APPROXIMATION AND REPRESENTATION THEOREMS IN L_p SPACES

A THESIS

Presented to

The Faculty of the Graduate Division

by

George Monroe Groome, Jr.

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Applied Mathematics

Georgia Institute of Technology

June, 1964

APPROXIMATION AND REPRESENTATION THEOREMS

IN L_p SPACES

Approved:

Date approved by Chairman: May 19, 1964

ACKNOWLEDGMENTS

I wish to thank my thesis advisor, Dr. E. R. Immel, not only for suggesting this topic but also for the guidance he has given me during its development. I am indebted to Dr. J. M. Osborn and Dr. H. A. Gersch for their reading of the manuscript and for their helpful suggestions in improving it.

TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

In applications of the theory of Lebesgue integration to the study of integral transforms, ordinary and partial differential equations, and integral equations, the need frequently arises to approximate functions in the $\mathcal{L}_p$ spaces by functions possessing certain regularity properties. In particular it is often desirable to approximate functions in the $\mathcal{L}_p$ spaces by step-functions, by continuous functions of compact support, and by differentiable functions. The most useful type of approximation is an approximation in the sense of the norm of $\mathcal{L}_p$. Thus one may be interested in showing that a family of "well-behaved" functions is dense in the space $\mathcal{L}_p$. Isolated occurrences of theorems of this general nature are to be found in the literature. However, most of these results deal only with the space $\mathcal{L}_1(E_1, \Lambda, \mu)$ where E_1 is one-dimensional Euclidean space and μ is Lebesgue measure. The object of Chapters II and III is to prove, under more general hypotheses on the measure space (X, Λ, μ), that various classes of functions are dense in $\mathcal{L}_p(X, \Lambda, \mu)$ for $1 \leq p < \infty$.

In Chapter II it is shown that the class of simple functions is dense in $\mathcal{L}_1(X, \Lambda, \mu)$ for an arbitrary measure space (X, Λ, μ). The measure space is then specialized to (E_q, Λ, μ) where μ is a Lebesgue-Stieltjes measure in E_q. The concept of Lebesgue-Stieltjes measure is defined and the measure-theoretic results necessary to the understanding of Chapters II and III are presented. It is then shown that the family of continuous functions of compact support, the family of step-functions, and the family of polynomials of compact support are dense in $\mathscr{L}_1$ (E_q, Λ, μ).

The theorems of Chapter II are extended to $\mathscr{L}_p$ spaces for $1 \leq p < \infty$ in Chapter III. The separability of the metric spaces $\mathscr{L}_p$ (E_q, Λ, μ) for $1 \leq p < \infty$ and μ a Lebesgue-Stieltjes measure in E_q is then deduced. Finally, the family of functions which are in C^∞ and are of compact support is proved to be dense in $\mathscr{L}_p$ (E_q, Λ, μ) for $1 \leq p < \infty$ and μ Lebesgue measure in E_q.

The object of Chapter IV is to deduce an integral representation for all bounded linear functionals on $\mathscr{L}_p$ (X, Λ, μ) for $1 \leq p < \infty$. Most of the theorems giving representations for bounded linear functionals on Banach or Hilbert spaces are due to F. Riesz, and theorems of this type are referred to as Riesz representation theorems. In order to deduce the desired theorems, certain results from measure theory are needed. The Radon-Nikodym theorem is proved through the use of the Riesz representation theorem for bounded linear functionals on the Hilbert space $\mathscr{L}_2$ (X, Λ, μ). This proof is particularly interesting in the context of Chapter IV since the Radon-Nikodym theorem is the key to the proof of the Riesz representation theorem for $\mathscr{L}_p$ (X, Λ, μ). The desired integral representation for bounded linear functionals is then proved under mild restrictions on the measure space (X, Λ, μ). As a corollary to the Riesz representation theorem, it is shown that the Banach space of all bounded linear functionals on $\mathscr{L}_p$ (X, Λ, μ) for $1 < p < \infty$ is isometrically isomorphic to the space $\mathscr{L}_q(X, \Lambda, \mu)$ where $\frac{1}{p} + \frac{1}{q} = 1$.

CHAPTER II

APPROXIMATIONS OF SUMMABLE FUNCTIONS

In this chapter certain measure-theoretic concepts are briefly discussed. In particular the notion of Lebesgue-Stieltjes measure in E_q (q-dimensional Euclidean space) is introduced. It is then shown that summable functions on arbitrary measure spaces can be approximated, in a certain average sense, by simple functions. It is further shown that Lebesgue-Stieltjes summable functions can be approximated, in this average sense, by continuous functions, step-functions, and polynomials.

Definition 2.1. If X is a nonempty set, Λ is a σ-algebra of sets of X, and μ is a measure with domain Λ, the triple (X, Λ, μ) is called a measure space.

Remark. In this and the succeeding chapters, it will always be assumed, unless the contrary is specifically stated, that Λ is complete for the measure μ. However, no further restrictions on (X, Λ, μ) are tacitly assumed. In particular (X, Λ, μ) is not generally assumed to be σ-finite.

In order to obtain the desired approximation theorems, the measure space must be specialized.

Definition 2.2. Let $\mathcal{B}$ denote the Borel σ-algebra of sets in E_q (the minimal σ-algebra containing the family of all open sets). A Borel measure in E_q is a measure $\bar{\mu}$ defined on $\mathcal{B}$ and such that $\bar{\mu}(K) < \infty$

for every compact set $K \subset E_q$.

Let Λ be the family defined as follows:

A set $A \subset E_q$ is in Λ if and only if there are sets E, M, and N such that $A = E \cup N$, $E, M \in \mathcal{B}$, $N \subset M$, and $\bar{\mu}(M) = 0$.

For each $A = E \cup N \in \Lambda$ with E and N as above, define the set function μ by

$$\mu(A) = \bar{\mu}(E) .$$

Theorem 2.3. The family Λ is a σ-algebra of sets in E_q, μ is a measure on Λ, and Λ is complete for μ.

A proof may be found in Halmos (cf. [4], p. 55).

Remark. The measure μ is called the completion of $\bar{\mu}$.

Definition 2.4. If a measure μ is the completion of a Borel measure $\bar{\mu}$ in E_q, and if Λ is the family described above, then the measure μ with domain Λ is called a Lebesgue-Stieltjes measure in E_q.

It is clear that Lebesgue measure in E_q is a Lebesgue-Stieltjes measure. Several important properties of Lebesgue measure are also shared by the more general Lebesgue-Stieltjes measures. One such property is regularity, which is now discussed.

Definition 2.5. Let $\mathcal{C}$ denote the family of all compact subsets of E_q and let $\mathcal{U}$ denote the family of all open sets in E_q. Let (E_q, Λ, μ)

be a (not necessarily complete) measure space.

(i) A set $A \in \Lambda$ is <u>outer regular</u> with respect to μ if

$$\mu(A) = \inf\{\mu(U) : A \subset U \in \mathcal{U}\} .$$

(ii) A set $A \in \Lambda$ is <u>inner regular</u> with respect to μ if

$$\mu(A) = \sup\{\mu(C) : A \supset C \in \mathcal{C}\} .$$

A set $A \in \Lambda$ is <u>regular</u> if it is both inner and outer regular. A measure μ is <u>regular</u> if every set $A \in \Lambda$ is regular.

<u>Theorem 2.6.</u> Every Borel measure $\bar{\mu}$ in E_q is regular.

The proof may be found in Halmos (cf. [4], p. 228).

<u>Corollary 2.7.</u> A Lebesgue-Stieltjes measure in E_q is regular.

<u>Proof</u>. Let μ be a Lebesgue-Stieltjes measure in E_q with domain Λ and let $A \in \Lambda$. By definition of Λ there are sets E, M, and N with $E, M \in \mathcal{B}$ such that $A = E \cup N$, $N \subset M$, $\mu(A) = \bar{\mu}(E)$, and $\mu(N) = \bar{\mu}(M) = 0$. If $\mu(A) < \infty$, Theorem 2.6 implies that for any $\varepsilon > 0$ there exist sets U_1 and U_2 in $\mathcal{U}$ such that $U_1 \supset E$, $U_2 \supset M \supset N$,

$$\mu(A) = \bar{\mu}(E) > \bar{\mu}(U_1) - \frac{\varepsilon}{2} = \mu(U_1) - \frac{\varepsilon}{2} ,$$

and

$$0 = \bar{\mu}(M) > \bar{\mu}(U_2) - \frac{\varepsilon}{2} = \mu(U_2) - \frac{\varepsilon}{2} .$$

Thus $$A = E \cup N \subset U_1 \cup U_2 \in \mathcal{U} \text{ and}$$

$$\mu(A) > \mu(U_1) + \mu(U_2) - \varepsilon \geq \mu(U_1 \cup U_2) - \varepsilon .$$

Hence, if $\mu(A) < \infty$, then

$$\mu(A) = \inf \left\{\mu(U) : A \subset U \varepsilon \mathcal{U}\right\}.$$

If $\mu(A) = \infty$, outer regularity of A follows at once from the fact that $E_q \varepsilon \mathcal{U}$.

For inner regularity let A, E, and N be as before. Suppose first that $\mu(A) < \infty$. Let $\varepsilon > 0$ be given. By Theorem 2.6 there is a set $C \varepsilon \mathcal{C}$ such that $C \subset E$ and

$$\bar{\mu}(C) > \bar{\mu}(E) - \varepsilon .$$

But $A \supset E \supset C$ and

$$\mu(C) = \bar{\mu}(C) > \bar{\mu}(E) - \varepsilon = \mu(A) - \varepsilon .$$

Thus, if $\mu(A) < \infty$,

$$\mu(A) = \sup \left\{\mu(C) : A \supset C \varepsilon \mathcal{C}\right\}.$$

If $\mu(A) = \bar{\mu}(E) = +\infty$, then let $M > 0$ be given. By the regularity of $\bar{\mu}$, there is a set $C \varepsilon \mathcal{C}$ such that $C \subset E \subset A$ and

$$\bar{\mu}(C) = \mu(C) > M .$$

Thus

$$\mu(A) = \sup \left\{\mu(C) : A \supset C \varepsilon \mathcal{C}\right\} = +\infty$$

and μ is inner regular. ∎

At this stage it is thus known that if μ is a Lebesgue-Stieltjes measure in E_q with domain Λ, then:

(i) Λ is a σ-algebra containing all Borel sets.

(ii) Λ is complete for μ.

(iii) For each compact set $C \subset E_q$, $\mu(C) < \infty$.

(iv) μ is totally σ-finite.

(v) μ is a regular measure.

In this and the succeeding chapter, most of the results concerning approximations of summable functions will be proved for functions summable relative to a Lebesgue-Stieltjes measure in E_q. In the proofs of these theorems, essential use will be made of properties (i) - (v). However, aside from these, no other results from the theory of Lebesgue-Stieltjes measure will be used. For this reason nothing essential to the later development would be lost if one assumed that properties (i) - (v) were the defining properties of a Lebesgue-Stieltjes measure in E_q. The earlier discussion which has been outlined could then be ignored.

Other approaches to Lebesgue-Stieltjes measures appear in the literature.

The most common approaches are through interval functions [13] or through distribution functions [10], [11]. A discussion of the logical interrelations between the various approaches will not be attempted here. For a discussion of this subject, reference may be made to Morgan (cf. [9]).

<u>Definition 2.8</u>. Let (X, Λ, μ) be a measure space. A function f is a <u>measurable simple function</u> on X if and only if there are pairwise disjoint measurable sets $A_1, \ldots, A_n$ and distinct complex numbers $a_1, \ldots, a_n$ such that, for each $x \in X$,

$$f(x) = \sum_{i=1}^{n} a_i K_{A_i}(x)$$

where

$$K_{A_i}(x) = \begin{cases} 1 & \text{if} \quad x \in A_i, \\ 0 & \text{if} \quad x \notin A_i. \end{cases}$$

Theorem 2.9. Let (X, Λ, μ) be an arbitrary measure space, and let f be an extended-real-valued function defined on X (i.e., $f : X \longrightarrow E_1^*$) which is summable over X. For any given $\varepsilon > 0$ there exists a measurable simple function $g : X \longrightarrow E_1$ such that

$$\int_X |f - g| \, d\mu < \varepsilon .$$

Moreover, g is summable over X, and

$$\mu \left\{ x : g(x) \neq 0 \right\} < \infty .$$

Proof. Let $\varepsilon > 0$ be given. Define the functions f^+ and f^- as follows: for $x \in X$

$$f^+(x) = \begin{cases} f(x) & \text{if} \quad f(x) \geq 0 \\ 0 & \text{if} \quad f(x) < 0 \end{cases} ;$$

$$f^-(x) = \begin{cases} 0 & \text{if} \quad f(x) \geq 0 \\ -f(x) & \text{if} \quad f(x) < 0 \end{cases} .$$

Then $f = f^+ - f^-$. By a well-known property of measurable functions (cf. Theorem 1, Appendix), there exist sequences $\left\{g_n^+\right\}$ and $\left\{g_n^-\right\}$ of nonnegative measurable simple functions on X such that

$$g_1^+ \leq g_2^+ \leq \cdots \leq g_n^+ \leq \cdots \leq f^+ ,$$

$$g_1^- \leq g_2^- \leq \cdots \leq g_n^- \leq \cdots \leq f^- ,$$

$$\lim_{n \to \infty} g_n^+(x) = f^+(x) \quad \text{for each} \quad x \in X ,$$

and

$$\lim_{n \to \infty} g_n^-(x) = f^-(x) \quad \text{for each} \quad x \in X .$$

By the Lebesgue dominated convergence theorem, since $|f^+ - g_n^+| \leq |f^+|$ and $|f^- - g_n^-| \leq |f^-|$,

$$\lim_{n \to \infty} \int_X |f^+ - g_n^+| \, d\mu = 0$$

and

$$\lim_{n \to \infty} \int_X |f^- - g_n^-| \, d\mu = 0 .$$

Choose integers N_1 and N_2 such that

$$\int_X |f^+ - g_{N_1}^+| \, d\mu < \frac{\varepsilon}{2}$$

and

$$\int_X |f^- - g_{N_2}^-| \, d\mu < \frac{\varepsilon}{2} .$$

Then $g = g_{N_1}^+ - g_{N_2}^-$ is a measurable simple function on X and

$$\int_X |f - g|d\mu \leq \int_X |f^+ - g_{N_1}^+| \, d\mu + \int_X |f^- - g_{N_2}^-| \, d\mu$$

$$< \varepsilon \ .$$

Since $g_{N_1}^+$ and $g_{N_2}^-$ are summable over X (by comparison with f^+ and f^- respectively), g is summable over X. Now let $A_1, A_2, \ldots, A_n$ be pairwise disjoint measurable sets such that

$$g = \sum_{i=1}^{n} a_i K_{A_i}$$

where $a_i \neq 0$ for any i. If $\mu\left\{x : g(x) \neq 0\right\} = \mu\left[\bigcup_{i=1}^{n} A_i\right] = \sum_{i=1}^{n} \mu(A_i) = \infty$, then $\mu(A_i) = \infty$ for some i. But, it then follows that

$$\int_X |g| \, d\mu \geq \int_{A_i} |a_i| \, d\mu = |a_i| \, \mu(A_i) = \infty \ ,$$

and g is not summable, a contradiction. ∎

In the case of an arbitrary measure space, little more can be said about approximations. Thus, in the remainder of this chapter, attention will be restricted to a Lebesgue-Stieltjes measure in E_q as defined earlier in this chapter.

<u>Definition 2.10</u>. Let $f : E_q \to E_1^*$ be given. The function f has <u>compact support</u> if and only if there exists a compact set $K \subset E_q$ such that

$$f(x) = 0 \quad \text{for every} \quad x \in E_q - K \ .$$

The set K is called a <u>support</u> for f.

Theorem 2.11. Let μ be a Lebesgue-Stieltjes measure in E_q and let $f : E_q \to E_1^*$ be μ-summable over E_q. For any $\varepsilon > 0$ there exists a function $g : E_q \to E_1$ which is continuous on E_q, which has compact support, and which is such that

$$\int_{E_q} |f - g| \, d\mu < \varepsilon .$$

Proof. The argument is in three parts.

Case 1. Suppose that $f = K_A$, where A is a bounded measurable set. Let I be a compact interval such that $I \supset A$ and let $\varepsilon > 0$ be given. By the regularity of μ, there exists an open set $G \supset A$ such that

$$\mu(G) < \mu(A) + \frac{\varepsilon}{2} .$$

It may be supposed that $G \subset I^o$ (the interior of I) since otherwise G could be replaced with $G \cap I^o$. Again by regularity there exists a closed set $F \subset A$ such that

$$\mu(F) > \mu(A) - \frac{\varepsilon}{2} .$$

Thus

$$\mu(G - F) = \mu(G) - \mu(F) < \varepsilon .$$

For each $x \in E_q$, let

$$h(x) = \operatorname{dist}(x, G^c) = \inf \left\{ |x - y| : y \in G^c \right\}$$

where $G^c = E_q - G$. If $x \in G^c$, then $h(x) = 0$. Moreover, if x is such that $h(x) = 0$, then for any $\delta > 0$, there exists $y \in G^c$ such that $|y - x| < \delta$. But then x is a limit point of the closed set G^c; that is, $x \varepsilon G^c$. Thus $h(x) = 0$ if and only if $x \varepsilon G^c$. Now let n be a positive integer. By definition of infimum, if $x \varepsilon I$, there exists $z \varepsilon G^c$ such that

$$|x - z| < h(x) + \frac{1}{n} .$$

It $y \varepsilon I$, then

$$h(y) \leq |y - z| \leq |y - x| + |x - z| < |y - x| + h(x) + \frac{1}{n} .$$

Thus

$$h(y) - h(x) < |y - x| + \frac{1}{n} .$$

If x and y are interchanged in this argument, it follows that

$$h(x) - h(y) < |y - x| + \frac{1}{n} .$$

Since this is true for each positive integer n,

$$|h(x) - h(y)| \leq |y - x| \quad \text{for all } x, y \varepsilon I .$$

Thus h is continuous on I. But h vanishes outside $G \subset I^o$, so that $h(x) = 0$ for x in $E_q - I$ or x in the boundary of I. Hence h is continuous on all of E_q.

The set F found above is compact, and therefore h assumes its minimum value λ on F. Let $x^* \varepsilon F$ be such that $h(x^*) = \lambda = \min\{h(x) : x \varepsilon F\}$. Since $x^* \notin G^c$, $h(x^*) = \lambda > 0$. For each $x \varepsilon E_q$, define

$$\tilde{h}(x) = \frac{1}{\lambda} \min \{ \lambda, h(x) \} .$$

Note that $\tilde{h}(x) = 1$ if $x \varepsilon F$ and $\tilde{h}(x) = 0$ if $x \varepsilon G^c$. As the minimum of two continuous functions, $\tilde{h}$ is continuous on E_q and is hence measurable. Since h is nonnegative, $0 \leq \tilde{h}(x) \leq 1$ for every $x \varepsilon E_q$. Thus

$$\int_{E_q} |\tilde{h} - K_A| d\mu = \int_F |\tilde{h} - K_A| d\mu + \int_{E_q - G} |\tilde{h} - K_A| d\mu + \int_{G-F} |\tilde{h} - K_A| d\mu$$

$$= \int_{G-F} |\tilde{h} - K_A| d\mu \leq \int_{G-F} 1 \, d\mu = \mu(G - F) < \varepsilon .$$

Since $\tilde{h}$ has compact support I, the proof for Case 1 is complete.

Case 2. Suppose that f is a finite-valued measurable simple function vanishing outside a compact interval I. Thus let $A_1, \ldots, A_n$ be pairwise disjoint measurable sets such that $\bigcup_{j=1}^{n} A_j \subset I$ and let

$$f = \sum_{j=1}^{n} a_j K_{A_j} .$$

Let $\varepsilon > 0$ be given. The argument of Case 1 applies to K_{A_j} for each j. Thus, for each j, let h_j be a function, continuous on E_q and vanishing outside I, such that

$$\int_{E_q} |h_j - K_{A_j}| \, d\mu < \frac{\varepsilon}{(|a_j| + 1)\, n} .$$

Therefore,

$$\int_{E_q} \left| \sum_{j=1}^{n} a_j h_j - f \right| d\mu \leq \int_{E_q} \sum_{j=1}^{n} |a_j| \left| h_j - K_{A_j} \right| d\mu$$

$$\leq \sum_{j=1}^{n} |a_j| \frac{\varepsilon}{(|a_j| + 1)\, n} < \varepsilon .$$

Moreover, $h = \sum_{j=1}^{n} a_j h_j$ is continuous on E_q and has compact support I.

Case 3. Let f be summable over E_q. Since f is almost everywhere finite, there is no loss of generality in assuming that f is finite-valued. For $n = 1,2,\ldots,$ let

$$W_n = \left\{ x : |x_i| \leq n, \quad i = 1,2,\ldots, q \right\} .$$

Define the sequence $\left\{ f_n \right\}$ as follows: for $n = 1,2,\ldots$ and $x \in E_q$, let

$$f_n(x) = f(x)\, K_{W_n}(x) .$$

Then

$$\lim_{n\to\infty} f_n(x) = f(x)$$

and

$$|f_n(x)| \leq |f(x)|$$

for each $x \in E_q$. By the Lebesgue dominated convergence theorem,

$$\lim_{n\to\infty} \int_{E_q} |f - f_n| \, d\mu = 0 .$$

Let $\varepsilon > 0$ be given and choose N such that

$$\int_{E_q} |f - f_N| \, d\mu < \frac{\varepsilon}{3} .$$

For this fixed N, $f_N(x) = 0$ for $x \in E_q - W_N$ where W_N is a closed interval. By Theorem 2.9 there is a finite-valued measurable simple function h such that

$$\int_{E_q} |f_N - h| \, d\mu < \frac{\varepsilon}{3}$$

(clearly h may be assumed to vanish outside W_N since $f_N(x) = 0$ for $x \in E_q - W_N$).

By Case 2 a function g continuous on E_q with compact support I can be chosen so that

$$\int_{E_q} |h - g| \, d\mu < \frac{\varepsilon}{3} .$$

Thus

$$\int_{E_q} |f - g| \, d\mu \leq \int_{E_q} |f - f_N| \, d\mu + \int_{E_q} |f_N - h| \, d\mu$$

$$+ \int_{E_q} |h - g| \, d\mu$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon .$$

The function g is therefore of the desired type. ∎

Other approximation theorems can now be deduced from Theorem 2.11. Certain preliminary results are necessary.

<u>Definition 2.12</u>. A complex-valued function f defined on E_q is a <u>step-function</u> if and only if there are finitely many pairwise disjoint finite intervals $I_1, \ldots, I_n$ in E_q and complex numbers $a_1, \ldots, a_n$ such that

$$f = \sum_{i=1}^{n} a_i K_{I_i} .$$

Theorem 2.13. Let J be a finite closed interval in E_q. There is a countable collection $\mathcal{S}$ of step-functions, defined on E_q and vanishing outside J, with the following property. For any continuous complex-valued function f on J and any $\varepsilon > 0$, there exists a function $g \in \mathcal{S}$ such that

$$|f(x) - g(x)| < \varepsilon$$

for each $x \in J$.

Proof. Let J be a compact interval in E_q. Consider the countable family $\Pi = \{P_1, \ldots, P_n, \ldots\}$ of partitions of J where P_n divides J into 2^{qn} equal subintervals $J_{n1}, J_{n2}, \ldots, J_{n2^{qn}}$. Let $\mathcal{S}_n (n = 1,2,\ldots)$ be the family of all step-functions defined on E_q which vanish on $E_q - J$ and which have constant rational real and imaginary parts on each of the sets $J_{n1}, J_{n2}, \ldots, J_{n2^{qn}}$. Each $\mathcal{S}_n$ contains only a countable number of functions. If $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$, then $\mathcal{S}$ is also a countable collection of step-functions which vanish on $E_q - J$.

Let $\varepsilon > 0$ be given, and let f be any complex-valued function continuous on J. Then f is uniformly continuous on J, so there exists a $\delta > 0$ such that

$$|f(x) - f(y)| < \frac{\varepsilon}{2}$$

if $x, y \in J$ and $|x - y| < \delta$. From Π select a partition P_n for which

$$\sup \left\{ |x - y| : x, y \in J_{nk}, \quad k = 1, \ldots, 2^{qn} \right\} < \delta.$$

From each of the intervals $J_{nk} (k = 1, \ldots, 2^{qn})$ select a point x_k. Define the step-function h by

$$h(x) = \begin{cases} f(x_k) & \text{if } x \in J_{nk} \ (k = 1, \ldots, 2^{qn}) , \\ 0 & \text{if } x \notin J . \end{cases}$$

From $\mathscr{S}$ select a function g such that

$$|h(x) - g(x)| < \frac{\varepsilon}{2}$$

for each $x \in E_q$. Then, if $x \in J$, $x \in J_{nk}$ for exactly one k; thus, for each such x,

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - h(x)| + |h(x) - g(x)| \\ &< |f(x) - f(x_k)| + \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

since $|x - x_k| < \delta.$ ∎

<u>Theorem 2.14</u>. Let μ be a Lebesgue-Stieltjes measure in E_q and let $f : E_q \to E_1^*$ be μ-summable over E_q. For any $\varepsilon > 0$ there exists a step-function $g : E_q \to E_1$ such that

$$\int_{E_q} |f - g| \, d\mu < \varepsilon .$$

<u>Proof</u>. Let $\varepsilon > 0$ be given. By Theorem 2.11 there is a continuous function $h : E_q \to E_1$ with compact support K such that

$$\int_{E_q} |f - h| \, d\mu < \frac{\varepsilon}{2} .$$

Let J be a compact interval containing K. Since h is continuous on E_q, h is continuous on J. By the preceding theorem there is a step-function $g : E_q \to E_1$ such that, for each $x \varepsilon J$,

$$|h(x) - g(x)| < \frac{\varepsilon}{2(\mu(J) + 1)}$$

and such that $g(x) = 0$ for $x \varepsilon E_q - J$. Thus, for $x \varepsilon E_q - J$,

$$|h(x) - g(x)| = 0 .$$

Therefore,

$$\int_{E_q} |f - g| \, d\mu \leq \int_{E_q} |f - h| \, d\mu + \int_{E_q} |h - g| \, d\mu$$

$$< \frac{\varepsilon}{2} + \int_J |h - g| \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \blacksquare$$

Theorem 2.11, together with the Weierstrass approximation theorem, can now be used to deduce still another approximation theorem.

<u>Theorem 2.15</u>. Let μ be a Lebesgue-Stieltjes measure in E_q and let $f : E_q \to E_1^*$ be μ-summable over E_q. For any $\varepsilon > 0$ there exists a function $g : E_q \to E_1$ and a compact set K such that:

(i) For $x = (x_1, x_2, \ldots, x_q) \varepsilon K$, $g(x) = p(x)$ where p is a polynomial in $x_1, x_2, \ldots, x_q$.

(ii) For $x \varepsilon E_q - K$, $g(x) = 0$.

(iii) $\int_{E_q} |f - g| \, d\mu < \varepsilon$.

Proof. Let $\varepsilon > 0$ be given. By Theorem 2.11 there is a function $h : E_q \to E_1$, continuous on E_q with compact support K, such that

$$\int_{E_q} |f - h| \, d\mu < \frac{\varepsilon}{2} .$$

Since h is continuous on the compact set K, the Weierstrass approximation theorem guarantees the existence of a polynomial p defined on E_q such that, for each $x \in K$,

$$|h(x) - p(x)| < \frac{\varepsilon}{2(\mu(K) + 1)} .$$

Now define a function g as follows:

$$\begin{aligned} g(x) &= p(x) \qquad \text{if } x \in K , \\ &= 0 \qquad \text{if } x \in E_q - K . \end{aligned}$$

The function g satisfies (i) and (ii). Clearly g is measurable. Thus

$$\int_{E_q} |f - g| \, d\mu \leq \int_{E_q} |f - h| \, d\mu + \int_{E_q} |h - g| \, d\mu$$

$$< \frac{\varepsilon}{2} + \int_K |h - p| \, d\mu < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \blacksquare$$

CHAPTER III

APPROXIMATIONS IN L_p SPACES

In this chapter the approximation theorems of Chapter II are extended to the L_p spaces $(1 \leq p < \infty)$. It is shown in addition that functions in $L_p(E_q, \Lambda, \mu)$, for μ Lebesgue measure, can be approximated by functions of compact support which have derivatives of all orders. This result is of considerable practical importance (cf. [6], [15]).

Certain of the notions of L_p space theory will first be surveyed briefly (details may be found in [5] or [16]). Recall that a measure space (X, Λ, μ) is tacitly assumed to be complete for the measure μ.

<u>Definition 3.1</u>. Let p be a real number such that $1 \leq p < \infty$. Let (X, Λ, μ) be a measure space. A complex-valued function f defined on X is in $L_p(X, \Lambda, \mu)$ if the functions $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are measurable (i.e., f is measurable) and $|f|^p$ is summable over X (relative to μ).

<u>Definition 3.2</u>. Let (X, Λ, μ) be a measure space. A complex-valued function f defined on X is in $L_\infty(X, \Lambda, \mu)$ if f is measurable and if there is a real number M such that

$$\mu\left\{x : |f(x)| > M\right\} = 0 .$$

The number M is called an <u>essential upper bound</u> for f.

<u>Definition 3.3</u>. For $f \varepsilon L_p(X, \Lambda, \mu)$, $1 \leq p < \infty$, the symbol $\|f\|_p$ denotes the number

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} .$$

For $f \varepsilon L_\infty(X, \Lambda, \mu)$, $\|f\|_\infty$ denotes the number

$$\|f\|_\infty = \inf \left\{ M : M \text{ is an essential upper bound for } f \right\} .$$

In order to circumvent certain difficulties arising from the fact that it is possible to have $\|f\|_p = \|g\|_p$ even though $f(x) = g(x)$ does not hold for every $x \varepsilon X$, the following definitions are customarily made.

<u>Definition 3.4</u>. Let $1 \leq p \leq \infty$ and let $f, g \varepsilon L_p(X, \Lambda, \mu)$. If $f(x) = g(x)$ almost everywhere on X, write $f \sim g$. Then define $\hat{f}$ by

$$\hat{f} = \left\{ g : g \varepsilon L_p(X, \Lambda, \mu), \ f \sim g \right\} .$$

<u>Definition 3.5</u>. Let $1 \leq p \leq \infty$. The set of all equivalence classes $\hat{f}$ for $f \varepsilon L_p(X, \Lambda, \mu)$ is denoted by $\mathcal{L}_p(X, \Lambda, \mu)$.

<u>Definition 3.6</u>. For $1 \leq p \leq \infty$ and $\hat{f} \varepsilon \mathcal{L}_p(X, \Lambda, \mu)$, define the <u>norm</u> $\|f\|_p$ to be the number

$$\|\hat{f}\|_p = \|f\|_p \qquad \text{for any } f \varepsilon \hat{f} .$$

With these definitions it is well-known that, for $1 \leq p \leq \infty$, $\mathcal{L}_p(X, \Lambda, \mu)$ is a normed linear space. Furthermore, if the function

$d : X \times X \to E_1$ is defined by

$$d(\hat{f}, \hat{g}) = \|\hat{f} - \hat{g}\|_p \tag{1}$$

for each $\hat{f}, \hat{g} \in \mathcal{L}_p(X, \Lambda, \mu)$, then d is a metric on $\mathcal{L}_p(X, \Lambda, \mu)$. It can then be shown (cf. [5], [16]) that, for $1 \leq p \leq \infty$, $\mathcal{L}_p(X, \Lambda, \mu)$ is a complete metric (linear) space.

In addition to these concepts, the standard inequalities of Hölder and Minkowski will be used.

Remark. The symbols $L_p(X, \Lambda, \mu)$ and $\mathcal{L}_p(X, \Lambda, \mu)$ will be shortened to L_p and $\mathcal{L}_p$, respectively, when no confusion seems possible. If only functions which are extended-real-valued are to be considered, then the spaces will be designated by real L_p and real $\mathcal{L}_p$.

The theorems of Chapter II will now be generalized.

<u>Theorem 3.7</u>. Let (X, Λ, μ) be an arbitrary measure space and let $1 \leq p < \infty$. If $f \in L_p(X, \Lambda, \mu)$, then for any $\varepsilon > 0$ there is a measurable simple function $g \in L_p(X, \Lambda, \mu)$ such that

(i) $\|f - g\|_p < \varepsilon$ and

(ii) $\mu \{x : g(x) \neq 0\} < \infty$.

<u>Proof</u>. Case 1. Suppose that f is nonnegative on X. Then there is a sequence $\{g_n\}$ of nonnegative measurable simple functions on X such that

$$g_1(x) \leq g_2(x) \leq \dots \leq f(x)$$

and

$$\lim_{n \to \infty} g_n(x) = f(x)$$

for each $x \in X$ (cf. Theorem 1, Appendix). Thus $|f - g_n|^p \leq |f|^p$ for each $n = 1,2,\ldots$. By the Lebesgue dominated convergence theorem

$$\lim_{n \to \infty} \int_X |f - g_n|^p \, d\mu = 0.$$

Let $\varepsilon > 0$ be given and choose an integer N such that

$$\int_X |f - g_N|^p \, d\mu < \varepsilon^p .$$

The function g_N is a simple function, and g_N is in real L_p since $0 \leq g_N \leq f$. By an argument analogous to that used in Theorem 2.9, it follows that

$$\mu \{x : g_N(x) \neq 0\} < \infty .$$

Case 2. Let f be an extended-real-valued function defined on X and write $f = f^+ - f^-$. Let $\varepsilon > 0$ be given. By Case 1 applied to f^+ and f^-, there are simple functions g_1 and g_2 in real L_p such that

$$\|f^+ - g_1\|_p < \frac{\varepsilon}{2} \quad \text{and} \quad \|f^- - g_2\|_p < \frac{\varepsilon}{2} .$$

Moreover, $\mu \{ x : g_1(x) \neq 0\} < \infty$ and $\mu \{x : g_2(x) \neq 0\} < \infty$. Therefore, by Minkowski's inequality

$$\|f - (g_1 - g_2)\|_p \leq \|f^+ - g_1\|_p + \|f^- - g_2\|_p < \varepsilon .$$

The function $g = g_1 - g_2$ is a measurable simple function with the desired properties.

Case 3. Let f be a complex-valued function on X. Let $\varepsilon > 0$ be given.

By Case 2 applied to $Re(f)$ and $Im(f)$, there are simple functions g_1 and g_2 in L_p such that

$$\|Re(f) - g_1\|_p < \frac{\varepsilon}{2} \text{ and } \|Im(f) - g_2\|_p < \frac{\varepsilon}{2} .$$

The functions g_1 and g_2 vanish outside sets of finite measure. By an argument analogous to that of Case 2, it follows that $g = g_1 + ig_2$ is a measurable simple function in L_p possessing properties (i) and (ii).∎

Attention now will be restricted to a Lebesgue-Stieltjes measure in E_q.

<u>Theorem 3.8</u>. Let $1 \leq p < \infty$ and let μ be a Lebesgue-Stieltjes measure in E_q. If $f \in L_p(E_q, \Lambda, \mu)$, then, for any $\varepsilon > 0$ there is a function $g \in L_p(E_q, \Lambda, \mu)$ such that

(i) g is continuous on E_q,

(ii) g has compact support, and

(iii) $\|f - g\|_p < \varepsilon$.

<u>Proof</u>. Case 1. Suppose first that f is in real L_p. For $n = 1,2,\ldots$ let

$$W_n = \left\{ x : |x_i| \leq n, \quad i = 1,2,\ldots q \right\}$$

and define f_n by

$$f_n(x) = f(x) \text{ if } x \in W_n \text{ and } |f(x)| \leq n ,$$
$$= 0 \quad \text{otherwise.}$$

For each x, $\lim_{n\to\infty} f_n(x) = f(x)$; thus $\lim_{n\to\infty} |f - f_n|^p = 0$. Since

$|f - f_n|^p \leq |f|^p$ for each n,

$$\lim_{n\to\infty} \int_{E_q} |f - f_n|^p \, d\mu = 0$$

by the Lebesgue dominated convergence theorem. Let $\varepsilon > 0$ be given and choose an integer N such that

$$\|f - f_N\|_p < \frac{\varepsilon}{2} . \tag{2}$$

Let this N be held fixed. Since f_N is summable over E_q, for any $\eta > 0$ there is, by Theorem 2.11, a function h, continuous on E_q and of compact support, such that

$$\|f_N - h\|_1 < \eta .$$

Now consider the function g defined by

$$\begin{aligned} g(x) &= \min\left\{h(x), N\right\} \text{ if } h(x) \geq 0 , \\ &= \max\left\{h(x), -N\right\} \text{ if } h(x) < 0 . \end{aligned}$$

It is asserted that g is continuous and that $|f_N(x) - g(x)| \leq |f_N(x) - h(x)|$ for each $x \in E_q$. The continuity is clear since h is continuous on E_q. For the inequality several possibilities must be considered. Let $y \in E_q$. If $|h(y)| \leq N$, then $g(y) = h(y)$ and there is nothing to prove. If $h(y) > N$, then $g(y) = N$. Since $|f_N(x)| \leq N$ for each x,

$$0 \leq g(y) - f_N(y) \quad \text{and}$$

$$\begin{aligned} |g(y) - f_N(y)| &= g(y) - f_N(y) < h(y) - f_N(y) \\ &< |h(y) - f_N(y)| . \end{aligned}$$

If $h(y) < -N = g(y)$, then $0 \leq f_N(y) - g(y)$ and

$$|f_N(y) - g(y)| = f_N(y) - g(y) < f_N(y) - h(y)$$
$$< |f_N(y) - h(y)| .$$

Thus $|f_N(x) - g(x)| \leq |f_N(x) - h(x)|$ for each $x \varepsilon E_q$; consequently

$$\|f_N - g\|_1 \leq \|f_N - h\|_1 < \eta .$$

By definition g has compact support, and $|g(x)| \leq N$ for each $x \varepsilon E_q$.

Since $1 \leq p < \infty$, $x^{p-1} \leq (2N)^{p-1}$ for each $x \varepsilon [0, 2N] \subset E_1$. But $0 \leq |f_N - g| \leq 2N$; thus

$$|f_N - g|^{p-1} \leq (2N)^{p-1}$$

or, equivalently,

$$|f_N - g|^p \leq (2N)^{p-1} |f_N - g| .$$

Therefore,

$$\int_{E_q} |f_N - g|^p \, d\mu \leq (2N)^{p-1} \|f_N - g\|_1 < (2N)^{p-1} \eta .$$

Now choose $\eta = (2N)^{1-p} (\frac{\varepsilon}{2})^p$. Then

$$\|f_N - g\|_p < \frac{\varepsilon}{2} . \tag{3}$$

By Minkowski's inequality and Inequalities (2) and (3), it follows that

$$\|f - g\|_p \leq \|f - f_N\|_p + \|f_N - g\|_p$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon .$$

Recall that g is continuous on E_q and of compact support.

Case 2. The extension to complex-valued functions f is immediately obtained if Case 1 is applied to $Re(f)$ and $Im(f)$. ∎

Theorem 3.9. Let $1 \leq p < \infty$ and let μ be a Lebesgue-Stieltjes measure in E_q. There is a countable collection $\mathcal{S}$ of step-functions with the following property. If $f \varepsilon L_p(E_q, \Lambda, \mu)$ and ε is a positive number, then there is a function $g \varepsilon \mathcal{S}$ such that

$$\|f - g\|_p < \varepsilon.$$

Proof. For $n = 1,2,\ldots$ let

$$W_n = \{x : |x_i| \leq n, \; i = 1,2,\ldots, q\}.$$

By Theorem 2.13, to each W_n there corresponds a countable family $\mathcal{S}_n$ of complex-valued step-functions which vanish outside W_n. Moreover, by the same theorem, it is known that $\mathcal{S}_n$ may be chosen in such a way that, for any complex-valued function h continuous on W_n, there is a function in $\mathcal{S}_n$ which approximates h uniformly on W_n. Now let $\mathcal{S} = \bigcup_{n=1}^{\infty} \mathcal{S}_n$ and let $f \varepsilon L_p$. By Theorem 3.8 there is, given $\varepsilon > 0$, a function $h \varepsilon L_p$ such that h is continuous on E_q, h has compact support K, and

$$\|f - h\|_p < \frac{\varepsilon}{2}.$$

Let N be an integer such that $K \subset W_n$ and let $\eta > 0$ be given. By definition of the family $\mathcal{S}_N$, there is a step-function $g \varepsilon \mathcal{S}_N$ (and hence in $\mathcal{S}$) such that

$$|h(x) - g(x)| < \eta$$

for each $x \varepsilon W_N$. Outside W_N note that $g(x) = h(x) = 0$. Hence

$$\int_{E_q} |h - g|^p \, d\mu = \int_{W_N} |h - g|^p \, d\mu < \eta^p \ \mu(W_N) \ .$$

Choose η such that $\eta[\mu(W_N)]^{\frac{1}{p}} < \frac{\varepsilon}{2}$. By Minkowski's inequality

$$\begin{aligned} \|f - g\|_p &\leq \|f - h\|_p + \|h - g\|_p \\ &< \frac{\varepsilon}{2} + \eta[\mu(W_N)]^{\frac{1}{p}} \\ &< \varepsilon \ . \blacksquare \end{aligned}$$

This theorem points out an important fact about the topology of the Banach space $\mathcal{L}_p \ (E_q, \Lambda, \ \mu)$.

<u>Definition 3.10</u>. Let (X, d) be a metric space. If there exists a set $D \subset X$ such that for any $\varepsilon > 0$ and any $x \varepsilon X$ there is a $y \varepsilon D$ for which $d(x, y) < \varepsilon$, then D is <u>dense</u> in (X, d). If there is a countable set $D \subset X$ which is dense in X, then (X, d) is <u>separable</u>.

<u>Corollary 3.11</u>. Let $1 \leq p < \infty$ and let μ be a Lebesgue-Stieltjes measure in E_q. The space $\mathcal{L}_p(X, \Lambda, \mu)$, with the metric d defined by Equation (1), is separable.

<u>Proof</u>. In view of Theorem 3.9, it remains only to show that each $g \ \varepsilon \ \mathcal{S}$ is in L_p. This is immediate since g vanishes outside a compact set and assumes only a finite number of finite values.$\blacksquare$

The most important theorem of this chapter will now be discussed.

Several lemmas will be given first in order to simplify the argument as much as possible.

Definition 3.12. Let f be a complex-valued function defined on E_q. If $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ have derivatives of all orders everywhere in E_q, then f is in C^∞ on E_q; i.e., $f \in C^\infty$ on E_q. The same terminology applies to real-valued functions.

Define the function $\varphi : E_q \to E_1$ by

$$\varphi(x) = e^{1/|x|^2 - 1} \qquad \text{if} \quad |x| < 1 \,, \tag{4}$$

$$= 0 \qquad \text{if} \quad |x| \geq 1 \,.$$

Lemma 3.13. The function φ defined in Equation (4) is in C^∞ on E_q. Furthermore, if μ is Lebesgue measure in E_q, then $\int_{E_q} |\varphi| d\mu > 0$.

Proof. Define the function $f : E_1 \to E_1$ by

$$f(t) = e^{\frac{1}{t}} \qquad \text{if} \quad t < 0 \,,$$

$$= 0 \qquad \text{if} \quad t \geq 0 \,.$$

Define the function $g : E_q \to E_1$ by

$$g(x) = |x|^2 - 1 \qquad \text{for each} \quad x \in E_q.$$

Then, for each $x \in E_q$,

$$\varphi(x) = f[g(x)] \,.$$

Clearly $g \in C^\infty$ on E_q; thus, in virtue of the chain rule, $\varphi \in C^\infty$ on E_q provided that $f \in C^\infty$ on E_1. It is easy to see that f has derivatives of all orders except perhaps at $t = 0$. In fact

$f'(t) = -\frac{1}{t^2} e^{\frac{1}{t}}$ if $t < 0$ and $f'(t) = 0$ if $t > 0$. To show that $f'(0)$ exists, let $h < 0$. Then

$$\frac{f(h) - f(0)}{h} = \frac{e^{\frac{1}{h}}}{h} .$$

Replace h by $-\frac{1}{k}$ to obtain

$$(-k)[f(-\tfrac{1}{k}) - f(0)] = -ke^{-k} .$$

From elementary analysis it is known that $\lim\limits_{t \to \infty} t^{\alpha} e^{-t} = 0$ for any fixed real α. Thus

$$\lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{k \to \infty} \frac{f(-\frac{1}{k}) - f(0)}{-\frac{1}{k}}$$

$$= -\lim_{k \to \infty} k e^{-k} = 0 .$$

Evidently

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = 0$$

and hence $f'(0) = 0$. For $n = 2, 3, \ldots$ the proof that $f^{(n)}(0)$ exists and is zero is analogous. Thus $f \varepsilon C^{\infty}$ on E_1. It follows that $\varphi \varepsilon C^{\infty}$ on E_q.

By definition of φ, $\varphi(x) > 0$ for $|x| < 1$ and $\varphi(x) = 0$ for $|x| \geq 1$. Since the set $D = \{x : |x| < 1\}$ has positive Lebesgue measure,

$$\int_{E_q} |\varphi| \, d\mu = \int_D \varphi \, d\mu > 0 . \blacksquare$$

Let μ be Lebesgue measure in E_q. Define the function $u : E_q \rightarrow E_1$ by

$$u(x) = \frac{\varphi(x)}{\|\varphi\|_1} \quad \text{for each} \quad x \in E_q \tag{5}$$

where φ is the function defined in Equation (4). Then $\|u\|_1 = 1$, $u \in C^\infty$ on E_q by Lemma 3.13, and u and each of its derivatives are zero for $x \in E_q - \{x : |x| \leq 1\}$. Thus u and each of its derivatives are uniformly continuous on E_q.

In Lemmas 3.14 and 3.15 it is assumed that μ is Lebesgue measure in E_q.

<u>Lemma 3.14</u>. Let $g : E_q \rightarrow E_1$ be continuous on E_q and have compact support K. Let u be the function defined by Equation (5). For any $h > 0$ define the function $u_h : E_q \rightarrow E_1$ by

$$u_h(x) = \int_{E_q} g(x - hy)\, u(y)\, d\mu(y) \tag{6}$$

for each $x \in E_q$. Then, for any $h > 0$, $u_h \in C^\infty$ on E_q and u_h has compact support $K(h)$.

<u>Proof</u>. Let $h > 0$ be fixed. It must first be shown that u_h is well defined. Define the function $F_h : E_q \times E_q \rightarrow E_1$ by

$$F_h(x, y) = g(x - hy)\, u(y) .$$

The function defined on $E_q \times E_q$ by $g(x - hy)$ is continuous since it is a composition of continuous functions. The function defined by $u(y)$

is also continuous on $E_q \times E_q$, and thus F_h is continuous. Hence, for fixed x, F_h is measurable. Now, since g vanishes outside K and since $u(y) = 0$ for $|y| > 1$, $F_h(x, y) = 0$ unless there exists a y with $|y| \leq 1$ and an x such that $x - hy \varepsilon K$. Suppose $F_h(x,y) \neq 0$ and let $z = x - hy \varepsilon K$. Then

$$\begin{aligned} \text{dist.}\ (x, K) &= \inf \left\{ d(x, \tilde{x}) : \tilde{x} \varepsilon K \right\} \\ &\leq d(x, z) = d(z + hy , z) \\ &\leq h|y| \leq h . \end{aligned}$$

Let $K(h) = \left\{ x : \text{dist.}\ (x, K) \leq h \right\}$. It follows that $F_h(x, y) = 0$ unless $x \varepsilon K(h)$ and $|y| \leq 1$. Therefore, u_h defined by

$$u_h(x) = \int_{E_q} F_h(x, y)\, d\mu(y)$$

is finite-valued and vanishes outside the compact set $K(h)$.

It must now be shown that $u_h \varepsilon C^\infty$ on E_q. Let $z = x - hy$. By a linear change of variable (cf. [8], p. 196), it follows that

$$\begin{aligned} u_h(x) &= h^{-q} \int_{E_q} g(z)\, u(\tfrac{x-z}{h})\, d\mu(z) \\ &= h^{-q} \int_{E_q} g(y)\, u(\tfrac{x-y}{h})\, d\mu(y) . \end{aligned}$$

Let $e_1 = (1, 0, 0, \ldots, 0)$ be a unit vector in E_q. For some real number $k \neq 0$, consider the expression

$$H_k(x) = \frac{U_h(x + ke_1) - U_h(x)}{k}$$

$$= h^{-q} \int_{E_q} g(y) \left[\frac{u\left(\frac{x + ke_1 - y}{h} \right) - u\left(\frac{x-y}{h}\right)}{k} \right] d\mu(y).$$

Since $u \varepsilon C^{\infty}$ on E_q, Taylor's formula (cf. [1], p. 124) asserts that for each $x \varepsilon E_q$ and for each fixed $y \varepsilon E_q$,

$$u\left(\frac{x - y + ke_1}{h} \right) - u\left(\frac{x-y}{h}\right) = \frac{k}{h} D_1 u \left(\frac{x-y}{h}\right)$$

$$+ \frac{k^2}{2h^2} D_1^2 u \left(\frac{x - y + \theta k e_1}{h} \right)$$

for $0 < \theta < 1$. But, since $D_1^2 u$ is uniformly continuous on E_q, there is a number M such that, for each $x, y \varepsilon E_q$,

$$\left| \frac{u\left(\frac{x - y + ke_1}{h} \right) - u\left(\frac{x-y}{h}\right)}{k} - \frac{1}{h} D_1 u \left(\frac{x-y}{h}\right) \right| =$$

$$\left| \frac{k}{2h^2} D_1^2 u \left(\frac{x - y + \theta k e_1}{h} \right) \right| \leq \frac{|k|}{h^2} M .$$

Let $\varepsilon > 0$ be given and let $|k|$ be chosen such that $|k| M h^{-q-2} \|g\|_1 < \varepsilon$ (recall that $h > 0$ is fixed). Thus, for such values of k and for each $x \varepsilon E_q$,

$$\left| H_k(x) - h^{-q} \int_{E_q} g(y) \frac{D_1 u \left(\frac{x-y}{h}\right)}{h} d\mu(y) \right| \leq$$

$$h^{-q} \int_{E_q} |g(y)| \left| \frac{u\left(\frac{x - y + ke_1}{h}\right) - u\left(\frac{x-y}{h}\right)}{k} - \frac{D_1 u\left(\frac{x-y}{h}\right)}{h} \right| d\mu(y)$$

$$\leq h^{-q} \frac{|k|}{h^2} M \|g\|_1 < \varepsilon .$$

This shows that for each $x \in E_q$

$$D_1 u_h(x) = h^{-q} \int_{E_q} g(y) \frac{D_1 u\left(\frac{x-y}{h}\right)}{h} d\mu(y).$$

To show that $D_1 u_h$ is continuous on E_q, let $\eta > 0$ be given. Let $x \in E_q$ and let $\delta > 0$ be such that for each fixed $y \in E_q$

$$\left| D_1 u\left(\frac{x-y}{h}\right) - D_1 u\left(\frac{z-y}{h}\right) \right| < h^{q+1} \frac{\eta}{\|g\|_1 + 1}$$

whenever $z \in E_q$ and $\left|\frac{x-z}{h}\right| < \delta$ (this makes use of the uniform continuity of $D_1 u$). Then, for $|x - z| < \delta h$,

$$|D_1 u_h(x) - D_1 u_h(z)| = h^{-q} \left| \int_{E_q} g(y) \left[\frac{D_1 u\left(\frac{x-y}{h}\right) - D_1 u\left(\frac{z-y}{h}\right)}{h} \right] d\mu(y) \right|$$

$$\leq h^{-q} h^{q+1} \frac{\eta}{\|g\|_1 + 1} \int_{E_q} \frac{|g(y)|}{h} d\mu(y) < \eta .$$

A similar argument shows that $D_j u_h$ exists and is continuous for $j = 2, 3, \ldots, q$. This is sufficient to ensure that u_h is differentiable on E_q.

In a similar manner it can be shown that u_h possesses derivatives of arbitrary order on E_q and that these derivatives can be calculated by differentiation inside the integral sign. ∎

Lemma 3.15. Let $g : E_q \to E_1$ be continuous on E_q and have compact support K. For any $h > 0$ let the function $u_h : E_q \to E_1$ be defined by Equation (6). Then, if $1 \leq p < \infty$,

$$\lim_{h \to 0} \| g - u_h \|_p = 0 .$$

Proof. The function g is uniformly continuous on E_q. Thus, for any given $\eta > 0$, there is a $\delta > 0$ such that, if $x, y \in E_q$ and $|hy| < \delta$, $|g(x) - g(x - hy)| < \eta$. Recall that

$$\int_{E_q} u \, d\mu = \int_{E_q} |u| \, d\mu = 1$$

and that $u(y) = 0$ unless $y \in F = \{ y : |y| \leq 1 \}$. Thus, if $y \in F$ and $h < \delta$, then $|hy| \leq h < \delta$ and $|g(x) - g(x - hy)| < \eta$. In Lemma 3.14 it was shown that u_h, for each $h > 0$, has compact support $K(h) = \{ x : \text{dist}\,(x, K) \leq h \}$. Note that $K(h_1) \subset K(h_2)$ if $h_1 \leq h_2$. Thus, since $g(x) = 0$ for $x \in E_q - K$,

$$\int_{E_q} |g - u_h|^p \, d\mu = \int_{K \cup K(h)} |g - u_h|^p \, d\mu .$$

Let $x \in E_q$ and let $h < \min(\delta, 1)$. Then $h < \delta$, $K(h) \subset K(1)$, and

$$\int_{E_q} |g - u_h|^p \, d\mu = \int_{K \cup K(1)} |g(x) - \int_{E_q} g(x - hy)\, u(y)\, d\mu(y)|^p \, d\mu(x)$$

$$= \int_{K \cup K(1)} | \int_{E_q} [g(x) - g(x - hy)]\, u(y)\, d\mu(y)|^p \, d\mu(x)$$

$$\leq \int_{K \cup K(1)} \left[\int_F |g(x) - g(x - hy)| \, |u(y)\, d\mu(y) \right]^p \, d\mu(x)$$

$$\leq \int_{K \cup K(1)} \eta^p \left[\int_F |u(y)| \, d\mu(y) \right]^p \, d\mu(x)$$

$$\leq \eta^p \mu(K \cup K(1)) .$$

Thus, for any given $\varepsilon > 0$, choose η such that $\eta^p \mu(K \cup K(1)) < \varepsilon^p$. Then, for $h < \min(\delta, 1)$,

$$\|g - u_h\|_p < \varepsilon . \blacksquare$$

The proof of the desired approximation theorem is now straightforward.

Theorem 3.16. Let $1 \leq p < \infty$ and let μ be Lebesgue measure in E_q. If $f \in L_p(E_q, \Lambda, \mu)$, then for any $\varepsilon > 0$ there is a function $U \in L_p(E_q, \Lambda, \mu)$ such that

(i) $U \in C^\infty$ on E_q,

(ii) U has compact support, and

(iii) $\|f - U\|_p < \varepsilon$.

Proof. Case 1. Suppose f is in real L_p. Let $\varepsilon > 0$ be given. By Theorem 3.8 there exists a function $g : E_q \to E_1$ such that g is continuous on E_q, g has compact support K, and

$\|f - g\|_p < \frac{\varepsilon}{2}$. For any $h > 0$ consider the function u_h defined by Equation (6). Each function $u_h \in C^\infty$ on E_q and each has compact support $K(h)$ by Lemma 3.14. Furthermore, by Lemma 3.15, there is a $\delta > 0$ such that $\|g - u_h\|_p < \frac{\varepsilon}{2}$ whenever $0 < h < \delta$. Let h_1 be a positive number such that $h_1 < \delta$ and define $U = u_{h_1}$. Then $U \in C^\infty$ on E_q, U has compact support, and

$$\|f - U\|_p \leq \|f - g\|_p + \|g - U\|_p < \varepsilon .$$

Clearly U is in real L_p since it is bounded and has compact support.

Case 2. If $f \in L_p$, then apply Case 1 to $Re(f)$ and $Im(f)$. ∎

No mention has, as yet, been made of the possibility of approximating functions in L_∞. Indeed, results similar to those of Theorem 3.8 and Corollary 3.11 do not hold for $L_\infty(E_q, \Lambda, \mu)$ if μ is an arbitrary Lebesgue-Stieltjes measure in E_q. Consider the following example. Let μ be Lebesgue measure in E_1. For each $a \in E_1$ let f_a be the characteristic function of the set $\{x : x \in E_1, x > a\}$. Then $\|f_a\|_\infty = 1$ for each $a \in E_1$; thus each $f_a \in L_\infty$. Suppose that E is a subset of L_∞ which is dense in L_∞; i.e., suppose that for any $\varepsilon > 0$ and any $f \in L_\infty$ there is a $g \in L_\infty$ such that $\|f - g\|_\infty < \varepsilon$. Thus, for $\varepsilon = \frac{1}{2}$ and for each $a \in E_1$, there must be a function $g_a \in E$ such that $\|f_a - g_a\|_\infty < \frac{1}{2}$. These functions are all necessarily distinct since, if $\|f_a - g_a\|_\infty < \frac{1}{2}$ and $b \in E_1$ with $b \neq a$,

$$\|f_b - g_a\|_\infty \geq \|f_b - f_a\|_\infty - \|f_a - g_a\|_\infty > 1 - \frac{1}{2} = \frac{1}{2} .$$

Hence E is uncountable. Moreover, if g is any function such that $\|f_a - g\|_\infty < \frac{1}{2}$, then $g(x) < \frac{1}{2}$ for almost all $x \leq a$ and $g(x) > \frac{1}{2}$ for almost all $x > a$. Thus g cannot be continuous. Therefore, if μ is Lebesgue measure in E_1, then

(i) $L_\infty(E_1, \Lambda, \mu)$ is not separable, and

(ii) no family of continuous functions is dense in $L_\infty(E_1, \Lambda, \mu)$.

Hence Theorem 3.8 and Corollary 3.11 cannot be extended to $L_\infty(E_q, \Lambda, \mu)$ for all Lebesgue-Stieltjes measures.

Finally, it should be noted that Theorem 2.15 could be extended to $L_p(E_q, \Lambda, \mu)$ for $1 \leq p < \infty$. The proof would be quite simple if Theorem 3.8 were used.

CHAPTER IV

REPRESENTATION THEOREMS IN L_p SPACES

This chapter is primarily concerned with establishing the Riesz representation theorem for bounded linear functionals defined on $\mathcal{L}_p$ spaces $(1 \leq p < \infty)$. The proof makes use of the Radon-Nikodym theorem in an essential way. Thus the Radon-Nikodym theorem is deduced first. The proof given is based on the Riesz representation theorem for linear functionals on a Hilbert space.

Definition 4.1. Let V be a linear space over the field K of complex numbers (or over the field R of real numbers). A complex-valued (or real-valued) function F defined on V is called a linear functional if

$$F(\alpha x + \beta y) = \alpha F(x) + \beta F(y)$$

for every $x, y \in V$ and every $\alpha, \beta \in K$ (or R).

Definition 4.2. If V is a normed linear space over K (or R), if F is a linear functional on V, and if there is a real number M such that $|F(x)| \leq M \|x\|$ for each $x \in V$, then F is called a bounded linear functional. If F is a bounded linear functional on V, the number $\|F\|$ defined by

$$\|F\| = \inf \left\{ M : |F(x)| \leq M \|x\| \text{ for every } x \in V \right\} \qquad (1)$$

is called the norm of F.

<u>Definition 4.3</u>. Let V be a normed linear space over K (or R). The collection of all bounded linear functionals on V is denoted by V^* and is called the <u>conjugate</u> (or <u>dual</u>) <u>space</u> of V.

<u>Theorem 4.4</u>. If V is a complete normed linear space (Banach space) over K (or R), then V^* is also a complete normed linear space over K (or R) with the norm defined by Equation (1). Moreover, for each $F \in V^*$ and each $x \in V$, $|F(x)| \leq \|F\| \cdot \|x\|$.

The proof is straightforward and will not be given here.

<u>Theorem 4.5</u>. A linear functional F on a normed linear space V is bounded if and only if it is continuous.

<u>Proof</u>. If there is a number M such that $|F(x)| \leq M \|x\|$ for every $x \in V$, then $|F(x) - F(y)| = |F(x - y)| \leq M\|x - y\|$ for each $x, y \in V$. Thus F is continuous on V. Conversely, suppose that F is continuous on V. If F is not bounded, for each positive integer n there is an $x_n \in V$ such that $|F(x_n)| > n\|x_n\|$. Since $F(\theta) = 0$, $x_n \neq \theta$ for any n (θ is the zero element of V). Thus $y_n = \dfrac{x_n}{n\|x_n\|} \in V$, and for each n

$$|F(y_n)| = \frac{1}{n\|x_n\|} |F(x_n)| > 1 . \tag{2}$$

However, $\|y_n\| = \frac{1}{n}$ and, by the continuity of F, $\lim_{n \to \infty} |F(y_n)| = |F(\theta)| = 0$. This contradicts Inequality (2). ∎

A special case of the Riesz representation theorem for bounded linear functionals on a Hilbert space will now be stated. The proof of Riesz's theorem for any Hilbert space is quite elementary and may be found in Halmos (cf. [3], p. 31). The theorem in question was also proved by M. Frechét and is often called the Riesz-Frechét theorem.

<u>Theorem 4.6</u>. (Riesz representation theorem). Let (X, Λ, μ) be a (complete) measure space. Let F be a (complex) bounded linear functional on the Hilbert space $\mathscr{L}_2\,(X, \Lambda, \mu)$. Then there is a unique $\hat{g} \in \mathscr{L}_2(X, \Lambda, \mu)$ such that, if $\hat{f} \in \mathscr{L}_2(X, \Lambda, \mu)$ and $g \in \hat{g}$, $f \in \hat{f}$,

$$F(\hat{f}) = \int_X f\,\bar{g}\,d\mu$$

($\bar{g}$ is the complex conjugate of g).

In preparation for the Radon-Nikodym theorem, certain measure-theoretic concepts will now be discussed.

<u>Definition 4.7</u>. Let X be a nonempty set and let Λ be a σ-ring of subsets of X such that $\bigcup\{A : A \in \Lambda\} = X$. A <u>signed measure</u> is an extended-real-valued, countably additive set function μ defined on Λ such that $\mu(\varphi) = 0$ and such that μ assumes at most one of the value $+\infty$ and $-\infty$. A <u>complex measure</u> is a set function μ defined on Λ such that, for each $A \in \Lambda$, $\mu(A) = \mu_1(A) + i\mu_2(A)$ where μ_1 and μ_2 are signed measures on Λ.

A result of fundamental importance in the theory of signed measures is the Jordan decomposition of a signed measure. This result will now be stated for the case of interest here.

<u>Definition 4.8</u>. Let μ be a totally finite signed measure on a σ-algebra Λ of subsets of X. For every $A \in \Lambda$ let

$$\mu^+(A) = \sup\left\{\mu(E) : A \supset E \in \Lambda\right\} \quad \text{and}$$

$$\mu^-(A) = -\inf\left\{\mu(E) : A \supset E \in \Lambda\right\} .$$

<u>Theorem 4.9</u>. (Jordan decomposition of a signed measure). Let μ be a totally finite signed measure on a σ-algebra Λ of subsets of X. Then μ^+ and μ^- are totally finite measures on Λ and $\mu = \mu^+ - \mu^-$.

The proof may be found in Halmos (cf. [4], pp. 122-3) or in Hewitt (cf. [5], pp. 274-6).

<u>Definition 4.10</u>. Let (X, Λ, μ) be a measure space and let ν be a complex measure on Λ. The complex measure ν is <u>absolutely continuous</u> with respect to μ, in symbols $\nu \ll \mu$, if $\nu(A) = 0$ whenever $A \in \Lambda$ and $\mu(A) = 0$.

<u>Theorem 4.11</u>. Let (X, Λ, μ) be a measure space.

(i) If ν is a totally finite signed measure on Λ and $\nu = \nu^+ - \nu^-$, then $\nu \ll \mu$ if and only if $\nu^+ \ll \mu$ and $\nu^- \ll \mu$.

(ii) If ν is a complex measure on Λ, then $\nu \ll \mu$ if and only if $Re(\nu) \ll \mu$ and $Im(\nu) \ll \mu$.

<u>Proof</u>. (i) Let $\nu = \nu^+ - \nu^-$ be a totally finite signed measure on Λ. If $\nu^+ \ll \mu$ and $\nu^- \ll \mu$, then clearly $\nu \ll \mu$. Thus suppose $\nu \ll \mu$. Let $A \in \Lambda$ and suppose $\mu(A) = 0$. Since Λ is complete for μ, $\mu(E) = 0$ for every $E \in \Lambda$ such that $E \subset A$. Thus $\nu(E) = 0$ for every such $E \in \Lambda$. By Definition 4.8

$$\nu^+(A) = \sup\left\{\nu(E) : A \supset E \in \Lambda\right\} .$$

Hence $\nu^+(A) = 0$ and $\nu^+ \ll \mu$. Similarly, from the definition of ν^-, it follows that $\nu^- \ll \mu$.

(ii) If ν is a complex measure on Λ and $A \in \Lambda$, then $\nu(A) = 0$ if and only if $\mathrm{Re}(\nu(A)) = 0$ and $\mathrm{Im}(\nu(A)) = 0$. ∎

<u>Definition 4.12</u>. If (X, Λ, μ) is a measure space, and if f and g are two functions defined on X such that $\mu(\{x : f(x) \neq g(x)\}) = 0$, then $f = g$ <u>modulo</u> μ.

<u>Theorem 4.13</u>. (Radon-Nikodym theorem). Let (X, Λ, μ) be a totally finite measure space. If ν is a totally finite signed measure on Λ which is absolutely continuous with respect to μ, then there exists a finite-valued μ-summable function g on X such that

$$\nu(A) = \int_A g \, d\mu \quad \text{for every } A \in \Lambda.$$

The function g is unique in the sense that if

$$\nu(A) = \int_A h \, d\mu \quad \text{for every } A \in \Lambda,$$

then $g = h$ modulo μ.

<u>Proof</u>.[1] Case 1. First suppose that ν is a totally finite measure on Λ such that $\nu \ll \mu$. Define the totally finite measure λ on Λ by $\lambda(A) = \mu(A) + \nu(A)$ for each $A \in \Lambda$. If $\lambda(A) = 0$ and $B \subset A$, then $\mu(A) = 0$ and $\nu(A) = 0$. Since Λ is complete for μ, $\mu(B) = 0$. Since $\nu \ll \mu$, it follows that $\nu(B) = 0$. Thus $\lambda(B) = 0$ and Λ is complete for λ. For any $f \in L_2(X, \Lambda, \lambda)$

---

[1] This proof is due to J. von Neumann [12].

$$\int_X |f|\, d\lambda \leq \|f\|_2 \, \|1\|_2 = \|f\|_2 \, [\lambda(X)]^{\frac{1}{2}} < \infty \tag{3}$$

by Hölder's inequality and the total finiteness of λ. Thus $L_2(X, \Lambda, \lambda) \subset L_1(X, \Lambda, \lambda)$. If f is any λ-summable function, then

$$\int |f|\, d\mu \leq \int_X |f|\, d\mu + \int_X |f|\, d\nu = \int_X |f|\, d\lambda < \infty$$

and, similarly,

$$\int_X |f|\, d\nu < \infty .$$

For each $\hat{f} \in \mathcal{L}_2 (X, \Lambda, \lambda)$ define the functional F by

$$F(\hat{f}) = \int_X f\, d\nu \quad \text{for any } f \in \hat{f} .$$

By Inequality (3) it follows that

$$|F(\hat{f})| \leq \int_X |f|\, d\nu \leq \int_X |f|\, d\lambda \leq [\lambda(X)]^{\frac{1}{2}} \|f\|_2$$

for each $\hat{f} \in \mathcal{L}_2 (X, \Lambda, \lambda)$. Thus F is a bounded linear functional on $\mathcal{L}_2 (X, \Lambda, \lambda)$. By Theorem 4.6 there exists a unique $\hat{g} \in \mathcal{L}_2 (X, \Lambda, \lambda)$ such that

$$F(\hat{f}) = \int_X f \bar{g}\, d\lambda \quad \text{for each } \hat{f} \in \mathcal{L}_2 (X, \Lambda, \lambda).$$

Hence there is a function $g \in L_2 (X, \Lambda, \lambda)$ such that

$$\int_X f\, d\nu = \int_X f \cdot \bar{g}\, d\lambda \quad \text{for each } f \in L_2(X, \Lambda, \lambda) . \tag{4}$$

Moreover, g is unique modulo λ.

For any set $A \varepsilon \Lambda$, $K_A \varepsilon L_2 (X, \Lambda, \lambda)$; thus

$$\int_X K_A \, d\nu = \int_X K_A \bar{g} \, d\lambda$$

$$= \int_A \mathrm{Re}(g) \, d\lambda - i \int_A \mathrm{Im}(g) \, d\lambda .$$

Therefore

$$\int_A \mathrm{Im}(g) \, d\lambda = 0 \quad \text{for every } A \varepsilon \Lambda.$$

It follows that $\mathrm{Im}(g) = 0$ modulo λ (cf. Theorem 3, Appendix). Redefine $\mathrm{Im}(g)$ on the set where it is nonzero so that $\mathrm{Im}(g(x)) = 0$ for each $x \varepsilon X$. Now consider the sets $A_1 = \{x : g(x) < 0\}$ and $A_2 = \{x : g(x) \geq 1\}$. It will be shown that $\mu(A_1) = \mu(A_2) = 0$. From Equation (4) it follows that

$$\int_X K_{A_j} \, d\nu = \int_X K_{A_j} \bar{g} \, d\lambda$$

$$= \int_X K_{A_j} g \, d\mu + \int_X K_{A_j} g \, d\nu$$

for $j = 1,2$. Since each term is finite,

$$\int_X K_{A_j} (1 - g) d\nu = \int_X K_{A_j} g \, d\mu \quad \text{for } j = 1,2 . \tag{5}$$

On A_1, $g(x) < 0$; thus

$$0 \geq \int_{A_1} g \, d\mu = \int_{A_1} (1 - g) d\nu \geq 0 .$$

It follows that $\int_{A_1} g\, d\mu = 0 = \int_{A_1} (-g)\, d\mu$. Hence $\mu(A_1) = 0$ (cf. Theorem 2, Appendix). On A_2, $g(x) \geq 1$; by Equation (5)

$$0 \geq \int_{A_2} (1 - g)\, d\nu = \int_{A_2} g\, d\mu \geq \mu(A_2) \geq 0 .$$

Thus $\mu(A_2) = 0$. Since $A_1 \cap A_2 = \Phi$ and since $\nu \ll \mu$,

$$\lambda(A_1 \cup A_2) = \mu(A_1 \cup A_2) + \nu(A_1 \cup A_2) = 0 .$$

Hence $0 \leq g(x) < 1$ for almost all $x \in X$ (relative to λ). Redefine g on a set of zero measure in such a way that $0 \leq g(x) < 1$ for all $x \in X$.

Let f be any nonnegative measurable function on X. Define the sequence $\{f_n\}$ by

$$f_n(x) = f(x) \quad \text{if} \quad f(x) \leq n ,$$

$$= 0 \quad \text{if} \quad f(x) > n .$$

for each $x \in X$ and each $n = 1,2,\ldots$. For $n = 1,2,\ldots$, $f_n \in L_2(X, \Lambda, \lambda) \subset L_1(X, \Lambda, \lambda)$, and thus f_n is summable with respect to each of λ, μ, and ν. By Equation (4)

$$\int_X f_n\, d\nu = \int_X f_n\, g\, d\mu + \int_X f_n\, g\, d\nu ,$$

and therefore

$$\int_X f_n\, (1 - g)\, d\nu = \int_X f_n\, g\, d\mu \text{ for } n = 1,2,\ldots .$$

Since $\{f_n(1-g)\}$ and $\{f_n g\}$ are nondecreasing sequences of nonnegative measurable functions, it follows by the monotone convergence theorem that

$$\int_X f(1-g)\,dv = \int_X f\,g\,d\mu \tag{6}$$

(each integral may have the value $+\infty$).

Let $g_o = \frac{g}{1-g}$. The function g_o is nonnegative and finite-valued since $0 \leq g(x) < 1$ for each $x \varepsilon X$. Moreover, g_o is measurable. For any $A \varepsilon \Lambda$ let $f = \frac{K_A}{1-g}$. Then f is nonnegative and measurable; by Equation (6)

$$\int_X \frac{K_A}{1-g}(1-g)\,dv = \int_X \frac{K_A}{1-g}\,g\,d\mu .$$

Thus $v(A) = \int_A g_o\,d\mu$ for each $A \varepsilon \Lambda$.

Since $v(X) < \infty$, it follows that

$$\int_X g_o\,d\mu = v(X) < \infty,$$

and g_o is thus μ-summable. This completes the proof in Case 1 except for uniqueness. This will be handled in the more general case.

Case 2. Suppose that v is a totally finite signed measure on Λ such that $v \ll \mu$. By Theorem 4.9, $v = v^+ - v^-$ where v^+ and v^- are totally finite measures on Λ. By Theorem 4.11 it follows that $v^+ \ll \mu$ and $v^- \ll \mu$. Thus by Case 1 there are finite-valued μ-summable functions g^+ and g^- such that

$$\nu^+(A) = \int_A g^+ \, d\mu$$

and

$$\nu^-(A) = \int_A g^- \, d\mu \qquad \text{for each } A \in \Lambda$$

Hence for each $A \in \Lambda$,

$$\nu(A) = \int_A (g^+ - g^-) \, d\mu \, .$$

To show uniqueness, suppose that g and h are any two finite-valued μ-summable functions for which

$$\nu(A) = \int_A g \, d\mu = \int_A h \, d\mu \quad \text{for each } A \in \Lambda.$$

Then, since ν is totally finite ,

$$\int_A (g - h) \, d\mu = 0 \quad \text{for each } A \in \Lambda.$$

Thus $g = h$ modulo μ (cf. Theorem 3, Appendix). ∎

The Radon-Nikodym theorem can be extended to a totally σ-finite space (X, Λ, μ) if ν is a totally σ-finite measure on Λ. However, generalizations of this nature are not needed here. It is, however, necessary to extend Theorem 4.13 to allow ν to be a complex measure.

<u>Corollary 4.14</u>. Let (X, Λ, μ) be a totally finite measure space. If ν is a totally finite complex measure on Λ which is absolutely continuous with respect to μ, then there exists a complex-valued μ-summable

function g on X such that

$$\nu(A) = \int_A g \, d\mu \quad \text{for each } A \in \Lambda.$$

The function g is unique modulo μ.

Proof. By definition of a complex measure, there are totally finite signed measures ν_1 and ν_2 on Λ such that $\nu = \nu_1 + i \, \nu_2$. If $\nu \ll \mu$, then $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$. By Theorem 4.13 there are finite-valued μ-summable functions g_1 and g_2 such that

$$\nu_1(A) = \int_A g_1 \, d\mu$$

and

$$\nu_2(A) = \int_A g_2 \, d\mu \quad \text{for each } A \in \Lambda.$$

The function $g = g_1 + i \, g_2$ clearly serves as a suitable function for ν. Since g_1 and g_2 are unique modulo μ and since ν_1 and ν_2 are unique, g is unique modulo μ. ∎

Let (X, Λ, μ) be a measure space. Let p be a real number such that $1 \leq p < \infty$, and define the number q as follows:

$$q = \frac{p}{p-1} \quad \text{if } 1 < p < \infty \quad \left(\text{i.e., } \frac{1}{p} + \frac{1}{q} = 1\right);$$

$$q = \infty \qquad \text{if } p = 1 .$$

With q related to p in this way, the study of the space $\mathcal{L}_p(X, \Lambda, \mu)$ leads, in a natural way, to consideration of the space $\mathcal{L}_q(X, \Lambda, \mu)$. This relationship can be seen, for example, in Hölder's inequality. The remaining theorems of this chapter point to another natural relationship between $\mathcal{L}_p$ and $\mathcal{L}_q$.

Theorem 4.15. Let (X, Λ, μ) be an arbitrary measure space and let $1 \leq p < \infty$. For every $\hat{g} \in \mathcal{L}_q(X, \Lambda, \mu)$, $1 < q \leq \infty$, the functional F defined for $\hat{f} \in \mathcal{L}_p(X, \Lambda, \mu)$ by

$$F(\hat{f}) = \int_X f\ g\ d\mu \tag{7}$$

is a bounded linear functional. Moreover, $\|\hat{g}\|_q = \|F\|$ for $1 < q < \infty$.

If (X, Λ, μ) is totally σ-finite, then $\|g\|_\infty = \|F\|$.

Proof. Let F be defined on $\mathcal{L}_p$ by Equation (7). If $1 \leq p < \infty$ and $\hat{f} \in \mathcal{L}_p$, then

$$|F(\hat{f})| \leq \int_X |f|\ |g|\ d\mu \leq \|f\|_p\ \|g\|_q$$

by Hölder's inequality. Thus F is a bounded linear functional on $\mathcal{L}_p$ for $1 \leq p < \infty$ (the linearity of F follows from that of the integral). Moreover, $\|F\| \leq \|g\|_q$. It remains to show that $\|g\|_q \leq \|F\|$.

Define the measurable function h on X by

$$\begin{aligned} h(x) &= \frac{|g(x)|}{g(x)} && \text{if } 0 < |g(x)| < \infty \\ &= 1 && \text{if } |g(x)| = 0 \text{ or } \infty. \end{aligned} \tag{8}$$

Let (X, Λ, μ) be an arbitrary measure space and let $1 < p < \infty$. Let $f = h|g|^{q-1}$. The function f is measurable, and

$$\|f\|_p = \left(\int_X |g|^{(q-1)p} \, |h|^p \, d\mu\right)^{\frac{1}{p}} = \|g\|_q^{\frac{q}{p}} .$$

Thus $f \in L_p$ and $\hat{f}$ is in the domain of F. Furthermore, since $E = \{x : |g(x)| = \infty\}$ is of measure zero,

$$F(\hat{f}) = \int_X |g|^{q-1} \, h \, g \, d\mu = \int_{X-E} |g|^{q-1} \, hg \, d\mu$$

$$= \int_{X-E} |g|^q \, d\mu = \|g\|_q^{\,q} .$$

Hence

$$\|g\|_q^{\,q} = |F(\hat{f})| \leq \|F\| \, \|f\|_p = \|F\| \, \|g\|_q^{\frac{q}{p}} .$$

If $\|g\|_q \neq 0$, then $\|F\| \geq \|g\|_q^{\,q - \frac{q}{p}} = \|g\|_q$. If $\|g\|_q = 0$, then it is evident that $\|F\| \geq \|g\|_q$.

Now let $p = 1$, $q = \infty$ and let (X, Λ, μ) be a totally σ-finite measure space. If $\mu(X) = 0$, then $\|g\|_\infty = 0 \leq \|F\|$. Thus suppose that $\mu(X) > 0$. Suppose also that there is an $\varepsilon > 0$ such that $\|g\|_\infty > \|F\| + \varepsilon$. If $E = \left\{ x : |g(x)| \geq \|F\| + \frac{\varepsilon}{2} \right\}$, then $\mu(E) > 0$ since otherwise $\|g\|_\infty \leq \|F\| + \varepsilon$. Let $A_1, A_2, \ldots$ be sets of finite positive measure such that $X = \bigcup_{i=1}^{\infty} A_i$. Then there is an integer k such that $0 < \mu(E \cap A_k) < \infty$. For some such k let $A = E \cap A_k$ and let $f = K_A$. Then $f \in L_1$, and

$$\|fg\|_1 = \int_A |g| \, d\mu \geq \int_A \left(\|F\| + \frac{\varepsilon}{2}\right) \, d\mu$$

$$\geq \left(\|F\| + \frac{\varepsilon}{2}\right) \mu(A) .$$

If h is the function defined by Equation (8), then $|h(x)| = 1$ for each $x \in X$. Thus $|f| h \in L_1$, and

$$\int_X |fg| \, d\mu = \int_X |f| hg \, d\mu = F(\widehat{|f|h})$$

$$\leq \|F\| \, \|fh\|_1 = \|F\| \, \mu(A) .$$

Hence, if $\|g\|_\infty > \|F\| + \varepsilon$,

$$(\|F\| + \tfrac{\varepsilon}{2}) \, \mu(A) \leq \|fg\|_1 \leq \|F\| \, \mu(A) .$$

Since $0 < \mu(A) < \infty$, it must follow that $\|F\| + \frac{\varepsilon}{2} \leq \|F\|$. Thus there is no $\varepsilon > 0$ for which $\|g\|_\infty > \|F\| + \varepsilon$; i.e., $\|g\|_\infty \leq \|F\|$. ∎

Theorem 4.16. (Riesz representation theorem). Let (X, Λ, μ) be an arbitrary measure space and let p be a number such that $1 < p < \infty$. If $F \in \mathcal{L}_p^*(X, \Lambda, \mu)$, then there exists a unique $\hat{g} \in \mathcal{L}_q(X, \Lambda, \mu)$ such that

(i) for every $\hat{f} \in \mathcal{L}_p(X, \Lambda, \mu)$

$$F(\hat{f}) = \int_X fg \, d\mu, \quad \text{and}$$

(ii) $\|F\| = \|\hat{g}\|_q$.

Remark. The proof to be given here is essentially based on that to be found in [14].

Proof. Case 1. Suppose that (X, Λ, μ) is such that $A \in \Lambda$ only if $\mu(A) = +\infty$ or $\mu(A) = 0$. Let f be defined on X and suppose that $A = \{x : |f(x)| > 0\}$ has nonzero measure. For $n = 1,2,\ldots,$ let

$A_n = \left\{x : |f(x)| > \frac{1}{n}\right\}$. There is some integer N such that $\mu(A_N) \neq 0$; for, if not,

$$\mu(A) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{i=1}^{\infty} \mu(A_n) = 0.$$

Thus $\mu(A_N) = \infty$, and

$$\|f\|_p^{\,p} \geq \int_{A_N} |f|^p \, d\mu > \frac{1}{N^p}\,\mu(A_N) = \infty .$$

Hence $f \notin L_p$. It follows that $f \in L_p$ only if $|f(x)| = 0$ almost everywhere. Therefore, $\mathcal{L}_p = \{\theta\}$, $\mathcal{L}_p^{\,*} = \{\theta\}$, and $\hat{g} = \theta \in \mathcal{L}_q = \{\theta\}$ satisfies (i) and (ii).

Case 2. Suppose that (X, Λ, μ) is such that there exists an $A \in \Lambda$ for which $0 < \mu(A) < \infty$. Let F be a (fixed) member of $\mathcal{L}_p^{\,*}$. Now (A, Λ_A, μ_A) is a totally finite measure space if Λ_A is the family of measurable subsets of A and $\mu_A(E) = \mu(E)$ for each $E \in \Lambda_A$. If $B \in \Lambda_A$, then $K_B \in L_p(X, \Lambda, \mu)$. Thus define the complex-valued set function λ on Λ_A by

$$\lambda(B) = F(\hat{K}_B) \quad \text{for each } B \in \Lambda_A.$$

Let $B_1, B_2, \ldots$ be a disjoint sequence of measurable subsets of A, and let $B = \bigcup_{i=1}^{\infty} B_i$. If h_n is the characteristic function of

$$\bigcup_{i=1}^{n} B_i \quad \text{for } n = 1, 2, \ldots,$$

then $0 \leq K_B - h_n \leq K_B$ and $\lim_{n \to \infty} h_n = K_B$. By the Lebesgue dominated convergence theorem, $\lim_{n \to \infty} \|K_B - h_n\|_p = 0$. By Theorem 4.5 F is continuous on $\mathscr{L}_p$. Thus

$$\begin{aligned}\lambda(B) &= F(\hat{K}_B) = \lim_{n\to\infty} F(\hat{h}_n) \\ &= \lim_{n\to\infty} F\left(\hat{K}_{\bigcup_{i=1}^{n} B_i}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} F(\hat{K}_{B_i}) \\ &= \sum_{i=1}^{\infty} \lambda(B_i)\ .\end{aligned}$$

Hence λ is countably additive on Λ_A. Since $\lambda(\Phi) = F(\theta) = 0$ and $\lambda(A) < \infty$, λ is a totally finite complex measure on Λ_A. Moreover, if $B \varepsilon \Lambda_A$ and $\mu(B) = 0$, then

$$\lambda(B) = F(\hat{K}_B) = F(\theta) = 0\ .$$

It follows that $\lambda << \mu_A$. By Corollary 4.14 there exists a complex-valued summable function g_A on A such that

$$\lambda(B) = F(\hat{K}_B) = \int_B g_A\, d\mu_A \quad \text{for each } B \varepsilon \Lambda_A\ .$$

The function g_A is unique modulo μ. Define g_A to be zero on $X - A$. Then

$$F(\hat{K}_B) = \int_X K_B g_A\, d\mu \quad \text{for each } B \varepsilon \Lambda_A\ .$$

By the uniqueness of g_A, if $A' \varepsilon \Lambda$ and $\mu(A') < \infty$, then $g_{A'} = g_A$ modulo μ on $A \cap A'$.

Let $A \varepsilon \Lambda$ with $0 < \mu(A) < \infty$. Let $A_1, \ldots, A_n$ be a disjoint sequence of subsets of A such that $\mu(A_i) > 0$ for $i = 1, \ldots, n$. Let

$G = \left\{ f : f = \sum_{i=1}^{n} a_i K_{A_i} \text{ for some complex numbers } a_1, \ldots, a_n \right\}$. Then $G \subset L_p$ and, for each $f \in G$,

$$F(\hat{f}) = \sum_{i=1}^{n} a_i F(\hat{K}_{A_i}) = \sum_{i=1}^{n} a_i \int_X K_{A_i} g_A \, d\mu$$

$$= \sum_{i=1}^{n} a_i \int_{A_i} g_A \, d\mu \, .$$

Define the function $\tilde{f} \in G$ by

$$\tilde{f} = \sum_{j=1}^{n} \left| \frac{F(\hat{K}_{A_j})}{\mu(A_j)} \right|^{q-1} e^{-i\theta_j} K_{A_j} \, ,$$

where θ_j is the argument of $F(\hat{K}_{A_j})$, with the understanding that $\arg(0) = 0$. Since $K_{A_i} K_{A_j} = 0$ if $j \neq k$,

$$\|\tilde{f}\|_p^p = \int_X \sum_{j=1}^{n} \left| \frac{F(\hat{K}_{A_j})}{\mu(A_j)} \right|^{(q-1)p} |K_{A_j}|^p \, d\mu$$

$$= \sum_{j=1}^{n} \left| \frac{F(\hat{K}_{A_j})}{\mu(A_j)} \right|^{q} \mu(A_j) \, .$$

Thus, if $\|\tilde{f}\|_p \neq 0$,

$$\|F\| \geq \frac{|F(\tilde{f})|}{\|\tilde{f}\|_p} = \frac{\left| \sum_{j=1}^{n} \left| \frac{F(\hat{K}_{A_j})}{\mu(A_j)} \right|^{q-1} e^{-i\theta_j} F(\hat{K}_{A_j}) \right|}{\left[\sum_{j=1}^{n} |F(\hat{K}_{A_j})|^q (\mu(A_j))^{1-q} \right]^{\frac{1}{p}}} \geq$$

$$\geq \frac{\left| \sum_{j=1}^{n} \left| F(\hat{K}_{A_j}) \right|^q (\mu(A_j))^{1-q} \right|}{\left[\sum_{j=1}^{n} |F(\hat{K}_{A_j})|^q (\mu(A_j))^{1-q} \right]^{\frac{1}{p}}}$$

$$\geq \left[\sum_{j=1}^{n} |F(\hat{K}_{A_j})|^q (\mu(A_j))^{1-q} \right]^{\frac{1}{q}} \tag{9}$$

If $\|\tilde{f}\|_p = 0$, then $F(\hat{K}_{A_j}) = 0$ for $j = 1,\ldots, n$ by definition of $\tilde{f}$. In this case Inequality (9) merely asserts that $\|F\| \geq 0$.

Let A be a fixed set of finite measure, and let g_A be defined as before. For $n = 1,2,\ldots$ define the function g_n on X as follows:

$$\mathrm{Re}(g_n(x)) = \min\left(\frac{k}{n}, n\right) \quad \text{if} \quad 0 \leq \frac{k}{n} \leq \mathrm{Re}(g_A(x)) < \frac{k+1}{n}$$

for some integer k;

$$\mathrm{Re}(g_n(x)) = \max\left(-\frac{k}{n}, -n\right) \quad \text{if} \quad 0 \leq \frac{k}{n} \leq \mathrm{Re}(-g_A(x)) < \frac{k+1}{n}$$

for some integer k;

$\mathrm{Im}\,(g_n(x))$ is defined analogously.

Thus $\lim_{n\to\infty} g_n(x) = g_A(x)$ for each $x \in X$. Note that each g_n is a simple function, and each is thus of the form

$$g_n = \sum_{i=1}^{p_n} C_i^n A_i^n \quad \text{where} \quad \bigcup_{i=1}^{p_n} A_i^n \subset A$$

If $\mu(A_i^n) = 0$ for any i and n, redefine g_n to be zero on A_i^n.

In this way choose integers M_n $(n = 1,2,\ldots)$ and relabel the disjoint sets A_i^n in such a way that $\mu(A_i^n) > 0$ for $i = 1,2,\ldots, M_n$ and $n = 1,2,\ldots$. Let $x_i^n \varepsilon A_i^n$ for $i = 1,\ldots, M_n$ and $n = 1,2,\ldots$. Then, for $i = 1,\ldots, M_n$ and $n = 1,2,\ldots$,

$$\left| \mathrm{Re} \left(\int_{A_i^n} g_A \, d\mu \right) \right| = \left| \int_{A_i^n} \mathrm{Re}(g_A) \, d\mu \right| \geq \left| \int_{A_i^n} \mathrm{Re}(g_n) \, d\mu \right|$$

$$\geq \left| \mathrm{Re}(g_n(x_i^n)) \right| \mu(A_i^n) .$$

Similarly,

$$\left| \mathrm{Im} \left(\int_{A_i^n} g_A \, d\mu \right) \right| \geq |\mathrm{Im}\, (g_n(x_i^n))| \, \mu(A_i^n) .$$

Hence, for $i = 1,\ldots, M_n$ and $n = 1,2,\ldots,$

$$\left| \int_{A_i^n} g_A \, d\mu \right| \geq \left\{ \left[\mathrm{Re}(g_n(x_i^n)) \right]^2 + \left[\mathrm{Im}\, (g_n(x_i^n)) \right]^2 \right\}^{\frac{1}{2}} \mu(A_i^n)$$

$$\geq |g_n \, (x_i^n)| \, \mu(A_i^n) .$$

By Inequality (9) it follows that, for $n = 1,2,\ldots,$

$$\|F\| \geq \left[\sum_{i=1}^{M_n} |F(\hat{K}_{A_i^n})|^q \, (\mu(A_i^n))^{1-q} \right]^{\frac{1}{q}}$$

$$\geq \left[\sum_{i=1}^{M_n} \left| \int_{A_i^n} g_A \, d\mu \right|^q \, (\mu(A_i^n))^{1-q} \right]^{\frac{1}{q}}$$

$$\geq \left[\sum_{i=1}^{M_n} |g_n(x_i^n)|^q \, \mu(A_i^n) \right]^{\frac{1}{q}}$$

$$\geq \left[\sum_{i=1}^{M_n} \int_X |g_n(x_i^n)|^q K_{A_i^n} \, d\mu \right]^{\frac{1}{q}}$$

$$\geq \|g_n\|_q .$$

An application of Fatou's lemma results in the inequality

$$\|F\| \geq \liminf \|g_n\|_q \geq \|g_A\|_q .$$

Now, for each $A \varepsilon \Lambda$ for which $\mu(A) < \infty$, let $H(A) = \|g_A\|_q^q$.
Let

$$U = \sup \left\{ H(A) : \mu(A) < \infty \right\} \leq \|F\|^q .$$

If $A, B \varepsilon \Lambda$, $A \subset B$, and $\mu(B) < \infty$, then $g_A = g_B$ modulo μ on A and $g_A = 0$ on $X - A$. Thus $H(A) \leq H(B)$. Hence a sequence $\{ A_i \}$ of sets of finite measure may be chosen so that $A_1 \subset A_2 \subset \ldots$ and $\lim_{n \to \infty} H(A_n) = U$. Let $T = \bigcup_{n=1}^{\infty} A_n$ and let $g = \lim_{n \to \infty} g_{A_n}$ (since $g_{A_n} = g_{A_{n+1}}$ modulo μ on A_n, $\lim_{n \to \infty} g_{A_n}$ exists almost everywhere relative to μ and vanishes on $X - T$). Since $\left\{ |g_{A_n}|^q \right\}$ is a nondecreasing sequence of measurable functions, it follows from the monotone convergence theorem that

$$\int_X |g|^q \, d\mu = \lim_{n \to \infty} \|g_{A_n}\|_q^q = \lim_{n \to \infty} H(A_n) = U < \infty .$$

Thus $g \varepsilon L_q$. Furthermore, if $B \subset A_n$ for some n and $B \varepsilon \Lambda$, then

$$F(\hat{K}_B) = \int_X K_B \, g_{A_n} \, d\mu .$$

Inasmuch as $|K_B g_{A_n}| \le K_B \; |g| \; \varepsilon \; L_1$, the Lebesgue dominated convergence theorem implies that

$$F(\hat{K}_B) = \lim_{n\to\infty} \int_X K_B \, g_{A_n} \, d\mu = \int_X K_B \, g \, d\mu \, . \tag{10}$$

Suppose $C \; \varepsilon \; \Lambda$ is such that $T \cap C = \Phi$. Then

$$\lim_{n\to\infty} H(A_n \cup C) = \lim_{n\to\infty} [H(A_n) + H(C)] = U + H(C) \, .$$

By definition of U, $H(C) = 0$. Now let $A_o \; \varepsilon \; \Lambda$ be a set of finite measure for which $H(A_o) > 0$. Thus $A_o \cap T \neq \Phi$, and $K_{A_o} = K_{A_o \cap T} + K_{A_o - T}$. Since $T \cap (A_o - T) = \Phi$, $H(A_o - T) = 0$ and

$$F(\hat{K}_{A_o - T}) = \int_X K_{A_o - T} \, g_{A_o - T} \, d\mu = 0$$

Thus $F(\hat{K}_{A_o}) = F(\hat{K}_{A_o \cap T})$. Note that $K_{A_o \cap T} = \lim_{n\to\infty} K_{A_o \cap A_n}$ since $A_1 \subset A_2 \subset \ldots$. Also observe that $|K_{A_o \cap T} - K_{A_o \cap A_n}|^p \le |K_{A_o \cap T}|^p$. By the Lebesgue dominated convergence theorem, $\lim_{n\to\infty} \|K_{A_o \cap T} - K_{A_o \cap A_n}\|_p = 0$.

Since $A_o \cap A_n \subset A_n$, it follows from Equation (10) and the above discussion that

$$F(\hat{K}_{A_o}) = F(\hat{K}_{A_o \cap T}) = \lim_{n\to\infty} F(\hat{K}_{A_o \cap A_n})$$

$$= \lim_{n\to\infty} \int_X K_{A_o \cap A_n} \, g \, d\mu \, .$$

Moreover, $|K_{A_o \cap A_n} \, g| \le K_{A_o} \, |g| \; \varepsilon \; L_1$ because $\mu(A_o) < \infty$ and $g \; \varepsilon \; L_q$.

Another application of the dominated convergence theorem results in the equation

$$F(\hat{K}_{A_o}) = \int_X K_{A_o \cap T}\, g \, d\mu = \int_X K_{A_o}\, g \, d\mu \tag{11}$$

because $g = 0$ on $X - T$. Now let $A_o \varepsilon \Lambda$ be a set of finite measure for which $H(A_o) = 0$. As noted previously, $g_{A_o} = g_{A_n}$ modulo μ on $A_o \cap A_n$. Hence $g_{A_o} = g$ modulo μ on $A_o \cap T$. Since $H(A_o) = \|g_{A_o}\|_q^q = 0$,

$$\int_X K_{A_o}\, g \, d\mu = \int_X K_{A_o \cap T}\, g \, d\mu = \int_X K_{A_o \cap T}\, g_{A_o} = 0 \, .$$

Moreover, $F(\hat{K}_{A_o}) = \int_X K_{A_o} g_{A_o} \, d\mu = 0$. Hence Equation (11) is valid for any $A_o \varepsilon \Lambda$ which has finite measure.

Let $f = \sum_{i=1}^{n} a_i K_{B_i} \varepsilon L_p$ where $a_i \neq 0$ for any i. Since $\|f\|_p^p \geq |a_i| \, \mu(B_i)$ for $i = 1, \ldots, n$, $\mu(B_i) < \infty$ for $i = 1, \ldots, n$. By Equation (11) it follows that

$$F(\hat{f}) = \sum_{i=1}^{n} a_i \, F(\hat{K}_{B_i}) = \sum_{i=1}^{n} a_i \int_X K_{B_i}\, g \, d\mu$$

$$= \int_X f \, g \, d\mu \, . \tag{12}$$

Now let $f \varepsilon L_p$. By Theorem 3.7 there is a sequence $\{f_n\}$ of measurable simple functions such that $\lim_{n \to \infty} \|f - f_n\|_p = 0$ and $\mu \{x : f_n(x) \neq 0\} < \infty$ for $n = 1, 2, \ldots$. By Equation (12), for $n = 1, 2, \ldots$,

$$F(\hat{f}_n) = \int_X f_n \, g \, d\mu .$$

Since F is continuous on $\mathscr{L}_p$, $\lim_{n\to\infty} F(\hat{f}_n) = F(\hat{f})$. The functional G defined on $\mathscr{L}_p$ by

$$G(\hat{f}) = \int_X f \, g \, d\mu \quad \text{for } \hat{f} \in \mathscr{L}_p$$

is also continuous by Theorem 4.15. Thus

$$\lim_{n\to\infty} \int_X f_n \, g \, d\mu = \int_X f \, g \, d\mu .$$

Hence, for each $\hat{f} \in \mathscr{L}_p$,

$$F(\hat{f}) = \int_X f \, g \, d\mu .$$

Since $g \in L_q$, it follows from Theorem 4.15 that $\|g\|_q = \|F\|$.

To show that the $\hat{g} \in \mathscr{L}_q$ determined by g is unique, suppose that $g_1, g_2 \in L_q$ and

$$F(\hat{f}) = \int_X f g_1 d\mu = \int_X f g_2 d\mu \quad \text{for each } \hat{f} \in \mathscr{L}_p.$$

From Theorem 4.15 it follows immediately that $\|F\| = \|g_1\|_q = \|g_2\|_q$. Thus $\hat{g}_1 = \hat{g}_2$. ∎

To deduce a result analogous to that of Theorem 4.16 for $\mathscr{L}_1^*$, it is necessary to restrict the measure space (X, Λ, μ) further. An example to show that Theorem 4.16 cannot be extended without modification to $\mathscr{L}_1^*$ may be found in [14]. It is sufficient, however, to require that (X, Λ, μ) be totally σ-finite.

<u>Theorem 4.17</u>. Let (X, Λ, μ) be a totally σ-finite measure space. If $F \in \mathcal{L}_1^*(X, \Lambda, \mu)$, then there exists a unique $\hat{g} \in \mathcal{L}_\infty(X, \Lambda, \mu)$ such that

(i) for every $\hat{f} \in \mathcal{L}_1 (X, \Lambda, \mu)$

$$F(\hat{f}) = \int_X fg \, d\mu, \text{ and}$$

(ii) $\|F\| = \|\hat{g}\|_\infty$.

<u>Proof</u>. Case 1. Suppose that $\mu(X) < \infty$. In Theorem 4.16 it was shown that, for each $F \in \mathcal{L}_p^*$ $(1 < p < \infty)$, there is a complex-valued summable function g such that

$$F(\hat{K}_A) = \int_X K_A \, g \, d\mu \quad \text{for each } A \in \Lambda .$$

The corresponding result for $F \in \mathcal{L}_1^*$ follows from this since the assumption that $p > 1$ was not used in that portion of the proof of Theorem 4.16. Thus suppose that $F \in \mathcal{L}_1^*$ and that $g \in L_1$ is such that

$$F(\hat{K}_A) = \int_X K_A \, g \, d\mu \quad \text{for each } A \in \Lambda.$$

If $f \in L_1$ is a simple function, then it follows that

$$F(\hat{f}) = \int_X f \, g \, d\mu .$$

Let $g_1 = \text{Re}(g)$ and $g_2 = \text{Im}(g)$. For each f in the real space L_1, write

$$F(\hat{f}) = F_1(\hat{f}) + i \, F_2(\hat{f})$$

where F_1 and F_2 are real-valued functionals. The functionals F_1 and F_2 are bounded linear functionals on real $\mathcal{L}_1$. It follows that, for any simple function f in real L_1,

$$F_1(\hat{f}) = \int_X f\, g_1\, d\mu \quad \text{and} \tag{13}$$

$$F_2(\hat{f}) = \int_X f\, g_2\, d\mu \ .$$

Let $P = \{x : g_1(x) \geq 0\}$ and let f be any nonnegative function in real L_1. Let $\{f_n\}$ be a nondecreasing sequence of nonnegative measurable simple functions such that $f_1(x) \leq f_2(x) \leq \ldots \leq f(x)$ and $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \varepsilon X$ (cf. Theorem 1, Appendix). For $n = 1,2,\ldots,$ $|f - f_n|K_P \leq |f|$; by the Lebesgue dominated convergence theorem, $\lim_{n \to \infty} \int_X K_P\, |f - f_n|\, d\mu = 0$. Since F_1 is continuous on $\mathcal{L}_1$, $\lim_{n \to \infty} F_1(\widehat{K_P f_n}) = F_1(\widehat{K_P f})$. The sequence $\{f_n\, |g_1|\}$ is a nondecreasing sequence of nonnegative measurable functions. Furthermore, for $x \varepsilon P$ $0 \leq f_n(x)\, g_1(x)$. By the monotone convergence theorem and Equation (13),

$$\int_X K_P f g_1\, d\mu = \int_P f\, g_1\, d\mu = \lim_{n \to \infty} \int_P f_n g_1\, d\mu$$

$$= \lim_{n \to \infty} F_1(\widehat{f_n K_P}) = F_1(\widehat{K_P f}) < \infty \ .$$

Now $X - P = \{x : g_1(x) < 0\}$. For any nonnegative function f in real L_1, an analogous argument shows that

$$F_1(\widehat{fK_{X-P}}) = \int_X fK_{X-P}\, g_1\, d\mu > -\infty \ .$$

For each nonnegative f in real L_1, it thus follows that $fg_1 \varepsilon L_1$, and

$$F_1(\hat{f}) = F_1(\widehat{fK_P}) + F_1(\widehat{fK_{X-P}})$$

$$= \int_P fg_1 \, d\mu + \int_{X-P} fg_1 \, d\mu$$

$$= \int_X fg_1 \, d\mu \ .$$

For any f in real L_1, write $f = f^+ - f^-$. Then

$$F_1(\hat{f}) = F_1(\widehat{f^+}) - F_1(\widehat{f^-}) = \int_X f^+ g_1 d\mu - \int_X f^- g_1 d\mu$$

$$= \int_X fg_1 \, d\mu \ , \text{ and } fg_1 \ \varepsilon \ L_1 \ .$$

In the same way it follows that

$$F_2(\hat{f}) = \int_X fg_2 \, d\mu$$

for each f in real L_1. Thus, for each f in real L_1,

$$F(\hat{f}) = F_1(\hat{f}) + i \ F_2(\hat{f}) = \int_X f(g_1 + ig_2) \, d\mu$$

$$= \int_X f \ g \ d\mu \ .$$

Finally, for any $f \ \varepsilon \ L_1$,

$$F(\hat{f}) = F(\widehat{Re(f)}) + iF \ (\widehat{Im(f)})$$

$$= \int_X fg \ d\mu \ , \tag{14}$$

and $f\,g \in L_1$.

As is the proof of Theorem 4.15, define the measurable function h by

$$h(x) = \frac{|g(x)|}{g(x)} \quad \text{if } 0 < |g(x)| < \infty,$$

$$= 1 \quad \text{if } |g(x)| = 0 \text{ or } \infty.$$

Let $f_1 = |g|h$. Then, since $g \in L_1$ and $|h| = 1$, $f_1 \in L_1$. Thus, since $\|h\|_1 = \mu(X) < \infty$,

$$\int_X |g|^2\, d\mu = \int_X f_1 g\, d\mu = F(\hat{f}_1) \leq \|F\|\, \|f_1\|_1$$

$$\leq \|F\| \int_X |g|\, d\mu = \|F\| \int_X h\, g\, d\mu$$

$$\leq \|F\|\, F(\hat{h}) \leq \|F\|^2\, \|h\|_1 \leq \|F\|^2\, \mu(X)\ .$$

Therefore $g \in L_2$, and the function $f_2 = |g|^2\, h \in L_1$. For each positive integer n, define $f_n = |g|^n\, h$. By induction it follows that

$$\int_X |g|^n\, d\mu \leq \|F\|^n\, \mu(X)\ .$$

For each positive integer k, let $A_k = \{x : |g(x)| \geq k\}$. Then

$$k^n\, \mu(A_k) \leq \int_{A_k} |g|^n\, d\mu \leq \int_X |g|^n\, d\mu$$

$$\leq \|F\|^n\, \mu(X).$$

If $\mu(X) > 0$, then, for fixed k and for $n = 1,2,\ldots,$

$$\left[\frac{\mu(A_k)}{\mu(X)}\right]^{\frac{1}{n}} \leq \frac{\|F\|}{k} .$$

Thus, if k is an integer for which $\mu(A_k) > 0$,

$$\lim_{n \to \infty} \left[\frac{\mu(A_k)}{\mu(X)}\right]^{\frac{1}{n}} = 1 \leq \frac{\|F\|}{k} .$$

However, if $\mu(A_k) > 0$ and $k > \|F\|$, the preceding inequality is contradicted. Thus $\mu(A_k) = 0$ for each $k > \|F\|$. Hence $\mu\{x : |g(x)| > \|F\|\} = 0$ and $\|g\|_\infty \leq \|F\|$. If $\mu(X) = 0$ and $\|g\|_\infty \leq \|F\|$. If $\mu(X) = 0$, then $\|g\|_\infty = 0 \leq \|F\|$. Finally, it follows from Equation (14) and Hölder's inequality that $\|F\| \leq \|g\|_\infty$. It now follows from Equation (14) and Theorem 4.15 that the equivalence class $\hat{g}$ induced by g is unique.

Case 2. Suppose that (X, Λ, μ) is totally σ-finite. Let $\{A_n\}$ be an increasing sequence of sets of finite measure such that $X = \bigcup_{n=1}^{\infty} A_n$. For each n, (A_n, Λ_n, μ) is a totally finite measure space if Λ_n denotes the family of all measurable subsets of A_n. Thus, for $n = 1,2,\ldots,$ there is a unique $\hat{g}_n \in \mathcal{L}_\infty$ such that

$$F(\widehat{fK_{A_n}}) = \int_X fK_{A_n} g_n d\mu \quad \text{for each } f \in L_1 \tag{15}$$

and $\|F\| \geq \|g_n\|_\infty$ (the restriction of F to functions vanishing outside A_n is a functional having norm less than or equal to the norm of F). Moreover, since $A_n \subset A_{n+1}$ for $n = 1,2,\ldots,$

$$F(\widehat{fK_{A_n}}) = F\left(\widehat{fK_{A_n} K_{A_{n+1}}}\right)$$

$$= \int_X fK_{A_n} K_{A_{n+1}} g_{n+1} \, d\mu$$

$$= \int_{A_n} f \, g_{n+1} \, d\mu$$

for each $f \varepsilon L_1$. By Equation (15) and the uniqueness of g_n, it follows that $g_n = g_{n+1}$ modulo μ on A_n. Thus $\lim_{n\to\infty} g_n(x) = g_k(x)$ for almost all $x \varepsilon A_k$ $(k = 1,2,\ldots)$. Let $g(x) = \lim_{n\to\infty} g_n(x)$ where the limit exists and set $g(x) = 0$ elsewhere. Since $\|g_n\|_\infty \leq \|F\|$ for $n = 1,2,\ldots,$ $|g_n(x)| \leq \|F\|$ for almost all $x \varepsilon X$. Thus $\lim_{n\to\infty} |g_n(x)| = |g(x)| \leq \|F\|$ for almost all $x \varepsilon X$ and $\|g\|_\infty \leq \|F\|$. Now, since $g = g_k$ modulo μ on A_k, it follows from Equation (15) that

$$F(\widehat{fK_{A_k}}) = \int_X fK_{A_k} \, g \, d\mu \quad \text{for each } f \varepsilon L_1$$

and $k = 1,2,\ldots$. Now $|fK_{A_k} g| \leq |fg|$, $\lim_{k\to\infty} fK_{A_k} g = fg$, and $|f - fK_{Ak}| \leq |f|$. Moreover, for $f \varepsilon L_1$

$$\|fg\|_1 \leq \|f\|_1 \, \|g\|_\infty < \infty \, .$$

By the Lebesgue dominated convergence theorem,

$$\lim_{k\to\infty} \int_X fK_{A_k} \, g \, d\mu = \int f \, g \, d\mu$$

and

$$\lim_{k \to \infty} \int_X |f - fK_{A_k}| \, d\mu = 0 .$$

Since F is continuous on $\mathcal{L}_1$,

$$F(\hat{f}) = \lim_{k \to \infty} F(\widehat{fK_{A_k}})$$

$$= \lim_{k \to \infty} \int_X fK_{A_k} g \, d\mu = \int_X f \, g \, d\mu$$

for each $f \in L_1$. By Hölder's inequality $\|F\| \leq \|g\|_\infty$. Since the reverse inequality was previously obtained, $\|F\| \leq \|g\|_\infty$. Since the reverse inequality was previously obtained, $\|F\| = \|g\|_\infty$. The uniqueness of $\hat{g}$ follows immediately from Theorem 4.15. ∎

The Riesz representation theorem cannot be extended to linear functionals on $\mathcal{L}_\infty$ (X, Λ, μ) even if μ is Lebesgue measure and $\mu(X) < \infty$. An example illustrating this may be found in Zaanen (cf. [17], pp. 201-2).

For an arbitrary measure space (X, Λ, μ) and $1 < p < \infty$, the spaces $\mathcal{L}_p^*$ and $\mathcal{L}_q$ are closely related. For (X, Λ, μ) a totally σ-finite measure space, there is a similar relationship between $\mathcal{L}_1^*$ and $\mathcal{L}_\infty$. The verification of these statements is now quite simple.

<u>Definition 4.18</u>. Let B_1 and B_2 be Banach (complete normed linear) spaces. An <u>isometric isomorphism</u> of B_1 into B_2 is a one-to-one linear transformation Φ of B_1 into B_2 such that $\|\Phi(x)\|_2 = \|x\|_1$ for every $x \in B_1$. If there is an isometric isomorphism of B_1 onto B_2, then B_1 and B_2 are <u>isometrically isomorphic</u>.

<u>Theorem 4.19</u>. If (X, Λ, μ) is an arbitrary measure space and $1 < p < \infty$, then $\mathcal{L}_p^*(X, \Lambda, \mu)$ and $\mathcal{L}_q$ (X, Λ, μ) are isometrically isomorphic.

If (X, Λ, μ) is a totally σ-finite measure space, then $\mathcal{L}_1^*(X, \Lambda, \mu)$ and $\mathcal{L}_\infty(X, \Lambda, \mu)$ are isometrically isomorphic.

Proof. For $1 \leq p < \infty$ and $1 < q \leq \infty$, define the transformation Φ from $\mathcal{L}_p^*$ to $\mathcal{L}_q$ as follows: for $F \in \mathcal{L}_p^*$, $\Phi(F) = \hat{g}$ if and only if

$$F(\hat{f}) = \int_X f\, g\, d\mu \quad \text{for every } \hat{f} \in \mathcal{L}_p. \tag{16}$$

The transformation Φ is well-defined on all of $\mathcal{L}_p^*$ by Theorems 4.16 and 4.17: for each $F \in \mathcal{L}_p^*$ $(1 \leq p < \infty)$, there is a unique $\hat{g} \in \mathcal{L}_q$ $(1 < q \leq \infty)$ which satisfies Equation (16). If $\Phi(F_1) = \Phi(F_2) = \hat{g}$, then clearly $F_1 = F_2$. Thus Φ is one-to-one. The linearity of Φ is evident, and it follows from Theorem 4.15 that

$$\|\Phi(F)\| = \|\hat{g}\|_q = \|F\| \quad \text{for each } F \in \mathcal{L}_p^*$$

$(1 \leq p < \infty)$. Since each $\hat{g} \in \mathcal{L}_q$ generates an $F \in \mathcal{L}_p^*$, the transformation is onto $\mathcal{L}_q$. Thus, $\mathcal{L}_p^*$ and $\mathcal{L}_q$ are isometrically isomorphic for $1 \leq p < \infty$ and $1 < q \leq \infty$ if (X, Λ, μ) is restricted in the way indicated. ∎

APPENDIX

THEOREMS CITED IN TEXT

Theorem 1. Let (X, Λ, μ) be a measure space and let f be a nonnegative measurable function on X. Then there exists a sequence $\{f_n\}$ of nonnegative measurable simple functions on X such that

(i) $f_1(x) \leq f_2(x) \leq \ldots \leq f(x)$ for each $x \varepsilon X$, and

(ii) $\lim_{n \to \infty} f_n(x) = f(x)$ for each $x \varepsilon X$.

Theorem 2. Let (X, Λ, μ) be a measure space. Let f be a function which is summable over a set $A \varepsilon \Lambda$. If f is positive almost everywhere on A, and if

$$\int_A f \, d\mu = 0, \text{ then } \mu(A) = 0 .$$

Theorem 3. Let (X, Λ, μ) be a measure space. Let f be a function which is summable over X. If $\int_A f \, d\mu = 0$ for every $A \varepsilon \Lambda$, then $f(x) = 0$ for almost all $x \varepsilon X$.

INDEX OF SYMBOLS

C^{∞}	infinitely differentiable functions
E_1^*	extended real line
E_q	q-dimensional Euclidean space
$f : A \rightarrow B$	function with domain A and range a subset of B
$\hat{f} \varepsilon \mathcal{L}_p$	equivalence class of functions (cf. p. 21)
$Im(f)$	imaginary part of f
K	complex numbers
K_A	characteristic function of A
L_p	space of functions f with $\lvert f\rvert^p$ summable (cf. p. 20)
$\mathcal{L}_p$	Banach space corresponding to L_p (cf. p. 21)
$\lVert \cdot \rVert$	norm in a normed linear space
$\lVert \cdot \rVert_p$	norm in $\mathcal{L}_p$
$v^+ - v^-$	Jordan decomposition of a signed measure v (cf. 42)
$v << \mu$	complex measure v is absolutely continuous relative to μ
R	real numbers
$Re(f)$	real part of f
V^*	conjugate (or dual) of V (cf. p. 40)
(X, Λ, μ)	measure space (cf. p. 3)

BIBLIOGRAPHY

1. Apostol, T. M., Mathematical Analysis, Reading, Mass., Addison-Wesley, 1957.

2. Dunford, N. and Schwartz, J. T., Linear Operators, I, New York, Interscience Publishers, 1958.

3. Halmos, P. R., Introduction to Hilbert Space and the Theory of Spectral Multiplicity, New York, Chelsea, 1951.

4. Halmos, P. R., Measure Theory, New York, D. Van Nostrand, 1950.

5. Hewitt, E., Theory of Functions of a Real Variable, New York, Holt, Rinehart, and Winston, 1960.

6. Hörmander, L., Linear Partial Differential Operators, New York, Academic Press, 1963.

7. McShane, E. J., Integration, Princeton, Princeton University Press, 1944.

8. McShane, E. J. and Botts, T. A., Real Analysis, New York, D. Van Nostrand, 1959.

9. Morgan, F. H., The Correspondence Theorem, M. S. Thesis (in process), Georgia Institute of Technology, 1964.

10. Munroe, M. E., Introduction to Measure and Integration, Cambridge, Mass., Addison-Wesley, 1953.

11. von Neumann, J., Functional Operators, Princeton, Princeton University Press, 1950.

12. von Neumann, J., "On Rings of Operators, III," Annals of Mathematics, 41, 1940, pp. 94-161.

13. Saks, S., Theory of the Integral, Warszawa-Lwow, Monografje Matematyczne, 1937.

14. Schwartz, J., "A Note on the Space L_p^*," Proceedings of the American Mathematical Society, 2, 1951, pp. 270-275.

15. Sobolev, S. L., Applications of Functional Analysis in Mathematical Physics, Providence, R. I., American Mathematical Society, 1963.

16. Taylor, A. E., An Introduction to Functional Analysis, New York, Wiley, 1958.

17. Zaanen, A. C., An Introduction to the Theory of Integration, Amsterdam, North Holland, 1958.