that could be payers in period 2 (under updated priors) - yet, the firm has no choice but to seed a fraction of that higher valuation population (because seeding is uniform). As such, unseeded customers do not update their prior all the way to the correct value of the quality (one would need $k = 1$ for that, which would essentially erase all profit) and also the firm can no longer get revenue in period 2 from some of the high type customers. On the other hand, under TLF, all customers update their priors upwards to the correct value of the quality factor during period 1 trial. Moreover, the trial version is not offered under perpetual license - all customers remain in the pool of potential adopters in period 2. As such, the firm can collect revenue in period 2 from more high type customers under TLF compared to S. Hence, TLF ends up dominating S in that region.

The dominance of TLF extends well beyond α_S all the way to $\bar{\alpha} \sim 0.41$. For $\alpha > \alpha_s$, S defaults to *CE-PL* (it is not optimal to seed any consumers with a perpetual license). While both *CE-PL* and *CE-SUB* strategies become progressively more profitable with higher α, under each of these models the firm is still considerably constrained by the priors and cannot price too high upfront. While both of these models rely gradually less and less on *new* period 2 adopters (which have not adopted in period 1) as consumer priors increase, the firm gives up on this component of revenue considerably faster under *CE-PL* than under

CE-SUB, as can be seen in panel (d) of Figure 2.1. Under *CE-PL*, social learning would have to more than double the priors for a period 1 non-adopter to even consider period 2 adoption. As α increases, under *CE-PL*, for the firm to ensure enough WOM thrust for period 2 adoption to occur, it would have to induce enough period 1 adoption, which would put downward pressure on the price it charges. On the other hand, as α increases, under *CE-SUB*, the firm continues to make use of *both* self and social learning as long as $\alpha < 1$ ($N^*_{2,CE-SUB} > N^*_{1,CE-SUB}$ for all $\alpha \in (0, 1)$). This added flexibility allows *CE-SUB* to overtake *CE-PL* when $\alpha \in (\alpha^\dagger, 1)$ with $\alpha^\dagger \sim 0.17$.[14]

It is this same flexibility that eventually enables *CE-SUB* to flip the tables and dominate TLF once customers only moderately or slightly underestimate the initial value of the product ($\alpha \in [\bar{\alpha}, 1)$). Under TLF it is optimal to have precisely half of the population paying for adoption for one period. As α increases in this region, under *CE-SUB*, we see from panels (c) and (d) of Figure 2.1 that the firm will optimally induce a little less than half of the population to pay for adoption in period 1 and a little more than half of the population to pay for adoption in period 2 (with $N^*_{1,CE-SUB} + N^*_{2,CE-SUB} > 1$ for all $\alpha \in [\bar{\alpha}, 1)$). With higher priors, the firm is able to charge a high enough per period subscription price (for most of this region we have $p^*_{CE-SUB} \geq p^*_{TLF}/2$), which ensures that from two periods it will collect more revenue than under TLF.

If consumers initially overestimate the product ($\alpha \geq 1$), then *CE-PL* strategy has the upper hand as it relies only on period 1 adoption, charging consumers for two periods *before* they get a chance to update their priors through learning. On the other hand, both *TLF* and *CE-SUB* rely on consumer valuation learning, which (under either self learning

[14] $\alpha^\dagger$ is defined in implicit form in the Appendix, in the proof of Proposition C.2.

or social learning) in this case, leads to a downward calibration of priors and, implicitly of the consumer's willingness to pay.[15]

In terms of social welfare, TLF dominates. Since price represents an internal transfer, social welfare amounts to consumer realized benefits from using the service. Under TLF, all customers get to use the product in period 1, and the top half of them (in terms of valuation) pay for it also in period 2. None of the other models achieve an aggregate product use similar to TLF. What this translates to is that the firm will choose a socially optimal strategy only when consumer priors are low (i.e., $\alpha \in (0, \bar{\alpha})$).

2.4 Individual Depreciation

A central finding from Proposition 1 above is that, S, as a way to leverage free offerings to incentivize paid adoptions, is dominated by other strategies. In particular, we show that TLF, as an alternative "free" business model, is more profitable than S when $k^* > 0$, which is partly due to the one-way adjustment of consumer valuation – all consumers' initial priors are underestimations. However, even if consumers' priors are going up, there can be loss of value from period 1 to period 2, which is not captured in the baseline model. One common instance of loss of value is value depreciation through use, also referred to as individual depreciation (Dou et al. 2017).

The presence of individual depreciation can originate from multiple sources, including consumer satiation, diminishing frequency of usage, or simply when the consumer's interest is diverted to other alternatives. The individual depreciation occurs widely in the

[15]While under *CE-SUB* the firm gets some paid adoption in period 1 (from selling one-period subscriptions), all those subscribers will downgrade their priors via self-learning and also WOM will push downwards he priors of the non-adopters in period 1. Hence, under *CE-SUB* the firm would experience churn in period 2, without a single new adopter.

intertemporal use of information goods and services, as demonstrated by Han et al. (2016). They found that, among mobile applications, the satiation level is highest for the portal search tools (users access these apps quickly and briefly), and the lowest for communication apps (users tend to continue communicating via mobile apps without growing tired of them). Additionally, it has been documented that, users tend to lose interest in many installed apps relatively quickly and the retention rates drop to single-digit percentages for most apps after only one month (Statista 2021a). In somewhat related context, in the case of video games, Shiller (2013) reports that consumers may tire quickly of playing, with their valuation decays from \$80 in the first month of use to just a couple of dollars by the 6^{th} month. The extant literature has documented that the presence of consumer-side value depreciation (Ishihara and Ching 2019) and investigates adjustments to the business model design that incorporate this effect (Dou et al. 2017).

To capture this effect, we propose an adjustment to our baseline model. More specifically, for period 1 adopters, the period 2 valuation scales downwards by a factor $\lambda \in (0, 1)$. We initially explore a scenario with an exogenous depreciation rate (in section 2.4.1). We then expand this analysis by endogenizing the depreciation rate (in section 2.4.2), to capture efforts by the firm to strategically adjust the valuation depreciation through new features/content. As an example of how to intervene and endogenize individual depreciation rate, a video-streaming platform can keep expanding the catalog of the titles to retain the existing adopters. Game developers can inject new value into their products through the production of DLCs (Dey et al. 2013).

2.4.1 Exogenous Individual Depreciation

Table 2.2 demonstrates our approach to formalize the individual depreciation. The impacts of the individual depreciation are captured in the following way: On one hand, the adopters' period 1 consumer valuation drops proportionally by $1-\lambda$ (thus, the residual valuation in period 2 is a fraction λ of the period 1 valuation). On the other hand, the valuation of period 1 non-adopters is not affected by depreciation (and can only be impacted by WOM effects). Nevertheless, we can fully characterize the equilibrium outcome even though the closed-form solutions are intractable in most cases.

Table 2.2: Consumer perceived utility with the individual depreciation

	Before release	Beginning of period 2
CE-PL	All consumers: $u_0 = a_0\theta(1+\lambda) - p$	Non-installed base at the end of period 1: $u_2 = a_2\theta - p$
CE-SUB	All consumers: $u_0 = a_0\theta - p$	Installed base at the end of period 1: $u_2 = a_2\theta\lambda - p$ All other consumers: $u_2 = a_2\theta - p$
S	Unseeded consumers: $u_0 = a_0\theta(1+\lambda) - p$	Non-installed base at the end of period 1 $u_2 = a_2\theta - p$
TLF	All consumers: $u_0 = a_0\theta$	All consumers: $u_2 = a_2\theta\lambda - p$

Proposition 2. *In the presence of individual depreciation, the firm's dominant strategy is:*

(i) CE-PL, if $0 < \alpha_1(\lambda) < \alpha$*;*

(ii) TLF, $\lambda_t < \lambda \leq 1$ *and* $0 < \alpha < \alpha_2(\lambda) < 1$*;*

(iii) Otherwise,

(a) CE-SUB, if $\alpha > \alpha_t$ *and* $\lambda > \lambda_x(\alpha)$*;*

(b) S, otherwise.

Functions $\alpha_1(\lambda)$, $\alpha_2(\lambda)$, $\lambda_x(\alpha)$, and exogenous thresholds α_t and λ_t are defined in the Appendix. On the social welfare, TLF yields the highest among all models.

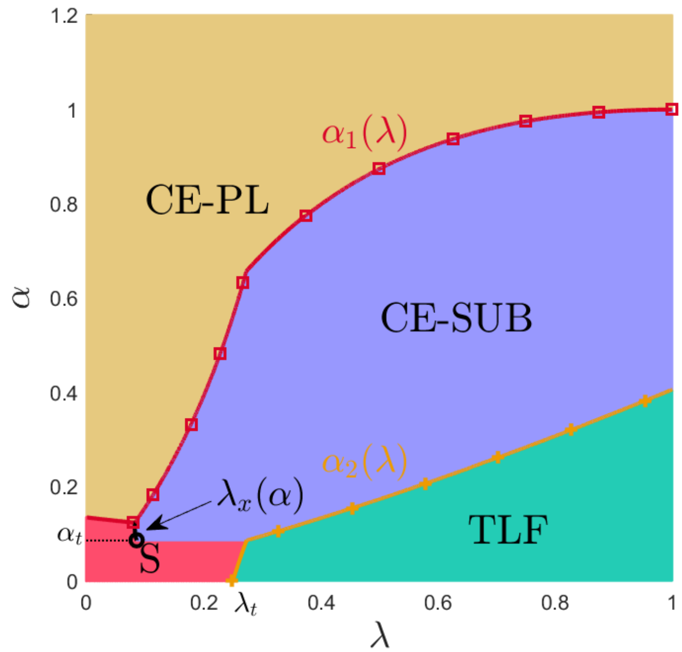


Figure 2.2: Individual Depreciation Scenario - Optimal Business Model

Proposition 2 is illustrated in Figure 2.2. Comparing Propositions 1 and 2, we see that individual depreciation plays a non-trivial role in determining which strategy is dominant in various regions of the parameter space. In contrast to Proposition 1, a major difference in Proposition 2 is that *S* shows up (the red region in the left-bottom corner) where both the individual depreciation and the prior underestimation are severe.[16] On one hand, with strong individual depreciation (i.e., a small λ), *TLF* is no longer profitable because free-trial users have little residual valuation in period 2, and hence little willingness to pay in

[16]Note that a smaller λ indicates a smaller period-2 residual valuation of period-1 adopters. Thus, the individual depreciation is severe when λ is close to zero.

period 2. On the other hand, even under severe individual depreciation (low λ), if the initial valuation priors are not too low (large α), *CE-PL* still outperforms *S* because the WoM generated from the seeded consumers are not important if the initial estimation is not too biased (i.e., a large α). While neither of these one-on-one strategy hierarchies are surprising, it is considerably less intuitive and more complex to evaluate how *CE-SUB* is impacted by depreciation and how that impacts its ability to dominate other strategies in various regions of the parameter space. In particular, in Figure 2.2, the blue region represents the optimality region for *CE-SUB*.

We further illustrate in Figure 2.3, for $\lambda = 0.26$, the sensitivity of price, demand and profit w.r.t. alpha. First, as α increases horizontally, the price and demand under *CE-PL* can be piece-wise increasing. The firm considers two strategies: (1) make profits solely from period 1 adoption (i.e., price such that there are no period 2 adopters), or (2) capitalize on adoption in both periods. The latter takes advantage of WOM, whereas the former does not. It also leads to the non-smooth boundary between *CE-PL* and *CE-SUB* ($\alpha_1(\lambda)$ in Figure 2.2): When α is relatively small, strategy (1) under *CE-PL* and *CE-SUB* determine the boundary between two models. When α is relatively large, strategy (2) under *CE-PL* and *CE-SUB* determine the the boundary between two models. Second and perhaps more interestingly, in Figure 2.3, *S* and *TLF*, as strategies with free offerings, work differently from each other. α is simply irrelevant to *TLF* with or without depreciation. All consumers learn the true value of the product via the free trial. By contrast, for *S* to dominate, it is important that it induces paid adoption solely in period 2, as shown in panel (d) of Figure 2.3. Specifically, our results suggest that the cutoff between *S* and *CE-SUB* turns out to be complex with two possible boundaries. The first one, α_t, between certain λ values ($\lambda > \alpha$),

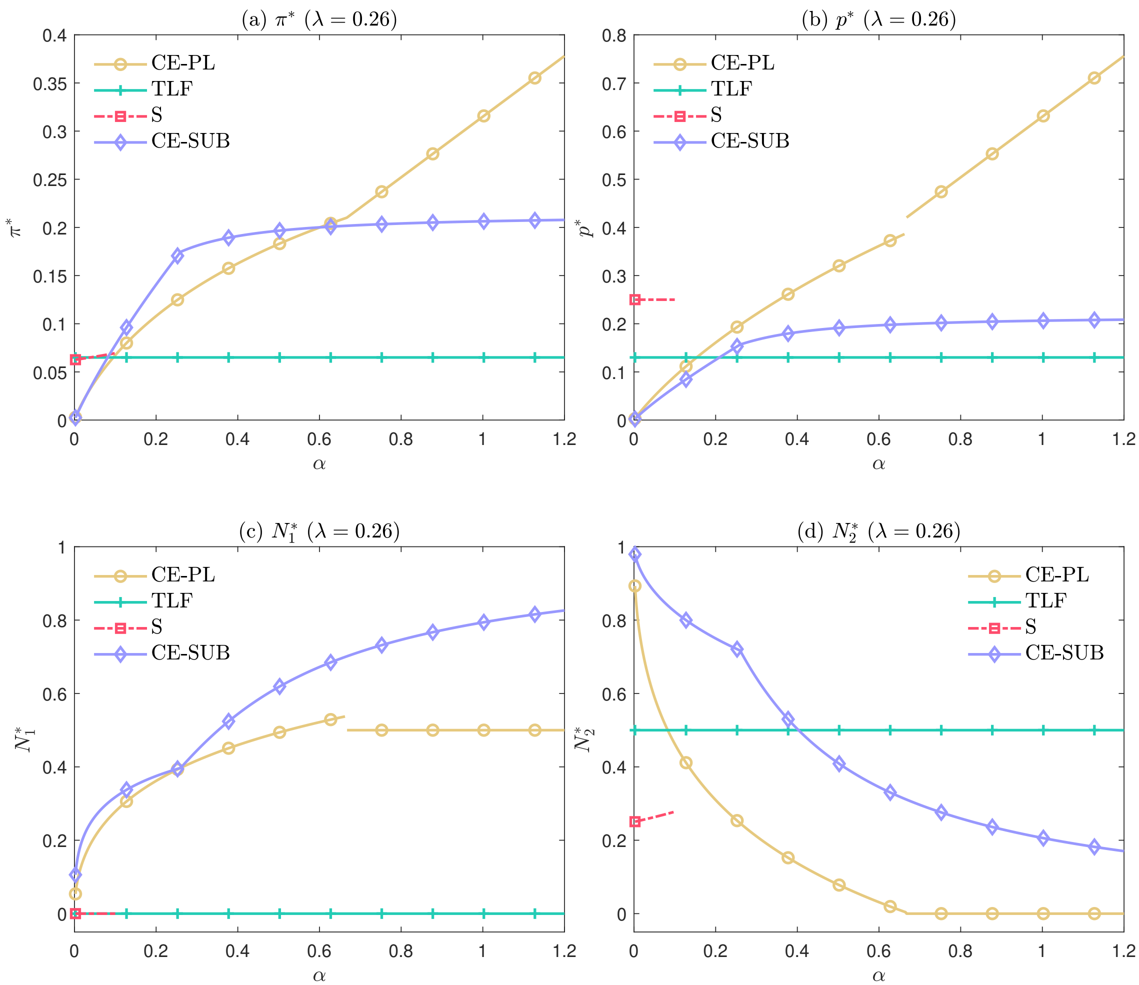


Figure 2.3: Optimal price and profit under each model with depreciation ($\lambda = 0.26$)

is a constant. This is because when depreciation is less severe than the underestimation in priors (i.e., $\lambda > \alpha$), all period 1 subscribers will renew their subscription in period 2 since the use-based update (increase) of their priors offsets the loss of value due to depreciation. Thus, a change in λ does not affect the number of subscribers in both periods. λ does not affect either *SUB* or *S* in this case. The second one, λ_x, exists for α low enough but not too low. When individual depreciation is strong enough (small λ), there is significant subscriber churn in period 2 under CE-SUB. Hence *CE-SUB* becomes less profitable when both λ and α are smaller, and eventually loses to *S* in the left bottom corner of Figure 2.2. Our findings also reveal that both the boundary between *CE-SUB* and *CE-PL* ($\alpha_1(\lambda)$) and the boundary between *CE-SUB* and *TLF* ($\alpha_2(\lambda)$) exhibit an increasing trend with respect to λ. The former is because, as we mentioned in section 2.3.3, under *CE-SUB*, the firm has more flexibility in pricing since it can leverage both self and social learning. However, as individual depreciation becomes more severe, this flexibility diminishes. The heightened depreciation leads to a higher churn rate among period 1 subscribers, consequently exerting more pressure on pricing. The latter is due to the fact that as individual depreciation lessens, *TLF* becomes more profitable due to an increased willingness to pay. However, the profit under *CE-SUB* does not change w.r.t λ (since $\alpha_2(\lambda)$ falls into the region $\lambda > \alpha$). Thus, as λ increases, *TLF* gradually gains advantage and eventually ends up dominating *CE-SUB* for low to moderate α.

2.4.2 Endogenous Individual Depreciation

Figure 2.2 offers an interesting implication that a firm's strategy choice depends on the degree of depreciation λ. A very small λ generally favors *CE-PL* and *S*, while *CE-SUB* and *TLF* can also emerge as optimal if λ is large. In this subsection, we push our analysis one step further to endogenize λ at a cost – What would the firm do if it could invest in altering λ, and more importantly, which strategy would it choose under the optimal λ? This question is relevant in the context of products or services that rely on the consumption of content, and for which consumers exhibit a certain propensity for freshness / newness. Included in this category are video streaming services (e.g., HBO Max, Apple TV+, Hulu+, Netflix), video games (which can get new downloadable content, i.e., DLCs), among others. Investments in content rejuvenation help the firm alleviate the players' fatigue and satiation over time. In our model, we consider a quadratic content rejuvenation cost $r\lambda^2$ with $r > 0$. Convex content production costs are commonplace in the literature, conveying increasing difficulty in adding novelty to an existing product or service catalog (e.g., Dou et al. 2013).

Endogenizing lambda further increases the complexity of the problem. For illustration purposes, we numerically derive the optimal strategy under different levels of cost factor r. When r is small, Indicating the ability to maintain a high λ (low depreciation) through relatively cheap content investments, the analysis is similar to the right-hand side of Figure 2.2 where λ is large. More interestingly, we focus on the non-trivial case in which r is relatively large. In this case, the firm faces a more difficult trade-off: incur high costs of content rejuvenation to maintain a low depreciation, or allow for more depreciation by investing less in content. We visualize the findings in Figure 2.4.

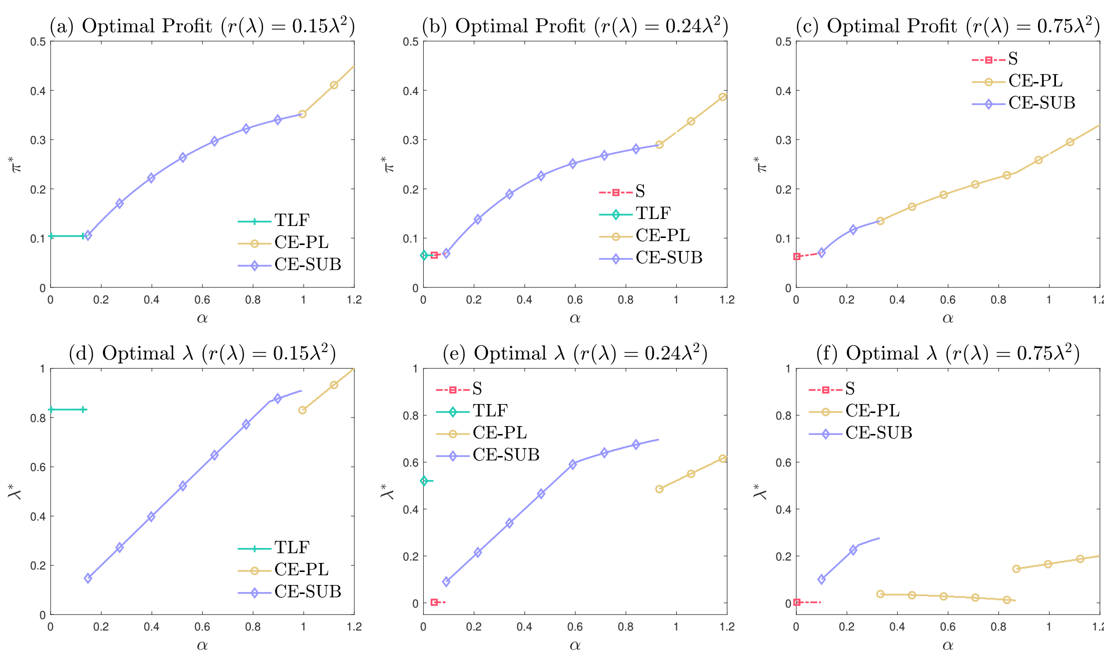


Figure 2.4: Optimal Profit and Depreciation Rate

It can be seen from panels (b) and (e) of Figure 2.4 that all four business models can emerge to be optimal – in the order *S*, *TLF*, *CE-PL*, and *CE-SUB*, as α increases, which is is consistent with Proposition 2. A somewhat surprising finding is that, when the cost of content rejuvenation is high, the optimal depreciation rate λ might be piecewise increasing for low and high α but decreasing for intermediate α. In other words, the firm might prefer investing less in λ when the underestimation is less severe, this is because the profit gain from getting more consumers by alternating a larger λ is offset by the cost of investing in λ. Thus, the firm prefers to choose a smaller λ as α increases. However, when the underestimation is severe (when α is close to 0), it is crucial to maintain a relatively high λ and once the firm completely gives up the period 2 profits (when α is close to 1 or even larger than 1), the optimal λ increases in α again.

Another interesting observation is that, for business models with free offerings (i.e., *TLF* and *S*), optimal λ values differ from each other significantly. Under *TLF*, the optimal λ is not affected by α because consumers update their priors solely via self-learning in free trials, but, as expected, it is impacted by r. In contrast, when S is optimal, paid adoption happens only in period 2 and, as such, depreciation plays no role in adoption. Interestingly, under *CE-SUB*, there is non-smooth point t the 45 degree line (see Figures 2.4 (d) to 2.4 (f)). This is because all subscribing consumers in period 1 will resume the subscription when $\lambda \geq \alpha$. Otherwise, consumers between $\theta \in [p/\alpha, p/\lambda]$ quit subscription in period 2. As a result, it is always optimal to set $\lambda = \alpha$. When α is large, maintaining $\lambda^* = \alpha$ becomes too costly, such that the firm chooses a λ^* which is slightly smaller than α.

To summarize, in the presence of endogenous individual depreciation, our implications on the optimality of *S* remain consistent. Further exploration suggests that the firm's strategy on individual depreciation is non-trivial, potentially exhibiting significant jumps and swifts in monotonicity as the firm selects the best model to use.

2.5 Adoption Costs

In this section, we revisit our baseline model but open up another complexity dimension by considering adoption costs for first-time users. On the user side, adoption costs are growing quickly among information goods, as software installation and configuration become increasingly sophisticated and time-consuming (e.g., a complete installation of Matlab R2022a comes with a storage requirement of 31.5 gigabytes, more than 5 times that for Matlab 2016b). This section shows that the dominant business model varies as the adoption costs become more significant. Unlike in the case of depreciation, adoption costs will impact every model, regardless of whether it relies on period 1 and/or period 2 paid adoption. However, as it turns out, when α is low, an increase in adoption costs impacts *TLF* more severely than *S*, allowing the latter to become optimal in a certain region of the parameter space.

To formalize, we incorporate the adoption cost, c, as a disutility parameter in the consumer's utility function at the time of the adoption. The updating process for consumer priors in the presence of adoption costs is summarized in Table 2.3. Although the optimal strategies are intractable for the most part, we are able to derive Proposition 3 (below).

Proposition 3. *In the presence of adoption costs, the firm's dominant strategy is:*

(i) Don't enter the market, if $\alpha \leq \frac{c}{2}$*;*

(ii) Enter the market, if $\alpha > \frac{c}{2}$*:*

Table 2.3: Consumer perceived utility in the presence of adoption costs

	Before release	Beginning of period 2
CE-PL	All consumers: $u_0 = 2a_0\theta - c - p$	Non-adopters in period 1: $u_2 = a_2\theta - c - p$
CE-SUB	All consumers: $u_0 = a_0\theta - c - p$	Adopters in period 1: $u_2 = a_2\theta - p$ Non-adopters in period 1: $u_2 = a_2\theta - c - p$
S	Seeded consumers: $u_0 = 2a_0\theta - c$ Unseeded consumers: $u_1 = 2a_0\theta - c - p$	Non-adopters in period 1: $u_2 = a_2\theta - c - p$
TLF	All consumers: $u_0 = a_0\theta - c$	Adopters in period 1: $u_2 = a_2\theta - p$ Non-adopters in period 1: $u_2 = a_2\theta - c$

(a) CE-SUB, if $\alpha_2(c) < \alpha < \alpha_1(c)$;

(b) TLF, $\alpha_3(c) < \alpha < \alpha_2(c)$;

(c) Otherwise,

i. S, if $\alpha < \alpha^\dagger(c)$ and $c < c^\ddagger(\alpha)$;

ii. CE-PL, otherwise.

Functions $\alpha_1(c)$, $\alpha_2(c)$, $\alpha_3(c)$, $\alpha_\dagger(c)$, and $c^\ddagger(\alpha)$ are defined in the Appendix. In terms of the social welfare, TLF yields the highest among all models.

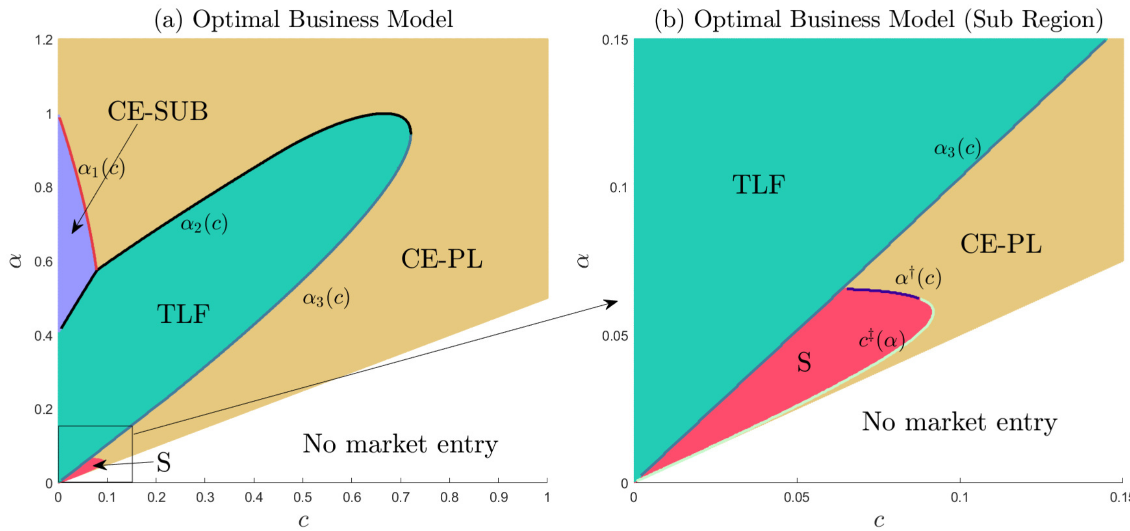


Figure 2.5: Optimal Pricing Model under Adoption Cost

We illustrate Proposition 3 in Figure 2.5. Panel (b) of the figure represents a magnification of the subregion from panel (a) corresponding to low c and low α. When $c \geq 2\alpha$, the is no way to jumpstart adoption under any of the models, including the ones with a free offering. When $alpha > c/2$, a quick takeaway is that, once again, pricing models can emerge as optimal. Starting from the left side where c is close to 0, *TLF*, *CE-SUB*, and *CE-PL* first dominate, which is consistent with the baseline model. Increasing c (moving towards right), *S* emerges as optimal briefly when both α and c are sufficiently small. Under an intermediate c, for a relatively low α (still satisfying $\alpha > c/2$) and high α, *CE-PL* dominates *TLF*, whereas for an intermediate α, the opposite occurs. For a very large c, *CE-PL* dominates everywhere. The switch among business models is due to the trade-off between social learning and period 1 profit: If the adoption cost is low and consumers severely underestimate the value of the product, *S* and *TLF* can effectively build an early user base, which helps push up the period 2 price through social learning. Specifically, in the region where *S* dominates, the firm gives up acquiring paid adopters in period 1 and

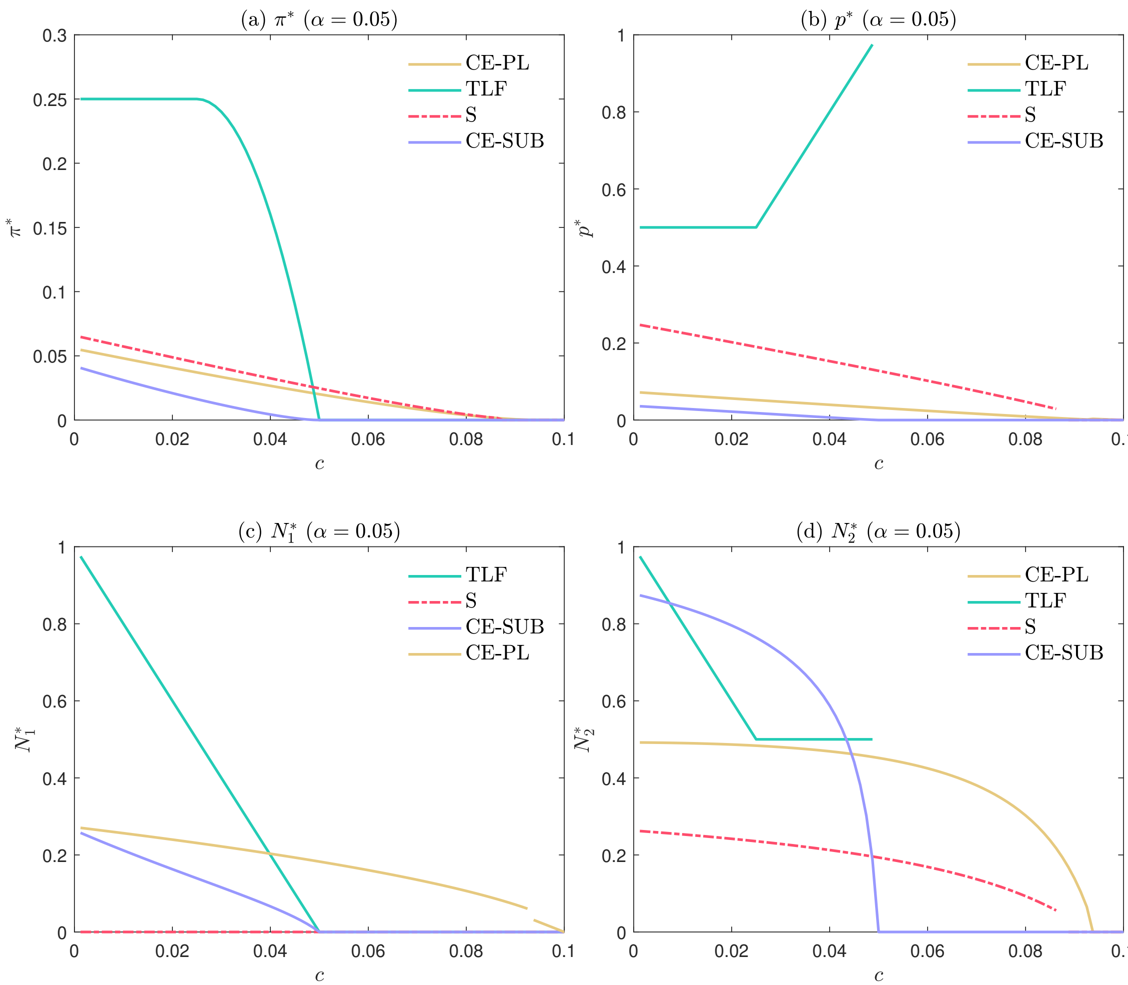


Figure 2.6: Optimal price and profit under each model with adoption costs ($\alpha = 0.05$)

simply provides giveaways instead (i.e., *S*), which pushes up the period 2 price and results in higher profits. In contrast, the market coverage of *TLF* in period 1 is undermined by the adoption cost because consumers might never try the product, even it is free. We further illustrate the dynamic among *TLF*, *S*, *CE-SUB*, and *CE-PL* in Figure 2.6.

First and foremost, for small α, Figure 2.6 illustrates that as the adoption cost (c) rises, the optimal model can transition from *TLF* to *S* and then to *CE-PL*. This occurs because, due to the relatively small value of α (signifying severe underestimation in consumer priors), the *CE-SUB* model is outperformed by the other three models. Customers only take into account the perceived benefits of one period, which are relatively insignificant compared to adoption costs, at the start of period 1.

Moreover, Figure 2.6 highlights a noteworthy distinction between the *S* and *TLF* strategies, both of which involve free offerings. The *TLF* strategy remains unaffected by underestimation but is sensitive to increasing adoption costs, causing its profits to plummet drastically beyond a certian point. In contrast, the *S* strategy exhibits less sensitivity to adoption costs, as demonstrated by the red curves on the left side of each panel in Figure 2.6. This occurs because, under the *S* strategy, seeded customers are more resistant to adoption costs since they receive a lifetime (i.e., 2 periods) license for free. However, in the case of *TLF*, despite being free during period 1, the benefits can be overshadowed by high adoption costs, leading more customers to opt against starting the free trial in the first place.

2.6 Extensions

We extend our model in multiple ways for robustness checks on the possibility of *S* to emerge as an optimal strategy. First, in Section 2.6.1, we generalize the model by considering individual depreciation and adoption costs simultaneously. Second, in Section 2.6.2, we relax the restrictions on the consumer priors by allowing some consumers to be overoptimistic in their priors, whereas others underestimate. We also consider alternative forms in our modeling components, such as the generalized WOM effects and targeted seeding in Sections 2.6.3 and 2.6.4, respectively. Collectively, all 4 extensions confirm that *S* can emerge as the optimal strategy in the presence of individual depreciation and/or adoption costs in a more general context.

2.6.1 Generalized Model

In this section we consider a model that combines exogenous individual depreciation and adoption costs (literally combining the frameworks for the models from Sections 2.4 and 2.5). Due to the high complexity of this model, closed form solutions for optimal strategies are analytically intractable. Instead, we run a numerical exploration over the 3-dimensional parameter space of alpha, c, lambda to identify the optimal strategies. We present in Figure 2.7 several slices of the outcomes under this parameter space, at three distinct and relatively small α values (0.01, 0.05, and 0.1).

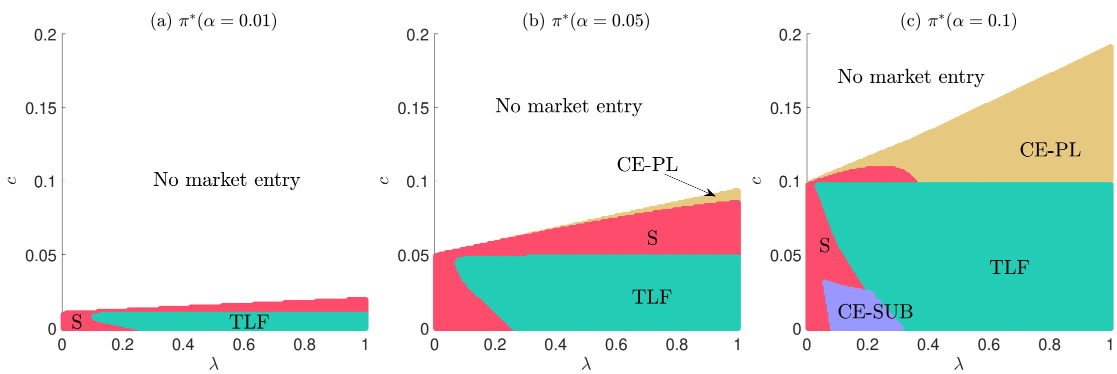


Figure 2.7: Optimal Pricing Model under Adoption Cost and Exogenous Individual Depreciation

The key takeaways from the numerical exploration in Figure 2.7 are three-fold and consistent with out prior findings. First, all four candidate models can emerge to be optimal. Second, the consumer prior (i.e., α) and adoption costs (i.e., c) is the major differentiating factor in determining the decide the dominance of *CE-PL* against others. In particular, when α is small and $c \leq \alpha$ (i.e., the bottom part of Figure 2.7, where consumers underestimate

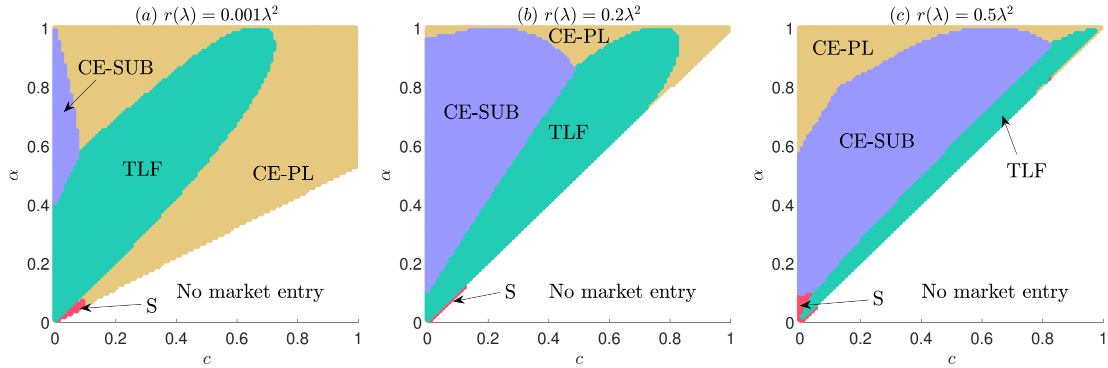


Figure 2.8: Optimal Pricing Model under Adoption Cost and Endogenous Individual Depreciation

the product significantly, and adoption costs are relatively small compared with consumer priors), *CE-PL* is always dominated and never shows up, suggesting that the absence of the free offering undermines the profitability of *CE-PL* under low α (the firm would have to price very low to jump start adoption and induce WOM effects). In contrast, *S* and *TLF* can effectively stimulate adoption in period 1 and charge a premium price in period 2. Third, in the region where α is mildly small but still higher than c, individual depreciation plays a significant role in determining the dominance between *S* and *TLF*, where a low depreciation rate (i.e., a high λ) favors *TLF* and a high depreciation rate (i.e., a low λ) favors *S*.

We also explore a setting of this generalized model with endogenous depreciation rate (directly extending the model in Section 2.4.2 by adding adoption cost to it). For this setup, we illustrate the optimal strategies in Figure 2.8, by presenting 3 different scenarios of content rejuvenation factor r (r = 0.001, r = 0.2, and r = 0.5). As expected, higher content investment costs increase the region of no market entry for the firm (as can be seen moving from panel (a) towards panel (c)). Resiliently, *S* is still the optimal choice at small α and small c in all three panels.

2.6.2 Heterogeneous priors

The baseline model has assumed that all consumers share the same prior factor α, implying that they all either underestimate or overestimate the product quality before period 1. This section relaxes this assumption by allowing for heterogeneity of priors, whereby some customers initially underestimate the value of the product whereas others overestimate it. We consider a Bernoulli distribution where a fraction τ of consumers (denoted by group H) initially overestimate the value of the product at level $a_{H0} = 2 - \alpha$ (with $\alpha \in (0, 1)$) and the other fraction $1 - \tau$ of consumers (denoted by group L) initially underestimate it at level $a_{L0} = \alpha$. Intuitively, when $\tau = 0$ or $\tau = 1$, this setup reduces back to the baseline model in Section 2.3. Again due to the analytical intractability, we explore this setup numerically to identify the optimal strategy, which is shown in Figure 2.9.

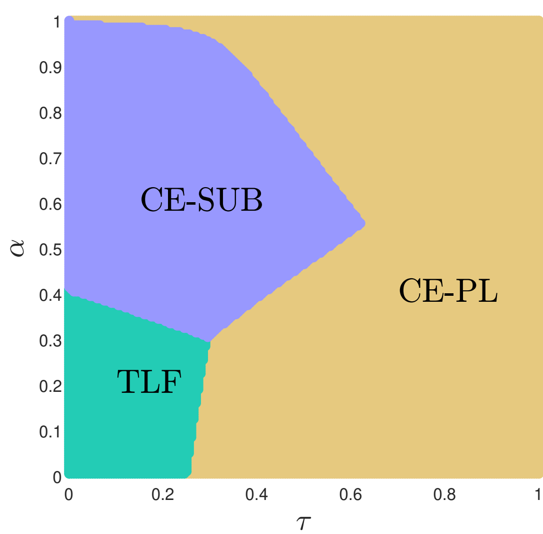


Figure 2.9: Optimal Business Strategy - Baseline Model with Consumer Prior Heterogeneity

In the absence of either individual depreciation or adoption cost, one interesting finding is that *CE-PL* can emerge to be optimal for $\alpha < 1$ (in contrast to Proposition 1), but *S* never shows up, still. It is due to that the profit focus of *S* is period 2, which is less affected by the distribution of the initial prior. In contrast, *CE-PL* can take better advantage of the existence of an overestimating subgroup of consumers by charging a premium price in period 1. Uniform seeding would lead to forfeiting revenue from some of these overestimating consumers in the first round, when they are willing to pay more (for two periods of product use).

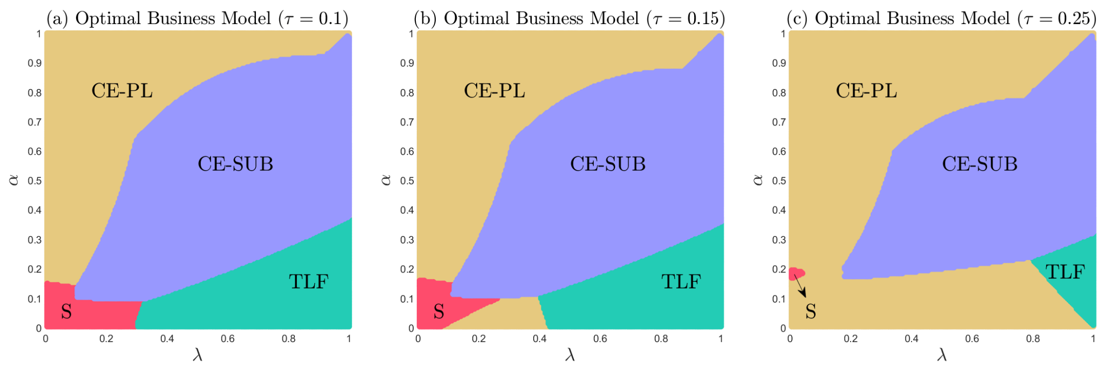


Figure 2.10: Optimal Business Strategy - Consumer Prior Heterogeneity with Depreciation

Adding the individual depreciation further corroborates the previous finding that all four models can emerge as the optimal strategy, as shown in Figure 2.10. With two consumer groups (L and H), what is new is that, as τ increases (i.e., there are more overestimating consumers), *CE-PL* starts to dominate under a small α (panels (b) and (c)). In these panels, under a small α, as the λ increases, the optimal strategy shifts from *S* to *CE-PL* and finally to *TLF*. In the region in which *CE-PL* emerges as the dominating strategy, with an increasing mass of overestimating consumers (in group H), the firm chooses to give up Group L and serves Group H only. *S* and *TLF* emerge as optimal when λ is small and when λ is large, respectively. However, under an intermediate λ, *CE-PL* achieves a greater profit because the advantage of *S* erodes as λ increases.

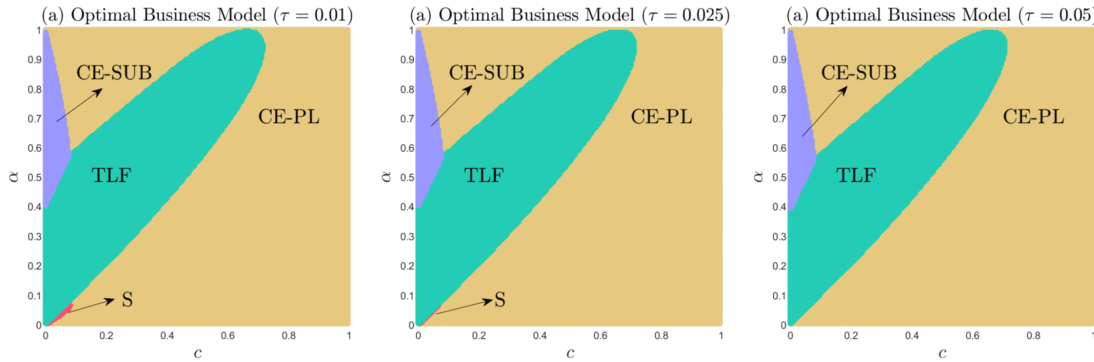


Figure 2.11: Optimal Business Strategy - Consumer Prior Heterogeneity with Adoption Costs

Similarly, we can show that our results with adoption costs from Section 2.5 are robust under the two-consumer-group setting as well. In particular, we compare and contrast outcomes for $\tau \in \{0.01, 0.025, 0.05\}$, which are presented in Figure 2.11. The results remain consistent in nature - *S* emerges as optimal when the majority of consumers underestimate the value of the product significantly and the adoption cost is small. However, different from the previous finding, under the case when $\tau > 0$, i.e., there always exist overestimating consumers, "No market entry" area does not exist when $c < 2(2 - alpha)$, and the corresponding region is dominated by *CE-PL*.

2.6.3 Generalized Social Learning Model

So far we have been working with $w = 1$ in which case the consumers are unlikely to either change their priors dramatically (when w is small) or stick to their priors stubbornly (when w is large when receiving WOM from period 1 adopters). In this extension, we relax this constraint. Consumers can either heavily rely on a limited collection of reviews, particularly when the product is a new launch (i.e., strong WOM effects), or firmly stick to the priors (i.e., weak WOM effects). We re-explore the model under different levels of WOM effects below.

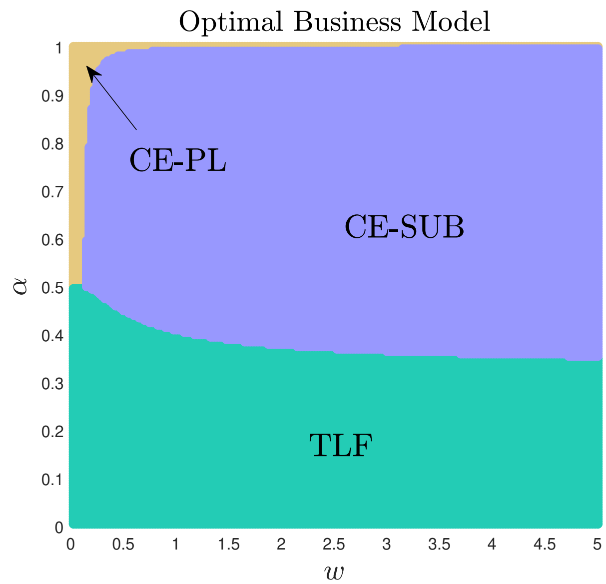


Figure 2.12: Optimal Business Model under No Depreciation and No Adoption Cost

Figure 2.12 presents the numerical results under the benchmark case (i.e.,$\lambda = 1$ and $c = 0$) over $(\alpha, w) \in \{(0,1) \times (0,5)\}$. As w moves from 1 to 0, indicating that consumers are more likely to stick with priors, with a greater α (vertically above), *CE-PL* emerges to be the optimal because its profit comes from period 1 only. Furthermore, it is worth mentioning that *S* is still dominated for all w because w does not affect *TLF*, even if w is large, unseeded consumers can only update to the true product quality, which is same as the perfect self-learning under *TLF*. However, due to the existence of the seeded group, *S* loses some consumers in period 2 comparing with *TLF*, thus, dominated by *TLF*. The results are consistent with Proposition 1. This is because consumers update priors solely based on self-learning under *TLF*.

Our further analysis suggests that the results under the varying w are also consistent

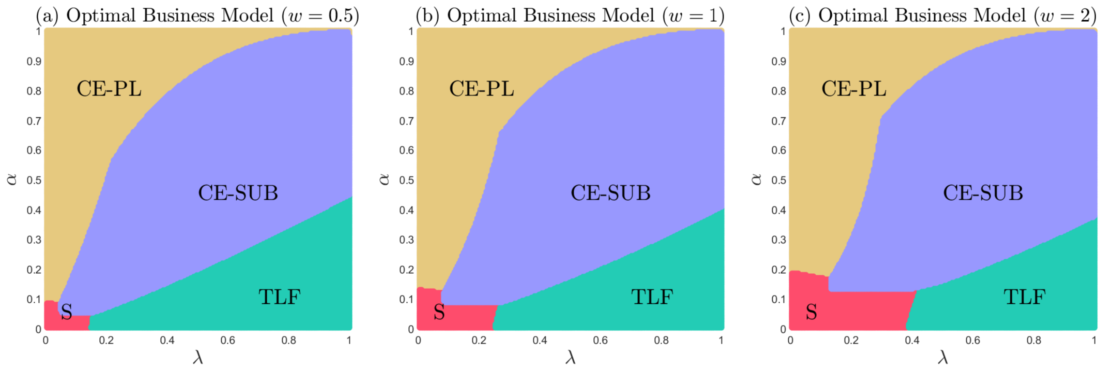


Figure 2.13: Optimal Business Model under Depreciation

with our prior findings when the depreciation and adoption costs are considered. In Figure 2.2, We consider the same setup as but under different levels of WOM effects (0.5, 1, 2). Serving as the benchmark, panel (b) of Figure 2.13 replicates Figure 2.2 with $w = 1$. An immediate observation from Figure 2.13 is that *S* is always optimal in the left-bottom corner where both α and λ are small. In addition, a greater w benefits *S*, expanding its dominating region (in red). This is because even a small group of seeded consumers can generate stronger WOM effects under a greater w. Consequently, a larger seeding ratio is no longer necessary. In contrast, varying w seems to impose less impact among the other three models (*CE-PL*, *CE-SUB*, *TLF*).

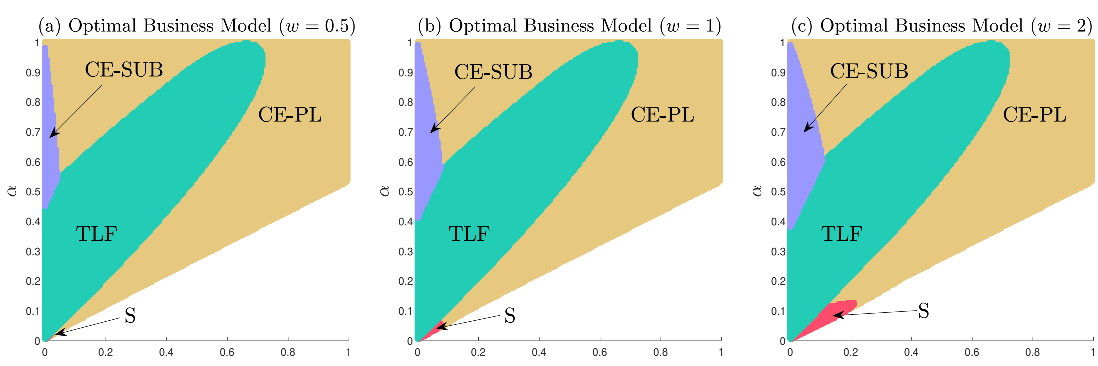


Figure 2.14: Optimal Business Model under Adoption Costs

When varying w, we also find that our earlier results remain robust in the presence of adoption costs (as illustrated in Figure 2.14). This is reflected by the similarity between Figure 2.14 and Figure 2.13. Again a greater w favors *S*, as its optimality region expands when w is larger. Unlike the previous case with individual depreciation, *CE-SUB* also benefits from a greater w when α is large but still below 1. This is because in this region *CE-PL* focuses solely on period 1 profit. In contrast, a greater w helps *CE-SUB* to secure more adopters in period 2. Therefore, a greater w is more helpful for *CE-SUB* when adoption costs are considered.

2.6.4 Targeted Seeding

In this section, we allow the firm to conduct *S* differently through targeting, instead of reaching consumers randomly with uniform seeding. Specifically, the firm is able to access consumer preference information, such that it can seed consumers selectively. In our model framework, it is intuitive to see that the optimal way to do this is to identify the consumers with low valuations and target them as the the seeded consumers.

Interestingly, although the profitability of seeding is clearly strengthened under targeted seeding, Proposition 4 suggests that it is still dominated in the baseline model (i.e.,$\lambda = 1$ and $c = 0$).

Proposition 4. *Under the baseline setup (*$c = 0, \lambda = 1$*), Seeding and Targeted Seeding are two dominated strategies.*

When depreciation presents, we replace *S* with *TS* and illustrate the results in Figure 2.15. It can be seen that *TS* dominates in a larger region as it erodes toward the region where *CE-PL* dominates in Figure 2.2. It is because that *CE-PL* is strictly dominated by *TS* because under *TS* the firm can always obtain better profit by seeding those low-end consumers who would not purchase in period 1 under *CE-PL*. In fact, the region of *TS* drills into the boundary between *TLF* and *CE-SUB*. Comparing the area of the dominant region of *S* and *TS*, we find that there are significant improvements for all levels of w. We use the proportion of the area dominated by *S* and *TS* to capture in increment: When $w = 0.5$, switching from *S* to *TS* leads to the region expands from 0.88% to 30.16%; When $w = 1$, switching from *S* to *TS* leads to the region expands from 2.39% to 36.30%; When $w = 2$, switching from *S* to *TS* leads to the region expands from 5.74% to 41.50%. It indicates that, the WOM effects can maximize the value of targeting capability for the firm.

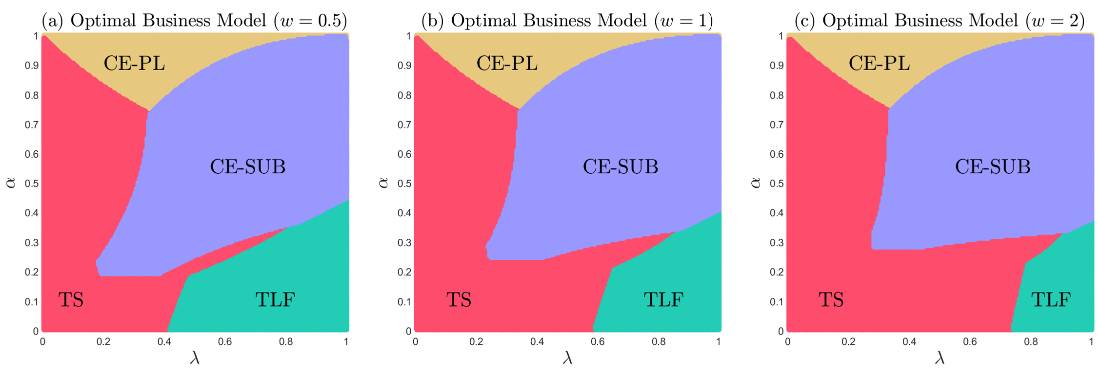


Figure 2.15: Optimal Business Model - Targeted Seeding

Not surprisingly, acquiring consumer information is unlikely to be free of cost. Therefore, it is worth exploring that, if *TS* is optimal, how much additional improvement can be achieved. We depict how the profit (π^*), price (p^*), number of adopters in period 1 (N_1^*), and number of adopters in period 2 (N_2^*) changes in α in Figure 2.16 ($w = 1$, $\lambda = 0.25$) and Figure 2.17 ($w = 1$, $\lambda = 0.75$). We find that under a low λ, *TS* posts a significantly higher price than other models if α is small. However, when both α and λ are large enough, the marginal benefits of adopting *TS* are much less glaring. It implies that firms need to carefully configure their strategies by balancing the costs of acquiring consumer information and the benefit of adopting *TS*.

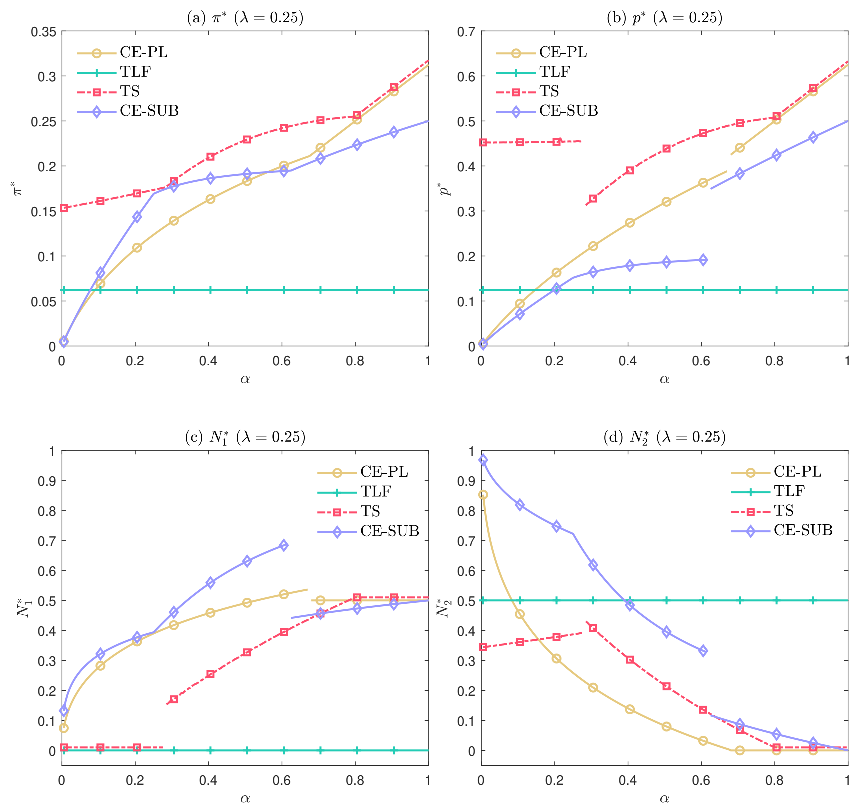


Figure 2.16: Comparative Stats - Targeted Seeding $\lambda = 0.25$

Under the model with adoption costs (illustrated by Figure 2.2), *TS* also expands the optimal region of *S* to some extent, but not as significant as above. When $w = 0.5$, switching from *S* to *TS* leads to the region expands from 0.01% to 0.39%; When $w = 1$, switching from *S* to *TS* leads to the region expands from 0.13% to 2.23%; When $w = 2$, switching from *S* to *TS* leads to the region expands from 0.69% to 5.40%. Due to the existence of adoption costs, *CE-PL* mainly focuses on period 1 and thus stays unaffected by WOM effects. Under this case, *TS* does make a difference because the low-end consumers' concern

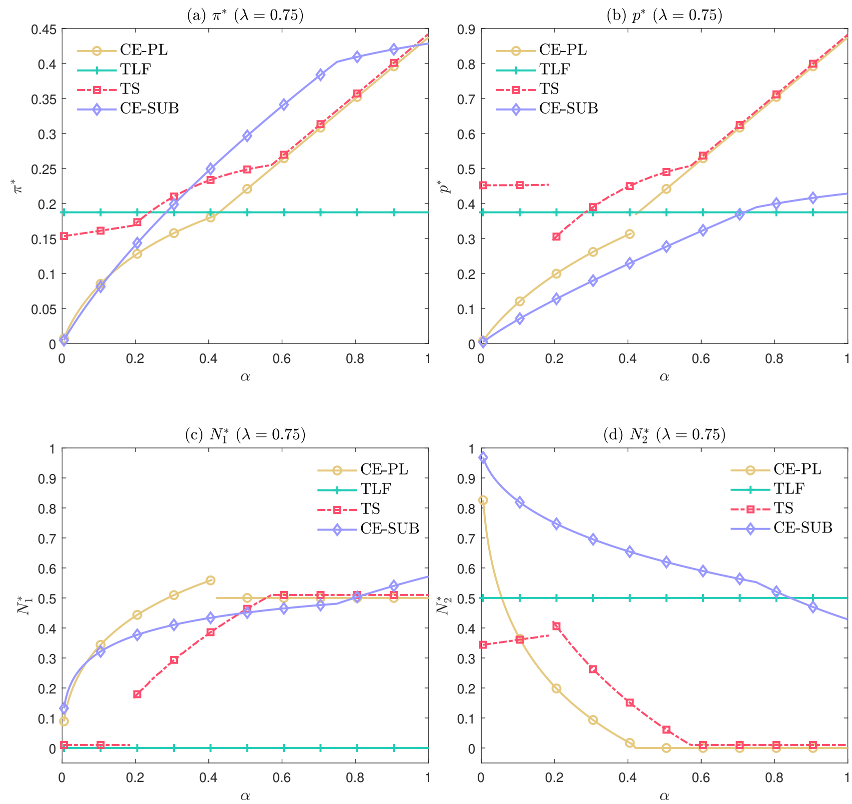


Figure 2.17: Comparative Stats - Targeted Seeding $\lambda = 0.75$

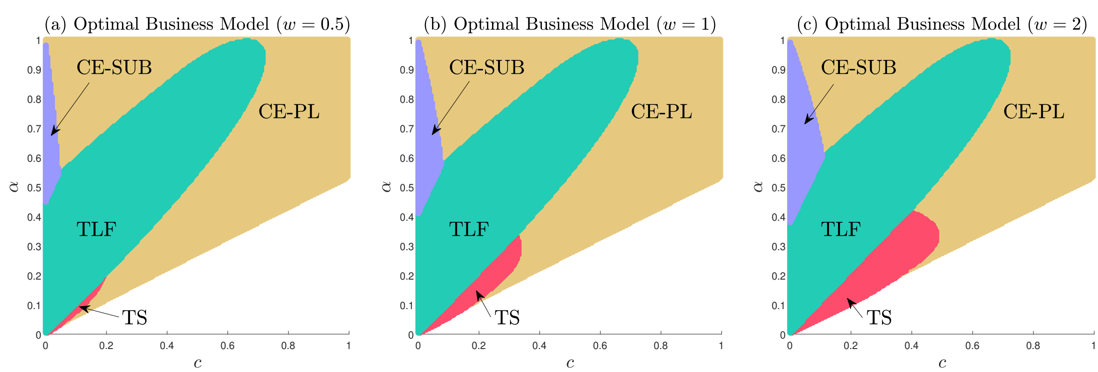


Figure 2.18: Optimal Business Model - Targeted Seeding

about the adoption costs even if it is free. Therefore, *CE-PL* still outperforms other models when both c and α are large enough.

2.7 Conclusions

This paper reexamines the free business models to advance the understanding of how firms can strategically manage consumer learning to improve profitability. As the recent literature and practice overwhelmingly proclaim free trials, we explore in particular whether seeding is still relevant. We identify two major factors that are understudied in the literature – individual depreciation and adoption cost. We show that ignoring those effects might lead to a biased understanding of the optimality of *S*. We also demonstrate that all the candidate strategies can emerge to be optimal in the presence of these two factors. In particular, a large depreciation and an adequately large adoption cost favors *S*, whereas the opposite case favors *TLF*. These findings are robust when we extend the model in different directions such as generalized WOM effects, generalized consumer priors, etc.

Our study offers timely practical implications for software providers. In the context of console or PC games, without releasing additional content (to retain engagement and reduce depreciation), as there is enough dispersion of priors (after all, games are experience goods, and thus it is difficult to know the precise value beforehand), the game developers should consider the seeding strategy. In recent years, indeed, we have witnessed an emergence of the game "streaming" phenomenon, whereby developers will offer the game for free to some players around market-release time (or very shortly before that) with the hope that these players will post their gameplay videos on social media channels and attract a substantial number viewers. On the other hand, if the individual depreciation is minimal, we tend to witness the greater implementation of the *CE-PL*, *TLF*, and *CE-SUB* models being applied (e.g., in the case of productivity software), which is consistent with our findings.

Our paper is also relevant to the platform and the digital content industry. Content providers such as Netflix, Hulu, and YouTube TV also need to invest in adding content regularly to keep the deprecation rate low. Under such cases, *TLF* is still the optimal strategy. Moreover, for professional software such as Matlab, Stata, Microsoft Office, etc, the major barrier that prevents adoption is the high adoption cost. Consumers need to spend time and effort getting familiar with the software. It would be interesting to test it empirically in future research.

CHAPTER 3

AN EMPIRICAL INVESTIGATION ON AMAZON'S SELF-PREFERENCING STRATEGY AND ITS LAUNCH OF PRIVATE-LABEL PRODUCTS

3.1 Introduction

An increasing number of platforms are functioning as both gatekeepers connecting third-party sellers and consumers and as sellers competing with third-party sellers by offering their own products. For instance, Amazon.com, the largest e-commerce platform in the U.S., enters the product space by launching private-label products such as Amazon Basics and Amazon Essentials, which rapidly gain popularity in the market (Fruhlinger 2019).

As the platform's private-label products gain prominence, concerns are raised about the platform potentially abusing its power at the expense of third-party sellers and consumers. For example, Amazon is reported to collect its sellers' data to launch competing products (Mattioli 2020) and rig the search results to promote its own products (Kalra and Stecklow 2021). Many regulators have highlighted the potential conflict of interest as a cause for antitrust concerns. U.S. Senator Elizabeth Warren stated that "Many big tech companies own a marketplace, where buyers and sellers transact, while also participating on the marketplace. This can create a conflict of interest that undermines competition." (Warren 2019). In contrast, Amazon has countered these reports by arguing that its private-label products improve customer experience and provide greater exposure for small businesses (Fung 2022).

Two key questions arise from this debate: Does Amazon employ a self-preferencing strategy? And if so, does the introduction of Amazon's private-label products positively or negatively impact third-party sellers' businesses? We address these questions in this paper. Utilizing data scrapped from Amazon.com, we document Amazon's self-preferencing strategy and examine the effects of the platform introducing its private-label products while engaging in self-preferencing.

In the first part of the paper, we present two pieces of evidence on Amazon's involvement in self-preferencing. We first show the direct evidence of self-preferencing by comparing the rank of Amazon's private-label products in the search results with third-party products. We find that Amazon's products are ranked higher than third-party products, even when accounting for other observable characteristics. The ranking distribution for Amazon's private-label products is also more right-skewed than for well-known third-party brands. The limitation of this direct evidence is that the ranking may reflect some unobserved qualities; it is possible that Amazon's private-label products are ranked higher because they genuinely have higher quality.

To control for unobserved product quality, we then leverage a scenario where Amazon becomes the seller of an existing third-party product (referred to as "Sold by Amazon" [SBA]). In this case, we can control for product quality since the product itself does not change. We find that when Amazon becomes the seller of an existing third-party product, the product immediately experiences a significant increase in sales, even though the product itself remains the same and there are no significant changes in product rating, price, or shipping fee. This result supports the hypothesis that Amazon promotes its own products more through its self-preferencing strategy.

In the second part of the paper, we then study the impacts of Amazon introducing its private-label products while employing the self-preferencing strategy for them. To determine the effects of launching private-label products on third-party products in the same category, we conduct a matched Difference-in-Differences analysis. For each affected category that experienced private-label product launches, we use a similar but unaffected category as the control group. We then compare the business outcomes between the two groups. We find that although Amazon favors its own products, the introduction of private-label products leads to a gradual increase in average sales and ratings of third-party products in the same category, while the average prices remain largely the same. These combined outcomes suggest an overall increase in consumer welfare.

We then conduct several analyses to investigate the underlying mechanisms that could drive the observed changes in the market. First, we find sellers with lower rating exit the platform. This suggests that sellers who provide better quality products and services are more resilient to the increased competition brought about by Amazon's private-label products. As a result, consumers can benefit from improved product quality at the same prices after Amazon introduces its private-label products.

Second, we find that newly launched third-party products become more differentiated from Amazon's offerings in terms of product design (using changes in product images as a proxy for changes in product design). This suggests that private-label launches stimulate innovation and increase variety in the category. Consumers can benefit from this change as they may be more likely to find a product that suits their preferences.

Third, we find that newly launched third-party products' descriptions become more similar to those of Amazon's private-label products. In the search data, we find that product rankings improve with the similarity between the focal product's description and that of the top product. This occurs because search outcomes are heavily influenced by the relevance between the product description and search queries. Since private-label products are more likely to appear at the top of the page, third-party sellers are incentivized to make their product descriptions more similar to private-label products in order to boost their products' searchability. Additionally, a closer examination of the product description texts reveals that private-label products' descriptions are generally more detailed and better structured, allowing customers to quickly become informed about the product features. Consequently, sellers may utilize private-label products' descriptions as guidance to enhance their own descriptions, increasing the likelihood of customers making an informed decision and leading to more similar descriptions.

Our findings demonstrate that although Amazon favors its own products in search, introducing private-labels does not necessarily harm consumers. Instead, it could increase consumer welfare by crowding out low-quality products and spurring product innovations. Furthermore, it can improve search outcomes by encouraging third-party sellers to enhance their product descriptions. These results offer valuable insights into the dynamics between e-commerce platforms and third-party sellers, as well as the potential implications of self-preferencing strategies on the market. Moreover, they contribute to the ongoing debate surrounding platform regulations.

3.2 Literature Review

Our paper contributes to three streams of literature: dual roles of the platform, platform's self-preferencing recommendation strategy, and third-party sellers' defense strategy.

First, we add to an expanding literature on dual roles of the platform. In the theoretical space, earlier works have studied the impact of banning the platform from selling their own products on consumer welfare and found mixed results (Hagiu et al. 2020, Lam and Liu 2021, Anderson and Bedre-Defolie 2021, Kang and Muir 2022, Lai et al. 2022). Other works have explored the types of product spaces the platform should enter, considering the popularity (Jiang et al. 2011), network effects (Hagiu and Spulber 2013, Hagiu and Wright 2015, Gautier et al. 2021), and shipping costs of the products (Etro 2021). In the empirical space, earlier works have studied the impacts of platforms becoming sellers in various contexts. For example, He et al. (2020) find that when a Chinese omnichannel platform started to sell the same packaged goods offered by third-party sellers in offline stores, it decreased offline demand for third-party stores, but had no effect on online demand. In the software market, Li and Agarwal (2017) find that Facebook's integration of Instagram harms small third-party applications but increases the demand for large third-party applications. Edelman and Lai (2016) document the negative effect when Google enters the travel market by promoting its own flight search service.

Specifically, some studies focus on Amazon's dual roles. For example, Gutierrez Gallardo (2021) build a structural model to estimate the welfare consequences of several regulatory interventions. They find that banning Amazon from selling any products can benefit third-party sellers at the expense of lower consumer welfare. Zhu and Liu (2018) document the effect of Amazon selling existing third-party products, which are the most relevant to our work. They find that when Amazon started to sell existing products, it increases product demand, but discourages affected third-party sellers from subsequently growing business on the platform. Our paper finds similar results when studying the impacts of Amazon selling existing products but we enrich the literature by studying the effect when Amazon sells its private-label product, which has become a rising concern in antitrust and platform regulation but lacks empirical evidence.

Second, there is an active stream of literature addressing concerns regarding platforms' self-preferencing recommendation algorithms. Previous theoretical literature has examined how platforms can strategically adopt self-preferencing algorithms to promote their private-label products (Hagiu and Jullien 2011, Zhou and Zou 2021, Long and Amaldoss 2022). These studies have also investigated antitrust agencies' proposals for search neutrality, which would ban platforms' self-preferencing algorithms, and found that it may harm consumers due to weakened price competition (Zou and Zhou 2022). Empirically, researchers have found that Amazon's algorithm for deciding Buy Box sellers favors Amazon's own products but may still improve consumer surplus if it aligns with customer preferences (Lee and Musolff 2021, Lam 2022). Moreover, Farronato et al. (2023) find that Amazon private-label products are ranked higher than third-party products when controlling for observables. The limitation of this evidence is that the difference in rank may arise from differences in product quality, which cannot be directly observed. We address this limitation by studying the scenario in which Amazon becomes the seller of an existing third-party product, allowing us to control for product quality and provide new evidence for this study.

Third, we contribute to a limited body of literature studying third-party sellers' defense strategies in response to platform entry. Using data from a Chinese e-commerce platform, Li et al. (2021) find that when third-party sellers choose to enter a platform's private-label product space, they are more likely to target areas with low prices, high demand, and lower logistics costs. Foerderer et al. (2018) and Wen and Zhu (2019) find that Google's entry into the Google Play Store increases the number of updates for other software developers in the affected category and forces developers to shift efforts to new apps. Our paper expands the literature by examining how third-party sellers respond to Amazon's introduction of private-label products and shed light on the effects on consumer welfare. To our best knowledge, this is the first paper that studies third-party sellers' responses to launches of Amazon brands.

3.3 Background and Data

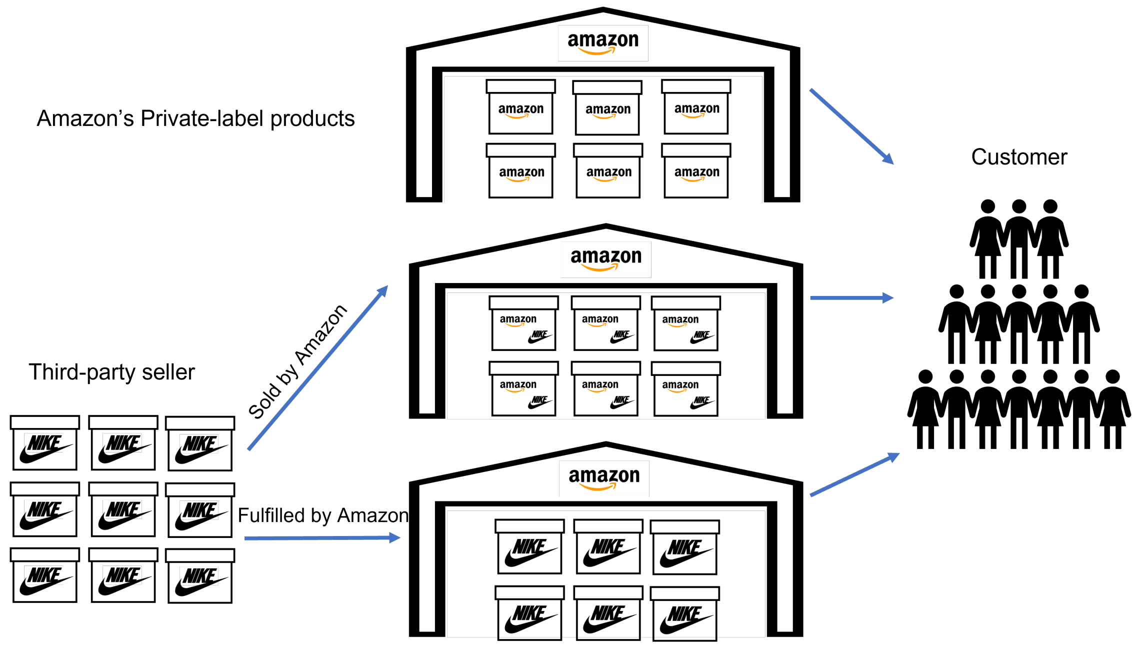
3.3.1 Amazon's Involvement in the Marketplace

In this paper, we focus our analysis on Amazon.com, the leading retail e-commerce platform in the United States.[1] With its extensive range of products, large customer base, and significant market share, Amazon has emerged as a dominant force in the U.S. online shopping landscape. In 2022, the company generated net sales revenue of $315.88 billion, accounting for 37.8% of the retail e-commerce market share in the United States(eMarketer 2022).

Amazon has experienced significant growth and transformation over the years, evolving from a simple online marketplace into a multifaceted platform offering a variety of services and products. While it continues to serve as a platform for third-party sellers, it also competes with them by selling its own private-label products. Figure 3.1 presents a diagram illustrating Amazon's multifaceted involvement (in addition to its role as gatekeeper).

[1] Amazon has different marketplace all over the world, for example, Amazon.co.uk (United Kingdom), Amazon.de (Germany), Amazon.ca (Canada), etc. We focus on the largest marketplace, i.e., Amazon.com (United States).

Figure 3.1: Amazon's involvement



Notes: This figure shows a diagram of Amazon's different involvement on Amazon.com.

Amazon's most direct involvement on Amazon.com is the sale of its private-label products. Over the years, Amazon has introduced its own private-label brands, such as Amazon Basics and Amazon Essentials, which compete with third-party sellers' products on the platform. These private-label products are designed and manufactured by Amazon and are sold exclusively on Amazon.com. When Amazon sells its private-label products, it earns the difference between the product's retail price and the cost of goods sold (COGS), which includes manufacturing costs, shipping, handling, and other operational expenses.

Besides Amazon's private-label products (PLs hereafter), Amazon also acts as a reseller by sourcing products from other suppliers or retail partners and sells them directly to consumers. In this case, Amazon takes charge of pricing, shipping, customer service, refunds, and returns for those products. The products are labeled as "Sold by Amazon" (SBA) on Amazon.com. Strictly speaking, PLs are also "Sold by Amazon." To distinguish these two scenarios in this paper, we separate PLs from SBA products. When we discuss SBA in this paper, we are referring only to products manufactured by third-party suppliers, with Amazon serving as the reseller.

In addition to the above roles, Amazon also provides fulfillment services, known as "Fulfilled by Amazon" (FBA). On Amazon.com, all products using FBA services are labeled as "Ships from Amazon." In this case, third-party sellers contract Amazon for fulfillment. They send their inventory in bulk to Amazon fulfillment centers, where they are stored until sale. Upon sale, Amazon handles shipping, customer service, refunds, and returns for the products, following Amazon's own processes and policies. Both Amazon's PLs and SBA products use FBA services.[2]

We summarize the key characteristics of PL, SBA, and FBA in Table 3.1. In this paper, we focus on SBA and PL, which are products owned by Amazon.

3.3.2 Data

Product Data

We use publicly available product-level data from Keepa.com for our analysis. Keepa is a third-party website that tracks Amazon's product information worldwide on a daily basis. As of March 2023, the database includes over 3.4 billion products sold on Amazon in 12 countries (USA, UK, Germany, France, Japan, Canada, Italy, Spain, India, Mexico, Brazil,

[2] Another well-known label related to fulfillment on Amazon.com is "Prime", i.e., whether the products are eligible for 2-day Prime shipping for Prime members. All products using FBA services are eligible for Prime.

Table 3.1: Summary of Amazon's involvement on Amazon.com

	Amazon Private-label	Sold by Amazon (SBA)	Fulfilled by Amazon(FBA)
Manufacturing	Yes	No	No
Pricing	Yes	Yes	No
Fulfillment	Yes	Yes	Yes
Prime-eligible	Yes	Yes	Yes

Notes: This table shows the summary of Amazon's involvement on Amazon.com. It mainly splits into three categories, Amazon's PL, SBA, and FBA.

and Netherlands). In our paper, we focus on products in the U.S. market (which refers to Amazon.com) from 2016 to 2022.

For each product, Keepa tracks two types of information. First, it collects cross-sectional information including the product's unique identifier (ASIN), brand, manufacturer, title, description, category information, images, etc. Second, it collects high-frequency panel data for product key characteristics such as price, rating, sales rank, seller information, etc. Amazon does not publish the actual sales number on the website; instead, they provide the sales rank of each product in its root category. For example, a pair of women's cowboy boots shows the sales rank in the entire "Clothing, Shoes & Jewelry" category. However, we are able to back out the approximated sales number using Jungle Scout[3], a website that provides the mapping between sales rank and actual sales number in each category.

As it is impossible to download all product data, we focus on the most popular cate-

[3]https://www.junglescout.com/

gories on Amazon.com. Amazon adopts a category tree system with up to six levels of categories for each product (from level 1 to level 6). Level 1 (also known as root category) is the most general category (such as Automotive, Electronics, Home & Kitchen, etc), whereas level 6 is the most specific category. Each product has at least four levels of category information available. Around 80% of products have five levels, and less than 50% of products have six levels. For example, women's cowboy boots have a category tree with five levels: "Clothing, Shoes & Jewelry → Women → Shoes → Boots → Knee-High". Since not all products have level 5 or level 6 category information and level 4 is specific enough to classify products, we collect data from level 4 category.

We focus on the top 2000 level 4 categories that have the most products. This information can be accessed from Keepa's category object that tracks the category information on Amazon, including the number of products, children categories, parent categories, etc. For each category, we collect data from the 100 bestseller products. Furthermore, since we are focusing on Amazon's self-preferencing and the impact of introducing PLs, we only collect categories that have a mix of Amazon PLs and third-party products. Specifically, we exclude categories such as Alexa Skills because only Amazon's products are in this category. We also exclude categories mainly containing digital goods such as software, digital games, books, videos, and music since these products are mainly offered by Amazon. Also, we only focus on products with the condition "new"; we do not investigate the used goods market. We eventually obtain 173,555 products that entered the platform between 2016 and 2022. Among these products, there are 125,037 third-party products that are only sold by third-party sellers, which means, during the observed window, Amazon never becomes the seller of these products. There are 47,580 SBAs, which means, during the observed window, these products are labeled as "Sold by Amazon" at some point. Additionally, there are 938 PLs manufactured by Amazon. We show the proportion of each type of products across root categories in Table3.2.

Table 3.2: Summary Statistics

Category	Avg Price	Avg Rating	Avg Sales Rank	SBA (%)	PL(%)
Arts, Crafts & Sewing	18.57	4.45	63153.10	14.40%	0.13%
Beauty & Personal Care	22.73	4.35	54349.78	36.08%	0.13%
Clothing, Shoes & Jewelry	29.48	4.39	70144.72	22.43%	1.60%
Electronics	67.21	4.30	112258.50	22.22%	0.47%
Health & Household	23.76	4.38	62804.01	34.38%	1.22%
Home & Kitchen	46.83	4.42	146318.94	23.79%	0.72%
Industrial & Scientific	37.64	4.44	140045.91	25.37%	0.16%
Office Products	37.61	4.45	54579.51	28.61%	1.92%
Patio, Lawn & Garden	56.25	4.34	67929.27	21.55%	0.19%
Pet Supplies	27.07	4.32	34847.49	25.07%	0.36%
Sports & Outdoors	37.65	4.39	142538.19	35.80%	0.13%
Tools & Home Improvement	46.41	4.41	96908.97	32.26%	0.61%

Notes: This table shows the summary statistics and proportion of SBA and PL across root categories in our data sample.

Search Data

In addition to product information from Keepa, we scrape Amazon's search page data to analyze the search rank of different types of products. We use various keywords under an anonymous IP address to ensure there is no influence from personal recommendations. To guarantee the keywords match the popular categories we collect from Keepa, we use the "Context Free Name" for all products in the top 2000 level 4 categories in our dataset. Amazon.com publicly provides the context-free name for each category at each level. The context-free name is able to accurately identify the keyword that can lead to products in the corresponding category. For example, the "Context Free Name" for the level 4 category "Clothing, Shoes & Jewelry $\rightarrow$ Women $\rightarrow$ Shoes $\rightarrow$ Boots" is "Women's Boots".

Once consumers search on Amazon, the ranking algorithm returns different numbers of pages depending on the available products that are relevant to the keywords. Although Amazon might return numerous products in the search results, consumers do not click past the first results page in around 72% of searches according to Farronato et al. (2023). Furthermore, only half of the products on the first page are actually seen by consumers. Consumers complete one-third of their purchases in three minutes or less on average (Bezos 2021). Thus, the competition is only relevant on the first few pages of the search results. To ensure our data covers the majority of products in the consumers' choice set, we scrape the products displayed on the first two pages for each keyword. On each page, Amazon typically displays 15 rows of products with 4 slots per row. We define the rank of a product using a zero-based index, starting from the top left corner of the search results page and moving from left to right and top to bottom. For example, the first row of products has ranks 0, 1, 2, and 3, while the second row has ranks 4, 5, 6, and 7, and so on. To investigate whether Amazon gives its products an advantage in the organic search results, we exclude all sponsored ads and only keep the organic results.

3.4 Evidence of Amazon's Self-preferencing Strategy

In this section, we present direct and indirect evidence demonstrating that Amazon employs a self-preferencing strategy. We first provide direct evidence by comparing Amazon's private-label (PL) and "Sold by Amazon" (SBA) products with third-party products in search results. We find that both Amazon's PL and SBA products tend to rank higher (occupy better positions) in the search results even when controlling for product characteristics. The limitation of this direct evidence is that we cannot exclude the influence of unobserved product qualities. To address this concern, we present indirect evidence by analyzing the case where Amazon begins selling an existing third-party product. This approach allows us to control for unobserved product qualities and further supports our hypothesis of Amazon's engagement in self-preferencing.

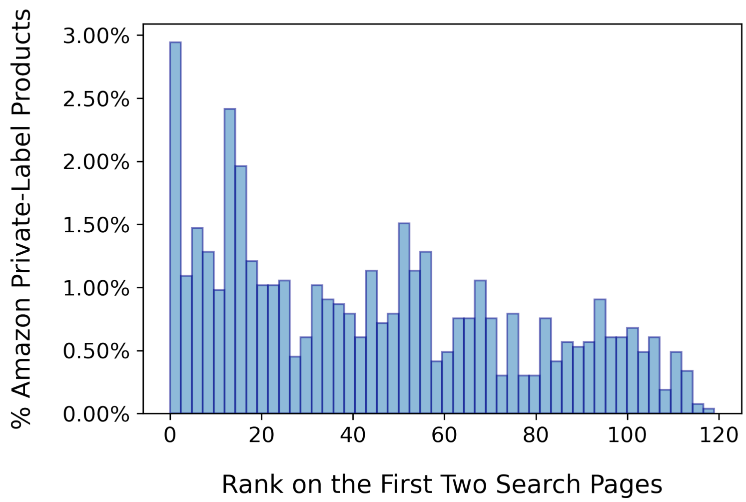
3.4.1 Direct evidence

The most direct way to see if Amazon favors their own products on search pages is to compare the ranking distribution of their own products vs third-party products.

Ranking Distribution of Amazon's Private-label Products

First, we check the rankings of PLs in search outcomes. Figure 3.2 shows the histogram of their rankings. In our data, 1,114 PLs show up on the search pages. Among these products, 8.44% are listed in the first row; 31.78% are listed in the top 5 rows; 68.22% are listed on the first page. As we can see, the distribution is highly right-skewed, i.e., PLs are concentrated at the top of the search results.

Figure 3.2: Ranking of Amazon's Private-label Products (PL) in Search Results

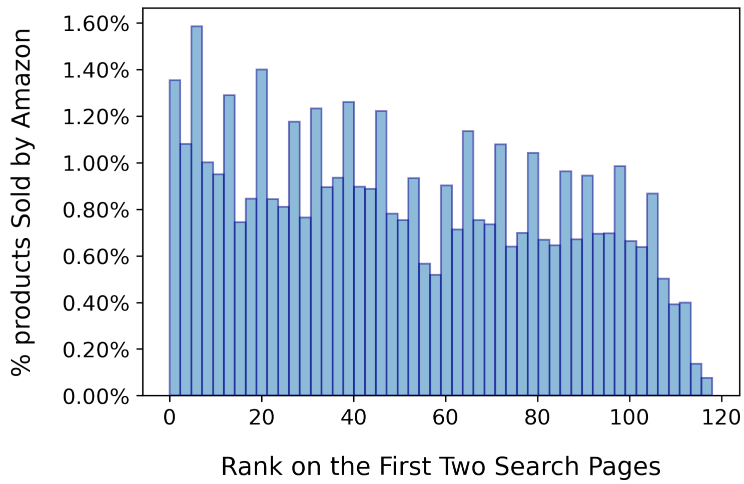


Notes: This figure shows the distribution of the ranking of Amazon's private-label products. 8.44% of PL are listed in the first row; 31.78% of PL are listed in the top 5 rows; 68.22% of PL are listed in the first page.

Ranking Distribution of Sold by Amazon Products (SBA)

Second, we check the ranking position of SBA. Figure 3.3 shows the distribution. In our data, 20,508 SBA show up. 4.35% are listed in the first row; 22.04% are listed in the top 5 rows; 58.95% are listed on the first page. As we can see, SBAs are more evenly distributed across the ranking positions than Amazon's PLs, but they are still right-skewed in ranking.

Figure 3.3: Ranking of Sold by Amazon Products (SBA) in Search Results



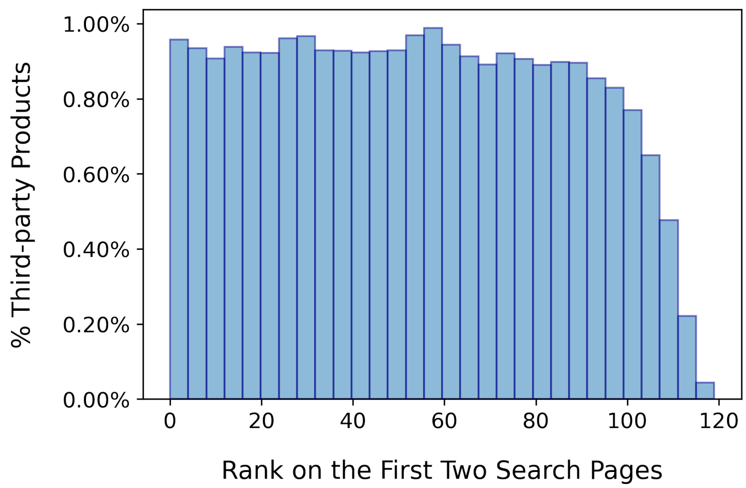
Notes: This figure shows the distribution of the ranking of Sold by Amazon (SBA). 4.35% SBA are listed in the first row; 22.04% SBA are listed in the top 5 rows; 58.95% SBA are listed in the first page.

Ranking Distribution of Third-party Products

Third, we check the ranking position of third-party products. Figure 3.4 shows the distribution. In our search data, 80,011 third-party products show up. Among them, 3.80% products are listed in the first row; 18.49% products are listed in the top 5 rows; 55.95% products are listed on the first page. As we can see, third-party products are more evenly distributed across the ranking positions than PL and SBA. We further adopt the Kolmogorov–Smirnov test and find that the ranking distributions between Amazon PL and third-party products are significantly different (with p-value 1.43×10^{-20}). The ranking distributions between SBAs and third-party products are significantly different as well (with p-value 6.09×10^{-43}).

However, the difference in ranking distribution could be attributed to disparities in product quality and brand recognition. PL and SBA may indeed have higher quality and brand

Figure 3.4: Ranking of all Third-party Products in Search Results



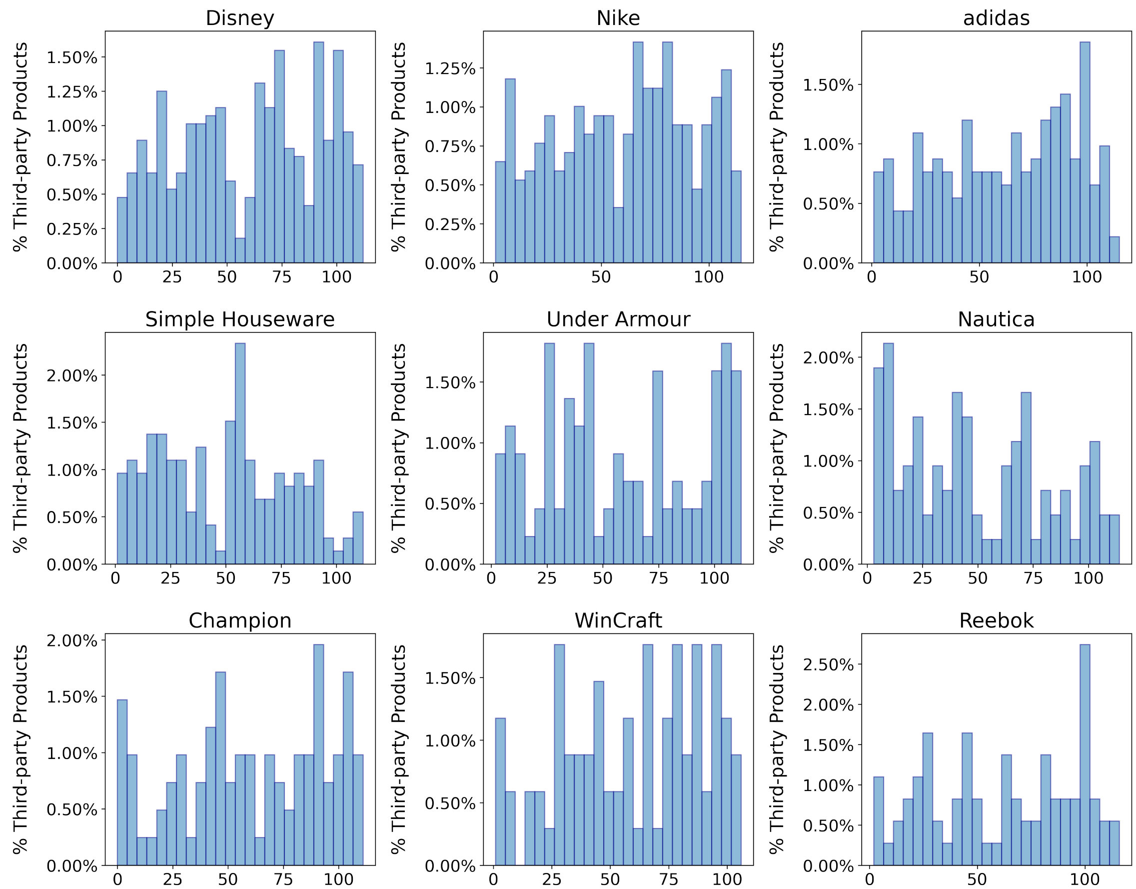
Notes: This figure shows the distribution of the ranking of all third-party products. 3.80% products are listed in the first row; 18.49% products are listed in the top 5 rows; 55.95% products are listed in the first page.

loyalty, resulting in higher ranks. Therefore, we examine the distribution of some popular third-party brands, such as Nike, Adidas, Disney, etc in Figure 3.5. We can see that, despite these third-party brands being already popular and of high quality, almost all of their rankings are evenly distributed on the first two pages. This observation suggests that Amazon may indeed favor its PL and SBA products in search rankings.

Ranking Distribution after Controlling for Observables

So far, we find that Amazon's products (SBA and PL) appear more prominently in search results. However, it is possible that the prominence may come from product features such as price, rating, etc. In this section, we further compare the ranking distribution between Amazon's products (SBA and PL) and third-party products by running the following OLS

Figure 3.5: Ranking of Popular Third-party Brands' Products in Search Results



Notes: This figure shows the ranking distribution of famous third-party brands. Third-party products are more evenly distributed in the search results.

regressions to control for observables:

$$y_{ij} = \beta A_{ij} + \gamma X_{ij} + \epsilon_{ij},$$

where y_{ij} denotes the rank of product i in the search result using search keyword j. The dummy variable A_{ij} denotes whether the product is Amazon's (when comparing SBA with third-party products, $A_{ij} = 1$ denotes SBA product; when comparing PL with third-party products, $A_{ij} = 1$ denotes PL product). X_{ij} captures the observables including search keyword fixed effects, product price, product rating, etc.

Table 3.3 presents the coefficient estimates for the dummy variable A_{ij}. Model (1) confirms that, under the same search keyword, SBA are ranked 3.172 positions higher than third-party products without controlling for observable characteristics. Model (2) shows that after controlling for observables, SBA are still ranked 2.963 higher than third-party products. The results from these two models suggest that SBA are given additional prominence in search results that cannot be explained by other observables such as price, rating.

However, it is possible that Amazon tends to choose recognizable third-party brands and make them SBA. To control for that, we compare the ranking distribution of SBA and third-party products under same brand. For example, the potential observations might include Nike shoes sold by Amazon and Nike shoes sold by third-party sellers. Model (3) and (4) show the results. We find that within the same brand, SBA are ranked 4.694 positions higher than third-party products without controlling for observables and are ranked 4.144 positions higher than third-party products after controlling for observables. Comparing model (3) and (4) with model (1) and (2), we find that after controlling for brand, SBA receive even greater prominence in search results.

We conduct similar analysis by comparing PL with third-party products in model (5) and model (6). The results show that PL are ranked 10.416 positions higher than third-party products without controlling for observables and 10.646 positions higher than third-party products after controlling for observables. It confirms that Amazon PL are more favored in search results comparing with third-party products. Our results in model (5) and model (6) are consistent with Farronato et al. (2023).

Table 3.3: Regression results of Product Ranking

	(1) SBA	(2) SBA	(3) SBA (same brand)	(4) SBA (same brand)	(5) PL	(6) PL
isSBA	-3.172***	-2.953***	-4.694***	-4.144***		
	(0.239)	(0.242)	(0.597)	(0.594)		
isPL					-10.416***	-10.646***
					(1.379)	(1.378)
Price		-0.0001		-0.455***		-0.028**
		(0.002)		(0.009)		(0.004)
Rating		-3.896***		-24.591***		-2.937***
		(0.527)		(2.343)		(0.707)
Keyword FE	Yes	Yes	Yes	Yes	Yes	Yes
R^2	0.272	0.273	0.414	0.425	0.273	0.274
Adj. R^2	0.262	0.262	0.400	0.410	0.255	0.256
N	96659	96659	8214	8214	54610	54610

Notes: This table shows the regression results of product rankings. Model (1) shows that SBA are ranked 3.172 positions higher than third-party products on average; Model 2 shows that after controlling for product key characteristics such as price and rating, SBA are ranked 2.953 positions higher than third-party products; Model 3 shows that when products are from the same brand and have the same search keyword, SBA are ranked 4.694 positions higher than third-party products on average; Model 4 shows that after controlling for product key characteristics such as price and rating, SBA are ranked 4.144 positions higher than third-party products within same brand; Model 5 shows that Amazon PL are ranked 10.416 positions higher than third-party products on average; Model 6 shows that Amazon PL are ranked 10.646 positions higher than third-party products even when controlling for product price and ratings.

Overall, these results suggest that SBA and PL tend to receive higher rankings than third-party products in search, even after controlling for factors such as brand, price, and rating. This evidence suggests that Amazon may promote SBA and PL more heavily on search page. However, as mentioned in Farronato et al. (2023), the limitation of this analysis is that the difference in ranking distribution may arise from the unobserved quality differences between Amazon's and third-party sellers' products that are not reflected in brand, price, and rating. To address this concern, we provide additional indirect evidence in the following section by analyzing products that were initially sold by third-party sellers but later became sold by Amazon. This analysis ensures that the product itself remains the same, thus controlling for the unobserved quality, so the only thing that drives the change in sales outcomes is the change of ownership.

3.4.2 Indirect Evidence

In this section, we show evidence of Amazon's self-preferencing strategy from a different angle by analyzing the scenario in which Amazon becomes the seller of an existing third-party product. That is, one product is sold by third-party seller first, and then during our observing window, Amazon becomes the seller of the product (the third-party product becomes SBA product) at some point. This event study is particularly valuable as it allows us to control for the unobserved product quality, which is a major concern when analyzing search rankings. Our hypothesis is that if Amazon adopts the self-preferencing strategy, it will improve the ranking of a product once Amazon becomes its seller, leading to increased sales. Ideally, we should directly use the rank change of a product once Amazon becomes its seller to study the self-preferencing strategy. However, we do not have access to the panel data for search rank. Thus, we use the change in sales rank data as a proxy for search rank change and provide the indirect evidence for the existence of self-preferencing.

Model Free Analysis

During the sample period, 34,809 products were initially sold by third-party sellers and later sold by Amazon. We keep products that we can observe 6 months before and after Amazon becomes its seller in the main analysis, which leaves us with 5,186 products. We first show the raw change in business outcomes when Amazon becomes the seller in Figure 3.6. Panel (a) shows that after Amazon becomes the seller of a product that was previously sold by a third-party seller, the sales soar immediately. Panel (b) and (c) show that the Buybox price and product rating remain the same as before. These two panels rule out the possibility that increased sales arise from decreased prices or increased product quality and service quality. Moreover, even if these characteristics may play a role in changing sales, they should have a more gradual impact (it takes time to change the product quality, etc). However, we can see that there is an immediate jump in sales when Amazon becomes the seller, which suggests that the change is likely to arise from the improved ranking in search results. This is because changing product ranking can be implemented instantly (as long as the product is labeled as sold by Amazon in the ranking algorithm) and has a drastic and direct impact on consumers' choices.[4] This combined model-free evidence implies that Amazon adopts a self-preferencing strategy to promote its own product.

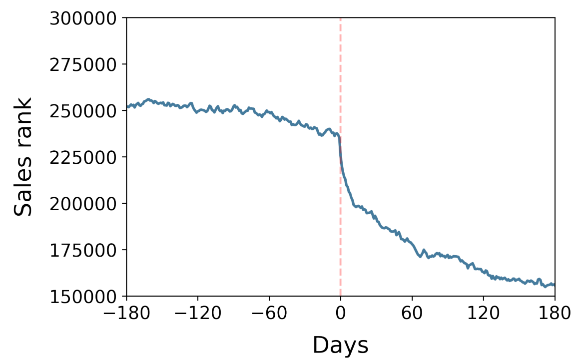
Matched Diff-in-diff Analysis

Next, we employ a matched difference-in-differences framework to formally estimate the effects on products when Amazon becomes its seller. Products that were initially sold by

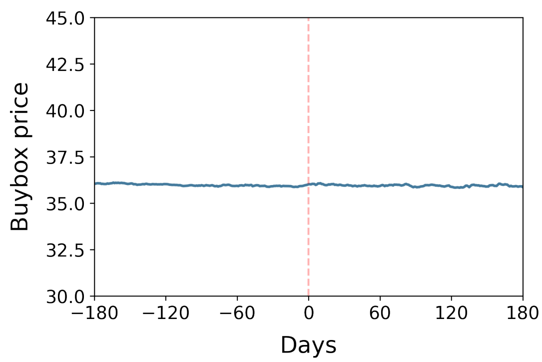
[4]Consumers on Amazon usually add a product to the cart within three minutes of search (Bezos 2021)

Figure 3.6: Impacts on the Product when Amazon Becomes its Seller (Model Free)

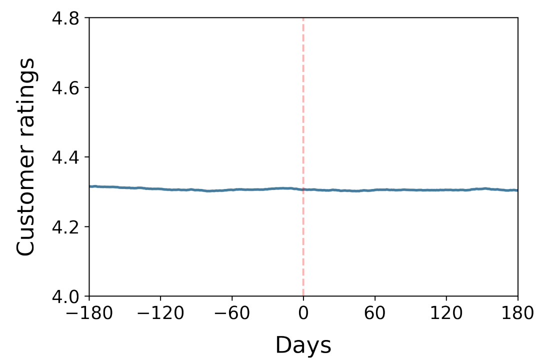
(a) Sales Rank



(b) Buybox Price



(c) Rating



Notes: This figure shows the model free analysis of the raw change when Amazon becomes the seller of an existing product. We use the relative day before and after Amazon's entry as x-axis. We use key product characteristics such as sales rank, buybox price, and rating as y-axis.

third party sellers and then sold by Amazon during our observational window serve as the treated group. Products that were only sold by third party sellers during the whole sample period serve as the control group.

We first check whether the two groups' products have similar characteristics prior to Amazon becoming the seller. Table 3.4 shows summary statistics of key characteristics of products in both groups. We find that treatment and control groups are significantly different from each other: Amazon tends to choose products with higher prices, lower ratings, and higher sales rank (lower sales). To ensure balance along observed characteristics, we implement a matching procedure to isolate the control group products that resemble the treated ones in every observed aspect except for Amazon becoming the seller.

We employ matching methods as follows. First, we only keep the treated products that we can observe for at least six months before and after Amazon becomes its seller. It leaves us with 5,186 products in the treatment group. For each treated product, we then identify all control products that can be observed six months before and after this treated product's Amazon entry date in our data. This ensures that the matched pair have the same observational window.

Second, we impose an restriction that control products should belong to the same parent category (level 3) as the treated product, but belong to a different subcategory (level 4) from the treated product. This ensures that the control products are similar enough to serve as a proxy for treated products' counterfactual, but not that similar to become a close competitor or direct substitutes. For example, as shown in Figure 3.7, the donor pool of a treated product in women's boots category can come from other categories within women's shoes, such as athletic, fashion sneakers, flats, etc.

Then, we calculate the Scaled Euclidean distance between each treated product and eligible control product in terms of price, sales rank, and ratings across the six months prior to Amazon becoming the seller. Finally, we use the one-nearest-neighbor (with replacement) algorithm to match treated product to its closest control counterparts. We also impose a caliper that puts an absolute maximum on the Euclidean distance to avoid bad matches. After the matching process, 3,308 treated products are matched to 2,639 unique products in the control group. The descriptive statistics of the matched results are shown in

Table 3.4: Comparison of product characteristics for control and treatment groups

# Treated 4200	(1)			(2)			(3)		
# Control 53177	Buybox Price			Rating			Sales Rank		
Month	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*
-6	35.84	30.36	7.11***	4.32	4.39	-6.92***	265859.69	193552.70	9.14***
-5	35.86	30.40	7.10***	4.32	4.39	-7.05***	269462.57	189910.78	10.16***
-4	35.84	30.44	7.00***	4.31	4.39	-7.68***	265787.41	186534.70	10.08***
-3	35.73	30.47	6.87***	4.31	4.39	-8.02***	263686.87	182420.55	10.23***
-2	35.75	30.50	6.86***	4.31	4.39	-8.04***	261571.95	179022.93	10.35***
-1	35.69	30.53	6.75***	4.32	4.39	-7.86***	255569.60	175890.23	10.20***

Notes: This table shows that products in treatment and control groups are significantly different on a set of key characteristics prior to Amazon becoming the seller of the products.

Table 3.5: Comparison of matched product characteristics for control and treatment groups

# Treated 3308	(1)			(2)			(3)		
# Control 2639	Buybox Price			Rating			Sales Rank		
Month	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*
-6	25.32	25.12	0.31	4.34	4.37	-2.47	120592.65	111586.72	1.79
-5	25.37	25.13	0.37	4.33	4.37	-2.36	121258.16	111740.71	1.86
-4	25.40	25.12	0.44	4.33	4.36	-2.47	119980.62	111987.04	1.56
-3	25.39	25.14	0.39	4.33	4.36	-2.64	118024.99	110730.39	1.41
-2	25.42	25.14	0.42	4.33	4.36	-2.57	118965.73	110564.49	1.60
-1	25.43	25.19	0.37	4.33	4.36	-2.32	118558.77	110888.48	1.45

Notes: This table shows that after matching, the differences between products in control and treatment groups are insignificant on a set of key characteristics prior to Amazon becoming the seller.

Figure 3.7: Matching - Category Selection for Control Group

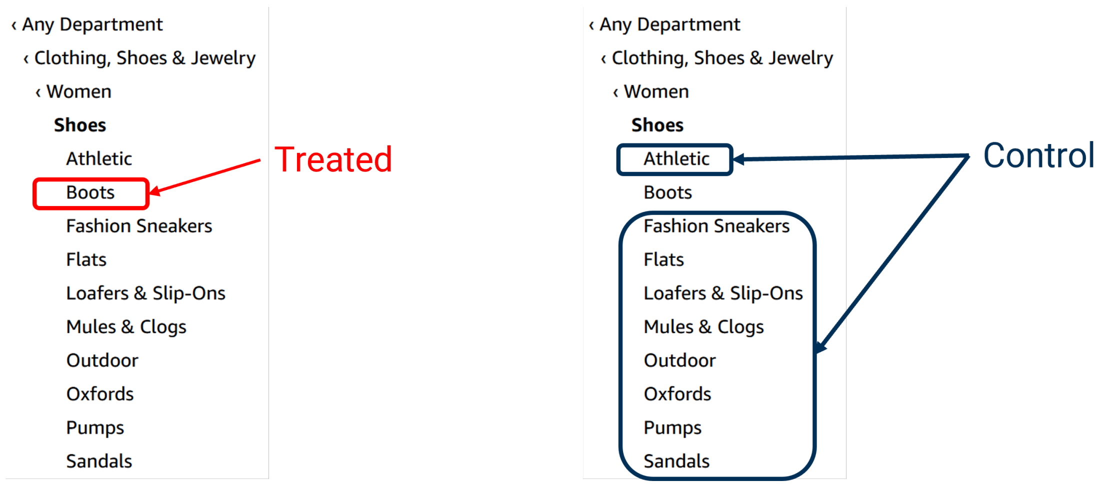


Table 3.5. The covariance are well balanced between the treated and control groups after matching.

Now we compare the average outcomes of treated products and matched control products to estimate the impacts of Amazon becoming the seller of an existing product. We run the following difference-in-differences regression:

$$y_{jt} = \sum_{m=-6}^{6} \beta_m (I_j \times \tau_m) + \omega_m \tau_m + \gamma_j + \lambda_t + \epsilon_{jt}, \tag{3.1}$$

where y_{jt} is the monthly product feature for product j at time t. I_j is an indicator variable that equals to 1 for treated products. τ_m is an indicator of m-th month since Amazon becomes the seller. β_m is our coefficient of interest. It represents the treatment effect in the τ-th month since Amazon becomes the seller. γ_j are product fixed effects, λ_t are year-month fixed effects.

Figure 3.8 shows our main estimation results. In Panel (a), we observe a significant and immediate decrease in product's sales rank, which means there is an immediate increase in product sales after Amazon becomes the seller. The sales rank decreases from around 110,000 to 30,000, which is equivalent to quantity sales increases from around 270 units per month to 930 units per month.[5] However, in Panels (b) and (c), the buybox price and rating of the product do not change significantly after Amazon's entry. Combining these three plots, we find although the product remains the same and there is no discernible change in product prices, quality and service quality, sales of the product soared after Amazon became the seller of the product. It is consistent with the hypothesis that Amazon runs a self-preferencing algorithm to give their own product an advantage in search results, which instantly generates a big and positive impact on sales.

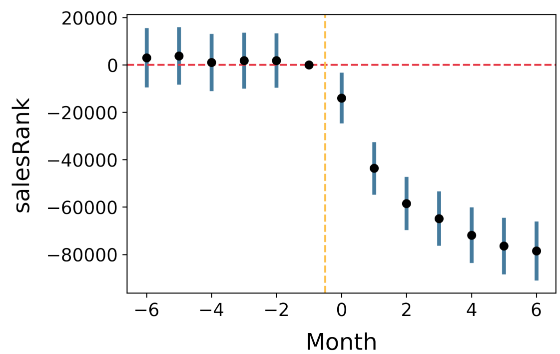
3.4.3 Robustness Tests

A caveat of the above analysis that matches on observables is that it may not be able to control for time-invariant and time-variant unobservables. It is possible that the products chosen by Amazon are systematically different from other products. It is also possible that Amazon strategically chooses the timing to become the seller. Furthermore, customers may favor the products after it becomes SBA since Amazon offers premier fulfillment services (FBA). To account for the above mentioned endogeneity concerns, we conduct several robustness tests in this section.

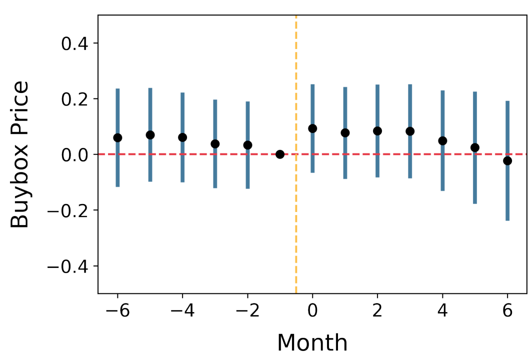
[5] Amazon does not officially provide the measurement to transfer sales rank to quantity sales. However, there are bunch of third-party tools provide the measurement on Amazon. We use one of the most famous tools: Jungle Scout (https://www.junglescout.com/estimator/) to transfer sales rank to monthly quantity sales.

Figure 3.8: Impacts on the Product when Amazon Becomes its Seller (Matched Diff-in-Diff)

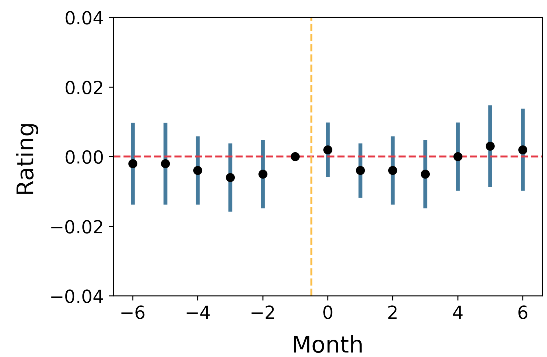
(a) Sales Rank



(b) Buybox Price



(c) Rating



Notes: This figure shows the event study when Amazon becomes the seller of existing products. Each point is an estimate of effect β_m in m-th month. We use one month before Amazon becomes the seller ($m = -1$) as the benchmark. 95% confidence intervals constructed using standard errors clustered at the product level are also displayed.

Selection on Unobservables

In our main analysis, we assume that we can control for the unobserved characteristics of products by conditioning on a rich set of observed characteristics. However, if products chosen by Amazon are systematically different from other products in a way that is not controlled for by the observable in the matching but affects the sales, our estimated effects will be biased. To check this, we replicate the main analysis using only treated products. Because products started being sold by Amazon in different periods, we use this timing variation to match early treated products with late treated products and use the latter as the control group. For example, if a product was sold by Amazon from Jan 2022, we then find products that were not sold by Amazon until six months later (July 2022) and use them as control sellers. We adopt the same matching method in the previous section and show the covariance balance check before and after matching in Table F.1 and Table F.2, respectively.

Figure 3.9 shows the matched diff-in-diff results using a within-treated group sample. We can see that the results are very similar to our main results. This provides further support that our matching procedure is free from selection on unobservables.

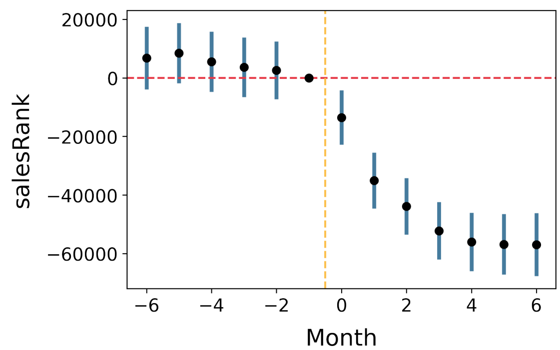
Endogenous Entry Time

Another potential concern of the analysis is that Amazon's entry date (ie the date it became a seller of an existing product) may be endogenous. In particular, they may be more likely to enter during holiday seasons which would bias our estimates of impacts on sales upwards. To alleviate this concern, we look at the distribution of Amazon entry dates. The figure is shown in the Appendix F. We can see that entry dates are mostly uniformly distributed across months. This suggests that Amazon's entry timing is not carefully calibrated and is spread throughout the whole year. Indeed, before amazon enters into a product space, they need to negotiate with the product manufacturers and arrange the logistics, which are not purely controlled by Amazon itself, introducing an element of randomness in their entry timing.

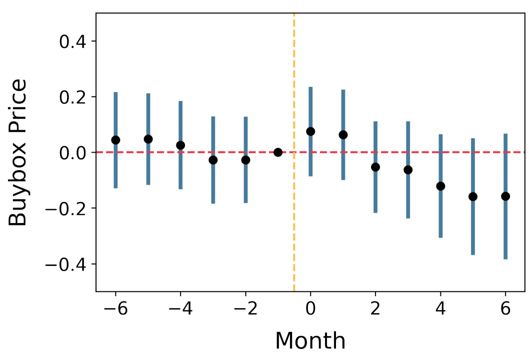
Furthermore, we replicate the event study by cohorts, that is, we looked at the effects

Figure 3.9: Impacts on the Product after Amazon Becomes its Seller (Matched Diff-in-Diff within treated)

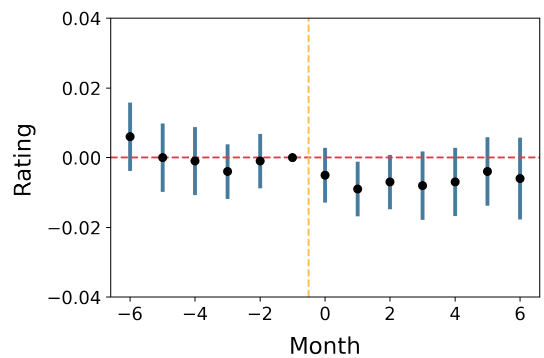
(a) Sales Rank



(b) Buybox Price



(c) Rating



Notes: This figure shows the event study of when Amazon becomes the seller of existing products. The matching process is conducing within the treated group, whereas a later treated product can serve as the control product for an earlier treated product. Each point is an estimate of effect β_m in m-th month. We use one month before Amazon becomes the seller ($m = -1$) as the benchmark. 95% confidence intervals constructed using standard errors clustered at the product level are also displayed.

when Amazon entered in different months of the year. We show the results in the Appendix F and find that no matter whether Amazon entered in peak season like December or off-season like April, we can see similar effect trends.

Change in Shipping Options

When Amazon becomes the seller of a product, it could change the logistics, e.g., storing the product in its own fulfillment center and enabling faster delivery. This change in inventory management and delivery options may have an impact on customers' choices and further affects the sales of the products. To control for the change in logistics, we replicate our analysis on a subsample of the data: we look at products that were already fulfilled by Amazon (FBA) before Amazon becomes its seller. That is, for these products, the inventory inbound and outbound remains the same when Amazon becomes its seller. We show the replicated event study results in the Appendix F. The results are very close to those from the main sample. This analysis rules out the possibility that the change in sales is primarily driven by the change in logistics.

3.5 The Effects of Launching Private-label Products

In the above section, we document evidence that Amazon engages in self-preferencing. In this section, we then explore the impacts of Amazon introducing its private-label products while employing the self-preferencing strategy.

3.5.1 Change in Third-parties' Business Outcomes

In our dataset, 362 level-4 categories experienced the introduction of PLs during the sample period. To maintain a consistent observation period, we only consider level-4 categories where we can observe data for 6 months before and after Amazon launches its private-label products. This leaves us with 312 categories for our analysis.

To estimate the effects of Amazon's introduction of PL on other third-party products in the same category, we employ a matched difference-in-differences (DiD) framework. Categories that initially did not have any PL but later had them during our observation window are considered as the treated group. On the other hand, categories that did not have any PL throughout the entire observation window are considered as the control group.

Before the estimation, we first examine whether the categories in the treated and control groups have similar characteristics prior to the introduction of PL. Table 3.6 presents summary statistics of key characteristics for categories in both groups. We observe that the treatment and control groups are significantly different from each other: Amazon tends to launch products in categories with higher average ratings and lower average sales rank (indicating higher sales).

To ensure that our comparison is balanced along the observed characteristics, we implement a matching procedure to identify control group categories that closely resemble the treated ones in every observed aspect, except for the introduction of PL. We discuss the detailed matching process in the Appendix G.

After matching, 198 treated categories are matched to 179 unique control group categories. The descriptive statistics of the matched results are shown in Table 3.7. Covariates are well balanced between the treated and control groups after matching, ensuring a more accurate estimation of the effects of Amazon's private-label product launches.

Now we compare the average outcomes of treated categories and matched control categories to estimate the impacts of Amazon introducing PL on third-party products in the

Table 3.6: Comparison of categories in control and treatment groups (before matching)

# Treated 312	(1)			(2)			(3)			(4)		
# Control 2017	Buybox Price			Rating			Sales Rank			# Sellers		
Month	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*
-6	37.97	39.30	-0.67	4.37	4.34	2.72***	111699.63	175212.40	-8.45***	7.23	6.92	1.02
-5	38.17	39.30	-0.57	4.37	4.34	2.82***	111631.82	173945.22	-8.41***	7.21	6.91	0.98
-4	38.09	39.38	-0.66	4.38	4.35	2.20**	112129.20	172849.56	-8.12***	7.17	6.88	0.95
-3	38.22	39.41	-0.60	4.38	4.35	2.58**	110742.78	171184.46	-7.87***	7.06	6.85	0.72
-2	37.88	39.44	-0.79	4.38	4.35	2.48**	105539.99	170380.57	-9.10***	7.00	6.80	0.67
-1	37.99	39.49	-0.78	4.39	4.36	2.77***	102714.35	169910.67	-10.20***	6.90	6.77	0.44

Notes: This table shows that categories in treatment and control groups are significantly different on a set of key characteristics prior to Amazon introducing PL products.

Table 3.7: Comparison of categories in control and treatment groups (after matching)

# Treated 198	(1)			(2)			(3)			(4)		
# Control 179	Buybox Price			Rating			Sales Rank			# Sellers		
Month	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*
-6	26.67	25.51	0.79	4.42	4.41	0.39	68241.69	68112.48	0.02	7.07	6.78	0.61
-5	26.83	25.49	0.91	4.42	4.42	0.49	69508.59	69019.85	0.08	7.01	6.73	0.59
-4	26.85	25.52	0.90	4.43	4.42	0.42	68525.18	68734.56	-0.04	6.99	6.72	0.57
-3	26.76	25.49	0.85	4.43	4.43	0.27	68090.77	67287.83	0.14	6.92	6.69	0.50
-2	26.68	25.40	0.87	4.43	4.43	0.15	66387.69	67064.50	-0.12	6.85	6.63	0.47
-1	26.77	25.50	0.86	4.44	4.43	0.28	66991.74	66612.84	0.07	6.79	6.55	0.52

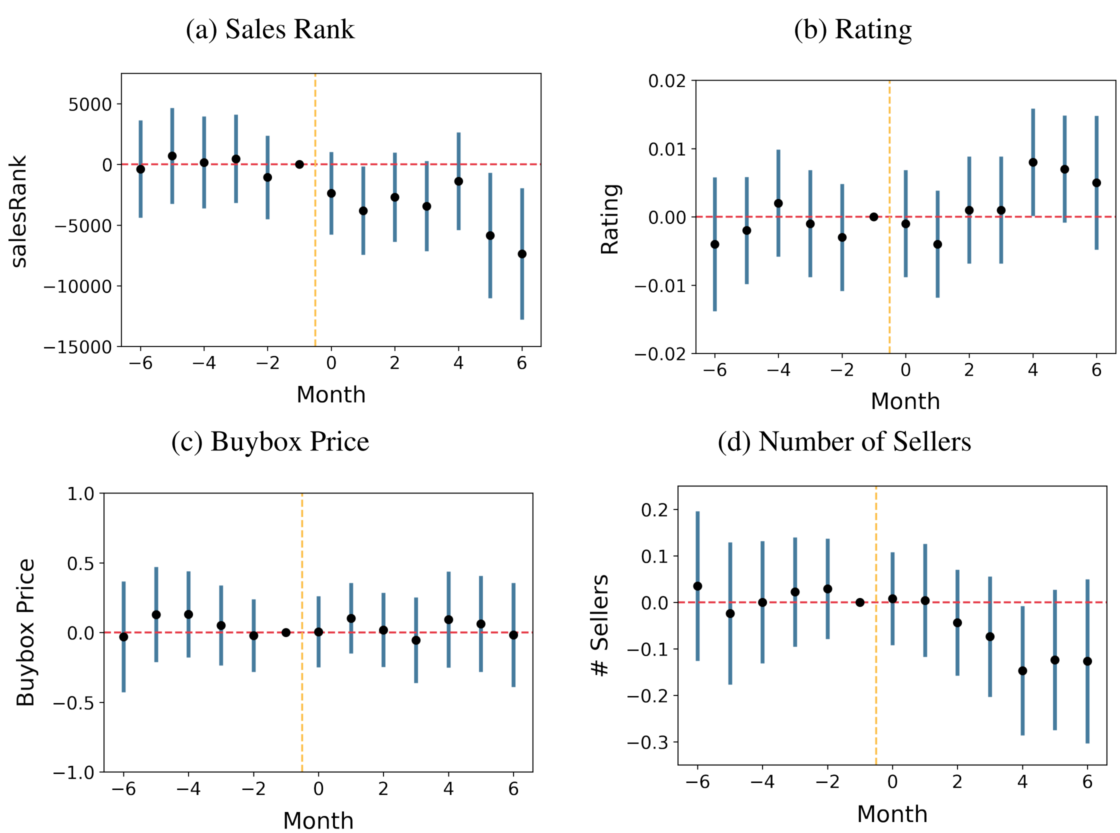
Notes: This table shows that after matching, the differences between products in control and treatment groups are insignificantly on a set of key characteristics prior to Amazon becoming the seller.

same category. We run the following difference-in-differences regression:

$$y_{jt} = \sum_{m=-6}^{6} \beta_m (I_j \times \tau_m) + \omega_m \tau_m + \gamma_j + \lambda_t + \epsilon_{jt}, \tag{3.2}$$

where y_{jt} is the average product features across third-party products in category j at time t. I_j is an indicator variable that equals to 1 for treated category. τ_m is an indicator for the m-th month since Amazon introduced private-label products in this category. β_m is our coefficient of interest, representing the treatment effect in the τ-th month since Amazon introduces its product. γ_j are category fixed effects, accounting for time-invariant category-specific characteristics. λ_t are year-month fixed effects, controlling for any time-specific factors that could affect all categories.

Figure 3.10: Impacts of Amazon Introducing PL on third-party sellers in the same category



Notes: This figure shows the event study of when Amazon introduced PL. Each point is an estimate of effect β_m in m-th month. We use one month before Amazon introduces PL ($m = -1$) as the benchmark. 95% confidence intervals constructed using standard errors clustered at the product level are also displayed.

Figure 3.10 presents our main estimation results. In Panel (a), we observe a gradual decrease in third-party products' average sales rank when Amazon introduced private labels, suggesting that this introduction potentially leads to higher sales for other products in the same category. Specifically, the sales rank decreases from around 67,500 to 60,000, which is equivalent to the quantity sales per month increasing from 450 to 570 units. Panel (b) shows that the average rating across third-party sellers in the same category increases in the long run, implying a higher product quality. Panel (c) shows that the buy box price does not change significantly before and after Amazon's product launch, indicating the introduction has a minimal impact on the price level within the category. Panel (d) shows that the number of sellers in the treated category declines over time, suggesting that more sellers are leaving the platform in the treated category.

Overall, the introduction of PL appears to positively impact consumer welfare by offering improved product quality and stable prices. These benefits can enhance consumers' shopping experiences and their satisfaction with the products available on the platform. For sellers, on the one hand, the sales and rating increase for the third-party products in the same category as PL. On the other hand, more sellers left the platform due to intensified competition, so the impact on sellers is mixed.

3.5.2 Possible Mechanisms

The most interesting finding from above is that although Amazon favors its own PL products in search, we observe an increase in sales for third-party sellers. Thus, in this section, we carry out several analyses to explore the possible mechanisms driving this change. We find that launched PLs displace lower-rated sellers, stimulate innovation and variety in product designs, and serve as valuable guidance for third-party sellers to enhance their searchability through improving product descriptions. These factors could potentially lead to higher sales and ultimately an increase in consumer welfare, with prices being largely unchanged.

Sellers with Lower Rating Exit the Platform

In Section 3.5.1, we found that the number of sellers in the treated category declines after Amazon introduces PL. To further explore this phenomenon, we compare the Amazon Seller Rating of those who exit the platform and those who remain after Amazon launches its private-label products.

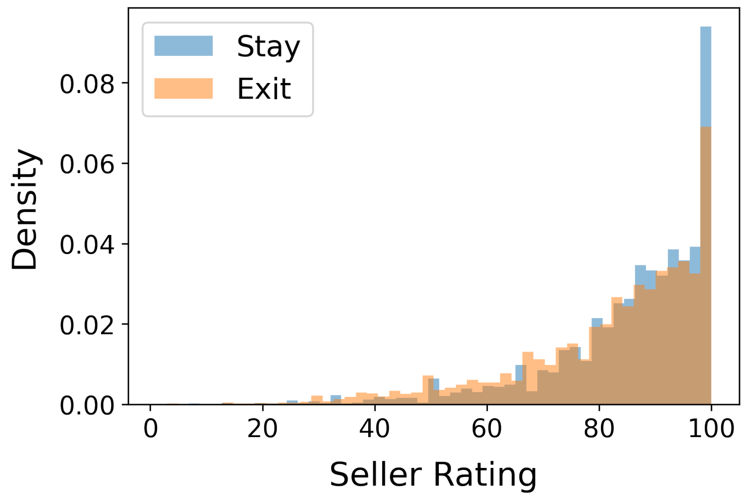
According to Amazon Seller Central[6], the Amazon Seller Rating is a metric employed by Amazon to assess the performance of third-party sellers on its platform. This rating system is designed to assist consumers in making informed purchasing decisions and to encourage sellers to deliver excellent customer service. The Amazon Seller Rating takes into account several factors, such as customer feedback, order defect rate, shipping performance, and policy compliance. Ratings are represented on a scale of 1 to 100, with 100 being the highest possible rating.

In the treatment group, 46,399 sellers exit the platform within six months of Amazon launching its private-label products, while 127,556 sellers remain on the platform. Figure 3.11 displays a histogram comparing the rating distribution of sellers who leave the platform versus those who stay. We can see that sellers with lower ratings are more likely to exit the platform. T-test shows that sellers who left the platform typically have ratings that

[6]https://sellercentral.amazon.com/

are 3.13 points (with p-value < 0.001) lower than those who continue operating on the platform.

Figure 3.11: Distribution of seller ratings - Exit vs. Stay



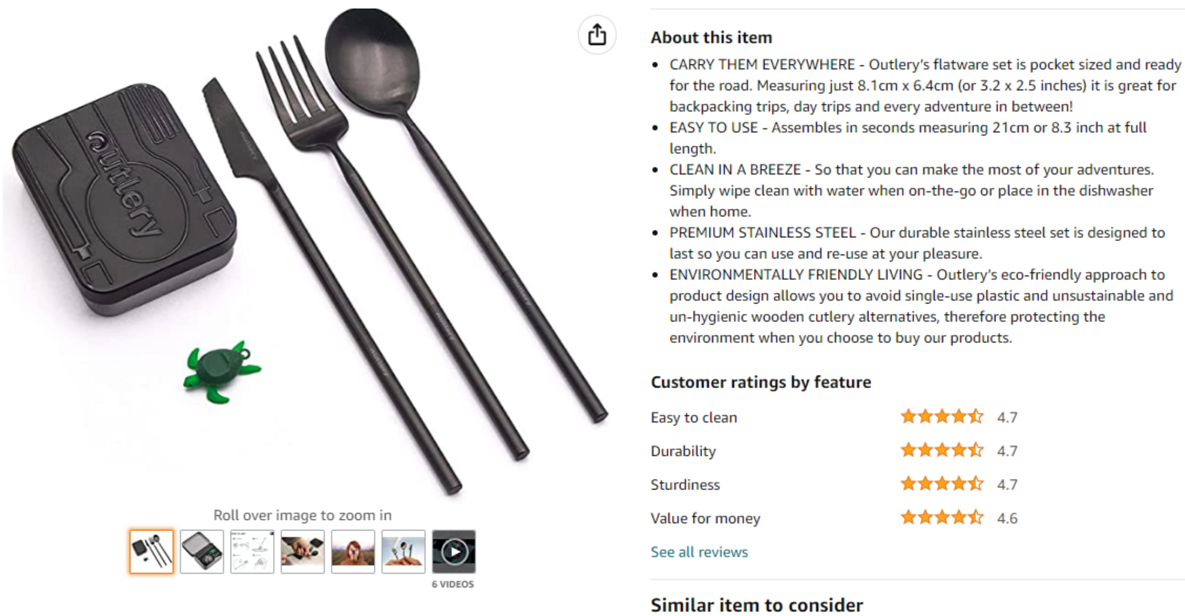
Notes: The figure shows the distribution of seller ratings for two groups. The sellers who exit the platform within 6 months after Amazon introducing PL are shown in orange. The sellers who stay on the platform for at least 6 months after Amazon introducing PL are shown in blue.

This difference in seller ratings could partly explain why the average product rating increased after Amazon introduced PL. As the competition becomes fierce, sellers may be incentivized to enhance their products and customer service in order to maintain their presence on the platform. Those with low ratings are forced to cease operations, resulting in superior products and services for consumers.

More Variety in Product Designs

Next, we investigate whether Amazon's introduction of PL stimulates the change in the design of third-party products. We collect all the product images of each product as shown in Figure 3.12. We then use the change in product images as a proxy to measure the change in product designs.

Figure 3.12: Product description and images



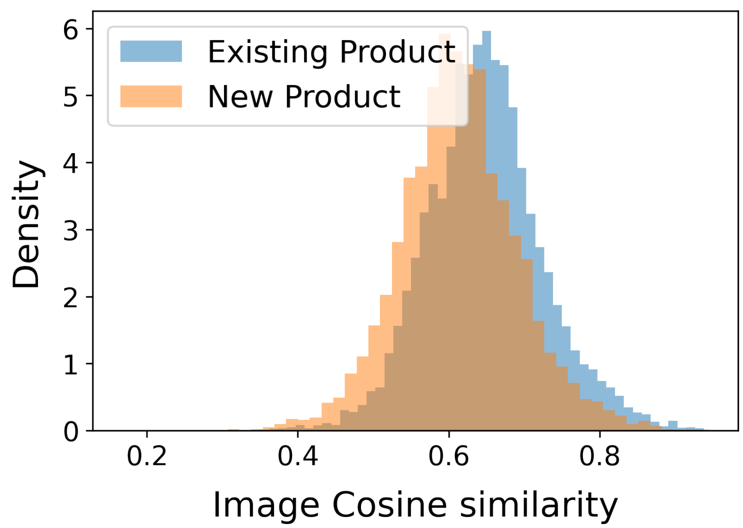
To measure the change in product images, the key process is to measure the change in image similarity before and after Amazon introducing PL in the same category. Thus, we split the third-party products into two groups. One group is the products that already exist when Amazon launches PL (existing products); the other group is the products that are newly launched after Amazon launches PL (new products).

Then for each product in each group, we compute the image similarity between that product's images and PL's images in the same category. This ensures that all the image similarities are computed using the same benchmark, here, the Amazon PL. To compute the image similarity, we use the off-the-shelf computer vision deep learning model to extract the image features and then use cosine similarity to measure the similarity among images.

The detailed methodology is explained in Appendix H.

We show the distribution of the cosine similarity between each third-party product and Amazon PL by group in Figure 3.13. It shows that new products tend to have lower similarity than existing products, suggesting that new products are more differentiated. T-test shows that the difference in product images increases by $0.0348/0.6486 = 5.36\%$ (with p-value < 0.001).

Figure 3.13: Distribution Comparison for image similarity - Existing vs. New products



Notes: The figure shows the distribution of image similarity for two groups. The existing products are before Amazon introducing PL are shown in blue. The newly launched products are before Amazon introducing PL are shown in orange.

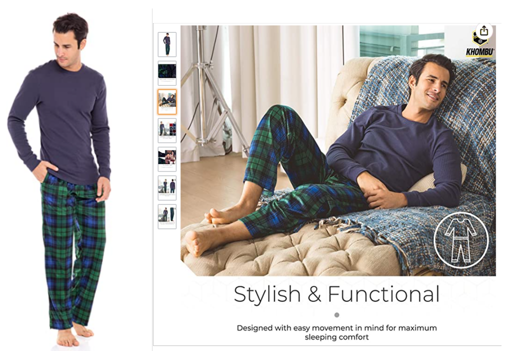
We show an example of the Pajamas category in Figure 3.17. Panel (a) depicts the product image before Amazon launches PL. Panel (b) shows the Amazon Basics image. Panel (c) shows images of the newly added product after the launch of Amazon Basics. Amazon Basics' product is displayed in the same way as the product already existed before its launch. However, the newly launched product shows quite different designs and images compared to Amazon's product. We think a potential underlying mechanism is that third-party sellers try to differentiate from Amazon's products by designing and displaying the product in a new way. This increase in product variety may further drive more demand from customers, resulting in increased sales.

Figure 3.14: An example of the product image change

(a) Pre Amazon Entry Product

(b) Amazon's Product

(c) Post Amazon Entry Product



Notes: This figure shows three different products' description in Pajamas category. Panel (a) is the product before Amazon's entry. Panel (b) is Amazon Basics' product. Panel (c) is the product after Amazon's entry.

Enhanced Searchability of the Products

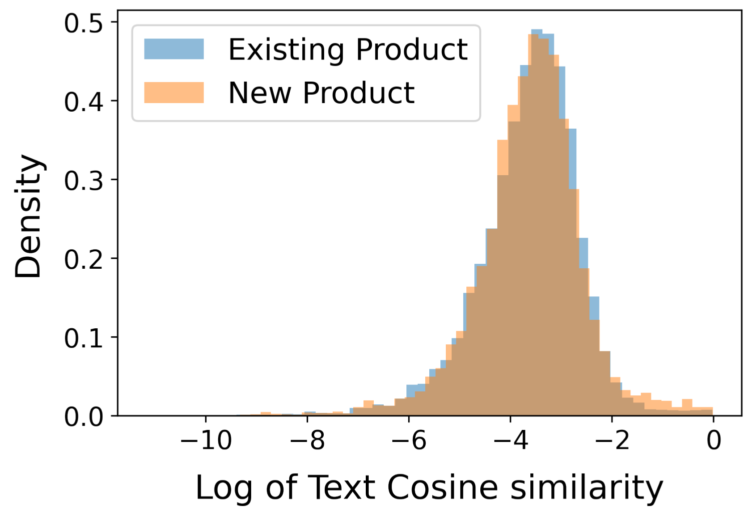
Finally, we investigate how the introduction of PL and the use of self-preferencing strategy affect third-party sellers' search engine optimization (SEO) strategy. It is well-documented that the easiest way to improve the relevancy on Amazon's search results is by matching product content with search keywords (Baldwin 2021, Breslin 2023). The product content that Amazon uses to match keywords is a combination of product titles, descriptions, and bullet points. Thus, we combine the text from Amazon's product title, description, and bullet points (which refer to the "About this item" section in Figure 3.12) as the product text description.

Ideally, we should select the most searchable product in each category as the benchmark, and then compute the cosine similarity between this product and all other products in the same category. However, we do not observe which product has the highest searchability under Amazon's algorithm. Alternatively, in previous section, we already found that PLs tend to be ranked at the top of the search results. Also, as Amazon itself has the most knowledge on its ranking algorithm, it can best utilize the text description and make sure it covers all the possible search keywords with high relevancy. Thus, we use PL in each category as the benchmark product, and calculate the cosine similarity in the description text between PL and third-party sellers' products that were launched before and after the introduction of PLs.[7]

We show the distribution of the cosine similarity among these two groups in Figure 3.15. It shows that new products tend to have higher similarity than existing products, suggesting that new products' text description is more similar to Amazon's. T-test also shows that the similarity in product text description increases by $0.0058/0.0413 = 14.04\%$ (with p-value < 0.001). We further check whether this change can make third-party sellers better-off.

[7] We first create the vector representations of the text description using "Term frequency-inverse document frequency" (TF-IDF) and then use cosine similarity to measure the similarity between different products. The methodology is discussed in detail in the Appendix I.

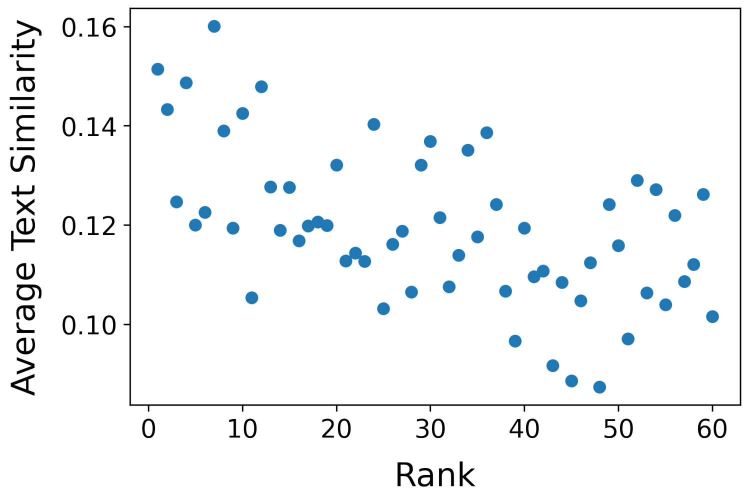
Figure 3.15: Distribution Comparison for text similarity - Existing vs. New products



Notes: The figure shows the distribution of log of text similarity for two groups. The existing products are before Amazon introducing PL are shown in blue. The newly launched products are before Amazon introducing PL are shown in orange.

First, it is important to check whether text description plays an critical role in search result. To verify this, we look at the search data and find that the text description does have an impact on product ranking in the search outcome. We use the first product in each search result (rank = 0) as the benchmark product since it is the most relevant product from Amazon's ranking algorithm. We calculate the cosine similarity of text description between this first product and all other products on the first page. We show the average text similarity between the top position and the rest other positions on the first search page in Figure 3.16. We observe a decreasing trend in text similarity as the rank number increases, which suggests that when the product is more similar to the first product in text descriptions, it is more likely to get a higher rank in the search result. Thus, with Amazon's self-preferencing strategy favoring PL, third-party sellers adjust their SEO strategy and make the product text description more similar to Amazon's. In this way, they can potentially get better position in search results.

Figure 3.16: Relationships between product texts description and product ranks in search



Notes: The figure shows the average text similarity for each ranking position in search results.

Second, we verify that Amazon's text description does contain more information and possibly achieves higher relevancy in search results. We show the example of the category Car Floor Mat in Figure 3.17. Panel (a) depicts the existing product's description before Amazon launches its private-label product. Panel (b) shows the Amazon Basics product. Panel (c) shows the newly added product after the launch of Amazon Basics. Before PL launches, the third party product description is written in a less well-organized way. Although it explains the main feature of the product, it lacks detailed information such as dimensions, material, etc. Amazon Basics' description is more specific and well-organized. After PLs are introduced, the newly entered products have similar product description as the PLs.

As we can see, after Amazon introduces PL, third-party sellers in the same category also get more knowledge about how to improve their SEO strategy. Amazon's product serves as the guideline for third-party sellers to write the product description. Thus, Amazon's introducing of PL also enhance the searchability of third-party sellers. In the meantime, customers are also able to find more relevant products on Amazon through search and make more informed decisions.

To summarize, we find three possible mechanisms that may explain the change in observed outcomes like sales and ratings. First, introducing PL intensifies the competition and crowds out low-quality products. Second, newly added products tend to be more different from Amazon's products in product design to differentiate themselves. This increased variety can improve consumer welfare by providing more options for consumers to choose from based on their unique needs and preferences. Third, newly added products use Amazon private label products as a leading example when crafting the product description. With better-structured and more specific product details, consumers are more likely to find a good match through search and make more informed decisions.

3.6 Conclusion

In this paper, we first document the existence of Amazon's self-preferencing strategy. Our direct evidence shows that PL and SBA are ranked higher in search than third party products. Our indirect evidence controls for the unobserved product quality and further supports the self-preference hypothesis.

Upon proving the existence of the self-preferencing strategy, we investigate the effects of Amazon introducing PL with the use of this strategy, which has become a rising concern from regulators. We find that after Amazon launches PL such as Amazon Basics and

Figure 3.17: An example of the product description change

(a) Pre Amazon Entry Product

About this item

- Our FormFit Design process perfectly forms each liner to the detailed contours of your specific ride.
- Our exclusive StayPut Cleats help keep your liners securely in place.
- Our sporty liner material is rugged and can stand up to abuse like no other.
- Made in the USA.

(b) Amazon's Product

About this item

- 3-piece set of front and back floor mats; protects vehicle floors from snow, dirt, mud, spills, and more
- Made of PVC that bends easily; can be trimmed to fit your car; Gray color
- Non-skid design won't slip or slide around on the floor; cleans easily with water
- Front Mat:17.5"×28" Rear Mat: 16.9"×52.2"
- Backed by an Amazon Basics limited one-year warranty

(c) Post Amazon Entry Product

About this item

- Trim able design to fit your vehicle for ultimate protection Waterproof and stain resistant
- Tall outer ridges prevent fluids from leaking onto carpets
- Heavy nibbed backing secures mats in place
- Easy to clean - vacuum or use soap and water; air dry
- Dimensions: Front Mat 27.5" x 18.5" Rear Mat 52" x 16" Mats are designed to be trim able to fit your vehicle

Notes: This figure shows three different products' description in Car Floor Mat category. Panel (a) is the product before Amazon's entry. Panel (b) is Amazon Basics' product. Panel (c) is the product after Amazon's entry.

Amazon Essentials in specific categories, the sales and rating of other third-party products within the same category increase in the long run, while the price does not change.

Additionally, we find that more low-rating sellers exit the platform in affected categories.

We further examine the underlying mechanisms for the observed changes. We find that the newly launched products have a more differentiated design than Amazon's. Also, third-party sellers enhanced the searchability of their products by referring to PL's product descriptions. These factors could ultimately lead to more sales and therefore an increase in consumer welfare, with prices being largely unchanged. These findings offer valuable insights into the implication of a platform selling its own products on third-party sellers and consumers and shed light on the ongoing debate surrounding platform regulations.

We acknowledge that other mechanisms could also contribute to the observed increase in sales for third-party sellers. For instance, Bairathi et al. (2022) find that platform endorsement led to increased search and purchases not only for endorsed services but also for unendorsed ones. This increase is mainly driven by an increase in overall quality perception of the services offered on the platform. It is possible that Amazon favoring its own products could similarly drive increased search and purchase activity for other products within the same category by increasing customers' perception of product quality on Amazon. This analysis would require data on changes in the website traffic when Amazon launched PLs and we leave this for future research.

Appendices

APPENDIX A

SUMMARY OF KEY NOTATIONS OF CHAPTER 2

Table A.1: Summary of key notation.

Symbol	Description
θ	Consumer type
a	True product quality
a_t	Consumers' perceived valuation at the beginning of period t, with $t \in \{1, 2\}$
α	Level of deviation in valuation in ex-ante (prior) beliefs
λ	Use-based value depreciation
c	Adoption costs incurred by consumers
w	Strength of WOM effects
p	Price of the product
k	Seeding ratio under S model
π	Firm's profit
N_t	Size of paying population in period t, with $t \in \{1, 2\}$
$N_{t,total}$	Size of installed base (including paying, free trial, and/or seeded consumers) in period t, with $t \in \{1, 2\}$

APPENDIX B

PROOFS OF RESULTS FOR THE BASELINE SETUP OF CHAPTER 2

We first present the optimal strategies under each of the business models separately. The solutions for pricing and profit for *CE-PL* and *S* are reproduced from Niculescu and Wu (2014) for readers' convenience.

Proposition 5. [*expanded from Proposition 1 in Niculescu and Wu (2014) to include social welfare*] *Under CE-PL model, the firm's optimal pricing strategy, profit, and ensuing social welfare are:*

	$0 < \alpha < 13 - 4\sqrt{10}$	$13 - 4\sqrt{10} \leq \alpha$
p^*_{CE-PL}	$\frac{2\alpha}{1-\alpha}\left(1 - \sqrt{\frac{2\alpha}{1+\alpha}}\right)$	α
π^*_{CE-PL}	$\frac{2\alpha(\sqrt{1+\alpha}-\sqrt{2\alpha})^2}{(1-\alpha)^2}$	$\frac{\alpha}{2}$
SW^*_{CE-PL}	$1 - \frac{1+2\alpha+2\alpha^2}{2(1+\alpha)(\sqrt{1+\alpha}+\sqrt{2\alpha})^2}$	$\frac{3}{4}$
Paid Adoption	in both periods	only in period 1

Proof. See Proposition 1 in Niculescu and Wu (2014) for the derivation of p^*_{CE-PL} and π^*_{CE-PL}. The social welfare derivation follows trivially. □

Proposition 6. *Under CE-SUB model, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	$0 < \alpha \leq 1$	$1 < \alpha \leq 3$	$\alpha > 3$
p^*_{CE-SUB}	$\tilde{p}$	$\frac{\alpha}{1+\alpha}$	$\frac{\alpha}{2}$
π^*_{CE-SUB}	$\tilde{p}\left(1-\frac{\tilde{p}}{\alpha}+1-\frac{\tilde{p}}{1+\tilde{p}-\frac{\tilde{p}}{\alpha}}\right)$	$\frac{\alpha}{1+\alpha}$	$\frac{\alpha}{4}$
SW^*_{CE-SUB}	$1-\frac{1}{2}\left(\frac{\tilde{p}}{\alpha}\right)^2-\frac{1}{2}\left(\frac{\tilde{p}}{1+\tilde{p}-\frac{\tilde{p}}{\alpha}}\right)^2$	$\frac{\alpha^2+4\alpha+1}{2(1+\alpha)^2}$	$\frac{3}{8}$
Subscription (paid adoption)	in both periods	in both periods	only in period 1

where $\tilde{p}$ *is unique solution to the equation* $2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p = 0$ *on the interval* $(0, \alpha)$.

Proof. In period 1, consumers subscribe iff $\alpha\theta \geq p$. To make any profit, the firm is constrained to set $0 < p < \alpha$. The marginal adopter has type $\theta_1 = \frac{p}{\alpha}$ and the installed base in period 1 is $N_1 = 1 - \frac{p}{\alpha}$. All period 1 adopters learn the true quality of the product in the first period. In the beginning of period 2, the non-adopters from period 1 update their priors through social learning from $a_1 = \alpha$ to $a_2 = \alpha + (1-\alpha)\left(1-\frac{p}{\alpha}\right) = 1 + p - \frac{p}{\alpha}$. We have two cases:

- Case 1: $0 < \alpha \leq 1$.

  In this case, $a_1 \leq a_2 \leq a = 1$. All period 1 adopters will renew the subscription in period 2. The marginal customer type θ_2 satisfies $\theta_2 = \frac{p}{1+p-\frac{p}{\alpha}} \leq \theta_1$. Therefore, the number of adopters in period 2 is $N_2 = 1 - \frac{p}{1+p-\frac{p}{\alpha}}$. The firm's profit maximization problem becomes

$$\max_{0<p<\alpha} \pi_{CE-SUB} = \max_{0<p<\alpha} p\left(1-\frac{p}{\alpha}+1-\frac{p}{1+p-\frac{p}{\alpha}}\right).$$

Differentiating π_{CE-SUB} with respect to p we obtain:

$$\frac{\partial \pi_{CE-SUB}(p)}{\partial p} = \frac{2\alpha^3 - 2(\alpha-1)^2 p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p}{\alpha(\alpha+(\alpha-1)p)^2}.$$

The denominator is positive. Denote the numerator as $g(p) \triangleq 2\alpha^3 - 2(\alpha-1)^2 p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p$. Thus, the sign of $\partial \pi_{CE-SUB}(p)/\partial p$ is the same as the sign of $g(p)$. Differentiating $g(p)$ w.r.t. p, we obtain:

$$\frac{\partial g(p)}{\partial p} = -2(\alpha+(\alpha-1)p)\,(3(\alpha-1)p - (\alpha-3)\alpha).$$

We have two subcases:

- If $\alpha = 1$, then $\frac{\partial g(p)}{\partial p} = -2\alpha^2(3-\alpha) < 0$ for all $p \in (0,\alpha)$.
- If $\alpha < 1$, then, $\frac{\partial g(p)}{\partial p} = 0$ has two solutions, p_1 and p_2 on the real line, but they are both outside the interval $(0,\alpha)$. More precisely, $\alpha < p_1 = \frac{(3-\alpha)\alpha}{3(1-\alpha)} < p_2 = \frac{\alpha}{1-\alpha}$. Thus, when $\alpha < 1$, $\frac{\partial g(p)}{\partial p} < 0$ for all $p \in (0,\alpha)$.

Thus, when $\alpha \in (0,1]$, $g(p)$ is decreasing in p over $(0,\alpha)$. Given that $g(0) = 2\alpha^3 > 0 > g(\alpha) = -\alpha^4(1+\alpha)$, there exists a unique $\tilde{p} \in (0,\alpha)$ that satisfies $g(p) = 0$. Thus, $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} > 0$ when $p \in (0,\tilde{p})$ and $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} < 0$ when $p \in (\tilde{p},\alpha)$. As such $p^*_{CE-SUB} = \tilde{p}$ is the optimal price. The formulas for the optimal profit and associated social welfare follow trivially.

- Case 2: $\alpha > 1$.

In this case, $a_1 > a_2 > a = 1$. None of the period 1 non-adopters will subscribe in period 2 (they value in period 2 the product even less than in period 1). Also, *only part* of the period 1 adopters will renew the subscription in period 2. Since all adopters from period 1 updated their priors to $a_2 = a = 1$ The marginal customer type θ_2 satisfies $\theta_2 = \min\{1, p\}$. We have two subcases:

- Case 2-i: $0 < p < 1$.

  Then $\theta_2 = p$ and $N_2 = 1 - p$. The firm's profit maximization problem becomes:

$$\max_{0<p<1} \pi_{CE-SUB} = \max_{0<p<1} p\left(1 - \frac{p}{\alpha} + 1 - p\right).$$

  We have $\frac{\partial^2 \pi_{CE-SUB}(p)}{\partial p^2} < 0$. From FOC, we obtain the following interior solution $p^*_{CE-SUB} = \frac{\alpha}{1+\alpha}$. This leads to $\pi^*_{CE-SUB} = \frac{\alpha}{1+\alpha}$, $SW^*_{CE-SUB} = \frac{\alpha^2+4\alpha+1}{2(1+\alpha)^2}$.

- Case 2-ii: $1 \leq p < \alpha$.

  Then $\theta_2 = 1$ and $N_2 = 0$, i.e., *none* of the period 1 adopters will renew the subscription in period 2. The firm's profit maximization is simplified to:

$$\max_{1\leq p<\alpha} \pi_{CE-SUB} = \max_{1\leq p<\alpha} p\left(1 - \frac{p}{\alpha}\right).$$

  We need to consider two subsequent subcases:

  * Case 2-ii-a: $1 < \alpha \leq 2$.

    Then we have a corner solution $p^*_{CE-SUB} = 1$, which yields $\pi^*_{CE-SUB} = \frac{\alpha-1}{\alpha}$ and $SW^*_{CE-SUB} = \frac{1}{2} - \frac{1}{2\alpha^2}$.

  * Case 2-ii-b: $\alpha > 2$.

    Then we have an interior solution $p^*_{CE-SUB} = \frac{\alpha}{2}$, which yields $\pi^*_{CE-SUB} = \frac{\alpha}{4}$

and $SW^*_{CE-SUB} = \frac{3}{8}$.

If $1 < \alpha \leq 3$, $\frac{\alpha}{1+\alpha} \geq \max\{\frac{\alpha-1}{\alpha}, \frac{\alpha}{4}\}$. If $\alpha > 3$, then $\frac{\alpha}{4} > \frac{\alpha}{1+\alpha} > \frac{\alpha-1}{\alpha}$. Comparing π^*_{CE-SUB} values among subcases, the results follow immediately. □

Proposition 7. *Under TLF model, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are $p^*_{TLF} = \frac{1}{2}$, $\pi^*_{TLF} = \frac{1}{4}$, and $SW^*_{TLF} = \frac{7}{8}$, respectively.*

Proof. Under *TLF*, all customers get the product for free in period 1, i.e., $N_{1,total} = 1$ (but the number of paying customers is $N_1 = 0$) . Consequently, in period 2, all customers update their prior on quality to $a_2 = a = 1$. Thus. customers purchase the product if and only if their types satisfy $\theta \geq p$, yielding $N_2 = 1 - p$. The firm's profit maximization problem becomes:

$$\max_{0<p<1} \pi_{TLF} = \max_{0<p<1} p(1-p),$$

which, in turn, yields $p^*_{TLF} = \frac{1}{2}$ and $\pi^*_{TLF} = \frac{1}{4}$. The social welfare is $\int_0^1 \theta d\theta = \frac{1}{2}$ for period 1 and $\int_{\frac{1}{2}}^1 \theta d\theta = \frac{3}{8}$ for period 2, which gives $SW^*_{TLF} = \frac{7}{8}$. □

Proposition 8. [*expanded from Proposition 2 in Niculescu and Wu (2014) to include social welfare*] *Under S model, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

where $\alpha_S \approx 0.065$ is the unique solution to the equation $f_S(\alpha) = 0$ over the interval $(0, 13 - 4\sqrt{10})$, with $f_S(\alpha) \triangleq \frac{1}{16(1-\alpha)} - \frac{2\alpha}{(\sqrt{1+\alpha}+\sqrt{2\alpha})^2}$.

	$0<\alpha<\alpha_S$	$\alpha_S \le \alpha < 13-4\sqrt{10}$	$\alpha \ge 13-4\sqrt{10}$
k_S^*	$\frac{1-2\alpha}{2(1-\alpha)}$	0	0
p_S^*	$\frac{1}{4}$	$\frac{2\alpha}{1-\alpha}\left(1-\sqrt{\frac{2\alpha}{1+\alpha}}\right)$	α
π_S^*	$\frac{1}{16(1-\alpha)}$	$\frac{2\alpha(\sqrt{1+\alpha}-\sqrt{2\alpha})^2}{(1-\alpha)^2}$	$\frac{\alpha}{2}$
SW_S^*	$\frac{11-16\alpha}{16(1-\alpha)}$	$1-\frac{1+2\alpha+2\alpha^2}{2(1+\alpha)(\sqrt{1+\alpha}+\sqrt{2\alpha})^2}$	$\frac{3}{4}$
Paid adoption	only in period 2	in both periods	only in period 1

Proof. See Proposition 2 in Niculescu and Wu (2014) for the derivation of p_S^* and π_S^*. The social welfare derivation follows trivially. □

Lemma 1. *If* $0<\alpha\le 1$*, then* $\frac{\alpha(\alpha+3)}{4(\alpha+1)} \le \pi^*_{CE-SUB} \le \frac{\alpha(\alpha+1)}{3\alpha+1}$.

Proof. [Derivation of the lower bound]

$$\pi^*_{CE-SUB} = \max_{0<p<\alpha} \pi_{CE-SUB}(p) \ge \pi_{CE-SUB}(p)\left.\right|_{p=\alpha/2} = \frac{\alpha(\alpha+3)}{4(\alpha+1)}.$$

[Derivation of the upper bound]

Recall from the proof of Proposition 6 that $\tilde{p}$ satisfies $g(\tilde{p})=0$ and $g(p)$ is decreasing in p over $(0,\alpha)$. Given that $g(\frac{\alpha}{2}) = \frac{1}{4}(1-\alpha)\alpha^3 > 0$, we have $\frac{\alpha}{2} < \tilde{p} < \alpha$. Also, it can be easily shown that the profit function satisfies:

$$p\left(1-\frac{p}{\alpha}+1-\frac{p}{-\frac{p}{\alpha}+p+1}\right) \le p\left(1-\frac{p}{\alpha}+1-\frac{p}{\frac{\alpha}{2}-\frac{1}{2}+1}\right), \quad \forall p \in \left(\frac{\alpha}{2},\alpha\right).$$

Denote $h(p) \triangleq p\left(1-\frac{p}{\alpha}+1-\frac{p}{\frac{\alpha}{2}-\frac{1}{2}+1}\right)$. Then, $\pi^*_{CE-SUB} \le h(\tilde{p})$. We next derive an upper bound for $h(\tilde{p})$. As $h(p)$ is a concave quadratic polynomial in p, we can use F.O.C to derive its maximum on $(\frac{\alpha}{2},\alpha)$. Setting $\frac{\partial h(p)}{\partial p} = 0$, we get the interior solution

$p_h^* = \frac{\alpha(\alpha+1)}{3\alpha+1} \in (\frac{\alpha}{2}, \alpha)$. Then, $\pi^*_{CE-SUB} \leq h(\tilde{p}) \leq h(p_h^*) = \frac{\alpha(\alpha+1)}{3\alpha+1}$. $\square$

Proof of Proposition 1.

We have two cases:

- Case 1: $0 < \alpha < 1$.

  [Firm's optimal strategy] We have several subcases:

  - Case 1-i: $0 < \alpha \leq \frac{1}{2}$.

    Then, it can be easily seen that $\pi^*_{TLF} \geq \max\{\pi^*_{CE-PL}, \pi^*_S\}$. So we are left to compare $\pi^*_{TLF} = \frac{1}{4}$ with π^*_{CE-SUB}. We define

$$\begin{aligned}\Delta(\tilde{p}(\alpha), \alpha) &\triangleq \pi^*_{CE-SUB} - \pi^*_{TLF} \\ &= \tilde{p}(\alpha)\left(1 - \frac{\tilde{p}(\alpha)}{\alpha} + 1 - \frac{\tilde{p}(\alpha)}{-\frac{\tilde{p}(\alpha)}{\alpha} + \tilde{p}(\alpha) + 1}\right) - \frac{1}{4},\end{aligned}$$

    where $\tilde{p}(\alpha)$ was defined in Prop. 6. Form the Envelope theorem, for $\alpha \in (0, \frac{1}{2}]$, we obtain:

$$\frac{\partial \Delta(\tilde{p}(\alpha), \alpha)}{\partial \alpha} = \tilde{p}(\alpha)^2 \left(\frac{1}{\alpha^2} + \frac{\tilde{p}(\alpha)}{(\alpha + (\alpha - 1)\tilde{p}(\alpha))^2}\right) > 0.$$

    Thus, $\Delta(\tilde{p}(\alpha), \alpha)$ is increasing in α for $\alpha \in (0, \frac{1}{2}]$. From Lemma 1, we see that $\Delta(\tilde{p}(\alpha), \alpha)\Big|_{\alpha=\frac{1}{2}} = \pi^*_{CE-SUB}\Big|_{\alpha=1/2} - \frac{1}{4} > \frac{\alpha(\alpha+3)}{4(\alpha+1)}\Big|_{\alpha=1/2} - \frac{1}{4} = \frac{7}{24} - \frac{1}{4} > 0$. Moreover, from Lemma 1, we have $\lim_{\alpha \downarrow 0} \Delta(\tilde{p}(\alpha), \alpha) = \lim_{\alpha \downarrow 0} \pi^*_{CE-SUB} - \frac{1}{4} \leq \lim_{\alpha \downarrow 0} \frac{\alpha(\alpha+1)}{3\alpha+1} - \frac{1}{4} = -\frac{1}{4} < 0$. Hence, there exists a unique $\bar{\alpha} \in (0, \frac{1}{2})$ such that

$\Delta(\tilde{p}(\alpha), \alpha)\,|_{\alpha=\bar{\alpha}} = 0.$

Thus, *TLF* is the dominating strategy on the interval $(0, \bar{\alpha})$, whereas *CE-SUB* is the dominant strategy on the interval $[\bar{\alpha}, \frac{1}{2}]$.

– Case 1-ii: $\frac{1}{2} < \alpha < 1$.

  Then, it can be easily seen that $\pi^*_{CE-PL} > \pi^*_{TLF}$ and $\pi^*_{CE-PL} = \pi^*_S$ (more precisely, *S* defaults to *CE-PL*). Thus, we only have to compare π^*_{CE-PL} and π^*_{CE-SUB}. Using Lemma 1, we have $\pi^*_{CE-SUB} \geq \frac{\alpha(\alpha+3)}{4(\alpha+1)} > \frac{\alpha}{2} = \pi^*_{CE-PL}$. Thus, *CE-SUB* is the dominant strategy.

[Social welfare comparison] It can be shown with relative ease, through direct comparisons of closed form solutions, that $SW^*_{TLF} = \frac{7}{8} \geq \max\{SW^*_{CE-PL}, SW^*_S\}$ for all $\alpha \in (0, 1)$. Thus, we only have to compare SW^*_{TLF} with SW^*_{CE-SUB}. We have shown in the proof of Lemma 1 that $\tilde{p}(\alpha) \in (\frac{\alpha}{2}, \alpha)$. It is straightforward to see that:

$$SW^*_{CE-SUB} = 1 - \frac{1}{2}\left(\frac{\tilde{p}}{\alpha}\right)^2 - \frac{1}{2}\left(\frac{\tilde{p}}{1 + \tilde{p} - \frac{\tilde{p}}{\alpha}}\right)^2 < 1 - \frac{1}{2}\left(\frac{\tilde{p}}{\alpha}\right)^2 < \frac{7}{8} = SW^*_{TLF}.$$

Thus, *TLF* yields the highest social welfare.

- Case 2: $\alpha \geq 1$.

  It can be seen from Propositions 5-8, by comparing profits and social welfare values, that *CE-PL* is the dominant strategy in terms of the profit[1] and *TLF* is the dominant

[1]We point out that *S* defaults to *CE-PL* in this region as $k^* = 0$.

strategy in terms of the social welfare. $\square$

APPENDIX C

PROOFS OF RESULTS FOR THE SETUP WITH EXOGENOUS INDIVIDUAL DEPRECIATION OF CHAPTER 2

We first present the optimal strategies under each of the business models separately.

Proposition 9. *Under CE-PL model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	(a) $0 < \alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$	(b) Otherwise
$p^*_{\textit{CE-PL}}$	$\frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$	$\frac{1}{2}\alpha(1+\lambda)$
$\pi^*_{\textit{CE-PL}}$	$\frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$	$\frac{1}{4}\alpha(1+\lambda)$
$SW^*_{\textit{CE-PL}}$	$\tilde{SW}_{\textit{CE-PL}}$	$\frac{3(\lambda+1)}{8}$
Paid adoption	in both periods	only in period 1

where $\tilde{SW}_{\text{CE-PL}} = \frac{1}{2}\left(1+\lambda - \frac{\alpha^2(\lambda+1)^2}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+\alpha\right)^2} - \frac{\lambda}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+1\right)^2}\right).$

Proof. In period 1, consumers purchase the product iff $(1+\lambda)\alpha\theta \geq p$. To make any profit, the firm is constrained to trigger adoption in period 1 (otherwise, no customer would update their priors and there will also be no adopters in period 2 either). To achieve that, the firm has to set price $p \in (0, (1+\lambda)\alpha)$. The marginal adopter has type $\theta_1 = \frac{p}{(1+\lambda)\alpha}$ and the installed base in period 1 is $N_1 = 1 - \theta_1 = 1 - \frac{p}{(1+\lambda)\alpha}$.

At the beginning of period 2, the consumers who did not adopt in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + N_1 (1 - a_1) = \alpha + (1-\alpha)\left(1 - \frac{p}{\alpha(1+\lambda)}\right) = 1 + \frac{(\alpha-1)p}{\alpha(1+\lambda)}.$$

In period 2, new consumers purchase the product if their type θ satisfies $a_2\theta \geq p$. It immediately follows the marginal potential consumer in period 2 has type $\theta_2 = \frac{p}{1+\frac{(\alpha-1)p}{\alpha(1+\lambda)}}$. We have new adopters in period 2 iff $0 \leq \theta_2 < \theta_1$. We have two cases:

- Case 1: $0 < \alpha < 1$

  In this case, we have two subcases:

  - Case 1-i: $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} < \alpha(1+\lambda)$.

    Then we have $0 < \theta_2 < \theta_1$. Then, $N_2 = \theta_1 - \theta_2 > 0$. In this case, the firm's profit maximization problem becomes:

    $$\max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}} \pi_{CE-PL} = \max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}} p\left(1 - \frac{p}{1+\frac{(\alpha-1)p}{\alpha(\lambda+1)}}\right).$$

    It can be shown that $\frac{\partial^2 \pi_{CE-PL}}{\partial p^2} < 0$ for $p \in (0, \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha})$. Thus, it is sufficient to solve FOC:

    $$\frac{\partial \pi_{CE-PL}}{\partial p} = \frac{\alpha^2(\lambda+1)^2 + (1-\alpha)p^2(\alpha\lambda+1) - 2\alpha(\lambda+1)p(\alpha\lambda+1)}{(\alpha\lambda+\alpha+(\alpha-1)p)^2} = 0.$$

Without constrains, the FOC yields two solutions:

$$p_1 = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)},$$
$$p_2 = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}.$$

It can be shown that $p_1 > \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$. Comparing p_2 with $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, we get two subcases:

* Case 1-i-a: $\alpha(\lambda+1)(\alpha\lambda+1) < 1$

  Then $0 < p_2 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$. It immediately follows that $p^*_{CE-PL} = p_2 = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$,
  and $\pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$.

* Case 1-i-b: $\alpha(\lambda+1)(\alpha\lambda+1) \geq 1$.

  Then $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} \leq p_2$. In this case, we have the corner solution $p^*_{CE-PL} = \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, $\pi^*_{CE-PL} = \frac{\alpha^2\lambda(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)^2}$.

– Case 1-ii: $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} \leq p < \alpha(\lambda+1)$.

Then $\theta_2 \geq \theta_1$. In this case, $N_2 = 0$; adoption takes place only in period 1. The firm's profit maximization problem becomes:

$$\max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} \leq p < (1+\lambda)\alpha} \pi_{CE-PL} = \max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} \leq p < (1+\lambda)\alpha} p\left(1-\frac{p}{\alpha(1+\lambda)}\right).$$

Since the function is quadratic, it is sufficient to use FOC. Unconstrained, FOC yields the following solution:

$$p_3 = \frac{1}{2}(\alpha + \alpha\lambda) < (1+\lambda)\alpha.$$

Comparing p_3 with $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, we have two subcases:

* Case 1-ii-a: $\alpha + 2\alpha\lambda > 1$.

  Then $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha} < p_3 < \alpha(\lambda+1)$, and, thus, $p^*_{CE-PL} = p_3 = \frac{1}{2}(\alpha + \alpha\lambda)$ and $\pi^*_{CE-PL} = \frac{1}{4}(\alpha + \alpha\lambda)$;

* Case 1-ii-b: $\alpha + 2\alpha\lambda \leq 1$.

  Then $p_3 \leq \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$. Then, we have the corner solution $p^*_{CE-PL} = \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{1-\alpha}$, $\pi^*_{CE-PL} = \frac{\alpha^2\lambda(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)^2}$.

Comparing case 1-i and case 1-ii, we can get the optimal solution and the associated social welfare for case 1:

- If $0 < \alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$,
  then $p^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$,
  $\pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$, and
  $SW^*_{CE-PL} = \frac{1}{2}\left(1+\lambda - \frac{\alpha^2(\lambda+1)^2}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+\alpha\right)^2} - \frac{\lambda}{\left(\alpha\lambda+\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+1\right)^2}\right)$;
- If $5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)} \leq \alpha < 1$, then $p^*_{CE-PL} = \frac{1}{2}(\alpha + \alpha\lambda)$, $\pi^*_{CE-PL} = \frac{1}{4}(\alpha + \alpha\lambda)$, and $SW^*_{CE-PL} = \frac{3(\lambda+1)}{8}$.

- Case 2: $\alpha \geq 1$

  In this case, $a_1 > a_2 > a = 1$. None of the period 1 non-adopters will purchase in period 2. The firm's profit maximization problem is:

$$\max_{0<p<(1+\lambda)\alpha} \pi_{CE-PL} = p\left(1 - \frac{p}{\alpha(1+\lambda)}\right).$$

  Since the profit is quadratic in p, we can derive the solution from FOC. We get $p^*_{CE-PL} = \frac{1}{2}\alpha(1+\lambda)$, $\pi^*_{CE-PL} = \frac{1}{4}\alpha(1+\lambda)$, and $SW^*_{CE-PL} = \frac{3(\lambda+1)}{8}$. $\square$

Proposition 10. *Under CE-SUB model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	(a) $0 < \alpha \leq \lambda \leq 1$	(b) $\lambda < \alpha \leq \alpha^\dagger$	(c) $\alpha^\dagger < \alpha \leq 1$	(d) $1 < \alpha \leq \max\{3\lambda, 1\}$	(e) $\alpha > \max\{3\lambda, 1\}$
p^*_{CE-SUB}	p_a	p_b	$\frac{\alpha}{\sqrt{\alpha}+1}$	$\frac{\alpha\lambda}{\alpha+\lambda}$	$\frac{\alpha}{2}$
π^*_{CE-SUB}	$\pi_{CE-SUB,a}$	$\pi_{CE-SUB,b}$	$\frac{\alpha}{\left(\sqrt{\alpha}+1\right)^2}$	$\frac{\alpha\lambda}{\alpha+\lambda}$	$\frac{\alpha}{4}$
SW^*_{CE-SUB}	$SW_{CE-SUB,a}$	$SW_{CE-SUB,b}$	$\frac{2\sqrt{\alpha}+1}{2\left(\sqrt{\alpha}+1\right)^2}$	$\frac{1}{2}\left(1+\lambda-\frac{\lambda\left(\alpha^2+\lambda\right)}{(\alpha+\lambda)^2}\right)$	$\frac{3}{8}$
Paid adoption	in both periods	in both periods	in both periods	in both periods	only in period 1

where:

- $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$ *is the unique solution to the equation* $G_{SUB,a}(p) \triangleq 2\alpha^3 - 2(\alpha-1)^2p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p = 0$ *over the interval* $(0, \alpha)$*,*

$\pi_{CE-SUB,a} = p_a \left(2 - \frac{p_a}{\alpha} - \frac{p_a}{1+p_a-\frac{p_a}{\alpha}}\right)$,
and $SW_{CE-SUB,a} = \frac{1}{2}\left(1 + \lambda - \frac{\lambda p_a^2}{\alpha^2} - \frac{p_a^2}{\left(1+p_a-\frac{p_a}{\alpha}\right)^2}\right)$*;*

- $p_b \in \left(\frac{\lambda}{2}, \lambda\right)$ *is the unique solution to the equation* $G_{SUB,b}(p) \triangleq 2\alpha^2\lambda + (\alpha-1)p^2(\alpha(\lambda-4)-2\lambda) - 2(\alpha-1)^2p^3 + 2\alpha p(\alpha(\lambda-1)-2\lambda) = 0$ *over the interval* $(0, \lambda)$*,* $\pi_{CE-SUB,b} = p_b \left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1+p_b-\frac{p_b}{\alpha}}\right)$*, and* $SW_{CE-SUB,b} = \frac{1}{2}\left(1 + \lambda - \frac{p_b^2}{\lambda} - \frac{1}{\left(1+p_b-\frac{p_b}{\alpha}\right)^2}\right)$*; and*

- *threshold* $\alpha^\dagger$ *is defined in implicit form in the proof, in equation* (C.1).

Proof. In period 1, customers subscribe iff $\alpha\theta \geq p$. To make any profit, the firm is constrained to set $0 < p < \alpha$. The marginal adopter has type $\theta_1 = \frac{p}{\alpha}$ and the installed base in period 1 is $N_1 = 1 - \frac{p}{\alpha}$. All period 1 adopters learn the true quality of the product in the first period. At the beginning of period 2, the period 1 non-adopters update their priors via social learning from $a_1 = \alpha$ to $a_2 = \alpha + (1-\alpha)\left(1 - \frac{p}{\alpha}\right) = 1 + p - \frac{p}{\alpha}$. We have two cases:

- Case 1: $0 < \alpha \leq 1$.

  In this case, $a_1 \leq a_2 \leq a = 1$. The marginal customer type for period 1 non-adopters at the beginning of period 2 is $\theta_2 = \frac{p}{1+p-\frac{p}{\alpha}} < \theta_1$. Thus, all customers with types $\theta \in [\theta_2, \theta_1)$ are new adopters in period 2 (i.e., fresh subscribers). For period 1 adopters, while their valuation of the product increased, due to individual depreciation, there is a limited residual value that they can extract in period 2. These past adopters make another decision at the beginning of period 2 on whether to renew subscription or abandon the product. A period 1 adopter with type θ will renew subscription in period 2 iff $p \leq \lambda\theta$. We get several subcases:

  – Case 1-i: $\alpha \leq \lambda \leq 1$.

Then $p/\lambda \leq \theta_1$. All period 1 subscribers renew the subscription in period 2. The profit maximization becomes:

$$\max_{0<p<\alpha} \pi_{CE-SUB} = \max_{0<p<\alpha} p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{1 + p - \frac{p}{\alpha}}\right).$$

It can be shown that SOC is satisfied ($\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$). Hence, FOC is sufficient to determine the optimal price:

$$\frac{\partial \pi_{CE-SUB}}{\partial p} = \frac{2\alpha^3 - 2(\alpha-1)^2 p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p}{\alpha(\alpha + (\alpha-1)p)^2}.$$

When solving FOC ($\frac{\partial \pi_{CE-SUB}}{\partial p} = 0$), it is enough to look at the numerator.

Denote $G_{SUB,a}(p) \triangleq 2\alpha^3 - 2(\alpha-1)^2 p^3 + (\alpha-6)(\alpha-1)\alpha p^2 + 2(\alpha-3)\alpha^2 p = 0$.

It can be easily shown that $G_{SUB,a}(p)$ is decreasing in $\big(-\infty, \frac{(3-\alpha)\alpha}{3(1-\alpha)}\big)$, increasing in $\big(\frac{(3-\alpha)\alpha}{3(1-\alpha)}, \frac{\alpha}{1-\alpha}\big)$, and decreasing in $\big(\frac{\alpha}{1-\alpha}, +\infty\big)$. Moreover, $\alpha < \frac{(3-\alpha)\alpha}{3(1-\alpha)} < \frac{\alpha}{1-\alpha}$.

Evaluating $G_{SUB,a}(p)$ at various threshold points allows us to further narrow the bounds for p_a:

$$G_{SUB,a}(0) > G_{SUB,b}\left(\frac{\alpha}{2}\right) > 0 > G_{SUB,a}(\alpha) > G_{SUB,a}\left(\frac{(3-\alpha)\alpha}{3(1-\alpha)}\right),$$
$$G_{SUB,a}\left(\frac{\alpha}{1-\alpha}\right) < 0.$$

Thus, $G_{SUB,a}(p) = 0$ has a unique solution $p_a \in \big(\frac{\alpha}{2}, \alpha\big)$ over the real line, which is also the price value maximizing the profit in this region. More precisely, $\frac{\partial \pi_{CE-SUB}}{\partial p} > 0$ for $p \in (0, p_a)$ and $\frac{\partial \pi_{CE-SUB}}{\partial p} < 0$ for $p \in (p_a, \alpha)$. The formulas for the optimal profit and associated social welfare follow trivially.

- Case 1-ii: $\lambda < \alpha \leq 1$.

  We explore two subcases:

  * Case 1-ii-a: $\lambda < p < \alpha$.

    Then $p/\lambda > 1 > \theta_1$. In this case, all period 1 subscribers (customers with type $\theta \in [\theta_1, 1]$) unsubscribe in period 2. The profit maximization problem becomes:

$$\max_{\lambda<p<\alpha} \pi_{CE-SUB} = \max_{\lambda<p<\alpha} p\left(1 - \frac{p}{\alpha} + \frac{p}{\alpha} - \frac{p}{1+p-\frac{p}{\alpha}}\right) = \max_{\lambda<p<\alpha} p\left(1 - \frac{p}{1+p-\frac{p}{\alpha}}\right).$$

    It can be shown that $\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$. Thus, FOC is sufficient to determine the optimal price. Solving the unconstrained FOC:

$$\frac{\partial \pi_{CE-SUB}}{\partial p} = \frac{\alpha^2 + p^2 - \alpha(p+2)p}{(\alpha + (\alpha-1)p)^2} = 0,$$

    we get two candidate solutions:

$$p_1 = \frac{\alpha}{1-\sqrt{\alpha}} \text{ and } p_2 = \frac{\alpha}{1+\sqrt{\alpha}}.$$

    It immediately follows that $p_1 > \alpha$ and $p_2 < \alpha$. Thus, p_2 is the only feasible candidate against the upper bound. Comparing p_2 and λ (the lower bound), we get two subcases:

· Case 1-ii-a1: $\frac{\alpha}{1+\sqrt{\alpha}} \leq \lambda$.

Then $p^*_{CE-SUB} \downarrow \lambda$, which is a corner solution and is weakly dominated by the case when $p \leq \lambda$ (case 1-ii-b).

· Case 1-ii-a2: $\frac{\alpha}{1+\sqrt{\alpha}} > \lambda$.

We point out that this subcase is feasible only when $0 < \lambda < \frac{1}{2}$ and $\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2} < \alpha \leq 1$. Then $p^*_{CE-SUB} = \frac{\alpha}{\sqrt{\alpha}+1}$, $\pi^*_{CE-SUB} = \frac{\alpha}{\left(\sqrt{\alpha}+1\right)^2}$, and $SW^*_{CE-SUB} = \frac{2\sqrt{\alpha}+1}{2\left(\sqrt{\alpha}+1\right)^2}$.

$*$ Case 1-ii-b: $p \leq \lambda$.

Then $1 \geq p/\lambda > \theta_1$. In this case, period 1 subscribers with type $\theta \in [\theta_1, p/\lambda)$ unsubscribe in period 2. The profit maximization problem becomes:

$$\begin{aligned}\max_{p\leq\lambda} \pi_{CE-SUB} &= \max_{p\leq\lambda} p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{\lambda} + \frac{p}{\alpha} - \frac{p}{1+p-\frac{p}{\alpha}}\right) \\ &= \max_{p\leq\lambda} p\left(2 - \frac{p}{\lambda} - \frac{p}{1+p-\frac{p}{\alpha}}\right).\end{aligned}$$

It can be shown that $\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$. Thus, FOC is sufficient to determine the optimal price. The FOC of the profit function is:

$$\begin{aligned}\frac{\partial \pi_{CE-SUB}}{\partial p} &= \frac{2\alpha^2\lambda + (\alpha-1)p^2(\alpha(\lambda-4)-2\lambda) - 2(\alpha-1)^2p^3 + 2\alpha p(\alpha(\lambda-1)-2\lambda)}{\lambda(\alpha+\alpha p - p)^2} \\ &= 0.\end{aligned}$$

Denote $G_{SUB,b}(p) \triangleq 2\alpha^2\lambda + (\alpha-1)p^2(\alpha(\lambda-4)-2\lambda) - 2(\alpha-1)^2p^3 + 2\alpha p(\alpha(\lambda-1)-2\lambda)$. It can be easily shown that $G_{SUB,b}(p)$ is decreasing in $\big(-$

$\infty, \frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)}\big)$, increasing in $\big(\frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)}, \frac{\alpha}{1-\alpha}\big)$, and decreasing in $\big(\frac{\alpha}{1-\alpha}, +\infty\big)$. Moreover, under the current case, $\lambda < \frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)} < \frac{\alpha}{1-\alpha}$.

Evaluating $G_{SUB,b}(p)$ at various threshold points allows us to further narrow the bounds for p_b. In particular, since we are in the case $\lambda < \alpha \leq 1$, we have:

$$G_{SUB,b}(0) > G_{SUB,b}\left(\frac{\lambda}{2}\right) > 0 > G_{SUB,b}(\lambda) > G_{SUB,b}\left(\frac{\alpha(1-\lambda)+2\lambda}{3(1-\alpha)}\right),$$
$$G_{SUB,b}\left(\frac{\alpha}{1-\alpha}\right) < 0.$$

Thus, $G_{SUB,b}(p) = 0$ has a unique solution $p_b \in \big(\frac{\lambda}{2}, \lambda\big)$ over the real line, which is also the price value maximizing the profit in this region. More precisely, $\frac{\partial \pi_{CE-SUB}}{\partial p} > 0$ for $p \in (0, p_b)$ and $\frac{\partial \pi_{CE-SUB}}{\partial p} < 0$ for $p \in (p_b, \lambda)$. The formulas for the optimal profit and associated social welfare follow trivially.

We next need to compare the optimal profits under cases 1-ii-a2 and 1-ii-b for the region in which we can simultaneously have $\lambda < \alpha \leq 1$ and $\frac{\alpha}{1+\sqrt{\alpha}} > \lambda$. As mentioned above (under the discussion of case 1-ii-a2), that region is characterized by $0 < \lambda < \frac{1}{2}$ and $\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2} < \alpha \leq 1$. Define the difference between the optimal profits under cases 1-ii-b and 1-ii-a2 as:

$$\Xi(p_b(\alpha,\lambda),\alpha,\lambda) \triangleq p_b(\alpha,\lambda)\left(2 - \frac{p_b(\alpha,\lambda)}{\lambda} - \frac{p_b(\alpha,\lambda)}{1+p_b(\alpha,\lambda)-\frac{p_b(\alpha,\lambda)}{\alpha}}\right) - \frac{\alpha}{(\sqrt{\alpha}+1)^2},$$

where $p_b(\alpha,\lambda)$ is the unique solution mentioned in case 1-ii-b. From the Envelope theorem, and using $p_b < \lambda < \frac{\alpha}{1+\sqrt{\alpha}}$, we obtain:

$$\begin{aligned}\frac{\partial \Xi(p_b(\alpha,\lambda),\alpha,\lambda)}{\partial \alpha} &= \frac{p_b(\alpha,\lambda)^3}{(\alpha-(1-\alpha)p_b(\alpha,\lambda))^2} - \frac{1}{\left(\sqrt{\alpha}+1\right)^3} \\ &= \frac{p_b(\alpha,\lambda)^3(1+\sqrt{\alpha})^3 - (\alpha-(1-\alpha)p_b(\alpha,\lambda))^2}{(\alpha-(1-\alpha)p_b(\alpha,\lambda))^2(1+\sqrt{\alpha})^3} \\ &< \frac{\alpha^3 - (\alpha-(1-\alpha)\frac{\alpha}{1+\sqrt{\alpha}})^2}{(\alpha-(1-\alpha)p_b(\alpha,\lambda))^2(1+\sqrt{\alpha})^3} = 0.\end{aligned}$$

Thus, $\Xi(p_b(\alpha,\lambda),\alpha,\lambda)$ is decreasing in α.

Note that, since $p_b(\cdot)$ maximizes the profit under case 1-ii-b, it also maximizes $\Xi(p,\cdot)$ under the feasible region and it is a strictly interior solution. As such, since $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) > \Xi(p,\alpha,\lambda)\,|_{p=\lambda}$ for all $\alpha \in \left(\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}, 1\right]$ when $\lambda < \frac{1}{2}$. By applying this inequality, the fact that $\frac{\alpha}{(1+\sqrt{\alpha})^2} = \frac{\lambda^2}{\alpha}$ when $\alpha = \frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}$, and a few algebraic manipulations of the grouped expressions, we get the sign of Ξ at that lower boundary for α:

$$\begin{aligned}\Xi(p_b(\alpha,\lambda),\alpha,\lambda)\Big|_{\alpha=\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}),0<\lambda<\frac{1}{2}} &> \Xi(p,\alpha,\lambda)\Big|_{p=\lambda,\alpha=\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2},0<\lambda<\frac{1}{2}} \\ &= \lambda\left(2-\frac{\lambda}{\lambda}-\frac{\lambda}{1+\lambda-\frac{\lambda}{\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}}}\right) \\ &\quad -\frac{2\lambda}{\lambda+2+\sqrt{\lambda^2+4\lambda}} \\ &= 0.\end{aligned}$$

At the upper boundary, when $\alpha = 1$, we can directly solve $p_b(1,\lambda) = \frac{\lambda}{1+\lambda}$. Thus,

$\Xi(p_b(1,\lambda),1,\lambda) = \frac{\lambda}{1+\lambda} - \frac{1}{4}$. It immediately follows that:

$$\Xi(p_b(1,\lambda),1,\lambda) \begin{cases} < 0 & \text{if } 0 < \lambda < \frac{1}{3}, \\ \geq 0 & \text{if } \frac{1}{3} \leq \lambda < \frac{1}{2}. \end{cases}$$

Given that $\Xi(p_b(\alpha,\lambda),\alpha,\lambda)$ is decreasing in α, then, $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) \geq 0$ when $\frac{1}{3} \leq \lambda < \frac{1}{2}$ and $\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2} < \alpha \leq 1$. As such, in this region, profit under case 1-ii-b dominates profit under case 1-ii-a2.

However, when $0 < \lambda < \frac{1}{3}$, we have a single crossing. In other words, there exits a unique threshold $\tilde{\alpha}^\dagger \in \left(\frac{\lambda(\lambda+2+\sqrt{\lambda^2+4\lambda})}{2}, 1\right]$ such that $\Xi(p_b(\tilde{\alpha}^\dagger,\lambda),\tilde{\alpha}^\dagger,\lambda) = 0$, $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) > 0$ for all $\alpha \in (\lambda, \tilde{\alpha}^\dagger)$, and $\Xi(p_b(\alpha,\lambda),\alpha,\lambda) < 0$ for all $\alpha \in (\tilde{\alpha}^\dagger, 1]$. Denote $\alpha^\dagger$ as:

$$\alpha^\dagger \triangleq \begin{cases} \tilde{\alpha}^\dagger, & \text{if } 0 < \lambda < \frac{1}{3}, \\ 1, & \text{if } \frac{1}{3} \leq \lambda \leq 1. \end{cases} \tag{C.1}$$

Then, we obtain that Case 1-ii-b dominates Case 1-ii-a when $\lambda < \alpha < \alpha\dagger$ and Case 1-ii-a dominates Case 1-ii-b when $\alpha^\dagger \leq \alpha \leq 1$. Defining $\alpha^\dagger$ as in (C.1) ensures that region $\alpha^\dagger \leq \alpha \leq 1$ vanishes if feasibility conditions are not met.

- Case 2: $\alpha > 1$.

  In this case, $a_1 > a_2 > a = 1$. None of period 1 non-adopters will subscribe in period 2. Also, *only part* of the period 1 adopters will renew the subscription in period 2 because of tandem pressure from both the downward updating of the valuation and the individual depreciation. The marginal subscriber in period 2 has type $\theta_2 = \min\left\{1, \frac{p}{\lambda}\right\} > \theta_1$. We have two subcases:

– Case 2-i: $0 < p < \lambda$.

  Then, we have $\theta_2 = \frac{p}{\lambda}$ and $N_2 = 1 - \frac{p}{\lambda}$. The firm's profit maximization problem becomes:

$$\max_{0<p<\lambda} \pi_{CE-SUB} = \max_{0<p<\lambda} p\left(1 - \frac{p}{\alpha} + 1 - \frac{p}{\lambda}\right).$$

  We have $\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$. From FOC, we obtain the following interior solution $p^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$, $\pi^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$.

– Case 2-ii: $\lambda \leq p < \alpha$.

  Then, we have $\theta_2 = 1$ and $N_2 = 0$. The firm's profit maximization problem becomes:

$$\max_{\lambda\leq p<\alpha} \pi_{CE-SUB} = \max_{\lambda\leq p<\alpha} p\left(1 - \frac{p}{\alpha}\right).$$

  We have two subcases:

  * Case 2-ii-a: $\alpha \leq 2\lambda$.

    This case is feasible only if $\frac{1}{2} < \lambda < 1$. Then, $p^*_{CE-SUB} = \lambda$ and $\pi^*_{CE-SUB} = \lambda\left(1 - \frac{\lambda}{\alpha}\right)$. However, we do notice that $\left(1 - \frac{\lambda}{\alpha}\right) < \frac{\alpha\lambda}{\alpha+\lambda}$. As such, case 2-i dominates case 2-ii-a and we do not have to consider case 2-ii-a going further.

  * Case 2-ii-b: $\alpha > \max\{2\lambda, 1\}$.

    Then, $p^*_{CE-SUB} = \frac{\alpha}{2}$ and $\pi^*_{CE-SUB} = \frac{\alpha}{4}$.

Comparing profits under cases 2-i and 2-ii-b, we get:

* If $\alpha \leq 3\lambda$, then $p^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$, $\pi^*_{CE-SUB} = \frac{\alpha\lambda}{\alpha+\lambda}$, and $SW^*_{CE-SUB} = \frac{1}{2}\Big(1 + \lambda - \frac{\lambda(\alpha^2+\lambda)}{(\alpha+\lambda)^2}\Big)$. We point out that this case is only feasible when $\lambda > \frac{1}{3}$. This is why we define this region as $1 < \alpha \leq \max\{1, 3\lambda\}$ in the text of the proposition and we point out this region vanishes when $\lambda < \frac{1}{3}$.
* If $\alpha > \max\{1, 3\lambda\}$, $p^*_{CE-SUB} = \frac{\alpha}{2}$, $\pi^*_{CE-SUB} = \frac{\alpha}{4}$, and $SW^*_{CE-SUB} = \frac{3}{8}$. □

Proposition 11. *Under TLF model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are given by $p^*_{TLF} = \frac{\lambda}{2}$, $\pi^*_{TLF} = \frac{\lambda}{4}$, and $SW^*_{TLF} = \frac{3\lambda}{8} + \frac{1}{2}$.*

Proof. Under *TLF*, all consumers get the product for free in period 1, i.e., $N_{1,total} = 1$ (but the number of paying customers is $N_1 = 0$). Consequently, in period 2, all customers update their prior on quality to $a_2 = a = 1$. Taking into account depreciation, customers purchase the product in period 2 iff their types satisfy $\theta\lambda \geq p$. The firm's profit maximization problem is:

$$\max_{0<p<\lambda} \pi = p\left(1 - \frac{p}{\lambda}\right),$$

which yields $p^*_{TLF} = \frac{\lambda}{2}$, $\pi^*_{TLF} = \frac{\lambda}{4}$, and $SW^*_{TLF} = \frac{3\lambda}{8} + \frac{1}{2}$. □

Proposition 12. *Under S model, in the presence of exogenous individual depreciation, the firm's optimal pricing strategy, corresponding profit, and ensuing social welfare are:*

	(a) $0 < \alpha < \alpha^{\ddagger}$	(b) $\alpha^{\ddagger} \leq \alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$	(c) $\alpha \geq 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$
k_S^*	$\frac{1-2\alpha}{2(1-\alpha)}$	0	0
p_S^*	$\frac{1}{4}$	$\frac{\alpha(\lambda+1)\left(\alpha\lambda+1-\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)(\alpha\lambda+1)}$	$\frac{1}{2}\alpha(1+\lambda)$
π_S^*	$\frac{1}{16(1-\alpha)}$	$\frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$	$\frac{1}{4}\alpha(1+\lambda)$
SW_S^*	$\frac{4\lambda+7-8\alpha(\lambda+1)}{16(1-\alpha)}$	$\tilde{SW}_{CE-PL}$	$\frac{3(1+\lambda)}{8}$
Paid adoption	only in period 2	in both periods	only in period 1

where threshold $\alpha^{\ddagger}$ is the unique solution to equation $\alpha(32\alpha(\lambda+1)(8\alpha(\lambda+1)-6\lambda-7)+32\lambda+33)-1=0$ over the interval $\left(\frac{2\lambda(6\lambda+13)+14-(\lambda+1)\sqrt{48\lambda(3\lambda+5)+97}}{48(\lambda+1)^2}, \frac{1}{4(\lambda+1)}\right)$.

Proof. First, we point out that *CE-PL* is a particular case of *S* with seeding ratio set to zero. Throughout the proof, we will show that in certain regions *CE-PL* dominates *S* with non-zero seeding ratio - that is equivalent to saying that the optimal seeding ration will be 0 in those regions (i.e., *S* defaults to *CE-PL*).

If $\alpha \geq 1$, seeding brings no benefit as any social learning calibrates perceived valuations downwards, and, as such, *S* defaults to *CE-PL*.

Thus, we are left to explore the non-trivial case of $0 < \alpha < 1$. We have two cases:

- Case 1: $0 < p < (1+\lambda)\alpha$.

  There are paying adopters in period 1 (potentially alongside seeded customers if $k > 0$). The marginal paying customer in period 1 has type $\theta_1 = \frac{p}{\alpha(1+\lambda)}$. Then, the total number of adopters in period 1 is $N_{1,total} = (1-k)\left(1 - \frac{p}{\alpha(1+\lambda)}\right) + k$. In period 2, the potential customers who have not adopted in period 1 update their prior beliefs via social learning as follows:

$$\begin{aligned}a_2 &= a_1 + N_{1,total}\,(1-a_1)\\ &= \alpha + (1-\alpha)\left((1-k)\left(1-\frac{p}{\alpha(1+\lambda)}\right)+k\right)\\ &= 1-\frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}.\end{aligned}$$

A customer of type θ who has not adopted in period 1 (via paying for license or through the seeding program) will adopt in period 2 iff $\theta_1 > \theta \geq \theta_2 = \frac{p}{1-\frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}}$. Comparing θ_1 and θ_2, we have:

$$\theta_1 > \theta_2 \quad \Longleftrightarrow \quad 0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}.$$

We have two cases:

- Case 1-i: $\alpha + \alpha\lambda \geq 1$.

  In this case, we have $\theta_2 > \theta_1$ for any $k \in [0,1)$. There are no paying adopters in period 2. The firm's profit maximization problem becomes:

$$\max_{0<p<(1+\lambda)\alpha,\,0\leq k<1} \pi_S = \max_{0<p<(1+\lambda)\alpha,\,0\leq k<1} p(1-k)\left(1-\frac{p}{\alpha(1+\lambda)}\right).$$

  It trivially follows that $k_S^* = 0$. *S* defaults to *CE-PL*.

- Case 1-ii: $\alpha + \alpha\lambda < 1$.

We have:

$$\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} < (1+\lambda)\alpha \quad \Longleftrightarrow \quad 0 \leq k < \frac{\alpha\lambda}{1-\alpha}.$$

Subsequently, we have several subcases:

* Case 1-ii-a: $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$ and $0 \leq k < \frac{\alpha\lambda}{1-\alpha}$.
  In this case, $\theta_2 \leq \theta_1$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been seeded in period 1, adopt in period 2. The firm's profit maximization problem becomes:

$$\max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)},\, 0\leq k<\frac{\alpha\lambda}{1-\alpha}} \pi_S = \max_{0<p<\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)},\, 0\leq k<\frac{\alpha\lambda}{1-\alpha}} p(1-k)\left(1-\frac{p}{1-\frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}}\right).$$

  Taking first order derivative of the profit w.r.t. p, we get:

$$\frac{\partial\pi_S}{\partial p} = \frac{(1-k)\left[\alpha^2(\lambda+1)^2 + (1-\alpha)(1-k)p^2(\alpha\lambda+(\alpha-1)k+1) - 2\alpha(\lambda+1)p(\alpha\lambda-(1-\alpha)k+1)\right]}{((1-\alpha)(1-k)p-\alpha(\lambda+1))^2}.$$

  The denominator is always positive. We define the numerator as a function:

$$\eta(p) \triangleq \alpha^2(\lambda+1)^2 + (1-\alpha)(1-k)p^2(\alpha\lambda-(1-\alpha)k+1) - 2\alpha(\lambda+1)p(\alpha\lambda-(1-\alpha)k+1).$$

  $\eta(p)$ is convex in p. Solving in unconstrained form the equation $\eta(p) = 0$, we obtain two candidate solutions:

$$p_1 = \frac{\alpha(\lambda+1)(\alpha\lambda-(1-\alpha)k+1) - \sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda-(1-\alpha)k+1)}}{(1-\alpha)(1-k)(\alpha\lambda-(1-\alpha)k+1)},$$
$$p_2 = \frac{\alpha(\lambda+1)(\alpha\lambda-(1-\alpha)k+1) + \sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda-(1-\alpha)k+1)}}{(1-\alpha)(1-k)(\alpha\lambda-(1-\alpha)k+1)}.$$

It can be easily shown that $p_1 > 0$ and $p_2 > \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Moreover:

$$p_1 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \quad\iff\quad \frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)} < k.$$

We need to consider several subcases:

· Case 1-ii-a-I: $\alpha(1+\lambda)(1+\alpha\lambda) \geq 1$.

□ Case 1-ii-a-I1: $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$ and $0 \leq k \leq \frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)}$.
In this case, $p_1 \geq \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. As such, $\eta(p) > 0$ for all $0 < p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Thus, $\pi_S(p)$ is strictly increasing in p in this region and the profit in this case is strictly dominated by the profit under Case 1-ii-b.

□ Case 1-ii-a-I2: $0 \leq p < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$ and $\frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)} < k < \frac{\alpha\lambda}{1-\alpha}$.
In this case, $p_1 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. As such, $\eta(p) > 0$ for all $p \in (0, p_1)$ and $\eta(p) < 0$ for all $p \in \left(p_1, \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}\right)$. Thus, $p_S^* = p_1$. The profit function can be simplified to:

$$\pi_S = \frac{-2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda-(1-\alpha)k+1)} + \alpha(\lambda+1)(2\alpha\lambda+\alpha-(1-\alpha)k+1)}{(1-\alpha)^2(1-k)}.$$

It is straightforward to show that $\alpha > 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$ in this case, which corresponds to the second case under *CE-PL*. For any $k \in \left(\frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)}, \frac{\alpha\lambda}{1-\alpha}\right)$, it can be easily shown that $\pi_S(k) < \frac{1}{4}\alpha(1+\lambda) = \pi^*_{CE-PL}$. Therefore, this case is sub-optimal, as it is dominated by not seeding anyone.

- Case 1-ii-a-II: $\alpha(1+\lambda)(1+\alpha\lambda) < 1$.

  In this case, it immediately follows that $\frac{\alpha(1+\lambda)(1+\alpha\lambda)-1}{\alpha(1-\alpha)(1+\lambda)} < 0 \leq k$. Thus, $p_1 < \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Similar to case 1-ii-a-I2, we have $p_S^* = p_1$. Following the same steps in Case 1-ii-a, we get $k_S^* = 0$. *S* defaults to *CE-PL*.

* Case 1-ii-b: $\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \leq p < (1+\lambda)\alpha$ and $0 \leq k < \frac{\alpha\lambda}{1-\alpha}$.
  In this case, $\theta_2 \geq \theta_1$. There are no new adopters in period 2. The firm's profit maximization problem becomes:

$$\max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \leq p<(1+\lambda)\alpha,\, 0\leq k<\frac{\alpha\lambda}{1-\alpha}} \pi_S = \max_{\frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)} \leq p<(1+\lambda)\alpha,\, 0\leq k<\frac{\alpha\lambda}{1-\alpha}} p(1-k)\left(1-\frac{p}{\alpha(1+\lambda)}\right).$$

  It trivially follows that $k_S^* = 0$. *S* defaults to *CE-PL*.

* Case 1-ii-c: $0 < p < (1+\lambda)\alpha$ and $\frac{\alpha\lambda}{1-\alpha} \leq k < 1$.

  In this case, $p < (1+\lambda)\alpha \leq \frac{\alpha(\lambda+1)(1-\alpha-\alpha\lambda)}{(1-\alpha)(1-k)}$. Then, $\theta_2 \leq \theta_1$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been seeded in period 1, adopt in period 2. The firm's profit maximization problem becomes:

$$\max_{0<p<(1+\lambda)\alpha,\frac{\alpha\lambda}{1-\alpha}\leq k<1} \pi_S = \max_{0<p<(1+\lambda)\alpha,\frac{\alpha\lambda}{1-\alpha}\leq k<1} p(1-k)\left(1-\frac{p}{1-\frac{(1-\alpha)(1-k)p}{\alpha(1+\lambda)}}\right).$$

  Following the same steps as in Case 1-ii-a, we get the same two solutions, p_1 and p_2, to the equation $\eta(p) = 0$. It can be easily shown that $p_1 > 0$ and

$p_2 > (1+\lambda)\alpha$. Moreover:

$$p_1 < (1+\lambda)\alpha \quad \Longleftrightarrow \quad \frac{\alpha\lambda}{1-\alpha} \leq k < \min\left\{\frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)}, 1\right\}.$$

We have several subcases:

- Case 1-ii-c-I: $\alpha\lambda + \sqrt{\alpha(\lambda+1)(\alpha\lambda+\alpha+4)} + \alpha < 2$.

  Then, it can be shown that $\frac{\alpha\lambda}{1-\alpha} < \frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)} < 1$.

  □ Case 1-ii-c-I1: $\frac{\alpha\lambda}{1-\alpha} \leq k < \frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)}$.
  In this case we have $p_1 < (1+\lambda)\alpha$. Then, we have the interior solution $p_S^* = p_1$. The profit is simplified to:

$$\pi_S = \frac{\alpha(\lambda+1)\left(-2\sqrt{\alpha(\lambda+1)(\alpha\lambda-(1-\alpha)k+1)} + (2\alpha\lambda+\alpha-(1-\alpha)k+1)\right)}{(1-\alpha)^2(1-k)}.$$

  It is straightforward to show that $0 < \alpha < 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$ in this case, which corresponds to the first case under *CE-PL*. The first order derivative w.r.t. k satisfies $\frac{\partial\pi_S}{\partial k} < 0$. Hence, we have corner solution $k_S^* = \frac{\alpha\lambda}{1-\alpha}$. The optimal profit is simplified to:

$$\begin{aligned}\pi_S^* &= \frac{\alpha(\lambda+1)\left(\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)(1-\alpha-\alpha\lambda)} \\ &< \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} = \pi_{CE-PL}^*.\end{aligned}$$

Therefore, this case is sub-optimal.

☐ Case 1-ii-c-I2: $\frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)} \leq k < 1$.

In this case, we have $p_1 \geq (1+\lambda)\alpha$. We can see that, for any k in this region, $\pi_S(p)$ is strictly increasing in p and the profit in this case is strictly dominated by the profit under Case 2.

* Case 1-ii-c-II: $\alpha\lambda + \sqrt{\alpha(\lambda+1)(\alpha\lambda+\alpha+4)} + \alpha \geq 2$.

  Then, it can be shown that $\frac{\alpha\lambda}{1-\alpha} < 1 \leq \frac{-\alpha+\alpha\lambda+\sqrt{\alpha(1+\lambda)(4+\alpha+\alpha\lambda}}{2(1-\alpha)}$. As such, when $\frac{\alpha\lambda}{1-\alpha} \leq k \leq 1$, we have $p_1 < (1+\lambda)\alpha$, and, thus, we have the interior solution $p_S^* = p_1$. Following the same step in Case 1ii-c-I1, we get $k_S^* = \frac{\alpha\lambda}{1-\alpha}$ and, following the same reasoning, it can be shown that this case is sub-optimal as well.

In summary, we have shown that Case 1 either defaults to *CE-PL* or is strictly dominated by *CE-PL*.

- Case 2: $p \geq (1+\lambda)\alpha$.

  In this case, there are only seeded consumers in period 1 (i.e., no unseeded customer is willing to pay for the product based on priors). Hence, $N_{1,total} = k$. At the beginning of period 2, the un-seeded customers update their priors to $a_2 = \alpha + (1-\alpha)k$. The firm's profit maximization problem becomes:

$$\max_{p\geq(1+\lambda)\alpha,\,0\leq k<1} \pi_S = \max_{p\geq(1+\lambda)\alpha,\,0\leq k<1} p(1-k)\left(1 - \frac{p}{\alpha - \alpha k + k}\right).$$

The profit is concave in p. The first order derivative w.r.t p is:

$$\frac{\partial \pi_S}{\partial p} = \frac{(1-k)(\alpha + (1-\alpha)k - 2p)}{\alpha - \alpha k + k}.$$

From FOC, the uncontrained optimizer is $\bar{p} = \frac{\alpha+(1-\alpha)k}{2}$. We have:

$$\begin{aligned} \bar{p} \geq (1+\lambda)\alpha \quad &\iff \quad k \geq \frac{2\alpha\lambda + \alpha}{1-\alpha}, \\ \frac{2\alpha\lambda + \alpha}{1-\alpha} < 1 \quad &\iff \quad \alpha(\lambda+1) < \frac{1}{2}. \end{aligned}$$

We get two subcases:

– Case 2-i: $\alpha(\lambda+1) < \frac{1}{2}$.

  Then $\frac{2\alpha\lambda+\alpha}{1-\alpha} < 1$.

  * Case 2-i-a: $0 \leq k < \frac{2\alpha\lambda+\alpha}{1-\alpha}$.

    Then $\bar{p} < (1+\lambda)\alpha$. As such, we have the corner solution $p_S^* = \alpha(\lambda+1)$. The firm's profit maximization problem becomes:

$$\max_{0 \leq k < \frac{2\alpha\lambda+\alpha}{1-\alpha}} \pi_S = \max_{0 \leq k < \frac{2\alpha\lambda+\alpha}{1-\alpha}} \alpha(\lambda+1)(1-k)\left(1 - \frac{\alpha(\lambda+1)}{\alpha - \alpha k + k}\right).$$

    We have:

$$\frac{\partial \pi_S}{\partial k} = -\frac{\alpha(\lambda+1)\left(-\alpha\lambda - \alpha + (\alpha - \alpha k + k)^2\right)}{(\alpha - \alpha k + k)^2}.$$

Solving the unconstrained equation $\frac{\partial \pi_S}{\partial k} = 0$, we obtain two candidate solutions:

$$k_1 = \frac{-\alpha - \sqrt{\alpha(\lambda+1)}}{1-\alpha} < 0 < k_2 = \frac{-\alpha + \sqrt{\alpha(\lambda+1)}}{1-\alpha}.$$

π_S is decreasing in k on $(-\infty, k_1)$, increasing on (k_1, k_2), and then decreasing on (k_2, ∞). Comparing k_2 and $\frac{2\alpha\lambda+\alpha}{1-\alpha}$, we get two subcases:

· Case 2-i-a-I: $\frac{1}{4} < \alpha(\lambda+1) < \frac{1}{2}$.

Then, $k_2 < \frac{2\alpha\lambda+\alpha}{1-\alpha}$. Thus, we get the interior solution $k_S^* = k_2$ and $\pi_S^* = \frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2}$.

· Case 2-i-a-II: $0 < \alpha(\lambda+1) \leq \frac{1}{4}$.

Then, $k_2 \geq \frac{2\alpha\lambda+\alpha}{1-\alpha}$. Thus, π_S is increasing in k on the entire region and this case gets dominated by case 2-i-b-II.

∗ Case 2-i-b: $\frac{2\alpha\lambda+\alpha}{1-\alpha} \leq k < 1$.

In this case, $\bar{p} \geq (1+\lambda)\alpha$. Thus, we have the interior solution $p_S^* = \bar{p} = \frac{\alpha+(1-\alpha)k}{2}$. The firm's profit maximization problem becomes:

$$\max_{\frac{2\alpha\lambda+\alpha}{1-\alpha} \leq k < 1} \pi_S = \max_{\frac{2\alpha\lambda+\alpha}{1-\alpha} \leq k < 1} \frac{(1-k)(\alpha + k(1-\alpha))}{4}.$$

The profit function is concave in k. We have:

$$\frac{\partial \pi_S}{\partial k} = \frac{1}{4}(1 - 2\alpha - 2(1-\alpha)k).$$

Solving the unconstrained equation $\frac{\partial \pi_S}{\partial k} = 0$, we obtain the candidate solution $\bar{k}_S = \frac{1-2\alpha}{2(1-\alpha)} < 1$. We also have:

$$\frac{2\alpha\lambda + \alpha}{1-\alpha} \leq \bar{k}_S \quad \iff \quad \alpha(\lambda+1) \leq \frac{1}{4}.$$

We get two subcases:

· Case 2-i-b-I: $\frac{1}{4} < \alpha(\lambda+1) < \frac{1}{2}$.

Then, $\frac{2\alpha\lambda+\alpha}{1-\alpha} > \bar{k}_S$. As such, we get the corner solution $k^* = \frac{2\alpha\lambda+\alpha}{1-\alpha}$. Substituting k_S^*, we obtain $p_S^* = \alpha(\lambda+1)$ and $\pi_S^* = \frac{\alpha(\lambda+1)(1-2\alpha(\lambda+1))}{2(1-\alpha)}$.

· Case 2-i-b-II: $\alpha(\lambda+1) \leq \frac{1}{4}$.

Then, $\frac{2\alpha\lambda+\alpha}{1-\alpha} > \bar{k}_S$ and we get the interior solution $k_S^* = \bar{k}_S = \frac{1-2\alpha}{2(1-\alpha)}$. Substituting k_S^*, we obtain $p_S^* = \frac{1}{4}$ and $\pi_S^* = \frac{1}{16(1-\alpha)}$.

Comparing Cases 2-i-a and 2-i-b, we get:

∗ If $\alpha(\lambda+1) \leq \frac{1}{4}$, $p_S^* = \frac{1}{4}$, $k_S^* = \frac{1-2\alpha}{2(1-\alpha)}$, $\pi_S^* = \frac{1}{16-16\alpha}$.

* If $\frac{1}{4} < \alpha(\lambda + 1) < \frac{1}{2}$, then $p_S^* = \alpha(\lambda + 1)$ under both 2-i-a-I and 2-i-b-I. Comparing the profits directly, it can be shown that Case 2-i-a-I dominates. Thus, $k_S^* = k_2$ and $\pi_S^* = \frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2}$.

[Comparison between *S* and *CE-PL*]

Under both cases (i.e., when $\alpha(\lambda+1) < \frac{1}{2}$), we get $0 < \alpha < 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$. Therefore, in this region $\pi_{CE-PL}^* = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$. We have two regions to compare:

* If $\alpha(\lambda + 1) \leq \frac{1}{4}$, then $\pi_S^* = \frac{1}{16-16\alpha}$. It can be shown that:

$$\pi_S^* > \pi_{CE-PL}^* \iff \begin{cases} \alpha(\lambda+1) \leq \frac{1}{4}, \text{ and} \\ \\ 32\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)} + 1 > \alpha(16\alpha(\lambda+1)(2\lambda+1) + 16\lambda + 17). \end{cases} \tag{C.2}$$

When *S* dominates *CE-PL*, we have:

$$SW_S^* = k_S^* \int_0^1 (1+\lambda)\theta d\theta + (1-k_S^*) \int_{\frac{p_S^*}{\alpha-\alpha k_S^*+k_S^*}}^1 \theta d\theta = \frac{4\lambda+7-8\alpha(\lambda+1)}{16(1-\alpha)}.$$

Let us better understand the region characterized under condition (C.2).

· If $0 < \alpha \leq \frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}$, then $\alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17)-1 \leq 0$. Then, the inequality $32\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}+1 > \alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17)$ is always satisfied.

· If $\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}} < \alpha \leq \frac{1}{4(\lambda+1)}$,[1] then $\alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17)-1>0$. In this region, we have:

$$32\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}+1 > \alpha(16\alpha(\lambda+1)(2\lambda+1)+16\lambda+17)$$
$$\iff \Gamma(\alpha) \triangleq \alpha(32\alpha(\lambda+1)(8\alpha(\lambda+1)-6\lambda-7)+32\lambda+33)-1>0.$$

Solving $\frac{\partial\Gamma(\alpha)}{\partial\alpha}=0$, we get two solutions:

$$\tilde{\alpha}_1 = \frac{2\lambda(6\lambda+13)+14-(\lambda+1)\sqrt{48\lambda(3\lambda+5)+97}}{48(\lambda+1)^2},$$
$$\tilde{\alpha}_2 = \frac{2\lambda(6\lambda+13)+14+(\lambda+1)\sqrt{48\lambda(3\lambda+5)+97}}{48(\lambda+1)^2}.$$

It can be easily shown that $0 < \frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}} < \tilde{\alpha}_1 < \frac{1}{4(\lambda+1)} < \tilde{\alpha}_2, \Gamma(\tilde{\alpha}_1) > 0, \Gamma\left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}\right) > 0, \Gamma\left(\frac{1}{4(\lambda+1)}\right) < 0, \Gamma(\tilde{\alpha}_2) < 0$. In terms of monotonicity, $\Gamma(\alpha)$ is increasing on $\left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \tilde{\alpha}_1\right)$ and then decreasing on $\left(\tilde{\alpha}_1, \frac{1}{4(\lambda+1)}\right]$. Therefore, there exits a unique $\alpha^\ddagger \in \left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \frac{1}{4(\lambda+1)}\right)$, such that $\Gamma(\alpha^\ddagger)=0$. Thus, when $\alpha \in \left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \alpha^\ddagger\right)$, *S* dominates *CE-PL*, and when $\alpha \in \left[\alpha^\ddagger, \frac{1}{4(\lambda+1)}\right)$, *CE-PL* dominates *S*.

Moreover, the range for $\alpha^\ddagger$ can be further narrowed to:

$$\alpha^\ddagger \in \left(\tilde{\alpha}_1, \frac{1}{4(\lambda+1)}\right) \subset \left(\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}}, \frac{1}{4(\lambda+1)}\right].$$

[1]We also verified that $\frac{2}{16\lambda+17+\sqrt{32\lambda(12\lambda+23)+353}} < \frac{1}{4(\lambda+1)}$ to make sure this region exists.

Thus, putting the two subregions together, *S* dominates *CE-PL* when $0 < \alpha < \alpha^{\ddagger}$ and *CE-PL* dominates *S* when $\alpha \in \left[\alpha^{\ddagger}, \frac{1}{4(\lambda+1)}\right)$.

* If $\frac{1}{4} \leq \alpha(\lambda+1) < \frac{1}{2}$, it can be shown that:

$$\begin{aligned}\pi_S^* &= \frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2} \\ &< \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} = \pi_{CE-PL}^*.\end{aligned}$$

– Case 2-ii: $\alpha(\lambda+1) \geq \frac{1}{2}$.

Then $\frac{2\alpha\lambda+\alpha}{1-\alpha} \geq 1 > k$. As such $\bar{p} < (1+\lambda)\alpha$. Thus, we have the corner solution $p_S^* = \alpha(\lambda+1)$. Following the same steps as Case 2-i, we get threshold values $k_1 < 0 < k_2$. Comparing k_2 with 1, we have two subcases:

* Case 2-ii-a: $\frac{1}{2} \leq \alpha(\lambda+1) < 1$.

Then $k_S^* = k_2 = \frac{-\alpha+\sqrt{\alpha(\lambda+1)}}{1-\alpha}$ and $\pi_S^* = \frac{\alpha(\lambda+1)\left(1+\alpha((1-\alpha)\lambda+\alpha)-2(1-\alpha)\sqrt{\alpha(\lambda+1)}\right)}{(1-\alpha)^2}$. It can be shown that:

$$\pi_S^* < \begin{cases} \frac{\alpha(\lambda+1)(2\alpha\lambda+\alpha+1)-2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}}{(1-\alpha)^2} = \pi_{CE-PL}^* \\ \text{, if } \frac{1}{2(\lambda+1)} \leq \alpha < 5+8\lambda-4\sqrt{(\lambda+1)(4\lambda+1)}, \\ \frac{1}{4}\alpha(\lambda+1) = \pi_{CE-PL}^* \\ \text{, if } 5+8\lambda-4\sqrt{(\lambda+1)(4\lambda+1)} \leq \alpha < \frac{1}{\lambda+1} \end{cases}$$

Thus, *S* is dominated by *CE-PL* in this region.

* Case 2-ii-b: If $\alpha(\lambda+1) \geq 1$.

  Then, $k_S^* = 1$ and $\pi_S^* = 0$. This case is clearly suboptimal - in this region *S* is obviously dominated by *CE-PL* since the latter generates non-zero profit when $\alpha \geq \frac{1}{\lambda+1}$.[2] $\square$

Proof of Proposition 2.

When $\underline{\alpha \geq 1}$, by directly comparing profits and social welfare values from Propositions 9-12, it can be easily seen that *CE-PL* is always the dominant strategy for the firm, whereas *TLF* is always the strategy that yields the highest social welfare.

The bulk of the proof, below, is addressing the considerably more complex case $\underline{0 < \alpha < 1}$.

Let us define:

$$\alpha_1(\lambda) \triangleq \begin{cases} \alpha^{\ddagger}(\lambda) & \text{, if } 0 \leq \lambda < \lambda_1, \\ \alpha_a(\lambda) & \text{, if } \lambda_1 \leq \lambda < \lambda_2, \\ \alpha_b(\lambda) & \text{, if } \lambda_2 \leq \lambda \leq 1, \end{cases}$$

and

$$\alpha_2(\lambda) \triangleq \begin{cases} \alpha_c(\lambda) & \text{, if } \frac{1}{4} \leq \lambda < \lambda_3, \\ \alpha_d(\lambda) & \text{, if } \lambda_3 \leq \lambda \leq 1, \end{cases}$$

where $\alpha^{\ddagger}(\lambda)$ was defined in Prop 12, and functions $\alpha_a(\cdot)$, $\alpha_b(\cdot)$, $\alpha_c(\cdot)$, $\alpha_d(\cdot)$, as well as

[2]We have $5 + 8\lambda - 4\sqrt{(\lambda+1)(4\lambda+1)} < \frac{1}{\lambda+1} \leq \alpha$. In this region, $\pi_{CE-PL}^* = \frac{1}{4}\alpha(\lambda+1)$.

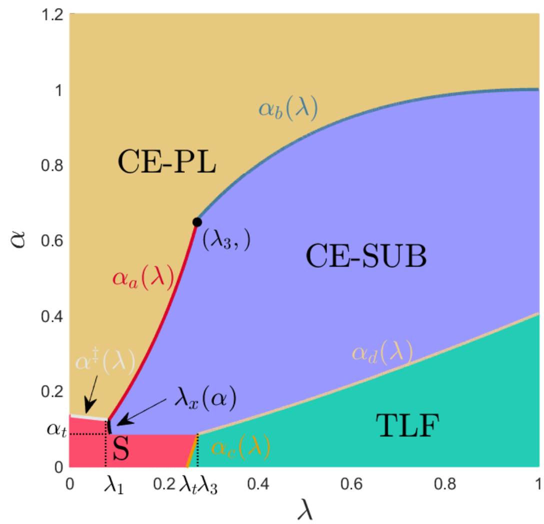


Figure C.1: Individual Depreciation Scenario - Optimal Business Model - Marked Boundaries

constant thresholds λ_1, λ_2, and λ_3 are defined and further analyzed below. For ease of identification, Figure C.1 contains the illustration of these boundaries and thresholds (this is a more detailed version of Figure 2.2 from the main body).

- **Monotonicity of** $\alpha^{\ddagger}(\lambda)$.

  As discussed in the text and proof of Prop. 12, $\alpha^{\ddagger}(\lambda)$ represents the boundary between the regions where *S* dominates *CE-PL* and the region where *CE-PL* dominates *S* (i.e., the region in which *S*, under optimality, requires $k^* = 0$, effectively defaulting to *CE-PL*). We have shown that $\alpha^{\ddagger}(\lambda)$ exists, it is unique (thus, it is well defined for all $\lambda \in (0, 1)$) and it satisfies:

$$\frac{\alpha^{\ddagger}(\lambda)(\lambda+1)\left(2\alpha^{\ddagger}(\lambda)\lambda+\alpha^{\ddagger}(\lambda)+1-2\sqrt{\alpha^{\ddagger}(\lambda)(\lambda+1)(\alpha^{\ddagger}(\lambda)\lambda+1)}\right)}{(1-\alpha^{\ddagger}(\lambda))^2}-\frac{1}{16-16\alpha^{\ddagger}(\lambda)}=0.$$

Define:

$$\Psi_{\ddagger}(\alpha,\lambda) \triangleq \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} - \frac{1}{16(1-\alpha)}.$$

We have $\Psi_{\ddagger}(\alpha^{\ddagger}(\lambda),\lambda)=0$. At the same time, for all $\alpha,\ \lambda \in (0,1)$, it can be shown that:

$$\frac{\partial\Psi_{\ddagger}(\alpha,\lambda)}{\partial\alpha} = \frac{\left(\alpha(8\lambda+7)^2+16\lambda+15\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} - 48\alpha(\lambda+1)^2 - 16\alpha^2(4\lambda+1)(\lambda+1)^2}{16(1-\alpha)^3\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}} \tag{C.3}$$

$$> 0, \tag{C.4}$$

$$\frac{\partial\Psi_{\ddagger}(\alpha,\lambda)}{\partial\lambda} = \frac{\alpha\left(4\alpha\lambda+3\alpha+1-\frac{\alpha(\lambda+1)(4\alpha\lambda+\alpha+3)}{\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}}\right)}{(1-\alpha)^2} \tag{C.5}$$

$$> 0. \tag{C.6}$$

Therefore, $\frac{\partial\alpha^{\ddagger}(\lambda)}{\partial\lambda} = -\frac{\frac{\partial\Psi_{\ddagger}(\alpha,\lambda)}{\partial\lambda}}{\frac{\partial\Psi_{\ddagger}(\alpha,\lambda)}{\partial\alpha}} < 0$. Hence, $\alpha^{\ddagger}(\lambda)$ is decreasing in λ.

- **Definition of λ_1, λ_2, and $\alpha_a(\lambda)$. Monotonicity of $\alpha_a(\lambda)$.**

  We know that when $\alpha \geq \alpha^{\ddagger}$, *CE-PL* dominates *S*. In this same region ($\underline{\alpha \geq \alpha^{\ddagger}}$), let us further compare profits under *CE-PL* and *CE-SUB* strategies.

  – First, the following two inequalities can be easily shown:

$$\frac{\alpha}{(\sqrt{\alpha}+1)^2} < \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2},\quad \forall\lambda\in(0,1),\ \alpha\in(\lambda,1),$$
$$\frac{\alpha}{(\sqrt{\alpha}+1)^2} < \frac{1}{4}\alpha(1+\lambda),\quad \forall\lambda\in(0,1),\ \max\{5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)},\lambda\} < \alpha < 1.$$

  Thus, given that $\lambda < \alpha^{\dagger}$, we see that in the region $\alpha^{\dagger} < \alpha \leq 1$ (third case in Prop.

10) we have $\pi^*_{CE-SUB} < \pi^*_{CE-PL}$.

– We further compare π^*_{CE-PL} under the first case in Prop. 9 and π^*_{CE-SUB} under the second case in Prop. 10. As we stay within region $\alpha \geq \alpha^\ddagger$, we look at the parameter region at the intersection among regions $\alpha \geq \alpha^\ddagger$, $\lambda < \alpha \leq \alpha^\dagger$, and $\alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$. Since $\alpha^\ddagger < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$, it can immediately follows that this is a non-empty region. In this region, define the difference between optimal profits under *CE-SUB* and *CE-PL* as:

$$\Psi_a(\alpha,\lambda) \triangleq p_b\left(1 - \frac{p_b}{\lambda} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha}}\right) - \frac{\alpha(\lambda+1)\left(2\alpha\lambda + \alpha + 1 - 2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}.$$

Let's next try to understand the monotonicity of $\Psi_a(\alpha,\lambda)$ with respect to α and λ. After taking derivatives and applying the Envelope theorem with respect to π^*_{CE-SUB}, given that $p_b(\alpha,\lambda)$ represents the maximizing price for *CE-SUB*, we obtain:

$$\begin{aligned}
\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha} =& \frac{2\alpha(\lambda+1)\left(2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} - 2\alpha\lambda + \alpha + 1\right)}{(1-\alpha)^3} \\
&- \frac{(\lambda+1)\left(\alpha\left(-\frac{(\lambda+1)(4\alpha\lambda+3)}{\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}} + 4\lambda + 2\right) + 1\right)}{(1-\alpha)^2} + \frac{p_b^3}{(\alpha-(1-\alpha)p_b)^2}, \\
\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \lambda} =& \frac{p_b^2}{\lambda^2} - \frac{\alpha\left(-\frac{\alpha(\lambda+1)(4\alpha\lambda+\alpha+3)}{\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}} + 2\alpha(\lambda+1) + 2\alpha\lambda + \alpha + 1\right)}{(1-\alpha)^2}.
\end{aligned}$$

We know from the proof of Prop. 10 that $p_b \in \left(\frac{\lambda}{2}, \lambda\right)$. Using these additional bounds on p_b, it is easy to get $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \lambda} > 0$. Therefore, for any given α, when we increase λ there can be at most one crossing point that separates the optimality regions for *CE-SUB* and *CE-PL*, and, moreover, the crossing (if it exists) can only be from *CE-PL* to *CE-SUB* as λ increases in this region of the parameter space.

Next, let's check the sign of $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha} = \frac{q_1}{q_2}$, where:

$$
\begin{aligned}
q_1 = &p_b^3(1-\alpha)^3\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)} \\
&+ p_b^2\alpha(1-\alpha)^2(\lambda+1)^2\left((\lambda+1)(4\alpha^2\lambda+\alpha^2+3\alpha)+(\alpha(4\lambda+3)+1)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right) \\
&- 2p_b\alpha^2(1-\alpha)(\lambda+1)^2\left((\lambda+1)(4\alpha^2\lambda+\alpha^2+3\alpha)-(\alpha(4\lambda+3)+1)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right) \\
&+ \alpha^3(\lambda+1)^2\left((\lambda+1)(4\alpha^2\lambda+\alpha^2+3\alpha)-(\alpha(4\lambda+3)+1)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right), \\
q_2 = &(1-\alpha)^3\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}(\alpha+\alpha p_b-p_b)^2 > 0.
\end{aligned}
$$

Therefore, the sign of $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha}$ is the same as the sign of the numerator, q_1.

Recall from Prop. 10 that p_b is the unique solution to the equation $G_{SUB,b}(p) = 2\alpha^2\lambda - 2(1-\alpha)^2p^3 - (1-\alpha)p^2(\alpha(\lambda-4)-2\lambda) - 2\alpha p(\alpha(1-\lambda)+2\lambda) = 0$. We use this property of p_b (i.e., $G_{SUB,b}(p_b) = 0$) to reduce the expression of q_1 from a cubic polynomial in p_b to a quadratic one, as follows:

$$
\begin{aligned}
q_1 = &\alpha(\lambda+1) \\
&\times\left(p_b^2\frac{1}{2}(1-\alpha)^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)-\left(\alpha\left(8\lambda^2+15\lambda+2\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)\right. \\
&\quad + p_b(1-\alpha)\alpha\left(\left(\alpha\left(8\lambda^2+15\lambda+5\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}-2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)\right) \\
&\quad \left. + \alpha^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)-\left(\alpha\left(4\lambda^2+8\lambda+3\right)+1\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)\right).
\end{aligned}
$$

Denote:

$$
\begin{aligned}
A &\triangleq \frac{1}{2}(1-\alpha)^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)-\left(\alpha\left(8\lambda^2+15\lambda+2\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right),\\
B &\triangleq (1-\alpha)\alpha\left(\left(\alpha\left(8\lambda^2+15\lambda+5\right)+2\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}-2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)\right),\\
C &\triangleq \alpha^2\left(2\alpha(\lambda+1)^2(4\alpha\lambda+\alpha+3)-\left(\alpha\left(4\lambda^2+8\lambda+3\right)+1\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right).
\end{aligned}
$$

Then $\frac{q_1}{\alpha(\lambda+1)} = Ap_b^2 + Bp_b + C$. Define quadratic function $H_{SUB,PL}(p) \triangleq Ap^2 + Bp + C$. In this range of the parameter space, it can be shown that:

$$
\begin{aligned}
B^2-4AC = (1-\alpha)^2\alpha^4(\lambda+1)& \\
&\times\Big(2\left(\lambda^2-\lambda-2\right)(4\alpha\lambda+\alpha+3)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\\
&\quad-((\alpha\lambda+1)(\alpha(\lambda(\lambda(8\lambda-1)-28)-13)+2(\lambda-2)))\Big) > 0.
\end{aligned}
$$

Hence, there are two real solutions of $H_{SUB,PL}(p) = 0$, namely:

$$
p_{H1} = -\frac{B+\sqrt{B^2-4AC}}{2A} \quad \text{and} \quad p_{H2} = \frac{-B+\sqrt{B^2-4AC}}{2A}.
$$

It can be shown that $p_{H1} < p_{H2} < \frac{\alpha}{1-\alpha}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. 10, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞).

It can be proved directly that $G_{SUB,b}(p_{H1}) > 0 = G_{SUB,b}(p_b) > G_{SUB,b}(p_{H2})$. Hence, $p_{H1} < p_b < p_{H2}$.

Furthermore, it can be shown that $A > 0$, which indicates that $H_{SUB,PL}(p)$ is convex. Therefore, $H_{SUB,PL}(p_b) < 0$. Thus, $\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha} < 0$. Therefore, for any given λ, when we increase α, there can be at most one crossing point that separates the optimality regions for *CE-SUB* and *CE-PL*, and, moreover, the crossing (if it exists) can be only from *CE-SUB* to *CE-PL* as α increases.

So far, we proved that a threshold (crossing) boundary between optimality regions for *CE-SUB* and *CE-PL* within this particular region of the parameter space (at the intersection among regions $\alpha \geq \alpha^{\ddagger}$, $\lambda < \alpha \leq \alpha^{\dagger}$, and $\alpha < 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$) is unique for every λ and for every α (i.e., if we look vertically or horizontally), *if it exists*. Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space.

We look at two particular delimiting boundaries for this region, namely $\alpha = \alpha^{\dagger}$ and $\alpha = \lambda$ and examine the sign of $\Psi_a(\alpha, \lambda)$ along these boundaries.

* On the boundary $\alpha = \alpha^{\dagger}(\lambda)$, since we are under condition $\alpha < 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)} < 1$, by definition of $\alpha^{\dagger}$, we obtain:

$$\begin{aligned}\Psi_a(\alpha, \lambda)\Big|_{\alpha=\alpha^{\dagger}(\lambda)} &= \frac{\alpha}{(\sqrt{\alpha}+1)^2} - \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2} \\ &< 0.\end{aligned}$$

* On the boundary $\alpha = \lambda$, we obtain:

$$\Psi_a(\alpha,\lambda)\Big|_{\alpha=\lambda} = p_b\left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1+p_b-\frac{p_b}{\lambda}}\right) - \frac{\lambda(\lambda+1)\left(2\lambda^2+\lambda+1-2\sqrt{\lambda(\lambda+1)(\lambda^2+1)}\right)}{(1-\lambda)^2}.$$

We point out that we could have written the profit for *CE-SUB* in terms of p_a at the boundary when $\alpha = \lambda$ - however, on that boundary, whether we write the profit in terms of p_a or p_b, we obtain the same profit because on that particular line, $p_a = p_b$ (as they satisfy the same implicit equation). Bringing all the terms to a common denominator, we can write $\Psi_a(\alpha,\lambda)\,|_{\alpha=\lambda} = \frac{q_3}{q_4}$, where:

$$\begin{aligned} q_3 =& (1-\lambda)^3 p_b^3 - (3-\lambda)(1-\lambda)^2\lambda p_b^2 \\ &+ (1-\lambda)\lambda^2\left(2\lambda^3+3\lambda^2+3-2\sqrt{\lambda(\lambda+1)^3\left(\lambda^2+1\right)}\right)p_b \\ &- \lambda^2\left(2\lambda^4+3\lambda^3+2\lambda^2-2\sqrt{\lambda^3(\lambda+1)^3\left(\lambda^2+1\right)}+\lambda\right), \\ q_4 =& (1-\lambda)^2\lambda(\lambda+\lambda p_b - p_b) > 0. \end{aligned}$$

Therefore, the sign of $\Psi_a(\alpha,\lambda)\,|_{\alpha=\lambda}$ is the same as the sign of the numerator, q_3.

We use $G_{SUB,b}(p_b) = 0$ to reduce the expression of q_3 from a cubic polynomial in p_b to a quadratic one, as follows:

$$\begin{aligned}
q_3 =& \frac{\lambda^2}{2} \\
& \times \left((1-\lambda)^2 p_b^2 + 2\left(1-\lambda^2\right)\left(2\lambda^2 - 2\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)} + \lambda\right) p_b \right. \\
& \qquad \left. + 4\lambda(\lambda+1)\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)} - 2\lambda^2\left(2\lambda^2+3\lambda+3\right)\right).
\end{aligned}$$

Denote:

$$\begin{aligned}
& D \triangleq (1-\lambda)^2, \\
& E \triangleq 2\left(1-\lambda^2\right)\left(2\lambda^2 - 2\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)} + \lambda\right), \\
& F \triangleq 4\lambda(\lambda+1)\sqrt{\lambda(\lambda+1)\left(\lambda^2+1\right)} - 2\lambda^2\left(2\lambda^2+3\lambda+3\right).
\end{aligned}$$

Then $\frac{2q_3}{\lambda^2} = Dp_b^2 + Ep_b + F$. Define quadratic function $\tilde{H}_{SUB,PL}(p) \triangleq Dp^2 + Ep + F$. In this range of the parameter space, it can be shown that $E^2 - 4DF > 0$. Hence, there are two real solutions to the equation $\tilde{H}_{SUB,PL}(p) = 0$, namely:

$$p_{\tilde{H}1} = \frac{-E - \sqrt{E^2 - 4DF}}{2D} \quad \text{and} \quad p_{\tilde{H}2} = \frac{-E + \sqrt{E^2 - 4DF}}{2D}.$$

It can be shown that $\frac{\lambda}{2} < p_{\tilde{H}1} < \lambda < p_{\tilde{H}2}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. 10, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p_{\tilde{H}1}) < 0 = G_{SUB,b}(p_b)$. Hence, $p_b < p_{\tilde{H}1} < p_{\tilde{H}2}$. Furthermore, it can be shown that $D > 0$, which indicates that $\tilde{H}_{SUB,PL}(p)$ is convex.

Therefore, $\tilde{H}_{SUB,PL}(p_b) > 0$. Hence, in this region of the parameter space:

$$\Psi_a(\alpha,\lambda)\Big|_{\alpha=\lambda} > 0.$$

Thus, $\Psi_a(\alpha,\lambda)\Big|_{\alpha=\alpha^\dagger(\lambda)} < 0$ and $\Psi_a(\alpha,\lambda)\Big|_{\alpha=\lambda} > 0$. Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_a(\lambda)}$, which separates the optimality regions for *CE-SUB* and *CE-PL*, and which falls between boundaries $\alpha = \alpha^\dagger$ and $\alpha = \lambda$. It satisfies:

$$\frac{\alpha_a(\lambda)(\lambda+1)\left(2\alpha_a(\lambda)\lambda + \alpha_a(\lambda) + 1 - 2\sqrt{\alpha_a(\lambda)(\lambda+1)(\alpha_a(\lambda)\lambda+1)}\right)}{(1-\alpha_a(\lambda))^2}$$
$$= p_b\left(1 - \frac{p_b}{\lambda} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha_a(\lambda)}}\right).$$

Since existence and uniqueness are satisfied, $\alpha_a(\lambda)$ is properly defined as a function. Moreover, since $\Psi_a(\alpha_a(\lambda),\lambda) = 0$, by differentiation w.r.t. λ, we obtain $\frac{\partial \alpha_a(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_a(\alpha,\lambda)}{\partial \alpha}} > 0$. Hence, $\alpha_a(\lambda)$ is increasing.

Since both boundaries $\alpha^\ddagger$ and $\alpha = 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$ are decreasing in λ and $\alpha_a(\lambda)$ is increasing in lambda and strictly between the lines $\alpha^\dagger$ and $\alpha = \lambda$, then there exists a unique intersection point between $\alpha_a(\lambda)$ and $\alpha^\ddagger$, and a unique intersection point between $\alpha_a(\lambda)$ and $\alpha = 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$.

* Define $\{\lambda_1, \alpha^\ddagger(\lambda_1)\}$ as the unique intersection between $\alpha_a(\lambda)$ and $\alpha^\ddagger$. Then, λ_1

satisfies:

$$\frac{1}{16(1-\alpha^{\ddagger}(\lambda_1))} = \frac{\alpha^{\ddagger}(\lambda_1)(\lambda_1+1)(2\alpha^{\ddagger}(\lambda_1)\lambda_1+\alpha^{\ddagger}(\lambda_1)+1)}{(1-\alpha^{\ddagger})^2} - \frac{2\sqrt{\alpha^{\ddagger}(\lambda_1)^3(\lambda_1+1)^3(\alpha^{\ddagger}(\lambda_1+1)}}{(1-\alpha^{\ddagger})^2} = p_b\left(1-\frac{p_b}{\lambda_1}+1-\frac{p_b}{1+p_b-\frac{p_b}{\alpha^{\ddagger}(\lambda_1)}}\right).$$

More precisely, at $\{\lambda_1, \alpha^{\ddagger}(\lambda_1)\}$, we have:

$$\pi^*_{CE-PL}(\lambda_1, \alpha^{\ddagger}(\lambda_1)) = \pi^*_{CE-SUB}(\lambda_1, \alpha^{\ddagger}(\lambda_1)) = \pi^*_S(\lambda_1, \alpha^{\ddagger}(\lambda_1)).$$

* Define $\{\lambda_2, 5+8\lambda_2-4\sqrt{(1+\lambda_2)(1+4\lambda_2)}\}$ as the unique intersection between $\alpha_a(\lambda)$ and $\alpha = 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$. Then, λ_2 satisfies:

$$\frac{\left(5+8\lambda_2-4\sqrt{(1+\lambda_2)(1+4\lambda_2)}\right)(1+\lambda_2)}{4} = p_b\left(1-\frac{p_b}{\lambda_2}+1-\frac{p_b}{1+p_b-\frac{p_b}{5+8\lambda_2-4\sqrt{(1+\lambda_2)(1+4\lambda_2)}}}\right).$$

It immediately follows that $\lambda_1 < \lambda_2$ and $\alpha_a(\lambda)$ is properly defined and increasing on $\lambda \in [\lambda_1, \lambda_2]$. We show $\alpha_a(\lambda)$ in Figure C.2.

- **Definition and monotonicity of $\alpha_b(\lambda)$.**

  We further compare the second case under *CE-PL* and the second case under *CE-SUB* at the intersection among regions $5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)} \leq \alpha < 1$ and $\lambda < \alpha \leq \alpha^{\dagger}$.

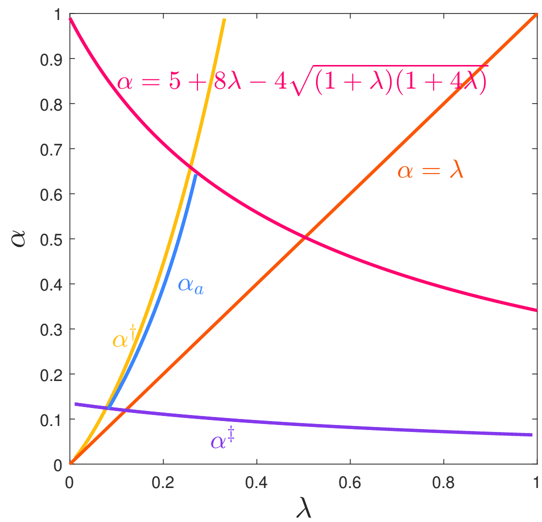


Figure C.2: Specification of $\alpha_a(\lambda)$

Denote the profit difference between *CE-SUB* and *CE-PL* in this region as:

$$\Psi_b(\alpha, \lambda) \triangleq p_b \left(1 - \frac{p_b}{\lambda} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha}}\right) - \frac{\alpha(1+\lambda)}{4}.$$

Then, using Envelope theorem (since p_b maximizes π_{CE-SUB}), we have :

$$\frac{\partial \Psi_b(\alpha, \lambda)}{\partial \alpha} = \frac{p_b^3}{(\alpha - (1-\alpha)p_b)^2} - \frac{1+\lambda}{4},$$
$$\frac{\partial \Psi_b(\alpha, \lambda)}{\partial \lambda} = \frac{p_b^2}{\lambda^2} - \frac{\alpha}{4} > \frac{(\frac{\lambda}{2})^2}{\lambda^2} - \frac{\alpha}{4} > 0.$$

Next, let's check the sign of $\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha} = \frac{q_5}{q_6}$, where:

$$q_5 \triangleq -\alpha^2\lambda - \alpha^2 + 4p_b^3 + p_b^2\left(-\alpha^2\lambda - \alpha^2 + 2\alpha\lambda + 2\alpha - \lambda - 1\right) + p_b\left(-2\alpha^2\lambda - 2\alpha^2 + 2\alpha\lambda + 2\alpha\right),$$

$$q_6 \triangleq 4(\alpha + \alpha p_b - p_b)^2 > 0.$$

Thus, the sign of $\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha}$ is the same as the sign of the numerator, q_5. We use $G_{SUB,b}(p_b) = 0$ to reduce the expression of q_5 from a cubic polynomial in p_b to a quadratic one, as follows:

$$
\begin{aligned}
q_5 = &-p_b^2(1-\alpha)(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+11)+3\lambda-1) \\
&+ p_b 2\alpha(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+5)+3\lambda-1) \\
&+ \alpha^2(-(2-\alpha)\alpha(\lambda+1)-3\lambda+1).
\end{aligned}
$$

Denote:

$$
\begin{aligned}
J &\triangleq -(1-\alpha)(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+11)+3\lambda-1), \\
K &\triangleq 2\alpha(\alpha(-(3-\alpha)\alpha(\lambda+1)+\lambda+5)+3\lambda-1), \\
L &\triangleq \alpha^2(-(2-\alpha)\alpha(\lambda+1)-3\lambda+1).
\end{aligned}
$$

Then, $q_5 = Jp_b^2 + Kp_b + L$. Define the quadratic function $\bar{H}_{SUB,PL}(p) \triangleq Jp^2 + Kp + L$. In this range of the parameter space, it can be shown that $K^2 - 4JL > 0$. Hence, there are two real solutions to the equation $\bar{H}_{SUB,PL}(p) = 0$, namely:

$$
p_{\bar{H}1} = \frac{-K - \sqrt{K^2 - 4JL}}{2J} \quad \text{and} \quad p_{\bar{H}2} = \frac{-K + \sqrt{K^2 - 4JL}}{2J}.
$$

It can be shown that $p_{\bar{H}2} < \lambda < p_{\bar{H}1}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. 10, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p_{\bar{H}2}) > 0 =$

$G_{SUB,b}(p_b)$. Hence, $p_{\bar{H}2} < p_b < \lambda < p_{\bar{H}1}$. Furthermore, it can be shown that $J < 0$, which indicates that $\bar{H}_{SUB,PL}(p)$ is concave. Therefore, $\bar{H}_{SUB,PL}(p_b) > 0$. Hence, in this region of the parameter space:

$$\frac{\partial \Psi_b(\alpha, \lambda)}{\partial \alpha} > 0.$$

So far, we proved that a threshold (crossing) boundary between optimality regions for *CE-SUB* and *CE-PL* within this particular region of the parameter space (at the intersection among regions $5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)} \leq \alpha < 1$ and $\lambda < \alpha \leq \alpha^{\dagger}$) is unique for every λ and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = \alpha^{\dagger}$ and $\alpha = \lambda$ (boundary in limit) and examine the sign of $\Psi_b(\alpha, \lambda)$ along these boundaries.

- On the boundary $\alpha = \alpha^{\dagger}(\lambda)$, since we are under condition $5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)} \leq \alpha < 1$, by definition of $\alpha^{\dagger}$, we obtain:

$$\Psi_b(\alpha, \lambda)\Big|_{\alpha=\alpha^{\dagger}(\lambda)} = \frac{\alpha}{(\sqrt{\alpha}+1)^2} - \frac{\alpha(1+\lambda)}{4} < 0.$$

- On the boundary $\alpha = \lambda$ (boundary in limit), we obtain:

$$\Psi_b(\alpha,\lambda)\Big|_{\alpha=\lambda} = p_b\left(2-\frac{p_b}{\lambda}-\frac{p_b}{1+p_b-\frac{p_b}{\lambda}}\right)-\frac{\lambda(1+\lambda)}{4}.$$

Again, we remind the reader we could have written the profit for $CE-SUB$ in terms of p_a at the boundary when $\alpha = \lambda$ - however, on that boundary, whether we write the profit in terms of p_a or p_b, we obtain the same profit because on that particular line, $p_a = p_b$. Bringing all the terms to a common denominator, we can write $\Psi_b(\alpha,\lambda)\,|_{\alpha=\lambda} = \frac{q_7}{q_8}$, where:

$$\begin{aligned} q_7 =& -\lambda^4-\lambda^3-4\lambda p_b^3+4p_b^3+4\lambda^2p_b^2-12\lambda p_b^2-\lambda^4p_b+9\lambda^2p_b, \\ q_8 =& \, 4\lambda(\lambda+\lambda p_b-p_b)>0. \end{aligned}$$

Therefore, the sign of $\Psi_b(\alpha,\lambda)\,|_{\alpha=\lambda}$ is the same as the sign of the numerator, q_7.

We use $G_{SUB,b}(p_b)=0$ to reduce the expression of q_7 from a cubic polynomial in p_b to a quadratic one, as follows:

$$q_7 = \frac{\lambda^2\left(\lambda^3+3\lambda+2(1-\lambda)p^2-(3-\lambda)(\lambda+1)^2p\right)}{1-\lambda}.$$

Denote:

$$R \triangleq 2(1-\lambda),$$

$$S \triangleq -(3-\lambda)(\lambda+1)^2,$$

$$T \triangleq \lambda^3 + 3\lambda.$$

Then $\frac{(1-\lambda)q_7}{\lambda^2} = Rp_b^2 + Sp_b + T$. Define quadratic function $\hat{H}_{SUB,PL}(p) \triangleq Rp^2 + Sp + T$. In this range of the parameter space, it can be shown that $S^2 - 4RT > 0$. Hence, there are two real solutions to the equation $\hat{H}_{SUB,PL}(p) = 0$, namely:

$$p_{\hat{H}1} = \frac{-S - \sqrt{S^2 - 4RT}}{2R} \quad \text{and} \quad p_{\hat{H}2} = \frac{-S + \sqrt{S^2 - 4RT}}{2R}.$$

It can be shown that $\frac{\lambda}{2} < p_{\hat{H}1} < \lambda < p_{\hat{H}2}$. Recall that p_b is the unique solution of $G_{SUB,b}(p) = 0$. Moreover, from the proof of Prop. 10, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p_{\hat{H}1}) < 0 = G_{SUB,b}(p_b)$. Hence, $p_b < p_{\hat{H}1} < p_{\hat{H}2}$. Furthermore, $R > 0$, which indicates that $\hat{H}_{SUB,PL}(p)$ is convex. Therefore, $\hat{H}_{SUB,PL}(p_b) > 0$. Hence, in this region of the parameter space:

$$\Psi_b(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0.$$

Thus, $\Psi_b(\alpha, \lambda)\Big|_{\alpha=\alpha^\dagger(\lambda)} < 0$ and $\Psi_b(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0$. Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_b(\lambda)}$, which separates the optimality regions for *CE-SUB* and *CE-PL*, and which falls between boundaries $\alpha = \alpha^{\dagger}$ and $\alpha = \lambda$ (with the 3 lines converging when $\lambda \to 1$). It satisfies:

$$p_b \left(1 - \frac{p_b}{\lambda} + 1 - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha_b(\lambda)}}\right) = \frac{\alpha_b(\lambda)(1+\lambda)}{4}.$$

Since existence and uniqueness are satisfied, $\alpha_b(\lambda)$ is properly defined as a function. Moreover, since $\Psi_b(\alpha_b(\lambda), \lambda) = 0$, by differentiation w.r.t. λ, we obtain $\frac{\partial \alpha_b(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_b(\alpha,\lambda)}{\partial \alpha}} > 0$. Hence, $\alpha_b(\lambda)$ is increasing in λ.

As $\alpha = 5+8\lambda-4\sqrt{(1+\lambda)(1+4\lambda)}$ is decreasing in λ, there exists a unique intersection point between $\alpha = 5 + 8\lambda - 4\sqrt{(1+\lambda)(1+4\lambda)}$ and $\alpha_b(\lambda)$. Defining this point as $\{\lambda_{2,b}, \alpha_b(\lambda_{2,b})\}$, we can immediately see that $\{\lambda_2, \alpha_a(\lambda_2)\}$ and $\{\lambda_{2,b}, \alpha_b(\lambda_{2,b})\}$ satisfy exactly the same conditions. Due to the uniqueness properties discussed above, we have $\lambda_{2,b} = \lambda_2$ and $\alpha_a(\lambda_2) = \alpha_b(\lambda_2)$.

Thus, $\alpha_b(\lambda)$ is properly defined and increasing on $[\lambda_2, 1)$.

Moreover, we extend the domain of α_b to include 1. We define asymptotically $\alpha_b(1) = 1 = lim_{\lambda\uparrow 1}\alpha_b(\lambda)$.

- **Definition and monotonicity of $\alpha_c(\lambda)$.**

  Next, we compare *S* and *TLF* in the region $0 < \alpha < \alpha^{\ddagger}$. Denote the profit difference between *S* and *TLF* strategies in this region as:

$$\Psi_c(\alpha, \lambda) \triangleq \frac{1}{16(1-\alpha)} - \frac{\lambda}{4}.$$

Then:

$$\frac{\partial \Psi_c(\alpha, \lambda)}{\partial \alpha} > 0 > \frac{\partial \Psi_c(\alpha, \lambda)}{\partial \lambda}.$$

Therefore, a threshold (crossing) boundary between optimality regions for *S* and *TLF* within this particular region of the parameter space ($0 < \alpha < \alpha^{\ddagger}$) is unique for every λ and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\lambda = 0$ and $\lambda = 1$ and examine the sign of $\Psi_c(\alpha, \lambda)$ along these boundaries.

– On the boundary $\lambda = 0$, we obtain:

$$\Psi_c(\alpha, \lambda)\Big|_{\lambda=0} = \frac{1}{16(1-\alpha)} > 0.$$

– On the boundary $\lambda = 1$, as $\alpha < \alpha^{\ddagger} < \frac{3}{4}$, we obtain:

$$\Psi_c(\alpha, \lambda)\Big|_{\lambda=1} = \frac{1}{16(1-\alpha)} - \frac{1}{4} < 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_c(\lambda)}$, which separates the optimality regions for *S* and *TLF*. It satisfies:

$$\frac{1}{16(1-\alpha_c(\lambda))} - \frac{\lambda}{4} = 0.$$

We obtain: $\alpha_c(\lambda) \triangleq 1 - \frac{1}{4\lambda}$. Also, $\alpha_c(\lambda)$ is increasing in λ.

As $\alpha^{\ddagger}(\lambda)$ is decreasing in λ, there exists a unique intersection point between $\alpha^{\ddagger}(\lambda)$ and $\alpha_c(\lambda)$. Defining this intersection point as $\{\lambda_c, \alpha_c(\lambda_c)\}$, we can numerically get that $\lambda_c \approx 0.2789$. Then, $\alpha_c(\lambda)$ is properly defined and increasing on $\lambda \in [\frac{1}{4}, \lambda_c]$.

- **Definition of $\alpha_d(\lambda)$ and λ_3. Monotonicity of $\alpha_d(\lambda)$.**

  We further compare π^*_{CE-SUB} under the first case in Prop. 10, i.e., $0 < \alpha \leq \lambda \leq 1$, and π^*_{TLF}.

  We focus first on the case $\lambda < 1$. In this region, define the difference between optimal profits under *CE-SUB* and *TLF* as:

$$\Psi_d(\alpha, \lambda) = p_a \left(2 - \frac{p_a}{\alpha} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha}} \right) - \frac{\lambda}{4}.$$

  Let's next try to understand the monotonicity of $\Psi_d(\alpha, \lambda)$ with respect to α and λ. After taking derivatives and applying the Envelope theorem with respect to π^*_{CE-SUB}, given that $p_a(\alpha, \lambda)$ represents the maximizing price for *CE-SUB* in this region, we obtain:

$$\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \alpha} = p_a^2 \left(\frac{1}{\alpha^2} + \frac{p_a}{(\alpha-(1-\alpha)p_a)^2} \right) > 0,$$
$$\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \lambda} = -\frac{1}{4} < 0.$$

Therefore, for each λ (α) there can be *at most one* crossing point that separates the optimality regions for *CE-SUB* and *TLF* in this region as we move α (λ). Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space ($0 < \alpha \leq \lambda$).

We look at two particular delimiting boundaries for this region, namely $\alpha \to 0$ and $\alpha = \lambda$ and examine the sign of $\Psi_d(\alpha, \lambda)$ along these boundaries.

- On the boundary $\alpha \to 0$, under $CE-SUB$, the firm can only jump start adoption through a subscription rate $p_a \to 0$. Thus, $\lim_{\alpha \downarrow 0} \pi^*_{CE-SUB} = 0$. Hence:

$$\lim_{\alpha \downarrow 0} \Psi_d(\alpha,\lambda) = 0 - \frac{\lambda}{4} \leq 0.$$

- On the boundary $\alpha = \lambda$, we obtain:

$$\Psi_d(\alpha,\lambda)\Big|_{\alpha=\lambda} = \frac{4(1-\lambda)p_a^3 - 4(3-\lambda)\lambda p_a^2 + (9-\lambda)\lambda^2 p_a - \lambda^3}{4\lambda(\lambda+\lambda p_a - p_a)}.$$

Bringing all the terms to a common denominator, we can write $\Psi_d(\alpha,\lambda)\Big|_{\alpha=\lambda} = \frac{q_9}{q_{10}}$, where:

$$
\begin{aligned}
q_9 &= 4(1-\lambda)p_a^3 - 4(3-\lambda)\lambda p_a^2 + (9-\lambda)\lambda^2 p_a - \lambda^3, \\
q_{10} &= 4\lambda(\lambda + \lambda p_a - p_a) > 0.
\end{aligned}
$$

The second inequality holds because $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$ and, in this region, $\alpha < \lambda$. Therefore, the sign of $\Psi_d(\alpha, \lambda)\Big|_{\alpha=\lambda}$ is the same as the sign of the numerator, q_9.

We use $G_{SUB,a}(p_a) = 0$ to reduce the expression of q_9 from a cubic polynomial in p_a to a quadratic one, as follows:

$$
q_9 = \frac{\lambda^2 \left(\lambda(\lambda+3) + 2(1-\lambda)p_a^2 + (-3 + (\lambda-6)\lambda)p_a\right)}{1-\lambda}.
$$

Denote:

$$
\begin{aligned}
U &\triangleq 2(1-\lambda), \\
V &\triangleq -3 + (\lambda - 6)\lambda, \\
W &\triangleq \lambda(\lambda+3).
\end{aligned}
$$

Then $\frac{(1-\lambda)q_9}{\lambda^2} = Up_a^2 + Vp_a + W$. Define quadratic function $H_{SUB,TLF}(p) \triangleq Up^2 + Vp + W$. In this range of the parameter space, it can be shown that $V^2 - 4UW > 0$. Hence, there are two real solutions to the equation $H_{SUB,TLF}(p) = 0$, namely:

$$
\tilde{\tilde{p}}_{H1} = \frac{-V - \sqrt{V^2 - 4UW}}{2U} \quad \text{and} \quad \tilde{\tilde{p}}_{H2} = \frac{-V + \sqrt{V^2 - 4UW}}{2U}.
$$

It can be shown that $\frac{\alpha}{2} < \tilde{\tilde{p}}_{H1} < \alpha < \tilde{\tilde{p}}_{H2}$. Recall that p_a is the unique solution of $G_{SUB,a}(p) = 0$. Moreover, from the proof of Prop. 10, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\tilde{\tilde{p}}_{H1}) < 0 = G_{SUB,a}(p_a)$. Hence, $p_a < \tilde{\tilde{p}}_{H1} < \tilde{\tilde{p}}_{H2}$. Furthermore, $U > 0$, which indicates that $H_{SUB,TLF}(p)$ is convex. Therefore, $H_{SUB,TLF}(p_a) > 0$.

Hence, in this region of the parameter space:

$$\Psi_d(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0.$$

Thus, $\Psi_d(\alpha, \lambda)\Big|_{\alpha\to 0} < 0$ and $\Psi_d(\alpha, \lambda)\Big|_{\alpha=\lambda} > 0$. Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_d(\lambda)}$, which separates the optimality regions for ***CE-SUB*** and ***TLF*** (i.e., and $\Psi_d(\alpha_d(\lambda), \lambda) = 0$), which falls between boundaries $\alpha = 0$ and $\alpha = \lambda$. It satisfies:

$$p_a \left(2 - \frac{p_a}{\alpha_d(\lambda)} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha_d(\lambda)}} \right) = \frac{\lambda}{4}.$$

Since existence and uniqueness are satisfied, $\alpha_d(\lambda)$ is properly defined as a function. In terms of the domain of $\alpha_d(\lambda)$, given that $\Psi_d(\alpha, \lambda)\Big|_{\alpha\to 0} = 0 - \frac{\lambda}{4} \leq 0$, the intersection of $\alpha_d(\lambda)$ and $\alpha = 0$ (x-axis) line can only happen when $\lambda = 0$. Hence $\alpha_d(\lambda)$ is well defined on $(0, 1)$ domain. Moreover, since $\Psi_d(\alpha_d(\lambda), \lambda) = 0$, by differentiation w.r.t. λ, we obtain $\frac{\partial \alpha_d(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \alpha}} > 0$. Hence, $\alpha_d(\lambda)$ is increasing in λ.

For the case $\lambda = 1$, we have $\alpha_d(1) = \lim_{\lambda\to 1} \alpha_d(\lambda) = \tilde{\alpha}$, where $\tilde{\alpha}$ was defined in Proposition 1.

Next we check whether $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ have a crossing point. First, let's check that α_c and α_d are defined in overlapping regions. $\alpha_c(\lambda)$ is defined on $\lambda \in \left[\frac{1}{4}, \lambda_c\right]$, where $\lambda_C \approx 0.2789$. It is easy to check that $\alpha_c(\lambda) = 1 - \frac{1}{4\lambda} < \lambda$. Thus, any point $\{\lambda, \alpha_c(\lambda)\}$ with $\lambda \in \left[\frac{1}{4}, \lambda_c\right]$ falls inside the bigger region $0 < \alpha \leq \lambda \leq 1$, which is also the region where $\alpha_d(\lambda)$ is defined.

Both $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ are increasing in λ, as previously proved. In this region (i.e., $\lambda \in \left[\frac{1}{4}, \lambda_c\right]$, with $\lambda_C \approx 0.2789$), using $\alpha_d(\lambda) < \lambda$ and $p_a\left(\alpha_d(\lambda), \lambda\right) \in \left(\frac{\alpha_d(\lambda)}{2}, \alpha_d(\lambda)\right)$, it can be shown that:

$$\frac{\partial \alpha_d(\lambda)}{\partial \lambda} = -\left.\frac{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_d(\alpha,\lambda)}{\partial \alpha}}\right|_{\alpha=\alpha_d(\lambda)} = \frac{1}{4p_a^2\left(\frac{1}{\alpha_d(\lambda)^2} + \frac{p_a}{(\alpha_d(\lambda)-(1-\alpha_d(\lambda))p_a)^2}\right)} < \frac{1}{4\lambda^2} = \frac{\partial \alpha_c(\lambda)}{\partial \lambda}.$$

Therefore, there can be *at most one* intersection point between $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ in this region.

Given that α_d is defined on $(0, 1]$ and α_c is defined on $\left[\frac{1}{4}, \lambda_c\right]$, with $(\lambda_c, \alpha_c(\lambda_c))$ being on $\alpha^{\ddagger}$ line, for $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$ to intersect, it is sufficient to show that $\alpha_d\left(\frac{1}{4}\right) \geq \alpha_c\left(\frac{1}{4}\right)$ and $\alpha_d(\lambda_c) \leq \alpha_c(\lambda_c)$. Since α_d is increasing, it can be immediately seen that:

$$\alpha_d\left(\frac{1}{4}\right) \geq 0 = \alpha_c\left(\frac{1}{4}\right).$$

Moreover, through numerical derivation, it can be shown that:

$$\alpha_d(\lambda) - \alpha_c(\lambda)\Big|_{\lambda=\lambda_c} \approx 0.0882 - 0.1036 < 0.$$

Therefore, there exists one unique intersection point between $\alpha_c(\lambda)$ and $\alpha_d(\lambda)$, which we define as $\{\lambda_3, \alpha_c(\lambda_3)\}$. Then, we have:

$$\pi^*_{CE-SUB}(\lambda_3, \alpha_c(\lambda_3)) = \pi^*_S(\lambda_3, \alpha_c(\lambda_3)) = \pi^*_{TLF}(\lambda_3, \alpha_c(\lambda_3)).$$

More precisely, λ_3 satisfies:

$$\lambda_3 \in \left[\frac{1}{4}, \lambda_c\right] \quad \text{and} \quad \frac{1}{16(1-\alpha_c(\lambda_3))} = p_a\left(2 - \frac{p_a}{\alpha_c(\lambda_3)} - \frac{p_a}{1 + p_a - \frac{p_a}{\alpha_c(\lambda_3)}}\right) = \frac{\lambda_3}{4}.$$

Also, we can numerically get $\lambda_3 \approx 0.272 < \lambda_c$. $\{\lambda_3, \alpha_c(\lambda_3)\}$ falls into the region $0 < \alpha < \alpha^\ddagger$.

Since $\alpha_d(\lambda)$ is properly defined on $(0, 1]$, obviously it is also properly defined on $[\lambda_3, 1]$.

- **Definition of threshold constant α_t.**

We further compare π^*_{CE-SUB} under the first case in Prop. 10 and π^*_S under the first case in Prop 12. Specifically, we look at the parameter region at the intersection of constraints $0 < \alpha < \alpha^\ddagger$ and $0 < \alpha \le \lambda$. In this region, define the difference between optimal profits under *CE-SUB* and *S* as:

$$\Psi_t(\alpha,\lambda) \triangleq p_a\left(2-\frac{p_a}{\alpha}-\frac{p_a}{1+p_a-\frac{p_a}{\alpha}}\right)-\frac{1}{16(1-\alpha)}.$$

Since p_a is the unique solution of $G_{SUB,a}(p_a)=0$, p_a does not depend on λ. Therefore, $\Psi_t(\alpha,\lambda)$ does not depend on λ. After taking derivatives and applying the Envelope theorem with respect to π^*_{CE-SUB}, given that $p_a(\alpha,\lambda)$ represents the maximizing price for *CE-SUB* in this region, we obtain:

$$\begin{aligned}
\frac{\partial\Psi_t(\alpha,\lambda)}{\partial\alpha} &= p_a^2\left(\frac{1}{\alpha^2}+\frac{p_a}{(\alpha-(1-\alpha)p_a)^2}\right)-\frac{1}{16(1-\alpha)^2},\\
\frac{\partial\Psi_t(\alpha,\lambda)}{\partial\lambda} &= 0.
\end{aligned}$$

Let's check the sign of $\frac{\partial\Psi_t(\alpha,\lambda)}{\partial\alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial\Psi_t(\alpha,\lambda)}{\partial\alpha}=\frac{q_{11}}{q_{12}}$, where:

$$\begin{aligned}
q_{11} =& 16(1-\alpha)^4p_a^4+16\alpha(3\alpha-2)(1-\alpha)^2p_a^3+15\alpha^2(1-\alpha)^2p_a^2+2\alpha^3(1-\alpha)p_a-\alpha^4,\\
q_{12} =& 16(1-\alpha)^2\alpha^2(\alpha+\alpha p_a-p_a)^2>0.
\end{aligned}$$

Therefore, the sign of $\frac{\partial\Psi_t(\alpha,\lambda)}{\partial\alpha}$ is the same as the sign of the numerator, q_{11}.

Recall from Prop. 10 that p_a is the unique solution to the cubic equation $G_{SUB,a}(p_a)=0$. We use this property of p_a to reduce the expression of q_{11} from a quartic polynomial in p_a to a quadratic one, as follows:

$$q_{11} = \left(4\alpha^4 - 16\alpha^3 - 5\alpha^2 + 2\alpha + 15\right)\alpha^2 p_a^2$$
$$+ 2\left(4\alpha^3 - 8\alpha^2 + 3\alpha - 15\right)\alpha^3 p_a + \left(8\alpha^2 - 8\alpha + 15\right)\alpha^4.$$

Denote:

$$X \triangleq \alpha^2\left(4\alpha^4 - 16\alpha^3 - 5\alpha^2 + 2\alpha + 15\right),$$
$$Y \triangleq 2\alpha^3\left(4\alpha^3 - 8\alpha^2 + 3\alpha - 15\right),$$
$$Z \triangleq \alpha^4\left(8\alpha^2 - 8\alpha + 15\right).$$

Then $q_{11} = Xp_a^2 + Yp_a + Z$. Define quadratic function $H_{SUB,S}(p) \triangleq Xp^2 + Yp + Z$. In this range of the parameter space, it can be shown that:

$$Y^2 - 4XZ = 16\alpha^8(\alpha(\alpha(-4(\alpha-6)\alpha - 15) + 4) + 55) > 0.$$

Hence, there are two real solutions of $H_{SUB,S}(p) = 0$, namely:

$$\bar{\bar{p}}_{H1} = \frac{-Y - \sqrt{Y^2 - 4XZ}}{2X} \quad \text{and} \quad \bar{\bar{p}}_{H2} = \frac{-Y + \sqrt{Y^2 - 4XZ}}{2X}.$$

It can be shown that $\bar{\bar{p}}_{H1} < \bar{\bar{p}}_{H2} < \frac{\alpha}{1-\alpha}$. From the proof of Prop. 10, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\bar{\bar{p}}_{H1}) < 0$. Hence, $p_a < \bar{\bar{p}}_{H1} < \bar{\bar{p}}_{H2}$.

Furthermore, it can be shown that $X > 0$, which indicates that $H_{SUB,S}(p)$ is convex. Therefore, $H_{SUB,PL}(p_a) > 0$. Thus, $\frac{\partial \Psi_t(\alpha,\lambda)}{\partial \alpha} > 0$. Thus, for any given λ, when we increase α, there can be *at most one* crossing point that separates the optimality regions for *CE-SUB* and *S*, and, moreover, the crossing (if it exists) can be only from *S* to *CE-SUB* as α increases. Such a separating threshold line, if it exists, has to be horizontal (i.e., constant for any λ for which it exists in this region) since $\Psi_t(\alpha, \lambda)$ is independent of λ (because p_a is independent of λ).

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space ($0 < \alpha < \min\{\lambda, \alpha^\ddagger\}$). Since we established that such a threshold will be a horizontal line cutting through this region, it is enough to show that it exists at a particular λ. Consider $\{\lambda_4, \alpha_4\}$, with $\lambda_4 = \alpha_4$, to be the unique intersection between lines $\alpha = \lambda$ and $\alpha = \alpha^\ddagger$.[3] Numerical analysis reveals that $\lambda_4 = \alpha_4 \approx 0.1195$. We examine the sign of $\Psi_t(\alpha, \lambda)$ at boundary points $\{\lambda_4, 0\}$ and $\{\lambda_4, \alpha_4\}$:

$$\Psi_t(\alpha, \lambda)\Big|_{\lambda=\lambda_4,\, \alpha\downarrow 0} = 0 - \frac{1}{16} < 0,$$

$$\Psi_t(\alpha, \lambda)\Big|_{\lambda=\alpha=\lambda_4} \approx 0.0909 - 0.0710 > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary which separates the optimality regions for *S* and *CE-SUB* and which does not change with λ. And it is straight forward that the boundary line goes through the point $\{\lambda_3, \alpha_c(\lambda_3)\}$ since

[3]we know that the intersection is unique because $\alpha = \lambda$ is increasing in λ, whereas $\alpha^\ddagger$ is decreasing in λ.

it is the point when $\pi^*_S = \pi^*_{CE-SUB}$. We define this threshold as constant $\underline{\alpha_t \triangleq \alpha_c(\lambda_3)}$. This horizontal boundary extends from $\{\alpha_t, \alpha_t\}$ to $\{\lambda_3, \alpha_t\}$.

- **Definition and monotonicity of $\lambda_x(\alpha)$.**

  Finally, we compare π^*_{CE-SUB} under the second case in Prop. 10 and π^*_S under the first case in Prop. 12. More specifically, we explore the parameter space at the intersection of constraints $\lambda < \alpha \leq \alpha^\dagger$ and $0 < \alpha < \alpha^\ddagger$. We denote the difference between optimal profits under strategies *CE-SUB* and *S* as:

$$\Psi_x(\alpha, \lambda) \triangleq p_b \left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha}} \right) - \frac{1}{16(1-\alpha)}.$$

  As p_b maximizes π^*_{CE-SUB}, using Envelope theorem, we get:

$$\begin{aligned}
\frac{\partial \Psi_x(\alpha, \lambda)}{\partial \alpha} &= \frac{p_b^3}{(\alpha - (1-\alpha)p_b)^2} - \frac{1}{16(1-\alpha)^2} \\
\frac{\partial \Psi_x(\alpha, \lambda)}{\partial \lambda} &= \frac{p_b^2}{\lambda^2} > 0.
\end{aligned}$$

  As it turns out, in this range of the parameter space, $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \alpha}$ changes signs. As such, it is not possible to characterize the threshold between *S* and *CE-SUB* as a function of λ (there exist values of λ for which increasing α leads to multiple crossings between optimality regions for *S* and *CE-SUB*).

  Nevertheless, moving horizontally, given that $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda} > 0$, a threshold (crossing) boundary between optimality regions for *CE-SUB* and *S*, within this particular region of the parameter space, is unique for every α, *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:

– First, we consider points on the boundary $\alpha = \alpha^{\dagger}(\lambda)$. Note that, in this region, we have $\lambda < \alpha < \alpha^{\ddagger}(\lambda)$ (since we consider the intersection of constraints $\lambda < \alpha \leq \alpha^{\dagger}$ and $0 < \alpha < \alpha^{\ddagger}$). Given that $\alpha^{\ddagger}$ is decreasing, as shown above, it means that in this region we have $\lambda < \alpha^{\ddagger}(\lambda) \leq \alpha^{\ddagger}(0) \approx 0.1352 < \frac{1}{3}$.[4] From Prop. 10, given that in this parameter region we have $\lambda < \frac{1}{3}$, we consequently get $\alpha^{\dagger} = \tilde{\alpha}^{\dagger}$ (see equation (C.1)), which satisfies $\Xi(\alpha, \lambda) = 0$ and $\frac{\partial \Xi(\alpha,\lambda)}{\partial \alpha} < 0$. Similarly, from the Envelope theorem, we get:

$$\frac{\partial \Xi(\alpha, \lambda)}{\partial \lambda} = \frac{p_b^2}{\lambda^2} > 0.$$

Therefore,

$$\frac{\partial \alpha^{\dagger}(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Xi(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Xi(\alpha,\lambda)}{\partial \alpha}}\bigg|_{\alpha=\alpha^{\dagger}(\lambda)} > 0.$$

Thus, $\alpha^{\dagger}(\lambda) = \tilde{\alpha}^{\dagger}(\lambda)$ is strictly increasing in λ in this region. Therefore, it is invert-

[4] We can achieve the same conclusion the following way. The intersection point between boundaries $\alpha = \alpha^{\ddagger}(\lambda)$ (decreasing) and $\alpha = \lambda$ (increasing), which occurs approximately at $\{0.1195, 0.1195\}$, achieves the maximum λ for this region, which is smaller than $\frac{1}{3}$.

ible. On the boundary $\alpha = \alpha^{\dagger}(\lambda)$, we obtain.

$$\Psi_x(\alpha,\lambda)\Big|_{\lambda=\alpha^{\dagger-1}(\alpha)} = \frac{\alpha}{(\sqrt{\alpha}+1)^2} - \frac{1}{16(1-\alpha)} < 0. \tag{C.7}$$

- Next, we consider the intersection point between $\alpha = \alpha^{\ddagger}$ and $\alpha = \lambda$, which is $\{0.1195, 0.1195\}$. At this point, we obtain:

$$\Psi_x(\alpha,\lambda)\Big|_{\alpha=\lambda=0.1195} = 0.0909 - 0.0710 > 0. \tag{C.8}$$

Therefore, in this parameter region, there exists a sub-region where $\Psi_x < 0$ (*S* dominates *CE-SUB*) and a sub-region where $\Psi_x > 0$ (*CE-SUB* dominates *S*). Given that, for any λ, as we increase α, there can be at most one crossing point between optimality regions for *S* and *CE-SUB*, then there exists a unique threshold boundary, which we define as $\underline{\lambda_x(\alpha)}$, which separates the optimality regions for *CE-SUB* and *S*. It satisfies:

$$\frac{1}{16(1-\alpha)} = p_b\left(2 - \frac{p_b}{\lambda_x(\alpha)} - \frac{p_b}{1+p_b-\frac{p_b}{\alpha}}\right).$$

Let's next examine the domain of $\lambda_x(\alpha)$. For that purpose, we look at the monotonicity of $\Psi_x(\alpha,\lambda)$ in terms of λ on two particular boundaries:

- First, we consider the line $\alpha = \lambda$ (boundary in limit). On this line, we get:

$$\Psi_x(\alpha,\lambda)\Big|_{\lambda=\alpha} = p_b\left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1+p_b-\frac{p_b}{\lambda}}\right) - \frac{1}{16(1-\lambda)}.$$

Again, we reminder the reader that we could have written the profit for *CE-SUB* in terms of p_a at the boundary when $\alpha = \lambda$ - however, on this boundary, whether we write the profit in terms of p_a or p_b, we obtain the same profit because on that particular line, $p_a = p_b$. Then,

$$\frac{\partial \Psi_x(\alpha, \lambda)}{\partial \lambda}\bigg|_{\lambda=\alpha} = p_b^2 \left(\frac{1}{\lambda^2} + \frac{p_b}{(\lambda + (\lambda - 1)p_b)^2}\right) - \frac{1}{16(\lambda - 1)^2}.$$

Let's check the sign of $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}\Big|_{\lambda=\alpha}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda} = \frac{q_{13}}{q_{14}}$, where:

$$\begin{aligned} q_{13} =& 16(1-\lambda)^4 p_b^4 + 16\lambda(3\lambda - 2)(1-\lambda)^2 p_b^3 \\ &+ 15\lambda^2(1-\lambda)^2 p_b^2 + 2\lambda^3(1-\lambda)p_b - \lambda^4, \\ q_{14} =& 16(1-\lambda)^2\lambda^2(\lambda + \lambda p_b - p_b)^2 > 0. \end{aligned}$$

Therefore, the sign of $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}$ is the same as the sign of the numerator, q_{13}.

Recall from Prop. 10 that p_b is the unique solution to the cubic equation $G_{SUB,b}(p_b) = 0$. We use this property of p_b to reduce the expression of q_{13} from a quartic polynomial in p_b to a quadratic one, as follows:

$$\begin{aligned} q_{13} =& \left(4\lambda^4 - 16\lambda^3 - 5\lambda^2 + 2\lambda + 15\right) p_b^2 + 2\lambda \left(4\lambda^3 - 8\lambda^2 + 3\lambda - 15\right) p_b \\ &+ \lambda^2 \left(8\lambda^2 - 8\lambda + 15\right). \end{aligned}$$

Denote:

$$A_{x,1} \triangleq \left(4\lambda^4 - 16\lambda^3 - 5\lambda^2 + 2\lambda + 15\right),$$

$$B_{x,1} \triangleq 2\lambda\left(4\lambda^3 - 8\lambda^2 + 3\lambda - 15\right),$$

$$C_{x,1} \triangleq \lambda^2\left(8\lambda^2 - 8\lambda + 15\right).$$

Then $q_{13} = A_{x,1}p_b^2 + B_{x,1}p_b + C_{x,1}$. Define quadratic function $H^{\Diamond}_{SUB,S}(p) \triangleq A_{x,1}p^2 + B_{x,1}p + C_{x,1}$. In this range of the parameter space, it can be shown that:

$$B_{x,1}^2 - 4A_{x,1}C_{x,1} = 16\lambda^4(\lambda(\lambda(-4(\lambda-6)\lambda - 15) + 4) + 55) > 0.$$

Hence, there are two real solutions to the equation $H^{\Diamond}_{SUB,S}(p) = 0$, namely:

$$p^{\Diamond}_{H1} = \frac{-B_{x,1} - \sqrt{B_{x,1}^2 - 4A_{x,1}C_{x,1}}}{2A_{x,1}} \quad \text{and} \quad p^{\Diamond}_{H2} = \frac{-B_{x,1} + \sqrt{B_{x,1}^2 - 4A_{x,1}C_{x,1}}}{2A_{x,1}}.$$

It can be shown that $p^{\Diamond}_{H1} < p^{\Diamond}_{H2} < \frac{\alpha}{1-\alpha}$ when $\lambda < \frac{1}{3}$ (which is satisfied in this region of the parameter space, as per the above argument). From the proof of Prop. 10, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(p^{\Diamond}_{H1}) < 0$. Hence, $p_b < p^{\Diamond}_{H1} < p^{\Diamond}_{H2}$.

Furthermore, it can be shown that $A_{x,1} > 0$, which indicates that $H^{\Diamond}_{SUB,S}(p)$ is convex. Therefore, $H^{\Diamond}_{SUB,S}(p_b) > 0$. Thus, $\left.\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}\right|_{\lambda=\alpha} > 0$. Thus, on the portion of boundary $\alpha = \lambda$ within this particular region ($0 < \alpha < \alpha^{\ddagger}$), as we increase λ (or, equivalently, as we increase α), there can be *at most one* crossing point that separates the optimality regions for *S* and *CE-SUB*, and, moreover, the crossing (if it exists) can be only from *S* to *CE-SUB* as λ increases.

On the asymptotic boundary $\lambda = \alpha$, when $\alpha \to 0$, $\pi_S^* > \pi_{CE-SUB}^*$ (as per inequality (C.7), given that $\lim_{\lambda \downarrow 0} \alpha^\dagger = 0$); when $\alpha \to \alpha^\ddagger$, $\pi_S^* < \pi_{CE-SUB}^*$ (as per inequality (C.8)). Therefore, there exists a unique intersection point between $\lambda_x(\alpha)$ and $\lambda = \alpha$ within this region. And it is straight forward to see that the intersection point is $\{\alpha_t, \alpha_t\}$ since it is the point when $\pi_S^* = \pi_{CE-SUB}^*$. More precisely, when $\lambda_x(\alpha_t) = \alpha_t$.

– Second, we consider the boundary $\alpha = \alpha^\ddagger(\lambda)$ (boundary in limit). On this line, we can get:

$$\Psi_x(\alpha^\ddagger(\lambda), \lambda) = p_b \left(2 - \frac{p_b}{\lambda} - \frac{p_b}{1 + p_b - \frac{p_b}{\alpha^\ddagger(\lambda)}} \right) - \frac{1}{16(1 - \alpha^\ddagger(\lambda))}.$$

Differentiating with respect to λ, and using Envelope theorem as p_b is maximizing π_{CE-SUB}, we obtain:

$$\frac{\partial \Psi_x}{\partial \lambda} = \left(\frac{p_b^3}{(\alpha^\ddagger - (1 - \alpha^\ddagger) p_b)^2} - \frac{1}{16(1 - \alpha^\ddagger)^2} \right) \frac{\partial \alpha^\ddagger}{\partial \lambda} + \frac{p_b^2}{\lambda^2},$$

where $\frac{\partial \alpha^{\ddagger}(\lambda)}{\partial \lambda} = -\frac{\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \lambda}}{\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \alpha}}$, with $\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \alpha}$ and $\frac{\partial \Psi_{\ddagger}(\alpha,\lambda)}{\partial \lambda}$ expressions derived in equations (C.3) and (C.5).

Let's check the sign of $\frac{\partial \Psi_x(\alpha^{\ddagger}(\lambda),\lambda)}{\partial \lambda}$. Bringing all the terms to a common denominator, we can write $\frac{\partial \Psi_x(\alpha^{\ddagger}(\lambda),\lambda)}{\partial \lambda} = \frac{q_{15}}{q_{16}}$, where,

$$\begin{aligned} q_{16} = &-(1-\alpha)\lambda^2(\alpha + \alpha p_b - p_b)^2 \\ &\times \Big(16\alpha(\lambda+1)^2(4\alpha\lambda + \alpha + 3) \\ &+ \left(-\alpha(8\lambda+7)^2 - 16\lambda - 15\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\Big) > 0, \end{aligned}$$

and q_{15}, via degree reduction (since, as per Prop. 10, p_b is the unique solution to the cubic equation $G_{SUB,b}(p_b) = 0$), can be simplified from a quartic polynomial in p_b to a quadratic function $q_{15} = A_{x,2}p_b^2 + B_{x,2}p_b + C_{x,2}$ with:

$$
\begin{aligned}
A_{x,2} \triangleq \frac{1}{4}(1-\alpha)\lambda^2 & \left(4\alpha(\lambda+1)\left(\alpha^3\left(48\lambda^2-104\lambda-29\right)-2\alpha^2\left(64\lambda^2+28\lambda+51\right)\right.\right. \\
& \left. + \alpha\left(64\lambda^2-16\lambda-29\right)+48(\lambda+1)\right)+\left(\alpha^3\left(-192\lambda^2+320\lambda+347\right)\right. \\
& + \alpha^2\left(512\lambda^2+576\lambda+301\right) \\
& \left.\left. -4\alpha\left(64\lambda^2+80\lambda+35\right)-64\lambda-60\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right), \\
B_{x,2} \triangleq \frac{1}{2}\alpha\lambda & \left(4\alpha(\lambda+1)\left(\alpha^3\left(48\lambda^3+8\lambda^2+15\lambda+4\right)\right.\right. \\
& - 2\alpha^2\left(64\lambda^3+28\lambda^2+9\lambda-6\right) \\
& \left. + \alpha\lambda\left(64\lambda^2-16\lambda-29\right)+48\lambda(\lambda+1)\right) \\
& + \left(-\left(\alpha^3\left(192\lambda^3+128\lambda^2+53\lambda+49\right)\right)\right. \\
& + \alpha^2\left(512\lambda^3+576\lambda^2+189\lambda-15\right)-4\alpha\lambda\left(64\lambda^2+80\lambda+35\right) \\
& \left.\left. -4\lambda(16\lambda+15)\right)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right), \\
C_{x,2} \triangleq \frac{1}{2}\alpha^2\lambda^2 & \left(2\alpha^3\left(96\lambda^3+148\lambda^2+59\lambda+7\right)\right. \\
& - \alpha^2\left(48\lambda^2\left(4\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+3\right)\right. \\
& + \lambda\left(200\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+6\right) \\
& \left. + 43\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}+128\lambda^3-10\right) \\
& + \alpha\left(32\lambda^2\left(4\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}-3\right)\right. \\
& + 16\lambda\left(11\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}-12\right) \\
& \left. +85\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}-96\right) \\
& \left. +2(16\lambda+15)\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right),
\end{aligned}
$$

where, for simplicity of notation, we dropped the superscript and used α instead of $\alpha^{\ddagger}(\lambda)$. Since $q_{16} > 0$, the sign of $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda}$ is the same as the sign of the numerator, q_{15}.

Define quadratic function $\check{H}_{SUB,S}(p) \triangleq A_{x,2}p^2 + B_{x,2}p + C_{x,2}$. In this range of the parameter space, it can be shown that:

$$B_{x,2}^2 - 4A_{x,2}C_{x,2} > 0.$$

Hence, there are two real solutions to the equation $\check{H}_{SUB,S}(p) = 0$, namely:

$$\check{p}_{H1} = \frac{-B_{x,2} - \sqrt{B_{x,2}^2 - 4A_{x,2}C_{x,2}}}{2A_{x,2}} \quad \text{and} \quad \check{p}_{H2} = \frac{-B_{x,2} + \sqrt{B_{x,2}^2 - 4A_{x,2}C_{x,2}}}{2A_{x,2}}.$$

It can be shown that $\check{p}_{H1} < \check{p}_{H2} < \frac{\alpha}{1-\alpha}$ when $\lambda < \frac{1}{3}$ (which is satisfied in this region of the parameter space, as per the above argument). From the proof of Prop. 10, we know that $G_{SUB,b}(p) > 0$ on $(-\infty, p_b)$ and $G_{SUB,b}(p) < 0$ on (p_b, ∞). It can be proved directly that $G_{SUB,b}(\check{p}_{H1}) < 0$. Hence, $p_b < \check{p}_{H1} < \check{p}_{H2}$.

Furthermore, it can be shown that $A_{x,2} > 0$, which indicates that $\check{H}_{SUB,S}(p)$ is convex. Therefore, $\check{H}_{SUB,S}(p_b) > 0$. Thus, $\frac{\partial \Psi_x(\alpha,\lambda)}{\partial \lambda} > 0$. Hence, on the line $\alpha = \alpha^{\ddagger}(\lambda)$, when we increase λ, there can be *at most one* crossing point that separates the optimality regions for *S* and *CE-SUB*, and, moreover, the crossing (if it exists) can be only from *S* to *CE-SUB* as λ increases.

As $\alpha^\dagger$ is increasing in λ (and spanning the entire interval $(0,1]$) and $\alpha^\ddagger$ is decreasing in λ, there exists a unique intersection point between $\alpha^\dagger$ and $\alpha^\ddagger$. Defining this point as $\{\lambda_{x,1}, \alpha^\ddagger(\lambda_{x,1})\}$, with $\alpha^\ddagger(\lambda_{x,1}) = \alpha^\dagger(\lambda_{x,1})$.

On the asymptotic boundary $\alpha = \alpha^\ddagger$, when $\alpha \to \alpha^\ddagger(\lambda_{x,1})$, $\pi_S^* > \pi_{CE-SUB}^*$ (as per inequality (C.7)); when $\alpha \to \lambda$, $\pi_S^* < \pi_{CE-SUB}^*$ (as per inequality (C.8)). Therefore, there exists a unique intersection point between $\lambda_x(\alpha)$ and $\alpha^\ddagger$. We define this point as $\{\lambda_x(\alpha_x), \alpha_x\}$. At this point, we have $\pi_S^* = \pi_{CE-PL}^* = \pi_{CE-SUB}^*$. As such, it can be easily seen that $\lambda_x(\alpha_x) = \lambda_1$.

As $\lambda_x(\alpha)$ only intersects once boundaries $\alpha = \lambda$ and $\alpha = \alpha^\ddagger(\lambda)$, it means that $\lambda_x(\alpha)$ is properly defined on $\alpha \in (\alpha_t, \alpha_x)$, as $\{\alpha, \lambda_x(\alpha)\}$ stays inside this region of the parameter space for all $\alpha \in (\alpha_t, \alpha_x)$.

Thus, we completely characterized lines α_1, α_2, and λ_x, (in particular, segments, $\alpha^\ddagger(\lambda)$, $\alpha_a(\cdot)$, $\alpha_b(\cdot)$, $\alpha_c(\cdot)$, $\alpha_d(\cdot)$, as well as constant thresholds λ_1, λ_2, λ_3, α_x), as well as threshold α_t.

Comparison of $\alpha_1(\lambda)$ and $\alpha_2(\lambda)$:

All segments in $\alpha_1(\lambda)$ on $[0,1]$ (i.e., $\alpha^\ddagger$ on $[0,\lambda_1)$, α_a on $[\lambda_1,\lambda_2)$, and α_b on $[\lambda_2, 1]$ satisfy $\alpha_1(\lambda) \geq \lambda$ (with equality happening only when $\lambda = 1$). At the same time, all segments of

$\alpha_2(\lambda)$ on $\left[\frac{1}{4}, 1\right]$ (i.e., α_c on $\left[\frac{1}{3}, \lambda_3\right)$ and α_d on $[\lambda_3, 1]$) satisfy $\alpha_2(\lambda) < \lambda$. Thus, we have:

$$\alpha_1(\lambda) > \alpha_2(\lambda) \quad \forall \lambda \in \left[\frac{1}{4}, 1\right].$$

Derivation of the dominating strategy in the entire region $0 < \alpha < 1$:

- When $\lambda \leq \alpha < 1$, it is easy to show, via direct comparison, that $\pi^*_{CE-PL} > \pi^*_{TLF}$. Therefore, *TLF* is suboptimal in this region. Then, by the definition of $\alpha_1(\lambda)$ and $\lambda_x(\alpha)$, and in light of the earlier analysis, we get:
  - When $\alpha_1(\lambda) \leq \alpha < 1$ and $\lambda \leq \alpha < 1$, *CE-PL* is the dominating strategy;
  - When $\lambda \leq \alpha < \alpha_1(\lambda)$, we have two subcases:
    * When $\lambda \leq \alpha < \alpha_1(\lambda)$ and $0 \leq \lambda < \lambda_x(\alpha)$, then *S* is the dominant strategy;
    * When $\lambda_x(\alpha) \leq \lambda \leq \alpha < \alpha_1(\lambda)$, then *CE-SUB* is the dominant strategy.
- When $0 < \alpha < \lambda$, we first show that *CE-PL* is always dominated:
  - When $0 < \alpha \leq \frac{\lambda\left(4+5\lambda-4(1+\lambda)\sqrt{\lambda}\right)}{16+24\lambda-7\lambda^2-16\lambda^3}$ and $0 < \alpha < \lambda$, then it can be shown that $\pi^*_{TLF} = \frac{\lambda}{4} \geq \pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$. Hence, in this region, *CE-PL* is dominated.
  - When $\frac{\lambda\left(4+5\lambda-4(1+\lambda)\sqrt{\lambda}\right)}{16+24\lambda-7\lambda^2-16\lambda^3} < \alpha < 5 + 8\lambda - 4\sqrt{(\lambda+1)(4\lambda+1)}$ and $0 < \alpha < \lambda$, then $\pi^*_{TLF} = \frac{\lambda}{4} < \pi^*_{CE-PL} = \frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2}$. Therefore, in this region, *CE-PL* dominates *TLF*. Define the difference between optimal profits under *CE-SUB*

and *CE-PL* as:

$$\Psi_e(\alpha,\lambda) \triangleq p_a\left(2-\frac{p_a}{\alpha}-\frac{p_a}{1+p_a-\frac{p_a}{\alpha}}\right) \\ -\frac{\alpha(\lambda+1)\left(2\alpha\lambda+\alpha+1-2\sqrt{\alpha(\lambda+1)(\alpha\lambda+1)}\right)}{(1-\alpha)^2},$$

Bringing all the terms to a common denominator, we can write $\Psi_e(\alpha,\lambda) = \frac{q_{17}}{q_{18}}$, where:

$$q_{17} = (1-\alpha)^3 p_a^3 + (3-\alpha)\alpha(1-\alpha)^2 p_a^2 \\ -\alpha(1-\alpha)\left(2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}+\alpha^2\left(-2\lambda^2-3\lambda+1\right)-\alpha(\lambda+3)\right)p_a \\ +\alpha^2\left(2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}-\alpha^2\left(2\lambda^2+3\lambda+1\right)-\alpha(\lambda+1)\right),$$

$$q_{18} = (1-\alpha)^2\alpha(\alpha+\alpha p_a-p_a) > 0,$$

where $q_{18} > 0$ is due to the fact that $p_a \in \left(\frac{\alpha}{2},\alpha\right)$. Therefore, the sign of $\Psi_e(\alpha,\lambda)$ is the same as the sign of q_{17}. Recall from Prop. 10 that p_a is the unique solution to the cubic equation $G_{SUB,a}(p_a) = 0$. We use this property of p_a to reduce the expression of q_{17} from a cubic polynomial in p_a to a quadratic one, as follows:

$$q_{17} = \frac{\alpha}{2} \\ \times\Big((1-\alpha)^2\alpha p_a^2 \\ +2(1-\alpha)\left(\alpha\lambda(\alpha(2\lambda+3)+1)-2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}\right)p_a \\ -2\alpha^3(\lambda(2\lambda+3)+2)+4\alpha\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}-2\alpha^2\lambda\Big).$$

Denote:

$$A_e \triangleq (1-\alpha)^2\alpha,$$
$$B_e \triangleq 2(1-\alpha)\left(\alpha\lambda(\alpha(2\lambda+3)+1) - 2\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)}\right),$$
$$C_e \triangleq -2\alpha^3(\lambda(2\lambda+3)+2) + 4\alpha\sqrt{\alpha^3(\lambda+1)^3(\alpha\lambda+1)} - 2\alpha^2\lambda.$$

Then $\frac{2}{\alpha} \times q_{17} = A_e p_a^2 + B_e p_a + C_e$. Define quadratic function $\tilde{H}^{\Diamond}_{SUB,PL}(p) \triangleq A_e p^2 + B_e p + C_e$. In this range of the parameter space, it can be shown that:

$$B_e^2 - 4A_eC_e > 0.$$

Hence, there are two real solutions to the equation $\tilde{H}^{\Diamond}_{SUB,PL}(p) = 0$, namely:

$$\tilde{p}^{\Diamond}_{H1} = \frac{-B_e - \sqrt{B_e^2 - 4A_eC_e}}{2A_e} \quad \text{and} \quad \tilde{p}^{\Diamond}_{H2} = \frac{-B_e + \sqrt{B_e^2 - 4A_eC_e}}{2A_e}.$$

It can be shown that $\tilde{p}^{\Diamond}_{H1} < \tilde{p}^{\Diamond}_{H2} < \frac{\alpha}{1-\alpha}$. From the proof of Prop. 10, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\tilde{p}^{\Diamond}_{H1}) < 0$. Hence, $p_a < \tilde{p}^{\Diamond}_{H1} < \tilde{p}^{\Diamond}_{H2}$.

Furthermore, since $A_e > 0$, $\tilde{H}^{\Diamond}_{SUB,PL}(p)$ is convex. Therefore, $\tilde{H}^{\Diamond}_{SUB,PL}(p_a) > 0$. Thus, $\Psi_e(\alpha, \lambda) > 0$, meaning that, in this region, *CE-PL* is dominated by *CE-SUB*.

- When $5 + 8\lambda - 4\sqrt{(\lambda+1)(4\lambda+1)} \leq \alpha < 1$, we show that *CE-PL* is dominated by

CE-SUB. In this region, define the difference between optimal profits under *CE-SUB* and *CE-PL* as:

$$\Psi_f(\alpha,\lambda) \triangleq p_a\left(2-\frac{p_a}{\alpha}-\frac{p_a}{1+p_a-\frac{p_a}{\alpha}}\right)-\frac{1}{4}\alpha(\lambda+1).$$

Bringing all the terms to a common denominator, we can write $\Psi_f(\alpha,\lambda) = \frac{q_{19}}{q_{20}}$, where:

$$q_{19} = p_a^3(4-4\alpha)+4p_a^2(\alpha-3)\alpha+p_a\alpha^2(-\alpha(\lambda+1)+\lambda+9)-\alpha^3(\lambda+1),$$

$$q_{20} = 4\alpha(\alpha+\alpha p_a-p_a) > 0,$$

where $q_{20} > 0$ is due to the fact that $p_a \in \left(\frac{\alpha}{2},\alpha\right)$. Therefore, the sign of $\Psi_f(\alpha,\lambda)$ is the same as that of q_{19}. Recall from Prop. 10 that p_a is the unique solution to the cubic equation $G_{SUB,a}(p_a) = 0$. We use this property of p_a to reduce the expression of q_{19} from a cubic polynomial in p_a to a quadratic one, as follows:

$$q_{19} = A_f p_a^2 + B_f p_a + C_f,$$

with

$$A_f \triangleq 2(1-\alpha),$$

$$B_f \triangleq \alpha(\alpha-6-(2-\alpha)\lambda)+\lambda-3,$$

$$C_f \triangleq \alpha(\alpha-1)\lambda+\alpha+3).$$

Define quadratic function $\check{H}^{\Diamond}_{SUB,PL}(p) \triangleq A_f p^2 + B_f p + C_f$. In this range of the parameter space, it can be shown that:

$$B_f^2 - 4A_fC_f = (\alpha((\alpha-2)\lambda+\alpha-6)+\lambda-3)^2 - 8(1-\alpha)\alpha((\alpha-1)\lambda+\alpha+3)$$
$$> 0.$$

Hence, there are two real solutions to the equation $\check{H}^{\Diamond}_{SUB,PL}(p) = 0$, namely:

$$\check{p}^{\Diamond}_{H1} = \frac{-B_f - \sqrt{B_f^2 - 4A_fC_f}}{2A_f} \quad \text{and} \quad \check{p}^{\Diamond}_{H2} = \frac{-B_f + \sqrt{B_f^2 - 4A_fC_f}}{2A_f}.$$

It can be shown that $\check{p}^{\Diamond}_{H1} < \check{p}^{\Diamond}_{H2} < \frac{\alpha}{1-\alpha}$. From the proof of Prop. 10, we know that $G_{SUB,a}(p) > 0$ on $(-\infty, p_a)$ and $G_{SUB,a}(p) < 0$ on (p_a, ∞). It can be proved directly that $G_{SUB,a}(\check{p}^{\Diamond}_{H1}) < 0$. Hence, $p_a < \check{p}^{\Diamond}_{H1} < \check{p}^{\Diamond}_{H2}$.

Furthermore, since $A_f > 0$, $\check{H}^{\Diamond}_{SUB,PL}(p)$ is convex. Therefore, $\check{H}^{\Diamond}_{SUB,PL}(p_a) > 0$. Thus, $\Psi_f(\alpha, \lambda) > 0$, meaning that, in this region, *CE-PL* is dominated by *CE-SUB*.

Since *CE-PL* is always dominated when $0 < \alpha < \lambda$, in this region we only need to compare *CE-SUB*, *TLF*, and *S*. By the definition of $\alpha_2(\lambda)$ and α_t, and in light of the earlier analysis, in the region $0 < \alpha < \lambda$ we get:

– If $\max\{\alpha, \frac{1}{4}\} < \lambda \leq 1$ and $0 < \alpha < \alpha_2(\lambda)$, then *TLF* is the dominating strategy;

– Else, if $\alpha_t \leq \alpha < \lambda$, then *CE-SUB* is the dominant strategy;

– Else, *S* is the dominant strategy.

This completes the mapping of dominant strategy to the parameter space (we discussed the case $\alpha \geq 1$ at the very beginning of the proof).

Social welfare comparison.

It can be shown with relative ease, through direct comparisons of closed form solutions, that $SW^*_{TLF} = \frac{3\lambda}{8} + \frac{1}{2} \geq \max\{SW^*_{CE-PL}, SW^*_{S}\}$. Thus, we only have to compare SW^*_{TLF} with $SW_{CE-SUB,a}$, $SW_{CE-SUB,b}$, and $\frac{2\sqrt{\alpha}+1}{2\left(\sqrt{\alpha}+1\right)^2}$ for $\alpha \in (0,1)$. From Prop 10, we know that $p_a \in \left(\frac{\alpha}{2}, \alpha\right)$ and $p_b \in \left(\frac{\lambda}{2}, \lambda\right)$. It is straightforward to see that:

$$\begin{aligned}
SW_{CE-SUB,a} &= \frac{1}{2}\left(1+\lambda-\frac{\lambda p_a^2}{\alpha^2}-\frac{p_a^2}{\left(1+p_a-\frac{p_a}{\alpha}\right)^2}\right) \\
&< \frac{1}{2}\left(1+\lambda-\frac{\lambda p_a^2}{\alpha^2}\right) < \frac{3\lambda}{8}+\frac{1}{2} = SW^*_{TLF}. \\
SW_{CE-SUB,b} &= \frac{1}{2}\left(1+\lambda-\frac{p_b^2}{\lambda}-\frac{1}{\left(1+p_b-\frac{p_b}{\alpha}\right)^2}\right) \\
&< \frac{1}{2}\left(1+\lambda-\frac{p_b^2}{\lambda}\right) < \frac{3\lambda}{8}+\frac{1}{2} = SW^*_{TLF}. \\
\frac{2\sqrt{\alpha}+1}{2\left(\sqrt{\alpha}+1\right)^2} &< \frac{1}{2} < \frac{3\lambda}{8}+\frac{1}{2} = SW^*_{TLF}.
\end{aligned}$$

Thus, *TLF* yields the highest social welfare when $\alpha \in (0,1)$. This completes the social welfare analysis since we discussed the case $\alpha \geq 1$ at the very beginning of the proof. $\square$

APPENDIX D

PROOFS OF RESULTS FOR THE SETUP WITH ADOPTION COSTS OF CHAPTER 2

We first present the optimal strategies under each of the business models separately. As the proofs require defining a lot of parameters, in the interest of avoiding notation abuse, we add a subscript D to some of the newly defined parameters (to distinguish from parameters used in the previous proofs)

Proposition 13. *Under CE-PL model, in the presence of adoption costs, the firm's optimal pricing strategy, the corresponding profit, and ensuing social welfare are:*

	$0 < \alpha < 13 - 4\sqrt{10}$			$\alpha \geq 13 - 4\sqrt{10}$	
	(a) $0 \leq c < c^\dagger$	(b) $c^\dagger \leq c < 2\alpha$	$c \geq 2\alpha$	$0 \leq c < 2\alpha$	$c \geq 2\alpha$
p^*_{CE-PL}	$\frac{\alpha^2 c - c + 2\alpha\left(1+\alpha-\sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2}$	$\frac{1}{2}(2\alpha - c)$	-	$\frac{1}{2}(2\alpha - c)$	-
π^*_{CE-PL}	$\frac{2\alpha+\alpha^2(c+6)-c-4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2}$	$\frac{(c-2\alpha)^2}{8\alpha}$	-	$\frac{(c-2\alpha)^2}{8\alpha}$	-
SW^*_{CE-PL}	$\tilde{SW}_{CE-PL,E}$	$\frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$	-	$\frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$	-
Paid adoption	in both periods	only in period 1	none	only in period 1	none

where

$$\tilde{SW}_{CE-PL,D} = \frac{\left(8\alpha^3 + 8\alpha^2 + 4\alpha - \left(4\alpha^3 - 6\alpha + 2\right)c\right)\sqrt{(\alpha+1)(2\alpha-(1-\alpha)c)}}{2(1-\alpha)^2(\alpha+1)(2\alpha-(1-\alpha)c)} + \frac{2\alpha^4\left(c^2+c-4\right) - \alpha^2(5c(c+1)+14) + 2\alpha(c+1)^2 + (c-1)c}{2(1-\alpha)^2(\alpha+1)(2\alpha-(1-\alpha)c)},$$

and threshold $c^\dagger(\alpha)$ is the unique solution to the equation $\Phi_{PL,D}(\alpha, c) = 0$ over the interval

$\left(c_L^{\dagger}, \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}\right)$, *with*

$$
\begin{aligned}
\Phi_{PL,D}(\alpha,c) \triangleq (1-\alpha)^4 c^4 & \\
& + 8(1-\alpha)^2(\alpha(2-3\alpha)+1)\alpha c^3 \\
& + \left(16(\alpha(2-3\alpha)+1)^2\alpha^2 + 8(\alpha-1)^2((\alpha-14)\alpha-3)\alpha^2\right)c^2 \\
& + \left(32\alpha^3(\alpha(2-3\alpha)+1)((\alpha-14)\alpha-3) - 1024(\alpha-1)\alpha^4(\alpha+1)\right)c \\
& - 2048(\alpha+1)\alpha^5 + 16\alpha^4((\alpha-14)\alpha-3)^2,
\end{aligned}
$$

and

$$
c_L^{\dagger} \triangleq \begin{cases} \left(6 - \frac{4}{1-\alpha}\right)\alpha, & \textit{if } 0 < \alpha < \frac{1}{3}, \\ 0, & \textit{if } \frac{1}{3} \le \alpha \le 13 - 4\sqrt{10}. \end{cases}
$$

Proof. In period 1, consumers with type θ purchase the product iff $2\alpha\theta - c \ge p$. To make any profit, the firm is constrained to trigger adoption in period 1 (otherwise, no customer would update their priors and there will also be no adopters in period 2 either). To achieve that, the firm has to set price $p \in (0, 2\alpha - c)$. Thus, it immediately follows that the firm can make profit iff $0 \le c < 2\alpha$. As such, the firm does not enter the market if $c \ge 2\alpha$.

In the remaining part of the proof we focus on the scenario $0 \le c < 2\alpha$. In period 1, the marginal adopter has type $\theta_1 = \frac{c+p}{2\alpha}$ and the installed base is $N_1 = 1 - \theta_1 = 1 - \frac{c+p}{2\alpha}$.

At the beginning of period 2, the consumers who did not adopt in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + (1 - a_1) N_1 = \frac{1}{2}\left(2 + c + p - \frac{c+p}{\alpha}\right).$$

In period 2, new consumers purchase the product if their type θ satisfies $a_2\theta - c \geq p$. The marginal potential consumer in period 2 has type $\theta_2 = \frac{c+p}{\frac{1}{2}\left(2+c+p-\frac{c+p}{\alpha}\right)}$. We have new adopters in period 2 iff $0 \leq \theta_2 < \theta_1$. We have two cases:

- Case 1: $0 < \alpha < 1$.

  In this case, we have three subcases:

  - Case 1-i: $0 \leq c < \frac{2\alpha - 4\alpha^2}{1-\alpha}$, $0 < p < \frac{2\alpha + \alpha c - 4\alpha^2 - c}{1-\alpha}$.

    Then we have $0 < \theta_2 < \theta_1$. Then, $N_2 = \theta_1 - \theta_2 > 0$. In this case, the firm's profit maximization problem becomes:

    $$\max_{0<p<\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}} \pi_{CE-PL} = \max_{0<p<\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}} p\left(1 - \frac{c+p}{\frac{1}{2}\left(2+c+p-\frac{c+p}{\alpha}\right)}\right).$$

    It can be shown that $\frac{\partial^2 \pi_{CE-PL}}{\partial p^2} < 0$ for $p \in \left(0, \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}\right)$. Thus, it is sufficient to solve FOC:

    $$\frac{\partial \pi_{CE-PL}}{\partial p} = \frac{-\alpha^2\left((c+p)^2 + 4(p-1)\right) - 4\alpha(c+p) + (c+p)^2}{(c+p-\alpha(c+p+2))^2} = 0.$$

Without constrains, the FOC yields two solutions:

$$p_{1,D} = \frac{\alpha^2 c - c + 2\alpha\left(1 + \alpha + \sqrt{(\alpha+1)(\alpha(c+2) - c)}\right)}{1-\alpha^2},$$
$$p_{2,D} = \frac{\alpha^2 c - c + 2\alpha\left(1 + \alpha - \sqrt{(\alpha+1)(\alpha(c+2) - c)}\right)}{1-\alpha^2}.$$

It can be shown that $p_{1,D} > \max\left\{\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}, p_{2,D}\right\}$ and $p_{2,D} > 0$. Comparing $p_{2,D}$ with $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, we have three subcases:

* Case 1-i-a: $0 < \alpha < \frac{1}{2}\left(\sqrt{3}-1\right)$, $0 < c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$.

  Then $0 < p_{2,D} < \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, and it immediately follows that $p^*_{CE-PL} = p_{2,D} = \frac{\alpha^2 c-c+2\alpha\left(1+\alpha-\sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2}$,
  and $\pi^*_{CE-PL} = \frac{2\alpha+\alpha^2(c+6)-c-4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2}$.

* Case 1-i-b: $0 < \alpha < \frac{1}{2}\left(\sqrt{3}-1\right)$, $\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha} \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$.

  Then $p_{2,D} \geq \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. In this case, we have $p^*_{CE-PL} \to \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. This case is suboptimal as optimal pricing is pushed into case 1-ii.

* Case 1-i-c: $\frac{1}{2}\left(\sqrt{3}-1\right) \leq \alpha < 1$.

  Then $p_{2,D} \geq \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. In this case, we have $p^*_{CE-PL} \to \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, and $\pi^*_{CE-PL} = \frac{\alpha(2\alpha(1-2\alpha)-(1-\alpha)c)}{(1-\alpha)^2}$. This case is suboptimal as optimal pricing is pushed into case 1-ii.

– Case 1-ii: $0 \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$, $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p < 2\alpha - c$.

Then we have $\theta_2 \geq \theta_1$. In this case, $N_2 = 0$; adoption takes place only in period 1.

The firm's profit maximization problem becomes:

$$\max_{\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}\leq p<2\alpha-c} \pi_{CE-PL} = \max_{\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}\leq p<2\alpha-c} p\left(1-\frac{c+p}{2\alpha}\right).$$

Since the profit function is quadratic concave in p, it is sufficient to use FOC. Unconstrained, FOC yields the following solution:

$$p_{3,D} = \frac{1}{2}(2\alpha - c).$$

It is obvious that $p_{3,D} < 2\alpha - c$. Comparing $p_{3,D}$ with $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, we have three subcases:

- ∗ Case 1-ii-a: $0 < \alpha < \frac{1}{3}$, $0 \leq c < \left(6 - \frac{4}{1-\alpha}\right)\alpha$.

  Then $0 < p_{3,D} < \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$. Then, we have the corner solution $p^*_{CE-PL} = \frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha}$, which is dominated by case 1-i-a (at the corner solution we have $\theta_1 = \theta_2$).

- ∗ Case 1-ii-b: $0 < \alpha < \frac{1}{3}$, $\left(6 - \frac{4}{1-\alpha}\right)\alpha \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$.

  Then $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p_3 < 2\alpha - c$. Thus, $p^*_{CE-PL} = p_{3,D} = \frac{1}{2}(2\alpha - c)$ and $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$.

- ∗ Case 1-ii-c: $\frac{1}{3} \leq \alpha < 1$, $0 \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$.

  Then $\frac{2\alpha+\alpha c-4\alpha^2-c}{1-\alpha} \leq p_{3,D} < 2\alpha - c$. Thus, $p^*_{CE-PL} = p_{3,D} = \frac{1}{2}(2\alpha - c)$ and $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$.

– Case 1-iii: $\frac{2\alpha-4\alpha^2}{1-\alpha} \leq c < 2\alpha$.

Then $\theta_2 \geq \theta_1$. In this case, $N_2 = 0$; adoption takes place only in period 1. The firm's profit maximization problem becomes:

$$\max_{0<p<2\alpha-c} \pi_{CE-PL} = \max_{0<p<2\alpha-c} p\left(1 - \frac{c+p}{2\alpha}\right).$$

Since the profit function is quadratic concave in p, it is sufficient to use FOC. Unconstrained, FOC yields the same solution $p_{3,D} = \frac{1}{2}(2\alpha - c)$. Then, $p^*_{CE-PL} = p_3 = \frac{1}{2}(2\alpha - c)$ and $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$.

As cases 1-i-b, 1-i-c, and 1-ii-a are suboptimal, in order to determine the optimal strategy when $0 \le c < \frac{2\alpha-4\alpha^2}{1-\alpha}$ we are left to compare cases 1-i-a to cases and 1-ii-b and 1-ii-c. When $0 < \alpha < \frac{1}{2}\left(\sqrt{3}-1\right)$ we have $\left(6 - \frac{4}{1-\alpha}\right)\alpha < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha} < \frac{2\alpha-4\alpha^2}{1-\alpha}$. Thus, we only need to explore two subregions:

– Comparison Subregion 1: $0 < \alpha < \frac{1}{3}$, $\left(6 - \frac{4}{1-\alpha}\right)\alpha \le c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$.

  In this region, denote the difference between profits under case 1-i-a and case 1-ii-b as:

$$\Delta_{PL,D}(\alpha, c) \triangleq \frac{2\alpha + \alpha^2(c+6) - c - 4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2} - \frac{(c-2\alpha)^2}{8\alpha}.$$

  Note that:

$$\begin{aligned} &\Delta_{PL,D}(\alpha, c) > 0 \\ \iff & 4\alpha^2((14-\alpha)\alpha + 3) - (1-\alpha)^2 - c^2 - 4\alpha(3\alpha+1)(1-\alpha)c \\ &> 32\alpha^2\sqrt{(\alpha+1)(2\alpha-(1-\alpha)c)}. \end{aligned}$$

It can be shown that the $4\alpha^2((14-\alpha)\alpha+3)-(1-\alpha)^2-c^2-4\alpha(3\alpha+1)(1-\alpha)c > 0$.

Thus, the sign of $\Delta_{PL,D}(\alpha, c)$ is same as the sign of $\Phi_{PL,D}(\alpha, c)$, where:

$$\Phi_{PL,D}(\alpha, c) \triangleq \left(4\alpha^2((14-\alpha)\alpha+3)-(1-\alpha)^2-c^2-4\alpha(3\alpha+1)(1-\alpha)c\right)^2 \tag{D.1}$$

$$\begin{aligned}
&\quad - 1024\alpha^4(\alpha+1)(2\alpha-(1-\alpha)c). \\
&= (1-\alpha)^4c^4 \\
&\quad + 8(1-\alpha)^2(\alpha(2-3\alpha)+1)\alpha c^3 \\
&\quad + \left(16(\alpha(2-3\alpha)+1)^2\alpha^2 + 8(\alpha-1)^2((\alpha-14)\alpha-3)\alpha^2\right)c^2 \\
&\quad + \left(32\alpha^3(\alpha(2-3\alpha)+1)((\alpha-14)\alpha-3)\right. \\
&\quad \left. -1024(\alpha-1)\alpha^4(\alpha+1)\right)c \\
&\quad - 2048(\alpha+1)\alpha^5 + 16\alpha^4((\alpha-14)\alpha-3)^2.
\end{aligned}$$

It can be shown that, in this region, $\frac{\partial \Phi_{PL,D}(\alpha,c)}{\partial c} < 0$. Next, we check the sign of $\Phi_{PL,D}(\alpha, c)$ at the two extremes in c :

$$\Phi_{PL,D}(\alpha, c)\Big|_{c=\left(6-\frac{4}{1-\alpha}\right)} = 256(1-2\alpha)^2\alpha^6 > 0,$$

$$\Phi_{PL,D}(\alpha, c)\Big|_{c=\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}} = 16(1-2\alpha)^2\alpha^7(\alpha(4(\alpha-1)\alpha-31)-32) < 0.$$

Thus, there exists a unique solution $c = c^\dagger(\alpha)$ to the equation $\Delta_{PL,D}(\alpha, c) = 0$ over the interval $\left(\left(6-\frac{4}{1-\alpha}\right), \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}\right)$, such that when $\left(6-\frac{4}{1-\alpha}\right) \leq c < c^\dagger$, case 1-i-a dominates case 1-ii-b; when $c^\dagger \leq c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$, case 1-ii-b dominates case

1-i-a.

- Comparison Subregion 2: $\frac{1}{3} \leq \alpha < \frac{1}{2}\left(\sqrt{3}-1\right)$, $0 \leq c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$.

  In this region, the difference between profits under case 1-i-a and case 1-ii-c is again given by $\Phi_{PL,D}(\alpha, c)$, as defined in equation (D.1). Following the same steps as above, we can show that in this region as well we have $\frac{\partial \Phi_{PL,D}(\alpha,c)}{\partial c} < 0$ and:

$$\Phi_{PL,D}(\alpha, c)\Big|_{c=\frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}} = 16(1-2\alpha)^2\alpha^7(\alpha(4(\alpha-1)\alpha-31)-32) < 0.$$

  Then, we look at the other extreme in c:

$$\Phi_{PL,D}(\alpha, c)\Big|_{c=0} = 16(1-\alpha)^2\alpha^4((\alpha-26)\alpha+9).$$

  It can be shown that:

$$\Phi_{PL,D}(\alpha, c)\Big|_{c=0} \begin{cases} > 0 & , \frac{1}{3} \leq \alpha < 13 - 4\sqrt{10}, \\ \leq 0 & , 13 - 4\sqrt{10} < \alpha < \frac{1}{2}\left(\sqrt{3}-1\right). \end{cases} \tag{D.2}$$

  Thus:

  * When $\frac{1}{3} \leq \alpha < 13 - 4\sqrt{10}$, there exists a unique solution $c = c^{\dagger}(\alpha)$[1] to the equation $\Delta_{PL,D}(\alpha, c) = 0$ over the interval $\left(0, \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}\right)$, such that when $0 \leq c < c^{\dagger}$, case 1-i-a dominates case 1-ii-c; when $c^{\dagger} \leq c < \frac{2\alpha(1-2\alpha(\alpha+1))}{1-\alpha}$, case 1-ii-c dominates case 1-i-a;
  * When $13 - 4\sqrt{10} < \alpha < \frac{1}{2}\left(\sqrt{3}-1\right)$, case 1-ii-c dominates case 1-i-a.

[1]We use the same notation as in the prior case, since the solution is to the same equation, but over a different range of α.

In summary, in case 1, when $0 < \alpha < 13 - 4\sqrt{10}$ and $0 \leq c < c^{\dagger}(\alpha)$, then we have $p^*_{CE-PL} = \frac{\alpha^2 c - c + 2\alpha\left(1+\alpha-\sqrt{(\alpha+1)(\alpha(c+2)-c)}\right)}{1-\alpha^2}$, $\pi^*_{CE-PL} = \frac{2\alpha+\alpha^2(c+6)-c-4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}{(1-\alpha)^2}$, and

$$SW^*_{CE-PL} = \frac{\left(8\alpha^3 + 8\alpha^2 + 4\alpha - \left(4\alpha^3 - 6\alpha + 2\right)c\right)\sqrt{(\alpha+1)(2\alpha - (1-\alpha)c)}}{2(1-\alpha)^2(\alpha+1)(2\alpha-(1-\alpha)c)} + \frac{2\alpha^4\left(c^2+c-4\right) - \alpha^2(5c(c+1)+14) + 2\alpha(c+1)^2 + (c-1)c}{2(1-\alpha)^2(\alpha+1)(2\alpha-(1-\alpha)c)}.$$

Otherwise, $p^*_{CE-PL} = \frac{1}{2}(2\alpha - c)$, $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$, and $SW^*_{CE-PL} = \frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$.

- Case 2: $\alpha \geq 1$.

  In this case, $a_1 > a_2 > a = 1$. None of the period 1 non-adopters will purchase in period 2. The profit maximization problem becomes:

$$\max_{0<p<2\alpha-c} \pi_{CE-PL} = \max_{0<p<2\alpha-c} p\left(1 - \frac{c+p}{2\alpha}\right).$$

  Then, $p^*_{CE-PL} = \frac{1}{2}(2\alpha - c)$, $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$, and $SW^*_{CE-PL} = \frac{(c-2\alpha)((4\alpha-1)c-6\alpha)}{16\alpha^2}$.

$\square$

Proposition 14. *Under CE-SUB model, in the presence of adoption costs, the firm's optimal pricing strategy, the corresponding profit, and ensuing social welfare are:*

- $0 < \alpha \leq 1$.

	$0 \leq c < \alpha$	$\alpha \leq c$
p^*_{CE-SUB}	$p_{a,D}$	-
π^*_{CE-SUB}	$\pi_{1,CE-SUB,D}$	-
SW^*_{CE-SUB}	$SW_{1,CE-SUB,D}$	-
Paid adoption	in both periods	none

where $p_{a,D}$ is the unique solution to the equation $G_{SUB,D}(p) = 0$ over the interval $\left[\frac{\alpha-c}{2}, \alpha - c\right)$ with

$$\begin{aligned}
G_{SUB,D}(p) \triangleq &-2(1-\alpha)^2p^3 + p^2(1-\alpha)((6-\alpha)\alpha + 5(\alpha-1)c) \\
&+ 2p\left((\alpha-3)\alpha^2 - 2(\alpha-1)^2c^2 + \alpha\left(\alpha^2 - 6\alpha + 5\right)c\right) \\
&+ (1-\alpha)^2c^3 + 2\alpha^3 + (3\alpha-5)\alpha^2c + \alpha\left(\alpha^2 - 5\alpha + 4\right)c^2,
\end{aligned}$$

and

$$\begin{aligned}
\pi_{1,CE-SUB,D} &= p_{a,D}\left(2 - \frac{c+p_{a,D}}{\alpha} - \frac{c+p_{a,D}}{1+c+p_{a,D}-\frac{c+p_{a,D}}{\alpha}}\right), \\
SW_{1,CE-SUB,D} &= 1 - c - \frac{(c+p_{a,D})^2}{2\alpha^2} - \frac{(c+p_{a,D})^2}{2\left(1+c+p_{a,D}-\frac{c+p_{a,D}}{\alpha}\right)^2} \\
&\quad + \frac{c(c+p_{a,D})}{1+c+p_{a,D}-\frac{c+p_{a,D}}{\alpha}}.
\end{aligned}$$

- $1 < \alpha \leq 2$.

	$0 \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$	$\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$	$\alpha \leq c$
p^*_{CE-SUB}	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Paid adoption	in period 1	in both periods	in both periods	none

where $SW_{2,CE-SUB,D} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.

- $2 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$.

	$0 \leq c < \alpha - 2$			$\alpha - 2 \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$	$\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$	$\alpha \leq c$
	$2 < \alpha \leq 3$	$3 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$					
		$0 \leq c < c_{1,SUB,D}$	$c_{1,SUB,D} \leq c < \alpha - 2$				
p^*_{CE-SUB}	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{\alpha-c}{2}$	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{(\alpha-c)^2}{4\alpha}$	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$	$SW_{2,CE-SUB,D}$	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Paid adoption	in both periods	in period 1	in both periods	in both periods	in both periods	in both periods	none

where $c_{1,SUB,D} = \alpha - \sqrt{\alpha+1} - 1$.

- $\frac{1}{2}\left(\sqrt{17}+3\right) \leq \alpha < 4\sqrt{2}+5$.

	$0 \le c < c_{1,SUB,D}$	$c_{1,SUB,D} \le c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$	$\frac{2(\alpha-1)\alpha}{3\alpha+1} \le c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha$	$\alpha \le c$
p^*_{CE-SUB}	$\frac{\alpha-c}{2}$	$\frac{2\alpha-c}{2(\alpha+1)}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(\alpha-c)^2}{4\alpha}$	$\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$\frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$	$SW_{2,CE-SUB,D}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Paid adoption	in period 1	in both periods	in both periods	in both periods	none

- $4\sqrt{2}+5 \le \alpha$.

	$0 \le c < c_{3,SUB,D}$	$c_{3,SUB,D} \le c < \frac{\alpha^2-\alpha}{\alpha+1}$	$\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha$	$\alpha \le c$
p^*_{CE-SUB}	$\frac{\alpha-c}{2}$	$\frac{c}{\alpha-1}$	$\frac{\alpha-c}{2}$	-
π^*_{CE-SUB}	$\frac{(\alpha-c)^2}{4\alpha}$	$\frac{2c(\alpha-c-1)}{(\alpha-1)^2}$	$\frac{(\alpha-c)^2}{2\alpha}$	-
SW^*_{CE-SUB}	$\frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$	$\frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$	$\frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$	-
Adoption	in period 1	in both periods	in both periods	none

where $c_{3,SUB,D} = \frac{(\alpha-1)\alpha(\alpha+3)-2\sqrt{2}\alpha(\alpha-1)}{\alpha(\alpha+6)+1}$.

Proof. In period 1, customers subscribe iff $\alpha\theta - c \ge p$. To make profit, the firm is constrained to set $0 < p < \alpha - c$. Thus, it immediately follows that the firm can make profit iff $0 \le c < \alpha$. As such, the firm does not enter the market if $c \ge \alpha$.

In the remaining part of the proof we focus on the scenario $0 \le c < \alpha$. In period 1, the marginal adopter has type $\theta_1 = \frac{c+p}{\alpha}$ and the installed base is $N_1 = 1 - \theta_1 = 1 - \frac{c+p}{\alpha}$.

At the beginning of period 2, the consumers who did not adopt in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + (1 - a_1)N_1 = 1 + p + c - \frac{c + p}{\alpha}.$$

In period 2, *new* consumers subscribe to the product/service if their type θ satisfies $a_2\theta - c \geq p$.

We have two cases:

- Case 1: $0 < \alpha \leq 1$.

  In this case, $a_1 \leq a_2 \leq a = 1$. The marginal customer type for period 1 non-adopters at the beginning of period 2 is $\theta_2 = \frac{c+p}{1+c+p-\frac{c+p}{\alpha}} < \theta_1$. Thus, all customers with types $\theta \in [\theta_2, \theta_1)$ are new adopters in period 2 (i.e., fresh subscribers). In the case of period 1 adopters (i.e., with type $\theta \in [\theta_1, 1]$), their valuation of the product updates upwards and there is no more adoption cost in period 2 (since adoption cost is a one-time cost). Thus, all adopters in period 1 continue to subscribe in period 2. The profit maximization problem becomes:

$$\begin{aligned}\max_{0<p<\alpha-c} \pi_{CE-SUB} &= \max_{0<p<\alpha-c} p(1 - \theta_1 + 1 - \theta_2) \\ &= \max_{0<p<\alpha-c} p\left(2 - \frac{c+p}{\alpha} - \frac{c+p}{1+c+p-\frac{c+p}{\alpha}}\right).\end{aligned}$$

  It can be shown that $\frac{\partial^2 \pi_{CE-SUB}}{\partial p^2} < 0$. Hence, FOC is sufficient to determine the optimal price. We have:

$$\frac{\partial \pi_{CE-SUB}}{\partial p} = \frac{G_{SUB,D}}{Q_{SUB,D}},$$

where:

$$\begin{aligned} G_{SUB,D}(p) \triangleq & -2(1-\alpha)^2 p^3 + p^2(1-\alpha)((6-\alpha)\alpha + 5(\alpha-1)c) \\ & + 2p\left((\alpha-3)\alpha^2 - 2(\alpha-1)^2 c^2 + \alpha\left(\alpha^2 - 6\alpha + 5\right)c\right) \\ & + (1-\alpha)^2 c^3 + 2\alpha^3 + (3\alpha-5)\alpha^2 c + \alpha\left(\alpha^2 - 5\alpha + 4\right)c^2, \\ Q_{SUB,D}(p) \triangleq & \ \alpha(\alpha - (1-\alpha)c - (1-\alpha)p)^2 > 0. \end{aligned}$$

Thus, when solving FOC $\left(\frac{\partial \pi_{CE-SUB}}{\partial p} = 0\right)$, it is enough to look at the numerator. We further have two cases:

– Case 1-i: $0 < \alpha < 1$.

  In this case, $G_{SUB,D}(p)$ is cubic in p and, thus, the equation $\frac{\partial G_{SUB,D}(p)}{\partial p} = 0$ has two solutions:

$$p_{1,SUB,D} = \frac{\alpha + \alpha c - c}{1-\alpha} \quad \text{and} \quad p_{2,SUB,D} = \frac{-\alpha^2 + 3\alpha + 2\alpha c - 2c}{3(1-\alpha)}.$$

  It can be shown that $p_{1,SUB,D} > \alpha - c$ and $p_{2,SUB,D} > \alpha - c$. Thus, $\frac{\partial G_{SUB,D}(p)}{\partial p} < 0$ for all $p \in (0, \alpha - c)$. Evaluating $G_{SUB,D}(p)$ at various threshold points, it can be shown that:

$$G_{SUB,D}(0) > G_{SUB,D}\Big(\frac{\alpha - c}{2}\Big) > 0 > G_{SUB,D}(\alpha - c).$$

Thus, $G_{SUB,D}(p) = 0$ has a unique solution $p_{a,D} \in \big[\frac{\alpha-c}{2}, \alpha - c\big)$ over the real line, which is also the optimal profit-maximizing price in this region ($p^*_{CE-SUB} = p_{a,D}$). More precisely, $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} > 0$ for $p \in (0, p_{a,D})$ and $\frac{\partial \pi_{CE-SUB}(p)}{\partial p} < 0$ for $p \in (p_{a,D}, \alpha - c)$. The formulas for the optimal profit and associated social welfare follow trivially.

- Case 1-ii: $\alpha = 1$.

  In this case, $G_{SUB,D}(p) = -2(c + 2p - 1)$. The equation $G_{SUB,D}(p) = 0$ has a unique solution $p_{a,D} = \frac{1-c}{2} \in \big[\frac{1-c}{2}, 1 - c\big)$. Therefore, $p^*_{CE-SUB} = p_{a,D}$.

- Case 2: $1 < \alpha$.

  In this case, $a_1 \geq a_2 \geq a = 1$. None of period 1 non-adopters will subscribe in period 2 as they revise downwards their perceived valuation of the product. On the other hand, period 1 subscribers, when exploring renewing their subscription for period 2, have to consider the tension between two opposing forces: (i) the downgrading in the perceived valuation (which by now has been calibrated to the real value through experience learning) and (ii) the reduction in adoption cost (the adoption cost is incurred only at adoption time and, as such, returning customers would no longer incur that cost in period 2). Thus, the marginal adopting customer type in period 2, θ_2, satisfies $\theta_2 = \max\{\theta_1, \min\{1, p\}\}$. Since $0 < p < \alpha - c$, comparing $\alpha - c$ with 1, we get three cases:

- Case 2-i: $0 \leq c < \alpha - 1, 0 < p < 1$.

  In this case, $\alpha - c > 1 > p$ and $\theta_2 = \max\{\theta_1, p\}$. Comparing θ_1 and p, we obtain two sub-cases:

  * Case 2-i-A: $0 < p \leq \frac{c}{\alpha-1}$.

    In this case, $\theta_1 \geq p$ and all period 1 subscribers continue to subscribe in period 2. Thus, the profit maximization problem becomes:

    $$\max_{0<p<\frac{c}{\alpha-1}} \pi_{CE-SUB} = \max_{0<p<\frac{c}{\alpha-1}} 2p\left(1 - \frac{c+p}{\alpha}\right).$$

    Since the profit is quadratic concave in p, it is sufficient to use FOC to derive optimal price. Unconstrained, FOC yields the following solution:

    $$p_{3,SUB,D} = \frac{\alpha - c}{2}.$$

    Comparing $p_{3,SUB,D}$ with $\frac{c}{\alpha-1}$, we obtain two sub-cases:

    · Case 2-i-A-I: $0 \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$.

      In this case, $p_{3,SUB,D} > \frac{c}{\alpha-1}$. Then, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$ and $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$.

    · Case 2-i-A-II: $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha - 1$.

      In this case, $p_{3,SUB,D} \leq \frac{c}{\alpha-1}$. Then, $p^*_{CE-SUB} = p_{3,SUB,D} = \frac{\alpha-c}{2}$ and $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$.

* Case 2-i-B: $\frac{c}{\alpha-1} < p < 1$.

  In this case, $\theta_2 = p > \theta_1$ and the profit maximization problem becomes:

$$\max_{\frac{c}{\alpha-1} \le p < 1} \pi_{CE-SUB} = \max_{\frac{c}{\alpha-1} \le p < 1} p\left(2 - \frac{c+p}{\alpha} - p\right).$$

  Since the profit function is quadratic concave in p, it is sufficient to use FOC to identify the optimal price. Unconstrained, FOC yields the following solution:

$$p_{4,SUB,D} = \frac{2\alpha - c}{2(\alpha+1)} < 1.$$

  Comparing $p_{4,SUB,D}$ with $\frac{c}{\alpha-1}$, we obtain two sub-cases:

  · Case 2-i-B-I: $0 \le c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.

    In this case, $p_{4,SUB,D} > \frac{c}{\alpha-1}$, $p^*_{CE-SUB} = p_{4,SUB,D} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$.

  · Case 2-i-B-II: $\frac{2(\alpha-1)\alpha}{3\alpha+1} \le c < \alpha - 1$.

    In this case, $p_{4,SUB,D} \le \frac{c}{\alpha-1}$, and $p^*_{CE-SUB} \to \frac{c}{\alpha-1}$. This case is suboptimal as we are pushed into case 2-i-A.

Since $\frac{\alpha^2-\alpha}{\alpha+1} > \frac{2(\alpha-1)\alpha}{3\alpha+1}$, comparing case 2-i-A (both subcases) against case 2-i-B-I and reorganizing, we get:

* Case 2-i-a: $0 \le c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.

  In this case, $\frac{2c(\alpha-c-1)}{(\alpha-1)^2} < \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, i.e., case 2-i-B-I dominates case 2-i-A-I, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$.

* Case 2-i-b: $\frac{2(\alpha-1)\alpha}{3\alpha+1} \le c < \frac{\alpha^2-\alpha}{\alpha+1}$.

In this case, as discussed above, case 2-i-B-II is dominated by 2-i-A-I. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$.

* Case 2-i-c: $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha - 1$.

  In this case, as discussed above, case 2-i-B-II is dominated by 2-i-A-II, Thus $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$.

– Case 2-ii: $0 \leq c < \alpha - 1$, $1 \leq p < \alpha - c$

In this case, $\theta_2 = 1$. There are no subscribers in period 2. The profit maximization problem becomes:

$$\max_{1\leq p<\alpha-c} \pi_{CE-SUB} = \max_{1\leq p<\alpha-c} p\left(1 - \frac{c+p}{\alpha}\right).$$

Since the function is quadratic, it is sufficient to use FOC. Unconstrained, FOC yields the following solution:

$$p_{3,SUB,D} = \frac{\alpha-c}{2} < \alpha - c.$$

Comparing $p_{3,SUB,D}$ with 1, we obtain three sub-cases:

* Case 2-ii-a: $1 < \alpha \leq 2$.

  In this case, $p_{3,SUB,D} \leq 1$, and thus $p^*_{CE-SUB} = 1$, $\pi^*_{CE-SUB} = 1 - \frac{c+1}{\alpha}$.

* Case 2-ii-b: $2 < \alpha$, $0 \leq c < \alpha - 2$.

  In this case, $p_{3,SUB,D} > 1$, $p^*_{CE-SUB} = p_{3,SUB,D} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$.

* Case 2-ii-c: $2 < \alpha$, $\alpha - 2 \leq c < \alpha - 1$.

In this case, $p_{3,SUB,D} \leq 1$, $p^*_{CE-SUB} = 1$, $\pi^*_{CE-SUB} = 1 - \frac{c+1}{\alpha}$.

- Case 2-iii: $\alpha - 1 \leq c < \alpha$, $0 < p < \alpha - c \leq 1$.

  In this case, $\frac{c}{\alpha-1} \geq 1 > p$. Thus, $\theta_1 > p$ and $\theta_2 = \theta_1$. The profit maximization problem becomes:

$$\max_{0 \leq p < \alpha - c} \pi_{CE-SUB} = \max_{0 \leq p < \alpha - c} 2p \left(1 - \frac{c+p}{\alpha}\right).$$

  It follows that $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$.

Let us summarize case 2 (and in particular compare 2.i and 2.ii cases). It is easy to see that $\alpha - 2 < \frac{\alpha^2-\alpha}{\alpha+1}$. Comparing $\alpha - 2$ and $\frac{2(\alpha-1)\alpha}{3\alpha+1}$, we get three cases:

- $1 < \alpha \leq 2$.

  It can be easily shown that case 2-i dominates case 2-ii-a when $c < \alpha - 1$. Combining with case 2-iii, we extend the region to $c < \alpha$.

- $2 < \alpha < \frac{1}{2}\left(\sqrt{17} + 3\right)$.

  In this case, we have $\alpha - 2 < \frac{2(\alpha-1)\alpha}{3\alpha+1} < \frac{\alpha^2-\alpha}{\alpha+1}$. We further have four sub-cases:

  * $0 \leq c < \alpha - 2$.

    In this region, denote the profit difference between case 2-i-a and case 2-ii-b as:

$$H_{1,SUB,D} \triangleq \frac{(2\alpha - c)^2}{4\alpha(\alpha+1)} - \frac{(\alpha - c)^2}{4\alpha} = \frac{-c^2 + 2(\alpha-1)c - (\alpha-3)\alpha}{4(\alpha+1)}.$$

The equation $H_{1,SUB,D} = 0$ has two solutions:

$$c_{1,SUB,D} = \alpha - \sqrt{\alpha+1} - 1 \quad \text{and} \quad c_{2,SUB,D} = \alpha + \sqrt{\alpha+1} - 1.$$

We have $c_{1,SUB,D} < \alpha - 2 < c_{2,SUB,D}$. Comparing $c_{1,SUB,D}$ with 0, we get:

- If $2 < \alpha \leq 3$, then $c_{1,SUB,D} < 0$ and $H_{1,SUB,D} \geq 0$ for all $c \in [0, \alpha - 2)$, i.e. case 2-i-a dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{2\alpha - c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, and $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.
- If $3 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$ and $0 \leq c < c_{1,SUB,D} = \alpha - \sqrt{\alpha+1} - 1$, then $H_{1,SUB,D} < 0$, i.e. case 2-ii-b dominates case 2-i-a. Thus, $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.
- If $3 < \alpha < \frac{1}{2}\left(\sqrt{17}+3\right)$ and $c_{1,SUB,D} \leq c < \alpha - 2$, then $H_{1,SUB,D} \geq 0$, i.e. case 2-i-a dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{2\alpha - c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, and $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.

* $\alpha - 2 \leq c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.

In this region, it can be shown that $\frac{(2\alpha-c)^2}{4\alpha(\alpha+1)} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-a dominates case 2-ii-c. Thus, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(c-2\alpha)^2}{4\alpha(\alpha+1)}$, and $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.

* $\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$.

In this region, it can be shown that $\frac{2c(\alpha-c-1)}{(\alpha-1)^2} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-b dominates case 2-ii-c. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.

* $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$.

In this region, it can be shown that, when $\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha - 1$, we have $\frac{(\alpha-c)^2}{2\alpha} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-c dominates case 2-ii-c, and $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$, $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$. Combining with case 2-iii, we extend the region to $\frac{\alpha^2-\alpha}{\alpha+1} \le c < \alpha$.

- $\alpha \ge \frac{1}{2}\left(\sqrt{17}+3\right)$.

  In this case, we have $\frac{2(\alpha-1)\alpha}{3\alpha+1} \le \alpha - 2 < \frac{\alpha^2-\alpha}{\alpha+1}$. We further have four sub-cases:

  * $0 \le c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$.

    Following the same steps as in the above case, it can be shown that $0 < c_{1,SUB,D}$ and $\frac{2(\alpha-1)\alpha}{3\alpha+1} < c_{2,SUB,D}$. It can be shown that $c_{1,SUB,D} < \frac{2(\alpha-1)\alpha}{3\alpha+1}$ iff $\alpha < 4\sqrt{2}+5$. We have the following sub-cases:

    · If $\frac{1}{2}\left(\sqrt{17}+3\right) \le \alpha < 4\sqrt{2}+5$, $0 \le c < c_{1,SUB,D}$, case 2-ii-b dominates case 2-i-a. Thus, $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, and $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.

    · If $\frac{1}{2}\left(\sqrt{17}+3\right) \le \alpha < 4\sqrt{2}+5$ and $c_{1,SUB,D} \le c < \frac{2(\alpha-1)\alpha}{3\alpha+1}$, case 2-i-a dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{2\alpha-c}{2(\alpha+1)}$, $\pi^*_{CE-SUB} = \frac{(2\alpha-c)^2}{4\alpha(\alpha+1)}$, $SW^*_{CE-SUB} = \frac{4\alpha^2(\alpha(\alpha+4)+1)+\left(\alpha^2(8\alpha+7)-1\right)c^2-4\alpha(2\alpha+1)\left(\alpha^2+1\right)c}{8\alpha^2(\alpha+1)^2}$.

    · If $\alpha \ge 4\sqrt{2}+5$, then $c_{1,SUB,D} \ge \frac{2(\alpha-1)\alpha}{3\alpha+1}$. Case 2-ii-b dominates case 2-i-a. $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, and $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.

  * $\frac{2(\alpha-1)\alpha}{3\alpha+1} \le c < \alpha - 2$.

In this region, denote the profit difference between case 2-i-b and case 2-ii-b as:

$$
\begin{aligned}
H_{2,SUB,D} &\triangleq \frac{2c(\alpha-c-1)}{(\alpha-1)^2} - \frac{(\alpha-c)^2}{4\alpha} \\
&= \frac{-\left(\alpha^2+6\alpha+1\right)c^2 + 2\alpha\left(\alpha^2+2\alpha-3\right)c - (\alpha-1)^2\alpha^2}{4(\alpha-1)^2\alpha}.
\end{aligned}
$$

The equation $H_{2,SUB,D} = 0$ has two solutions:

$$
\begin{aligned}
c_{3,SUB,D} &= \frac{(\alpha-1)\alpha(\alpha+3) - 2\sqrt{2}\alpha(\alpha-1)}{\alpha(\alpha+6)+1}, \\
c_{4,SUB,D} &= \frac{(\alpha-1)\alpha(\alpha+3) + 2\sqrt{2}\alpha(\alpha-1)}{\alpha(\alpha+6)+1}.
\end{aligned}
$$

It can be shown that $c_{3,SUB,D} < \alpha - 2 < c_{4,SUB,D}$. We have $c_{3,SUB,D} < \frac{2(\alpha-1)\alpha}{3\alpha+1}$ iff $\alpha < 4\sqrt{2}+5$. We get the following sub-cases:

· If $\frac{1}{2}\left(\sqrt{17}+3\right) \leq \alpha < 4\sqrt{2}+5$, then $c_{3,SUB,D} < \frac{2(\alpha-1)\alpha}{3\alpha+1}$. Thus, $H_{2,SUB,D} > 0$ for all $c \in \left[\frac{2(\alpha-1)\alpha}{3\alpha+1}, \alpha-2\right)$. Case 2-i-b dominates case 2-ii-b. We have $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.

· If $\alpha \geq 4\sqrt{2}+5$ and $\frac{2(\alpha-1)\alpha}{3\alpha+1} \leq c < c_{3,SUB,D}$, then case 2-ii-b dominates case 2-i-b. Thus, $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{4\alpha}$, and $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(4\alpha-1)c+3\alpha)}{8\alpha^2}$.

· If $\alpha \geq 4\sqrt{2}+5$ and $c_{3,SUB,D} \leq c < \alpha - 2$, then case 2-i-b dominates case 2-ii-b. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.

* $\alpha - 2 \leq c < \frac{\alpha^2-\alpha}{\alpha+1}$.

In this region, it can be shown that $\frac{2c(\alpha-c-1)}{(\alpha-1)^2} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-b dominates case 2-ii-c. Thus, $p^*_{CE-SUB} = \frac{c}{\alpha-1}$, $\pi^*_{CE-SUB} = \frac{2c(\alpha-c-1)}{(\alpha-1)^2}$, and $SW^*_{CE-SUB} = \frac{(\alpha-2)c^2}{(\alpha-1)^2} - c + 1$.

* $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$.

  In this region, it can be shown that when $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha - 1$, $\frac{(\alpha-c)^2}{2\alpha} > 1 - \frac{c+1}{\alpha}$, i.e. case 2-i-c dominates case 2-ii-c and $p^*_{CE-SUB} = \frac{\alpha-c}{2}$, $\pi^*_{CE-SUB} = \frac{(\alpha-c)^2}{2\alpha}$, $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(2\alpha-1)c+3\alpha)}{4\alpha^2}$. Combining with case 2-iii, we extend the region to $\frac{\alpha^2-\alpha}{\alpha+1} \leq c < \alpha$. $\square$

Proposition 15. *Under TLF model, in the presence of adoption costs, the firm's optimal pricing strategy, the corresponding profit, and ensuing social welfare are:*

	$0 < \alpha < 1$			$\alpha \geq 1$		
	$0 \leq c < \frac{\alpha}{2}$	$\frac{\alpha}{2} \leq c < \alpha$	$c \geq \alpha$	$0 \leq c < \frac{\alpha}{2}$	$\frac{\alpha}{2} \leq c < \alpha$	$c \geq \alpha$
p^*_{TLF}	$\frac{1}{2}$	$\frac{c}{\alpha}$	-	$\frac{1}{2}$	$\frac{c}{\alpha}$	-
π^*_{TLF}	$\frac{1}{4}$	$\frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}$	-	$\frac{1}{4}$	$\frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}$	-
SW^*_{TLF}	$\frac{\alpha c^2(\alpha+2(\alpha-1)c)}{2(\alpha+(\alpha-1)c)^2} - c + \frac{7}{8}$	$\frac{1}{2}c\left(\frac{(2\alpha-1)c}{\alpha^2} - 2\right) + \frac{7}{8}$	-	$\frac{2\alpha^4-(1-\alpha)^2c^4+2(1-\alpha)^2\alpha c^3+\left(\alpha^2-\alpha^4\right)c^2+2(\alpha-2)\alpha^3 c}{2\alpha^2(\alpha+\alpha c-c)^2}$	$\frac{(\alpha-c)(-(\alpha-1)c+\alpha)}{\alpha^2}$	-
Paid adoption	in both periods	in both periods	none	in both periods	in both periods	none

Proof. Under TLF, all customers get the product for free in period 1, but they incur adoption cost c. Thus, customers of type θ start the free trial iff $\alpha\theta \geq c$. It is straightforward to see that there is no adoption $0 < \alpha \leq c$. In the remaining part of the proof, we focus on the more interesting scenario in which adoption can take place, i.e. $0 \leq c < \alpha$.

The marginal adopter in period 1 has type $\theta_1 = \frac{c}{\alpha}$. The size of the adopter population in period 1 is $N_1 = 1 - \frac{c}{\alpha}$.

At the beginning of period 2, adopters in period 1 purchase the product iff $\theta \geq p$ (they already incurred the one-time adoption cost during the free trial in period 1). Period 1 adopters, through WOM, will help the non-adopters update their priors at the beginning of period 2 - however, the consumers who did not adopt in period 1 still have one free-trial period available to them and thus, regardless of how they update their priors, they will not contribute revenue to the firm. Hence, the only revenue can comes from consumers who took advantage of the free trial in period 1. Thus, the firm must set $p \in (0, 1)$. The marginal *paying* customer in period 2 has type $\theta_2 = \max\{\theta_1, p\}$. Comparing p with θ_1, we get two cases:

- Case 1: $0 < p < \frac{c}{\alpha} = \theta_1$.

  In this case, $\theta_2 = \theta_1$ and the profit maximization problem becomes:

  $$\max_{0<p<\frac{c}{\alpha}} \pi_{TLF} = \max_{0<p<\frac{c}{\alpha}} p\left(1 - \frac{c}{\alpha}\right).$$

  It follows that $p^*_{TLF} \uparrow \frac{c}{\alpha}$. This case is suboptimal as p^*_{TLF} is pushed into case 2 region.

- Case 2: $\frac{c}{\alpha} \leq p < 1$.

  In this case, $\theta_2 \geq \theta_1$ and the profit maximization problem becomes:

  $$\max_{\frac{c}{\alpha}\leq p<1} \pi_{TLF} = \max_{\frac{c}{\alpha}\leq p<1} p(1-p).$$

  We have two subcases:

  - Case 2-i: If $0 \leq c < \frac{\alpha}{2}$, $p^*_{TLF} = \frac{1}{2}$, $\pi^*_{TLF} = \frac{1}{4}$.
  - Case 2-ii: If $\frac{\alpha}{2} \leq c < \alpha$, $p^*_{TLF} = \frac{c}{\alpha}$, $\pi^*_{TLF} = \frac{c}{\alpha}\left(1 - \frac{c}{\alpha}\right)$.

To get the social welfare, we further consider the non-adopters in period 1. Non-adopters in period 1 update their priors via social learning from $a_1 = \alpha$ to:

$$a_2 = a_1 + N_1(1 - a_1) = 1 + c - \frac{c}{\alpha}.$$

For a period 1 non-adopter of type $\theta < \theta_1$ to adopt in period 2 under free trial, it must be the case that $\theta \geq \tilde{\theta}_2 \triangleq \frac{c}{1+c-\frac{c}{\alpha}}$. Comparing $\tilde{\theta}_2$ with θ_1, we further split cases 2-i and 2-ii each into two subcases as follows:

- Case 2-i-a: $0 \leq c < \frac{\alpha}{2}, 0 < \alpha < 1$.

  In this case, $\tilde{\theta}_2 < \theta_1$, $SW^*_{CE-SUB} = \frac{\alpha c^2(\alpha+2(\alpha-1)c)}{2(\alpha+(\alpha-1)c)^2} - c + \frac{7}{8}$.

- Case 2-i-b: $0 \leq c < \frac{\alpha}{2}$, $\alpha \geq 1$.

  In this case, $\tilde{\theta}_2 \geq \theta_1$, $SW^*_{CE-SUB} = \frac{1}{2}c\left(\frac{(2\alpha-1)c}{\alpha^2} - 2\right) + \frac{7}{8}$.

- Case 2-ii-a: $\frac{\alpha}{2} \leq c < \alpha, 0 < \alpha < 1$.

  In this case, $\tilde{\theta}_2 < \theta_1$, $SW^*_{CE-SUB} = \frac{2\alpha^4-(1-\alpha)^2c^4+2(1-\alpha)^2\alpha c^3+\left(\alpha^2-\alpha^4\right)c^2+2(\alpha-2)\alpha^3 c}{2\alpha^2(\alpha+\alpha c-c)^2}$.

- Case 2-ii-b: $\frac{\alpha}{2} \leq c < \alpha, \alpha \geq 1$.

  In this case, $\tilde{\theta}_2 \geq \theta_1$, $SW^*_{CE-SUB} = \frac{(\alpha-c)(-(\alpha-1)c+\alpha)}{\alpha^2}$. □

Proposition 16. *Under S model, in the presence of adoption costs, the firm's optimal seeding ratio, pricing strategy, the corresponding profit, and ensuing social welfare are: where:*

	$0 < c < 2\alpha$		$2\alpha \leq c$
	Region A (described below)	Otherwise	
p_S^*	$\frac{2\alpha-(7\alpha+1)c+t}{16\alpha}$	p^*_{CE-PL}	-
k_S^*	$\frac{-8\alpha^2+2\alpha+(\alpha-1)c+t}{4(1-\alpha)(2\alpha-c)}$	0	-
π_S^*	$\frac{(6\alpha+3(\alpha-1)c-t)(2\alpha-(7\alpha+1)c+t)^2}{128(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c+t)}$	π^*_{CE-PL}	-
SW_S^*	$\tilde{SW}_{S,D}$	SW^*_{CE-PL}	-
Paid adoption	in both periods	same as *CE-PL*	none

$$\begin{aligned}\tilde{SW}_{S,D} =& \frac{(2\alpha(1-c)+c)(2\alpha(1-4\alpha)+(\alpha-1)c+t)}{16(1-\alpha)\alpha^2} + \big((6\alpha+3(\alpha-1)c-t)\big(-16\alpha^3(c(16c+15) \\ & -18)+\alpha^2(c(c(86c+325)-228)+4(4t-3))+2\alpha(c(c(-46c+10t+15)-5t+6)-2t) \\ & +(2c-1)(c-t)(3c+t)\big)\big(4\alpha^2(56\alpha-3)+(\alpha(29\alpha+38)-3)c^2-2c(2\alpha(\alpha(4\alpha+47)-3) \\ & +(\alpha-1)t)+t^2-4\alpha(4\alpha+1)t)\big)\,/\,\big(2\big(4\alpha^2(64\alpha-3)-\big((\alpha-1)(43\alpha-3)c^2\big) \\ & +2c(2\alpha(\alpha(32\alpha-55)+3)-5\alpha t+t)+t^2-4\alpha t\big)^2(4(\alpha-1)(c-2\alpha))\Big),\end{aligned}$$

and $t = (2\alpha + 17\alpha c - c)(\alpha(c+2) - c)$*. Region A corresponds to parameters* α *and* c *satisfying:*

$$0 \leq c < c^{\ddagger}(\alpha) \quad , \textit{ if } 0 < \alpha < \tfrac{1}{16},$$

and

$$\tfrac{1}{16} \leq \alpha < \alpha^{\dagger}(c).$$

$c^{\ddagger}(\alpha)$ *and* $\alpha^{\dagger}(c)$ *are defined the proof below.*

Proof. First, we point out that *CE-PL* is a particular case of *S* with seeding ratio set to zero. Throughout the proof, we will show that in certain regions *CE-PL* dominates *S* with non-zero seeding ratio - that is equivalent to saying that the optimal seeding ratio will be 0 in those regions (i.e., *S* defaults to *CE-PL*).

If $\alpha \geq 1$, seeding brings no benefit as any social learning calibrates perceived valuations downwards, and, as such, *S* defaults to *CE-PL*.

Thus, we are left to explore the non-trivial case of $0 < \alpha < 1$. It is straightforward that the firm can make profit iff $0 \leq c < 2\alpha$. In the remaining part of the proof we focus on the scenario $0 \leq c < 2\alpha$. We have two cases:

- Case 1: $0 < p < 2\alpha - c$.

  In this case, there are paying adopters in period 1 (potentially alongside seeded customers if $k > 0$). The marginal paying adopter in period 1 has type $\theta_1 = \frac{c+p}{2\alpha}$. The marginal seeded adopter in period 1 has type $\theta_{seed} = \frac{c}{2\alpha}$ (unlike in the baseline model, in the scenario with adoption cost not all seeded customers adopt).

  Thus, the total number of adopters in period 1 is $N_{1,total} = k\left(1 - \frac{c}{2\alpha}\right) + (1-k)\left(1 - \frac{c+p}{2\alpha}\right) = \frac{2\alpha - c - p(1-k)}{2\alpha}$. In period 2, the potential customers who have not adopted in period 1 update their prior beliefs via social learning as follows:

  $$a_2 = a_1 + N_{1,total}(1 - a_1) = \alpha + \frac{(1-\alpha)(2\alpha - c - p(1-k))}{2\alpha}.$$

  A customer of type θ who has not adopted in period 1 (via paying for license or through the seeding program) will adopt in period 2 iff $\theta_1 > \theta \geq \theta_2 = \frac{c+p}{\alpha + \frac{(1-\alpha)(2\alpha - c - p(1-k))}{2\alpha}}$.

Comparing θ_1 and θ_2, we have:

$$\theta_1 > \theta_2 \quad \Longleftrightarrow \quad p < \frac{-4\alpha^2 + 2\alpha + \alpha c - c}{(1-\alpha)(1-k)}.$$

Comparing $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$ with 0, we have:

$$\frac{-4\alpha^2 + 2\alpha + \alpha c - c}{(1-\alpha)(1-k)} > 0 \quad \Longleftrightarrow \quad 0 < \alpha < \frac{1}{2} \quad \text{and} \quad 0 \leq c < \frac{2\alpha - 4\alpha^2}{1-\alpha} < 2\alpha.$$

Comparing $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$ with $2\alpha - c$, we have:

$$\frac{-4\alpha^2 + 2\alpha + \alpha c - c}{(1-\alpha)(1-k)} < 2\alpha - c \quad \Longleftrightarrow \quad 0 \leq k < \frac{2\alpha^2}{(1-\alpha)(2\alpha - c)} < 1.$$

Since in this case we consider $p \in (0, 2\alpha - c)$, we have four sub-cases:

- Case 1-i: $0 < \alpha < \frac{1}{2}$, $0 \leq c < \frac{2\alpha-4\alpha^2}{1-\alpha}$, $0 \leq k < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$.

  In this case, $0 < \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} < 2\alpha - c$. We have two sub-cases:

  * Case 1-i-a: $0 < p < \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$.

    In this case, $\theta_1 > \theta_2$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been successfully seeded in period 1, adopt in period 2.

    The firm's profit maximization problem becomes:

$$\begin{aligned}
&\max_{0<p<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)},\, 0\leq k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} \pi_S \\
=&\max_{0<p<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)},\, 0\leq k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} p(1-k)\left(1 - \frac{c+p}{\alpha + \frac{(\alpha-1)(-2\alpha+c-kp+p)}{2\alpha}}\right).
\end{aligned}$$

It can be shown that $\frac{\partial^2 \pi_S}{\partial p^2} < 0$. Thus, it is sufficient to solve FOC:

$$\begin{aligned}\frac{\partial \pi_S}{\partial p} = &\frac{(k-1)}{(2\alpha + (\alpha-1)c + p(\alpha - \alpha k + k - 1))^2} \\ &\times \left(\alpha^2\left(c^2 + 2c(k+1)p + p(k(4-kp) + p + 4) - 4\right)\right. \\ &\left.+2\alpha(c(2-2kp) + (k-1)p(kp-2)) - (c - kp + p)^2\right).\end{aligned}$$

Without constraints, the FOC yields two solutions:

$$\begin{aligned}p_{1,D,S} &= \frac{2\alpha + (\alpha-1)c + \frac{\sqrt{2}\sqrt{\alpha(\alpha(c+2)-c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}}{\alpha+(\alpha-1)k+1}}{(1-\alpha)(1-k)}, \\ p_{2,D,S} &= \frac{2\alpha + (\alpha-1)c - \frac{\sqrt{2}\sqrt{\alpha(\alpha(c+2)-c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}}{\alpha+(\alpha-1)k+1}}{(1-\alpha)(1-k)}.\end{aligned}$$

It can be shown that $p_{1,D,S} > \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$ and $p_{2,D,S} > 0$. Comparing $p_{2,D,S}$ with $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$, we have:

$$\begin{aligned}& p_{2,D,S} < \frac{-4\alpha^2 + 2\alpha + \alpha c - c}{(1-\alpha)(1-k)} \\ \iff\quad & 4\alpha^2 < \sqrt{2}\sqrt{\frac{\alpha(2\alpha + (\alpha-1)c)(2\alpha + (\alpha-1)ck)}{\alpha + (\alpha-1)k + 1}} \\ \iff\quad & 8\alpha^4 < \frac{\alpha(2\alpha + (\alpha-1)c)(2\alpha + (\alpha-1)ck)}{\alpha + (\alpha-1)k + 1} \\ \iff\quad & 8\alpha^4 + 8\alpha^3 + 2\alpha((1-\alpha)c - 2\alpha) \\ & \quad + k\left(8(\alpha-1)\alpha^3 + (\alpha-1)c((1-\alpha)c - 2\alpha)\right) < 0.\end{aligned}$$

Without constraints,

$$8\alpha^4+8\alpha^3+2\alpha((1-\alpha)c-2\alpha)+k\left(8(\alpha-1)\alpha^3+(\alpha-1)c((1-\alpha)c-2\alpha)\right)=0$$

yields one solution:

$$k_{1,D,S}=\frac{2\alpha(2\alpha(2\alpha(\alpha+1)-1)-\alpha c+c)}{(1-\alpha)\left(8\alpha^3+c^2-\alpha c(c+2)\right)}.$$

Notice that:

$$8(\alpha-1)\alpha^3+(\alpha-1)c((1-\alpha)c-2\alpha)>0 \quad\Longleftrightarrow\quad c(2\alpha+(\alpha-1)c)-8\alpha^3>0$$

$$k_{1,D,S}\geq 0 \quad\Longleftrightarrow\quad \left(c(2\alpha+(\alpha-1)c)-8\alpha^3\right)(2\alpha(2\alpha(\alpha+1)-1)-\alpha c+c)\leq 0,$$

$$k_{1,D,S}<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \quad\Longleftrightarrow\quad c(2\alpha+(\alpha-1)c)-8\alpha^3<0.$$

Then, we obtain four cases:

- Case 1-i-a-I: $c(2\alpha+(\alpha-1)c)-8\alpha^3\geq 0$.

  In this case, $p_{2,D,S}<\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$. Thus, $p_S^*=p_{2,D,S}$. The profit maximization problem becomes:

$$\begin{aligned}\max_{0\leq k<\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} & \;(-(\alpha-1)c(\alpha+(3\alpha-1)k+1)\\ & +2\sqrt{2}\sqrt{\alpha(2\alpha+(\alpha-1)c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}\\ & +2\alpha(-\alpha(k+3)+k-1))\,/\,\left((\alpha-1)^2(k-1)\right).\end{aligned}$$

It can be shown that $\frac{\partial \pi_S}{\partial k} < 0$. Thus, $k_S^* = 0$, *S* defaults to *CE-PL*.

- Case 1-i-a-II: $c(2\alpha+(\alpha-1)c)-8\alpha^3 < 0$, $2\alpha(2\alpha(\alpha+1)-1)+c-\alpha c \geq 0$, $0 \leq k < k_{1,D,S}$.

  In this case, $p_{2,D,S} > \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$, $p_S^* = \frac{2\alpha+\alpha c-2c}{-\alpha+\alpha k-k+2}$. The profit maximization problem becomes:

$$\max_{0\leq k<\frac{2\alpha(2\alpha(2\alpha(\alpha+1)-1)-\alpha c+c)}{(1-\alpha)\left(8\alpha^3+c^2-\alpha c(c+2)\right)}} \frac{(2\alpha-1)(k-1)(2\alpha+(\alpha-2)c)}{-\alpha+(\alpha-1)k+2}.$$

  It can be shown that $\Delta_{PL,D}(\alpha, c) < 0$ in this case, which corresponds to the second case under *CE-PL*. For any $k \in [0, k_{1,D,S})$, $\frac{(2\alpha-1)(k-1)(2\alpha+(\alpha-2)c)}{-\alpha+(\alpha-1)k+2} < \frac{(c-2\alpha)^2}{8\alpha} = \pi^*_{CE-PL}$. Therefore, this case is sub-optimal, as it is dominated by not seeding anymore.

- Case 1-i-a-III: $c(2\alpha+(\alpha-1)c)-8\alpha^3 < 0$, $2\alpha(2\alpha(\alpha+1)-1)+c-\alpha c \geq 0$, $k_{1,D,S} \leq k < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$.

  In this case, $p_{2,D,S} < \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$. Thus, $p_S^* = p_{2,D,S}$. It can be shown $\frac{\partial \pi}{\partial k} < 0$ as well. Therefore, $k_S^* = k_{1,D,S}$. It can be shown that $\Delta_{PL,D}(\alpha, c) < 0$ in this case, which corresponds to the second case under *CE-PL*. For any $k \in \left[k_{1,D,S}, \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\right)$, $\pi_S < \frac{(c-2\alpha)^2}{8\alpha} = \pi^*_{CE-PL}$. Therefore, this case is sub-optimal, as it is dominated by not seeding anymore.

- Case 1-i-a-IV: $c(2\alpha+(\alpha-1)c)-8\alpha^3 < 0$, $2\alpha(2\alpha(\alpha+1)-1)+c-\alpha c < 0$.

  In this case, $p_{2,D,S} < \frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)}$. Thus, $p_S^* = p_{2,D,S}$. Same as case 1-i-a-I, $\frac{\partial \pi_S}{\partial k} < 0$. Thus, $k_S^* = 0$, *S* defaults to *CE-PL*.

Thus, under case 1-i-a, *S* either defaults to *CE-PL* or is strictly dominated by

CE-PL.

* Case 1-i-b: $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \leq p < 2\alpha - c$.

  In this case, $\theta_1 \leq \theta_2$. There are no new adopters in period 2. The firm's profit maximization problem becomes:

$$\max_{\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \leq p < 2\alpha-c, 0 \leq k < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} \pi_S$$
$$= \max_{\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \leq p < 2\alpha-c, 0 \leq k < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}} p(1-k)\left(1 - \frac{c+p}{2\alpha}\right).$$

  It trivially follows that $k_S^* = 0$. *S* defaults to *CE-PL*.

- Case 1-ii: $0 < \alpha < \frac{1}{2}$, $0 \leq c < \frac{2\alpha - 4\alpha^2}{1-\alpha}$, $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < 1$.

  In this case, $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \geq 2\alpha - c$. $\theta_2 < \theta_1$. Customers with type $\theta \in [\theta_2, \theta_1)$, who have not been seeded in period 1, adopt in period 2. The firm's profit maximization problem becomes:

$$\max_{0<p<2\alpha-c, \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < 1} \pi_S$$
$$= \max_{0<p<2\alpha-c, \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < 1} p(1-k)\left(1 - \frac{c+p}{\alpha + \frac{(\alpha-1)(-2\alpha+c-kp+p)}{2\alpha}}\right).$$

  Similarly to case 1-i-a, it can be shown that $p_{1,D,S} > 2\alpha - c$ and $p_{2,D,S} > 0$. Comparing $p_{2,D,S}$ with $2\alpha - c$, we have:

$$
\begin{aligned}
& p_{2,D,S} < 2\alpha - c \\
\iff\quad & (\alpha-1)ck + 2\alpha(\alpha - \alpha k + k) < \sqrt{2}\sqrt{\frac{\alpha(2\alpha+(\alpha-1)c)(2\alpha+(\alpha-1)ck)}{\alpha+(\alpha-1)k+1}} \\
\iff\quad & ((\alpha-1)ck + 2\alpha(\alpha-\alpha k+k))^2 < \frac{2\alpha(2\alpha+(\alpha-1)c)(2\alpha+(\alpha-1)ck)}{\alpha+(\alpha-1)k+1} \\
\iff\quad & (1-\alpha)^2(c-2\alpha)^2k^2 + 2(1-\alpha)(2\alpha-c)\alpha ck + 4\alpha^2(c-\alpha(\alpha+2)) < 0.
\end{aligned}
$$

Without constraints, $(1-\alpha)^2(c-2\alpha)^2k^2+2(1-\alpha)(2\alpha-c)\alpha ck+4\alpha^2(c-\alpha(\alpha+2)) = 0$ yields two solutions:

$$
\begin{aligned}
k_{2,D,S} &= \frac{-\alpha c + \sqrt{\alpha^2(4\alpha(\alpha+2)+(c-4)c)}}{(1-\alpha)(2\alpha-c)} \\
k_{3,D,S} &= \frac{-\alpha c - \sqrt{\alpha^2(4\alpha(\alpha+2)+(c-4)c)}}{(1-\alpha)(2\alpha-c)}.
\end{aligned}
$$

It can be shown that $k_{3,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$ and $k_{2,D,S} > \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Comparing $k_{2,D,S}$ with 1, we obtain three cases:

* Case 1-ii-a: $\alpha\left(2\alpha+\sqrt{4\alpha(\alpha+2)+(c-4)c}-2\right)+c \geq 2\alpha c$.

  In this case, $k_{2,D,S} \geq 1$, i.e., $p_{2,D,S} < 2\alpha - c$. Thus, $p_S^* = p_{2,D,S}$. The profit maximization problem becomes:

$$
\begin{aligned}
\max_{\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\leq k<1} & (-(\alpha-1)c(\alpha+(3\alpha-1)k+1) \\
& +2\sqrt{2}\sqrt{\alpha(2\alpha+(\alpha-1)c)(\alpha+(\alpha-1)k+1)(2\alpha+(\alpha-1)ck)}+ \\
& 2\alpha(-\alpha(k+3)+k-1))\,/\,\left((\alpha-1)^2(k-1)\right).
\end{aligned}
$$

It can be shown that $\frac{\partial \pi_S}{\partial k} < 0$. Thus, $k_S^* = \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Therefore, this case is weakly dominated by case 1-i. Thus, this case is strictly dominated by *CE-PL*.

* Case 1-ii-b: $\alpha\left(2\alpha + \sqrt{4\alpha(\alpha+2)+(c-4)c} - 2\right) + c < 2\alpha c$, $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < k_{2,D,S}$.

  In this case, $p_{2,D,S} < 2\alpha - c$. Thus, $p_S^* = p_{2,D,S}$. Similarly as case 1-ii-a, we get $\frac{\partial \pi_S}{\partial k} < 0$. Thus, $k_S^* = \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Therefore, this case is also weakly dominated and is strictly dominated by *CE-PL*.

* Case 1-ii-c: $\alpha\left(2\alpha + \sqrt{4\alpha(\alpha+2)+(c-4)c} - 2\right) + c < 2\alpha c$, $k_{2,D,S} \leq k < 1$.

  In this case, $p_{2,D,S} \geq 2\alpha - c$. We can see that, for any k in this region, $\pi_S(p)$ is strictly increasing in p and the profit in this case is strictly dominated by the profit under Case 2.

– Case 1-iii: $0 < \alpha < \frac{1}{2}$, $\frac{2\alpha-4\alpha^2}{1-\alpha} \leq c < 2\alpha$.

  In this case, $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \leq 0$. $\theta_2 \geq \theta_1$, the profit maximization problem becomes:

$$\max_{0<p<2\alpha-c, 0\leq k\leq 1} \pi_S = \max_{0<p<2\alpha-c, 0\leq k\leq 1} p(1-k)\left(1 - \frac{c+p}{2\alpha}\right).$$

  It trivially follows that $k_S^* = 0$. *S* defaults to *CE-PL*.

– Case 1-iv: $\frac{1}{2} \leq \alpha < 1$.

  In this case, $\frac{-4\alpha^2+2\alpha+\alpha c-c}{(1-\alpha)(1-k)} \leq 0$. $\theta_2 \geq \theta_1$, the profit maximization problem becomes:

$$\max_{0<p<2\alpha-c, 0\leq k\leq 1} \pi_S = \max_{0<p<2\alpha-c, 0\leq k\leq 1} p(1-k)\left(1 - \frac{c+p}{2\alpha}\right).$$

  It trivially follows that $k_S^* = 0$. *S* defaults to *CE-PL*.

- Case 2: $p \geq 2\alpha - c$.

  In this case, there are only seeded consumers in period 1 (i.e., no unseeded customer is willing to pay for the product based on priors). Hence, $N_{1,total} = k(1 - \frac{c}{2\alpha})$. At the beginning of period 2, the un-seeded customers update their priors to:

$$a_2 = a_1 + N_{1,total}(1 - a_1) = \alpha + (1-\alpha)k\left(1 - \frac{c}{2\alpha}\right).$$

  The marginal paying customer in period 2 has type $\theta_2 = \frac{c+p}{\alpha+(1-\alpha)k\left(1-\frac{c}{2\alpha}\right)}$. Comparing θ_2 with 1, we obtain:

$$\theta_2 < 1 \quad \Longleftrightarrow \quad p < \alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right).$$

  Comparing $\alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right)$ with $2\alpha - c$, we have:

$$\alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right) \geq 2\alpha - c \quad \Longleftrightarrow \quad k \geq \frac{2\alpha^2}{(1-\alpha)(2\alpha - c)} > 0.$$

  Comparing $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$ with 1, we have:

$$\frac{2\alpha^2}{(1-\alpha)(2\alpha - c)} < 1 \quad \Longleftrightarrow \quad 0 < \alpha < \frac{1}{2} \quad \text{and} \quad 0 < c < \frac{2\alpha(1-2\alpha)}{1-\alpha}.$$

Thus, we obtain that:

$$\theta_2 < 1 \quad \Longleftrightarrow 0 < \alpha < \frac{1}{2}, 0 < c < \frac{2\alpha(1-2\alpha)}{1-\alpha}, \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < 1,$$
$$\text{and } 2\alpha - c \leq p < \alpha - \alpha k + k + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right).$$

Otherwise, $\theta_2 \geq 1$. There are no paying adopters in period 2, i.e., the firm does not make any profit.

When $\theta_2 < 1$, the firm's profit maximization problem becomes:

$$\max_{2\alpha-c\leq p<\alpha-\alpha k+k+\frac{1}{2}c\left(-\frac{k}{\alpha}+k-2\right),\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\leq k<1} \pi_S$$
$$= \max_{2\alpha-c\leq p<\alpha-\alpha k+k+\frac{1}{2}c\left(-\frac{k}{\alpha}+k-2\right),\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}\leq k<1} p(1-k)\left(1 - \frac{c+p}{\alpha + (1-\alpha)k\left(1-\frac{c}{2\alpha}\right)}\right).$$

Since it is quadratic in p, it is sufficient to use FOC. Taking the first order derivative of the profit w.r.t. p, we get:

$$\frac{\partial \pi_S}{\partial p} = \frac{2\alpha(k-1)(c+2p)}{(\alpha-1)ck + 2\alpha(\alpha - \alpha k + k)} - k + 1.$$

Without constraints, the FOC yields one solution:

$$p_{3,D,S} = \frac{\alpha c(k-2) - ck + 2\alpha(\alpha - \alpha k + k)}{4\alpha}.$$

It can be shown that $p_{3,D,S} < \alpha + \frac{1}{2}c\left(-\frac{k}{\alpha} + k - 2\right) - \alpha k + k$. Comparing $p_{3,D,S}$ with

$2\alpha - c$, we obtain:

$$p_{3,D,S} \geq 2\alpha - c \quad \Longleftrightarrow \quad k \geq \frac{2\alpha(3\alpha - c)}{(1-\alpha)(2\alpha - c)}.$$

It can be shown that $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} > \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Comparing $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$ with 1, we obtain:

$$\frac{2\alpha(3\alpha - c)}{(1-\alpha)(2\alpha - c)} < 1 \quad \Longleftrightarrow$$
$$\left(0 < \alpha < \frac{1}{3} \text{ and } c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}\right) \text{ or } \left(\frac{1}{3} < \alpha < \frac{1}{2} \text{ and } c > \frac{2\alpha(1-4\alpha)}{1-3\alpha}\right).$$

Comparing $\frac{2\alpha(1-4\alpha)}{1-3\alpha}$ with 0 and $\frac{2\alpha(1-2\alpha)}{1-\alpha}$, we obtain three cases:

- Case 2-i: $0 < \alpha < \frac{1}{4}$.

  In this case, $0 < \frac{2\alpha(1-4\alpha)}{1-3\alpha} < \frac{2\alpha(1-2\alpha)}{1-\alpha}$. We obtain three cases:

  * Case 2-i-a: $0 \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$, $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$.

    In this case, $p_{3,D,S} < 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. The profit maximization problem becomes:

$$\begin{aligned} &\max_{\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}} \pi_S \\ &= \max_{\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} \leq k < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}} (1-k)(2\alpha - c)\left(1 - \frac{2\alpha}{\alpha + \frac{(\alpha-1)k(c-2\alpha)}{2\alpha}}\right). \end{aligned}$$

    It can be shown that $\frac{\partial^2 \pi_S}{\partial k^2} < 0$. Hence, FOC is sufficient to determine the optimal seeding ratio. We have:

$$\frac{\partial \pi_S}{\partial k} = (2\alpha - c)\left(\frac{4\alpha^2(2\alpha + (\alpha-1)c)}{((\alpha-1)ck + 2\alpha(\alpha - \alpha k + k))^2} - 1\right) = 0.$$

Without constraints, FOC yields two solutions:

$$k_{4,D,S} = \frac{-2\alpha^2 + 2\alpha\sqrt{2\alpha + (\alpha-1)c}}{(1-\alpha)(2\alpha - c)},$$
$$k_{5,D,S} = \frac{-2\alpha^2 - 2\alpha\sqrt{(2\alpha + (\alpha-1)c)}}{(1-\alpha)(2\alpha - c)}.$$

It can be shown that $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$ and $k_{4,D,S} > \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Comparing $k_{4,D,S}$ with 1, we obtain two sub cases:

· Case 2-i-a-I: $\sqrt{\alpha(c+2)-c} + c < 4\alpha$.

In this case, $k_{4,D,S} < 1$, $p_S^* = 2\alpha - c$, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c + 2\alpha\left(-2\alpha + 2\sqrt{\alpha(c+2)-c} - 1\right) + c}{\alpha - 1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

· Case 2-i-a-II: $\sqrt{\alpha(c+2)-c} + c \geq 4\alpha$.

In this case, $k_{4,D,S} \geq 1$, $p_S^* = 2\alpha - c$, $k_S^* = 1$, $\pi_S^* = 0$. Thus, it is dominated by *CE-PL*.

∗ Case 2-i-b: $0 \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$, $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} \leq k < 1$.

In this case, $p_{3,D,S} \geq 2\alpha - c$. Thus, $p_S^* = p_{3,D,S}$. The firm's profit maximization problem becomes:

$$\max_{\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}\leq k\leq 1} \pi_S$$
$$= \max_{\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}\leq k\leq 1} \frac{(k-1)(\alpha c(k-2)-ck+2\alpha(\alpha-\alpha k+k))^2}{8\alpha\left(-\alpha(c+2)k+ck+2\alpha^2(k-1)\right)}.$$

We differentiate π_S w.r.t. k:

$$\frac{\partial \pi_S}{\partial k} = \big((c(2\alpha-\alpha k+k)+2\alpha((\alpha-1)k-\alpha)\big)\big((\alpha-1)c^2((\alpha-1)k(2k-1)$$
$$-2\alpha)-2\alpha c\left(\alpha^2+\alpha+4(\alpha-1)^2k^2+((7-5\alpha)\alpha-2)k\right)$$
$$+4\alpha^2((\alpha-1)k-\alpha)(-2\alpha+2(\alpha-1)k+1)\big)\big)$$
$$/\left(8\alpha((\alpha-1)ck+2\alpha(\alpha-\alpha k+k))^2\right).$$

It can be shown that: $\frac{c(2\alpha-\alpha k+k)+2\alpha((\alpha-1)k-\alpha)}{8\alpha((\alpha-1)ck+2\alpha(\alpha-\alpha k+k))^2} < 0$. Denote:

$$G_{D,S}(k) \triangleq -\big((\alpha-1)c^2((\alpha-1)k(2k-1)-2\alpha)-2\alpha c\left(\alpha^2+\alpha+4(\alpha-1)^2k^2\right.$$
$$\left.+((7-5\alpha)\alpha-2)k\right)+4\alpha^2((\alpha-1)k-\alpha)(-2\alpha+2(\alpha-1)k+1)\big)$$
$$=-2(1-\alpha)^2(2\alpha-c)^2k^2$$
$$+(\alpha-1)\left(4\alpha^2(4\alpha-1)+(\alpha-1)c^2+2\alpha(2-5\alpha)c\right)k$$
$$+2\alpha\left(2\alpha^2(1-2\alpha)+(\alpha-1)c^2+\alpha(\alpha+1)c\right).$$

It is straightforward that $G_{D,S}(k)$ is concave. Without constraints, $G_{D,S}(k)=0$

yields two solutions:

$$k_{6,D,S} = \frac{-8\alpha^2 + 2\alpha + (\alpha - 1)c + \sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}}{4(1-\alpha)(2\alpha - c)},$$
$$k_{7,D,S} = \frac{-8\alpha^2 + 2\alpha + (\alpha - 1)c - \sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}}{4(1-\alpha)(2\alpha - c)}.$$

It can be shown that $k_{7,D,S} < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$ and $k_{6,D,S} < 1$. Comparing $k_{6,D,S}$ with $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$, we obtain two sub-cases:

· Case 2-i-b-I: $\alpha(9c+2)+\sqrt{(2\alpha + (\alpha - 1)c)(2\alpha + (17\alpha - 1)c)} > 32\alpha^2+c$.

  In this case, $k_{6,D,S} > \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$. Denote $t = (2\alpha+17\alpha c-c)(\alpha(c+2)-c)$,

  We can further get:

  $p_S^* = \frac{2\alpha^2-2\alpha c+\alpha ck-ck-2\alpha^2k+2\alpha k}{4\alpha} = \frac{2\alpha-(7\alpha+1)c+t}{16\alpha}$,

  $k_S^* = k_{6,D,S} = \frac{-8\alpha^2+2\alpha+(\alpha-1)c+t}{4(1-\alpha)(2\alpha-c)}$,

  $\pi_S^* = \frac{(6\alpha+3(\alpha-1)c-t)(2\alpha-(7\alpha+1)c+t)^2}{128(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c+t)}$,

$$
\begin{aligned}
SW_S^* =& \frac{(2\alpha(1-c)+c)(2\alpha(1-4\alpha)+(\alpha-1)c+t)}{16(1-\alpha)\alpha^2} \\
&+ \left((6\alpha+3(\alpha-1)c-t)\left(-16\alpha^3(c(16c+15)-18)\right.\right. \\
&+ \alpha^2(c(c(86c+325)-228)+4(4t-3)) \\
&+ 2\alpha(c(c(-46c+10t+15)-5t+6)-2t) \\
&+(2c-1)(c-t)(3c+t))\left(4\alpha^2(56\alpha-3)+(\alpha(29\alpha+38)-3)c^2\right. \\
&\left.\left.-2c(2\alpha(\alpha(4\alpha+47)-3)+(\alpha-1)t)+t^2-4\alpha(4\alpha+1)t\right)\right) \\
&/\left(2\left(4\alpha^2(64\alpha-3)-\left((\alpha-1)(43\alpha-3)c^2\right)\right.\right. \\
&\left.+2c(2\alpha(\alpha(32\alpha-55)+3)-5\alpha t+t)+t^2-4\alpha t\right)^2 \\
&\left.(4(\alpha-1)(c-2\alpha))\right).
\end{aligned}
$$

· Case 2-i-b-II: $\alpha(9c+2)+\sqrt{(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)} \leq 32\alpha^2+c$.

In this case, $k_{6,D,S} \leq \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$, $\pi_S^* = \frac{(2\alpha-c)(2\alpha(4\alpha-1)-3\alpha c+c)}{(\alpha-1)(4\alpha-c)}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

* Case 2-i-c: $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \leq c < \frac{2\alpha(1-2\alpha)}{1-\alpha}$.

In this case, $p_{3,D,S} < 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. Following the same step in case 2-i-a, we obtain that $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} < k_{4,D,S} < 1$ and $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c+2\alpha\left(-2\alpha+2\sqrt{\alpha(c+2)-c}-1\right)+c}{\alpha-1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

- Case 2-ii: $\frac{1}{4} \leq \alpha < \frac{1}{3}$.

  In this case, $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \leq 0$. $p_{3,D,S} \leq 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. Following the same step in case 2-i-a, we obtain that $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} < k_{4,D,S} < 1$ and $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c+2\alpha\left(-2\alpha+2\sqrt{\alpha(c+2)-c}-1\right)+c}{\alpha-1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

- Case 2-iii: $\frac{1}{3} \leq \alpha < \frac{1}{2}$.

  In this case, $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \geq \frac{2\alpha(1-2\alpha)}{1-\alpha}$. Thus, $\frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)} \geq 1$. $p_{3,D,S} < 2\alpha - c$. Thus, $p_S^* = 2\alpha - c$. Following the same step in case 2-i-a, we obtain that $\frac{2\alpha^2}{(1-\alpha)(2\alpha-c)} < k_{4,D,S} < \frac{2\alpha(3\alpha-c)}{(1-\alpha)(2\alpha-c)}$ and $k_{5,D,S} < \frac{2\alpha^2}{(1-\alpha)(2\alpha-c)}$. Thus, $k_S^* = k_{4,D,S}$, $\pi_S^* = \frac{-\alpha c+2\alpha\left(-2\alpha+2\sqrt{\alpha(c+2)-c}-1\right)+c}{\alpha-1}$. It can be shown that under both the first and second case in *CE-PL*, we have $\pi_S^* < \pi_{CE-PL}^*$. Thus, it is dominated by *CE-PL*.

In summary, only under case 2-i-b-I, *S* can be optimal. We further explore the boundary between *S* and *CE-PL*. Recall that the condition for case 2-i-b-I is: $0 \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$, and $\alpha(9c+2)+\sqrt{(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)} > 32\alpha^2+c$. This region is only relevant to case (a) and case (b) under *CE-PL*. The last inequality can be rewrite as:

$$\sqrt{(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)} > 32\alpha^2+c-\alpha(9c+2). \tag{D.3}$$

We first check whether the R.H.S. is positive. Denote $H_{S,1}(\alpha,c) \triangleq 32\alpha^2+c-\alpha(9c+2) = 2\alpha(16\alpha-1)+(1-9\alpha)c$. We obtain two cases (we reorganize the case number to avoid it goes too deep):

- Case 1: $0 < \alpha < \frac{1}{9}$.

  In this case, $H_{S,1}(\alpha, c)$ is increasing in c, $32\alpha^2 + c - \alpha(9c + 2) > 0$ is equivalent to $c > \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $\frac{2\alpha(1-16\alpha)}{1-9\alpha} < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Comparing $\frac{2\alpha(1-16\alpha)}{1-9\alpha}$ with 0, we obtain two sub cases:

  - Case 1-i: $0 < \alpha < \frac{1}{16}$.

    In this case, $0 < \frac{2\alpha(1-16\alpha)}{1-9\alpha} < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. We obtain two sub cases:

    * Case 1-i-a: $0 \le c < \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.

      In this case, $32\alpha^2 + c - \alpha(9c+2) < 0$, the inequality D.3 is always satisfied. Recall that for *CE-PL*, the boundary between two cases is $c^\dagger(\alpha)$, where $c^\dagger(\alpha)$ is the unique solution to the equation $\Phi_{PL,D}(\alpha, c) = 0$ and $\Phi_{PL,D}(\alpha, c)$ is decreasing in c. It can be shown that $\Phi_{PL,D}(\alpha, c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, $\frac{2\alpha(1-16\alpha)}{1-9\alpha} < c^\dagger(\alpha)$. This case falls into the region of first case under *CE-PL*.

      Next, we compare the optimal profit between *S* and *CE-PL*. We first simplify the optimal profit under *S* as (move the square root to the numerator):

$$\begin{aligned}
\pi_S^* = \Big(&\sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)}\left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right)c^2\right.\\
&\left. + 4\alpha(9\alpha - 1)c\right) - \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right)c^3\right.\\
&\left. + 2\alpha\left(109\alpha^2 - 74\alpha - 3\right)c^2 + 4\alpha^2(37\alpha + 3)c\right)\Big)\\
&/\left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right).
\end{aligned}$$

      It can be shown that under this case, $\pi_S^* > \pi_{CE-PL}^*$. Thus, *S* dominates *CE-PL*.

* Case 1-i-b: $\frac{2\alpha(1-16\alpha)}{1-9\alpha} \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$.

  In this case, $32\alpha^2+c-\alpha(9c+2) \geq 0$. We take square both sides of the inequality D.3. After the simplification the inequality is equivalent to:

$$c^2 + (1-9\alpha)c + 2\alpha(8\alpha-1) < 0,$$

  which is equivalent to:

$$\frac{1}{2}\left(-\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) < c < \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right).$$

  It can be shown that:

$$\begin{aligned}\frac{1}{2}\left(-\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) &< \frac{2\alpha(1-16\alpha)}{1-9\alpha} \\ < \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) &< \frac{2\alpha(1-4\alpha)}{1-3\alpha}.\end{aligned}$$

  Thus, the inequality D.3 is equivalent to:

$$\frac{2\alpha(1-16\alpha)}{1-9\alpha} \leq c < \frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)$$

  . Then we compare *S* with *CE-PL*. We first check the relationship between $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)$ and $c^{\dagger}(\alpha)$.

  It can be shown that $\Phi_{PL,D}(\alpha,c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} > 0$ is equivalent to $\frac{1}{17} < \alpha < \frac{1}{16}$. We further get two sub cases:

  · Case 1-i-b-I: $0 < \alpha < \frac{1}{17}$.

In this case, we have $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) > c^{\dagger}(\alpha)$. Thus, we consider two regions:

Region 1: $\frac{2\alpha(1-16\alpha)}{1-9\alpha} \leq c < c^{\dagger}(\alpha)$**.**

In this region, denote the profit difference between *S* and *CE-PL* as:

$$
\begin{aligned}
H_{S,2} \triangleq \Big(& \sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2\right. \\
& \left.+4\alpha(9\alpha-1)c\right)-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3\right. \\
& \left.+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big) \\
& /\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right) \\
& -\frac{2\alpha+\alpha^2(c+6)-4\sqrt{\alpha^2(\alpha+1)(2\alpha+(\alpha-1)c)}-c}{(1-\alpha)^2}.
\end{aligned}
$$

Thus, the boundary between *S* and *CE-PL* satisfies: $H_{S,2}=0$. We simplify the equation $H_{S,2}=0$ by getting rid of the fraction and square root. We finally get $H_{S,2}=0$ is equivalent to $H_{S,3}=0$, where $H_{S,3}$ is defined as:

$$
\begin{aligned}
H_{S,3} \triangleq & 4(\alpha+1)(2\alpha-c)^2(2\alpha+(\alpha-1)c)\left(c^2-128\alpha^4(c+6)\right. \\
& +5\alpha^3(c(27c+92)-52)+\alpha^2(c(184-109c)+4) \\
& \left.-\alpha c(27c+4)\right)^2-\left(8\alpha^3(\alpha(\alpha(544\alpha+451)+30)\right. \\
& -1)+2(\alpha-1)^2(\alpha+1)(26\alpha-1)c^4-(\alpha-1)(\alpha(\alpha(5\alpha(27\alpha \\
& -134)-604)-14)+1)c^3+2\alpha(\alpha(\alpha(\alpha(\alpha(32\alpha-1147)+760) \\
& +1310)+72)-3)c^2+4\alpha^2(\alpha(\alpha(\alpha(448\alpha-1137) \\
& \left.-1279)-83)+3)c\right)^2.
\end{aligned}
$$

We can obtain that:

$$
\begin{aligned}
\frac{\partial H_{S,3}(\alpha, c)}{\partial \alpha} =& 2(1-\alpha)\big(64(\alpha-1)\alpha^5(\alpha(\alpha(32\alpha(192\alpha-287) \\
&+3653)-232)+3)+4(\alpha-1)^2(\alpha+1)(26\alpha-1) \\
&(\alpha(104\alpha+23)-27)c^8-4(\alpha-1)(\alpha(\alpha(\alpha(\alpha(5\alpha(3159\alpha \\
&+1238)-36154)-7416)+12637)-658)+6)c^7 \\
&+(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(124405\alpha-24086)-663362) \\
&+406518)+503028)-359002)+30266)-1414) \\
&+31)c^6-2(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(3\alpha(10\alpha(1584\alpha+1693) \\
&-197561)+329621)+774877)-651119)+108631) \\
&-10665)+375)-3)c^5+4\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(96\alpha(64\alpha \\
&+603)-216517)+132899)+637875)-745325) \\
&+236785)-33335)+1425)-15)c^4 \\
&-16\alpha^2(\alpha(\alpha(\alpha(\alpha(\alpha(2\alpha(16\alpha(384\alpha+395) \\
&+13645)+142805)-295747)+151714)-26660) \\
&+1285)-15)c^3+16\alpha^3(\alpha(\alpha(\alpha(3\alpha(\alpha(64\alpha(192\alpha \\
&+269)+25401)-108576)+221380)-46398)+2445) \\
&-30)c^2-32\alpha^4(\alpha(\alpha(\alpha(2\alpha(48\alpha(256\alpha+223)-54221) \\
&+84329)-20999)+1191)-15)c).
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H_{S,3}(\alpha,c)}{\partial c} =& 2(1-\alpha)^2(32\alpha^5(\alpha(\alpha(\alpha(32\alpha(64\alpha+119)-7417)\\
&+2777)-195)+3)-16(-26\alpha^3+\alpha^2+26\alpha-1)^2c^7\\
&+28(\alpha-1)\alpha(\alpha+1)(\alpha(\alpha(5\alpha(351\alpha+67)-4007)+323)\\
&-6)c^6-3(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(24881\alpha+16764)\\
&-151172)-13428)+156486)-18412)+1324)\\
&-60)+1)c^5+10\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(4320\alpha^2+8967\alpha\\
&-57883)-9445)+102601)-23019)+3311)-183)\\
&+3)c^4-8\alpha^2(\alpha(\alpha(\alpha(\alpha(\alpha(\alpha(64\alpha(16\alpha+179)\\
&-32993)+206)+159649)-76108)+15505)-930)\\
&+15)c^3+48\alpha^3(\alpha(\alpha(\alpha(\alpha(\alpha(32\alpha(32\alpha+65)+4601)\\
&+19957)-19506)+4954)-315)+5)c^2\\
&-16\alpha^4(\alpha(\alpha(\alpha(3\alpha(128\alpha(16\alpha+39)+9573)-46856)\\
&+14346)-960)+15)c)<0.
\end{aligned}
$$

As it turns out, in this range of the parameter space, $\frac{\partial H_{S,3}(\alpha,c)}{\partial \alpha}$ changes signs. As such, it is not possible to characterize the threshold between *S* and *CE-PL* as a function of c (there exist values of c for which increasing α leads to multiple crossings between optimality regions for *S* and *CE-PL*).

Nevertheless, moving horizontally, given that $\frac{\partial H_{S,3}(\alpha,c)}{\partial c} < 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and *S*, within this particular region of the parameter space, is unique for every α, *if it ex-*

ists.

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:

(1) First, we consider points on the boundary $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, *S* dominates *CE-PL* on $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.

(2) First, we consider points on the boundary

$c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$. It can be shown that

$H_{S,3}(\alpha, c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} < 0$. Thus, *CE-PL* dominates *S* on $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$.

Therefore, in Region 1, as we increase c, there can be at most one crossing point between optimality regions for *S* and *CE-PL*, then there exists a unique boundary, which we define as $c_a(\alpha)$, which separates the optimality regions for *S* and *CE-PL*. It satisfies:

$$H_{S,3}(\alpha, c_1(\alpha)) = 0.$$

Region 2: $c^{\dagger}(\alpha) \leq c < \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$**.**

In this region, denote the profit difference between *S* and *CE-PL* as:

$$
\begin{aligned}
H_{S,4} \triangleq \Big(& \sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2\right. \\
& \left.+4\alpha(9\alpha-1)c\right)-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3\right. \\
& \left.+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big) \\
& /\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right) \\
& -\frac{(c-2\alpha)^2}{8\alpha}.
\end{aligned}
$$

Thus, the boundary between *S* and *CE-PL* satisfies: $H_{S,4}=0$. We simplify the equation $H_{S,4}=0$ by getting rid of the fraction and square root. We finally get $H_{S,5}=0$ is equivalent to $H_{S,5}=0$, where $H_{S,5}$ is defined as:

$$
\begin{aligned}
H_{S,5} \triangleq & (2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)^3-(-4\alpha^2(16(\alpha-1)\alpha+1) \\
& +8(\alpha-1)c^3+(\alpha(23\alpha+10)-1)c^2+4\alpha(\alpha(24\alpha-5)+1)c)^2.
\end{aligned}
$$

We can obtain that:

$$
\begin{aligned}
\frac{\partial H_{S,5}(\alpha,c)}{\partial \alpha} =& -4(-4\alpha^2(16(\alpha-1)\alpha+1)+8(\alpha-1)c^3 \\
&+(\alpha(23\alpha+10)-1)c^2 \\
&\quad +4\alpha(\alpha(24\alpha-5)+1)c)(-4\alpha(8\alpha(4\alpha-3)+1) \\
&+4c^3+(23\alpha+5)c^2 \\
&\quad +2(2\alpha(36\alpha-5)+1)c)+(c+2)(2\alpha+(17\alpha-1)c)^3 \\
&\quad +3(17c+2)(2\alpha+(\alpha-1)c)(2\alpha+(17\alpha-1)c)^2>0.
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial H_{S,5}(\alpha,c)}{\partial c} =& 16(16\alpha^4(\alpha(6\alpha-5)(8\alpha-3)-1)-24(\alpha-1)^2c^5 \\
&-5(\alpha-1)(\alpha(23\alpha+10)-1)c^4 \\
&+8\alpha(\alpha(\alpha(89\alpha-137)+15)+1)c^3 \\
&+12\alpha^2(\alpha((133-53\alpha)\alpha-34)+2)c^2 \\
&+16\alpha^3(\alpha(\alpha(17-49\alpha)+10)-2)c)<0.
\end{aligned}
$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and *S* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular functions for this region, namely $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$ and $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$ and examine the sign of $H_{S,5}(\alpha, c)$ along these boundaries.

(1) On the boundary $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$, we obtain:

$H_{S,5}(\alpha, c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} < 0$. Thus, *CE-PL* dominates *S* on $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$.

(2) On the boundary $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$, we obtain:

$H_{S,5}(\alpha, c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, *S* dominates *CE-PL* on $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{c_b(\alpha)}$, which separates the optimality regions for *CE-PL* and *S*. It satisfies:

$$H_{S,5}(\alpha, c_b(\alpha)) = 0.$$

Also, it is straightforward that $\frac{\partial c_b(\alpha)}{\partial \alpha} = -\frac{\frac{\partial H_{S,5}(\alpha,c)}{\partial \alpha}}{\frac{\partial H_{S,5}(\alpha,c)}{\partial c}} > 0$. Hence, $c_b(\alpha)$ is increasing in α.

It can be shown that there are two intersection points between $c_b(\alpha)$ and $c^\dagger(\alpha)$, i.e., $(0, 0)$ and (c_x, α_x) (where $c_x \approx 0.0231$ and $\alpha_x \approx 0.0117$). Thus, $c_b(\alpha)$ is properly defined and increasing on $(0, \alpha_x)$.

It can be shown that $c_a(\alpha)$ is also passing through (c_x, α_x), thus, $c_a(\alpha)$ is properly defined on $\left(\alpha_x, \frac{1}{17}\right)$.

· Case 1-i-b-II: $\frac{1}{17} \leq \alpha < \frac{1}{16}$.

In this case, we have $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) \leq c^{\dagger}(\alpha)$. Thus, the profit difference between *S* and *CE-PL* is:

$$
\begin{aligned}
H_{S,2} \triangleq & \Big(\sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)\, c^2\right. \\
& \left.+4\alpha(9\alpha-1)c\right)-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3\right. \\
& \left.+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big) \\
& /\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right) \\
& -\frac{2\alpha+\alpha^2(c+6)-4\sqrt{\alpha^2(\alpha+1)(2\alpha+(\alpha-1)c)}-c}{(1-\alpha)^2}.
\end{aligned}
$$

Similarly, we can simplify $H_{S,2}(\alpha,c)$ and finally analyze $H_{S,3}(\alpha,c)$. Following the same step in case 1-i-b-I, we can get that in this region, $\frac{\partial H_{S,3}(\alpha,c)}{\partial c} < 0$. $\frac{\partial H_{S,3}(\alpha,c)}{\partial \alpha}$ changes signs. As such, it is not possible to characterize the threshold between *S* and *CE-PL* as a function of c (there exist values of c for which increasing α leads to multiple crossings between optimality regions for *S* and *CE-PL*).

Nevertheless, moving horizontally, given that $\frac{\partial H_{S,3}(\alpha,c)}{\partial c} < 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and *S*, within this particular region of the parameter space, is unique for every α, *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:

(1) First, we consider points on the boundary $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{c=\frac{2\alpha(1-16\alpha)}{1-9\alpha}} > 0$. Thus, *S* dominates *CE-PL* on $c = \frac{2\alpha(1-16\alpha)}{1-9\alpha}$.

(2) First, we consider points on the boundary $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$. It can be shown that $H_{S,3}(\alpha, c)\Big|_{c=\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right)} < 0$. Thus, *CE-PL* dominates *S* on $c = \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$.

Therefore, as we increase c, there can be at most one crossing point between optimality regions for *S* and *CE-PL*, which is defined as $c_a(\alpha)$ in case 1-i-b-I. Thus, we can further extend the domain of $c_a(\alpha)$ to $\left(\alpha_x, \frac{1}{16}\right)$.

– Case 1-ii: $\frac{1}{16} \leq \alpha < \frac{1}{9}$.

  In this case, $\frac{2\alpha(1-16\alpha)}{1-9\alpha} \leq 0 \leq c < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Therefore, $H_{S,1} \geq 0$. We square both sides of the inequality D.3 and follow the same step in case 1-i-b. The inequality D.3 is equivalent to:

$$0 \leq c < \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right).$$

Also, it can be shown that $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) < c^\dagger(\alpha)$. Therefore, in this region, the profit difference between *S* and *CE-PL* is:

$$
\begin{aligned}
H_{S,2} \triangleq \Big(& \sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2\right. \\
& \left.+4\alpha(9\alpha-1)c\right)-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3\right. \\
& \left.+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big) \\
& /\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right) \\
& -\frac{2\alpha+\alpha^2(c+6)-4\sqrt{\alpha^2(\alpha+1)(2\alpha+(\alpha-1)c)}-c}{(1-\alpha)^2}.
\end{aligned}
$$

Similarly, we can simplify $H_{S,2}(\alpha,c)$ and finally analyze $H_{S,3}(\alpha,c)$. Following the same step in case 1-i-b-I, we can get that in this region, $\frac{\partial H_{S,3}(\alpha,c)}{\partial\alpha} < 0$. $\frac{\partial H_{S,3}(\alpha,c)}{\partial c}$ changes signs. As such, it is not possible to characterize the threshold between *S* and *CE-PL* as a function of α (there exist values of α for which increasing c leads to multiple crossings between optimality regions for *S* and *CE-PL*).

Nevertheless, moving vertically, given that $\frac{\partial H_{S,3}(\alpha,c)}{\partial\alpha} < 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and *S*, within this particular region of the parameter space, is unique for every c, *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region.

We look at two particular cases for this region:

(1) First, we consider points on the boundary $\alpha = \frac{1}{16}$. It can be shown that

$H_{S,3}(\alpha,c)\Big|_{\alpha=\frac{1}{16}} > 0$. Thus, *S* dominates *CE-PL* on $\alpha = \frac{1}{16}$.

(2) Then, we consider points on the boundary $\alpha = \frac{1}{9}$. It can be shown that $H_{S,3}(\alpha,c)\Big|_{\alpha=\frac{1}{9}} < 0$. Thus, *CE-PL* dominates *S* on $\alpha = \frac{1}{9}$.

Therefore, as we increase α, there can be at most one crossing point between optimality regions for *S* and *CE-PL*, then there exists a unique boundary, which we define as $\alpha^{\dagger}(c)$, which separates the optimality regions for *S* and *CE-PL*. It satisfies:

$$H_{S,3}(\alpha^{\dagger}(c), c) = 0.$$

It is straightforward that the domain of $\alpha^{\dagger}(c)$ is $\big(0, c_a(\frac{1}{16})\big)$.

- Case 2: $\frac{1}{9} \leq \alpha < \frac{1}{4}$.

  In this case, $H_{S,1}(\alpha,c)$ is decreasing in c, $32\alpha^2 + c - \alpha(9c+2) > 0$ is equivalent to $c < \frac{2\alpha(1-16\alpha)}{1-9\alpha}$. It can be shown that $\frac{2\alpha(1-16\alpha)}{1-9\alpha} > \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Thus, $32\alpha^2+c-\alpha(9c+2) > 0$. We square both sides of the inequality D.3 and follow the same step in case 1-i-b. The inequality D.3 is equivalent to:

$$0 \leq c < \frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right).$$

  Comparing $\frac{1}{2}\left(\sqrt{17\alpha^2 - 10\alpha + 1} + 9\alpha - 1\right)$ with $\frac{2\alpha(1-4\alpha)}{1-3\alpha}$, we further get two cases:

  - Case 2-i: $\frac{1}{9} \leq \alpha < \frac{1}{17}\left(5 - 2\sqrt{2}\right)$.

In this case, $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) < \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. It can be shown that $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) < c^{\dagger}(\alpha)$. Therefore, in this region, the profit difference between *S* and *CE-PL* is:

$$
\begin{aligned}
H_{S,2} \triangleq \Big(& \sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2\right. \\
& \left.+4\alpha(9\alpha-1)c\right)-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3\right. \\
& \left.+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big) \\
& /\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right) \\
& -\frac{2\alpha+\alpha^2(c+6)-4\sqrt{\alpha^2(\alpha+1)(2\alpha+(\alpha-1)c)}-c}{(1-\alpha)^2}.
\end{aligned}
$$

It can be shown that $H_{S,3} < 0$, i.e., $H_{S,2} < 0$. Thus, *S* is dominated by *CE-PL*.

– Case 2-ii: $\frac{1}{17}\left(5-2\sqrt{2}\right) \leq \alpha < \frac{1}{4}$.

  In this case, $\frac{1}{2}\left(\sqrt{17\alpha^2-10\alpha+1}+9\alpha-1\right) \geq \frac{2\alpha(1-4\alpha)}{1-3\alpha}$. Comparing $\frac{2\alpha(1-4\alpha)}{1-3\alpha}$ with $c^{\dagger}(\alpha)$, we further have two cases:

  * Case 2-ii-a: $\frac{2\alpha(1-4\alpha)}{1-3\alpha} < c^{\dagger}(\alpha)$.

    The profit difference between *S* and *CE-PL* is:

$$
\begin{aligned}
H_{S,2} \triangleq \Big(& \sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)}\left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2\right. \\
& \left.+4\alpha(9\alpha-1)c\right)-\left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3\right. \\
& \left.+2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big) \\
& /\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right) \\
& -\frac{2\alpha+\alpha^2(c+6)-4\sqrt{\alpha^2(\alpha+1)(2\alpha+(\alpha-1)c)}-c}{(1-\alpha)^2}.
\end{aligned}
$$

It can be shown that $H_{S,2} < 0$. Thus, *S* is dominated by *CE-PL*.

* Case 2-ii-b: $\frac{2\alpha(1-4\alpha)}{1-3\alpha} \geq c^{\dagger}(\alpha)$.

  The profit difference between *S* and *CE-PL* is:

$$H_{S,4} \triangleq \Big(\sqrt{(2\alpha+17\alpha c-c)(\alpha(c+2)-c)} \left(4\alpha^2+\left(17\alpha^2-18\alpha+1\right)c^2 \right.$$
$$\left. +4\alpha(9\alpha-1)c\right) - \left(-8\alpha^3+\left(71\alpha^3-109\alpha^2+37\alpha+1\right)c^3 \right.$$
$$\left. +2\alpha\left(109\alpha^2-74\alpha-3\right)c^2+4\alpha^2(37\alpha+3)c\right)\Big)$$
$$/\left(64(1-\alpha)\alpha(2\alpha-c)(2\alpha+(\alpha-1)c)\right)$$
$$-\frac{(c-2\alpha)^2}{8\alpha}.$$

  It can be shown that $H_{S,4} < 0$. Thus, *S* is dominated by *CE-PL*.

To summarize, we define $c^{\ddagger}(\alpha)$ as:

$$c^{\ddagger}(\alpha) \triangleq \begin{cases} c_a(\alpha) & \text{, if } \alpha_x \leq \alpha < \frac{1}{16}, \\ c_b(\alpha) & \text{, if } 0 < \alpha < \alpha_x. \end{cases}$$

Then *S* dominates *CE-PL* if and only if:

$$0 \leq c < c^{\ddagger}(\alpha) \qquad \text{, if } 0 < \alpha < \tfrac{1}{16},$$

and

$$\tfrac{1}{16} < \alpha < \alpha^{\dagger}(c).$$

□

Proof of Proposition 3

When $\alpha \geq 1$, by directly comparing profits and social welfare values from Propositions 13-16, it can be easily seen that *CE-PL* is always the dominant strategy for the firm, whereas *TLF* is always the strategy that yields the highest social welfare.

The bulk of the proof, below, is addressing the considerably more complex case $\underline{0 < \alpha < 1}$.

Let us define:

$$\alpha_1(c) \triangleq \quad \alpha_f(c) \quad \text{, if } 0 \leq c < c_4$$

$$\alpha_2(c) \triangleq \begin{cases} \alpha_e(c) & \text{, if } 0 \leq c < c_4, \\ \alpha_a(c) & \text{, if } c_4 \leq c < c_1, \\ \alpha_b(c) & \text{, if } c_1 \leq c < c_2, \end{cases}$$

and

$$\alpha_3(c) \triangleq \begin{cases} \alpha_g(c) & \text{, if } 0 \leq c < c_5, \\ \alpha_d(c) & \text{, if } c_5 \leq c < c_3, \\ \alpha_c(c) & \text{, if } c_3 \leq c < c_2, \end{cases}$$

and

$$\alpha_4(c) \triangleq \begin{cases} \alpha_g(c) & \text{, if } 0 \leq c < c_5, \\ \alpha^{\dagger}(c) & \text{, if } c_5 \leq c < c^{\ddagger}(\frac{1}{16}), \end{cases}$$

where functions $\alpha_a(\cdot)$, $\alpha_b(\cdot)$, $\alpha_c(\cdot)$, $\alpha_d(\cdot)$, $\alpha_e(\cdot)$, $\alpha_f(\cdot)$, $\alpha_g(\cdot)$, as well as constant thresholds c_1, c_2, c_3, c_4, and c_5 are defined and further analyzed below. $\alpha^{\dagger}(c)$ and $c^{\ddagger}(\alpha)$ is defined in the

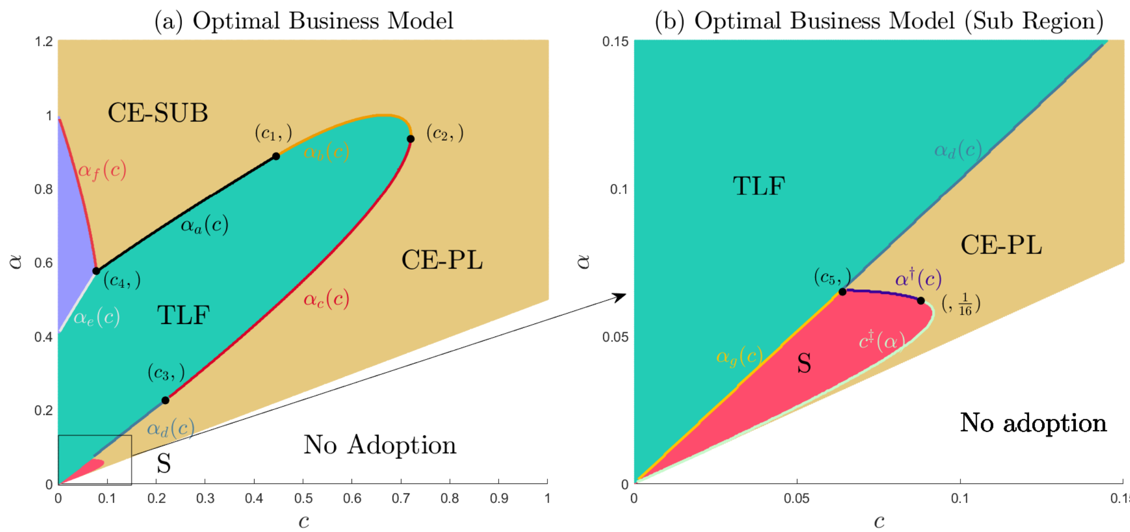


Figure D.1: Adoption Costs Scenario - Optimal Business Model - Marked Boundaries

Prop 16. For ease of identification, Figure D.1 contains the illustration of these boundaries and thresholds (this is a more detailed version of Figure 2.5 from the main body).

- **Definition of c_1 and $\alpha_a(c)$. Monotonicity of $\alpha_a(c)$.**

  We first compare *CE-PL* and *TLF* under the intersection of regions $0 \leq c < \frac{\alpha}{2}$ and $13 - 4\sqrt{10} \leq \alpha < 1$, it can immediately follws that this is a non-empty region. In this region, define the difference between optimal profits under *CE-PL* and *TLF* as:

$$\Psi_{a,D}(\alpha, c) \triangleq \frac{(c - 2\alpha)^2}{8\alpha} - \frac{1}{4}.$$

  We can obtain that:

$$\frac{\partial \Psi_{a,D}(\alpha, c)}{\partial \alpha} = \frac{1}{2} - \frac{c^2}{8\alpha^2} > 0,$$
$$\frac{\partial \Psi_{a,D}(\alpha, c)}{\partial c} = \frac{1}{4}\left(\frac{c}{\alpha} - 2\right) < 0.$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = 13 - 4\sqrt{10}$ and $\alpha = 1$ and examine the sign of $\Psi_{a,D}(\alpha, c)$ along these boundaries.

- On the boundary $\alpha = 13 - 4\sqrt{10}$, we obtain:

$$\Psi_{a,D}(\alpha, c)\Big|_{\alpha=13-4\sqrt{10}} = \frac{1}{72}\left(4\sqrt{10}+13\right)\left(c+8\sqrt{10}-26\right)^2 - \frac{1}{4} < 0.$$

- On the boundary $\alpha = 1$, we obtain:

$$\Psi_{a,D}(\alpha, c)\Big|_{\alpha=1} = \frac{1}{8}((c-4)c+2) > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_a(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{(c-2\alpha_a(c))^2}{8\alpha_a(c)} - \frac{1}{4} = 0.$$

Also, it is straightforward that $\frac{\partial \alpha_a(c)}{\partial c} = -\frac{\frac{\partial \Psi_{a,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{a,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\alpha_a(c)$ is increasing in c.

It is straightforward there is a unique intersection point between $\alpha_a(c)$ and $c = 0$, i.e., $(0, \frac{1}{2})$. Moreover, there is a unique intersection point between $\alpha_a(c)$ and $\alpha = 2c$, i.e., $(c_1 = \frac{4}{9}, \frac{8}{9})$. Thus, $\alpha_a(c)$ is properly defined and increasing on $[0, c_1)$.

- **Definition of c_2, c_3, $\alpha_b(c)$ and $\alpha_c(c)$. Monotonicity of $\alpha_c(c)$.**

  We then compare *CE-PL* and *TLF* under the intersection of regions $\frac{\alpha}{2} \le c < \alpha$ and the union of regions $0 < \alpha < 13 - 4\sqrt{10}, c^\dagger \le c < \alpha$ and $13 - 4\sqrt{10} \le \alpha < 1$ (In this union of regions, $\pi^*_{CE-PL} = \frac{(c-2\alpha)^2}{8\alpha}$). In this region, define the difference between optimal profits under *CE-PL* and *TLF* as:

$$\Psi_{b,D}(\alpha, c) \triangleq \frac{(c-2\alpha)^2}{8\alpha} - \frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}.$$

  We can obtain that:

$$\frac{\partial \Psi_{b,D}(\alpha, c)}{\partial \alpha} = \frac{1}{2} - \frac{c((\alpha+16)c - 8\alpha)}{8\alpha^3},$$
$$\frac{\partial \Psi_{b,D}(\alpha, c)}{\partial c} = \frac{(\alpha+8)c - 2\alpha(\alpha+2)}{4\alpha^2}.$$

  As it turns out, in this range of the parameter space, $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha}$ and $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial c}$ changes signs. As such, it is not possible to characterize the threshold between *CE-PL* and *TLF* as a function of c or α.

  Nevertheless, we first find the point that satisfies the equation $\Psi_{b,D}(\alpha, c) = 0$ (i.e., on the boundary between *CE-PL* and *TLF*) and has a vertical tangent line, i.e., $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha} = 0$. We can obtain that with in the above mentioned region, there is only one point that satisfies the condition, which is $\left(20\sqrt{5}-44, 4\left(\sqrt{5}-2\right)\right)$. We define $c_2 = 20\sqrt{5} - 44, 4\left(\sqrt{5}-2\right)$.

Next, we define $\alpha_b(c)$ and $\alpha_c(c)$ by splitting the boundary $\Psi_{b,D}(\alpha,c) = 0$
at $\left(20\sqrt{5}-44, 4\left(\sqrt{5}-2\right)\right)$.

It is straightforward that when $c > 20\sqrt{5} - 44$, $\Psi_{b,D}(\alpha,c) > 0$, i.e., *CE-PL* dominates *TLF*. Then, we focus on the case when $c \leq 20\sqrt{5} - 44$.

We first construct a line go through $\left(0, \frac{1}{2}\right)$ and $\left(20\sqrt{5}-44, 4\left(\sqrt{5}-2\right)\right)$. And it is straightforward that the expression of the line is: $\alpha_{l1} = \frac{\left(8\sqrt{5}-17\right)c}{8\left(5\sqrt{5}-11\right)} + \frac{1}{2}$.

If $\alpha_{l1} \leq \alpha < 2c$, we obtain that:

$$\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha} = \frac{1}{2} - \frac{c((\alpha+16)c - 8\alpha)}{8\alpha^3} > 0,$$
$$\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial c} = \frac{(\alpha+8)c - 2\alpha(\alpha+2)}{4\alpha^2}.$$

As it turns out, in this range of the parameter space, $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial c}$ changes signs. Nevertheless, moving vertically, given that $\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha} > 0$, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF*, within this particular region of the parameter space, is unique for every c, if *it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = \alpha_{l2} = \frac{1}{4}\left(\sqrt{5}+3\right)c$ and $\alpha = 1$, and examine the sign of $\Psi_{b,D}(\alpha, c)$ along these boundaries.

– On the boundary $\alpha = \alpha_{l2}$, we obtain:

$$\begin{aligned}\Psi_{b,D}(\alpha, c)\Big|_{\alpha=\alpha_{l2}} &= \frac{c^2\left(\left(9117-4077\sqrt{5}\right)c - 358912\sqrt{5} + 802608\right)}{64\left(\left(8\sqrt{5}-17\right)c + 20\sqrt{5} - 44\right)^2} \\ &\quad + \frac{64c\left(3881\sqrt{5}-8679\right)}{64\left(\left(8\sqrt{5}-17\right)c + 20\sqrt{5} - 44\right)^2} \\ &\quad + \frac{512\left(123-55\sqrt{5}\right)}{64\left(\left(8\sqrt{5}-17\right)c + 20\sqrt{5} - 44\right)^2} \\ &< 0.\end{aligned}$$

– On the boundary $\alpha = 1$, we obtain:

$$\Psi_{b,D}(\alpha, c)\Big|_{\alpha=1} = \frac{1}{8}(2-3c)^2 > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_b(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{(c-2\alpha_b(c))^2}{8\alpha_b(c)} - \frac{c\left(1-\frac{c}{\alpha_b(c)}\right)}{\alpha_b(c)} = 0.$$

It is easy to obtain that $(c_1, \frac{8}{9})$ is on $\alpha_b(c)$. Thus, $(c_1, \frac{8}{9})$ is the unique interaction point between $\alpha = 2c$ and $\alpha_b(c)$. Thus, $\alpha_b(c)$ is properly defined on $[c_1, c_2)$.

Then we construct a line go through $(0,0)$ and $\left(20\sqrt{5}-44, 4\left(\sqrt{5}-2\right)\right)$. And it is straightforward that the expression of the line is: $\alpha_{l2} = \frac{1}{4}\left(\sqrt{5}+3\right)c$.

We can obtain that when $\alpha_{l2} \leq \alpha < \alpha_{l1}$, $\Psi_{b,D}(\alpha, c) < 0$, *TLF* dominates *CE-PL*.

If $c \leq \alpha < \alpha_{l2}$, we obtain that:

$$\frac{\partial \Psi_{b,D}(\alpha, c)}{\partial \alpha} = \frac{1}{2} - \frac{c((\alpha+16)c - 8\alpha)}{8\alpha^3} < 0,$$
$$\frac{\partial \Psi_{b,D}(\alpha, c)}{\partial c} = \frac{(\alpha+8)c - 2\alpha(\alpha+2)}{4\alpha^2} > 0.$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = c$ and $\alpha = \alpha_{l2} = \frac{1}{4}\left(\sqrt{5}+3\right)c$ and examine the sign of $\Psi_{b,D}(\alpha, c)$ along these boundaries.

– On the boundary $\alpha = c$, we obtain:

$$\Psi_{b,D}(\alpha, c)\Big|_{\alpha=c} = \frac{c}{8} > 0.$$

– On the boundary $\alpha = \alpha_{l2}$, we obtain:

$$\Psi_{b,D}(\alpha, c)\Big|_{\alpha=\alpha_{l2}} = \frac{c}{4} - 5\sqrt{5} + 11 > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_c(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{(c - 2\alpha_c(c))^2}{8\alpha_c(c)} - \frac{c\left(1 - \frac{c}{\alpha_c(c)}\right)}{\alpha_c(c)} = 0.$$

Also, it is straightforward that $\frac{\partial \alpha_c(c)}{\partial c} = -\frac{\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{b,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\alpha_c(c)$ is increasing in c.

Moreover, by solving the system of equations $c^{\dagger} = 0$ and $\frac{(c-2\alpha_c(c))^2}{8\alpha_c(c)} - \frac{c\left(1-\frac{c}{\alpha_c(c)}\right)}{\alpha_c(c)} = 0$, we can get there is a unique intersection point (it is around $(0.2255, 0.2329)$) between $c^{\dagger}$ and $\alpha_c(c)$, denote it as $(c_3, \alpha_c(c_3))$. Thus, $\alpha_c(c)$ is properly defined and increasing on $[c_3, c_2)$.

- **Definition and Monotonicity of** $\alpha_d(c)$.

  We then compare *CE-PL* and *TLF* under the intersection of regions $\frac{\alpha}{2} \leq c < \alpha$ and $0 < \alpha < 13 - 4\sqrt{10}$, $0 \leq c < c^{\dagger}$, it can immediately follows that this is a non-empty region. In this region, define the difference between optimal profits under *CE-PL* and

TLF as:

$$\Psi_{d,D}(\alpha,c) \triangleq \frac{2\alpha+\alpha^2(c+6)-4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}-c}{(\alpha-1)^2} - \frac{c\left(1-\frac{c}{\alpha}\right)}{\alpha}.$$

First, we can obtain that in this region, when $\frac{\alpha}{2} \leq c < \frac{2\alpha}{3}$, $\Psi_{d,D}(\alpha,c) < 0$, i.e., *TLF* dominates *CE-PL*.

Next, we check the case when $\frac{2\alpha}{3} \leq c < \alpha$. We can further obtain that:

$$\begin{aligned}\frac{\partial\Psi_{d,D}(\alpha,c)}{\partial\alpha} = & \frac{-4\alpha-2\alpha^2(c+6)+8\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}+2c}{(\alpha-1)^3} \\ & + \frac{2\alpha(c+6)-\frac{4\alpha(\alpha(c+2)+1)}{\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}-4\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}+2}{(\alpha-1)^2} \\ & - \frac{c^2}{\alpha^3}+\frac{c(\alpha-c)}{\alpha^3} < 0, \\ \frac{\partial\Psi_{d,D}(\alpha,c)}{\partial c} = & \frac{c}{\alpha^2}+\frac{\alpha^2-\frac{2\left(\alpha^2-1\right)\alpha}{\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)}}-1}{(\alpha-1)^2}+\frac{c-\alpha}{\alpha^2} > 0.\end{aligned}$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-PL* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $c = \frac{2}{3}\alpha$ and $c = \alpha$ and examine the sign of $\Psi_{d,D}(\alpha,c)$ along these boundaries.

– On the boundary $c = \frac{2}{3}\alpha$, we obtain:

$$\Psi_{d,D}(\alpha, c)\Big|_{c=\frac{2}{3}\alpha} = \frac{2\alpha\left(\alpha(3\alpha+26) - 6\sqrt{6}\sqrt{\alpha(\alpha+1)(\alpha+2)} + 8\right) - 2}{9(\alpha-1)^2} < 0.$$

– On the boundary $c = \alpha$, we obtain:

$$\Psi_{d,D}(\alpha, c)\Big|_{c=\alpha} = \frac{\alpha\left(\alpha(\alpha+6) - 4\sqrt{\alpha(\alpha+1)^2} + 1\right)}{(\alpha-1)^2} > 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_d(c)}$, which separates the optimality regions for *CE-PL* and *TLF*. It satisfies:

$$\frac{2\alpha + \alpha^2(c+6) - 4\alpha\sqrt{(\alpha+1)(2\alpha+(\alpha-1)c)} - c}{(\alpha-1)^2} - \frac{c\left(1 - \frac{c}{\alpha}\right)}{\alpha} = 0.$$

Also, it is straightforward that $\frac{\partial \alpha_d(c)}{\partial c} = -\frac{\frac{\partial \Psi_{d,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{d,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\alpha_d(c)$ is increasing in c.

- **Definition and Monotonicity of** $\alpha_e(c)$.

  It is easy to obtain that when $c \leq \alpha < 2c$, $\pi_{TLF} > \pi_{CE-SUB}$, i.e., *TLF* dominates *CE-SUB*.

We then compare *CE-SUB* and *TLF* under the intersection of regions $\alpha \geq 2c$. In this region, define the difference between optimal profits under *CE-SUB* and *TLF* as:

$$\Psi_{e,D}(\alpha, c) \triangleq p_{a,D}\left(2 - \frac{c + p_{a,D}}{\alpha} - \frac{c + p_{a,D}}{1 + c + p_{a,D} - \frac{c+p_{a,D}}{\alpha}}\right) - \frac{1}{4}.$$

Then, using the Envelope theorem (since $p_{a,D} \in \left(\frac{\alpha-c}{2}, \alpha - c\right)$ maximize $\pi_{CE,SUB}$), we have:

$$\frac{\partial \Psi_{e,D}(\alpha,c)}{\partial \alpha} = \frac{p_{a,D}(c+p_{a,D})\left(\frac{c+p_{a,D}}{\left(-\frac{c+p_{a,D}}{\alpha}+c+p_{a,D}+1\right)^2}+1\right)}{\alpha^2} > 0,$$
$$\frac{\partial \Psi_{e,D}(\alpha,c)}{\partial c} = p_{a,D}\left(-\frac{1}{\alpha} - \frac{(1-\alpha)\alpha(c+p_{a,D})}{(\alpha+(\alpha-1)c+(\alpha-1)p_{a,D})^2} - \frac{1}{-\frac{c+p_{a,D}}{\alpha}+c+p_{a,D}+1}\right)$$
$$< 0.$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-SUB* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary does indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $\alpha = 2c$ and $\alpha = 2c + \frac{1}{2}$ and examine the sign of $\Psi_{e,D}(\alpha, c)$ along these boundaries.

- On the boundary $\alpha = 2c$, we obtain:

$$\Psi_{e,D}(\alpha,c)\Big|_{\alpha=2c} = p_{a,D}\left(-\frac{2c(c+p_{a,D})}{2c^2+2cp_{a,D}+c-p_{a,D}} - \frac{c+p_{a,D}}{2c}+2\right) - \frac{1}{4} < 0.$$

  The above inequality is satisfied for all $p \in \left(\frac{\alpha-c}{2}, \alpha - c\right)$.

– On the boundary $2c+\frac{1}{2}$, we obtain:

$$\begin{aligned}\Psi_{e,D}(\alpha,c)\Big|_{2c+\frac{1}{2}} &= p_{a,D}\left(-\frac{2(c+p_{a,D})}{4c+1}-\frac{c+p_{a,D}}{-\frac{2(c+p_{a,D})}{4c+1}+c+p_{a,D}+1}+2\right)-\frac{1}{4} \\ &\geq \frac{6c\left(2c\left(8c^2+2c+1\right)+1\right)+1}{8(4c+1)(2c(12c+7)+3)}(\text{Plug } p_{a,D}=\frac{1}{4}(2c+1)) \\ &> 0.\end{aligned}$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_e(c)}$, which separates the optimality regions for *CE-SUB* and *TLF*. It satisfies:

$$p_{a,D}\left(2-\frac{c+p_{a,D}}{\alpha_e(c)}-\frac{c+p_{a,D}}{1+c+p_{a,D}-\frac{c+p_{a,D}}{\alpha_e(c)}}\right)-\frac{1}{4}=0.$$

Also, it is straightforward that $\frac{\partial\alpha_e(c)}{\partial c}=-\frac{\frac{\partial\Psi_{e,D}(\alpha,c)}{\partial c}}{\frac{\partial\Psi_{e,D}(\alpha,c)}{\partial\alpha}}>0$. Hence, $\alpha_e(c)$ is increasing in c.

- **Definition of c_4 and $\alpha_f(c)$. Monotonicity of $\alpha_f(c)$.**

  We further compare *CE-SUB* with *CE-PL*. From the definition of $\alpha_a(c)$, we know that $\alpha_a(c)$ has a unique interaction point with y axis, i.e., $(0,\frac{1}{2})$. Given that $\alpha_a(c)$ is increasing in c, *CE-PL* can only have the possibility to become the dominant strategy when $\frac{1}{2}<\alpha<1$. Also, from the definition of $\alpha_e(c)$, we know that $\alpha_e(c)$ has a unique interaction point with $\alpha=1$, i.e., $(1-\frac{1}{\sqrt{2}},1)$. *CE-SUB* can only have the possibility to become the dominant strategy when $0\leq c<1-\frac{1}{\sqrt{2}}$. Thus, we only need to compare *CE-SUB* with *CE-PL* in the intersection of $\frac{1}{2}\leq\alpha<1, 0\leq c<1-\frac{1}{\sqrt{2}}$, and $\alpha\geq 2c$.

In this region, define the difference between optimal profits under *CE-SUB* and *CE-PL* as:

$$\Psi_{e,D}(\alpha, c) \triangleq p_{a,D}\left(2 - \frac{c + p_{a,D}}{\alpha} - \frac{c + p_{a,D}}{1 + c + p_{a,D} - \frac{c+p_{a,D}}{\alpha}}\right) - \frac{(c - 2\alpha)^2}{8\alpha}.$$

Then, using the Envelope theorem (since $p_{a,D} \in \left(\frac{\alpha-c}{2}, \alpha - c\right)$ maximize $\pi_{CE,SUB}$), we have:

$$\begin{aligned}
\frac{\partial\Psi_{f,D}(\alpha, c)}{\partial\alpha} &= \frac{c^2 + 8cp_{a,D} + 8p_{a,D}^2}{8\alpha^2} + \frac{p_{a,D}(c + p_{a,D})^2}{(\alpha + (\alpha - 1)c + (\alpha - 1)p_{a,D})^2} - \frac{1}{2},\\
\frac{\partial\Psi_{f,D}(\alpha, c)}{\partial c} &= -\frac{c}{4\alpha} + p_{a,D}\left(-\frac{1}{\alpha} + \frac{(\alpha - 1)\alpha(c + p_{a,D})}{(\alpha + (\alpha - 1)c + (\alpha - 1)p_{a,D})^2}\right.\\
&\qquad \left. - \frac{1}{-\frac{c+p_{a,D}}{\alpha} + c + p_{a,D} + 1}\right) + \frac{1}{2} < 0.
\end{aligned}$$

Therefore, a threshold (crossing) boundary between optimality regions for *CE-SUB* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, let's check the sign of $\frac{\partial\Psi_{f,D}(\alpha,c)}{\partial\alpha}$. Bring all the terms to a common denominator, we can write $\frac{\partial \Psi_{f,D}(\alpha,c)}{\partial \alpha} = \frac{q_{1,D}}{q_{2,D}}$, where:

$$
\begin{aligned}
q_{1,D} \triangleq\ & 8p_{a,D}^4(1-\alpha)^2 + 8p_{a,D}^3\left(\alpha(3\alpha-2)+3(\alpha-1)^2c\right) \\
& + p_{a,D}^2\left(-4\alpha^2\left(\alpha^2-2\alpha-1\right)+25(\alpha-1)^2c^2+16\alpha(3\alpha-2)c\right) \\
& + 2p_{a,D}\left(-4(\alpha-1)\alpha^3+5(\alpha-1)^2c^3+\alpha(13\alpha-9)c^2-4(\alpha-2)\alpha^3c\right) \\
& + (\alpha+(\alpha-1)c)^2\left(c^2-4\alpha^2\right), \\
q_{2,D} \triangleq\ & 8\alpha^2(\alpha+\alpha c-c+\alpha p_{a,D}-p_{a,D})^2 > 0.
\end{aligned}
$$

Thus, the sign of $\frac{\partial \Psi_{f,D}(\alpha,c)}{\partial \alpha}$ is the same as the sign of the numerator, $q_{1,D}$. We use $G_{SUB,D}(p_{a,D}) = 0$ to reduce the expression of $q_{1,D}$ from a quartic polynomial in $p_{a,D}$ to a quadratic one, as follows:

$$
\begin{aligned}
q_{1,D} = & \frac{1}{(1-\alpha)^2} \\
& \times \left(p_{a,D}^2(1-\alpha)\left(2\alpha^2(\alpha((\alpha-3)\alpha+10)+2)+(\alpha-1)^3c^2-4\alpha^2(1-\alpha)c\right)\right. \\
& + p_{a,D}\left(-4\alpha^3(\alpha((\alpha-4)\alpha+5)+2)-2(\alpha-1)^4c^3-2\alpha(5\alpha-1)(\alpha-1)^2c^2\right. \\
& \left.-4\alpha^2(\alpha((\alpha-4)\alpha+9)+2)(\alpha-1)c\right) \\
& + 4\alpha^4(\alpha+1)-(\alpha-1)^4c^4-2\alpha(3\alpha-1)(\alpha-1)^2c^3-\alpha^2(\alpha(2(\alpha-4)\alpha+15) \\
& \left.+3)(\alpha-1)c^2 -2(\alpha-5)\alpha^4(\alpha-1)c-8\alpha^3c\right).
\end{aligned}
$$

Denote:

$$
\begin{aligned}
A &= (1-\alpha)\left(2\alpha^2(\alpha((\alpha-3)\alpha+10)+2)+(\alpha-1)^3c^2-4\alpha^2(1-\alpha)c\right)\\
B &= -4\alpha^3(\alpha((\alpha-4)\alpha+5)+2)-2(\alpha-1)^4c^3-2\alpha(5\alpha-1)(\alpha-1)^2c^2\\
&\quad -4\alpha^2(\alpha((\alpha-4)\alpha+9)+2)(\alpha-1)c\\
C &= 4\alpha^4(\alpha+1)-(\alpha-1)^4c^4-2\alpha(3\alpha-1)(\alpha-1)^2c^3-\alpha^2(\alpha(2(\alpha-4)\alpha+15)+\\
&\quad 3)(\alpha-1)c^2-2(\alpha-5)\alpha^4(\alpha-1)c-8\alpha^3c.
\end{aligned}
$$

Then, $q_{1,D} = \frac{1}{(1-\alpha)^2}\times\left(Ap_{a,D}^2 + Bp_{a,D} + c\right)$. Define the quadratic function $H_{SUB,PL,D}(p) \triangleq Ap^2 + Bp + c$. In this range of the parameter space, it can be shown that $B^2 - 4AC > 0$ and $A > 0$. Hence, there are two real solutions to the equation $H_{SUB,PL,D}(p) = 0$, namely:

$$
p_{H1} = \frac{-B-\sqrt{B^2-4AC}}{2A} \quad \text{and} \quad p_{H2} = \frac{-B+\sqrt{B^2-4AC}}{2A}.
$$

It can be shown that $\frac{\alpha-c}{2} < p_{H1} < \alpha - c < p_{H2}$. Recall that $p_{a,D}$ is the unique solution of $G_{SUB,D}(p) = 0$. Moreover, from the proof of Prop. 14, we know that $G_{SUB,D}(p) > 0$ on $(\frac{\alpha-c}{2}, p_{a,D})$ and $G_{SUB,D}(p) < 0$ on $(p_{a,D}, \alpha - c)$. It can be proved directly that $G_{SUB,D}(p_{H1}) > 0 = G_{SUB,D}(p_{a,D})$. Hence, $\frac{\alpha-c}{2} < p_{H1} < p_{a,D} < \alpha - c < p_{H2}$. Furthermore, it can be shown that $A > 0$, which indicates that $\bar{H}_{SUB,PL}(p)$ is convex. Therefore, $H_{SUB,PL,D}(p) < 0$. Hence, in this region of the parameter space:

$$
\frac{\partial \Psi_{f,D}(\alpha, c)}{\partial \alpha} < 0.
$$

So far, we proved that a threshold (crossing) boundary between optimality regions for *CE-SUB* and *CE-PL* within this particular region of the parameter space is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $c = 0$ and $c = 1 - \frac{1}{\sqrt{2}}$ and examine the sign of $\Psi_{f,D}(\alpha, c)$ along these boundaries.

- On the boundary $c = 0$, it defaults to our basic model. And from Prop 1, we know that $\pi^*_{CE-SUB} > \pi^*_{CE-PL}$, i.e., $\Psi_{f,D}(\alpha, c)\Big|_{c=0} > 0$.
- On the boundary $c = 1 - \frac{1}{\sqrt{2}}$, we obtain:

$$\Psi_{f,D}(\alpha, c)\Big|_{c=1-\frac{1}{\sqrt{2}}} = p_{a,D}\left(\frac{-p_{a,D} + \frac{1}{\sqrt{2}} - 1}{\alpha} + \frac{1}{\frac{1}{\alpha} + \frac{1}{-p_{a,D} + \frac{1}{\sqrt{2}} - 1} - 1} + 2\right) - \frac{\left(4\alpha + \sqrt{2} - 2\right)^2}{32\alpha} < 0.$$

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_f(c)}$, which separates the optimality regions for *CE-SUB* and *CE-PL*. It satisfies:

$$p_{a,D}\left(2 - \frac{c + p_{a,D}}{\alpha_e(c)} - \frac{c + p_{a,D}}{1 + c + p_{a,D} - \frac{c + p_{a,D}}{\alpha_e(c)}}\right) - \frac{(c - 2\alpha)^2}{8\alpha} = 0.$$

Also, it is straightforward that $\frac{\partial \alpha_f(c)}{\partial c} = -\frac{\frac{\partial \Psi_{f,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{f,D}(\alpha,c)}{\partial \alpha}} < 0$. Hence, $\alpha_f(c)$ is decreasing in c.

As $\alpha_e(c)$ is increasing in c, there exists a unique intersection point between $\alpha_e(c)$ and $\alpha_f(c)$. Defining this point as $(c_4, \alpha_e(c_4))$. At this point, we get $\pi^*_{CE-SUB} = \pi^*_{CE-PL}$ (from the definition of $\alpha_f(c)$) and $\pi^*_{CE-SUB} = \pi^*_{TLF}$ (from the definition of $\alpha_e(c)$). Thus, $\pi^*_{CE-PL} = \pi^*_{TLF}$. $(c_4, \alpha_e(c_4))$ is also on $\alpha_a(c)$. $\alpha_f(c)$ is properly defined and decreasing on $[0, c_4)$.

- **Definition of c_5 and $\alpha_g(c)$. Monotonicity of $\alpha_g(c)$.**

  We further compare *TLF* with *S*. From Proposition 16, we know *S* dominates *CE-PL* when:

  $$0 \leq c < c^{\ddagger}(\alpha) \quad \text{, if } 0 < \alpha < \tfrac{1}{16},$$

  and

  $$\tfrac{1}{16} \leq \alpha < \alpha^{\dagger}(c).$$

  In this region, we further consider two regions:

  - Region 1: $0 \leq c < \frac{\alpha}{2}$.

    In this region, it can be shown that $\pi^*_S < \pi^*_{TLF}$, i.e., *TLF* dominates *S*.

  - Region 2: $\frac{\alpha}{2} \leq c < \alpha$.

In this region, define the difference between optimal profits under *S* and *TLF* as:

$$\Psi_{g,D}(\alpha, c) \triangleq \Big(\sqrt{(2\alpha + 17\alpha c - c)(\alpha(c+2) - c)} \left(4\alpha^2 + \left(17\alpha^2 - 18\alpha + 1\right) c^2 \right.$$
$$\left. +4\alpha(9\alpha - 1)c\right) - \left(-8\alpha^3 + \left(71\alpha^3 - 109\alpha^2 + 37\alpha + 1\right) c^3\right.$$
$$\left. +2\alpha \left(109\alpha^2 - 74\alpha - 3\right) c^2 + 4\alpha^2(37\alpha + 3)c\right)$$
$$/ \left(64(1-\alpha)\alpha(2\alpha - c)(2\alpha + (\alpha - 1)c)\right) - \frac{c\left(1 - \frac{c}{\alpha}\right)}{\alpha}.$$

First, it can be shown that when $\frac{\alpha}{2} \leq c < \frac{3\alpha}{4}$, $\Psi_{f,D}(\alpha, c) < 0$, i.e., *TLF* dominates *S*.

Then we focus on the region $\frac{3\alpha}{4} \leq c < \alpha$. We obtain that:

$$\frac{\partial \Psi_{g,D}(\alpha, c)}{\partial \alpha} < 0,$$
$$\frac{\partial \Psi_{g,D}(\alpha, c)}{\partial c} > 0.$$

Therefore, a threshold (crossing) boundary between optimality regions for *S* and *TLF* within this particular region is unique for every c and for every α (i.e., if we look vertically or horizontally), *if it exists*.

Next, we show that such a threshold boundary *does* indeed exist in this region of the parameter space. We look at two particular delimiting boundaries for this region, namely $c = \frac{3}{4\alpha}$ and $c = \alpha$ and examine the sign of $\Psi_{g,D}(\alpha, c)$ along these boundaries.

∗ On the boundary $c = \frac{3}{4\alpha}$, we obtain:

$$
\begin{aligned}
\Psi_{g,D}\Big|_{c=\frac{3}{4\alpha}} &= \frac{\alpha\left(639\alpha^2 - 51\sqrt{\alpha^2(3\alpha+5)(51\alpha+5)} + 330\alpha + 215\right)}{1280(\alpha-1)\alpha} \\
&\quad - \frac{5\sqrt{\alpha^2(3\alpha+5)(51\alpha+5)}}{1280(\alpha-1)\alpha} \\
&< 0.
\end{aligned}
$$

Thus, on the boundary, *TLF* dominates *S*.

* On the boundary $c = \alpha$, it is easy to get $\pi^*_{TLF} \to 0$, whereas $\pi^* S > 0$. Thus, $\pi^*_S > \pi^*_{TLF}$, *S* dominates *TLF*.

Therefore, in this parameter region, there exists a unique threshold boundary, which we define as $\underline{\alpha_g(c)}$, which separates the optimality regions for *S* and *TLF*. It satisfies:

$$\Psi_{g,D}(\alpha_g(c), c) = 0.$$

Also, it is straightforward that $\frac{\partial \alpha_g(c)}{\partial c} = -\frac{\frac{\partial \Psi_{g,D}(\alpha,c)}{\partial c}}{\frac{\partial \Psi_{g,D}(\alpha,c)}{\partial \alpha}} > 0$. Hence, $\alpha_g(c)$ is decreasing in c.

As $\alpha_g(c) > c$ and $c_1(\frac{1}{16}) \approx 0.0876 > \frac{1}{16}$, there exists a unique intersection point between $\alpha_g(c)$ and $\alpha^\dagger(c)$. Defining this point as $(c_5, \alpha_g(c_5))$. Thus, $\alpha_g(c)$ is properly defined and increasing on $[0, c_5)$.

Thus, we completely characterized lines $\alpha^\dagger(c)$, $c_a(\alpha)$, $\alpha_1(c)$, $\alpha_2(c)$, $\alpha_3(c)$ and $\alpha_4(c)$, (in particular, segments, $\alpha_a(c)$, $\alpha_b(c)$, $\alpha_c(c)$, $\alpha_d(c)$, $\alpha_e(c)$, $\alpha_f(c)$, $\alpha_g(c)$, as well as constant thresholds c_1, c_2, c_3, c_4, c_5).

Comparison of $\alpha_1(c)$ and $\alpha_2(c)$:

When $0 \leq c < c_4$, we already show that $\alpha_f(c)$ is decreasing in c and $\alpha_e(c)$ is increasing in c. Furthermore, they interact at the point $(c_4, \alpha_e(c)[\alpha_f(c)])$. Thus, we have $\alpha_f(c) \geq \alpha_e(c)$, i.e., $\alpha_1(c) > \alpha_2(c)$.

Comparison of $\alpha_2(c)$ and $\alpha_3(c)$:

The region where $\alpha_g(c)$ is defined is with in $0 < \alpha < \frac{1}{9} < \alpha_e(c)$. Thus, $\alpha_g(c) < \alpha_e(c)$. For the other segments of $\alpha_2(c)$ and $\alpha_3(c)$, they are all related to comparing *TLF* and *CE-PL*. Thus, by definition, $\alpha_2(c) > \alpha_3(c)$.

Derivation of the dominating strategy in the entire region $0 < \alpha < 1$:

- By the definition of $\alpha_f(c)$ and $\alpha_e(c)$, we know when $\alpha_2(c) \leq \alpha < \alpha_1(c)$, *CE-SUB* dominates *CE-PL* and *TLF*. Since *S* can only dominates *CE-PL* within a subregion in $0 < \alpha < \frac{1}{9}$. Thus, *CE-SUB* also dominates *TLF* as well when $\alpha_2(c) \leq \alpha < \alpha_1(c)$.
- By the definition of $\alpha_2(c)$ and $\alpha_3(c)$ (including the comparison between *TLF* with *CE-SUB*, *CE-PL*, and *S*), it is straightforward that *TLF* is the optimal strategy when $\alpha_3(c) \leq \alpha < \alpha_2(c)$.
- We have discussed in detail in Prop. 16 and we can get that *S* is the optimal strategy when $\alpha < \alpha_3(c)$ and $c < c^{\ddagger}(\alpha)$.

This completes the mapping of dominant strategy to the parameter space (we discussed the case $\alpha \geq 1$ at the very beginning of the proof). $\square$

APPENDIX E

PROOFS OF RESULTS FOR TARGETED SEEDING OF CHAPTER 2

We first present the optimal solution under Targeted Seeding in Prop. 17.

Proposition 17. *Under TS model, the firm's optimal pricing strategy and profit are:*

	$0 < \alpha < \frac{1}{5}\left(3-\sqrt{7}\right)$	$\frac{1}{5}\left(3-\sqrt{7}\right) \leq \alpha < \frac{1}{2}$	$\frac{1}{2} \leq \alpha \leq 1$
p^*_{TS}	p^*_{TS-a}	$\tilde{p}_{TS}$	p^*_{CE-PL}
π^*_{TS}	π^*_{TS-a}	$\max\{\pi^*_{TS-a}, \pi^*_{TS-b}\}$	π^*_{CE-PL}
Adoption	in period 2	in period 2 or in both periods	in period 1

where:

$$
\begin{aligned}
p^*_{TS-a} &= \frac{\left(-2\alpha+\sqrt{(\alpha-1)\alpha+1}+1\right)\left(\alpha+\sqrt{(\alpha-1)\alpha+1}+1\right)}{9(1-\alpha)},\\
\pi^*_{TS-a} &= \frac{\left(-2\alpha+\sqrt{(\alpha-1)\alpha+1}+1\right)\left(\alpha+\sqrt{(\alpha-1)\alpha+1}+1\right)}{27(1-\alpha)^2}\\
&\qquad\times\left(-\alpha-\sqrt{(\alpha-1)\alpha+1}+2\right),\\
p^*_{TS-b} &= 2\alpha\theta^*_1,\\
\pi^*_{TS-b} &= 2\alpha\theta^*_1\left(1-\frac{-\sqrt{(\alpha-1)\theta^*_1\left((\alpha-1)\theta^*_1-8\alpha+2\right)+1}+(\alpha-1)\theta^*_1+1}{2(\alpha-1)}\right),\\
\tilde{p}_{TS} &= \begin{cases} p^*_{TS-a} & \textit{if } \pi^*_{TS-a} > \pi^*_{TS-b} \\ p^*_{TS-b} & \textit{o/w} \end{cases},
\end{aligned}
$$

θ^*_1 *is the unique solution to the equation*

$$
\begin{aligned}
&8(\alpha-1)^2\left(1-\alpha^2\right)\theta_1^3+4(\alpha-1)\left(6\left(1-\alpha^2\right)-19\alpha\left(1-\alpha^2\right)\right)\theta_1^2\\
&+4(\alpha-1)(2(\alpha-2)\alpha(4\alpha-5)-6)\theta_1+4(2-\alpha)(\alpha-1)=0
\end{aligned}
$$

over the interval $[0,1]$.

Proof. It trivially follows that when $\alpha \geq 1$, *TS* defaults to *CE-PL*. If $0<\alpha<1$, in period 1, customer purchase iff $2\alpha\theta \geq p$. Under Targeted Seeding, the firm can choose two different strategies:

- Case 1: The firm sets $0 < p < 2\alpha$, there are both seeded customers and adopters in period 1;

- Case 2: The firm sets $p \geq 2\alpha$, under this case, there are only seeded customers in period

1.

Also, we can further derive that for period 2 adoption to occur, we need $0 < \alpha \leq \frac{1}{2}$. Otherwise, no amount of social learning can compensate for reducing the product life in half. Also, when $\frac{1}{2} \leq \alpha < 1$, there is only period 1 adoption strategy and *TS* defaults to *CE-PL* outcome with no seeding. Therefore, for *TS* to have a chance to be the optimal model, it is necessary that $0 < \alpha < \frac{1}{2}$.

Case 1: $0 < \alpha < \frac{1}{2}$ **and** $0 < p < 2\alpha$

We denote θ_1 and θ_2 as the marginal consumer in period 1 and period 2, respectively. We get: $\theta_1 = \frac{p}{2\alpha}$, and the number of adopters in period 1 is $\theta_2 + 1 - \theta_1$. Based on social learning, non-adopters in period 1 updates their valuation from $a_1 = \alpha$ to $a_2 = (1-\alpha)(\theta_2 + 1 - \theta_1) + \alpha$. In period 2, the marginal customer satisfies $a_2\theta_2 = p$, and it gives us:

$$\theta_2 \left((1-\alpha)(-\theta_1 + \theta_2 + 1) + \alpha\right) = 2\alpha\theta_1.$$

We solve θ_2 as a function of θ_1:

$$\theta_{21} = \frac{-\sqrt{(\alpha-1)\theta_1\left((\alpha-1)\theta_1 - 8\alpha + 2\right) + 1} + (\alpha-1)\theta_1 + 1}{2(\alpha-1)},$$
$$\theta_{22} = \frac{\sqrt{(\alpha-1)\theta_1\left((\alpha-1)\theta_1 - 8\alpha + 2\right) + 1} + (\alpha-1)\theta_1 + 1}{2(\alpha-1)}.$$

It can be shown that $\theta_{21} < 0$ and $\theta_{22} \in [0, \theta_1]$. Therefore:

$$\theta_2 = \frac{-\sqrt{(\alpha-1)\theta_1\left((\alpha-1)\theta_1 - 8\alpha + 2\right) + 1} + (\alpha-1)\theta_1 + 1}{2(\alpha-1)} \leq \theta_1.$$

The firm's profit maximization problem becomes:

$$\begin{aligned}\max_{0<p<2\alpha} \pi_{TS} &= \max_{0<p<2\alpha} (1-\theta_2)\, p \\ &= \max_{\theta_1} 2\alpha\theta_1 \left(1 - \frac{-\sqrt{(\alpha-1)\theta_1\left((\alpha-1)\theta_1 - 8\alpha + 2\right) + 1} + (\alpha-1)\theta_1 + 1}{2(\alpha-1)}\right).\end{aligned}$$

Maximizing by changing p is equivalent to maximizing by changing θ_1 since $p = 2\alpha\theta_1$. Differentiating $\pi(\theta_1)$ we obtain:

$$\frac{\partial \pi(\theta_1)}{\partial \theta_1} = \frac{\frac{(\alpha-1)\alpha\theta_1\left(-2\sqrt{(\alpha-1)\theta_1((\alpha-1)\theta_1-8\alpha+2)+1}+2(\alpha-1)\theta_1-12\alpha+3\right)+\alpha}{\sqrt{(\alpha-1)\theta_1((\alpha-1)\theta_1-8\alpha+2)+1}} + \alpha(2\alpha-3)}{\alpha-1}.$$

To solve the equation $\frac{\partial \pi(\theta_1)}{\partial \theta_1} = 0$, we define $M = \sqrt{(\alpha-1)\theta_1\left((\alpha-1)\theta_1 - 8\alpha + 2\right) + 1}$ and simplify the equation as:

$$(3-2\alpha)M - \left((\alpha-1)\theta_1\left(2(\alpha-1)\theta_1 - 12\alpha - 2M + 3\right) + 1\right) = 0,$$

so we have:

$$2(1-\alpha)(1-\alpha)\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 = M\left(-2(1-\alpha)\theta_1 - 2\alpha + 3\right).$$

It can be shown that $-2(1-\alpha)\theta_1 - 2\alpha + 3 > 0$. Therefore, the interior point solution can exist only if $2(1-\alpha)(1-\alpha)\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 > 0$. Solve $2(1-\alpha)(1-\alpha)\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 > 0$. By solving $2(1-\alpha)^2\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 > 0$. Solve $2(1-\alpha)^2\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 = 0$, we get:

$$\theta_{11} = \frac{12\alpha^2 - (\alpha-1)\sqrt{144\alpha^2 - 72\alpha + 1} - 15\alpha + 3}{2\left(2\alpha^2 - 4\alpha + 2\right)},$$
$$\theta_{12} = \frac{12\alpha^2 + (\alpha-1)\sqrt{144\alpha^2 - 72\alpha + 1} - 15\alpha + 3}{2\left(2\alpha^2 - 4\alpha + 2\right)}.$$

We have three subcases:

- Case 1-i: If $0 < \alpha \leq \frac{1}{12}\left(3 - 2\sqrt{2}\right)$, $144\alpha^2 - 72\alpha + 1 > 0$, there are two real solutions θ_{11} and θ_{12} for $2(1-\alpha)^2\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 = 0$. It trivially follows $\frac{\partial\pi(\theta_1)}{\partial\theta_1} > 0$, $\theta_1^* = 1$. Therefore, there is no adoption in period 1.

- Case 1-ii: If $\frac{1}{12}\left(3 - 2\sqrt{2}\right) < \alpha < \frac{1}{12}\left(2\sqrt{2} + 3\right)$, $144\alpha^2 - 72\alpha + 1 < 0$, there is no real solution for $2(1-\alpha)^2\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 = 0$. Recall that we want to find the solution of:

$$\frac{\frac{(\alpha-1)\alpha\theta_1\left(-2\sqrt{(\alpha-1)\theta_1((\alpha-1)\theta_1-8\alpha+2)+1}+2(\alpha-1)\theta_1-12\alpha+3\right)+\alpha}{\sqrt{(\alpha-1)\theta_1((\alpha-1)\theta_1-8\alpha+2)+1}} + \alpha(2\alpha-3)}{\alpha-1} = 0.$$

  We can simplify the equation as:

$$8(\alpha-1)^2\left(1-\alpha^2\right)\theta_1^3 + 4(\alpha-1)\left(6\left(1-\alpha^2\right) - 19\alpha\left(1-\alpha^2\right)\right)\theta_1^2$$
$$+ 4(\alpha-1)(2(\alpha-2)\alpha(4\alpha-5)-6)\theta_1 + 4(2-\alpha)(\alpha-1) = 0.$$

Denote the L.H.S as $H(\theta_1)$, and it can be shown that $H(0) < 0$. We further derive the FOC of $H(\theta_1)$:

$$\frac{\partial H(\theta_1)}{\partial \theta_1} = -24(\alpha-1)^2\left(\alpha^2-1\right)\theta_1^2 + 4(\alpha-1)\left(38\alpha\left(\alpha^2-1\right) - 12\left(\alpha^2-1\right)\right)\theta_1 + 4(\alpha-1)(2(\alpha-2)\alpha(4\alpha-5)-6).$$

By solving $\frac{\partial H(\theta_1)}{\partial \theta_1} = 0$, we can get:

$$\theta_{13} = \frac{19\alpha^4 - 25\alpha^3 - 13\alpha^2 + 25\alpha - 6}{6\left(\alpha^4 - 2\alpha^3 + 2\alpha - 1\right)} - \frac{\sqrt{409\alpha^8 - 1250\alpha^7 + 815\alpha^6 + 988\alpha^5 - 1417\alpha^4 + 334\alpha^3 + 193\alpha^2 - 72\alpha}}{6\left(\alpha^4 - 2\alpha^3 + 2\alpha - 1\right)},$$

$$\theta_{14} = \frac{19\alpha^4 - 25\alpha^3 - 13\alpha^2 + 25\alpha - 6}{6\left(\alpha^4 - 2\alpha^3 + 2\alpha - 1\right)} + \frac{\sqrt{409\alpha^8 - 1250\alpha^7 + 815\alpha^6 + 988\alpha^5 - 1417\alpha^4 + 334\alpha^3 + 193\alpha^2 - 72\alpha}}{6\left(\alpha^4 - 2\alpha^3 + 2\alpha - 1\right)}.$$

We split case 1-ii into two subcases:

- Case 1-ii-a: If $\frac{1}{12}\left(3-2\sqrt{2}\right) < \alpha < \frac{1}{818}\left(\sqrt{118321}+23\right)$, $409\alpha^8 - 1250\alpha^7 + 815\alpha^6 + 988\alpha^5 - 1417\alpha^4 + 334\alpha^3 + 193\alpha^2 - 72\alpha < 0$, there is no real solution for $\frac{\partial H(\theta_1)}{\partial \theta_1} = 0$. Under case 1-ii-a, it can be shown that $\frac{\partial H(\theta_1)}{\partial \theta_1} > 0$, and we already know $H(0) < 0$. By checking the sign of $H(1)$, we get two subcases:
  * If $\frac{1}{12}\left(3-2\sqrt{2}\right) < \alpha < \frac{1}{5}\left(3-\sqrt{7}\right)$, $H(1) < 0$, $H(\theta_1) < 0$, $\frac{\partial \pi(\theta_1)}{\partial \theta_1} > 0$, $\theta_1^* = 1$, such that there is no adoption in period 1;
  * If $\frac{1}{5}\left(3-\sqrt{7}\right) \leq \alpha < \frac{1}{818}\left(\sqrt{118321}+23\right)$, there exists a unique $\theta_1^* \in (0,1)$

such that $H(\theta_1^*) = 0$. It can be shown that when $\theta_1 \in [0, \theta_1^*)$, $H(\theta_1) < 0$, the profit π is increasing in θ_1; when $\theta_1 \in (\theta_1^*, 1]$, $H(\theta_1) > 0$, the profit π is decreasing in θ_1. Therefore, θ_1^* can get us the optimal profit.

– Case 1-ii-b: If $\frac{1}{818}\left(\sqrt{118321}+23\right) \leq \alpha < \frac{1}{12}\left(2\sqrt{2}+3\right)$, $409\alpha^8 - 1250\alpha^7 + 815\alpha^6 + 988\alpha^5 - 1417\alpha^4 + 334\alpha^3 + 193\alpha^2 - 72\alpha \geq 0$, there are two real solutions θ_{13} and θ_{14} for $\frac{\partial H(\theta_1)}{\partial \theta_1} = 0$. It can be shown that both θ_{13} and θ_{14} are negative. Therefore, $\frac{\partial H(\theta_1)}{\partial \theta_1} > 0$ for all $\theta_1 \in [0, 1]$. It immediately follows $H(\theta_1)$ is increasing in $\theta_1 \in [0, 1]$. And we already know $H(0) < 0$. It can be shown that $H(1) > 0$. Thus, there is a unique $\theta_1^* \in [0, 1]$ such that θ_1^* can get us the optimal profit.

- Case 1-iii: If $\frac{1}{12}\left(2\sqrt{2}+3\right) < \alpha < \frac{1}{2}$, there are two real solutions θ_{11} and θ_{12} for $2(1-\alpha)^2\theta_1^2 - 3(1-\alpha)\theta_1 + 12(1-\alpha)\alpha\theta_1 + 1 = 0$. It can be shown that $\theta_{11} < 0$ and $\theta_{12} < 0$. Following the steps under case 1-2, we can get $H(0) < 0$ and $H(1) > 0$. It can be shown that $H(\theta_1)$ is increasing in θ over the interval $[0, 1]$. Thus, there is a unique $\theta_1^* \in [0, 1]$.

In summary, under case 1, when $0 < \alpha < \frac{1}{5}\left(3-\sqrt{7}\right)$, there is no profit. When $\frac{1}{5}\left(3-\sqrt{7}\right) \leq \alpha < \frac{1}{2}$, there exists a unique $\theta_1^* \in [0, 1]$ such that:

$$8(\alpha-1)^2\left(1-\alpha^2\right)\theta_1^{*3} + 4(\alpha-1)\left(6\left(1-\alpha^2\right) - 19\alpha\left(1-\alpha^2\right)\right)\theta_1^{*2}$$
$$+ 4(\alpha-1)(2(\alpha-2)\alpha(4\alpha-5)-6)\theta_1^* + 4(2-\alpha)(\alpha-1) = 0.$$

Furthermore, when $\theta_1 \in [0, \theta_1^*)$, $H(\theta_1) < 0$, the profit π is increasing in θ_1; when $\theta_1 \in$

$(\theta_1^*, 1]$, $H(\theta_1) > 0$, the profit π is decreasing in θ_1. The optimal profit is:

$$\pi^*_{TS-b} = 2\alpha\theta_1^* \left(1 - \frac{-\sqrt{(\alpha-1)\theta_1^*\left((\alpha-1)\theta_1^* - 8\alpha + 2\right) + 1} + (\alpha-1)\theta_1^* + 1}{2(\alpha-1)}\right).$$

Case 2: $0 < \alpha < \frac{1}{2}$ **and** $p \geq 2\alpha$

Under case 2, the profit maximization problem becomes:

$$\max_{p \geq 2\alpha} \pi = \max_{p \geq 2\alpha} \left(1 - \theta_1\right) p = \max_{\theta_1} \theta_1 \left(1 - \theta_1\right) \left((1-\alpha)\theta_1 + \alpha\right).$$

It trivially follows:

$$\begin{aligned}
\theta_1^* &= \frac{-2\alpha + \sqrt{(\alpha-1)\alpha+1} + 1}{3 - 3\alpha}, \\
p^*_{TS-a} &= \frac{\left(-2\alpha + \sqrt{(\alpha-1)\alpha+1} + 1\right)\left(\alpha + \sqrt{(\alpha-1)\alpha+1} + 1\right)}{9(1-\alpha)}, \\
\pi^*_{TS-a} &= \frac{\left(-2\alpha + \sqrt{(\alpha-1)\alpha+1} + 1\right)\left(\alpha + \sqrt{(\alpha-1)\alpha+1} + 1\right)}{27(1-\alpha)^2} \\
&\quad \times \left(-\alpha - \sqrt{(\alpha-1)\alpha+1} + 2\right).
\end{aligned}$$

□

Proof of Proposition 4

Proof. From Prop. 1, we get *S* is dominated by *TLF*.

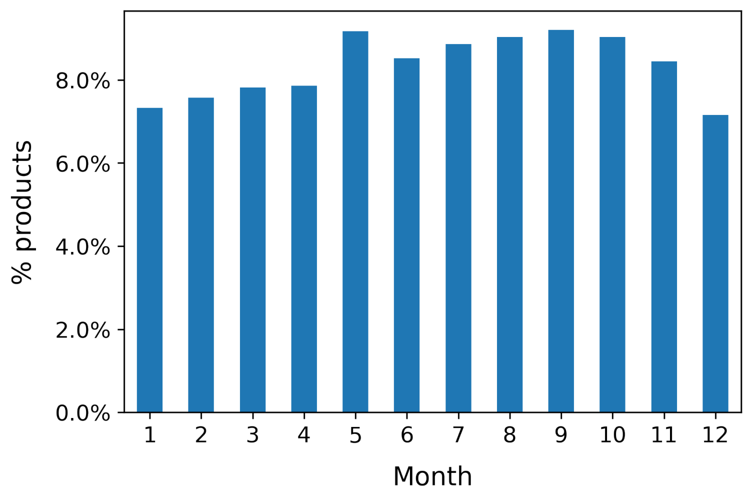
Comparing both case 1 and case 2 with *TLF*, we get when $0 < \alpha < \frac{1}{2}$, $\pi^*_{TS-a} < \frac{1}{4} = \pi^*_{TLF}, \forall\theta_1^* \in (0,1)$ and $\pi^*_{TS-b} < \frac{1}{4} = \pi^*_{TLF}$. *TLF* dominates *TS*. □

APPENDIX F

ROBUSTNESS TESTS OF CHAPTER 3

F.1 Entry Timing

Figure F.1: Distribution of the timing when Amazon takes over as the seller of the product

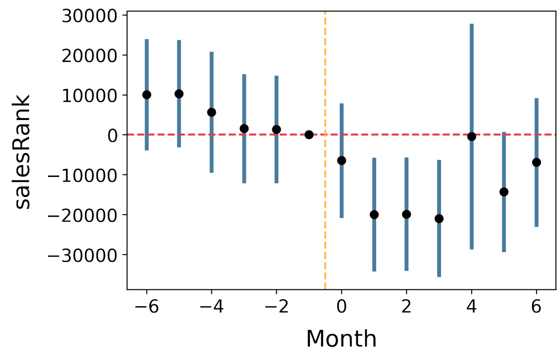


Notes: The figure shows the distribution of the timing when Amazon takes over as the seller of the product. There does not appear to be a discernible pattern in the timing of Amazon becoming the seller.

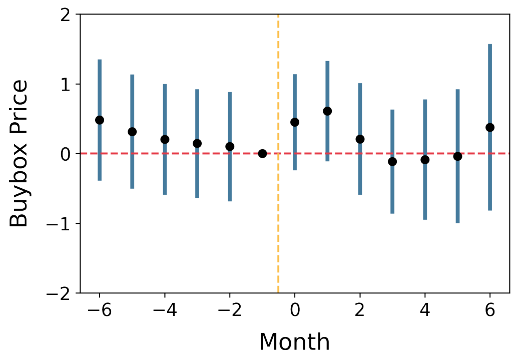
F.2 Cohort Analysis

Figure F.2: Impacts on Product Characteristics when Amazon Becomes its Seller Cohort Analysis (April)

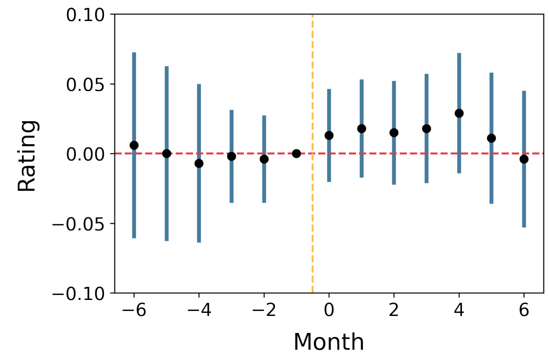
(a) Sales Rank



(b) Buybox Price



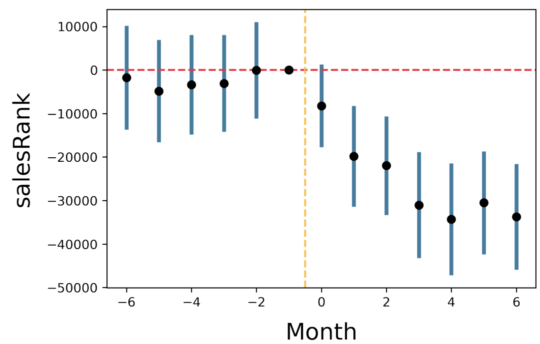
(c) Rating



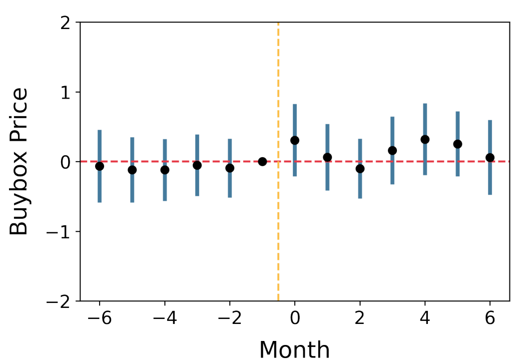
Notes: This figure shows the event study (cohort analysis) of when Amazon becomes the seller of existing products in April. Each point is an estimate of effect β_m in m-th month. We use one month before Amazon becomes the seller ($m = -1$) as the benchmark. 95% confidence intervals constructed using standard errors clustered at the product level are also displayed. The results are consistent with our main analysis.

Figure F.3: Impacts on Product Characteristics when Amazon Becomes its Seller Cohort Analysis (August)

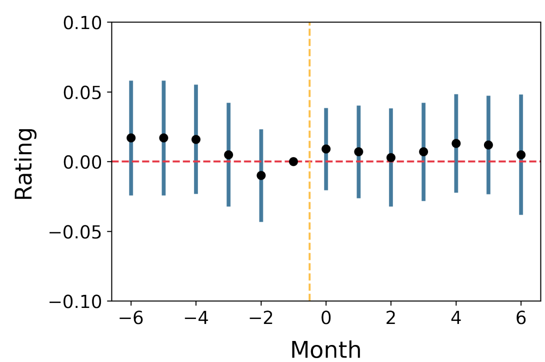
(a) Sales Rank



(b) Buybox Price



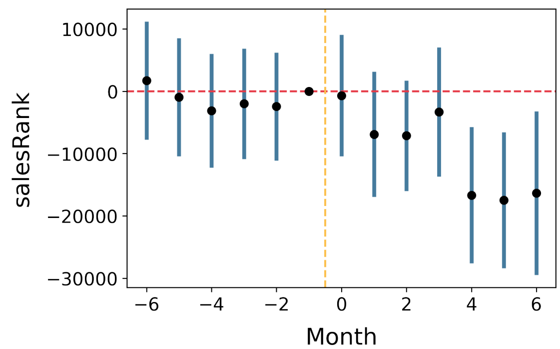
(c) Rating



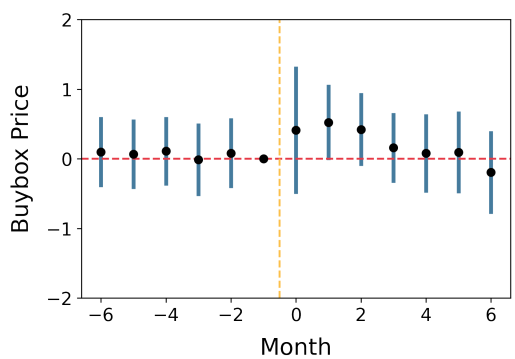
Notes: This figure shows the event study (cohort analysis) of when Amazon becomes the seller of existing products in August. Each point is an estimate of effect β_m in m-th month. We use one month before Amazon becomes the seller ($m = -1$) as the benchmark. 95% confidence intervals constructed using standard errors clustered at the product level are also displayed. The results are consistent with our main analysis.

Figure F.4: Impacts on Product Characteristics when Amazon Becomes its Seller Cohort Analysis (December)

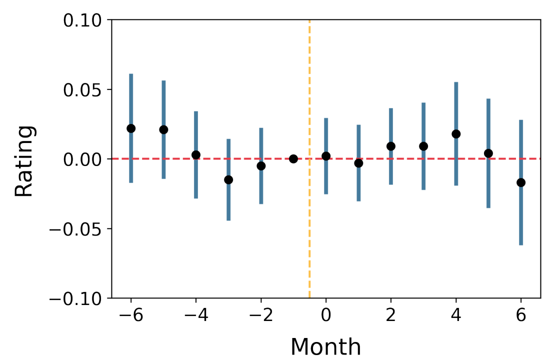
(a) Sales Rank



(b) Buybox Price



(c) Rating



Notes: This figure shows the event study (cohort analysis) of when Amazon becomes the seller of existing products in December. Each point is an estimate of effect β_m in m-th month. We use one month before Amazon becomes the seller ($m = -1$) as the benchmark. 95% confidence intervals constructed using standard errors clustered at the product level are also displayed. The results are consistent with our main analysis.

F.3 Selection on Unobservables

Table F.1: Products characteristics for control and treatment groups (within treated)

# Treated 4898	(1)			(2)			(3)		
# Control 5180	Buybox Price			Rating			Sales Rank		
Month	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*
-6	36.07	31.53	6.37***	4.31	4.31	0.07	254754.59	203804.56	7.29***
-5	36.07	31.53	6.38***	4.31	4.32	-0.02	256339.05	200712.30	8.03***
-4	36.04	31.52	6.34***	4.31	4.32	-1.24	253404.11	197692.61	7.98***
-3	36.00	31.53	6.29***	4.30	4.32	-1.75*	249551.05	193968.45	7.91***
-2	35.97	31.54	6.24***	4.31	4.32	-1.71*	246673.48	189879.50	8.13***
-1	35.98	31.54	6.26***	4.31	4.32	-1.67*	238754.23	185594.29	7.88***

Notes: This table shows that products in treatment and control groups are significantly different on a set of key characteristics prior to Amazon introducing PL.

Table F.2: Matched products characteristics in control and treatment groups (within treated)

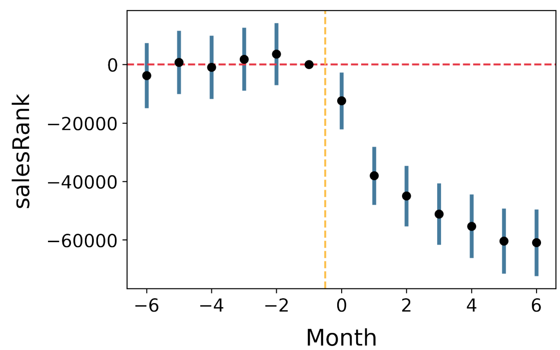
# Treated 3549	(1)			(2)			(3)		
# Control 2253	Buybox Price			Rating			Sales Rank		
Month	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*	Treat	Control	*t-stat.*
-6	25.63	25.23	0.61	4.32	4.33	-0.92	86685.58	81095.48	1.63
-5	25.64	25.19	0.68	4.32	4.34	-1.43	85668.05	80708.06	1.43
-4	25.61	25.20	0.63	4.31	4.34	-1.82	84385.14	80661.06	1.06
-3	25.62	25.25	0.56	4.31	4.34	-2.08	83712.70	79951.04	1.06
-2	25.62	25.27	0.53	4.31	4.34	-2.03	84870.13	79709.48	1.45
-1	25.67	25.30	0.56	4.32	4.34	-1.95	85383.27	79861.20	1.52

Notes: This table shows that after matching, the differences between products in control and treatment groups are insignificant on a set of key characteristics prior to Amazon introducing PL.

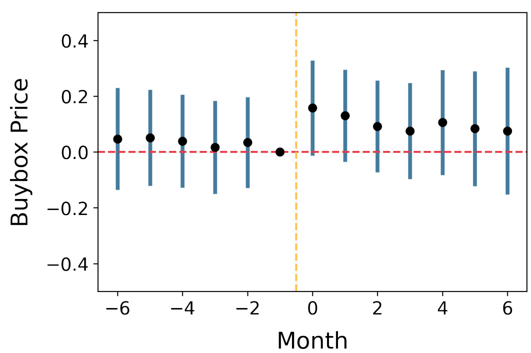
F.4 FBA

Figure F.5: Impacts on Product Sales Rank after Amazon Becomes its Seller (FBA)

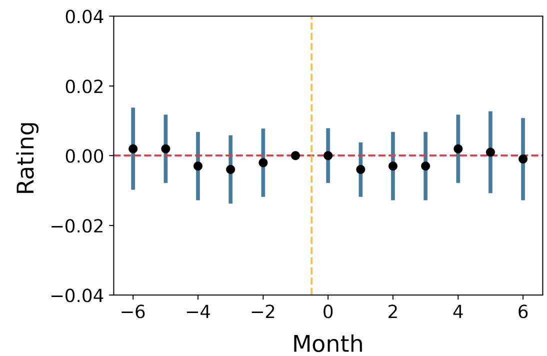
(a) Sales Rank



(b) Buybox Price



(c) Rating



Notes: This figure shows the event study of when Amazon becomes the seller of existing products that are already fulfilled by Amazon. Therefore, there is no change in fulfillment methods before and after Amazon becomes the seller of existing products. Each point is an estimate of effect β_m in m-th month. We use one month before Amazon becomes the seller ($m = -1$) as the benchmark. 95% confidence intervals constructed using standard errors clustered at the product level are also displayed. The results are consistent with our main analysis.

APPENDIX G

MATCHING PROCESS OF CHAPTER 3

We employ matching methods as follows. First, We keep only the treated categories that can be observed for at least six months before and after Amazon introduces its private-label product. This results in 312 categories in the treatment group. For each treated category, we then identify all control categories that can be observed at least six months before and after the treated category's Amazon entry date. This ensures that the matched pairs have the same observational window.

Second, We impose a restriction that control categories must belong to the same root category (level 1) as the treated category. This ensures that the control categories are similar enough to serve as a proxy for the counterfactual of the treated categories.

Third, We compute the Scaled Euclidean distance between each treated category and eligible control category in terms of category average price, average sales rank, average ratings, and average number of sellers over the six months prior to Amazon introducing its own product.

Finally, we use the one-nearest-neighbor (with replacement) algorithm to match each treated category to its closest control counterpart. We also impose a caliper that sets an absolute maximum on the Euclidean distance to avoid poor matches.

APPENDIX H

VECTOR REPRESENTATION OF IMAGES AND COSINE SIMILARITY ANALYSIS OF CHAPTER 3

The key process is to calculate the similarity between images as the proxy for the difference in product design.

We first extract image features using a deep learning model GoogLeNet Inception v3 (Szegedy et al. 2016). It is a widely used neural network for image classification and was originally trained on ImageNet dataset which contains more than 14 million annotated images with more than 1000 classes. In our dataset, we collect 123,878 images from 19,155 products. We adopt the transfer learning approach (Xia et al. 2017, Wang et al. 2019, Jignesh Chowdary et al. 2020) to use Inception v3 architecture to extract image features on the top layer of the neural network. The final output of the Inception V3 architecture is the score for images classes. However, for our analysis, we do not need the score to determine the class of images since we have category that can accurately determine the image class. Instead, we keep the top layer in the neural network since it is the most informative vector representation of the image. This process eventually transforms each image into a 2048-dimensional vector.

We then use the vectors to calculate the cosine similarity between images. The benefit of using cosine similarity is it measures the angle between two vectors rather than their magnitudes. This means that the metric is not sensitive to the scale of the vectors.

On Amazon, each product can have multiple images. To calculate the cosine similarity between products, we first calculate the cosine similarity matrix between each image of a third-party product and those of the Amazon's product in the same category. Specifically, for each Amazon product image m_j and a third-party product image m_p, the pairwise cosine similarity is computed as:

$$CS_{j,p} = \frac{m_j \cdot m_p'}{|m_j||m_p|}. \tag{H.1}$$

The cosine similarity score measures the angle between the vectors of two images and can range between 0 and 1. A score of 0 indicates that the two images have nothing in common, while a larger score suggests that the two images are more similar.

We use the largest similarity as a proxy for the similarity between this third-party product's image and Amazon product's images. That is, we find the closest images between Amazon's and the third-party's product. The purpose of this step is, the images of one product can be taken from different angles, different angles might lead to small cosine similarity even if two products are pretty close in the design. To address this concern, we only keep the max cosine similarity between a third-party image and a group of Amazon's images.

Then we calculate the average cosine similarity between all images of the third-party product and Amazon's products in the same category and use it as the cosine similarity between the two types of products.

APPENDIX I

VECTOR REPRESENTATION OF TEXT DESCRIPTIONS AND COSINE SIMILARITY ANALYSIS OF CHAPTER 3

We first create vector representations of the product description text using a method called "Term frequency-inverse document frequency" (TF-IDF) (Robertson 2004, Wang et al. 2019, Burtch et al. 2022). TF-IDF calculates statistics that represent the importance of a word or phrase within a document in a corpus. With this approach, the raw frequency of words is weighted by the uniqueness of the word. Specifically, if a word is highly unique and only appears in one document, in our case, in a single product's description, it maintains its raw frequency. However, if a word is commonly used across all products, its frequency decreases to a smaller value.

We use cosine similarity to measure the similarity in text descriptions between existing third-party products and newly added products following previous literature (Allan et al. 2003, Gentzkow et al. 2019, Kelly et al. 2021). The benefit of using cosine similarity is it measures the angle between two vectors rather than their magnitudes. This means that the metric is not sensitive to the scale of the vectors, making it suitable for comparing documents of different lengths or with different frequencies of terms.

We then compile all product descriptions into a matrix M_{jw}, where each row represents a product, and each column represents the total number of times word w occurs in that product. Next, we calculate the pairwise cosine similarity scores between Amazon's products and third-party products. Specifically, for each Amazon product m_j and a third-party product m_p, the pairwise cosine similarity is computed as:

$$CS_{j,p} = \frac{m_j \cdot m_p'}{|m_j||m_p|}. \tag{I.1}$$

The cosine similarity score measures the angle between the vectors of two products and can range between 0 and 1. A score of 0 indicates that the two products' descriptions have nothing in common, while a larger score suggests that the two products are more similar.

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