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About This Document

This resource contains curriculum for the distance education version of a course offered at the Georgia Institute of Technology, Math 1502, in Fall 2014. This distance education course explored linear algebra, infinite series, and differential equation concepts during lectures and recitations. Recitations are synchronous sessions that offer students an opportunity to apply and review course concepts, which they have been exposed to in lectures. Contained in this curriculum are materials for 26 recitations, available in PDF and presentation slide formats. The slide format is offered for teaching assistants to import directly into web-conferencing software. Slides contain activities that students would solve during recitations. The associated notes contain solutions to corresponding activities and are available in PDF format. A similar version of this work, that corresponds to activities conducted in the Spring 2014 semester is available through SMARTech at https://smartech.gatech.edu/handle/1853/52896

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For Further Information

Questions regarding this document can be directed to Greg Mayer (gsmayer@gmail.com), who would be happy to hear your suggestions on how to improve this document.

Schedule of Activities

The following table presents a list of topics that were explored in the recitation activities. Numbers in brackets correspond to section numbers in the course textbook (Lay, D., Linear Algebra and its Applications, Fourth Edition).

Week	Recitation	Topics	Chapters	Format
1	1	Introduction to Math 2401, Vector Parametric Representations of Curves	13.1	PPT
	2	Quadratic Surfaces, Vector Parametric Representations of Curves	12.6, 13.1	PPT
2	3	Quadratic Surfaces, Vector Parametric Representations of Curves	12.6, 13.1	PPT
	4	Projectile Motion, Path Length	13.2, 13.3	PPT
3	5	Projectile Motion, Path Length	13.2, 13.3	PPT
	6	Curvature & Normal Vectors, Tangential & Normal Components of Acceleration	13.4, 13.5	PPT
4	7	Quiz 1 Review	Review	PPT
	8	No Recitation - Quiz 1	NA	
5	9	Domain of Multivariable Function, Limits	14.1, 14.2	LaTeX
	10	Limits, Partial Derivatives, Chain Rule	14.2, 14.3, 14.4	LaTeX
6	11	The Gradient	14.5	LaTeX
	12	Tangent Planes, Absolute Min/Max	14.6, 14.7	LaTeX
7	13	Quiz 2 Review	Review	LaTeX
	14	No Recitation - Quiz 2	NA	
8	15	Lagrange Multipliers	14.8	LaTeX
	16	Lagrange Multipliers, Taylor Approx, Derivatives with Constrained Var	14.8, 14.9, 14.10	LaTeX
9	17	Integration over General Regions	15.2, 15.3	LaTeX
	18	Integration over General Regions	15.2, 15.3	LaTeX
10	19	Quiz 3 Review, Integration with Polar Coordinates	15.4	LaTeX
	20	No Recitation - Quiz 3		
11	21	No Recitation – Spring Break		
	22	No Recitation – Spring Break		
12	23	Triple Integrals in Rectangular Coordinates, Moments of Inertia and Mass	15.5, 15.6	LaTeX

	24	Integration in Cylindrical and Spherical Coordinates	15.7	LaTeX
13	25	Quiz 4 Review, Change of Variables	15.8	LaTeX
	26	No Recitation - Quiz 4		
14	27	Line Integrals; Vector Fields and Line Integrals, Work, Circulation, Flux	16.1, 16.2	PPT
	28	Vector Fields and Line Integrals, Work, Circulation, Flux; Path Independence	16.2, 16.3	PPT
15	29	Vector Fields and Line Integrals, Work, Circulation, Flux; Path Independence	16.2, 16.3	PPT
	30	Green's Theorem, Surface Area	16.4, 16.5	PPT
16	31	Surface Area, Surface Integrals	16.5, 16.6	PPT
	32	Final Exam Review, Stokes Theorem, Divergence Theorem	16.7, 16.8	PPT

Welcome Back!

This session is an opportunity to make sure that your computer is ready for recitations and to familiarize yourself with the software we are using.

Recitation 01: Welcome Back!

Today: Course Organization, Vector Representations of Curves (13.1)

Thursday: Quadratic Surfaces (12.6)

Start-of-Term Survey

Please fill out if you haven't already:

https://www.surveymonkey.com/s/Math2401-2015

Graded Recitation Activities This Semester

- details sent via email
- group work, in Adobe Connect, count towards your pop quiz grade

WebEx and Adobe Connect

1. WebEx for first two weeks
2. online survey to determine if we want to continue using WebEx
3. Adobe Connect for graded group work activities and pop quizzes

Other Announcements

- Piazza isn't set-up yet
- Tegrity is set-up, can view yesterday's lecture (let me know if you can't)
- Two MML HWs due Monday

Quiz and GRA Dates

Tentative Quiz Dates

- Quiz 1: Thursday, January 29
- Quiz 2: Thursday, February 19
- Quiz 3: Thursday, March 12
- Quiz 4: Thursday, April 9

GRAs: Tuesdays before quizzes

- Tue Jan 27
- Tue Feb 17
- Tue Feb 10
- Tue Apr 7

We may have additional GRAs.

Final Exam Exemption and Quizzes

- no mention of exemption in syllabus or course calendar
- the most difficult material in this course is at the end of the semester

Objectives

Throughout this course we find parametric representations of motion and use them to characterize motions.

Today's Learning Objectives
Characterize the two (or three) dimensional motion of an object, in parametric form, in terms of its
- velocity and acceleration
- unit tangent vector

Later in this course we'll use parametric representations of curves to calculate curvature, path length, momentum, and other ways of describing a motion.

I'm assuming you've seen parametric representation of curves in lecture.

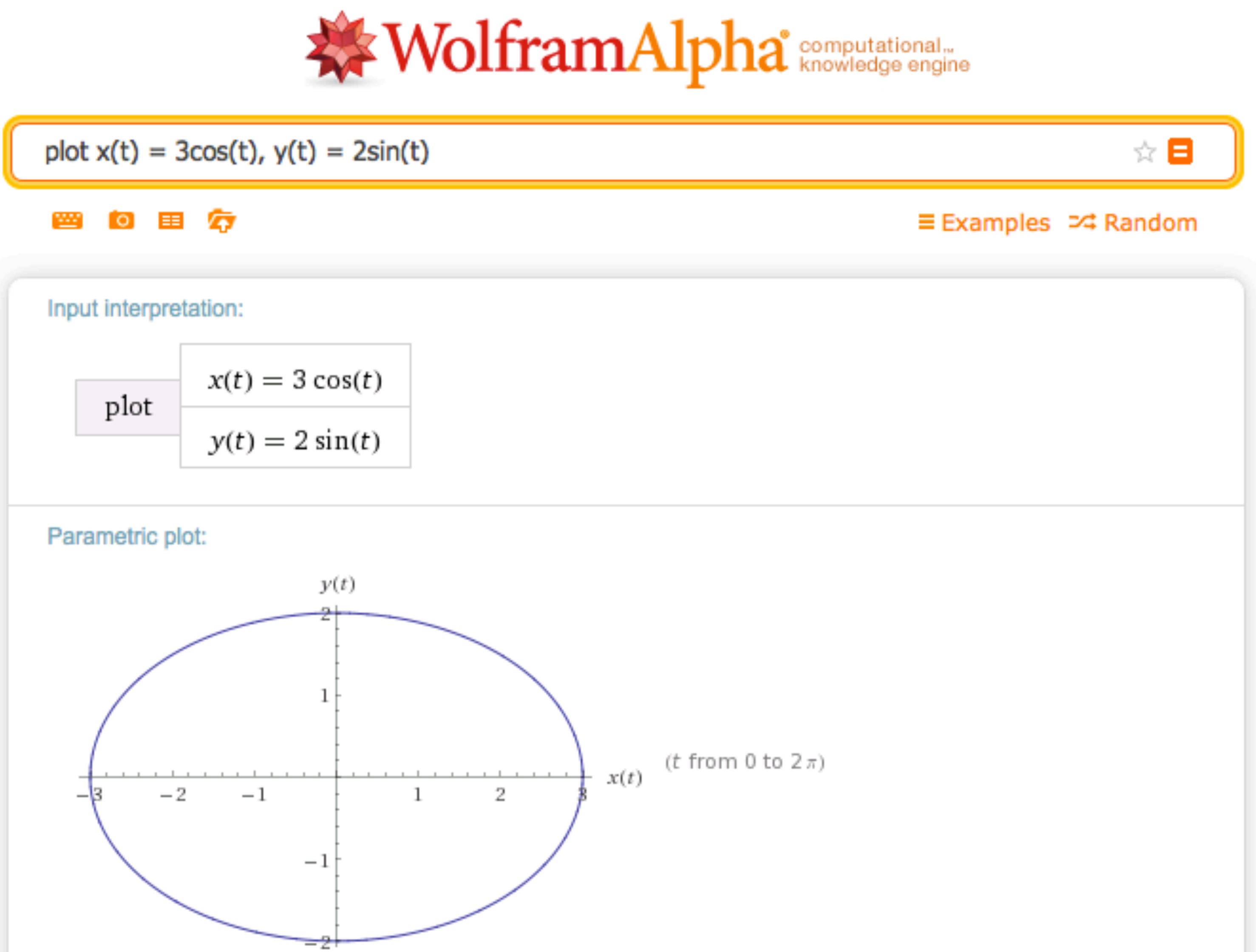
Parametric Representation

Find a parametric representation of the counterclockwise motion that travels along the curve $4x^2 + 9y^2 = 36$. Sketch the motion.

R01
5

Wolfram Alpha Syntax

This is the syntax you would use for plotting parametric curves in WolframAlpha.



Position, Velocity and Acceleration

The position of an object is given by the curve $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j}$, for all t.

a) Sketch the curve.
b) When are the position and velocity vectors perpendicular?
c) When do the position and acceleration vectors have the same direction?
d) Calculate the unit tangent vector for all t.

R01
7

Position and Velocity

The position of a particle is given by r(t). Describe situations where the following is true for all values of t.

$$\vec{r}\left(t\right)\cdot\frac{d\vec{r}}{dt}=0$$

Parametric Vector Representation

Find a parametric vector representation, r(t), of the curve that satisfies the following equations, and y increases when x is positive. Sketch the motion.

$$z = \sqrt{x^2 + y^2}, y = x$$

R019

Parametric Vector Representation

Find a parametric vector representation, r(t), of the curve that satisfies the following equations, and z decreases when x is positive. Sketch the motion.

$$z = \sqrt{4 - x^2 - y^2},\ y^2 + x^2 - 2y = 0$$

Recitation 02

Today: Vector Representations of Curves (13.1), Quadratic Surfaces (12.6)

Start-of-Term Survey

Please fill out if you haven't already:

https://www.surveymonkey.com/s/Math2401-2015

Last Recitation

- Find parametric representations of given curves
- Characterize motion of an object, in parametric form, in terms of its
  - velocity and acceleration
  - unit tangent vector

Today

- Identify and sketch quadratic surfaces given their algebraic equations

Don't Forget

Evidence of inappropriate behavior will be forwarded to the course instructors, and possibly also to the chair of the School of Mathematics and High school facilitators. Evidence will be reviewed to determine if further action is required. Such action could either result in the Georgia Tech's Office of Undergraduate Admissions being made aware of student behavior, and/or all students from a particular school moved to another section where interactions between students from different schools is not possible. Behavior is inappropriate if it can interpreted as hurtful or disrespectful. Students can request to be moved to another section at any time. Questions can be directed to the students teaching assistant and/or the course instructors at any time.

Quadratic Surfaces (12.6)

Sketch and describe the surface $5x^2 + 2y^2 - z^2 = -10$.

Quadratic Surfaces (12.6)

Sketch and describe the surface $5x^2 + 2y^2 - z^2 = -10$.

WolframAlpha computational... knowledge engine

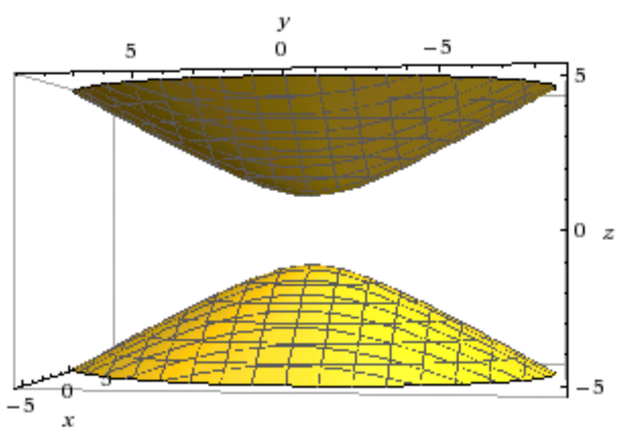
plot 5x^2 + 2y^2 – 6z^2 = –10

Examples Random

Input interpretation:

plot	$5x^2 + 2y^2 - 6z^2 = -10$

Surface plot:



Enable interactivity

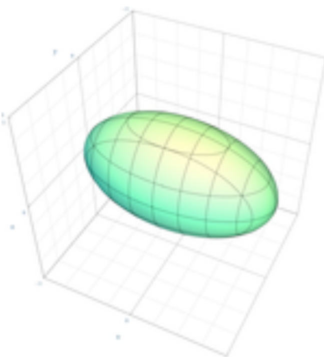
Geometric figure:

two-sheeted hyperboloid

Quadratic Surfaces

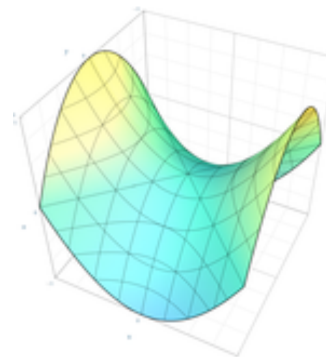
The textbook should list and describe every quadratic surface that you need to be familiar with (but the online textbook currently doesn't work). Wikipedia also has a page that lists and describes every possible quadratic surface (for our course):
http://en.wikipedia.org/wiki/Quadric

Below are four surfaces:

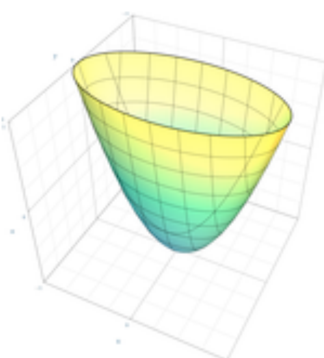


Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

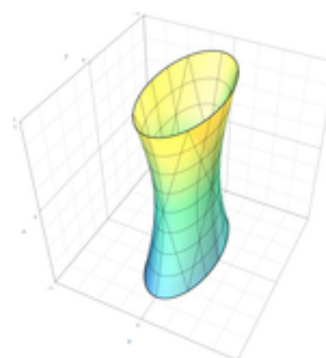


Hyperbolic paraboloid



Elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$$



Elliptic hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

R02
5

Quadratic Surfaces

Identify the correct answer.

The set of all points whose distance from the z-axis is 4 is the:

a) sphere of radius 4 centered on the z-axis
b) line parallel to the z-axis 4 units away from the origin
c) cylinder of radius 4 centered on the z-axis
d) plane $z = 4$

Parametric Vector Representation and Quadratic Surfaces

Find a parametric vector representation of the curve, r(t), that satisfies both quadratic surfaces. Sketch r(t) and both surfaces.

$$z = x^2 + y^2, \quad 5 = x^2 + y^2$$

R02
7

Quadratic Surfaces (12.6)

Consider the surface $z = Ax^2 + By^2$, where A and B are constants. Identify all possible surfaces for the following cases.

i) $A = B = 0$

ii) $AB > 0$

Parametric Vector Representation and Quadratic Surfaces

The following surfaces intersect along a curve, C. Find a) the projection of C onto the xy-plane and b) the parametric vector representation of the projection.

$$z = x^2 + y^2, \quad z = 2y + 3$$

Recitation 03

Today: Group Work on Vector Representations of Curves, Quadratic Surfaces

Hello from San Antonio! Your instructor and I are at a large annual math conference. I hope the wifi is going to hold up for our recitation this morning, many apologies if it doesn't. In case you're interested, this the conference website: http://jointmathematicsmeetings.org/jmm

Textbook: technical issues should be resolved now

Start-of-Term Survey
Please fill out if you haven't already (survey closes Wednesday at midnight):
https://www.surveymonkey.com/s/Math2401-2015

Today: Quadractic Surfaces and Parametric Vectors

- Find parametric representations of given curves
- Characterize motion of an object, in parametric form, in terms of its velocity and acceleration, unit tangent vector
- Identify and sketch quadratic surfaces given their algebraic equations

Group Work Questions

Complete each problem in small groups. The first four questions are from old Math 2401 quizzes (2013 and 2014).

1) Consider the twisted cubic $r(t) = ti + t^2j + t^3k$ and the plane $x + 2y + 3z = 34$.
    a) Where does the cubic intersect the plane?
    b) Find the cosine of the tangent to the curve and the normal to the plane.
2) Find the intersection of the surface $x^2 + 2y^2 = z$ and the plane $x - y = 5$. A parameterization would be fine.
3) Conisder the surface $x^2 - 6x + 4y + y^2 + 8z - z^2 = 4$.
    a) Find the center of the surface.
    b) Name the surface.
    c) Draw a picture of the surface, labelling the center and axes.
4) Conisder the surface $9x^2 - 18x - 16y + 4y^2 - 4z^2 = 11$.
    a) Find the center of the surface.
    b) Name the surface.
    c) Draw a picture of the surface, labelling the center and axes.
5) Create a vector function, $\mathbf{r}(t)$, on the interval $[0, 2\pi]$, that satisfies the conditions $\mathbf{r}(0) = a\mathbf{i}$, and as t increases from 0 to 2π, traces out an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, twice in a counterclockwise manner.

R03

3

1) Consider the twisted cubic $r(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ and the plane $x + 2y + 3z = 34$.

a) Where does the cubic intersect the plane?

b) Find the cosine of the tangent to the curve and the normal to the plane.

R03 2) Find the intersection of the surface $x^2 + 2y^2 = z$ and the plane $x - y = 5$. A
4 parameterization would be fine.

R03
5

3) Conisder the surface $x^2 - 6x + 4y + y^2 + 8z - z^2 = 4$.

a) Find the center of the surface.

b) Name the surface.

c) Draw a picture of the surface, labelling the center and axes.

R03
6

4) Conisder the surface $9x^2 - 18x - 16y + 4y^2 - 4z^2 = 11$.

a) Find the center of the surface.
b) Name the surface.
c) Draw a picture of the surface, labelling the center and axes.

R037

5) Create a vector function, **r**(t), on the interval [0, 2π], that satisfies the conditions **r**(0) = a**i**, and as t increases from 0 to 2π, traces out an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, twice in a counterclockwise manner.

Recitation 04

Today: Displacement, Velocity, Acceleration (13.2), Path Length (13.3)

Homework: Due Tonight and Monday

Learning Obectives for Today: Characterize motion of an object, in parametric form, in terms of its unit tangent vector, acceleration, path length (aka arc length).

Photo by Wikimedia Commons user Kreuzschnabel

R04

2

Particle Motion

Let $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

a) How is the unit tangent vector, $\mathbf{T}(t)$, defined mathematically?

b) Suppose $x = t^2$, $y = t^3$, $z = t^2$, and $t \geq 0$. Then what is the unit tangent vector when $t = 0$?

R04 **Differential Equation**

3 Solve the following initial value problem.

$$\vec{F}(t) = m\vec{r}\,''(t) = t\hat{i} + t^2\hat{j},\ \vec{r}(0) = \hat{i},\ \vec{v}(0) = \hat{k}.$$

R04 **Velocity and Acceleration**

4 What constant acceleration must a particle experience if it is to travel from (1,2,3) to (4,5,7) along the straight line joining the points, starting from rest, and covering the distance in 2 units of time?

R04

Velocity and Position

5 **r**(t) is the position of a moving particle.

a) Describe, in words, what **r**’ is parallel to.

b) Show that ||**r**(t)|| is constant iff **r** $\perp$ **r**’

R04
6

The Hanging Cable

The hanging cable, also referred to as a ______________ , has the shape:

Photo by Flickr user Robert Valencia

R04 A cable is suspended between two poles that are 10 m apart. Find the length of the cable, if
7 the cable’s shape is $y(x) = k\,[\cosh(x/k) - 1]$.

Recitation 05

Today's Topics

- Projectile Motion (13.2)
- Path Length and Tangential Vector (13.3)
- Curvature & Normal Vectors (13.4)

Today's Learning Obectives

- Apply vector function integration to determine path of projectiles
- Characterize motion of an object, in parametric form, in terms of its arc length and its tangential, normal and binormal vectors

Announcements

Survey Results: students want to collaborate, have trouble with technical issues and not knowing how to solve problems in group work. So lets use Adobe Connect, keep group size to 4 to 6, use group work on stuff covered from last assignments.

Thursday Recitation: 13.4, 13.5, Adobe Connect
Graded Recitation Activity: Next week during Tuesday recitation, question coming soon
HW Due Tomorrow: 13.4, 13.5
Quiz 1: Thur Jan 29
Office Hours: 7:30 pm – 8:30 pm, Wed Jan 21, Wed Jan 28
https://georgiatech.adobeconnect.com/distancecalculusofficehours

Send Your TA an Email
Explain, in an email, using your own words, what the following quantities represent:

- the unit tangent vector, $\mathbf{T}(t)$
- the curvature, κ

Try to send this email by the end of the day today. If you send your TA an email with a description of what these quantities represent, you will get a reply.

Helpful Formulas

Ideal Projectile Motion: $\vec{r}(t) = (v_0 \cos\alpha) t\hat{i} + \left((v_0 \sin\alpha) t - \frac{gt^2}{2}\right)\hat{j}$

v_0 is the ______________________ , and α is the _________________________________ .

max range: $R = \frac{v_0^2 \sin 2\alpha}{g}$ max height: $\frac{v_0^2 \sin^2 \alpha}{2g}$

Unit tangent vector T = ___________________

Principle unit normal vector N = ___________________

Binormal vector B = ____________________

R05
4

1) Ball Rolling off of a Table (Projectile Motion, 13.2)

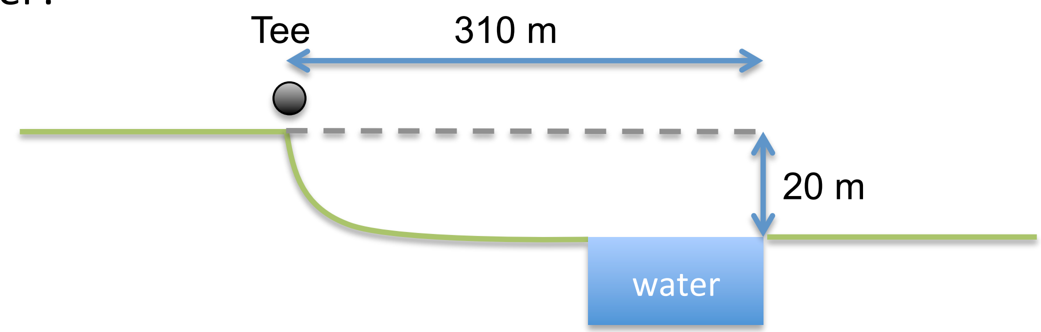
A ball rolls off a table 1 meter high with a speed of 0.5 m/s.

a) At what speed does the ball strike the floor?

b) Where does the ball strike the floor?

2) Golf Ball (Projectile Motion, 13.2)

A golfer can send a golf ball 300m across a level ground. From the tee in the figure, can the golfer clear the water?



3) Arc Length, Normal and Binormal Vectors (13.3, 13.4)

Consider the surfaces $x^2 + y^2 + z^2 = 4$, and $z^2 = x^2 + y^2$ for $z \geq 0$.

a) Find a parameterization for the intersection curve, **r**(t), of the two surfaces.

b) Sketch the two surfaces and their intersection.

c) Calculate the length of **r**(t).

d) Find the unit tangent, normal, and binormal vectors for **r**(t) at the point (sqrt(2) , 0, sqrt(2)).

e) Add the three vectors to your sketch.

R05 **1) Ball Rolling off of a Table (Projectile Motion, 13.2)**

5 A ball rolls off a table 1 meter high with a speed of 0.5 m/s.

a) At what speed does the ball strike the floor?

R05 **1) Ball Rolling off of a Table (Projectile Motion, 13.2)**

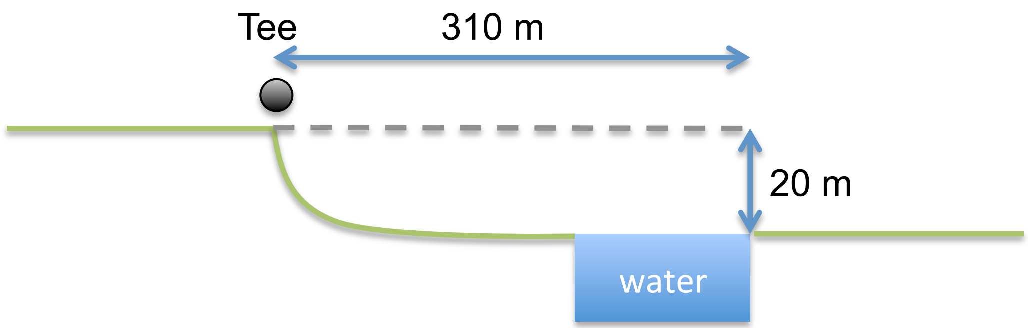
6 A ball rolls off a table 1 meter high with a speed of 0.5 m/s.

b) Where does the ball strike the floor?

R05
7

2) Golf Ball (Projectile Motion, 13.2)

A golfer can send a golf ball 300m across a level ground. From the tee in the figure, can the golfer clear the water?



R05

8

3) Arc Length, Normal and Binormal Vectors (13.3, 13.4)

Consider the surfaces $x^2 + y^2 + z^2 = 4$, and $z^2 = x^2 + y^2$ for $z \geq 0$.

a) Find a parameterization for the intersection curve, **r**(t), of the two surfaces.

b) Sketch the two surfaces and their intersection.

c) Calculate the length of **r**(t).

d) Find the unit tangent, normal, and binormal vectors for **r**(t) at the point (sqrt(2) , 0, sqrt(2)).

e) Add the three vectors to your sketch.

R05 **3) Arc Length, Normal and Binormal Vectors (13.3, 13.4)**

9 Consider the surfaces $x^2 + y^2 + z^2 = 4$, and $z^2 = x^2 + y^2$ for $z \geq 0$.

c) Calculate the length of $\mathbf{r}(t)$.

R05 **3) Arc Length, Normal and Binormal Vectors (13.3, 13.4)**

10 Consider the surfaces $x^2 + y^2 + z^2 = 4$, and $z^2 = x^2 + y^2$ for $z \geq 0$.

d) Find the unit tangent, normal, and binormal vectors for **r**(t) at the point (sqrt(2) , 0, sqrt(2)).

e) Add the three vectors to your sketch.

Recitation 06

Today's Topics:

- Curvature & Normal Vectors (13.4)
- Tangential and Normal Components of Acceleration (13.5)
- Veocity and Acceleration in Polar Coordinates (13.6)

Today's Learning Obectives

1. Given a motion of an object, in either parametric form or as a function of a single variable, calculate the
   - curvature
   - tangent, normal, and binormal vectors
   - acceleration (tangential and normal components)
   - torsion
2. Calculate the osculating, normal, and rectifying planes for a given curve **r**(t) at a given value of t

Helpful Formulas

principle normal vector: $\vec{N} = \dfrac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$

curvature: $\kappa = \dfrac{1}{|\vec{v}|}\left|\vec{T}'(t)\right|$

curvature: $\kappa = \dfrac{|f''(x)|}{\left[1+\left(f'(x)\right)^2\right]^{3/2}}$

acceleration: $\vec{a} = a_T\vec{T} + a_N\vec{N}$

$a_T = \dfrac{d}{dt}|\vec{v}|$

$a_N = \sqrt{|\vec{a}| + |a_T|}$

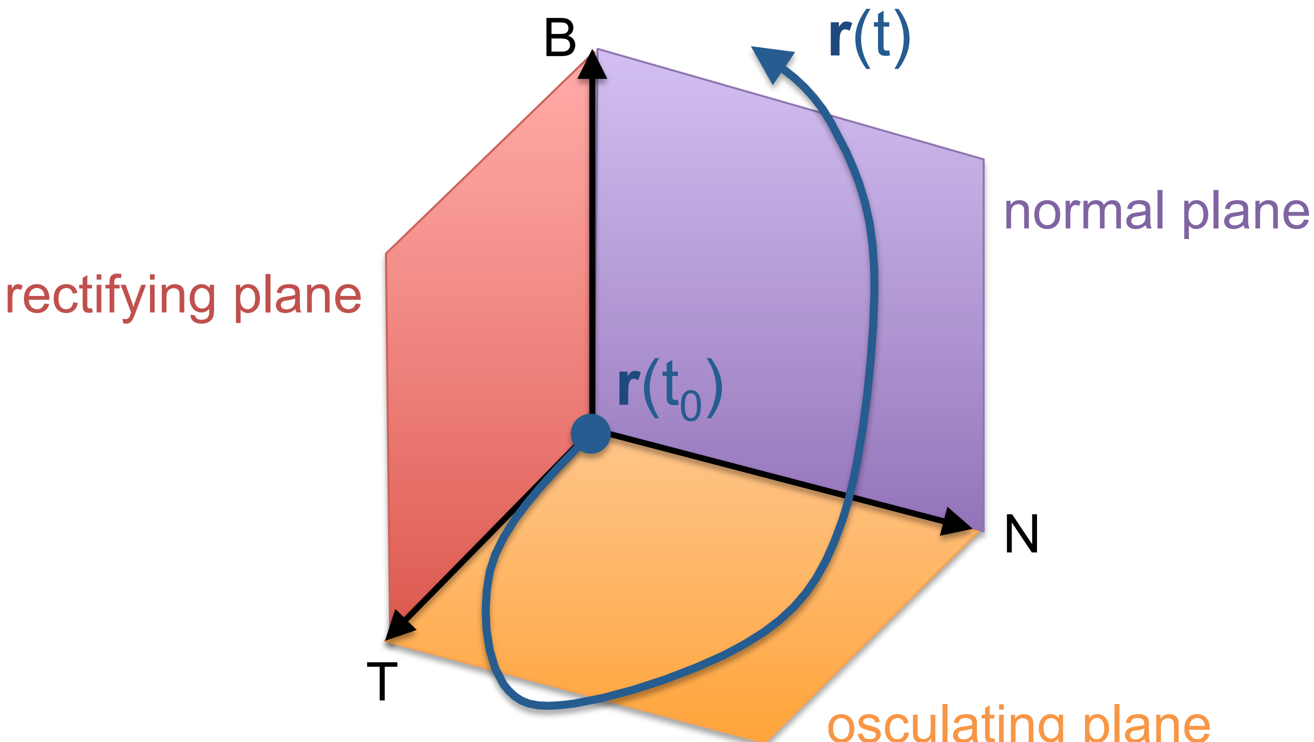
torsion: $\tau = \dfrac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{|\vec{v}\times\vec{a}|^2}$

Notes:

- *One of the above equations has an error, where is it?*
- *There are alternate expressions for these formulas. Above are the formulas that the textbook uses.*

Normal, Rectifying, and Osculating Planes

The geometry of the three planes determined by vectors **T**, **N**, and **B**, for curve **r**(t), at **r**(t_0).



If a motion, **r**(t), lies completely in a plane, then the binormal vector is ________________ .

Announcements

Graded Recitation Activity: Next week during Tuesday recitation, question sent
HW Due Tomorrow: 13.6
Quiz 1: Thur Jan 29
Office Hours: 7:30 pm – 8:30 pm, Wed Jan 28
https://georgiatech.adobeconnect.com/distancecalculusofficehours

Send Your TA an Email

Using your own words, describe

- the relationship between the curvature and the normal plane
- the relationship between the torsion and the osculating plane

Try to send an email with your answers by the end of the day today. If you send your TA an email with an answer to these questions you will get a response.
Hint: these relationships are described in the textbook.

Group Work Activity: Part (a)

There are four parts to the following question. Solve them in groups of 3 to 5 students.

Consider $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}$, $t = -\pi/2$.

a) Find **T**, **N**, and **B** at the given value of t. Is **B** constant for all values of t?

Group Work Activity: Parts (b) and (c)

Consider $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}$, $t = -\pi/2$.

b) Sketch $\mathbf{r}$ for $[0, 2\pi]$ and indicate the direction of motion.

c) Sketch $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the given value of t.

Group Work Activity: Part (d)

Consider $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}$, $t = -\pi/2$.

d) Find the equation of the normal plane at $t = -\pi/2$.

Message your TA when you've finished this question. Move on to the remaining questions after this if there is time.

True or False

a) Curvature is a scalar and can be any real number.

This statement is ______________ *because:*

b) Torsion is a scalar and can be any real number.

This statement is ______________ *because:*

c) If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the normal vector, $\mathbf{N}$, is given by $\mathbf{N} = \mathbf{n}/|\mathbf{n}|$, where $\mathbf{n} = -x'(t)\mathbf{i} + y'(t)\mathbf{j}$.

This statement is ______________ *because:*

Recitation 07

Today's Topics: Quiz 1 Review, Graded Recitation Activity 1

Quiz 1 Topics

12.6 Quadratic Surfaces

13.1 Vector Parametric Representations of Curves

13.2 Quadratic Surfaces

13.2 Projectile Motion

13.2 Path Length

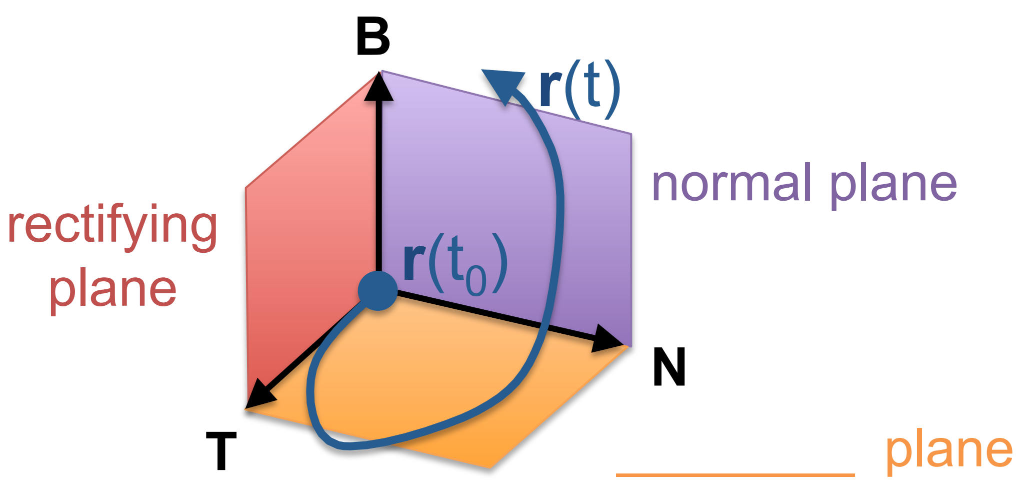
13.3 Curvature & Normal Vectors

13.5 Tangential & Normal Components of Acceleration

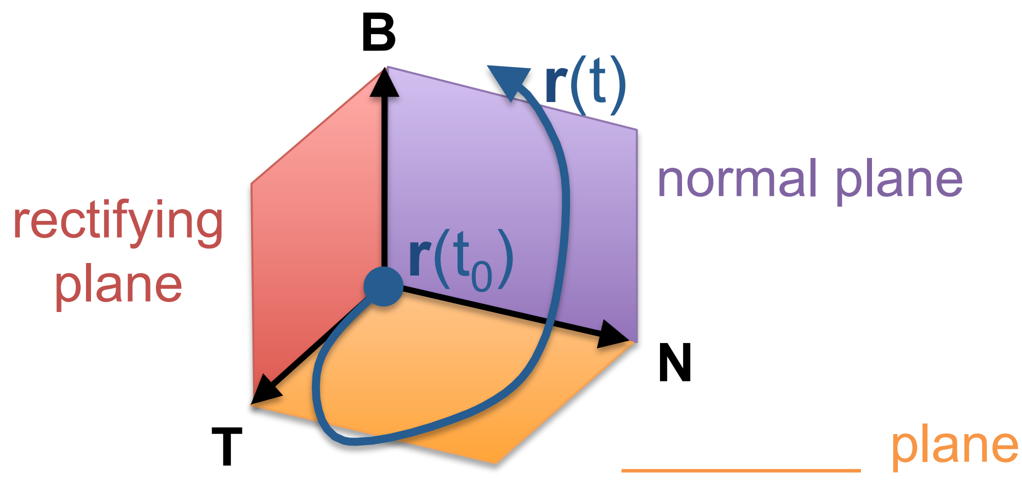
Quiz 1 Learning Objectives

You should be able to do the following for Quiz 1.

- Identify and sketch quadratic surfaces given their algebraic equations
- Develop parameteric representations of curves
- Integrate vector functions to determine projectile motion
- Characterize a motion, given in either parametric form **r**(t), or as a continuous function f(x), using:
  - vectors: velocity, acceleration, tangent, binormal
  - scalars: curvature, torsion, tanential & normal components of accel, arc length
  - planes: tangential, rectifying, ___________



Interpretations of Curvature and Torsion



Cuvature is the rate at which the __________________________ turns.

Torsion is the rate at which the ________________________ turns.

Helpful Formulas

Ideal Projectile Motion: $\vec{r}(t) = (v_0 \cos\alpha)t\hat{i} + \left((v_0 \sin\alpha)t - \frac{gt^2}{2}\right)\hat{j}$

max range: $R = \frac{v_0^2 \sin 2\alpha}{g}$ max height: $\frac{v_0^2 \sin^2\alpha}{2g}$

principle normal vector: $\vec{N} = \vec{T}'(t) \big/ \left|\vec{T}'(t)\right|$

binormal vector: $\vec{B} = \vec{N}'(t) \big/ \left|\vec{N}'(t)\right|$

curvature: $\kappa = \left|\vec{T}'(t)\right| \big/ \left|\vec{v}\right|$

curvature: $\kappa = \left|f''(x)\right| \Big/ \left[1 + \left(f'(x)\right)^2\right]^{3/2}$

acceleration: $\vec{a} = a_T\vec{T} + a_N\vec{N}$

$a_T =$ ________

$a_N =$ ________

torsion: $\tau = \frac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{\left|\vec{v} \times \vec{a}\right|^2}$

R07
5

Graded Group Work Activity

Instructions

- Every student in your group needs to write their name or initials on the board.
- You have 20 minutes to answer the questions below.
- For full marks, show at least three intermediate steps for each question.
- Answer each question on a different slide.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers
- You can use c = cos(t) and s = sin(t)

1) Tangential & Normal Components of Acceleration (4 points)

Let $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$ be a motion. Compute the tangential and normal components of the acceleration.

2) Arc Length (2 points)

Find the arc length, from 0 to t, of the curve $\mathbf{r}(t) = e^t\cos(t)\mathbf{i} + e^t\sin(t)\mathbf{j} + 5e^t\mathbf{k}$.

R07

6

1) Tangential & Normal Components of Acceleration (4 points)

Let $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$ be a motion. Compute the tangential and normal components of the acceleration.

R07
7

2) Arc Length (2 points)

Find the arc length, from 0 to t, of the curve $\mathbf{r}(t) = e^t\cos(t)\mathbf{i} + e^t\sin(t)\mathbf{j} + 5e^t\mathbf{k}$.

Message your TA when you've finished both questions., then move on to the remaining questions.

R07

8

Curvature and Torsion

This question has 4 parts. Consider the surfaces $z = x^2 + y^2$ and $y = 2$, for $z \geq 0$.

A) Find a parametric vector representation for their intersection.

B) Sketch the intersection and the 2 surfaces.

R07 9

Curvature and Torsion

This question has 4 parts. Consider the surfaces $z = x^2 + y^2$ and $y = 2$, for $z \geq 0$.

C) Calculate the curvature and identify on your sketch whre the curvature is maximized.

Curvature and Torsion

This question has 4 parts. Consider the surfaces $z = x^2 + y^2$ and $y = 2$, for $z \geq 0$.

D) Calculate the torsion of the intersecting curve and explain your answer.

Recitation 09

R09 Topics

14.1 Functions of Several Variables

14.2 Limits and Continuity

R09 Learning Objectives

By the end of today's session you should be able to

- Identify and sketch the domain of a function of several variables.
- Determine whether or not limits of functions of several variables exist.

While We're Waiting to Start

Consider the function

$$g(x, y) = \frac{\sqrt{y+1}}{x^2y + xy^2}.$$

For $g(x, y)$ to be defined and a real-valued function, what values of x and y can we allow?

Domain of a Function of Two Variables

Identify and sketch the domain of

$$g(x, y) = \frac{\sqrt{y+1}}{x^2 y + x y^2}.$$

Limits of a Function of Two Variables

Consider the function of two variables

$$f(x,y) = \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}.$$

We want to evaluate

$$\lim_{(x,y)\to(1,0)} f(x,y)$$

What strategies might we try to evaluate the desired limit?

Limits of a Function of Two Variables, Example 1

Evaluate

$$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}$$

Limits of a Function of Two Variables, Example 2

In groups of 3 to 5 students, evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}.$$

Definition of Limit

Evaluating limits along paths will not show that a given limit exists. To show that a limit exists, we can use the definition of limit.

The limit of $f(x, y)$ as (x, y) aproach (a, b) is L if for every number $\epsilon > 0$, there is a corresponding $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon \qquad \text{when } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

In other words, the distance between f and L can be made **arbitrarily** small by making the distance from (x, y) to (a, b) **sufficiently** small.

An Epsilon Delta Example

Evaluate, or show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}.$$

An Epsilon Delta Example

An Epsilon Delta Example

An Epsilon Delta Example

Conclusions: Evaluating Limits of Multivariable Functions

Suppose we need to evaluate a limit of a function of two variables

$$\lim_{(x,y)\to(a,b)} f(x,y).$$

If we know that $f(x,y)$ is continuous at (a,b), we can evaluate the limit with direct substitution. If we don't know that $f(x,y)$ is continuous at (a,b), we can either

- evaluate the limit along curves ($y = mx$, for example) to see if the limit **does not** exist, or
- we can use the definition of limit to prove that the limit **does** exist and determine what the limit is equal to.

Notes:

- evaluating a limit along curves cannot tell us that a given limit exists, it can only tell us whether it doesn't exist
- I'm assuming you're familiar with continuity for a function of several variables, but if you aren't it's on the next homework and isn't a diffcult concept.

Recitation 11

R11 Topics
14.5 The Gradient

R11 Learning Objectives
By the end of today's session you should be able to do the following.

- Compute gradients and directional derivatives.
- Provide geometric interpretations of gradients and directional derivatives.
- Describe the relationship between gradients and level curves.

While We're Waiting to Start
Consider $f(x, y) = y^2 e^{2x}$.

1. Find the direction of steepest ascent at $P(0, 1)$ and at $Q(0, -1)$.
2. Sketch the level curves of f, and the gradient vectors at P and Q.
3. Find the rate at which f is increasing in the direction $\vec{u} = \hat{i} - \hat{j}$ at P.

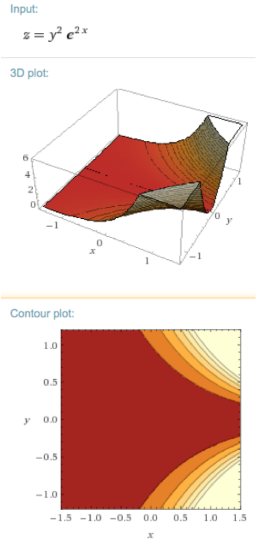
The Gradient and Directional Derivative

Consider $f(x, y) = y^2 e^{2x}$.

1. Find the direction of steepest ascent at $P(0, 1)$ and at $Q(0, -1)$.
2. Sketch the level curves of f, and the gradient vectors at P and Q.
3. Find the rate at which f is increasing in the direction $\vec{u} = \hat{i} - \hat{j}$ at P.

The Gradient and Directional Derivative

Wolfram Alpha's Plots of $f(x, y)$

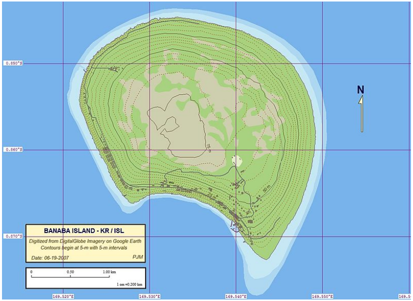


In case it helps see what is going on, to the left are plots of our function, $y^2 e^{2x}$, that WolframAlpha produces.

Notice that the contour plot gives a set of level curves.

Level Curves

If C is in the __________________ of $f(x, y)$, then the curve $C = f(x, y)$ is a **level curve** of $f(x, y)$. For functions of two variables, we can think of level curves as curves of constant **height** (analogous to topographic maps, that have curves of constant elevation).



In other words, a level curve is an intersection between $f(x, y)$ and the plane $z = C$. Level curves are a useful view of the overall behavior of a function.

Banaba Island image under a CCBY2.0 license, available from https://www.flickr.com/photos/evsmap

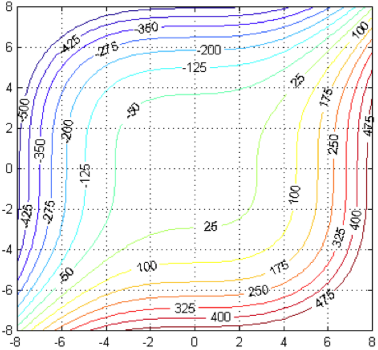
Level Curves and the Gradient

This following helps explain why the gradient is $\perp$ to level curves.

Let $C = g(x, y)$ be a level curve of $g(x, y)$. Show that ∇g is always perpendicular to the level curve.

A Conceptual Question: The Gradient

At which point does the gradient vector have the largest magnitude? Draw the gradient at this point.



1. (0,0)
2. (8,-8)
3. (6,-2)
4. (-4,-4)

Group Work Activities

Solve the following in groups of 3 to 5 students.

1. Find the directional derivative of $f = z\ln(x/y)$ at $(1,1,2)$ towards the point $(2,2,1)$ and provide a geometric interpretation of your answer.
2. For $z = 3xy - x^3 - y^3$, find the points where the gradient vector is the zero vector. Provide a geometric interpretation of your answer.
3. Suppose $\vec{F} = \nabla f(x,y) = (2x + \sin y)\hat{i} + (x\cos(y) - 2y)\hat{j}$. Find $f(x,y)$.

Question 1: A Directional Derivative

Find the directional derivative of $f = z \ln(x/y)$ at $(1, 1, 2)$ towards the point $(2, 2, 1)$. Provide a geometric interpretation of your answer.

Question 2: Zero Gradient

For $z = 3xy - x^3 - y^3$, find the points where the gradient vector is the zero vector. Provide a geometric interpretation of your answer.

Question 3: Constructing a Function From its Gradient

Suppose $\vec{F} = \nabla f(x, y) = (2x + \sin y)\hat{i} + (x\cos(y) - 2y)\hat{j}$. Find $f(x, y)$.

Recitation 12

R12 Topics
14.6 Tangent Planes and Differentials
14.7 Absolute Min/Max

R12 Learning Objectives
By the end of today's session you should be able to do the following.

- Find equations of tangent planes and normal lines of surfaces.
- Apply tangent planes and differentials to make approximations.
- Locate and classify critical points of surfaces.

Example 1
Consider the surface $x^2 + 4y^2 = z^2$.

1. Find the equation of the tangent plane at $P(3, 2, 5)$.
2. Find the equation of the normal line at P, and identify where the normal line intersects the xy-plane.
3. Sketch the surface and the normal line.

Example 1: Part 1

Consider the surface $x^2 + 4y^2 = z^2$. Find the equation of the tangent plane at $P(3, 2, 5)$.

Example 1: Part 2

Consider the surface $x^2 + 4y^2 = z^2$. Find the equation of the normal line at $P(3, 2, 5)$, and identify where the normal line intersects the xy-plane.

Example 1: Part 3

Consider the surface $x^2 + 4y^2 = z^2$. Sketch the surface and the normal line.

Tangent Planes and Differentials (14.6)

For a function of one variable, $y(x)$, we define the differential dy as

$$dy = \frac{dy}{dx}dx,$$

where dy is the change in height of the tangent line.

For a function of two variables, $z(x, y)$, we define the differential dz as

$$dz = \underline{\hspace{3cm}},$$

where dz is the change in height of the ____________ .

The equation of the tangent plane to $z = z(x, y)$ at the point $\vec{r}_0$ is

$$z = z_0 + \nabla z \cdot (\vec{r} - \vec{r}_0)$$

The vector $\vec{r} - \vec{r}_0$ is a vector in the tangent plane.

A Quick Calculation: Tangent Plane Approximation

Suppose $z_x(3,4) = 5$, $z_y(3,4) = -2$, and $z(3,4) = 6$. Assuming the function z is differentiable, what is the best estimate for $z(3.1, 3.9)$ using this information?

1. 6.3
2. 9
3. 6
4. 6.7

Estimating Change in Volume

Estimate, using the tangent plane approximation, the change in volume of a cylinder if its height is changed from 12.0 to 12.2 cm and the radius is changed from 8.0 to 7.7 cm. How much does the volume actually change?

Second Derivative Test (14.7)

Suppose f has continuous 2^{nd} order partial derivatives around some point $P(x_0, y_0)$, and that $\nabla f(x_0, y_0) = 0$. Let

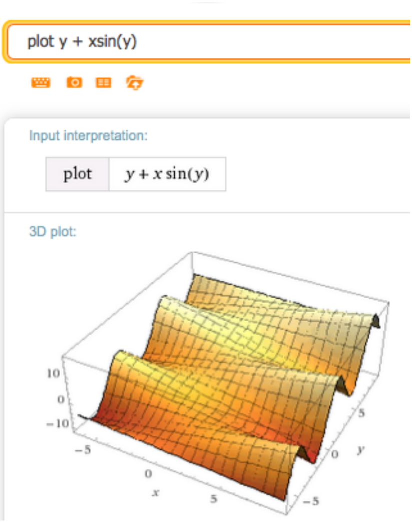
$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $D = 0$, then ____________.
If $D < 0$, then P is a saddle point.
If $D > 0$, then P is a maximum if $f_{xx} < 0$ and a minimum if $f_{xx} > 0$.

Optimization

Find the critical points of $f(x, y) = y + x\sin(y)$ and determine whether they correspond to local or absolute minimums or maximums of $f(x, y)$.

Surface Plot of $f(x, y) = y + x\sin(y)$



Group Work Activities

Solve the following in groups of 3 to 5 students.

1. Consider the function $f(x, y) = 3xy - x^3 - y^3$.
   1.1 Find the points where the gradient vector, $\nabla f(x, y)$, is the zero vector.
   1.2 Find the points where the tangent plane is horizontal.
   1.3 Find the critical points of $f(x, y)$. Classify these points as min, max, or saddle points.
2. Find an equation of the tangent plane and normal line to $z = (x^2 + y^2)^2$ at $P(1, 1, 4)$.

Question 1.1: Zero Gradient

For $f = 3xy - x^3 - y^3$, find the points where the gradient vector, $\nabla f(x, y)$, is the zero vector.

Questions 1.2 and 1.3

Consider the function $f(x, y) = 3xy - x^3 - y^3$. Find the points where the tangent plane is horizontal. Find the critical points of $f(x, y)$. Classify these points as min, max, or saddle points.

Question 2

Find an equation of the tangent plane and normal line to $z = (x^2 + y^2)^2$ at $P(1, 1, 4)$.

Recitation 16

R16 Topics

14.8 Lagrange Multipliers
14.9 Taylor's Formula for Two Variables
14.10 Partial Derivatives with Constrained Variables

R16 Learning Objectives

- Derive the least squares equations to fit the plane $Ax + By + C$ to a set of given points (14.8).
- Calculate a cubic approximation to a function of two variables at a specified point (14.9).
- Apply the chain rule to compute partial derivatives with intermediate variables (14.10).

While We're Waiting to Start

Let $L = f(U, V, S)$, and $S = 3UV$. Calculate or derive expressions for the following derivatives.

A) $\left(\dfrac{\partial S}{\partial V}\right)_U$ B) $\dfrac{dS}{dV}$

C) $\left(\dfrac{\partial L}{\partial V}\right)_U$ D) $\left(\dfrac{\partial L}{\partial V}\right)_{S,U}$

The Chain Rule with Intermediate Variables, Parts A and B

Let $L = f(U, V, S)$, and $S = 3UV$. Calculate or derive expressions for the following derivatives.

$$\text{A)} \quad \left(\frac{\partial S}{\partial V}\right)_U \qquad\qquad \text{B)} \quad \frac{dS}{dV}$$

The Chain Rule with Intermediate Variables, Parts C and D

Let $L = f(U, V, S)$, and $S = 3UV$. Calculate or derive expressions for the following derivatives.

$$\text{C)} \quad \left(\frac{\partial L}{\partial V}\right)_U \qquad\qquad \text{D)} \quad \left(\frac{\partial L}{\partial V}\right)_{S,U}$$

Taylor Approximation (14.9)

Calculate the cubic approximation to $f(x, y) = 4x\cos(y)$ near the origin. Complete this question in group work. *Note: this was a pop quiz in 2014*.

Approximation Error (14.9)

Use your results from the previous problem to find the quadratic approximation to $f(x, y) = 4x\cos(y)$ near the origin. Then estimate the error in the approximation if $|x| < 0.5$ and $|y| < 0.1$.

Least Squares (14.8)

The plane $z = Ax + By + C$ is to be fitted to a given set of points, (x_n, y_n, z_n). Derive the linear system of equations that, when solved, minimizes

$$E = \sum_{n=1}^{N} (Ax_n + By_n + C - z_n)^2.$$

Least Squares (continued)

Least Squares (continued)

Recitation 17

R17 Topics

15.2 Double Integrals over General Regions
15.3 Area by Double Integration

R17 Learning Objectives

- Construct a double integral that represents the area of a region bounded by a set of given curves in Cartesian coordinates.
- Change the order of integration of a double integral (Cartesian coordinates).

Today's Questions

1. Sketch the region bounded by the given curves and construct a double integral that represents its area.
   a) $y = \sqrt{x}$, $y = x^3$.
   b) $x = 5 - y$, $x = 2y - 1$, $y = 1$.
   c) $y = x - 6$, $y^2 = x$.
2. Change the order of integration for the following integrals.

$$\text{a)} \int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx dy \qquad \text{b)} \int_{2}^{1+e} \int_{0}^{\ln(x-1)} f(x,y) dy dx$$

Announcements, WolframAlpha Syntax

GRA3, Next Tuesday (5 points)
Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2+y^2)^2+xy^2=1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

Quiz 3: One Week from Thursday
Quiz 3 may cover 14.8 to 14.10, and 15.1 to 15.4. We'll see.

Wolfram Alpha Syntax for Double Integrals
You may want to use Wolfram Alpha to check your answers while completing your HW. Suppose that we want to determine the value of

$$\int_{-2}^{-1}\int_{0}^{x-1}(x^{2C}+y)dydx$$

The syntax we could use to compute this particular integral is the following.

```
integrate x^{2C}+y, x from -2 to -1 and y from 0 to (x-1)
```

1a) Area of a Region

Sketch the region bounded by $y = \sqrt{x}$, $y = x^3$ and construct a double integral that represents its area.

1b) Area of a Region

Sketch the region bounded by $x = 5 - y$, $x = 2y - 1$, $y = 1$, and construct a double integral that represents its area.

1c) Area of a Region

Sketch the region bounded by $y = x - 6$, $y^2 = x$, and construct a double integral that represents its area.

2a) Changing the Order of Integration

2a) Change the order of integration for the following integral.

$$\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx dy$$

2a) Changing the Order of Integration (continued)

2b) Changing the Order of Integration

Change the order of integration for the following integral.

$$\int_{2}^{1+e} \int_{0}^{\ln(x-1)} f(x,y)\,dy\,dx$$

3) Evaluating an Integral (if time permits)

Evaluate the following double integral.

$$\int_0^4 \int_y^4 e^{x^2}\, dx dy$$

Recitation 18

R18 Topics

15.2 Double Integrals over General Regions

15.3 Area by Double Integration

R18 Learning Objectives

- Construct a double integral that represents the area of a region bounded by a set of given curves in Cartesian coordinates.
- Change the order of integration of a double integral.
- Calculate the average value of a function of two variables.

Today's Questions

1. Change the order of integration.

$$\text{a)} \int_{-1}^{0}\int_{-\sqrt{y+1}}^{\sqrt{y+1}} dxdy \qquad \text{b)} \int_{2}^{1+e}\int_{0}^{\ln(x-1)} f(x,y)dydx$$

2. Construct a double integral that represents the volume of the solid enclosed by the cylinder $x^2+y^2=1$, the planes $z=y$, $x=0$, $z=0$, in the first octant.
3. Evaluate $\int_0^4 \int_y^4 e^{x^2} dxdy$.

Announcements

GRA3, Next Tuesday (5 points)
Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2+y^2)^2+xy^2=1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

Quiz 3: Next Thursday
Quiz 3 may cover 14.8 to 14.10, and 15.1 to 15.4. We'll see.

The Average Value of a Function (15.3)

The average value of a function, $f(x, y)$, over a region R, is given by

$$\text{Average value of } f \text{ over region } R = \frac{1}{\text{area of } R} \iint_R f(x, y)\,dA$$

This definition can be used to find the value of some double integrals quickly.

Example
Region R is the unit circle $\sqrt{x^2 + y^2} \leq 1$. The definite integral of $f = x + 1$ over R is equal to:

a) 0
b) 1
c) π
d) $\pi/4$

Conceptual Question Related to Double Integrals

Let region R be the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. The definite integral of x^3 over region R is equal to:

a) a positive number

b) a negative number

c) zero

d) a function of x

1a) Changing the Order of Integration

Change the order of integration.

$$\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx dy$$

1a) Changing the Order of Integration (continued)

1b) Changing the Order of Integration

Change the order of integration.

$$\int_{2}^{1+e}\int_{0}^{\ln(x-1)} f(x,y)\,dy\,dx$$

2) Volume of a Solid

Construct a double integral that represents the volume of the solid enclosed by the cylinder $x^2 + y^2 = 1$, the planes $z = 1 - y$, $x = 0$, $z = 0$, in the first octant.

3) Evaluating a Double Integral

Evaluate the following double integral.

$$\int_0^4 \int_y^4 e^{x^2}\, dx dy$$

Additional Exercises

1. Set up an integral that represents the volume of the solid enclosed by the planes $x = 1$, $y = 3$, the three coordinate planes, and $x^2 + 2y^2 + z = 1$.
2. Find the volume of the solid enclosed by $z = x^2 + y^2$, $y = x^2$ and $x = y^2$.

Recitation 19

R19 Topics
15.4 Double Integrals in Polar Coordinates
Quiz 3 Review

Quiz 3 Topics

- 14.08 Lagrange Multipliers
- 14.09 Taylor's Formula for Two Variables
- 14.10 Partial Derivatives with Constrained Variables
- 15.01 Iterated Integrals over Rectangles
- 15.02 Double Integrals over General Regions
- 15.03 Area by Double Integration
- 15.04 Double Integration in Polar Coordinates

Office Hours
I'll hold additional office hours and a review session:

- Quiz 3 Review Session $\forall$ Math 2401 students: Tue 5:30 - 7:00 pm, at https://georgiatech.adobeconnect.com/dcp-online-drop-in-tutor-center-2014-fall
- Quiz 3 Review Session $\forall$ QH8 students: Wed: 7:30 - 8:30 pm at https://georgiatech.adobeconnect.com/distancecalculusofficehours

Quiz 3 Learning Objectives

You should be able to do the following for Quiz 3.

- Solve constrained optimization problems using Lagrange multipliers (14.8).
- Calculate a Taylor approximation to a function of two variables at a point (14.9).
- Apply the chain rule to compute partial derivatives with intermediate variables (14.10).
- Construct a double integral that represents the area of a region bounded by a set of given curves in Cartesian or polar coordinates (15.1 to 15.4).
- Change the order of integration of a double integral (15.1 to 15.4).
- Calculate the average value of a function of two variables (15.3).

Volume of a Sphere

Identify the expressions that represent the volume of a sphere of radius R.

$$1) \quad 4\int_0^{\pi}\int_0^{R} r\sqrt{R^2-r^2}\,dr\,d\theta$$

$$2) \quad \int_0^{2\pi}\int_0^{R} \sqrt{R^2-r^2}\,dr\,d\theta$$

$$3) \quad 2\int_0^{2\pi}\int_0^{R} r\sqrt{R^2-r^2}\,dr\,d\theta$$

$$4) \quad \int_0^{2\pi}\int_0^{R/2} r\sqrt{R^2-r^2}\,dr\,d\theta$$

Volume of a Sphere (continued)

Graded Recitation Activity 3

Instructions

- Every student in your group needs to write their name or initials on the board.
- You have 10 minutes to answer the question below.
- For full marks, show at least one intermediate step.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Question (5 points, from last year's quiz)
Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2 + y^2)^2 + xy^2 = 1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

GRA3

Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2+y^2)^2+xy^2=1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

Converting Double Integral to Polar Coordinates

Convert to a double integral in polar coordinates *(from 2014 Quiz 2)*.

$$\int_0^2 \int_0^{\sqrt{4-(x-2)^2}} xy\,dy\,dx$$

Converting Double Integral to Polar Coordinates (continued)

Additional Exercise: Normal Distribution

Evaluate

$$I = \int_0^\infty e^{-x^2} dx$$

Additional Exercise: Integration in Polar Coordinates

Sketch the rose curve $r = 2\cos(2\theta)$ and find the area of one petal.

Recitation 23

R23 Topics

15.5 Triple Integrals in Rectangular Coordinates
15.6 Moments of Inertia and Mass

R23 Learning Objectives

- Construct a triple integral that represents the area of a region bounded by a set of given curves in Cartesian or cylindrical coordinates
- Change the order of integration of a triple integral
- Set-up integrals that represent moments of inertia and centres of mass of solids

Today's Questions

1. Set-up a triple integral that represents the volume bounded by the following surfaces. Set-up the integrals in at least two different ways.
   1.1 $y^2 + z^2 = 1$, and the planes $y = x$, $x = 0$, and $z = 0$.
   1.2 $z^2 = y$, and the planes $y + z = 2$, $x = 0$, $x = 2$, and $z = 0$.
2. Consider the region inside the curve $r = 2 + \sin(\theta)$. Set up the three integrals you need to find the x and y coordinates of the centroid of the region, assuming its density is $\delta(x, y)$. Express these integrals in polar coordinates. This is a question from a 2014 quiz.

Graded Recitation Activity 4: Next Tuesday

Instructions (same as before)

- Every student in your group needs to write their name or initials on the board.
- You have 15 minutes to answer both questions below.
- For full marks, show at least **two intermediate steps**.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Questions (5 points each, both questions are from old quizzes)

1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Triple Integrals, Example 1

Set-up a triple integral that represents the volume of the region bounded by $y^2 + z^2 = 1$, and the planes $y = x$, $x = 0$, and $z = 0$. Set-up the integral in at least two different ways.

Triple Integrals, Example 1 (continued)

Triple Integrals, Example 2

Set-up a triple integral that represents the volume of the region bounded by $z^2 = y$, and the planes $y + z = 2$, $x = 0$, $x = 2$, and $z = 0$. Set-up the integral in at least two different ways.

Triple Integrals, Example 2, Continued

Triple Integrals, Example 2, Continued

Centroid

Consider the region inside the curve $r = 2 + \sin(\theta)$. Set up the three integrals you need to find the x and y coordinates of the centroid of the region, assuming its density is $\delta(x, y)$. Express these integrals in polar coordinates. This is a question from a 2014 quiz.

Recitation 24

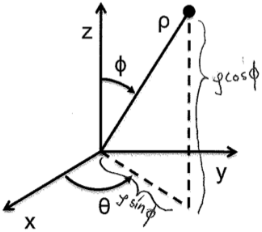
R24 Topics

15.7 Integration in Cylindrical and Spherical Coordinates

R24 Learning Objectives

- Construct a triple integral that represents the area of a region bounded by a set of given curves in cylindrical or spherical coordinates
- Change the order of integration of a triple integral

The Spherical Coordinate System



Fill in the blanks.

$x = \rho \cos\theta$ ____________

$y = \rho \sin\theta$ ____________

$z = \rho$ ____________

Graded Recitation Activity 4: Next Tuesday

Instructions (same as before)

- Every student in your group needs to write their name or initials on the board.
- You have 15 minutes to answer both questions below.
- For full marks, show at least **two intermediate steps**.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Questions (5 points each, both questions are from old quizzes)

1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Spherical Coordinates

Provide a geometric interpretation the surfaces $\rho \sin \phi = 1$ and $\rho \cos \phi = 1$.

1) A Triple Integral in Cylindrical Coordinates

Use cylindrical coordinates to set-up an integral that represents the volume of the solid bounded by $x^2 + y^2 + z^2 = 1$, and $z^2 = 3(x^2 + y^2)$.

2) A Triple Integral in Spherical Coordinates

Use spherical coordinates to set-up an integral that represents the volume of the solid bounded by $z = 0$, $x^2 + y^2 = 4$, and $z = 2\sqrt{x^2 + y^2}$.

3) A Triple Integral in Spherical Coordinates

Use spherical coordinates to set-up an integral that represents the volume of the solid in the first octant, between the surfaces $x^2 + y^2 = z^2$ and $z = \sqrt{2 - (x^2 + y^2)}$.

4) Triple Integrals

Set-up a triple integral that represents the volume of the solid bounded by $z = x^2 + y^2$, and the plane $y = z$. Use cylindrical coordinates.

Recitation 25

Quiz 4 Topics
15.3 to 15.8

Quiz 4 Learning Objectives

- Construct a triple integral that represents the area or volume of a region in Cartesian, polar, cylindrical, or spherical coordinates
- Change the order of integration, or coordinate system, for a triple integral
- Construct integrals that represent moments of inertia and centres of mass
- Identify a suitable transformation for a triple integral, and use that transform to find the area or volume of a given region

GRA4

1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Graded Recitation Activity 4

Instructions (same as before)

- Every student in your group needs to write their name or initials on the board.
- You have 15 minutes to answer both questions below.
- For full marks, show at least **two intermediate steps**.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Questions (5 points each, both questions are from old quizzes)

1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

GRA4.1

Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.

GRA4.2

Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Change of Variables

- After completing HW 15.8, you might be familiar with computing an integral, if you are given a transform.
- But if we were given an integral over a complicated region, and were not given a suitable transform, how could we find one?
- The basic idea is to find a transform that converts a complicated region into a simple one, such as a square, or a circle

1) Change of Variables

Show that the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is πab.

2) Change of Variables

Set-up an integral that represents the area of a region bounded by $x + y = 0$, $x + y = 1$, $x - y = 0$, $x - y = 2$.

2) Change of Variables (continued)

3) Triple Integrals

Set-up a triple integral that represents the volume of the solid bounded by $0 \leq x \leq 1$, $0 \leq y \leq \sqrt{1-x^2}$, and $\sqrt{x^2+y^2} \leq z \leq \sqrt{2-(x^2+y^2)}$.

4) Cylindrical

Set-up a triple integral that represents the volume of the solid bounded by $z = x^2 + y^2$, and the plane $y = z$. Use cylindrical coordinates.

5) Triple Integral

Set-up a triple integral that represents the volume of the solid bounded by $1 = x^2 + y^2$, above $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

5) Triple Integral (Alternate Solution)

Set-up a triple integral that represents the volume of the solid bounded by $1 = x^2 + y^2$, above $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

Recitation 27

Today's Topics

- 16.1 Line Integrals (brief review)
- 16.2 Vector Fields and Line Integrals, Work, Circulation, Flux

Learning Objectives

16.1 Set-up and evaluate a line integral to calculate the mass of a thin wire

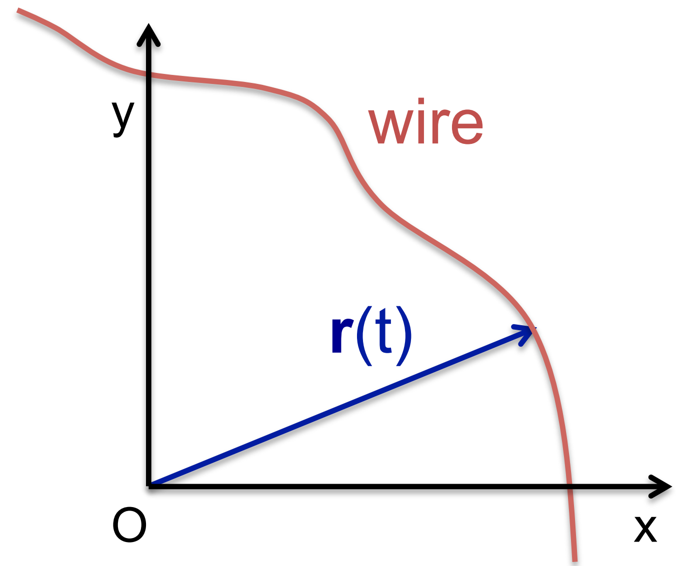
16.2 Set-up and evaluate a line integral that represents total work

16.1: Mass of a Thin Wire (a review of lecture material?)

How To Calculate Mass of a Wire

- position on wire given by parameterization, **r**(t)
- density of wire is δ = δ(r(t))
- length of a small piece of wire is Δs(**r**(t))
- we can approximate the total mass with:

$$M \approx$$



In the limit as Δs tends to zero,

$$M =$$

To compute total mass, we can show that:

$$M =$$

16.1: Mass of a Thin Wire

Compute the total mass of a wire whose density is given by $\delta = 3x^2 - 2y$, and whose shape is given by the line segment from the origin to the point (2,4).

16.2: Work (a review of lecture material?)

Work is the ________________ transferred to or from an object by

means of a ________________ acting on the ________________ .

16.2: Work Over a Straight Line Path

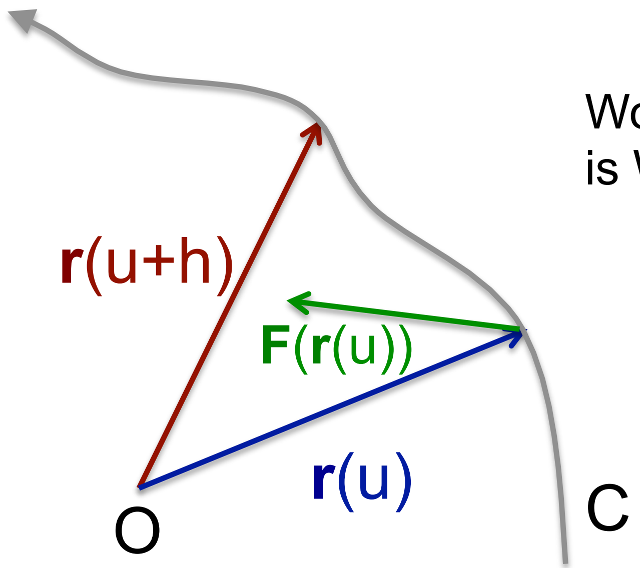
Force **F** is applied to an object as it moves from x = a to x = b along the x-axis.

	Applied Force	Work
Case 1	**F** = 4**i**	W =
Case 2	**F** = 4**i** – 2**j**	W =

we need to extend this concept to curved paths in R^3

16.2: Force Over a Curved Path

Force **F** applied to an object as it moves from **r**(u) to **r**(u + h) along curve C.



Work done by force **F** from **r**(u) to **r**(u+h) is W(u + h) – W(u).

	Applied Force	**Work**
Case 3	**F** = **F**(**r**(u))	W(u + h) – W(u) ≈

16.2: Calculating Work

Set up an integral that represents the total work.

a) **F** = (x + 2y)**i +** (2x + y)**j**, path is $y = x^2$ from (0,0) to (2,4).

b) **F** = (x - y) **i** – xy **j**, along the line from (2,3) to (1,2).

c) **F** = xy **i** – 2 **j** + 4z**k**, along the circular helix **r** = cos(u)**i** + sin(u)**j** + u**k**, from u = 0 to u = 2π.

Recitation 28

Today's Topics

- 16.2 Vector Fields and Line Integrals, Work, Circulation, Flux
- 16.3 Path Independence

Learning Objectives

16.2 Set-up, evaluate, and interpret integrals to calculate **circulation** and **flux**

16.3 Determine whether a vector field is **conservative**

Circulation

Cicrulation is a measure of the **flow** along a curve C, or **net velocity** along C.

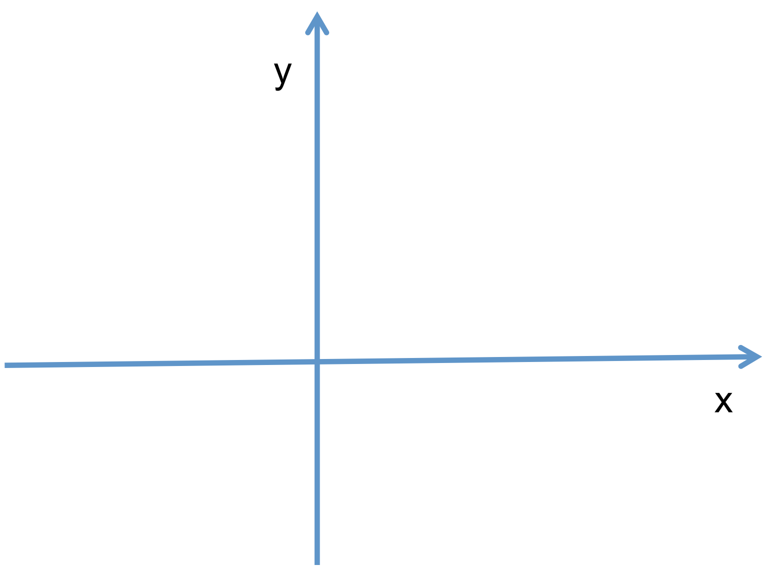
$$\text{circulation} = \Gamma = \int_C \vec{v}\left(\vec{r}\right)\cdot d\vec{r} = \int_a^b \vec{v}\left(\vec{r}\left(t\right)\right)\cdot \vec{r}\,'\left(t\right)dt$$

R28
2

16.2: Circulation

Sketch the velocity field for **v**, and calculate the **circulation** over curve C, where C is the circle of radius R.

$$\vec{v} = \begin{cases} 2\hat{\mathrm{i}},\ R \le y \le R \\ 0,\ \text{else} \end{cases}$$



For part a), the circulation is ______ because ______________ .

For part b), the circulation is ______ because ______________ .

Application of Circulation

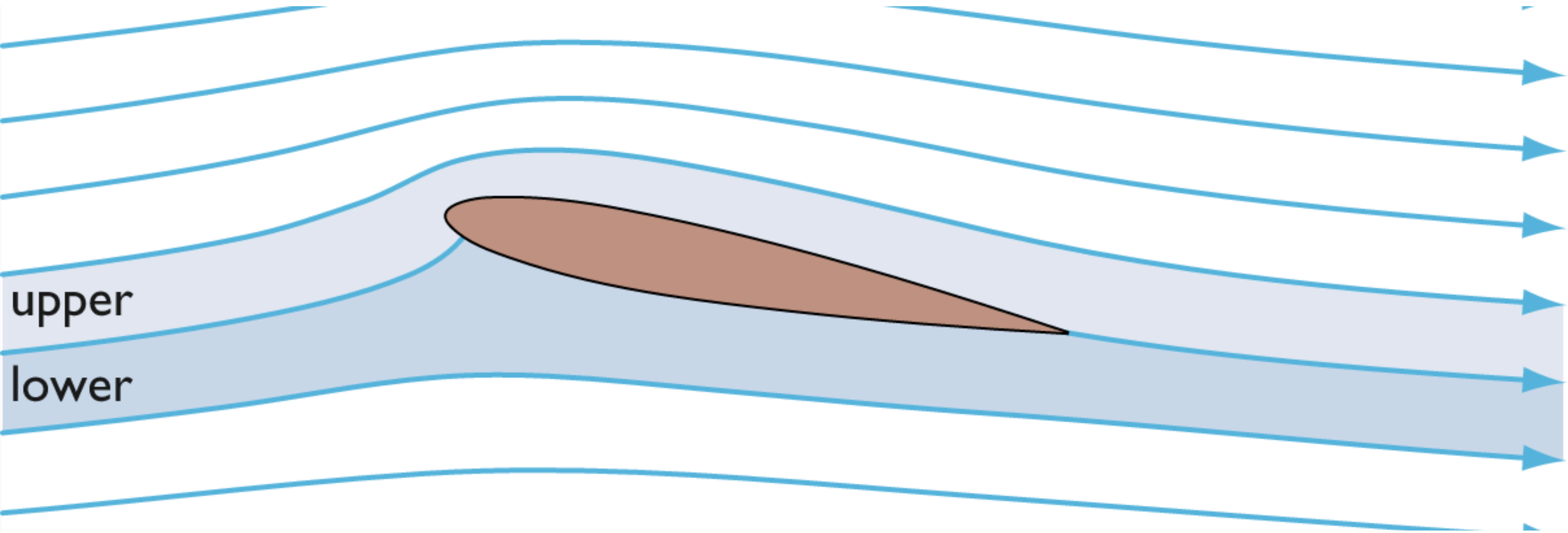
The circulation of a vector field **V** around a directed closed curve is

$$\text{circulation} = \Gamma = \int_{C} \vec{v}(\vec{r}) \cdot d\vec{r}$$

- Note the cross-sectional profile of the wing
- Take C to be a path around the wing, on its surface
- Upward lift force is proportional to circulation, Γ

16.2: An Application of Circulation

Take C to be a closed path around the wing on its surface



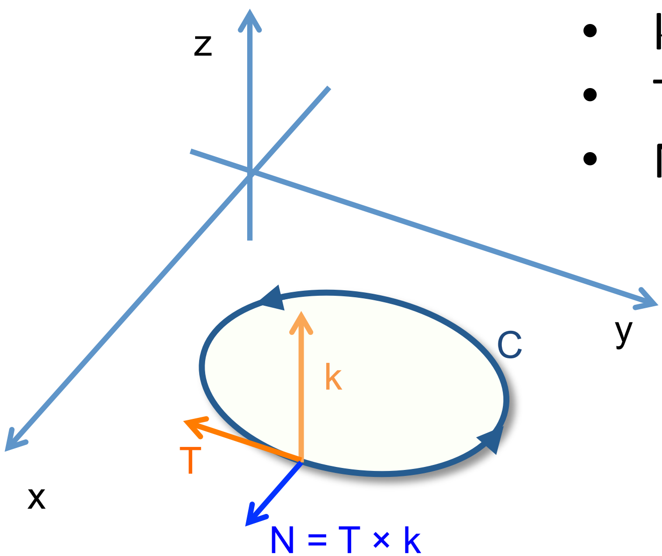
- Write Γ as $\Gamma = \Gamma_{upper} + \Gamma_{lower}$
- Γ_{upper} and Γ_{lower} have opposite signs
- the magnitude of **V** along the upper surface of the wing is greater than along the lower surface: net circulation is non-zero

R28

5

16.2: Flux Across a Closed Plane Curve

Suppose we have a curve C in the xy plane, and a flow field **v** = M(x,y)**i** + N(x,y)**k**. We want to measure the **net flow through C**.



counterclockwise motion

- k is the unit vector parallel to the z-axis
- T is the tangent vector
- N is the outward pointing unit normal vector of C

$$\text{flux} = \oint_C \vec{v} \cdot \vec{N}\, dt = \oint_C M\, dy - N\, dx$$

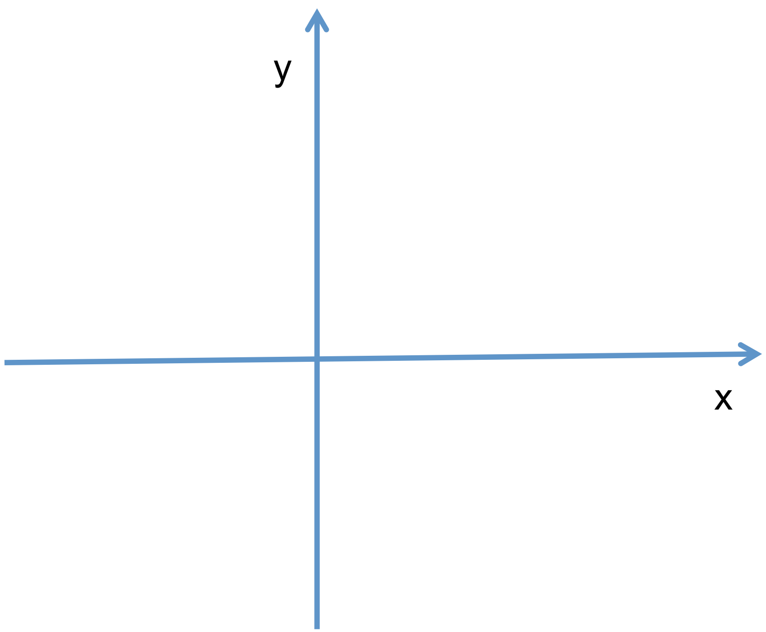
Note that:

- for a clockwise motion, we would instead use k × T
- later on, we will make a connection between flux and Green's theorem

16.2: Flux

Calculate the **flux** over curve C, where C is the circle of radius R.

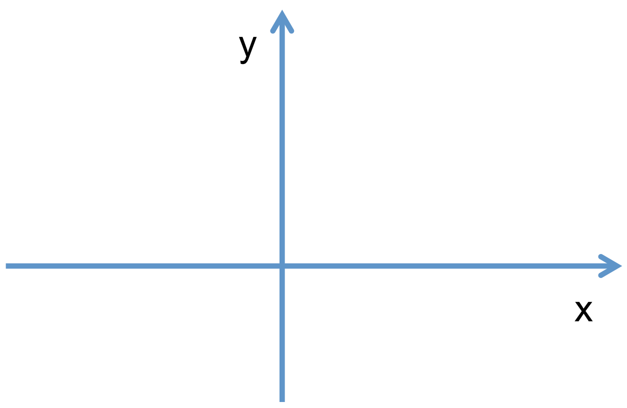
$$\vec{v} = \begin{cases} 2\hat{\imath}, & R \le y \le R \\ 0, & \text{else} \end{cases}$$



Therefore: the flux is ______ because _____________ .

16.2: Circulation and Flux

1) Sketch the velocity field for **v = -** x**i** **-** y**j**, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.

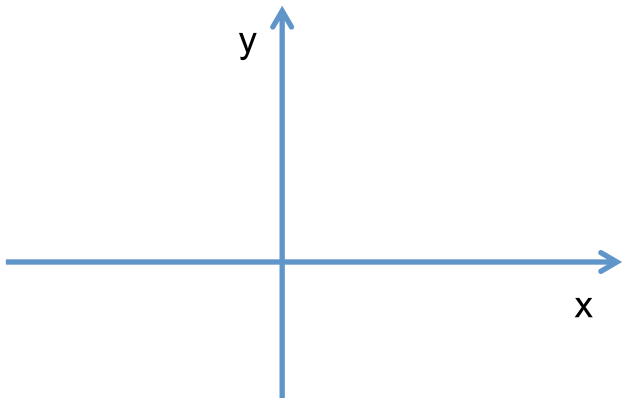


Therefore: the circulation is ______ because ______________ .

Therefore: the flux is ______ because ______________ .

16.2: Circulation and Flux

2) Sketch the velocity field for **v = -** y**i** + x**j**, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.



R28
9

16.3: Conservative Vector Fields

Recall the Pipe example.

a) Why was the circulation zero?

b) For any path that starts and ends at point A, and stays inside “the pipe”, the

circulation is _____________ .

c) For all paths that starts at A and ends at point B, the integral ______________ is the same.

In general: if **v** is a conservative vector field (or is path independent), then there exists a scalar field, S, s.t. _________ .

Recitation 29

Today's Topics

16.2 Vector Fields and Line Integrals, Work, Circulation, Flux

16.3 Path Independence

Learning Objectives

16.2 Set-up, evaluate, and interpret integrals to calculate **circulation** and **flux**

16.3 Determine whether a vector field is **conservative** and apply the FTLI

Circulation and Flux

Circulation is a measure of ____________________

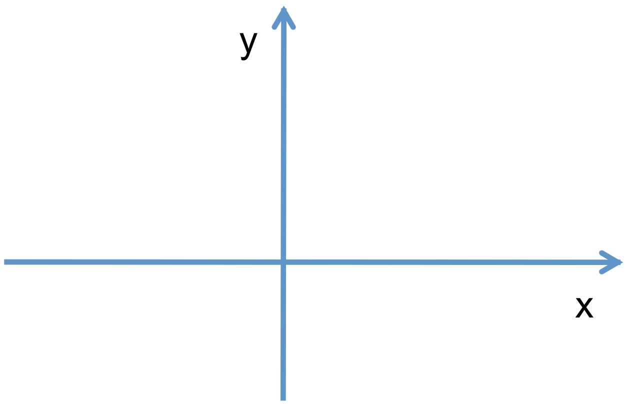
Flux is a measure of ___________________

$$\text{circulation} = \Gamma = \int_C \vec{v}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\text{flux} = \oint_C \vec{v} \cdot \vec{N}\,dt = \oint_C M\,dy - N\,dx$$

16.2: Circulation and Flux (review)

1) Sketch the velocity field for **v = -** x**i - **y**j**, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.



Therefore: the circulation is ______ because _____________ .

Therefore: the flux is ______ because _____________ .

16.3: Conservative Vector Fields

In general: if **F** is a conservative vector field (or is path independent), then there exists a scalar field, f, s.t. _________ , and

Example: Calculate total work from the force **F** = (x^2-y)**i** + $(y^2 - x)$**j**, over the path **r** = a cos(t)**i** + b sin(t)**j**, where $0 \leq t \leq 2\pi$.

16.3: Conservative Fields

Group work activity: determine whether the following fields are conservative

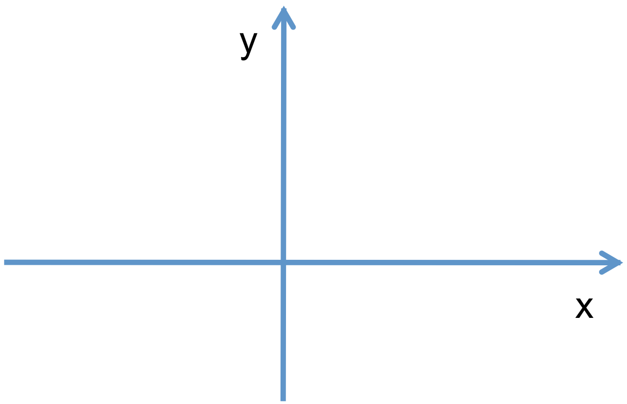
1) $\mathbf{v} = -x\mathbf{i} - y\mathbf{j}$

2) $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$

R29
5

16.2: Circulation and Flux

Group work activity: sketch the velocity field for $\mathbf{v} = -y\mathbf{i} + x\mathbf{j}$, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.



Conclusions

a) Circulation measures flow __________________ path C.

b) Flux measures the flow ______________________ of C.

c) If a flow is conservative, the line integral _______ is the same for any path C.

field name	velocity field equation	circulation	flux	is v conservative?
pipe	v = 2i for $-R \le y \le +R$, v = 0 otherwise			
	v = -xi - yj			
	v = -yi + xj			

Recitation 30

Today's Topics

16.4 Green's Theorem

16.5 Surfaces and Areas

Learning Objectives

16.4 Apply Green’s theorem to calculate area, flux, and circulation

16.5 Calculate the area of a surface given explicitly, implicitly, or parametrically

Green’s Theorem

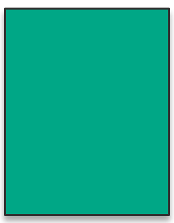
If R is a region that is ________________________________ , and M and N are scalar fields that are differentiable on R, and C is the boundary of R, then:

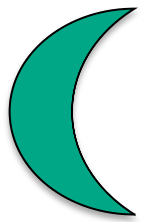
flux =

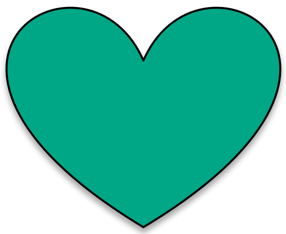
circulation =

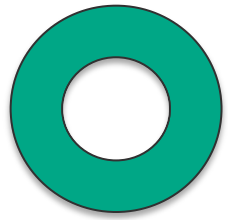
Green’s Theorem Example (from an old quiz)

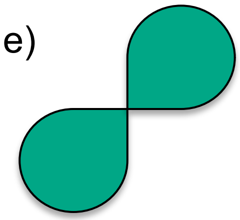
Below are five regions. For which regions can we apply Green’s Theorem?

a) 

b) 

c) 

d) 

e) 

Green's Theorem Example (from an old quiz)

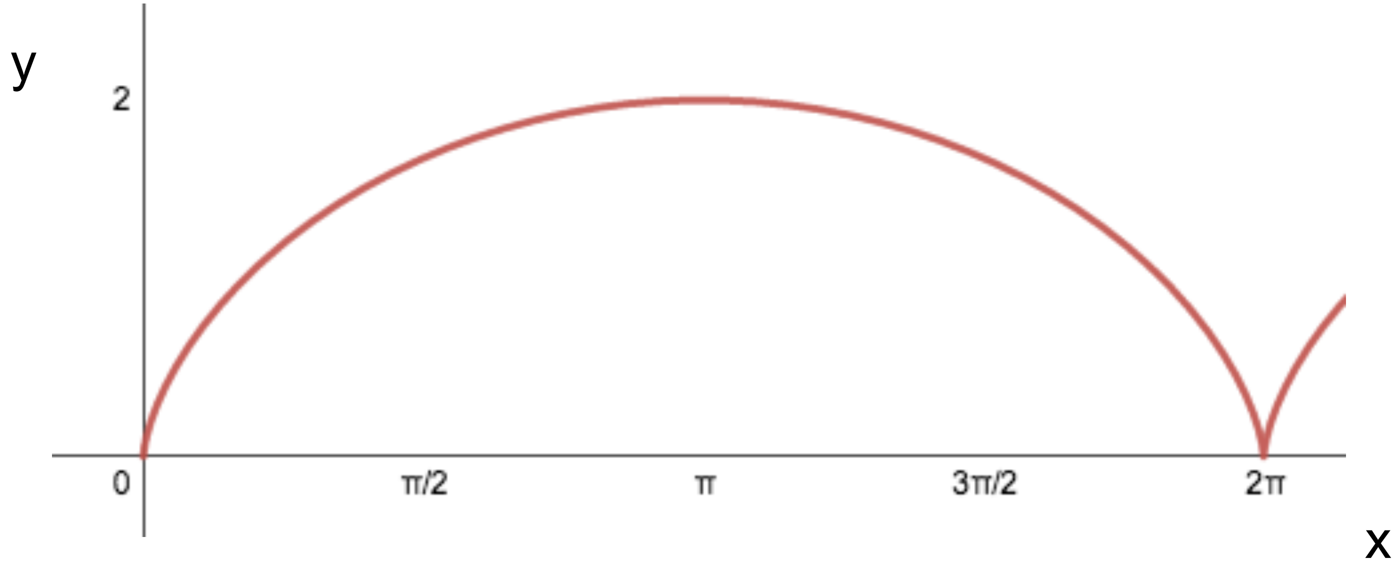
Find the circulation AND flux for the field $\mathbf{F} = 3x^2y^2\,\mathbf{i} + 2x^3y\,\mathbf{j}$ around the rectangle $0 \le x \le 2,\ 0 \le y \le 3$. Use Green's Theorem.

Green's Theorem Example (from an old quiz)

Let R be the region in the plane, inside the cardiod $r = 1 + \cos(\theta)$, and C its boundary Consider the line integral

$\int_C xy\,dx - xy^2\,dy$. Use Green's theorem to convert to an double integral, and express this as a double integral in polar coordinates with limits.

Additional Green's Theorem Example

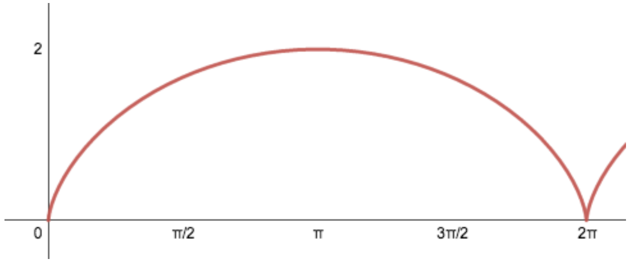


The curve traced by a point on a rolling wheel is

$$x(t) = t - \sin(t)$$
$$y(t) = 1 - \cos(t)$$

Additional Example: Green's Theorem

Find the area under one arch of the cycloid:
$x(t) = t - \sin(t)$, $y(t) = 1 - \cos(t)$



Additional Example: Green's Theorem

a) Evaluate $\oint_C y^2\,dx + 2xy\,dy$, C is one loop of $r = 2\sin 2\theta$

b) Change the integral so that it represents the area of one loop.

16.5 Surfaces and Areas

Surface area for a parameterized surface:

Your textbook has formulas for calcuatling the surface area for **implicit** and **explicit** surfaces, we probably won't have time to work on these in recitation.

16.5 Surfaces and Areas

a) What properties does a parametric representation of a surface need to have?

b) Find a parametric representation for the part of the plane $z = x + 2$ in the first octant and inside the cylinder $x^2 + y^2 = 1$.

Recitation 31

Today's Topics

16.5 Surfaces and Areas

16.6 Surface Integrals

Learning Objectives

16.5 Calculate the area of a surface given explicitly, implicitly, or parametrically

16.6 Calculate outward flux through a surface

16.6 Calculate the total mass and centroid of a thin surface (if time permits)

Course Logistics

1. Has a final exemption cutoff been announced?
2. What is the cutoff?
3. When is your final exam?

R31
2

16.5 Surfaces and Areas

Surface area for a parameterized surface:

Your textbook has formulas for calcuatling the surface area for **implicit** and **explicit** surfaces, we probably won't have time to work on these in recitation.

R31
3

16.5 Surface Area Example

Set up an integral that represents the surface area of $z = y^2$, for $0 \le x \le a$, $0 \le y \le b$.

16.5 Surface Area Example

Calculate the surface area of the part of the plane $x + 2y + z = 4$ that is inside the cylinder $x^2 + y^2 = 4$.

16.6 Surface Integrals

Suppose we want to characterize 3D flow through a pipe.

To calculate 2D flux across a **curve**, we used: $\text{flux} = \int_C \vec{v} \cdot \vec{n}\, du = \int_C M\, dy - N\, dx$

If our flow field, **v**, is 3D, we calculate flux across a **surface.**

R31
6

16.6 Flux Through a Surface

A fluid has velocity field $\mathbf{v} = y\mathbf{i} + \mathbf{j} + z\mathbf{k}$. Set up an integral that represents the flux through the paraboloid $z = 9 - (x^2 + y^2)/4$, if $x^2 + y^2 \leq 36$.

R31
7

16.6 Surface Integrals (this was a 2014 pop quiz question)

Set up a double integral that represents the flux of flow $\mathbf{F} = x\mathbf{i} + z\mathbf{k}$ thorugh the surface $z(x,y) = x^2 - y^2$, where $0 \leq x \leq 1$, $-1 \leq y \leq 1$.

16.6 Centroid of a Thin Surface (if time permits)

The mass density at any point on a thin surface $z^2 = x^2 + y^2$, $0 \le z \le 1$, is proportional to its distance to the z-axis.

a) Find the total mass of the surface.

b) Find the centroid of the surface.

16.5 Surface Area Parameterization (additional example)

Find parametric representations for the following surfaces.

a) the upper half of $4x^2 + 9y^2 + z^2 = 36$

b) the part of the plane $z = x + 2$ inside the cylinder of $x^2 + y^2 = 1$

Recitation 32

Today's Topics

Final Exam Review

16.7 Stokes Theorem

16.8 The Divergence Theorem

Learning Objectives

16.7 Use Stoke’s theorem to calculate either work, or circulation over a curve

16.8 Calculate flux through a surface using the divergence theorem

Final Exam Logistics

Review session: information sent via email

Questions during final: information sent via email

Studying for the Final Exam

There are two prep-finals available on T^2. Each of them have five questions that focus on specific areas of our textbook.

	Chapter 13	Chapter 14	Chapter 15	Chapter 16
Prep-Final A	P1			P2, P3, P4, P5
Prep-Final B	P1	P2	P3	P4, P5

Ways you may want to study:
- solve prep final questions
- re-do quizzes 1 through 4
- re-do MML problems
- memorize formulas (especially from Chapters 13 and 16)

PrepFinal Question A1

Find the speed, the tangential acceleration and the normal acceleration for the motion $\mathbf{r} = (t,t^2,t^2)$. Compute also the curvature of the corresponding curve as a function of t.

PrepFinal Question A2

Find the moment of inertia with respect to the x axis of a thin shell of mass δ that is in the first quadrant of the xy plane and bounded by the curve $r^2 = \sin 2\theta$.

PrepFinal Question A3

Compute the center of mass of a thin shell that is formed by the cone $(z - 2)^2 = x^2 + y^2$, $0 \leq z \leq 2$.

PrepFinal Question A4

Compute the line integral of the vector field $\mathbf{F} = (xyz + 1, x^2z, x^2y)e^{xyz}$ along the curve $\mathbf{r}(t) = (\cos t, \sin t, t)$, $0 \leq t \leq \pi$.

R327

PrepFinal Question A5

Use the divergence theorem to compute the outward flux of the vector field $\mathbf{F} = (x^2, y^2, z^2)$ through the cylindrical can that is bounded on the side by $x^2 + y^2 = 4$, bounded above by $z = 1$ and below by $z = 0$.

PrepFinal Question B1

Find the parametric equations of the line that is tangent to the curve $\mathbf{r}(t) = (e^t, \sin t, \ln(1 - t))$, at $t = 0$.

PrepFinal Question B2

Find the minimum cost area of a rectangular solid with volume 64 cubic inches, given that the top and sides cost 4 cents per square inch and the bottom costs 7 cents per square inch. Just set up the equations using Lagrange multipliers, you do not have to solve them.

PrepFinal Question B3

Compute the average of the function x^4 over the sphere centered at the origin whose radius is $R > 0$.

PrepFinal Question B4

Compute the flux $\int_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$, S where S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, $\mathbf{n}$ points toward the origin and $\mathbf{F} = (x(z-y),\ y(x-z),\ z(y-x))$.

R32

12

PrepFinal Question B5

Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve given by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = -y$, counterclockwise when viewed from above, and $\mathbf{F} = (x^2 + y, x + y, 4y^2 - z)$.

16.7 Stokes' Theorem

Curl describes the tendency a fluid has to ________________ at a specific point. Stokes' Theorem states that:

Note that curve C must be ___________

Stokes' theorem can be used to calculate ___________ and _______ .

Historical note: Stokes' theorem is named after Sir George Stokes, but was discovered by Sir William Thomson.

16.8 What is Divergence?

Divergence describes the tendency a fluid has to _________________.

Water is (approximately) an incompressible fluid. If you place your thumb at the end of a hose,

the speed of the water ______________ , because _______________ , or because _____________.

R32

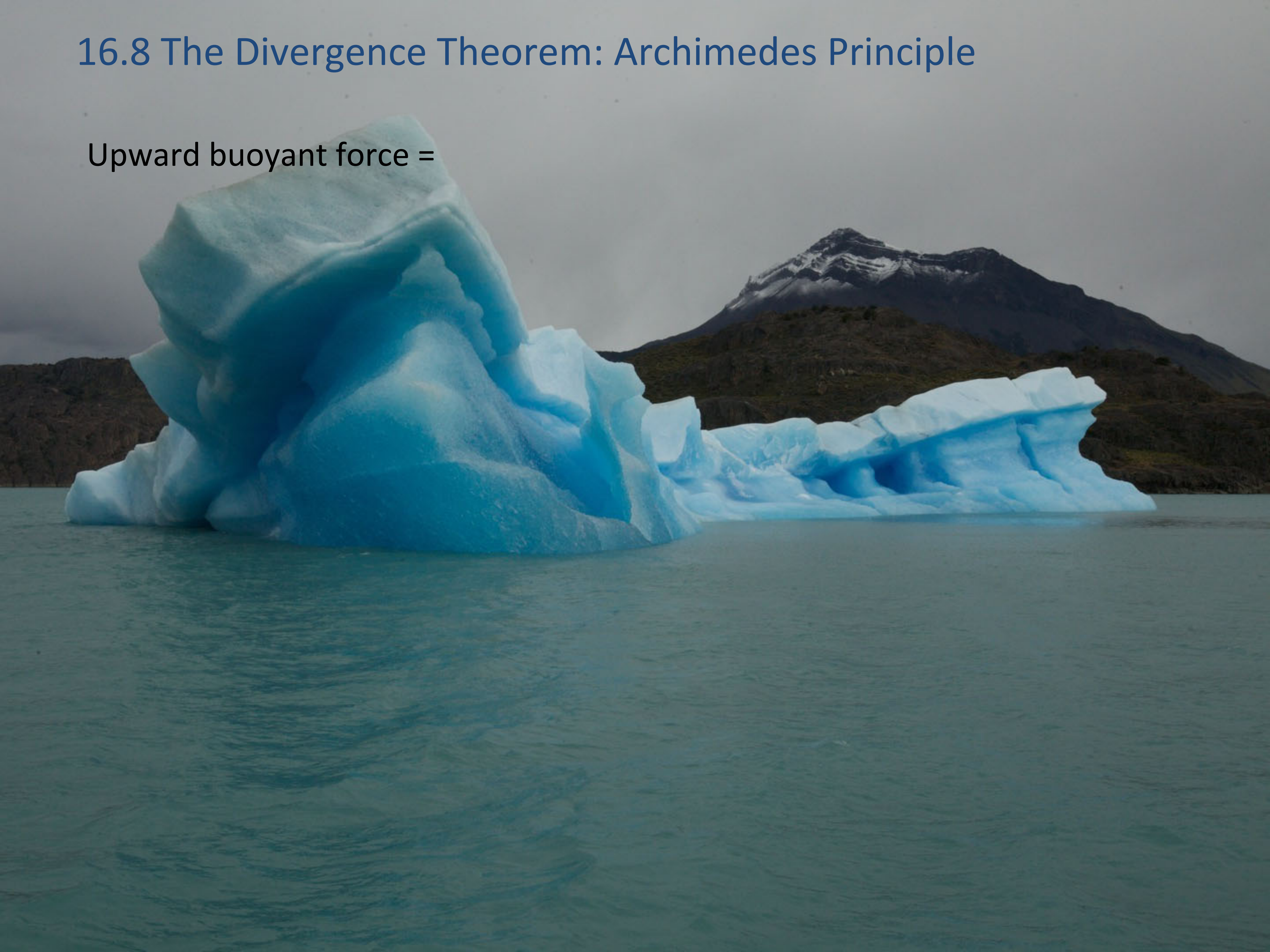
15

16.8 The Divergence Theorem

The divergence theorem states that

16.8 The Divergence Theorem: Archimedes Principle

Upward buoyant force =



16.8 Prove Archimedes Principle

16.8 Electric Charge

E = electric field. Then, Gauss's Law states that:

total charge = (ε_0)(flux of **E** through closed surface)

Find the total charge contained in a solid hemisphere if **E** = x**i** + y**j** + z**k**.

R01

Recitation 01: Welcome Back!

Vector Representations (13.1)

Today: Course and Recitation Organization, ~~Velocity and Acceleration (12.6)~~

Thursday: Quadratic Surfaces (12.6)

Start-of-Term Survey

Please fill out if you haven't already:

Graded Recitation Activities This Semester

- details sent via email
- group work, in Adobe Connect, count towards your pop quiz grade

WebEx and Adobe Connect

1. WebEx for first two weeks
2. online survey to determine if we want to continue using WebEx
3. Adobe Connect for graded group work activities and pop quizzes

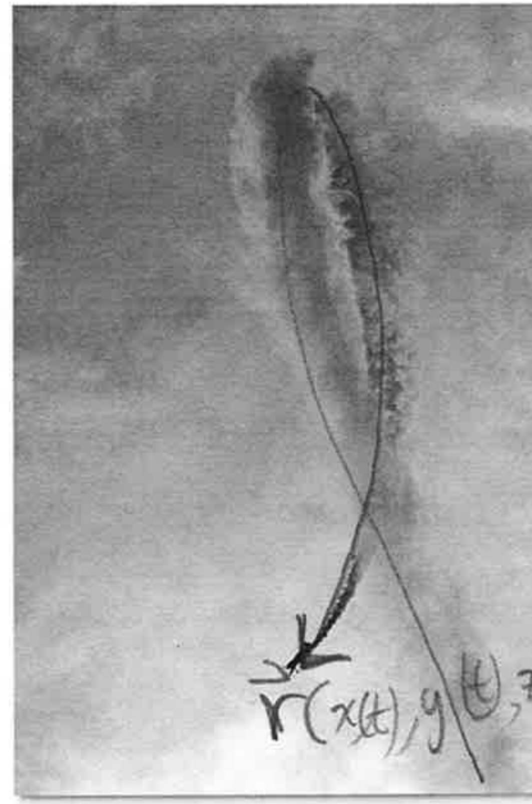
Other Announcements

- Piazza is set-up, link in T-Square
- Tegrity is set-up, can view yesterday's lecture
- Two MML HWs set-up, due Monday

R01

Objectives

Throughout this course we find parametric representations of motion and use them to characterize ~~a~~ motions,



$\vec{r}(x(t), y(t), z(t))$

Today's Learning Objectives

Characterize the two (or three) dimensional motion of an object, in parametric form, in terms of its

- velocity and acceleration
- tangent vector

Later in this course we'll look at curvature, path length, momentum, and other ways of describing a motion.

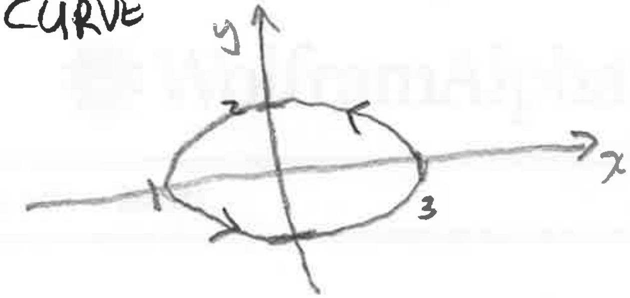
I'm assuming you've seen parametric representation of curves in lecture.

R01

Parametric Representation

Find a parametric representation of the counterclockwise motion that travels along the curve $4x^2 + 9y^2 = 36$. WANT: $\vec{r}(t) = \underline{x(t)}\hat{i} + \underline{y(t)}\hat{j}$

1) SKETCH CURVE



$$\frac{4x^2}{36} + \frac{9y^2}{36} = \frac{36}{36}$$

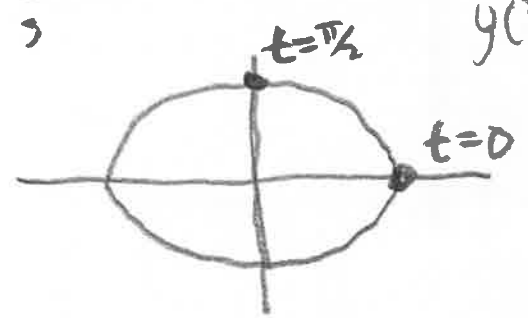
$$\boxed{\left(\frac{x}{3}\right)^2 + \left(\frac{y}{2}\right)^2 = 1} \quad (*)$$

2) LET $x = x(t) = 3\cos t$, $y = y(t) = 2\sin t$ } because this choice satisfies $(*)$ for all t.

$$\Rightarrow \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} = 3\cos t\,\hat{i} + 2\sin t\,\hat{j}$$

But is motion counterclockwise? Check:

Motion "STARTS" AT $t = 0$, $x(0) = 3$ ✓, $y(0) = 0$; when $t = \frac{\pi}{2}$, $x(\frac{\pi}{2}) = 0$ ✓, $y(\frac{\pi}{2}) = 2$



R01

Position, Velocity and Acceleration

The position of an object is given by the curve **r**(t) = sin(t)**i** + cos(t)**j**, for all t.

a) Sketch the curve.
b) When are the position and velocity vectors perpendicular?
c) When do the position and acceleration vectors have the same direction?
d) Calculate the unit tangent vector for all t.

a) $x(t) = \sin t$, $y(t) = \cos t$, and $x^2 + y^2 = \cos^2 + \sin^2 = 1$, so $\vec{r}$ traces a circle (clockwise?)

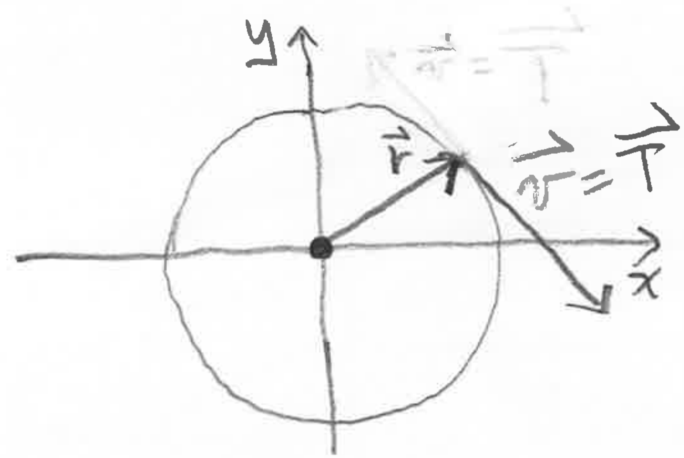
b) $\vec{r}'(t) = c\hat{i} - s\hat{j}$

$0 = \vec{r} \cdot \vec{r}' = (s\hat{i} + c\hat{j}) \cdot (c\hat{i} - s\hat{j}) = 0 \Rightarrow$ perpendicular $\forall t$.

$\Rightarrow$ for circular motion, $\vec{v} \perp \vec{r}$ $\forall t$.

c) $\vec{r}'' = -\vec{r}$, so anti-parallel $\forall t$, so never in same direction.

d) $\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \vec{v} = c\hat{i} - s\hat{j}$



R01

Position and Velocity

The position of a particle is given by r(t). Describe situations where the following is true for all values of t.

$$\vec{r}(t)\cdot\frac{d\vec{r}}{dt}=0$$

1) CIRCULAR MOTION

2) STATIONARY OBJECT (so that $\vec{r}'(t)=\vec{0}$)

R01
8

Parametric Vector Representation

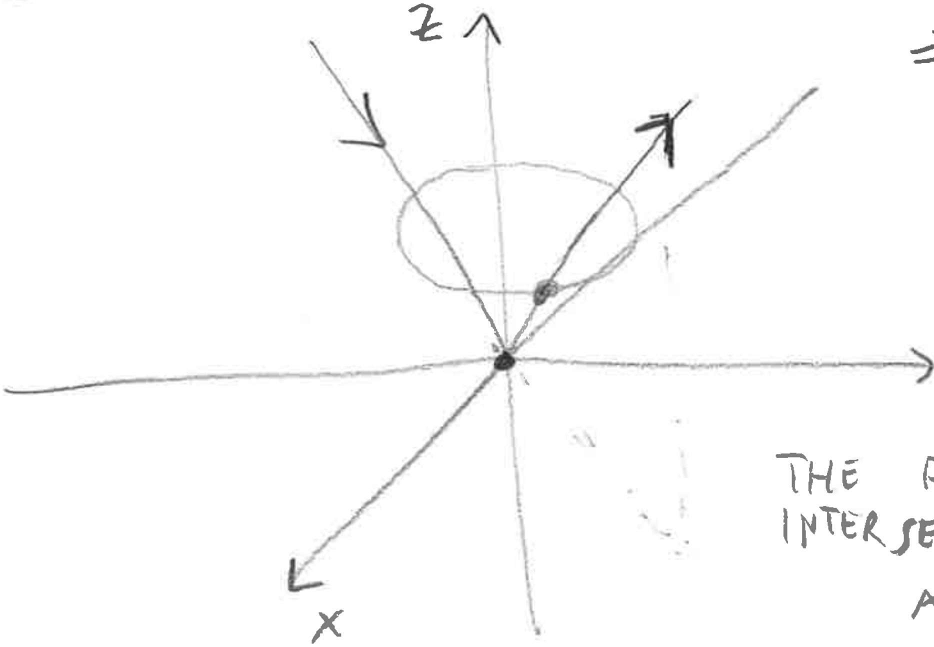
Find a parametric vector representation, r(t), of the curve that satisfies the following equations, and y increases when x is positive.

$$z = \sqrt{x^2 + y^2}, y = x \quad (*)$$

We want functions $x(t), y(t), z(t)$ that satisfy $(*)$.

Try: $x = t$, so $y = x = t$, $z = \sqrt{t^2+t^2} = \sqrt{2}|t|$

$$\Rightarrow \vec{r} = \begin{bmatrix} t \\ t \\ \sqrt{2}|t| \end{bmatrix}$$



THE PATH $\vec{r}(t)$ IS THE INTERSECTION OF PLANE $y = x$ AND CONE $z = \sqrt{x^2+y^2}$

R01

9

Parametric Vector Representation

Find a parametric vector representation, r(t), of the curve that satisfies the following equations, and z decreases when x is positive. ~~Sketch the motion.~~

$$\underbrace{z=\sqrt{4-x^2-y^2}}_{1},\ \underbrace{y^2+x^2-2y=0}_{2}$$

We want $x(t)$, $y(t)$, $z(t)$ that satisfy given equations, 1 and 2

Complete the square:

$$x^2+y^2-2y+1-1=0$$

$$x^2+(y-1)^2=1$$

$\Rightarrow$ $x=\cos t$, $y=\sin t+1$ then 2 is satisfied.

$$z=\sqrt{4-c^2-(1+s)^2}=\sqrt{2-2\sin t} \quad \Rightarrow \quad \vec{r}=\begin{bmatrix}\cos t\\ \sin t+1\\ \sqrt{2-2\sin t}\end{bmatrix}$$

CURVE IS INTERSECTION OF $\underbrace{x^2+y^2+z^2=4}_{\text{SPHERE}}$ AND $\underbrace{x^2+(y-1)^2=1}_{\text{CYLINDER}}$

R02
1

Recitation 02

Today: Vector Representations of Curves (13.1), Quadratic Surfaces (12.6)

Start-of-Term Survey

Please fill out if you haven't already:

https://www.surveymonkey.com/s/Math2401-2015

Last Recitation

- Find parametric representations of given curves
- Characterize motion of an object, in parametric form, in terms of its
  - velocity and acceleration
  - unit tangent vector

Today

- Identify and sketch quadratic surfaces given their algebraic equations

While Waiting to Start: Sketch and describe the surface $5x^2 + 2y^2 - z^2 = -10$.

R02

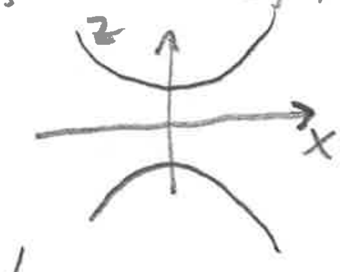
2

Quadratic Surfaces (12.6)

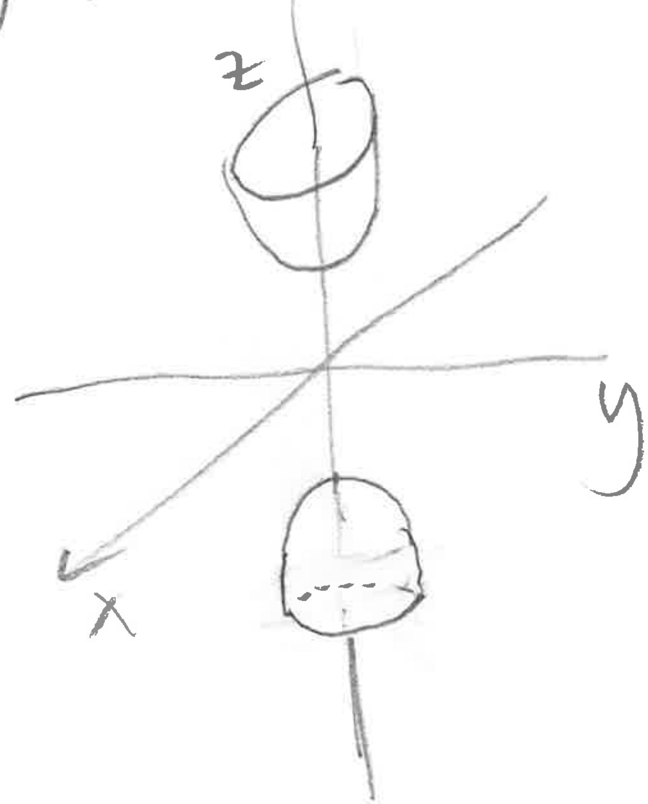
Sketch and describe the surface $5x^2 + 2y^2 - z^2 = -10$.

In the xy-plane, $z=0$, $5x^2+2y^2=-10 \Rightarrow$ inconsistent! So surface does not intersect zy-plane.

In the xz-plane, $y=0$, $5x^2-z^2=-10$ is a hyperbola:



In the yz-plane, $x=0$, $2y^2-z^2=-10$, hyperbola



NOTE: we could also consider level curves

R02

3

Quadratic Surfaces (12.6)

Sketch and describe the surface $5x^2 + 2y^2 - z^2 = -10$.

WolframAlpha computational knowledge engine

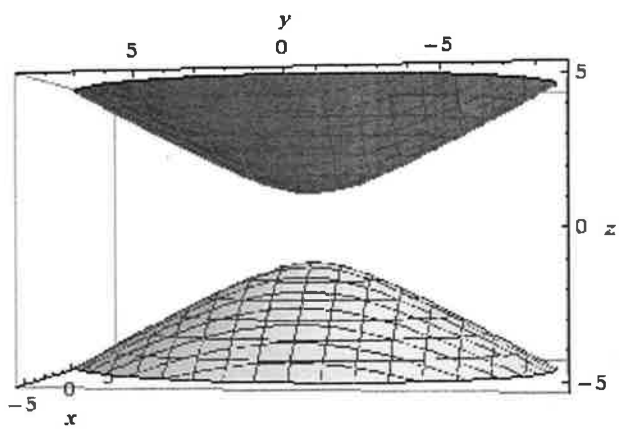
plot 5x^2 + 2y^2 – 6z^2 = –10

≡ Examples ⤮ Random

Input interpretation:

plot	$5x^2 + 2y^2 - 6z^2 = -10$

Surface plot:



Enable interactivity

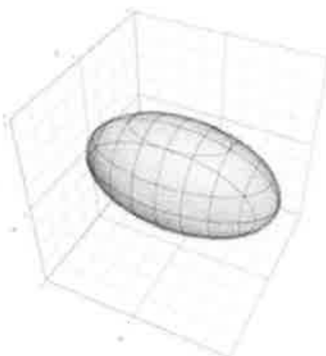
Geometric figure:

two-sheeted hyperboloid

Quadratic Surfaces

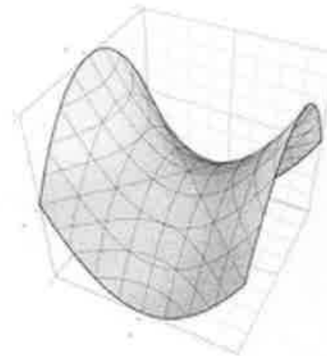
The textbook should list and describe every quadratic surface that you need to be familiar with (but the online textbook currently doesn't work). Wikipedia also has a page that lists and describes every possible quadratic surface (for our course):
http://en.wikipedia.org/wiki/Quadric

Below are four surfaces:



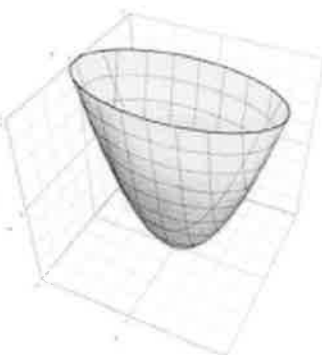
Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$



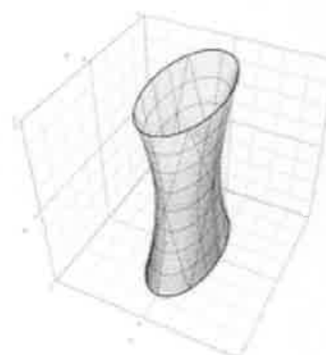
Hyperbolic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z}{c}=0$$



Elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$$



Elliptic hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

R02

5

Quadratic Surfaces

Identify the correct answer.

The set of all points whose distance from the z-axis is 4 is the:

a) sphere of radius 4 centered on the z-axis
b) line parallel to the z-axis 4 units away from the origin
c) cylinder of radius 4 centered on the z-axis
d) plane $z = 4$

a) is the set of all points whose distance from origin is 4
b) is a subset of points " " " z axis is 4
d) is set of points " " " xy plane is 4

R02
6

Parametric Vector Representation and Quadratic Surfaces

Find a parametric vector representation of the curve, r(t), that satisfies both quadratic surfaces. Sketch r(t) and both surfaces.

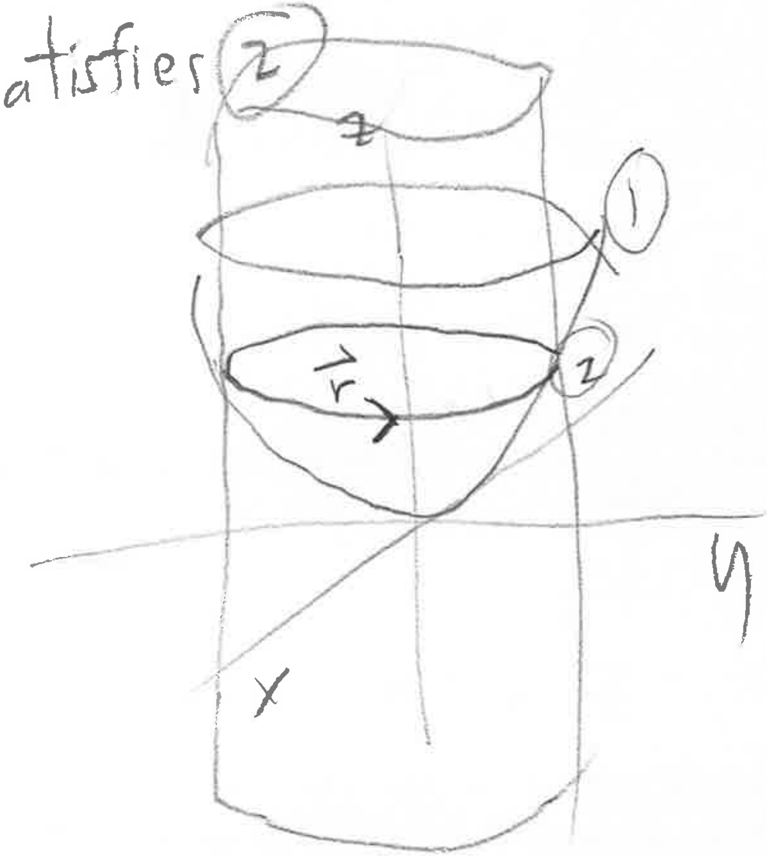
$$\underbrace{z = x^2 + y^2}_{1}, \quad \underbrace{5 = x^2 + y^2}_{2}$$

We need $\vec{r} = x(t)\hat{i} + y(t)\hat{j}$ that satisfies both surfaces.

Try: $\left.\begin{aligned} x(t) &= \sqrt{5}\cos t \\ y(t) &= \sqrt{5}\sin t \end{aligned}\right\}$ satisfies 2

Then $z(t)$ must be 5.

$$\Rightarrow \quad \vec{r}(t) = \begin{bmatrix} \sqrt{5}\cos t \\ \sqrt{5}\sin t \\ 5 \end{bmatrix}$$



NOTE: CAN ALSO CHOOSE $x(t) = \sqrt{5}\sin t$
$y(t) = \sqrt{5}\cos t$

R02
7

Quadratic Surfaces (12.6)

Consider the surface $z = Ax^2 + By^2$, where A and B are constants. Identify all possible surfaces for the following cases.

i) $A = B = 0$

ii) $AB > 0$

~~iii) $AB < 0$~~

i) $z = 0 + 0 = 0 \Rightarrow$ xy-plane.

ii) A & B POSITIVE : z MUST BE POSITIVE (ELLIPTIC PARABOLOID)

A & B NEGATIVE : z MUST BE NEGATIVE

WHY DOES IT LOOK LIKE THIS?
Fix z: if $z = 1$: $\underbrace{1 = Ax^2 + By^2}_{\text{ellipse}}$

R02

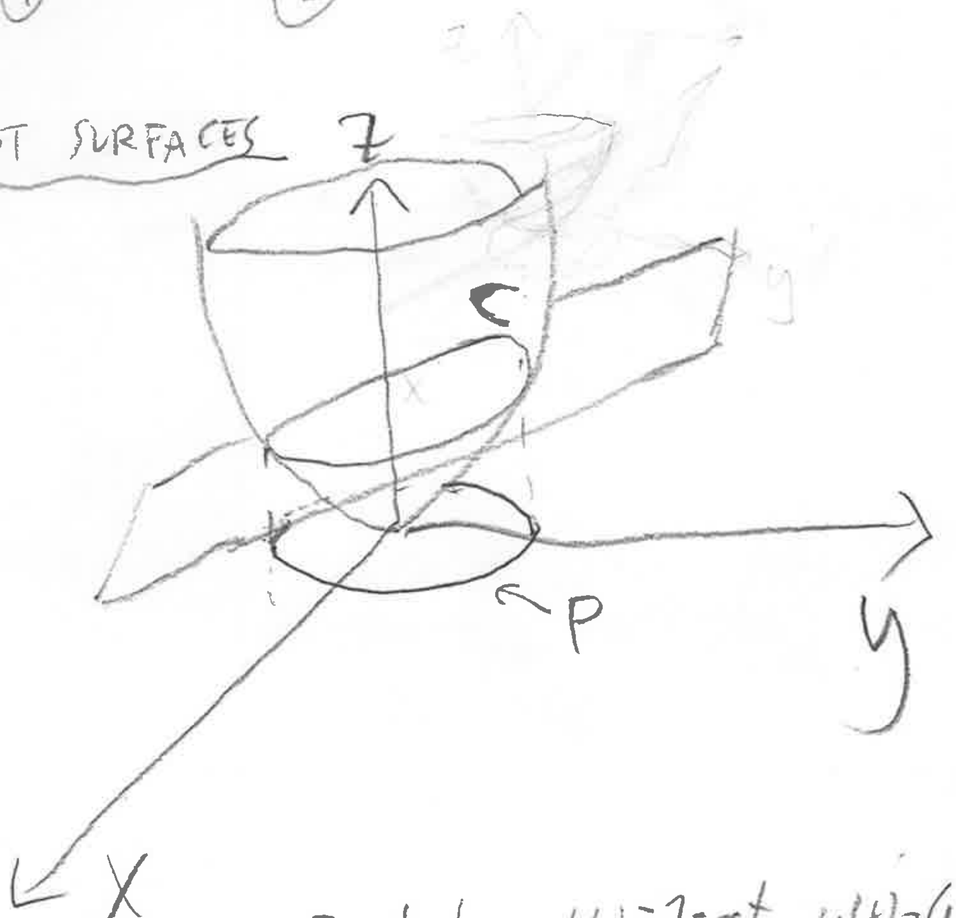
8

Parametric Vector Representation and Quadratic Surfaces

The following surfaces intersect along a curve C. Find a) the projection of the curve onto the xy-plane and b) its parametric vector representation.

$$\underbrace{z = x^2 + y^2}_{1}, \quad \underbrace{z = 2y + 3}_{2}$$

PLOT SURFACES



CURVE P IS SET OF POINTS $(x, y, 0)$ SUCH THAT

$$x^2 + y^2 = 2y + 3$$

$$x^2 + y^2 - 2y = 3$$

$$x^2 + y^2 - 2y + 1 - 1 = 3$$

$$x^2 + (y-1)^2 = 4$$

$\Rightarrow$ P is the circle, radius 2, centre $(0, 1)$

$\Rightarrow$ let $x(t) = 2\cos t$, $y(t) = (1 + 2\sin t)$

$\Rightarrow$ $\boxed{r = \begin{bmatrix} 2\cos t \\ 1 + 2\sin t \end{bmatrix} \text{ parameterization of } P}$

Recitation 03

Today: Group Work on Vector Representations of Curves, Quadratic Surfaces

- Hello from San Antonio! Your instructor and I are at a large annual math conference. I hope the wifi is going to hold up for our recitation this morning, many apologies if it doesn't. In case you're interested, this the conference website: http://jointmathematicsmeetings.org/jmm

Textbook: technical issues should be resolved now

Start-of-Term Survey
Please fill out if you haven't already (survey closes Wednesday at midnight):
https://www.surveymonkey.com/s/Math2401-2015

Today: Quadractic Surfaces and Parametric Vectors

- Find parametric representations of given curves
- Characterize motion of an object, in parametric form, in terms of its velocity and acceleration, unit tangent vector
- Identify and sketch quadratic surfaces given their algebraic equations

Group Work Questions

Complete each problem in small groups. The first four questions are from old Math 2401 quizzes (2013 and 2014).

1) Consider the twisted cubic $r(t) = ti + t^2j + t^3k$ and the plane $x + 2y + 3z = 34$.
    a) Where does the cubic intersect the plane?
    b) Find the cosine of the tangent to the curve and the normal to the plane.
2) Find the intersection of the surface $x^2 + 2y^2 = z$ and the plane $x - y = 5$. A parameterization would be fine.
3) Conisder the surface $x^2 - 6x + 4y + y^2 + 8z - z^2 = 4$.
    a) Find the center of the surface.
    b) Name the surface.
    c) Draw a picture of the surface, labelling the center and axes.
4) Conisder the surface $9x^2 - 18x - 16y + 4y^2 - 4z^2 = 11$.
    a) Find the center of the surface.
    b) Name the surface.
    c) Draw a picture of the surface, labelling the center and axes.
5) ~~How do the surfaces in questions 3 and 4 compare? How are they different?~~
6) 5) Create a vector function, $\mathbf{r}(t)$, on the interval $[0, 2\pi]$, that satisfies the conditions $\mathbf{r}(0) = a\mathbf{i}$, and as t increases from 0 to 2π, traces out an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, twice in a counterclockwise manner.

R03

3

1) Consider the twisted cubic $\vec{r}(t) = ti + t^2j + t^3k$ and the plane $x + 2y + 3z = 34$.

a) Where does the cubic intersect the plane?

b) Find the cosine of the tangent to the curve and the normal to the plane.

a) By definition, the plane is the set of all points s.t. $x+2y+3z=34$.
" " " vector $\vec{r}(t)$ is the curve $\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{j}$.
For some value(s) of t, $\vec{r}(t)$ will intersect the plane.
For a point on the curve to be in the plane, it must satisfy: $x(t) + 2y(t) + 3z(t) = 34$

$$t + 2(t^2) + 3(t^3) = 34$$

Guess & check: $t=2$ works $\left(2 + 2(4) + 3(8) = 34\right)$. So the cubic poly can be written as

$$(t-2)\underbrace{(3t^2+8t+17)}_{\text{complex roots}} = 0$$

FROM →

POLYNOMIAL LONG DIVISION

$$\begin{array}{r} 3t^2+8t+17 \\ t-2\overline{\big)\, t+2t^2+3t^3-34} \\ -6t^2+3t^3 \\ \hline t+8t^2-34 \\ -16t+8t-0 \\ \hline 17t-34 \end{array}$$

$\Rightarrow t=2$ is only intersection, and this is the point $(2, 2^2, 2^3)$.

b) $\cos\theta = \dfrac{\vec{r}'(2)\cdot N}{|\vec{r}'(2)||N|}$, $\vec{r}' = \begin{bmatrix}1\\4\\8\end{bmatrix}$, $\vec{N} = \begin{bmatrix}1\\2\\3\end{bmatrix}$.

R03 2) Find the intersection of the surface $x^2 + 2y^2 = z$ and the plane $x - y = 5$. A parameterization would be fine.

4

A parameterization of the plane is

$$x = t$$

$$y = x - 5 = t - 5$$

$\Rightarrow$ parameterization of intersection is

$$\vec{r} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$

For $z(t)$ use equation of surface $z(t) = x^2 + 2y^2$

$$= t^2 + 2(t-5)^2$$

$$= 3t^2 - 20t + 50$$

$$\Rightarrow \vec{r} = \begin{bmatrix} t \\ t-5 \\ 3t^2 - 20t + 50 \end{bmatrix}$$

R03

5

3) Conisder the surface $x^2 - 6x + 4y + y^2 + 8z - z^2 = 4$.

a) Find the center of the surface.

b) Name the surface.

c) Draw a picture of the surface, labelling the center and axes.

FIRST PUT SURFACE EQUATION INTO STANDARD FORM:

$$(x^2 - 6x + 9 - 9) + (y^2 + 4y + 4 - 4) - (z^2 - 8z + 16 - 16) = 4$$

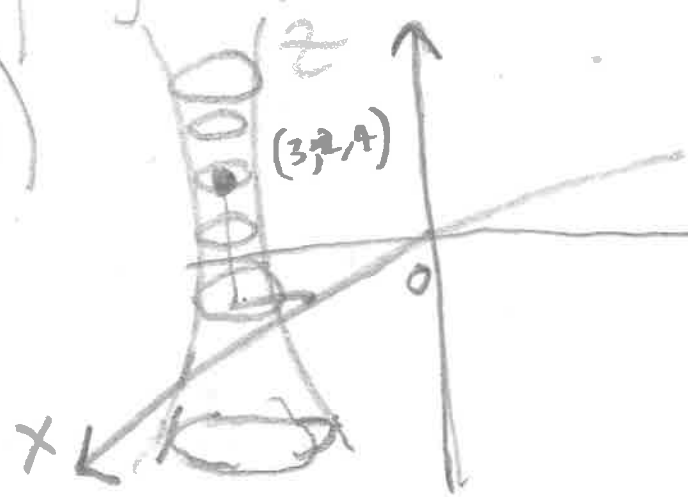
$$(x-3)^2 - 9 + (y+2)^2 - 4 - (z-4)^2 + 16 = 4$$

$$(x-3)^2 + (y+2)^2 - (z-4)^2 = 1$$

a) Centre is at $(3, -2, +4)$.

b) By inspection, surface is a hyperbolic paraboloid of 1 sheet

b)



R03

6

4) Conisder the surface $9x^2 - 18x - 16y + 4y^2 - 4z^2 = 11$.

a) Find the center of the surface.

b) Name the surface.

c) Draw a picture of the surface, labelling the center and axes.

EXPRESS IN STANDARD FORM:

$$9(x^2 - 2x + 1 - 1) + 4(y^2 - 4y + 4 - 4) - 4z^2 = 11$$

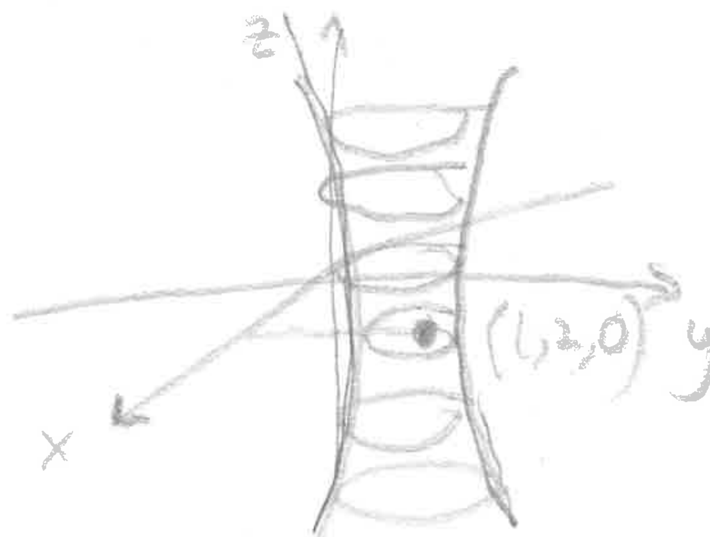
$$9(x-1)^2 + 4(y-2)^2 - 4z^2 = 11 + 9 + 16 = 36$$

$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{9} - \frac{z^2}{9} = 1$$

a) Centre is located at the point $(1,2,0)$.

b) Surface is a one-sheet hyperboloid

c)



R03 6) Create a vector function, **r**(t), on the interval [0, 2π], that satisfies the conditions
8 **r**(0) = a**i**, and as t increases from 0 to 2π, traces out an ellipse $b^2x^2 + a^2y^2 = a^2b^2$, twice in a counterclockwise manner.

We need an $\vec{r}(t)$ that satisfies

$$b^2x^2 + a^2y^2 = a^2b^2, \quad (*)$$

We can try: $x(t) = a\cos t$

$y(t) = b\sin t$

Does this satisfy (*)? Let's see: $b^2(a\cos t)^2 + a^2(b\sin t)^2 = a^2b^2$ ✓

Equation (*) satisfied. However, does

$$\vec{r}(t) = \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$$

trace out an elipse twice in a c.clockwise manner on $[0, 2\pi]$?

No. But this does:

$x(t) = a\cos(2t)$

$y(t) = b\sin(2t)$

R04

1

Recitation 04

Today: Displacement, Velocity, Acceleration (13.2), Path Length (13.3)

Homework: Due Tonight and Monday

Learning Obectives for Today: Characterize motion of an object, in parametric form, in terms of its unit tangent vector, acceleration, path length (aka arc length).

Photo by Wikimedia Commons user Kreuzschnabel

R04 **Particle Motion**

2 Let **r**(t) = x(t)**i** + y(t)**j** + z(t)**k**.

a) How is the unit tangent vector, **T**(t), defined mathematically?

b) Suppose $x = t^2$, $y = t^3$, $z = t^2$, and ~~t is any real number~~ $t \geq 0$, . Then what is the unit tangent vector when t = 0?

a) $\vec{T} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{r}'}{\|\vec{r}'\|}$

b) $\vec{v} = \begin{bmatrix} 2t \\ 3t^2 \\ 2t \end{bmatrix}$, $\|\vec{v}\| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2} = \sqrt{8t^2 + 9t^4}$

$\vec{T}(t) = \frac{2t\hat{i} + 3t^2\hat{j} + 2t\hat{k}}{\sqrt{8t^2 + 9t^4}}$ (leave as is to see if students point out that we can factor)

$\Rightarrow \vec{T}(0) = 0/0$ .... uh-oh! what does this mean? What can we do?

1) l'Hôspital works but is messy

2) factoring/simplifying is easier : $\vec{T} = \frac{2t\hat{i} + 3t^2\hat{j} + 2t\hat{k}}{t\sqrt{8 + 9t^4}}$

don't need abs sign because $t \geq 0$

$\vec{T}(0) = \frac{2\hat{i} + 3(0)\hat{j} + 2\hat{k}}{\sqrt{8+0}} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$\Rightarrow$ SOMETIMES NEED TO SIMPLIFY ALONG THE WAY

R04 **Differential Equation**

3 Solve the following initial value problem.

$$\vec{F}(t) = m\vec{r}\,''(t) = t\hat{i} + t^2\hat{j},\ \vec{r}(0) = \hat{i},\ \vec{v}(0) = \hat{k}.$$

Q) what are we trying to find? A) $\vec{r}(t)$.

$\vec{v} = \int r''(t)\,dt$

$= \int \begin{bmatrix} t \\ t^2 \\ 0 \end{bmatrix} dt$ ← get students to tell you that this is needed

$= \begin{bmatrix} t^2/2 + c_1 \\ t^3/3 + c_2 \\ c_3 \end{bmatrix}$ } students can tell you what these are

but $\vec{v}(0) = \hat{k}$, so $c_1 = c_2 = 0$, $c_3 = 1$

$\Rightarrow \vec{v} = \begin{bmatrix} t^2/2 \\ t^3/3 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{r} = \begin{bmatrix} t^3/6 + d_1 \\ t^4/12 + d_2 \\ t + d_3 \end{bmatrix}$, $d_1 = 1$, $d_2 = d_3 = 0$. $\Rightarrow \vec{r}(t) = \begin{bmatrix} t^3/6 + 1 \\ t^4/12 \\ t \end{bmatrix}$

R04 **Velocity and Acceleration**

4 What constant acceleration must a particle experience if it is to travel from (1,2,3) to (4,5,7) along the straight line joining the points, starting from rest, and covering the distance in 2 units of time?

Q) How can we start?

A) We need an $\vec{r}(t)$, which should be a straight line.

Acceleration, $\vec{a} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$. We want c_1, c_2, c_3.

$\Rightarrow \vec{v} = \begin{bmatrix} c_1 t + d_1 \\ c_2 t + d_2 \\ c_3 t + d_3 \end{bmatrix} = \vec{C}t + \vec{D}$

But $\vec{v}(0) = 0$, so $d_1 = d_2 = d_3 = 0$.

$\vec{v} = \vec{C}t$, $\vec{r} = \frac{\vec{C}t^2}{2} + \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$. $\vec{r}(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, so $\vec{r} = \frac{ct^2}{2} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and to be at (4,5,7) when t=2, $\begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}(2)^2 + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \left.\begin{array}{l} c_1 = 3/2 \\ c_2 = 3/2 \\ c_3 = 2 \end{array}\right\} \vec{a} = \begin{bmatrix} 3/2 \\ 3/2 \\ 2 \end{bmatrix}$

R04 **Velocity and Position**

5 $\mathbf{r}(t)$ is the position of a moving particle.

a) Describe, in words, what $\mathbf{r}'$ is parallel to.

b) Show that $||\mathbf{r}(t)||$ is constant iff $\mathbf{r} \perp \mathbf{r}'$

a) $\vec{r}'$ is parallel to the direction of motion (it points in the " " ")

b) If $\|\vec{r}\| = c$, c = constant.

then $\|\vec{r}\|^2 = c^2$

$$\Rightarrow \vec{r} \cdot \vec{r} = c^2$$

$$\Rightarrow \frac{d}{dt}(\vec{r} \cdot \vec{r}) = 0$$

$$\Rightarrow \vec{r}' \cdot \vec{r} + \vec{r} \cdot \vec{r}' = 0$$

$$\Rightarrow \vec{r} \cdot \vec{r}' = 0, \text{ so } \vec{r} \perp \vec{r}'.$$

NOTES

1) Logic goes both ways: iff easily shown.

2) MEANS THAT

a) IF CIRCULAR MOTION, $\vec{r}' \perp \vec{r}$,

b) IF $\vec{r}' \perp \vec{r}$, we have circular motion.

AWESOME.

R04 The Hanging Cable

6

(STUDENTS PROBABLY HAVEN'T SEEN HYPERBOLIC FUNCTIONS)

The hanging cable, also referred to as a caternary, has the shape:

$$y = k\left(\cosh\left(\frac{x}{k}\right) - 1\right) = k\left(e^{x/k} + e^{-x/k} - 1\right)$$

, k = constant related to stiffness of cable. As k increases, cable becomes more "stiff"

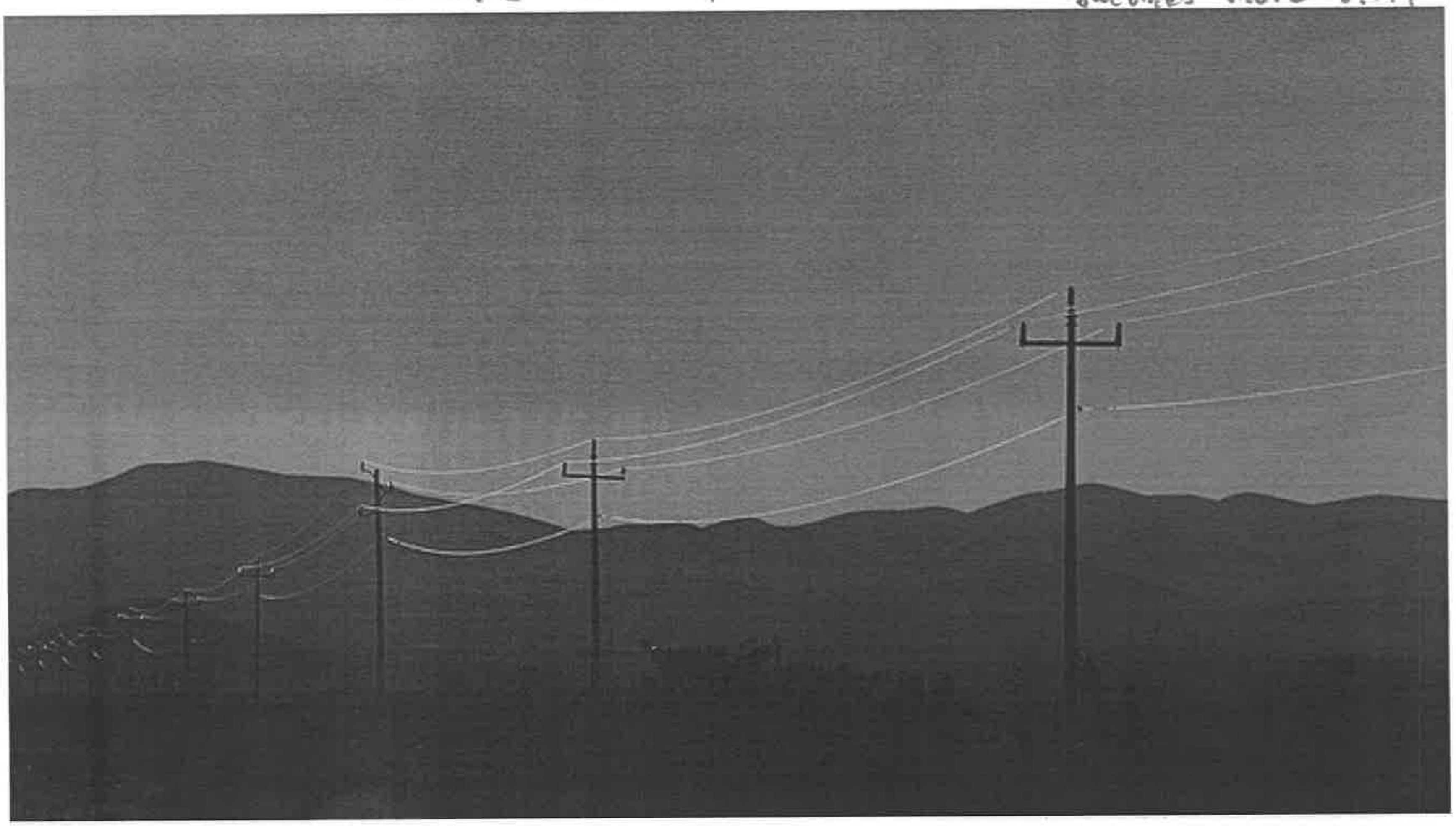


Photo by Flickr user Robert Valencia

R04 A cable is suspended between two poles that are 10 m apart. Find the length of the cable, if the cable's shape is $y(x) = k\,[\cosh(x/k) - 1]$. DO NOT INTEGRATE.

CAN CHOOSE BETWEEN TWO APPROACHES:

1) $L = \int \sqrt{1+(y')^2}\,dx$

2) $L = \int \sqrt{(y'(t))^2 + (x'(t))^2}\,dt = \int |v|\,dt$

LETS TRY APPROACH 2:

Let $x(t) = t$, $y(t) = k\left(e^{t/k} + e^{-t/k} - 1\right)$

Then $\vec{v} = \vec{r}\,' = \frac{d}{dt}\begin{bmatrix} t \\ k\cosh(t/k) - k \end{bmatrix} = \begin{bmatrix} 1 \\ k\left(\frac{1}{k}\left(e^{x/k} - e^{-x/k}\right) - 0\right) \end{bmatrix}$

$$L = \int_{-5}^{5} |\vec{v}| = \int \sqrt{1^2 + \left(e^{t/k} - e^{-t/k}\right)^2}\,dt$$

STOP HERE UNLESS STUDENTS WOULD LIKE TO CONTINUE

$$= \int \sqrt{1 + e^{2t/k} + e^{-2t/k} - e^0 - e^0}\,dt$$

$$= \int \sqrt{e^{2t/k} + e^{-2t/k} - 1}\,dt$$

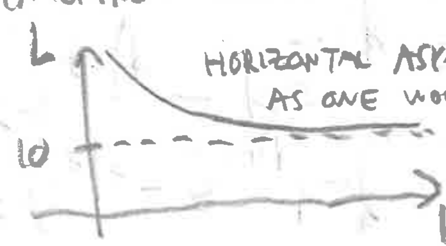
$$= \int_{-5}^{+5} \sqrt{\left(e^{t/k} + e^{-t/k}\right)^2}\,dt$$

$$= ke^{t/k} - ke^{-t/k}\Big|_{-5}^{5}$$

$$= k\left[\left(e^{5/k} - e^{-5/k}\right) - \left(e^{-5/k} - e^{5/k}\right)\right]$$

$$= 2k\left(e^{5/k} - e^{-5/k}\right)$$

INTERPRET RESULT!



HORIZONTAL ASYMPTOTE AT L=10, AS ONE WOULD EXPECT

Recitation 05

Today's Topics

- Projectile Motion (13.2)
- Path Length and Tangential Vector (13.3)
- Curvature & Normal Vectors (13.4)

Today's Learning Obectives

- Apply vector function integration to determine path of projectiles
- Characterize motion of an object, in parametric form, in terms of its arc length and its tangential, normal and binormal vectors

Announcements

Survey Results: students want to collaborate, have trouble with technical issues and not knowing how to solve problems in group work. So lets use Adobe Connect, keep group size to 4 to 6, use group work on stuff covered from last assignments.

Thursday Recitation: 13.4, 13.5, Adobe Connect
Graded Recitation Activity: Next week during Tuesday recitation, question coming soon
HW Due Tomorrow: 13.4, 13.5
Quiz 1: Thur Jan 29
Office Hours: 7:30 pm – 8:30 pm, Wed Jan 21, Wed Jan 28
https://georgiatech.adobeconnect.com/distancecalculusofficehours

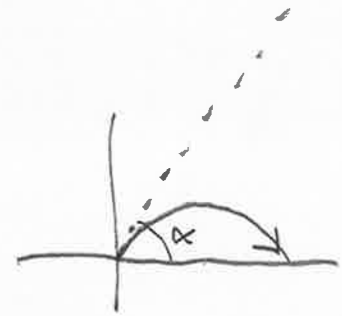
Send Your TA an Email
Explain, in an email, using your own words, what the following quantities represent:

- the unit tangent vector, $\mathbf{T}(t)$
- the curvature, κ

Try to send this email by the end of the day today. If you send your TA an email with a description of what these quantities represent, you will get a reply.

Helpful Formulas

Ideal Projectile Motion: $\vec{r}(t) = (v_0 \cos\alpha) t\hat{i} + \left((v_0 \sin\alpha) t - \frac{gt^2}{2} \right) \hat{j}$



v_0 is the initial velocity, and α is the angle from ground.

max range: $R = \frac{v_0^2 \sin 2\alpha}{g}$ max height: $\frac{v_0^2 \sin^2 \alpha}{2g}$

Unit tangent vector T = $\vec{r}\,' / \|\vec{r}\,'\|$

Principle unit normal vector N = $\vec{T}\,' / \|\vec{T}\,'\|$

Binormal vector B = $\vec{T} \times \vec{N}$

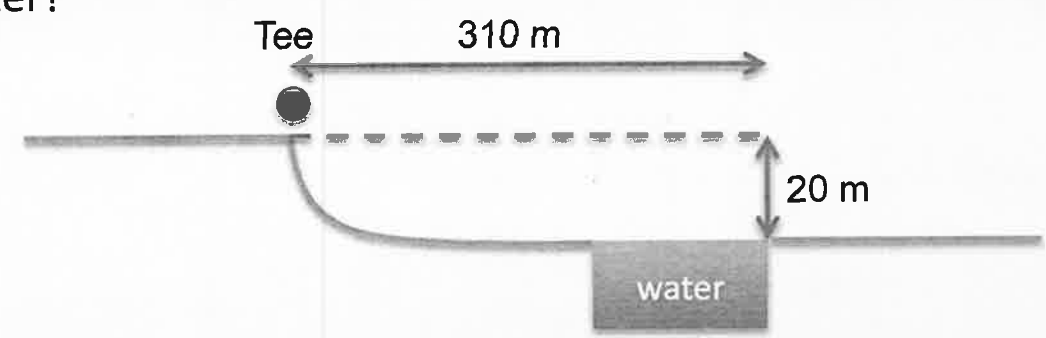
R05 **1) Ball Rolling off of a Table (Projectile Motion, 13.2)**

4 A ball rolls off a table 1 meter high with a speed of 0.5 m/s.

a) At what speed does the ball strike the floor?
b) Where does the ball strike the floor?

2) Golf Ball (Projectile Motion, 13.2)

A golfer can send a golf ball 300m across a level ground. From the tee in the figure, can the golfer clear the water?



3) Arc Length, Normal and Binormal Vectors (13.3, 13.4)

Consider the surfaces $x^2 + y^2 + z^2 = 4$, and $z^2 = x^2 + y^2$ for $z \geq 0$.

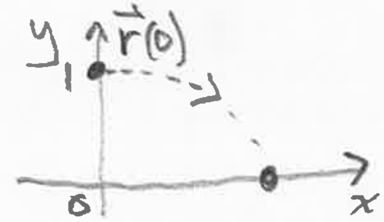
a) Find a parameterization for the intersection curve, **r**(t), of the two surfaces.
b) Sketch the two surfaces and their intersection.
c) Calculate the length of **r**(t).
d) Find the unit tangent, normal, and binormal vectors for **r**(t) at the point (sqrt(2) , 0, sqrt(2)).
e) Add the three vectors to your sketch.

R05 **1) Ball Rolling off of a Table (Projectile Motion, 13.2)**

5 A ball rolls off a table 1 meter high with a speed of 0.5 m/s.

a) What speed does the ball strike the floor?

b) Where does the ball strike the floor?



a) acceleration $= \vec{a}(t) = \begin{bmatrix} 0 \\ -g \end{bmatrix}$, so velocity $= \vec{v}(t) = \begin{bmatrix} c_1 \\ -gt + c_2 \end{bmatrix}$. But $\vec{v}(0) = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$,

so $c_1 = \frac{1}{2}$, $c_2 = 0$. Thus, $\vec{v}(t) = \begin{bmatrix} 1/2 \\ -gt \end{bmatrix}$. Position $= \vec{r}(t) = \begin{bmatrix} \frac{1}{2}t + d_1 \\ -gt^2/2 + d_2 \end{bmatrix}$.

But $\vec{r}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, so $d_1 = 0$, $d_2 = 1$.

Ball hits floor when $0 = -gt^2/2 + 1$, or $t = \sqrt{2/g}$

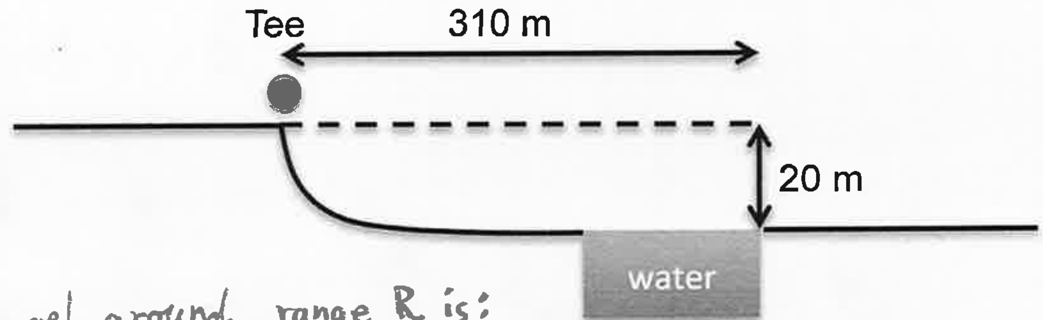
$\Rightarrow \left|\vec{v}\left(\sqrt{2/g}\right)\right| = \sqrt{(1/2)^2 + \left(g\left(\sqrt{2/g}\right)\right)^2} \approx 4.455\ldots$

b) $\vec{r}\left(\sqrt{\frac{2}{g}}\right) = \begin{bmatrix} \frac{1}{2}\sqrt{\frac{2}{g}} \\ -g\left(\sqrt{\frac{2}{g}}\right)^2/2 + 1 \end{bmatrix} \approx \begin{bmatrix} 0.22576\ldots \\ 0 \end{bmatrix}$

$\Rightarrow$ at point $(0.2, 0)$

R05 **2) Golf Ball (Projectile Motion, 13.2)**

6 A golfer can send a golf ball 300m across a level ground. From the tee in the figure, can the golfer clear the water?



$$\vec{a} = \begin{bmatrix} 0 \\ -g \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} c_1 \\ -gt + c_2 \end{bmatrix} = \begin{bmatrix} v_0 \cos\alpha \\ -gt + v_0 \sin\alpha \end{bmatrix}$$

Use range formula to get v_0. On level ground, range R is:

$$R = 300 = \frac{v_0^2 \sin 2\alpha}{g}$$

Range R maximized when $\alpha = \pi/4$, solving for v_0 yields $v_0 = \sqrt{300g} \approx 54.25$.

Velocity $\vec{v}$ becomes: $\vec{v}(t) = \begin{bmatrix} \sqrt{300g}\,(1/\sqrt{2}) \\ -gt + \sqrt{300g}\,\frac{1}{\sqrt{2}} \end{bmatrix}$.

Position $\vec{r}(t) = \begin{bmatrix} v_0 \cos\alpha\, t + d_1 \\ -g\frac{t^2}{2} + v_0 \sin\alpha\, t + d_2 \end{bmatrix}$, but we'll use $d_1 = d_2 = 0$.

x-component is 310 when: $310 = v_0 \cos\alpha\, t$, solving for t yields $t \approx 8.08\ldots$

We need y-component to be > -20 when $t \approx 8$: $-g(5.7)^2/2 + \sqrt{300g}\,(\sin\frac{\pi}{4})\, 5.7$

≈ -11

$\Rightarrow$ YAY! Golfer can clear water.

R05 **3) Arc Length, Normal and Binormal Vectors (13.3, 13.4)**

7 Consider the surfaces $x^2 + y^2 + z^2 = 4$, and $z^2 = x^2 + y^2$ for $z \geq 0$.

a) Find a parameterization for the intersection curve, **r**(t), of the two surfaces.
b) Sketch the two surfaces and their intersection.
c) Calculate the length of **r**(t).
d) Find the unit tangent, normal, and binormal vectors for **r**(t) at the point (sqrt(2) , 0, sqrt(2)).
e) Add the three vectors to your sketch.

a) $4 = x^2+y^2+z^2 = x^2+y^2+(x^2+y^2)$ (substitute $z^2 = x^2+y^2$)

$4 = 2x^2+2y^2$

$\Rightarrow$ let $x = x(t) = \sqrt{2}\cos t$, $y(t) = \sqrt{2}\sin t$, $z = \sqrt{2}$, $\Rightarrow \vec{r} = \begin{bmatrix} \sqrt{2}\,c \\ \sqrt{2}\,s \\ \sqrt{2} \end{bmatrix}$

b)

c) $L = \int_0^{2\pi} \sqrt{(x')^2 + (y')^2 + (z')^2}\,dt = \sqrt{2}\int_0^{2\pi} dt$

$= 2\sqrt{2}\,\pi$

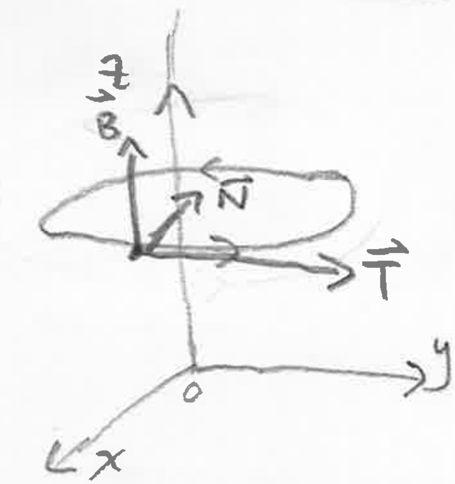
(Circumference of a circle is $2\pi r$, and radius $= \sqrt{2}$)

d) $\vec{T} = \dfrac{\vec{r}\,'}{\|\vec{r}\,'\|} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} -\sqrt{2}\,s \\ \sqrt{2}\,c \\ 0 \end{bmatrix}$, $\vec{T}(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ✓

$\vec{N} = \vec{T}\,'/|\vec{T}\,'| = \begin{bmatrix} -c \\ -s \\ 0 \end{bmatrix} / 1$, $\vec{N}(0) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ -s & c & 0 \\ -c & -s & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

e)



R06
1

Recitation 06

Today's Topics:

- Curvature & Normal Vectors (13.4)
- Tangential and Normal Components of Acceleration (13.5)
- ~~Vecocity and Acceleration in Polar Coordinates (13.6)~~

Today's Learning Obectives

1. Given a motion of an object, in either parametric form or as a function of a single variable, calculate the
   - curvature
   - tangent, normal, and binormal vectors
   - acceleration (tangential and normal components)
   - torsion
2. Calculate the osculating, normal, and rectifying planes for a given curve $\mathbf{r}(t)$ at a given value of t

Helpful Formulas

principle normal vector: $\vec{N} = \dfrac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$

curvature: $\kappa = \dfrac{1}{|\vec{v}|}\left|\vec{T}'(t)\right|$

curvature: $\kappa = \dfrac{|f''(x)|}{\left[1+\left(f'(x)\right)^2\right]^{3/2}}$

acceleration: $\vec{a} = a_T\vec{T} + a_N\vec{N}$

$a_T = \dfrac{d}{dt}|\vec{v}|$

$a_N = \cancel{\sqrt{|\vec{a}| + |a_T|}} = \sqrt{|\vec{a}|^2 - |a_T|^2}$

torsion: $\tau = \dfrac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$

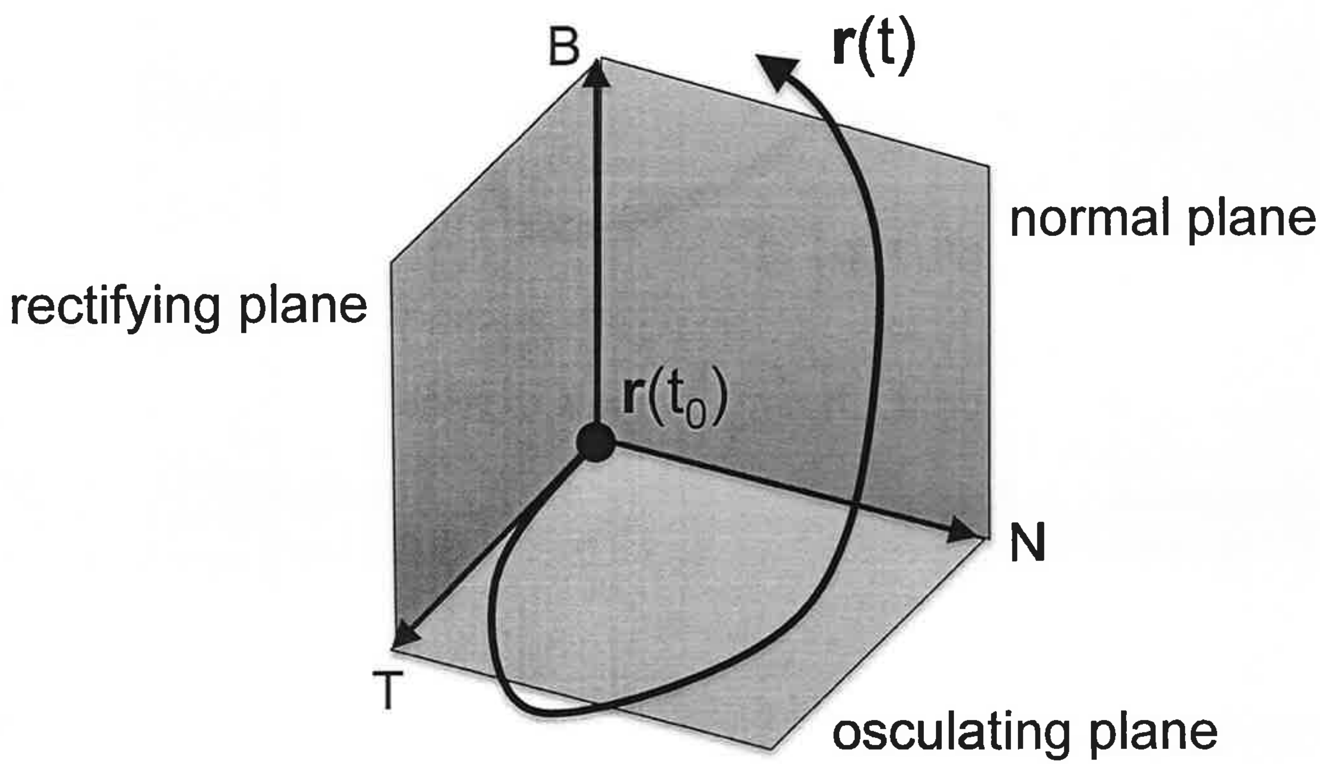
Notes:

- *One of the above equations has an error, where is it?*
- *There are alternate expressions for these formulas. Above are the formulas that the textbook uses.*

R06
3

Normal, Rectifying, and Osculating Planes

The geometry of the three planes determined by vectors **T**, **N**, and **B**, for curve **r**(t), at **r**(t_0).



If a motion, **r**(t), lies completely in a plane, then the binormal vector is CONSTANT.

Announcements

Graded Recitation Activity: Next week during Tuesday recitation, question sent
HW Due Tomorrow: 13.6
Quiz 1: Thur Jan 29
Office Hours: 7:30 pm – 8:30 pm, Wed Jan 28
https://georgiatech.adobeconnect.com/distancecalculusofficehours

Send Your TA an Email

Using your own words, describe

- the relationship between the curvature and the normal plane
- the relationship between the torsion and the osculating plane

Try to send an email with your answers by the end of the day today. If you send your TA an email with an answer to these questions you will get a response.
Hint: these relationships are described in the textbook.

R06
5

Group Work Activity: Part (a)

There are ~~three~~ four parts to the following question. Solve them in groups of 3 to 5 students.

Consider **r**(t) = sin(t)**i** + cos(t)**j** + **k**, t = -π/2.

a) Find **T**, **N**, and **B** at the given value of t.

$\vec{T} = \vec{v}/|\vec{v}| \Rightarrow \vec{v} = \begin{bmatrix} c \\ -s \\ 0 \end{bmatrix}, |\vec{v}| = 1$, so $\vec{T} = \begin{bmatrix} c \\ -s \\ 0 \end{bmatrix}$, so $\vec{T}(-\frac{\pi}{2}) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{N} = \begin{bmatrix} -s \\ -c \\ 0 \end{bmatrix}$ (using $\vec{N} = \vec{n}/|\vec{n}|$, $\vec{n} = \begin{bmatrix} s \\ c \\ 0 \end{bmatrix}$, or using $\vec{N} = \vec{T}'/|\vec{T}'|$)

$$\vec{B} = \vec{T} \times \vec{N} = \begin{vmatrix} i & j & k \\ c & -s & 0 \\ -s & -c & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ 0 \\ -c^2 - s^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\Rightarrow \vec{N}\left(-\frac{\pi}{2}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{B}\left(-\frac{\pi}{2}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

(NOTE THAT $\vec{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \; \forall t$)

Group Work Activity: Parts (b) and (c)

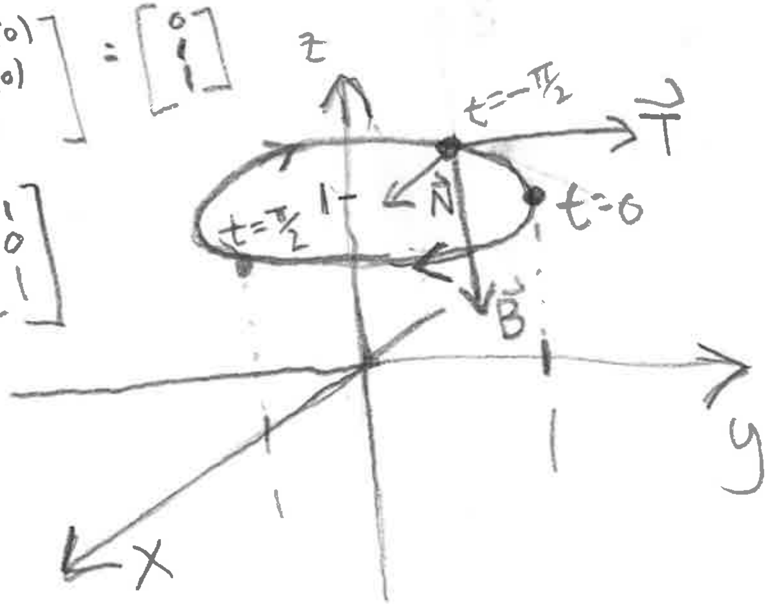
Consider $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}$, $t = -\pi/2$.

b) Sketch $\mathbf{r}$ for $[0, 2\pi]$ and indicate the direction of motion.

c) Sketch $\mathbf{T}$, $\mathbf{N}$, and $\mathbf{B}$ at the given value of t.

$$\vec{r}(0) = \begin{bmatrix} \sin(0) \\ \cos(0) \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{r}(\pi/2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$



$$\vec{T}\left(-\frac{\pi}{2}\right) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{N}\left(-\frac{\pi}{2}\right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{B}\left(-\frac{\pi}{2}\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

(NOTE THAT $\vec{B}$ IS A CONSTANT VECTOR.
WHY IS $\vec{B}$ CONSTANT?)

R06
7

Group Work Activity: Part (d)

Consider $\mathbf{r}(t) = \sin(t)\mathbf{i} + \cos(t)\mathbf{j} + \mathbf{k}$, $t = -\pi/2$.

d) Find the equation of the normal plane at $t = -\pi/2$.

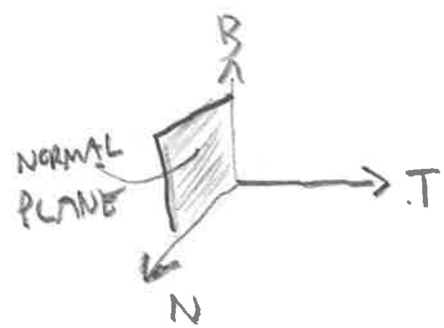
$\vec{T}$ is perpendicular to normal plane.

$\Rightarrow$ NORMAL PLANE IS:

$$\vec{T} \cdot (\vec{r} - \vec{r}_0) = 0, \quad \vec{r}_0 = \text{any point in plane.}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x - (-1) \\ y - (0) \\ z - (1) \end{bmatrix} = 0, \text{ or } y = 0.$$

(x and z are "free variables")



R06

8

True or False

a) Curvature is a scalar and can be any real number.

This statement is F *because:* $\kappa \geq 0$, so κ can't be any real number.

b) Torsion is a scalar and can be any real number.

This statement is T *because:* torsion is any real number

c) If **r**(t) = x(t)**i** + y(t)**j**, then the normal vector, **N**, is given by **N** = **n**/|**n**|, where **n** = -x'(t)**i** + y'(t)**j**.

This statement is F *because:*

$\vec{n} = \begin{bmatrix} -y' \\ x' \end{bmatrix}$. This is an alternate formula from one of the homework exercises

Recitation 07

Today's Topics: Quiz 1 Review, Graded Recitation Activity 1

Quiz 1 Topics

12.6 Quadratic Surfaces

13.1 Vector Parametric Representations of Curves

~~13.2 Quadratic Surfaces~~

13.2 Projectile Motion

13.3 ~~13.2~~ Path Length

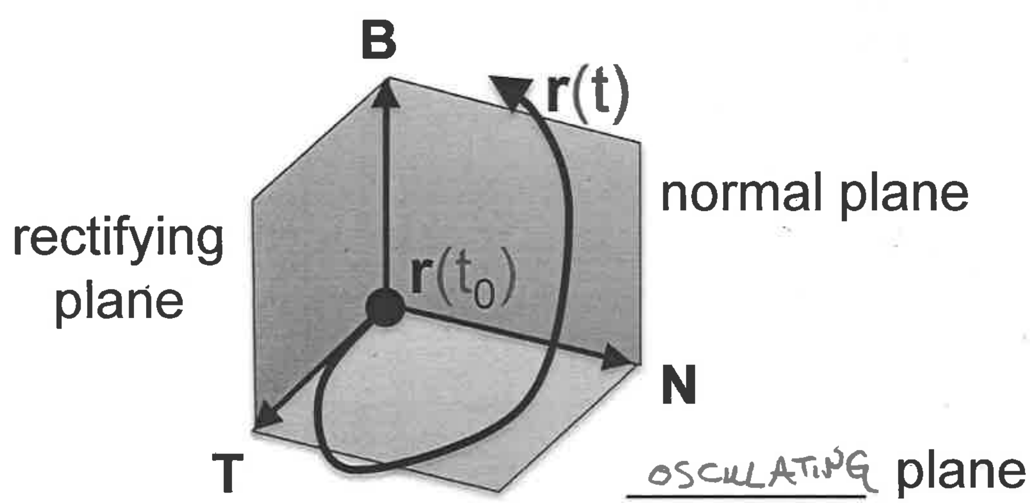
13.4 ~~13.3~~ Curvature & Normal Vectors

13.5 Tangential & Normal Components of Acceleration

Quiz 1 Learning Objectives

You should be able to do the following for Quiz 1.

- Identify and sketch quadratic surfaces given their algebraic equations
- Develop parameteric representations of curves
- Integrate vector functions to determine projectile motion
- Characterize a motion, given in either parametric form **r**(t), or as a continuous function f(x), using:
  - vectors: velocity, acceleration, tangent, binormal
  - scalars: curvature, torsion, tanential & normal components of accel, arc length
  - planes: tangential, rectifying, OSCULATING



R07

3

Helpful Formulas

Ideal Projectile Motion: $\vec{r}(t) = (v_0 \cos\alpha) t\hat{i} + \left((v_0 \sin\alpha)t - \dfrac{gt^2}{2}\right)\hat{j}$

max range: $R = \dfrac{v_0^2 \sin 2\alpha}{g}$ max height: $\dfrac{v_0^2 \sin^2 \alpha}{2g}$

principle normal vector: $\vec{N} = \vec{T}'(t) / |\vec{T}'(t)|$

binormal vector: $\vec{B} = \vec{N}'(t) / |\vec{N}'(t)|$

curvature: $\kappa = |\vec{T}'(t)| / |\vec{v}|$

curvature: $\kappa = |f''(x)| / \left[1 + (f'(x))^2\right]^{3/2}$

acceleration: $\vec{a} = a_T\vec{T} + a_N\vec{N}$

$a_T =$ ________

$a_N =$ ________

torsion: $\tau = \dfrac{\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}}{|\vec{v} \times \vec{a}|^2}$

R07

4

Graded Group Work Activity

Instructions

- Every student in your group needs to write their name or initials on the board.
- You have 20 minutes to answer the questions below.
- For full marks, show at least three intermediate steps for each question.
- Answer each question on a different slide.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.

1) Tangential & Normal Components of Acceleration (4 points)

Let $\mathbf{r}(t) = 2t\mathbf{i} + t\mathbf{j} + 2t^2\mathbf{k}$ be a motion. Compute the tangential and normal components of the acceleration.

2) Arc Length (3 points)

Find the arc length, from 0 to t, of the curve $\mathbf{r}(t) = e^t\cos(t)\mathbf{i} + e^t\sin(t)\mathbf{j} + 5e^t\mathbf{k}$.

R07 **1) Tangential & Normal Components of Acceleration (4 points)**

5 Let **r**(t) = 2t**i** + t**j** + 2t²**k** be a motion. Compute the tangential and normal components of the acceleration.

$\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 4t \end{bmatrix}$, $\vec{a} = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$, $|\vec{v}| = \sqrt{2^2+1^2+16t^2}$

TANGENTIAL

$$a_T = \frac{d}{dt}|\vec{v}| = \frac{d}{dt}\sqrt{2^2+1^2+4^2t^2} = \frac{1/2}{\sqrt{5+16t^2}} \cdot 32t$$

$$\boxed{a_T = \frac{16t}{\sqrt{5+16t^2}}}$$

NORMAL

can use $\sqrt{|\vec{a}|^2 - |a_T|^2}$, or can use $\frac{|\vec{v}\times\vec{a}|}{|\vec{v}|}$

1) $$a_N = \sqrt{(0+0+4^2) - \frac{(16t)^2}{5+16t^2}} = \boxed{\sqrt{16 - \frac{16^2t^2}{5+16t^2}}}$$

2) $$a_N = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|} = \frac{\left\| \begin{matrix} i & j & k \\ 2 & 1 & 4t \\ 0 & 0 & 4 \end{matrix} \right\|}{\sqrt{2^2+1^2+(4t)^2}} = \frac{|4\hat{i} - 8\hat{j} + 0\hat{k}|}{\sqrt{5+16t^2}} = \boxed{\frac{\sqrt{16+64}}{\sqrt{5+16t^2}}}$$

R07 **2) Arc Length ($\cancel{5}$ 2 points)**

6 Find the arc length, from 0 to t, of the curve **r**(t) = $e^t\cos(t)$**i** + $e^t\sin(t)$**j** + $5e^t$**k**.

$$\vec{v}(t) = \vec{r}\,'(t) = \begin{bmatrix} e^t c - e^t s \\ e^t s + e^t c \\ 5e^t \end{bmatrix}, \quad \cos t = c, \quad \sin t = s$$

$$L = \int_0^t |\vec{v}|\, d\tau = \int_0^t \left(2e^{2\tau}c^2 + 2e^{2\tau}s^2 + 25e^{2\tau}\right)^{\frac{1}{2}} d\tau$$

$$= \int_0^t \sqrt{27e^{2\tau}}\, d\tau$$

$$= \sqrt{27}\left(e^t - 1\right) = 3\sqrt{3}\left(e^t - 1\right)$$

Message your TA when you've finished both questions., then move on to the remaining questions.

R07

7

Curvature and Torsion

This question has 4 parts. Consider the surfaces z = $x^2 + y^2$ and y = 2, for z ≥ 0.

A) Find a parametric vector representation for the ~~ir~~ intersection. of the surfaces.

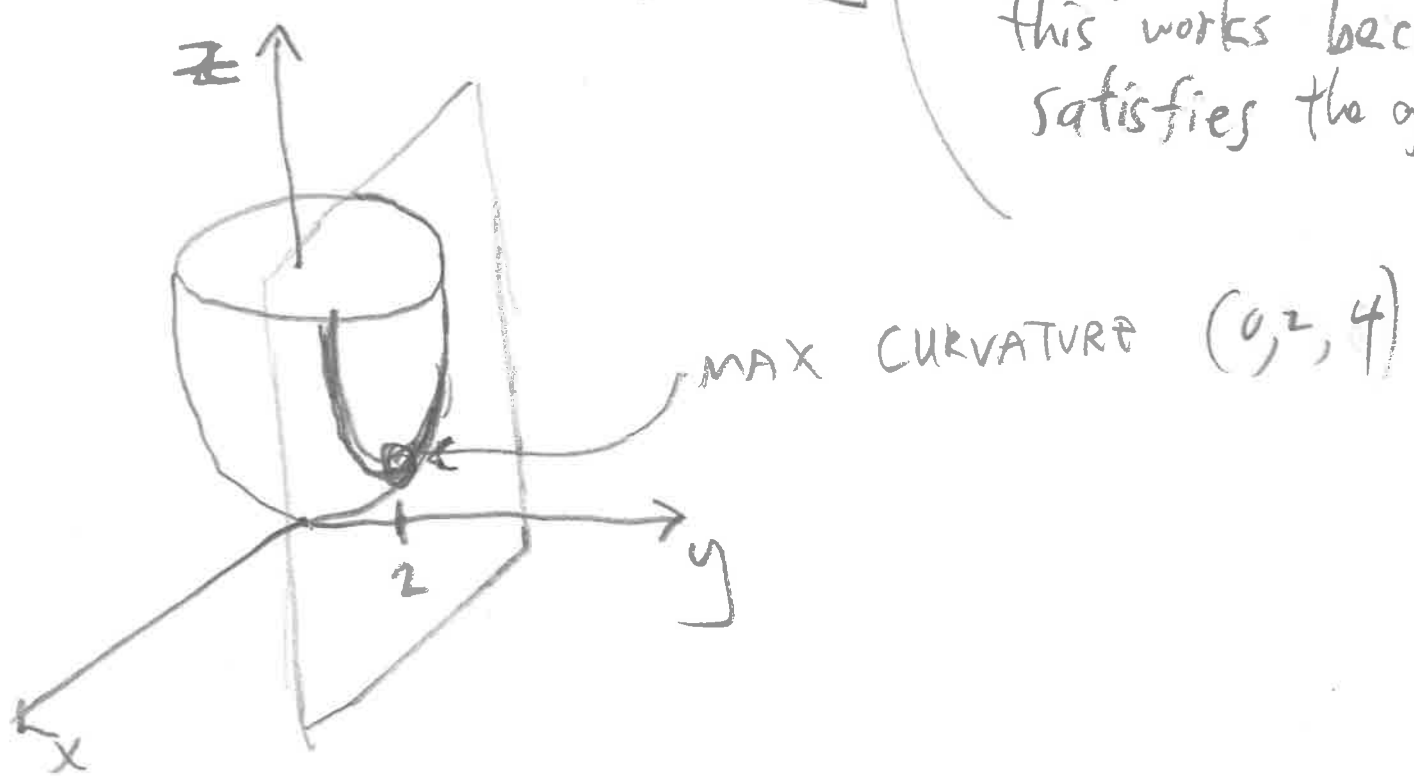
B) Sketch the intersection and the 2 surfaces.

A) Let $x = t$, y has to be 2, so $z = t^2 + 4$,

$$\Rightarrow \quad \vec{r}(t) = \begin{bmatrix} t \\ 2 \\ t^2+4 \end{bmatrix}$$

(We can use other parameterizations, but this "works" because it satisfies the given equations.)

B)



R07

8

Curvature and Torsion

This question has 4 parts. Consider the surfaces $z = x^2 + y^2$ and $y = 2$, for $z \geq 0$.

C) Calculate the curvature and identify on your sketch whre the curvature is maximized.

c) $K = \dfrac{|f''|}{[1+(f')^2]^{3/2}}$, where $f = z(x) = x^2 + 4$

$= \dfrac{2}{[1+(2x)^2]^{3/2}}$

$= \dfrac{2}{(1+4x)^{3/2}}$

$$\boxed{\text{MAX CURVATURE AT } (x,y,z) = (0,2,4)}$$

ALTERNATE: $K = \dfrac{|\vec{T}'(t)|}{|\vec{v}|}$ (more work)

R07

9

Curvature and Torsion

This question has 4 parts. Consider the surfaces $z = x^2 + y^2$ and $y = 2$, for $z \geq 0$.

D) Calculate the torsion of the intersecting curve and explain your answer.

$$\tau = \frac{\left\| \begin{matrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{matrix} \right\|}{\left\| \begin{matrix} i & j & k \\ 1 & 0 & 2t \\ 0 & 0 & 2 \end{matrix} \right\|} = \frac{0}{|-2|} = 0$$

τ is 0 because motion is planar.

Recitation 09

R09 Topics

14.1 Functions of Several Variables
14.2 Limits and Continuity

R09 Learning Objectives

By the end of today's session you should be able to

- Identify and sketch the domain of a function of several variables.
- Determine whether or not limits of functions of several variables exist.

While We're Waiting to Start

Consider the function

$$g(x,y) = \frac{\sqrt{y+1}}{x^2y + xy^2}.$$

For $g(x,y)$ to be defined and a real-valued function, what values of x and y can we allow?

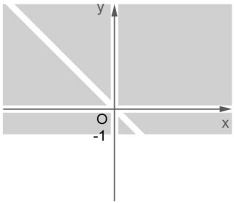
Domain of a Function of Two Variables

Identify and sketch the domain of

$$g(x,y) = \frac{\sqrt{y+1}}{x^2y + xy^2}.$$

Solution

For $g(x,y)$ to be defined, its denominator cannot be zero. This implies that $0 \neq x^2y + xy^2 = xy(x+y)$. Thus, $x \neq 0$, $y \neq 0$, and $y \neq -x$. The numerator of $g(x,y)$ also cannot be complex, which implies that $y + 1 \geq 0$, or that $y \geq -1$. The domain is the set $D = \{(x,y) | y \geq -1, x \neq 0, y \neq 0, y \neq -x\}$.



Limits of a Function of Two Variables

Consider the function of two variables

$$f(x, y) = \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}.$$

We want to evaluate

$$\lim_{(x,y)\to(1,0)} f(x, y)$$

What strategies might we try to evaluate the desired limit?

Solution

When we evaluate $f(x, y)$ at the limit point, we find $f(1, 0)$ is an indeterminant form of type $0/0$. It may be that f is not continuous at the point $(1, 0)$. In one dimension, we would use l'Hopsital's rule, or algebraic manipulation, to evaluate such a limit. But l'Hospitals rule only works for functions of one variable. So for this limit, we will try approaching the limit point along curves that pass through the limit point. In this case, we can try evaluating the limit along $y = m(x - 1)$.

Limits of a Function of Two Variables, Example 1

Evaluate

$$\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3}$$

Solution
Choose a function, $y(x)$, that passes through the given limit point $(1, 0)$. We can try $y = m(x-1)$, which passes through $(1, 0)$, and see what happens.

$$\begin{aligned}
\lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + y^2}{4(x-1)^2 + 9y^3} &= \lim_{(x,y)\to(1,0)} \frac{x(x-1)^3 + m^2(x-1)^2}{4(x-1)^2 + 9m^3(x-1)^3} \\
&= \lim_{(x,y)\to(1,0)} \frac{x(x-1) + m^2}{4 + 9m^3(x-1)} \\
&= \frac{m^2}{4}
\end{aligned}$$

Because the value of the limit depends on the path of approach, the limit does not exist.

Limits of a Function of Two Variables, Example 2

In groups of 3 to 5 students, evaluate the limit

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}.$$

Solution

Along the path $y = mx$, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{xm^2x^2}{x^2+m^4x^4} = \lim_{(x,y)\to(0,0)} \frac{m^2x}{1+m^4x^2} = 0.$$

We might be tempted to believe that this limit exists. But along the path $x = my^2$, we find

$$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4} = \lim_{(x,y)\to(0,0)} \frac{my^4}{m^2y^4+y^4} = \frac{m}{m^2+1}$$

Because the value of the limit depends on the path of approach, the limit does not exist.

Definition of Limit

Evaluating limits along paths will not show that a given limit exists. To show that a limit exists, we can use the definition of limit.

The limit of $f(x, y)$ as (x, y) aproach (a, b) is L if for every number $\epsilon > 0$, there is a corresponding $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon \qquad \text{when } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

In other words, the distance between f and L can be made **arbitrarily** small by making the distance from (x, y) to (a, b) **sufficiently** small.

An Epsilon Delta Example

Evaluate, or show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2}.$$

Solution

Along the path $y = mx$, we obtain

$$\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{3m^2x^3}{x^2(1+m^2)} = 0$$

Along the path $y = mx$, the limit is zero. We can also show that along the path $y = mx^2$, that the limit is also zero. So we are starting to suspect that this limit exists and that $L = 0$. Let $\epsilon > 0$. We want to find a $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon \qquad \text{when } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

We will do this on the next few slides.

An Epsilon Delta Example

We want to find a $\delta > 0$ such that

$$|f(x,y) - L| < \epsilon \qquad \text{when} \qquad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

The limit point is $(0,0)$, so $a = b = 0$. And we think the limit might equal zero, so we can try $L = 0$ and see what happens.

$$\left|\frac{3x^2y}{x^2+y^2} - 0\right| < \epsilon \qquad \text{when} \qquad 0 < \sqrt{x^2+y^2} < \delta$$

However,

$$\begin{aligned}\left|\frac{3x^2y}{x^2+y^2} - 0\right| &= \frac{3x^2|y|}{x^2+y^2} \\ &\leq \frac{3(x^2+y^2)|y|}{x^2+y^2} \\ &= 3|y| = 3\sqrt{y^2} \leq 3\sqrt{x^2+y^2}\end{aligned}$$

This result will suggest that we choose $\delta = \epsilon/3$. We see why on the next slide.

An Epsilon Delta Example

We have found that

$$|f(x,y) - L| = \left|\frac{3x^2y}{x^2+y^2} - 0\right| \leq 3\sqrt{x^2+y^2}$$

Choosing $\delta = \epsilon/3$, and letting $0 < \sqrt{x^2+y^2} < \delta$, we obtain

$$|f(x,y) - L| \leq 3\delta = 3(\epsilon/3) = \epsilon$$

Thus, given any ϵ, choosing $\delta = \epsilon/3$, and $0 < \sqrt{x^2+y^2} < \delta = \epsilon/3$, we can guarantee that $|f(x,y) - L| < \epsilon$.

Therefore, the limit exists and is equal to 0.

Conclusions: Evaluating Limits of Multivariable Functions

Suppose we need to evaluate a limit of a function of two variables

$$\lim_{(x,y)\to(a,b)} f(x,y).$$

If we know that $f(x,y)$ is continuous at (a,b), we can evaluate the limit with direct substitution. If we don't know that $f(x,y)$ is continuous at (a,b), we can either

- evaluate the limit along curves ($y = mx$, for example) to see if the limit **does not** exist, or
- we can use the definition of limit to prove that the limit **does** exist and determine what the limit is equal to.

Notes:

- evaluating a limit along curves cannot tell us that a given limit exists, it can only tell us whether it doesn't exist
- I'm assuming you're familiar with continuity for a function of several variables, but if you aren't it's on the next homework and isn't a diffcult concept.

Recitation 10

R10 Topics
14.2 Limits and Continuity
14.3 Partial Derivatives
14.4 The Chain Rule

R10 Learning Objectives
By the end of today's session you should be able to

- Determine whether or not limits of functions of several variables exist by evaluating the limit along paths or by using the formal definition of limit.
- Compute partial derivatives of multivariable functions using the chain rule.

While We're Waiting to Start
Calculate $f_y(1, -2, -1)$ for $f(x, y, z) = x^2 y e^{y/z}$.

A Partial Derivative

Calculate $f_y(1,-2,-1)$ for $f(x,y,z) = x^2 y e^{y/z}$.

Solution

$$\begin{aligned} f_y &= \frac{\partial f}{\partial y}\left(x^2 y e^{y/z}\right) \\ &= x^2 e^{y/z} + x^2 y e^{y/z}\left(\frac{\partial}{\partial y}\frac{y}{z}\right) \\ &= x^2 e^{y/z} + \frac{x^2 y e^{y/z}}{z} \end{aligned}$$

Thus, $f_y(1,-2,-1) = (1)^2 e^2 + \frac{(1)^2(-2)e^2}{-1} = 3e^2$.

A Conceptual Question

Select all options that are correct.

Given a function $f(x, y)$, to evaluate $\frac{\partial f}{\partial x}$ at the point (1,3), we can:

1. Differentiate f with respect to x and then set $x = 1, y = 3$.
2. Set $x = 1, y = 3$ and then differentiate f with respect to x.
3. Set $x = 1$ and then differentiate f with respect to x.
4. Set $y = 3$ and then differentiate f with respect to x.

Solution

The first option is acceptable and is the usual approach.

The second and third options would result in an answer of zero: we should differentiate with respect to the prescribed variable, x, and then set the variable equal to its value.

The fourth option is acceptable, because variables other than the one that we are differentiating are treated as constants.

Recall: Definition of Limit

Evaluating limits along paths will not show that a given limit exists. To show that a limit exists, we can use its formal definition.

The limit of $f(x, y)$ as (x, y) approach (a, b) is L if for every number $\epsilon > 0$, there is a corresponding $\delta > 0$ such that

$$|f(x, y) - L| < \epsilon \qquad \text{when } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$$

In other words, the distance between f and L can be made **arbitrarily** small by making the distance from (x, y) to (a, b) **sufficiently** small.

Epsilon Delta Definition of Limit

Use the definition of limit to show that the following exists and is equal to 0.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x+y}{x^2+1}$$

Solution

To apply the definition of limit, we start with $|f(x,y)-L|$, and work towards an expression that involves $\sqrt{(x-a)^2+(y-b)^2}$. We know are given that the limit is equal to zero, so we can use $L=0$. We also know that the limit point is $(0,0)$, so we can also use $a=b=0$.

$$\begin{aligned}
\Big|f(x,y)-L\Big| &= \left|\frac{x+y}{x^2+1}-0\right| \\
&= \frac{|x+y|}{|x^2+1|} \\
&\le \frac{|x+y|}{1} \quad \text{because } x^2+1\ge 1 \\
&= |x+y| \\
&\le |x|+|y| \quad \text{by the triangle inequality}
\end{aligned}$$

Epsilon Delta Definition of Limit

$$\begin{aligned}\Big|f(x,y) - L\Big| &\leq |x| + |y| \\ &= \sqrt{x^2} + \sqrt{y^2} \\ &\leq \sqrt{x^2+y^2} + \sqrt{x^2+y^2} \\ &= 2\sqrt{x^2+y^2}\end{aligned}$$

This result suggests that we choose $\delta = \epsilon/2$. By choosing $\delta = \epsilon/2$, and letting $0 < \sqrt{x^2+y^2} < \delta$, we obtain

$$|f(x,y) - L| \leq 2\sqrt{x^2+y^2} < 2\delta = 2(\epsilon/2) = \epsilon$$

Thus, given any ϵ, choosing $\delta = \epsilon/2$, and $0 < \sqrt{x^2+y^2} < \delta = \epsilon/2$, we can guarantee that $|f(x,y) - L| < \epsilon$.

Therefore, the limit exists and is equal to 0.

Group Work Activities

Solve the following in groups of 3 to 5 students.

1. Evaluate the following limit, or show that it does not exist.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}.$$

2. Evaluate the following limit, or show that it does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2}.$$

3. Calculate du/dt given that $u = x^2 - y^2$, $x = t^2 - 1$, and $y = 3\sin(\pi t)$. Simplification is not necessary.
4. The radius of a cylinder is decreasing at a rate of 2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume changing when the radius is 10 cm and the height is 100 cm?
5. Create a function, $f(x, y)$, that satisfies the following

$$\frac{\partial f(x,y)}{\partial x} = x^2 + y, \text{ and } \frac{\partial f(x,y)}{\partial y} = y^3 + x$$

Question 1: Limits

Evaluate, or show that the following limit does not exist.

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - y^2 - z^2}{x^2 + y^2 + z^2}.$$

Solution

Along the x-axis, $y = z = 0$, and the limit becomes

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 - 0 - 0}{x^2 + 0 + 0} = 1.$$

Along the y-axis, $x = z = 0$, and the limit becomes

$$\lim_{(x,y,z)\to(0,0,0)} \frac{0 - y^2 - 0}{0 + y^2 + 0} = -1.$$

Depending on which path we approach the limit point, we arrive at different values. Therefore the limit does not exist (DNE).

Question 2: Limits

Evaluate, or show that the following limit does not exist.

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}.$$

Solution

Along the line $y = mx$, the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{xmx}{x^2+m^2x^2} = \lim_{(x,y)\to(0,0)} \frac{mx^2}{x^2(1+m^2)} = \frac{m}{1+m^2}.$$

Depending on which path we approach the limit point, we arrive at different values. Therefore the limit does not exist (DNE).

Question 3: The Chain Rule

Calculate du/dt given that $u = x^2 - y^2$, $x = t^2 - 1$, and $y = 3\sin(\pi t)$. Simplification is not necessary.

Solution
We can approach this in two different ways. We can use the chain rule, as follows.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \\ &= 2x \cdot 2t + (-2y)(3\pi\cos(\pi t)) \\ &= 4t(t^2 - 1) - 6\sin(\pi t) \cdot 3\pi\cos(\pi t)\end{aligned}$$

An also substitute our known values for x and y first, and then differentiate.

$$\begin{aligned}\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(x^2 - y^2) &= \frac{\partial}{\partial t}\Big((t^2 - 1)^2 - (3\sin(\pi t))^2\Big) \\ &= 2(t^2 - 1)(2t) - 6\pi\sin(\pi t) \cdot 3\cos(\pi t)\end{aligned}$$

Question 4: The Chain Rule

The radius of a cylinder is decreasing at a rate of 2 cm/s while its height is increasing at a rate of 3 cm/s. At what rate is the volume changing when the radius is 10 cm and the height is 100 cm?

Solution

$$\begin{aligned} V &= \pi R^2 H \\ \frac{\partial V}{\partial t} &= \frac{\partial V}{\partial R}\frac{dR}{dt} + \frac{\partial V}{\partial H}\frac{dH}{dt} \\ &= \frac{\partial(\pi R^2 H)}{\partial R}(-2) + \frac{\partial(\pi R^2 H)}{\partial H}(3) \\ &= 2\pi RH(-2) + (\pi R^2)(3) \\ &= -4\pi RH + 3\pi R^2 \end{aligned}$$

When $R = 10$ and $H = 100$, we have

$$\frac{\partial V}{\partial t} = -4\pi \cdot 10 \cdot 100 + 3\pi(10)^2 = -4000\pi + 300\pi = -3700\pi.$$

Question 5: Partial Derivatives

Create a function, $f(x,y)$, that satisfies the following

$$\frac{\partial f(x,y)}{\partial x} = x^2 + y, \text{ and } \frac{\partial f(x,y)}{\partial y} = y^3 + x$$

Solution

A function whose derivative with respect to x is $x^2 + y$ is $f = \frac{x^3}{3} + xy + C(y)$, where C is some function of y. Differentiating with respect to y gives us $f_y = 0 + x + C'(y)$. Thus, by comparison, $C' = y^3$, and $C = \frac{y^4}{4}$. Thus

$$f(x,y) = \frac{x^3}{3} + xy + C(y) = \frac{x^3}{3} + xy + \frac{y^4}{4}.$$

Recitation 11

R11 Topics

14.5 The Gradient

R11 Learning Objectives

By the end of today's session you should be able to do the following.

- Compute gradients and directional derivatives.
- Provide geometric interpretations of gradients and directional derivatives.
- Describe the relationship between gradients and level curves.

While We're Waiting to Start

Consider $f(x, y) = y^2 e^{2x}$.

1. Find the direction of steepest ascent at $P(0, 1)$ and at $Q(0, -1)$.
2. Sketch the level curves of f, and the gradient vectors at P and Q.
3. Find the rate at which f is increasing in the direction $\vec{u} = \hat{i} - \hat{j}$ at P.

The Gradient and Directional Derivative

Consider $f(x, y) = y^2 e^{2x}$.

1. Find the direction of steepest ascent at $P(0, 1)$ and at $Q(0, -1)$.
2. Sketch the level curves of f, and the gradient vectors at P and Q.
3. Find the rate at which f is increasing in the direction $\vec{u} = \hat{i} - \hat{j}$ at P.

Solution

The direction of steepest ascent at any point is given by the gradient.

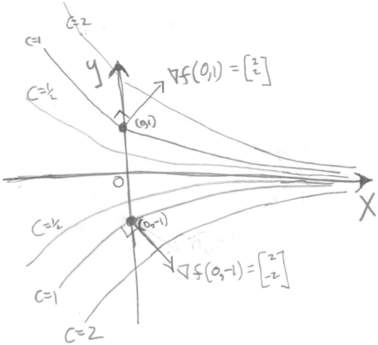
$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \end{bmatrix} = \begin{bmatrix} 2y^2 e^{2x} \\ 2y e^{2x} \end{bmatrix}$$

The direction of steepest ascent at P and Q are:

$$\nabla f(0, 1) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \qquad \nabla f(0, -1) = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

The level curves are obtained by setting $f(x, y) = C$, where C is a value in the range of f. $C = y^2 e^{-2x}$ implies $y = \pm\sqrt{C} e^{-x}$. We will plot the curves on the next slide.

The Gradient and Directional Derivative



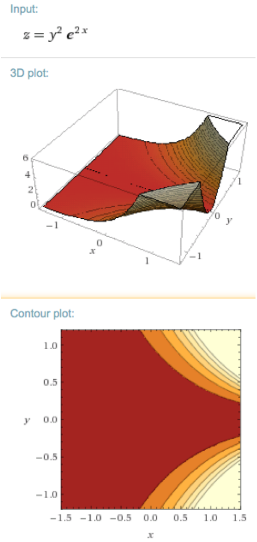
The gradient vectors at points $P(0,1)$ and $Q(0,-1)$ should be perpendicular to the level curves (apologies for the rough drawing).

The rate at which $f(x,y)$ is increasing at P in the direction $\vec{u} = \hat{i} - \hat{j}$ is given by the dot product:

$$\nabla f(0,1) \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 - 2 = 0$$

Thus, the rate of change of f in the direction of $\vec{u}$ is zero. Vector $\vec{u}$ points in the direction of a level curve of $f(x,y)$.

Wolfram Alpha's Plots of $f(x, y)$

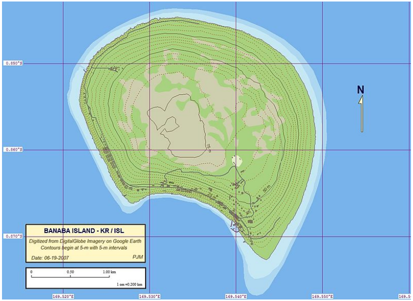


In case it helps see what is going on, to the left are plots of our function, $y^2 e^{2x}$, that WolframAlpha produces.

Notice that the contour plot gives a set of level curves.

Level Curves

If C is in the ____________________ of $f(x, y)$, then the curve $C = f(x, y)$ is a **level curve** of $f(x, y)$. For functions of two variables, we can think of level curves as curves of constant **height** (analogous to topographic maps, that have curves of constant elevation).



In other words, a level curve is an intersection between $f(x, y)$ and the plane $z = C$. Level curves are a useful view of the overall behavior of a function.

Banaba Island image under a CCBY2.0 license, available from https://www.flickr.com/photos/evsmap

Level Curves and the Gradient

This following helps explain why the gradient is $\perp$ to level curves.

Let $C = g(x, y)$ be a level curve of $g(x, y)$. Show that ∇g is always perpendicular to the level curve.

Solution

Let $\vec{r}(t)$ be a parameterization of the curve $g(x, y) = C$. A vector that is parallel to the curve at any t is $\vec{v}(t) = \vec{r}'(t)$. We will show that the gradient is perpendicular to $\vec{v}(t)$ for all t.

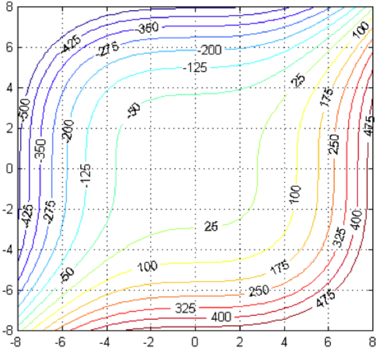
Because of our parameterization, $C = g(x, y) = g(x(t), y(t))$, and by the chain rule,

$$\frac{dg}{dt} = 0 = \frac{\partial g}{\partial x}\frac{dx}{dt} + \frac{\partial g}{\partial y}\frac{dy}{dt} = \begin{bmatrix} g_x \\ g_y \end{bmatrix} \cdot \begin{bmatrix} dx/dt \\ dy/dt \end{bmatrix} = \nabla g \cdot \vec{v}$$

Thus, the gradient is always perpendicular to the level curve $C = g(x, y)$.

A Conceptual Question: The Gradient

At which point does the gradient vector have the largest magnitude? Draw the gradient at this point.



1. (0,0)
2. (8,-8)
3. (6,-2)
4. (-4,-4)

Solution

The magnitude of the gradient is $|\nabla f| = \sqrt{f_x^2 + f_y^2}$. At $(6, -2)$, the contour lines are most closely packed: f is changing most rapidly at that point. The gradient points in the direction of steepest ascent and is perpendicular to the level curve at $(6, -2)$, so ∇f points to the right.

Group Work Activities

Solve the following in groups of 3 to 5 students.

1. Find the directional derivative of $f = z\ln(x/y)$ at $(1,1,2)$ towards the point $(2,2,1)$ and provide a geometric interpretation of your answer.
2. For $z = 3xy - x^3 - y^3$, find the points where the gradient vector, $\nabla z(x,y)$, is the zero vector. Provide a geometric interpretation of your answer.
3. Suppose $\vec{F} = \nabla f(x,y) = (2x + \sin y)\hat{i} + (x\cos(y) - 2y)\hat{j}$. Find $f(x,y)$.

Question 1: A Directional Derivative

Find the directional derivative of $f = z\ln(x/y)$ at $(1,1,2)$ towards the point $(2,2,1)$. Provide a geometric interpretation of your answer.

Solution

For clarity, I'm writing out more steps than are needed. We're using the Chain Rule a few times in this problem.

$$\begin{aligned}
\nabla f &= \frac{\partial}{\partial x}\Big(z\ln(x/y)\Big)\hat{i} + \frac{\partial}{\partial y}\Big(z\ln(x/y)\Big)\hat{j} + \frac{\partial}{\partial z}\Big(z\ln(x/y)\Big)\hat{k} \\
&= z\frac{\partial}{\partial x}\ln(x/y)\hat{i} + z\frac{\partial}{\partial y}\ln(x/y)\hat{j} + \ln(x/y)\frac{\partial}{\partial z}(z)\hat{k} \\
&= z\frac{1}{x/y}\frac{\partial}{\partial x}(x/y)\hat{i} + z\frac{1}{x/y}\frac{\partial}{\partial y}(x/y)\hat{j} + \ln(x/y)\hat{k} \\
&= \frac{z}{x}\hat{i} - \frac{z}{y}\hat{j} + \ln(x/y)\hat{k} \\
\nabla f(1,1,2) &= 2\hat{i} - 2\hat{j} + 0\hat{k}
\end{aligned}$$

On the next slide we will find the directional derivative and provide a geometric interpretation.

Question 1: A Directional Derivative (Continued)

Let the vector pointing from $(1, 1, 2)$ to $(2, 2, 1)$ be $\vec{u}$. The desired directional derivative is the dot product $\nabla f \cdot \vec{u}$.

$$\nabla f(1,1,2) \cdot \vec{u} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2-1 \\ 2-1 \\ 1-2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 0$$

Therefore, the directional derivative, at the point $(1, 1, 2)$, in the direction pointing towards $(2, 2, 1)$, is zero. Geometrically, this means that the value of the function f is not changing in the specified direction.

Question 2: Zero Gradient

For $z = 3xy - x^3 - y^3$, find the points where the gradient vector, $\nabla z(x, y)$, is the zero vector. Provide a geometric interpretation of your answer.

Solution

$$\nabla z = \begin{bmatrix} \frac{\partial}{\partial x} z \\ \frac{\partial}{\partial y} z \end{bmatrix} = \begin{bmatrix} 3y - 3x^2 \\ 3x - 3y^2 \end{bmatrix}$$

The gradient vector has zero magnitude when

$$0 = 3y - 3x^2$$
$$0 = 3x - 3y^2$$

Rearranging these equations yields the two curves $y = x^2$ and $x = y^2$. These curves intersect at two points, $(0, 0)$, and $(1, 1)$. Geometrically, these points correspond to points where the function $z(x, y)$ is flat. In other words, where its tangent plane is horizontal. These points **could** also indicate local minima/maxima.

Question 3: Constructing a Function From its Gradient

Suppose $\vec{F} = \nabla f(x,y) = (2x + \sin y)\hat{i} + (x\cos(y) - 2y)\hat{j}$. Find $f(x,y)$.

Solution

A function whose derivative with respect to x is $2x + \sin y$ is $f = x^2 + x\sin y + C(y)$, where C is some function of y. Differentiating with respect to y gives us $f_y = 0 + x\cos y + C'(y)$. Thus, by comparison, $C' = -2y$, and $C = -y^2$. Thus

$$f(x,y) = x^2 + x\sin y + C(y) = x^2 + x\sin y - y^2.$$

Recitation 12

R12 Topics

14.6 Tangent Planes and Differentials
14.7 Absolute Min/Max

R12 Learning Objectives

By the end of today's session you should be able to do the following.

- Find equations of tangent planes and normal lines of surfaces.
- Apply tangent planes and differentials to make approximations.
- Locate and classify critical points of surfaces.

Example 1

Consider the surface $x^2 + 4y^2 = z^2$.

1. Find the equation of the tangent plane at $P(3, 2, 5)$.
2. Find the equation of the normal line at P, and identify where the normal line intersects the xy-plane.
3. Sketch the surface and the normal line.

Example 1: Part 1

Consider the surface $x^2 + 4y^2 = z^2$. Find the equation of the tangent plane at $P(3, 2, 5)$.

Solution

The surface may be represented by the function $f(x, y, z) = x^2 + y^2 - z^2$. A normal vector at any point on the surface is given by the gradient $\nabla f(x, y, z)$.

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \\ \frac{\partial}{\partial z} f \end{bmatrix} = \begin{bmatrix} 2x \\ 8y \\ -2z \end{bmatrix} \quad \Rightarrow \quad \nabla f(3,2,5) = \begin{bmatrix} 6 \\ 16 \\ -10 \end{bmatrix}$$

The equation for the tangent plane is the dot product between a normal vector and a vector in the tangent plane.

$$0 = \nabla f(3,2,5) \cdot \begin{bmatrix} x-3 \\ y-2 \\ z-5 \end{bmatrix} = 6(x-3) + 16(y-2) - 10(z-5)$$

This simplifies to $3x + 8y - 5z = 0$.

Example 1: Part 2

Consider the surface $x^2 + 4y^2 = z^2$. Find the equation of the normal line at $P(3, 2, 5)$, and identify where the normal line intersects the xy-plane.

Solution

Recall that the scalar parametric equations for a line are given by $\vec{r}(t) = \vec{r}_0 + \vec{d}t$, where $\vec{r}_0$ is a point on the line, $\vec{d}$ is a direction vector. But ∇f is parallel to the normal line. So the normal line is given by

$$\vec{r} = \vec{r}_0 + \nabla f t = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 3t \\ 8t \\ -5t \end{bmatrix}$$

If you prefer, we could also write the normal line as:

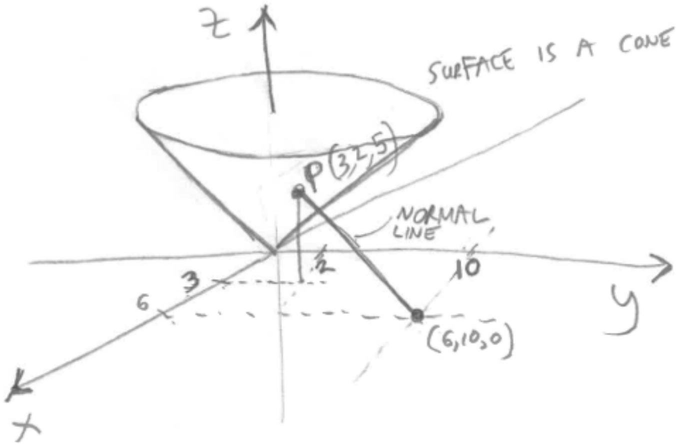
$$x = 3 + 3t, \quad y = 2 + 8t, \quad z = 5 - 5t.$$

The line intersects the xy-plane when $z = 0$, or when $t = 1$. Substituting $t = 1$ into the above equations yields the point $(6, 10, 0)$.

Example 1: Part 3

Consider the surface $x^2 + 4y^2 = z^2$. Sketch the surface and the normal line.

Solution



Tangent Planes and Differentials (14.6)

For a function of one variable, $y(x)$, we define the differential dy as

$$dy = \frac{dy}{dx} dx,$$

where dy is the change in height of the tangent line.

For a function of two variables, $z(x, y)$, we define the differential dz as

$$dz = \underline{\hspace{3cm}},$$

where dz is the change in height of the ____________ .

The equation of the tangent plane to $z = z(x, y)$ at the point $\vec{r}_0$ is

$$z = z_0 + \nabla z \cdot (\vec{r} - \vec{r}_0)$$

The vector $\vec{r} - \vec{r}_0$ is a vector in the tangent plane.

A Quick Calculation: Tangent Plane Approximation

Suppose $z_x(3,4) = 5$, $z_y(3,4) = -2$, and $z(3,4) = 6$. Assuming the function z is differentiable, what is the best estimate for $z(3.1, 3.9)$ using this information?

1. 6.3
2. 9
3. 6
4. 6.7

Solution

The correct answer is 6.7.

Since we are moving .1 units in the x direction, the function increases from 6 to approximately $6 + .1 * 5 = 6.5$. By similar reasoning, when we move in the y direction, the height is approximately $6.5 + (-.1)(-2) = 6.7$.

Estimating Change in Volume

Estimate, using the tangent plane approximation, the change in volume of a cylinder if its height is changed from 12.0 to 12.2 cm and the radius is changed from 8.0 to 7.7 cm. How much does the volume actually change?

Solution

Using $V = \pi R^2 H$, $R = 8$, $H = 12$, $dR = -0.3$, $dH = 0.2$, we obtain

$$\begin{aligned} dV &= \frac{\partial V}{\partial R} dR + \frac{\partial V}{\partial H} dH \\ &= (2\pi RH) dR + (\pi R^2) dH \\ &= 2\pi(8)(12)(-0.3) + \pi(8)^2(0.2) \\ &= -44.8\pi \\ &\approx -140.74 \end{aligned}$$

The estimate gives us a decrease in volume of about 140.74 cm^3. The actual change in volume is $V(12.2, 7.7) - V(12, 8)$ which, plugging everything into a calculator, is roughly 140.31 cm^3.

Second Derivative Test (14.7)

Suppose f has continuous 2^{nd} order partial derivatives around some point $P(x_0, y_0)$, and that $\nabla f(x_0, y_0) = 0$. Let

$$D = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $D = 0$, then ____________.
If $D < 0$, then P is a saddle point.
If $D > 0$, then P is a maximum if $f_{xx} < 0$ and a minimum if $f_{xx} > 0$.

Optimization

Find the critical points of $f(x,y) = y + x\sin(y)$ and determine whether they correspond to local or absolute minimums or maximums of $f(x,y)$.

Solution
The critical points are points where $\nabla f = \vec{0}$.

$$\vec{0} = \nabla f(x,y) = \sin y\hat{i} + (1 + x\cos y)\hat{j}$$

But $\sin y = 0$ implies that $y = N\pi$, where N is any integer. But $\cos(N\pi) = (-1)^N$, so $x = \pm 1$. The stationary points are located at the points $(-1, 2\pi N)$ and at $\big(1, 2\pi(N+1)\big)$.

To determine whether these points correspond to local min/max, we use the second derivatives test.

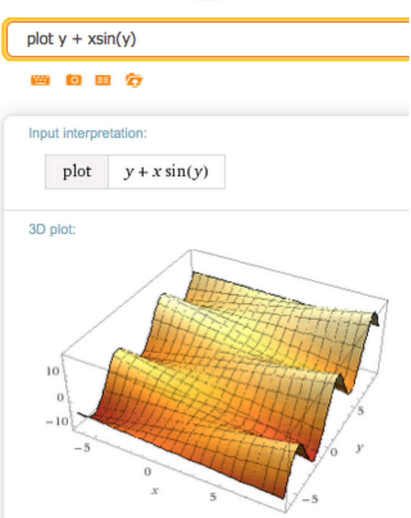
$$D = f_{xx}f_{yy} - f_{xy}^2 = 0 - \big(\cos(N\pi)\big)^2 = -1 < 0$$

All of the critical points correspond to saddle points. A plot of the surface, shown on the next slide, helps us see that this is the case.

Surface Plot of $f(x, y) = y + x\sin(y)$

Solution
Notice how there are no local min/max at the points $(-1, 2\pi N)$, $\left(1, 2\pi(N+1)\right)$.

In fact, the function has no local min/max values at all.



Group Work Activities

Solve the following in groups of 3 to 5 students.

1. Consider the function $f(x, y) = 3xy - x^3 - y^3$.
   1.1 Find the points where the gradient vector, $\nabla f(x, y)$, is the zero vector.
   1.2 Find the points where the tangent plane is horizontal.
   1.3 Find the critical points of $f(x, y)$. Classify these points as min, max, or saddle points.
2. Find an equation of the tangent plane and normal line to $z = (x^2 + y^2)^2$ at $P(1, 1, 4)$.

Question 1.1: Zero Gradient

For $f = 3xy - x^3 - y^3$, find the points where the gradient vector, $\nabla f(x, y)$, is the zero vector.

Solution

Note: this question was explored in the previous recitation.

$$\nabla f = \begin{bmatrix} 3y - 3x^2 \\ 3x - 3y^2 \end{bmatrix}$$

The gradient vector has zero magnitude when

$$0 = 3y - 3x^2$$
$$0 = 3x - 3y^2$$

Rearranging these equations yields the two curves $y = x^2$ and $x = y^2$. These curves intersect at two points, $(0, 0)$, and $(1, 1)$. These are the only two points where the gradient is zero.

Questions 1.2 and 1.3

Consider the function $f(x, y) = 3xy - x^3 - y^3$. Find the points where the tangent plane is horizontal. Find the critical points of $f(x, y)$. Classify these points as min, max, or saddle points.

Solution

The tangent plane is horizontal at points where $\nabla z(x, y)$ is the zero vector. We found these points to be $(0, 0)$, and $(1, 1)$.

These two points $(0, 0)$, and $(1, 1)$ **could** also indicate local minima/maxima. We use the second derivative test to tell us if they are.

$$D = f_{xx}f_{yy} - f_{xy}^2 = (9x)(9y) - (3)(3) = 81xy - 9$$

At $(0, 0)$, D is negative, so we have a saddle at $(0, 0)$.
At $(1, 1)$, D is positive, so we have a local maximum at $(1, 1)$ because f_{xx} is also positive.

Question 2

Find an equation of the tangent plane and normal line to $z = (x^2 + y^2)^2$ at $P(1, 1, 4)$.

Solution

Set $f(x, y, z) = (x^2 + y^2)^2 - z$.

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x} f \\ \frac{\partial}{\partial y} f \\ \frac{\partial}{\partial z} f \end{bmatrix} = \begin{bmatrix} 4x(x^2+y^2) \\ 4y(x^2+y^2) \\ -1 \end{bmatrix} \rightarrow \nabla f(1,1,4) = \begin{bmatrix} 8 \\ 8 \\ -1 \end{bmatrix}$$

Thus, the tangent plane is given by $8(x-1) + 8(y-1) - (z-4) = 0$, which simplifies to $8x + 8y - z = 12$. The normal line is given by the parametric equations

$$x = 1 + 8t, \quad y = 1 + 8t, \quad z = 4 - t.$$

Recitation 13

R13 Topics
GRA2, Quiz 2 Review

Quiz 2 Covers These Topics
13.6 Velocity and Acceleration in Polar Coordinates
14.2 Limits and Continuity
14.3 Partial Derivatives
14.4 The Chain Rule
14.5 Directional Derivatives, the Gradient
14.6 Tangent Planes, Differentials
14.7 Absolute Max/Min

Office Hours
I'll hold the usual additional office hours and drop-in session (same times and URLs as last quiz).

While We're Waiting to Start
Find the dimensions of a rectangular box of maximum volume such that the sum of its 12 lengths is a constant L.

Dimensions of a Rectangular Box

Find the dimensions of a rectangular box of maximum volume such that the sum of its 12 lengths is a constant L.

Solution

Letting the dimensions be a, b, and c, then $V = abc$. To incorporate the length constraint, we will eliminate c by using $4a + 4b + 4c = L$, or $c = L/4 - a - b$. The volume is

$$V = abc = ab(L/4 - a - b) = abL/4 - a^2b - ab^2$$
$$V_a = bL/4 - 2ab - b^2 = 0 \Rightarrow 2a + b = L/4$$
$$V_b = aL/4 - a^2 - 2ab = 0 \Rightarrow 2b + a = L/4$$

Solving these two questions yields $a = b = L/12$. Not surprisingly, $c = L/12$. From the geometrical nature of this problem, this critical point corresponds to a maximum.

Thus, the rectangular box is a cube with sides of length $L/12$.

Note that another approach to this problem would be to use Lagrange Multipliers, but we haven't explored that method yet in our course.

Quiz 2

Quiz 2 Learning Objectives

For Quiz 2, you should be able to do the following.

- Determine whether or not limits of functions of several variables exist by evaluating the limit along paths or by using the formal definition of limit.
- Compute partial derivatives of multi-variable functions using the chain rule.
- Compute gradients and directional derivatives.
- Provide geometric interpretations of gradients and directional derivatives.
- Describe the relationship between gradients and level curves and surfaces.
- Apply the gradient to find equations of tangent planes, normal lines and tangent lines of surfaces.
- Apply tangent planes and differentials to make approximations.
- Locate and classify critical points of surfaces.

Graded Recitation Activity 2

Instructions

- Every student in your group needs to write their name or initials on the board.
- You have 20 minutes to answer the questions below.
- For full marks, show at least two intermediate steps for each question.
- Answer each question on a different slide.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Question 1 (3 points)
Consider the surface $x^2yz + xy - y^2z^2 = -27$.

1. Find an equation of the tangent plane to the surface at the point $(1, 3, 2)$.
2. Find a parameterization of the normal line at the point $(1, 3, 2)$.

Question 2 (2 points)
Consider the surface $z = x^3y - x^2y^2$. Find a normal vector to z at $(2, 1, 4)$.

GRA2, Question 1 Part 1

Consider the surface $x^2yz + xy - y^2z^2 = -27$. Find an equation of the tangent plane to the surface at the point $(1, 3, 2)$.

Solution

Let $F(x, y, z) = x^2yz + xy - y^2z^2$. A vector that is perpendicular to this surface at any point is ∇F.

$$\nabla F(x,y,z) = \begin{bmatrix} \frac{\partial}{\partial x}F \\ \frac{\partial}{\partial y}F \\ \frac{\partial}{\partial z}F \end{bmatrix} = \begin{bmatrix} 2xyz + y \\ x^2z + x - 2yz^2 \\ x^2y - 2y^2z \end{bmatrix} \Rightarrow \nabla F(1,3,2) = \begin{bmatrix} 15 \\ -21 \\ -33 \end{bmatrix}$$

We now have a vector that is normal to the surface at $(1, 3, 2)$. The dot product between this vector, and any vector in the plane, is going to be zero.

$$0 = \nabla F(1,3,2) \cdot \begin{bmatrix} x-1 \\ y-3 \\ z-2 \end{bmatrix} = \begin{bmatrix} 15 \\ -21 \\ -33 \end{bmatrix} \cdot \begin{bmatrix} x-1 \\ y-3 \\ z-2 \end{bmatrix}$$

Thus, the tangent plane is given by $15(x-1) - 21(y-3) - 33(z-2) = 0$, which simplifies to $15x - 21y - 33z = -114$.

GRA2, Question 1 Part 2

Consider the surface $x^2yz + xy - y^2z^2 = -27$. Find a parameterization of the normal line at the point $(1, 3, 2)$.

Solution

The normal line is given by the parametric equations

$$x = 1 + 15t, \quad y = 3 - 21t, \quad z = 2 - 33t.$$

GRA2, Question 2

Consider the surface $z = x^3y - x^2y^2$. Find a normal vector to z at $(2, 1, 4)$.

Solution

Let $F(x, y, z) = x^3y - x^2y^2 - z$. Then the surface z has a normal vector given by the gradient ∇F.

$$\nabla F(x, y, z) = \begin{bmatrix} \frac{\partial}{\partial x}F \\ \frac{\partial}{\partial y}F \\ \frac{\partial}{\partial z}F \end{bmatrix} = \begin{bmatrix} 3x^2y - 2xy^2 \\ x^3 - 2x^2y \\ -1 \end{bmatrix}$$

$$\nabla F(2, 1, 4) = \begin{bmatrix} 3(2)^2(1) - 2(2)(1)^2 \\ 2^3 - 2(2)^2(1) \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix}$$

A vector that is normal to the surface is $[8, 0, -1]$. Another normal vector is $[-8, 0, 1]$.

Tangent Line

Find an equation for the tangent line to the curve of intersection of the surfaces $z = x^2 + y^2$ and $4x^2 + y^2 + z^2 = 9$ at $(-1, 1, 2)$.

Solution

Let $f = z - x^2 - y^2$ and $g = 4x^2 + y^2 + z^2 - 9$. Then the tangent line is perpendicular to both ∇f and ∇g. Vector $\vec{v} = \nabla f \times \nabla g$ is parallel to the desired tangent line (the textbook explains why in Section 14.6).

$$\nabla f(x, y, z) = \begin{bmatrix} -2x \\ -2y \\ 1 \end{bmatrix}, \quad \nabla f(-1, 1, 2) = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\nabla g(x, y, z) = \begin{bmatrix} 8x \\ 2y \\ 2z \end{bmatrix}, \quad \nabla g(-1, 1, 2) = \begin{bmatrix} -8 \\ 2 \\ 4 \end{bmatrix}$$

$$\nabla f(-1, 1, 2) \times \nabla g(-1, 1, 2) = \begin{vmatrix} i & j & k \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = \begin{bmatrix} -10 \\ 16 \\ -12 \end{bmatrix}$$

Parametric vector equations for the tangent line at $(-1, 1, 2)$ are

$$x = -1 - 10t, \quad y = 1 - 16t, \quad z = 2 - 12t.$$

Absolute Max/Min

Find the absolute maximum and minimum of the function $f(x, y) = 4xy^2 - x^2y^2 - xy^3$ in the closed triangle bounded by the lines $x = 0$, $y = 0$ and $y = 6 - x$.

Solution
We will first consider the boundaries of the triangular region, and then investigate the interior.

The Boundaries of the Triangular Region
There are three boundaries we must consider.

- Everywhere along $x = 0$, $f(0, y) = 0$.
- Everywhere along $y = 0$, $f(x, 0) = 0$.
- Along $y = 6 - x$, and $f(x, 6 - x) = -2x(x - 6)^2$. Taking the derivative and setting the result to zero gives us $0 = f_x(x, 6 - x) = -6(x^2 - 8x + 12) = -6(x - 2)(x - 6)$. This suggests that $x = 2$ and $x = 6$ could be min/max, so we can evaluate f at these points $f(2, 4) = -64$, and $f(6, 0) = 0$.

On the next slide, we will look at the interior of the region.

Absolute Max/Min

The Interior of the Region
$f_x = 4y^2 - 2xy^2 - y^3 = 0$ implies that either $y = 0$ or $y = 4 - 2x$. But $y = 0$ is not in the interior (it is along the boundary, which we've already looked at).
$f_y = 8xy - 2x^2y - 3xy^2 = 0$ implies that either $y = 0$ or $8x - 2x^2 - 3xy = 0$.
By substitution,

$$0 = 8x - 2x^2 - 3xy = 8x - 2x^2 - 3x(4 - 2x) = 4x(x - 1)$$

Thus, $x = 0$ or $x = 1$. Again, $x = 0$ is not in the interior of our region. When $x = 1$, $y = 4 - 2(1) = 2$. So for the interior, we need only consider the point $(1, 2)$, and $f(1, 2) = 4$

Putting everything together, we have:

$$\begin{aligned} f(0, y) &= 0 \\ f(x, 0) &= 0 \\ f(2, 4) &= -64 \\ f(1, 2) &= 4 \end{aligned}$$

The absolute maximum is $f(1, 2) = 4$ and the absolute minimum is $f(2, 4) = -64$.

Recitation 15

R15 Topics

14.8 Lagrange Multipliers (LM)

R15 Learning Objectives

- Solve constrained optimization problems using LM.
- Compare LM to other approaches that solve constriained optimization problems.

While We're Waiting to Start

A wire in the shape of a circle of radius 1 has temperature $T(x, y) = xy$.

1. Sketch the level curves of T.
2. Based on your sketch, where are ∇T, and the normal vector to the wire, parallel?
3. Find the hottest and coldest points on the wire using LM.
4. Describe another method of finding the hottest and coldest points, and why it may not work in more complex situations.

Constrained Temperature Optimization

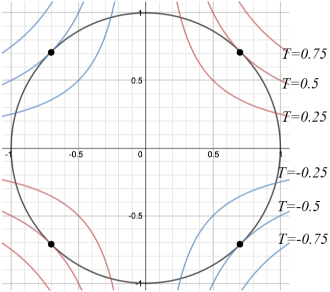
A wire in the shape of a circle of radius 1 has a temperature of $T(x, y) = xy$.

1. Sketch the level curves of T.
2. Based on your sketch, where are ∇T, and the normal vector to the wire, parallel?

Solution

The level curves have the form $C = xy$, or $y = C/x$, for constant C. The plot below shows the level curves for positive temperatures in red, negative in blue, and the wire in black. The four points where ∇T looks parallel to ∇g are also shown.

It would seem from our sketch that the hottest points occur at the points $(1/\sqrt{2}, 1/\sqrt{2})$ and $(-1/\sqrt{2}, -1/\sqrt{2})$, and the coldest points occur at $(-1/\sqrt{2}, 1/\sqrt{2})$ and $(1/\sqrt{2}, -1/\sqrt{2})$. It is at these points that ∇T seems to be parallel to ∇g, where $g(x, y) = x^2 + y^2 - 4$.



Constrained Temperature Optimization

A wire in the shape of a circle of radius 1 has a temperature of $T(x, y) = xy$. Find the hottest and coldest points on the wire using LM.

Solution
Let the constraint be $g(x, y) = x^2 + y^2 - 1 = 0$. The coldest and warmest points correspond to where the two gradients are parallel: $\nabla T = \lambda \nabla g$. The constant λ is an unknown parameter. Calculating the gradients gives us:

$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \lambda 2x \\ \lambda 2y \end{bmatrix}$$

Substitution yields $y = 2\lambda(2\lambda y) = 4\lambda^2 y$, which has the solutions $y = 0$ or $\lambda = \pm 1/2$. If $y = 0$, then $x = 0$, which is not a point on the wire. Thus, λ must be $\pm 1/2$, which means $y = \pm x$.

The constraint $x^2 + y^2 = 1$ implies we have four solutions, $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$, $(1/\sqrt{2}, -1/\sqrt{2})$, and $(-1/\sqrt{2}, 1/\sqrt{2})$.

Since T is positive in the first and third quadrants and negative in the other two, the first two points are the warmest points, and the other two are the coldest points on the wire.

Constrained Temperature Optimization

Describe another method of locating the hottest and coldest points, and why it may not work in more complex situations.

<u>Parametric Representations</u>

The constraint is specified by the unit circle, so we can identify a parametric representation of the constraint curve, with $x(t) = \cos t$, $y(t) = \sin t$. Then $g = 0$ is satisfied, and $T(x, y) = T(x(t), y(t))$. We can find the warmest and coldest points by solving $0 = \frac{d}{dt}T = \frac{d}{dt}\big(\cos t \sin t\big)$. This approach works for the given problem. But for more complicated constraints, $g(x, y)$, it may not be possible to find a parametric representation.

<u>Cross Product of the Gradients</u>

The cross product of two parallel vectors is the zero vector. Knowing that we need points where ∇T and ∇g are parallel, we can instead solve

$$\vec{0} = \begin{vmatrix} i & j & k \\ T_x & T_y & 0 \\ g_x & g_y & 0 \end{vmatrix} = (T_x g_y - T_y g_x)\hat{k} = (y^2 - x^2)\hat{k}$$

The rest of the solution is straightforward. This method is efficient because we have functions of two variables and did not need to introduce λ. But if we had functions of 3 variables, the resulting algebra could be tedious.

A Definition of the Method of LM

If point (x_0, y_0, z_0) ________________ a function $f(x, y, z)$, subject to $g(x, y, z) = 0$, then ∇f and ∇g are parallel at (x_0, y_0, z_0), and there exists a constant λ, such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$$

The scalar λ is called a *Lagrange multiplier*.

Note also that the above definition applies to when there is only one constraint, g. Your textbook also describes an approach for when there are two constraints: if we wish to minimize/maximize f subject to g and to h, then we solve

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

In this case, we have two Lagrange multipliers, λ and μ.

Solution
minimizes or maximizes

Test Your Understanding of LM

Where is the absolute maximum value of $f(x, y) = x + y$, subject to $xy = 9$, located?

1. (3,3)
2. (3,3) and (-3,-3)
3. (3,3), (-3,3), (3,-3), and (-3,-3)
4. There is no absolute maximum value.

Solution

There is no absolute maximum value of f subject to the given constraint. If we were to use LM, we would solve $\nabla f = \lambda \nabla g$, along with $xy = 9$. Calculating the gradients gives us

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} y \\ x \end{bmatrix}.$$

If $1 = \lambda y$, then $\lambda \neq 0$. And if $\lambda x = \lambda y$, then we can divide by λ to obtain $x = y$. Thus,

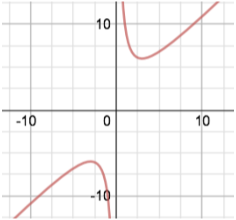
$$xy = 9 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3.$$

Thus, we have two points where the gradients are parallel, $(3, 3)$ and $(-3, -3)$. But we need to find the absolute maximum of $f(x, y)$. This problem is continued on the next slide.

Test Your Understanding of LM (Continued)

We have two points where the gradients are parallel, at $(3,3)$ and $(-3,-3)$. But what do these points correspond to? Are they local minima? Local maxima?

Maximizing $f(x,y)$ along the curve $xy=9$ implies that we are interested in values of f along $y=9/x$. Along this curve, our function becomes $f=x+9/x$, shown to the right. This function has critical points at $x=3$ and at $x=-3$. We can also see that $(-3,-3)$ corresponds to a local maximum, and $(3,3)$ corresponds to a local minimum.



But there is no absolute maximum, because $f\to\infty$ as $x\to\infty$ along the curve $xy=9$.

Conclusion: LM only gives us points where gradients are parallel. Extra work is needed to determine if these points are local/absolute min/max.

Group Work Activities

Solve the following in groups of 3 to 5 students.

1. Find the distance from the point $P(0,1)$ to the curve $x^2 = 4y$.
2. The volume of a cylindrical tank with hemispherical ends must be 100 cubic meters. What dimensions of the tank minimizes its surface area?

Distance From a Point to a Curve

1) Find the minimum distance from the point $P(0,1)$ to the curve $x^2 = 4y$.

Solution

We can minimize the square of the distance, $d(x,y) = x^2 + (y-1)^2$, subject to the constraint curve $g(x,y) = x^2 - 4y = 0$.

$$\nabla d(x,y) = \begin{bmatrix} 2x \\ 2(y-1) \end{bmatrix}, \quad \nabla g = \begin{bmatrix} 2x \\ -4 \end{bmatrix}$$

The minimum must occur where ∇d is parallel to ∇g. We can proceed by either solving $\nabla d = \lambda \nabla g$, or by using a cross product.

<u>Solve Using $\nabla d = \lambda \nabla g$</u>

We must solve the equations

$$2x = 2x\lambda$$
$$2(y-1) = -4\lambda$$

The first equation implies that either $x = 0$ or $\lambda = 1$. If $x = 0$, then from our constraint curve, $y = 0$. If $\lambda = 1$, then $y = -1$ but y can't be negative (because $x^2 = 4y$). We therefore have the point $(0,0)$. And $d(0,0) = 1$.

Distance From a Point to a Curve

Alternate Solution: Cross Product

The cross product of parallel vectors is zero, and we are looking for points where two vectors are parallel. We can also use a cross product to solve this problem.

$$\begin{aligned}
\vec{0} &= \begin{bmatrix} 2x \\ 2(y-1) \\ 0 \end{bmatrix} \times \begin{bmatrix} 2x \\ -4 \\ 0 \end{bmatrix} \\
&= \begin{vmatrix} i & j & k \\ 2x & 2(y-1) & 0 \\ 2x & -4 & 0 \end{vmatrix} \\
&= \Big(-8x - 4x(y-1) \Big)\hat{k} \\
&= (-4x - 4xy)\hat{k}
\end{aligned}$$

Thus, $-4x - 4xy = 0$, or $x(y+1) = 0$. As before, y can't be negative, so $x = 0$. And since $x^2 = 4y$, $x = y = 0$. The distance is $d(0,0) = 1$.

Minimizing Surface Area of a Tank

2) The volume of a cylindrical tank with hemispherical ends must be 100 cubic meters. What dimensions of the tank minimizes its surface area?

Solution

We want to minimize $S = 4\pi R^2 + 2\pi RL$, subject to $V = \frac{4}{3}\pi R^3 + \pi R^2 L = 100$. We could substitute one expression into the other to obtain a function of one variable which we can minimize, or we can use LM. To use LM, we set $g = g(R, L) = \frac{4}{3}\pi R^3 + \pi R^2 L - 100$. Then $\nabla S = \lambda \nabla g$ yields

$$\nabla S = \lambda \nabla V$$
$$\begin{bmatrix} 8\pi R + 2\pi L \\ 2\pi R \end{bmatrix} = \lambda \begin{bmatrix} 4\pi R^2 + 2\pi RL \\ \pi R^2 \end{bmatrix}$$

Thus, $\lambda = 2/R$, and

$$8\pi R + 2\pi L = (2/R)(4\pi R^2 + 2\pi RL)$$
$$4R + L = 4R + 2L$$

Thus, $L = 0$, the volume constraint gives $R = (75/\pi)^{1/3}$, and $S = 4\pi(75/\pi)^{2/3}$.

Recitation 16

R16 Topics

14.8 Lagrange Multipliers
14.9 Taylor's Formula for Two Variables
14.10 Partial Derivatives with Constrained Variables

R16 Learning Objectives

- Derive the least squares equations to fit the plane $Ax + By + C$ to a set of given points (14.8).
- Calculate a cubic approximation to a function of two variables at a specified point (14.9).
- Apply the chain rule to compute partial derivatives with intermediate variables (14.10).

While We're Waiting to Start

Let $L = f(U, V, S)$, and $S = 3UV$. Calculate or derive expressions for the following derivatives.

A) $\left(\dfrac{\partial S}{\partial V}\right)_U$ B) $\dfrac{dS}{dV}$

C) $\left(\dfrac{\partial L}{\partial V}\right)_U$ D) $\left(\dfrac{\partial L}{\partial V}\right)_{S,U}$

The Chain Rule with Intermediate Variables, Parts A and B

Let $L = f(U, V, S)$, and $S = 3UV$. Calculate or derive expressions for the following derivatives.

$$\text{A)} \quad \left(\frac{\partial S}{\partial V}\right)_U \qquad\qquad \text{B)} \quad \frac{dS}{dV}$$

Solution

A) The notation $\left(\frac{\partial S}{\partial V}\right)_U$ implies that V and U are independent variables, and that S is a dependent variable. Using $S = 3UV$, we obtain

$$\left(\frac{\partial S}{\partial V}\right)_U = \frac{\partial}{\partial V}(3UV) = 3U$$

B) The derivative dS/dV implies that S is a dependent variable, and V is an independent variable. U is not identified as either an independent or as a dependent variable, and so we must assume that U is an intermediate variable (U could be a function of V). Using the equation $S = 3UV$, we obtain

$$\frac{dS}{dV} = \frac{d}{dV}(3UV) = 3V\frac{dU}{dV} + 3U = 3VU' + 3U$$

The Chain Rule with Intermediate Variables, Parts C and D

Let $L = f(U, V, S)$, and $S = 3UV$. Calculate or derive expressions for the following derivatives.

C) $\left(\dfrac{\partial L}{\partial V}\right)_U$ D) $\left(\dfrac{\partial L}{\partial V}\right)_{S,U}$

Solution

C) V and U are identified as independent variables. S is an intermediate variable and could be a function of V, so

$$\begin{aligned}\left(\frac{\partial L}{\partial V}\right)_U &= \frac{\partial f}{\partial V} + \frac{\partial f}{\partial S}\frac{\partial S}{\partial V} \\ &= \frac{\partial f}{\partial V} + \frac{\partial f}{\partial S}3U\end{aligned}$$

D) V, U, and S are independent variables, so

$$\left(\frac{\partial L}{\partial V}\right)_{S,U} = \frac{\partial f}{\partial V}$$

If you want to check your results for parts C and D, it may help to substitute a function for $f(U, V, S)$ and see what happens, such as $f = 4U^2VS$. It may also help to use more familiar variables, so that $S = 3xy$ and $L = f(x, y, z)$.

Taylor Approximation (14.9)

Calculate the cubic approximation to $f(x,y) = 4x\cos(y)$ near the origin. Complete this question in group work. *Note: this was a pop quiz in 2014.*
Solution (below is a screen capture of my notes from 2014)

LET $C = \cos y$, $S = \sin y$

Find the cubic approximation of f(x,y) = 4x cos(y) near the origin.

DERIVATIVE	EVALUATE AT (0,0)
$f_x = 4C$	4
$f_y = -4xS$	0
$f_{xx} = 0$	0
$f_{xy} = -4S$	0
$f_{yy} = -4xC$	0
$f_{xxx} = 0$	0
$f_{xxy} = 0$	0
$f_{xyy} = -4C$	-4
$f_{yyy} = 4xS$	0

The cubic "term" has the form: $\frac{1}{3!}\left(f_{xxx}(0,0) + 3f_{xxy}(0,0) + 3f_{xyy}(0,0) + f_{yyy}(0,0)\right)$

$$\Rightarrow f \approx 0 + \frac{1}{1!}(4x + 0y) + \frac{1}{2!}(0x^2 + 0xy + 0y^2) + \frac{1}{3!}\left(0x^3 + 3\cdot 0x^2y - 3\cdot 4xy^2 + 0y^3\right)$$
$$= 4x - 2xy^2$$

Approximation Error (14.9)

Use your results from the previous problem to find the quadratic approximation to $f(x, y) = 4x\cos(y)$ near the origin. Then estimate the error in the approximation if $|x| < 0.5$ and $|y| < 0.1$.

Solution

Taylor's formula for a quadratic approximation is

$$f(x, y) = f(0, 0) + (xf_x + yf_y) + \frac{1}{2}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy})$$

Using our results from the previous problem, our quadratic approximation is $f = 4x$. The maximum error of this approximation is given by the next term in the expansion, which is

$$\begin{aligned}
|E(x, y)| &\le \left|\frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy})\right| \\
&= \frac{1}{3!}\left|(x^3 \cdot 0 + 3x^2 y \cdot 0 + 3xy^2 \cdot (-4) + y^3 \cdot 0)\right| \\
&= \frac{1}{3!}\left|(-12xy^2)\right| = 2|x|y^2.
\end{aligned}$$

Therefore, the desired error estimate is

$$|E(0.5, 0.1)| \le 2(0.5)(0.1)^2 = 0.01.$$

Least Squares (14.8)

The plane $z = Ax + By + C$ is to be fitted to a given set of points, (x_n, y_n, z_n). Derive the linear system of equations that, when solved, minimizes

$$E = \sum_{n=1}^{N} (Ax_n + By_n + C - z_n)^2.$$

Solution

We must find an expression that, given a set of data points, returns the values of A, B, and C that minimizes E. To minimize E, we take the derivatives of E with respect to the independent variables A, B, and C, and set these 3 equations to zero. In doing so, we can treat x_n, y_n, and z_n as constants.

$$\begin{aligned} 0 = \frac{\partial E}{\partial A} &= \frac{\partial}{\partial A} \sum_{n=1}^{N} (Ax_n + By_n + C - z_n)^2 \\ &= \sum_{n=1}^{N} 2(Ax_n + By_n + C - z_n) \frac{\partial}{\partial A} (Ax_n + By_n + C - z_n) \end{aligned}$$

But $\frac{\partial}{\partial A}(Ax_n + By_n + C - z_n) = x_n$, because $\frac{\partial}{\partial A}(By_n + C - z_n) = 0$. Remember that we are treating x_n, y_n, and z_n as constants.

Least Squares (continued)

We will also divide both sides of the equation by 2 to obtain the following.

$$
\begin{aligned}
0 = \frac{\partial E}{\partial A} &= 2\sum_{n=1}^{N}(Ax_n + By_n + C - z_n)\frac{\partial}{\partial A}(Ax_n + By_n + C - z_n) \\
&= 2\sum_{n=1}^{N}(Ax_n + By_n + C - z_n)x_n \\
&= \sum_{n=1}^{N} Ax_n x_n + \sum_{n=1}^{N} By_n x_n + C\sum_{n=1}^{N} x_n - \sum_{n=1}^{N} z_n x_n \\
\sum_{n=1}^{N} z_n x_n &= A\sum_{n=1}^{N}(x_n)^2 + B\sum_{n=1}^{N} y_n x_n + C\sum_{n=1}^{N} x_n \\
&= \begin{bmatrix} \sum\limits_{n=1}^{N}(x_n)^2 & \sum\limits_{n=1}^{N} x_n y_n & \sum\limits_{n=1}^{N} x_n \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix}
\end{aligned}
$$

In the last step above we expressed our sum as a vector product. A similar process for the derivatives E_B and E_C yields equations on the next slide.

Least Squares (continued)

Calculating the partial derivative E_B and setting it equal to zero gives us

$$\sum_{n=1}^{N} z_n y_n = \left[\begin{array}{ccc} \sum\limits_{n=1}^{N} x_n y_n & \sum\limits_{n=1}^{N} (y_n)^2 & \sum\limits_{n=1}^{N} y_n \end{array} \right] \left[\begin{array}{c} A \\ B \\ C \end{array} \right]$$

Likewise, $E_C = 0$ gives us the following.

$$\sum_{n=1}^{N} z_n = \left[\begin{array}{ccc} \sum\limits_{n=1}^{N} x_n & \sum\limits_{n=1}^{N} y_n & \sum\limits_{n=1}^{N} 1 \end{array} \right] \left[\begin{array}{c} A \\ B \\ C \end{array} \right]$$

Note that $\sum\limits_{n=1}^{N} 1 = N$. Putting our three vector product equations together gives us the linear system of equations that we were asked to find.

$$\left[\begin{array}{c} \sum z_n x_n \\ \sum z_n y_n \\ \sum z_n \end{array} \right] = \left[\begin{array}{ccc} \sum (x_n)^2 & \sum x_n y_n & \sum x_n \\ \sum x_n y_n & \sum (y_n)^2 & \sum y_n \\ \sum x_n & \sum y_n & N \end{array} \right] \left[\begin{array}{c} A \\ B \\ C \end{array} \right]$$

Recitation 17

R17 Topics

15.2 Double Integrals over General Regions
15.3 Area by Double Integration

R17 Learning Objectives

- Construct a double integral that represents the area of a region bounded by a set of given curves in Cartesian coordinates.
- Change the order of integration of a double integral (Cartesian coordinates).

Today's Questions

Sketch the region bounded by the given curves and construct a double integral that represents its area.

a) $y = \sqrt{x}$, $y = x^3$.

b) $x = 5 - y$, $x = 2y - 1$, $y = 1$.

c) $y = x - 6$, $y^2 = x$.

Announcements, WolframAlpha Syntax

GRA3, Next Tuesday (5 points)
Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2+y^2)^2+xy^2=1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

Quiz 3: One Week from Thursday
Quiz 3 may cover 14.8 to 14.10, and 15.1 to 15.4. We'll see.

Wolfram Alpha Syntax for Double Integrals
You may want to use Wolfram Alpha to check your answers while completing your HW. Suppose that we want to determine the value of

$$\int_{-2}^{-1}\int_{0}^{x-1}(x^{2C}+y)dydx$$

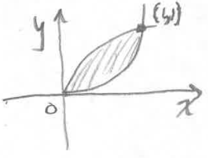
The syntax we could use to compute this particular integral is the following.

```
integrate x^{2C}+y, x from -2 to -1 and y from 0 to (x-1)
```

1a) Area of a Region

Sketch the region bounded by $y = \sqrt{x}$, $y = x^3$ and construct a double integral that represents its area.

Solution



We can either integrate with respect to (wrt) x first, or wrt y first. Either approach will let us express the area with one double integral.

Integrating wrt y first: the region of integration is the set of all points, (x, y), such that $0 \leq x \leq 1$, and $x^3 \leq y \leq \sqrt{x}$. A double integral that represents the area of the region is

$$\int_0^1 \int_{x^3}^{\sqrt{x}} dydx.$$

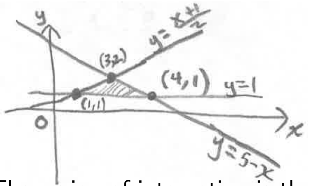
Alternatively, integrating wrt x first, we can express the region of integration as the set of all points, (x, y), such that $0 \leq y \leq 1$, and $y^2 \leq x \leq y^{1/3}$. A double integral that represents the area of the region is

$$\int_0^1 \int_{y^2}^{y^{1/3}} dxdy.$$

1b) Area of a Region

Sketch the region bounded by $x = 5 - y$, $x = 2y - 1$, $y = 1$, and construct a double integral that represents its area.

Solution



The shape of the region suggests that if we integrate wrt x first, then we can express the area with a single integral.

The region of integration is the set of all points, (x, y), such that $1 \leq y \leq 2$, and $2y - 1 \leq x \leq 5 - y$. A double integral that represents the area of the region is

$$\int_1^2 \int_{2y-1}^{5-y} dxdy.$$

Alternatively, we could also integrate wrt y first. This approach would require two integrals,

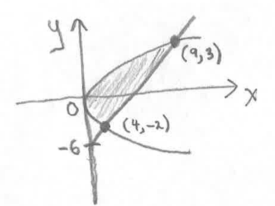
$$\int_1^3 \int_1^{\frac{x+1}{2}} dydx + \int_3^4 \int_1^{5-x} dydx.$$

1c) Area of a Region

Sketch the region bounded by $y = x - 6$, $y^2 = x$, and construct a double integral that represents its area.

Solution

Finding the intersection points requires solving $y^2 = y + 6$, which yields $y = -2$ and $y = 3$.



The shape of the region suggests that we integrate wrt x first. A double integral that represents the area of the region is

$$\int_{-2}^{3} \int_{y^2}^{y+6} dx dy.$$

Alternatively, we could also integrate wrt y first. It would require two integrals,

$$\int_{0}^{4} \int_{-\sqrt{x}}^{\sqrt{x}} dy dx + \int_{4}^{9} \int_{1}^{5-x} dy dx.$$

Recitation 18

R18 Topics

15.2 Double Integrals over General Regions
15.3 Area by Double Integration

R18 Learning Objectives

- Construct a double integral that represents the area of a region bounded by a set of given curves in Cartesian coordinates.
- Change the order of integration of a double integral.
- Calculate the average value of a function of two variables.

Today's Questions

1. Change the order of integration.

$$\text{a)} \int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx dy \qquad \text{b)} \int_{2}^{1+e} \int_{0}^{\ln(x-1)} f(x, y) dy dx$$

2. Construct a double integral that represents the volume of the solid enclosed by the cylinder $x^2 + y^2 = 1$, the planes $z = y$, $x = 0$, $z = 0$, in the first octant.
3. Evaluate $\int_0^4 \int_y^4 e^{x^2} dx dy$.

Announcements

GRA3, Next Tuesday (5 points)

Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2 + y^2)^2 + xy^2 = 1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

Quiz 3: Next Thursday

Quiz 3 may cover 14.8 to 14.10, and 15.1 to 15.4. We'll see.

The Average Value of a Function (15.3)

The average value of a function, $f(x, y)$, over a region R, is given by

$$\text{Average value of } f \text{ over region } R = \frac{1}{\text{area of } R} \iint_R f(x, y) dA$$

This definition can be used to find the value of some double integrals quickly.

Example
Region R is the unit circle $\sqrt{x^2 + y^2} \leq 1$. The definite integral of $f = x + 1$ over R is equal to:

a) 0
b) 1
c) π
d) $\pi/4$

Solution
The answer is c). The area of R is π. The average value of $1 + x$ over R is 1.

$$1 = \frac{1}{\pi} \iint_R (1 + x) dA \quad \Rightarrow \quad \iint_R (1 + x) dA = \pi.$$

Calculating this double integral by hand would have required many more steps.

Conceptual Question Related to Double Integrals

Let region R be the square $-1 \leq x \leq 1$, $-1 \leq y \leq 1$. The definite integral of x^3 over region R is equal to:

a) a positive number
b) a negative number
c) zero
d) a function of x

Solution
The answer is zero because the average value of f over R is zero. Alternatively, we can also argue that the double integral is zero because we are integrating an odd function (in x) over an interval that is symmetric about the y-axis.

Calculating the integral may help explain what this means.

$$\int_{-1}^{1}\int_{-1}^{1} x^3 dx dy = \int_{-1}^{1} \frac{x^4}{4}\bigg|_{-1}^{1} dy = \int_{-1}^{1} 0 \ dy = 0.$$

You may remember from integral calculus that for a function of one variable, the integral of an odd function over a symmetric interval is zero.

1a) Changing the Order of Integration

Change the order of integration.

$$\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} dxdy$$

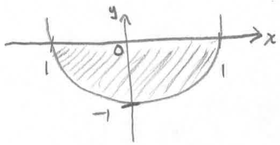
Solution

The inner integral tells us that $x \in \left[-\sqrt{y+1},\ \sqrt{y+1}\right]$. We can solve for y to more easily sketch the region of integration.

$$\begin{aligned} -\sqrt{y+1} \leq x &\leq \sqrt{y+1} \\ x^2 &\leq y+1 \\ y &\geq x^2 - 1 \end{aligned}$$

The above inequality tells us that we are interested in the region above the parabola $y = x^2 - 1$. The outer integral tells us that $-1 \leq y \leq 0$, so we are only interested in the region between $y = x^2 - 1$ and the x-axis. The rest of this problem is on the next slide.

1a) Changing the Order of Integration (continued)



Integrating wrt y first requires $y \in [x^2 - 1, 0]$, and $x \in [-1, 1]$. The integral becomes

$$\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} dx dy = \int_{-1}^{1} \int_{x^2-1}^{0} dy dx$$

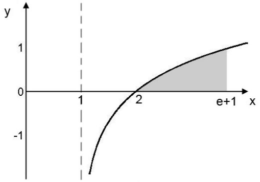
1b) Changing the Order of Integration

Change the order of integration.

$$\int_{2}^{1+e}\int_{0}^{\ln(x-1)} f(x,y)dydx$$

Solution

The region over which we are integrating $f(x,y)$ is the shaded area below.



The region is bounded by the lines $y=0$, $x=1+e$, and by the curve $y=\ln(x-1)$. Integrating wrt y last, values of y range from 0 to 1, and values of x range from $x=e^y+1$ to $x=1+e$. The double integral becomes

$$\int_{0}^{1}\int_{e^y+1}^{1+e} f(x,y)dxdy.$$

2) Volume of a Solid

Construct a double integral that represents the volume of the solid enclosed by the cylinder $x^2 + y^2 = 1$, the planes $z = 1 - y$, $x = 0$, $z = 0$, in the first octant.

Solution

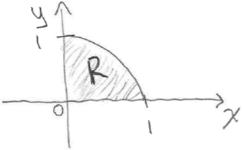
The solid lies under the surface $z = 1 - y$ and above the quarter circle R, with $0 \leq x \leq 1$, $0 \leq y \leq \sqrt{1 - x^2}$.

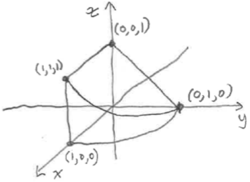
$$V = \iint_R f(x, y)dA = \int_0^1 \int_0^{\sqrt{1-x^2}} (1 - y)dydx$$

Alternatively, we could also integrate with respect to x first.

$$V = \iint_R f(x, y)dA = \int_0^1 \int_0^{\sqrt{1-y^2}} (1 - y)dxdy$$

In case it helps, sketches of region R and the solid are below.





3) Evaluating a Double Integral

Evaluate the following double integral.

$$\int_0^4 \int_y^4 e^{x^2}\, dx dy$$

Solution

The integral of e^{x^2} cannot be expressed in terms of elementary functions. What can we do to get around this problem?

The given integration region is bounded by the lines $y = 0$, $x = 4$, and $y = x$. Changing the order of integration, the double integral becomes

$$\begin{aligned}\int_0^4 \int_y^4 e^{x^2}\, dx dy &= \int_0^4 \int_0^x e^{x^2}\, dy dx \\ &= \int_0^4 y e^{x^2}\Big|_{y=0}^{y=x} dx \\ &= \int_0^4 x e^{x^2}\, dx = \frac{e^{x^2}}{2}\Big|_0^4 = \frac{e^{16}-1}{2}\end{aligned}$$

Changing the order of integration can sometimes make it easier to evaluate certain integrals.

Additional Exercises

1. Set up an integral that represents the volume of the solid enclosed by the planes $x = 1$, $y = 3$, the three coordinate planes, and $x^2 + 2y^2 + z = 1$.
2. Find the volume of the solid enclosed by $z = x^2 + y^2$, $y = x^2$ and $x = y^2$.

Solution

1. The solid lies under the surface $z = 1 - x^2 - 2y^2$ and above the rectangle R, with $0 \le x \le 1$, $0 \le y \le 3$.

$$\iint_R f(x, y)dA = \int_0^1 \int_0^3 (1 - x^2 - 2y^2)dydx$$

2. The curves $y = x^2$ and $x^2 = y$ intersect at (0,0) and at (1,1).

$$\begin{aligned}
\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 + y^2 dydx &= \int_0^1 \left(yx^2 + \frac{y^3}{3} \right) \Big|_{x^2}^{\sqrt{x}} dx \\
&= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - x^4 - \frac{x^6}{3} \right) dx \\
&= \left(\frac{2}{7} x^{7/2} + \frac{2}{15} x^{5/2} - \frac{1}{5} x^5 - \frac{1}{21} x^7 \right) \Big|_0^1 \\
&= \frac{2}{7} + \frac{2}{15} - \frac{1}{5} - \frac{1}{21} = 6/35
\end{aligned}$$

Recitation 19

R19 Topics
15.4 Double Integrals in Polar Coordinates
Quiz 3 Review

Quiz 3 Topics

- 14.08 Lagrange Multipliers
- 14.09 Taylor's Formula for Two Variables
- 14.10 Partial Derivatives with Constrained Variables
- 15.01 Iterated Integrals over Rectangles
- 15.02 Double Integrals over General Regions
- 15.03 Area by Double Integration
- 15.04 Double Integration in Polar Coordinates

Office Hours
I'll hold additional office hours and a review session:

- Quiz 3 Review Session $\forall$ Math 2401 students: Tue 5:30 - 7:00 pm, at https://georgiatech.adobeconnect.com/dcp-online-drop-in-tutor-center-2014-fall
- Quiz 3 Review Session $\forall$ QH8 students: Wed: 7:30 - 8:30 pm at https://georgiatech.adobeconnect.com/distancecalculusofficehours

Quiz 3 Learning Objectives

You should be able to do the following for Quiz 3.

- Solve constrained optimization problems using Lagrange multipliers (14.8).
- Calculate a Taylor approximation to a function of two variables at a point (14.9).
- Apply the chain rule to compute partial derivatives with intermediate variables (14.10).
- Construct a double integral that represents the area of a region bounded by a set of given curves in Cartesian or polar coordinates (15.1 to 15.4).
- Change the order of integration of a double integral (15.1 to 15.4).
- Calculate the average value of a function of two variables (15.3).

Volume of a Sphere

Identify the expressions that represent the volume of a sphere of radius R.

$$1) \quad 4\int_0^{\pi}\int_0^{R} r\sqrt{R^2-r^2}\,dr\,d\theta$$

$$2) \quad \int_0^{2\pi}\int_0^{R} \sqrt{R^2-r^2}\,dr\,d\theta$$

$$3) \quad 2\int_0^{2\pi}\int_0^{R} r\sqrt{R^2-r^2}\,dr\,d\theta$$

$$4) \quad \int_0^{2\pi}\int_0^{R/2} r\sqrt{R^2-r^2}\,dr\,d\theta$$

Solution: (1) and (3) are correct. In Cartesian coordinates, the volume of the sphere is

$$2\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \sqrt{R^2-(x^2+y^2)}\,dy\,dx$$

We multiply by 2 because the integral only represents the upper half of the sphere, whose height from the xy-plane is $R^2-(x^2+y^2)$. We must convert this integral from Cartesian to polar coordinates.

Volume of a Sphere (continued)

We need to do three things: convert the integrand to polar coordinates, identify the limits of integration, and change the differential (dxdy) to a polar representation, $rdrd\theta$.

Knowing that $x^2 + y^2 = r^2$, the integrand becomes $\sqrt{R^2 - r^2}$.

The projection of the volume onto the xy-plane is a circle of radius R, centered at the origin. So the points in the region have polar coordinates (r, θ) in the set $0 \leq \theta \leq 2\pi$, and $0 \leq r \leq R$.

Using these limits of integration our integral becomes

$$2 \int_0^{2\pi} \int_0^R r\sqrt{R^2 - r^2} drd\theta$$

Alternatively, we can use symmetry and use the limits $0 \leq \theta \leq \pi$, and $0 \leq r \leq R$, so the integral becomes

$$4 \int_0^{\pi} \int_0^R r\sqrt{R^2 - r^2} drd\theta$$

Graded Recitation Activity 3

Instructions

- Every student in your group needs to write their name or initials on the board.
- You have 10 minutes to answer the question below.
- For full marks, show at least one intermediate step.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Question (5 points, from last year's quiz)
Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2+y^2)^2+xy^2=1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

GRA3

Suppose we wanted to locate all the minimums and maximums of x^2y^2 subject to $(x^2+y^2)^2 + xy^2 = 1$. Reduce this problem to the problem of solving three equations in three unknowns. Do not solve the equations you derive.

Solution: *a screen capture of hand-written solutions are below.*

Let $f(x,y) = x^2y^2$.
Let $g(x,y) = (x^2+y^2)^2 + xy^2 - 1$.
Min and max occur at the solutions to:

$$\nabla f = \lambda \nabla g$$

$$\begin{bmatrix} 2xy^2 \\ 2x^2y \end{bmatrix} = \lambda \begin{bmatrix} 2(x^2+y^2)\cdot 2x + y^2 \\ 2(x^2+y^2)2y + 2xy \end{bmatrix}$$

Our three equations are

$$2xy^2 = \lambda\left[4x(x^2+y^2) + y^2\right]$$

$$2x^2y = \lambda\left[4y(x^2+y^2) + 2xy\right]$$

$$(x^2+y^2)^2 + xy^2 = 1$$

Converting Double Integral to Polar Coordinates

Convert to a double integral in polar coordinates *(from 2014 Quiz 2)*.

$$\int_0^2 \int_0^{\sqrt{4-(x-2)^2}} xy\,dy\,dx$$

Solution: *the 1st part of a screen capture of hand-written solutions are below.*

WE ARE GIVEN:

$x \in [0,2]$

$y \in [0, \sqrt{4-(x-2)^2}]$

TO UNDERSTAND THE SHAPE OF THE REGION OF INTEGRATION, LET'S WORK WITH OUR LIMITS FOR Y, AND CONVERT THEM TO POLAR.

$$0 \le y \le \sqrt{4-(x-2)^2}$$

$$0 \le y^2 \le 4-(x-2)^2$$

$$\Rightarrow (x-2)^2 + y^2 \le 4, \quad y \ge 0$$

$$\Rightarrow x^2 - 4x + y^2 \le 0, \quad y \ge 0$$

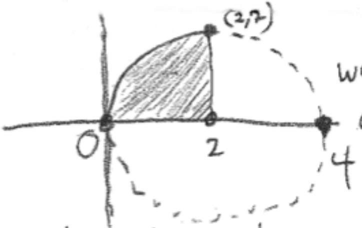
$$\Rightarrow r^2 - 4r\cos\theta \le 0, \quad \theta \in [0, \pi]$$

$$\Rightarrow \boxed{r \le 4\cos\theta, \quad \theta \in [0, \pi]}$$

$y \ge 0$ in polar becomes $\theta \in [0, \pi]$

Converting Double Integral to Polar Coordinates (continued)

Solution: *the 2nd part of a screen capture of hand-written solutions are below.*

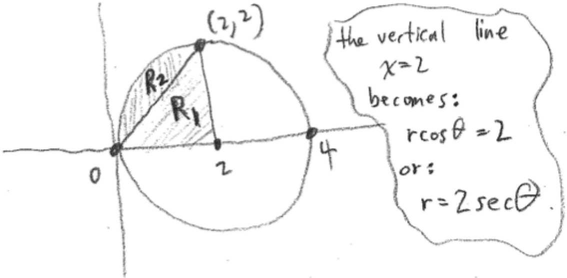


we only need $\theta \in [0, \pi/2]$

we are only looking at quarter of a circle because $x \in [0, 2]$

Converting Double Integral to Polar Coordinates (continued)

Solution: *the 3rd part of a screen capture of hand-written solutions are below.*



R_1

$\theta \in [0, \pi/4]$

$r \in [0, \frac{2}{\cos\theta}]$

R_2

$\theta \in [\pi/4, \pi/2]$

$r \in [0, 4\cos\theta]$

Converting Double Integral to Polar Coordinates (continued)

Solution: *the 4th part of a screen capture of hand-written solutions are below.*

$$\int_0^2 \int_0^{\sqrt{4-(x-2)^2}} xy\,dy\,dx = \int_0^{\pi/4} \int_0^{2\sec\theta} \sin\theta\cos\theta\, r^3\,dr\,d\theta$$

$$+ \int_{\pi/4}^{\pi/2} \int_0^{4\cos\theta} \sin\theta\cos\theta\, r^3\,dr\,d\theta$$

Additional Exercise: Normal Distribution

Evaluate

$$I = \int_0^\infty e^{-x^2} dx$$

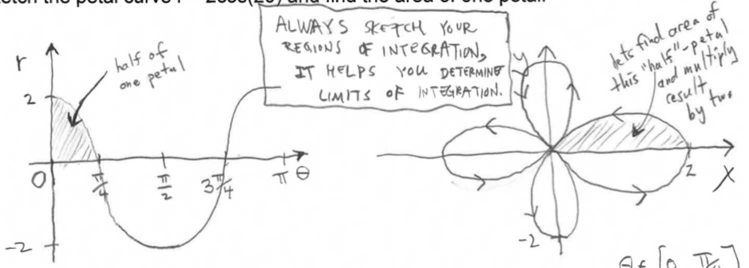
Solution

$$\begin{aligned}
I^2 &= \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy \\
&= \int_0^\infty \int_0^\infty e^{-x^2-y^2} dx dy \\
&= \lim_{a\to\infty} \int_0^{\pi/2} \int_0^a r e^{-r^2} dr d\theta \\
&= \lim_{a\to\infty} \int_0^{\pi/2} \frac{-1}{2} e^{-r^2} \Big|_0^a d\theta \\
&= \frac{-1}{2} \lim_{a\to\infty} \int_0^{\pi/2} (e^{-a^2} - 1) d\theta \\
I^2 &= \pi/4 \\
I &= \sqrt{\pi/4}
\end{aligned}$$

Additional Exercise: Integration in Polar Coordinates

Sketch the rose curve $r = 2\cos(2\theta)$ and find the area of one petal.
Solution: *a screen capture of hand-written solutions are below*.

Sketch the petal curve r = 2cos(2θ) and find the area of one petal.



THE BOUNDS OF INTEGRATION FOR THE "HALF" PETAL ARE $\theta \in [0, \pi/4]$, $r \in [0, 2\cos(2\theta)]$

$$\Rightarrow \text{AREA} = 2\int_0^{\pi/4}\int_0^{2\cos(2\theta)} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[(2\cos 2\theta)^2/2\right]d\theta$$

$$= 4\int\left(\tfrac{1}{2}+\tfrac{1}{2}\cos 4\theta\right)d\theta$$

$$= 2\left(\theta + \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/4} = \pi/2$$

(why is there an r in the integrand?)

Recitation 23

R23 Topics

15.5 Triple Integrals in Rectangular Coordinates
15.6 Moments of Inertia and Mass

R23 Learning Objectives

- Construct a triple integral that represents the area of a region bounded by a set of given curves in Cartesian or cylindrical coordinates
- Change the order of integration of a triple integral
- Set-up integrals that represent moments of inertia and centres of mass of solids

Today's Questions

1. Set-up a triple integral that represents the volume bounded by the following surfaces. Set-up the integrals in at least two different ways.
   1.1 $y^2 + z^2 = 1$, and the planes $y = x$, $x = 0$, and $z = 0$.
   1.2 $z^2 = y$, and the planes $y + z = 2$, $x = 0$, $x = 2$, and $z = 0$.
2. Consider the region inside the curve $r = 2 + \sin(\theta)$. Set up the three integrals you need to find the x and y coordinates of the centroid of the region, assuming its density is $\delta(x, y)$. Express these integrals in polar coordinates. This is a question from a 2014 quiz.

Graded Recitation Activity 4: Next Tuesday

Instructions (same as before)

- Every student in your group needs to write their name or initials on the board.
- You have 15 minutes to answer both questions below.
- For full marks, show at least **two intermediate steps**.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Questions (5 points each, both questions are from old quizzes)

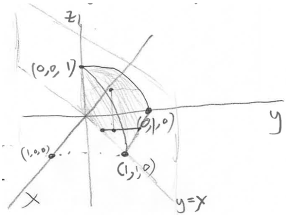
1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Triple Integrals, Example 1

Set-up a triple integral that represents the volume of the region bounded by $y^2 + z^2 = 1$, and the planes $y = x$, $x = 0$, and $z = 0$. Set-up the integral in at least two different ways.

Solution: $dzdydx$

We could choose the integration order $dzdydx$. The solid is shown below.



We chose to integrate wrt x last, so $x \in [0, 1]$.
Then, for any given value of x in $[0, 1]$, $y \in [x, 1]$.
Then, for any $y \in [x, 1]$, $z \in [0, \sqrt{1 - y^2}]$.
The volume of the solid, V, is equal to the triple integral

$$V = \int_0^1 \int_x^1 \int_0^{\sqrt{1-y^2}} dzdydx.$$

Triple Integrals, Example 1 (continued)

Solution: $dxdzdy$
We could also use the integration order $dxdzdy$.
We decided to integrate wrt y last, so $y \in [0, 1]$.
Then, for any given value of y in $[0, 1]$, $z \in [0, \sqrt{1 - y^2}]$.
Then, for any $z \in [0, \sqrt{1 - y^2}]$, $x \in [0, 2]$.
The volume is the triple integral:

$$V = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y dxdzdy$$

Note:

- Using only Cartesian coordinates, there are six integration orders that can be considered (dxdydz, dxdzdy, dydxdz, dydzdx, dzdxdy, dzdydx).
- Regardless of how we set up our integral, we should obtain the same value for V, which in this case happens to be $1/3$.
- WolframAlpha syntax for evaluating the above triple integral is

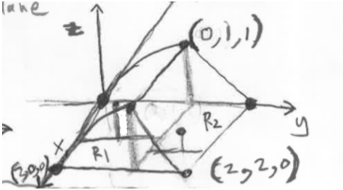
```
\int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^y dxdzdy
```

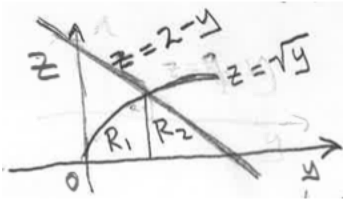
Triple Integrals, Example 2

Set-up a triple integral that represents the volume of the region bounded by $z^2 = y$, and the planes $y + z = 2$, $x = 0$, $x = 2$, and $z = 0$. Set-up the integral in at least two different ways.

Solution

If we were to choose $dzdydx$, then we would need to break up our volume into two regions. The curves $z = 2 - y$ and $z = y^2$ are shown below, along with regions R1 and R2.





Triple Integrals, Example 2, Continued

Volume of Region R1 with dzdydx
We chose to integrate wrt x last, so $x \in [0, 2]$.
Then, for any given value of x in $[0, 2]$, $y \in [0, 1]$.
Then, for any $y \in [0, 1]$, $z \in [0, \sqrt{y}]$.

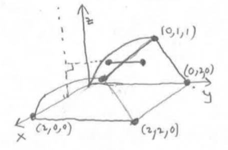
Volume Region R2 with dzdydx
We chose to integrate wrt x last, so $x \in [0, 2]$.
Then, for any x in $[0, 2]$, $y \in [1, 2]$.
Then, for any $y \in [1, 2]$, $z \in [0, 2 - y]$.

Thus, the total volume is the sum of the two triple integrals:

$$\begin{aligned} V &= \iiint_{R_1} dV + \iiint_{R_2} dV \\ &= \int_0^2 \int_0^1 \int_0^{\sqrt{y}} dzdydx + \int_0^2 \int_1^2 \int_0^{2-y} dzdydx \end{aligned}$$

Triple Integrals, Example 2, Continued

Solution: dydzdx



With the integration order dydzdx, we do not need to break up the solid into two regions. We are integrating wrt x last, so $x \in [0, 2]$. Then, for any x in $[0, 2]$, $z \in [0, 1]$. Then, for any $z \in [0, 1]$, $y \in [z^2, 2 - z]$.

Thus, the total volume is the triple integral:

$$V = \int_0^2 \int_0^1 \int_{z^2}^{2-z} dydzdx$$

Note:

- Regardless of how we set up our integral, we should obtain the same value for V, which in this case happens to be $7/3$.
- WolframAlpha syntax for evaluating the above triple integral is

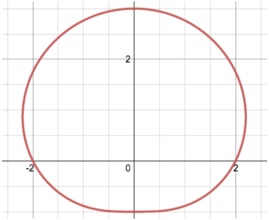
```
\int_0^2 \int_0^1 \int_{z^2}^{2-z}dydzdx
```

Centroid

Consider the region inside the curve $r = 2 + \sin(\theta)$. Set up the three integrals you need to find the x and y coordinates of the centroid of the region, assuming its density is $\delta(x, y)$. Express these integrals in polar coordinates. This is a question from a 2014 quiz.

Solution

A plot of the curve is shown below.



The mass of the solid, m, is

$$m = \int_0^{2\pi} \int_0^{2+\sin(\theta)} \delta(r, \theta) \; r dr d\theta$$

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of the region are

$$m\bar{x} = \int_0^{2\pi} \int_0^{2+\sin(\theta)} \delta(r, \theta) \; r^2 \cos(\theta) dr d\theta$$

$$m\bar{y} = \int_0^{2\pi} \int_0^{2+\sin(\theta)} \delta(r, \theta) \; r^2 \sin(\theta) dr d\theta$$

Recitation 24

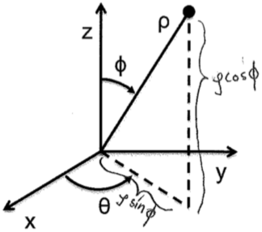
R24 Topics

15.7 Integration in Cylindrical and Spherical Coordinates

R24 Learning Objectives

- Construct a triple integral that represents the area of a region bounded by a set of given curves in cylindrical or spherical coordinates
- Change the order of integration of a triple integral

The Spherical Coordinate System



Fill in the blanks.

$x = \rho \cos\theta$ ____________

$y = \rho \sin\theta$ ____________

$z = \rho$ ____________

Graded Recitation Activity 4: Next Tuesday

Instructions (same as before)

- Every student in your group needs to write their name or initials on the board.
- You have 15 minutes to answer both questions below.
- For full marks, show at least **two intermediate steps**.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Questions (5 points each, both questions are from old quizzes)

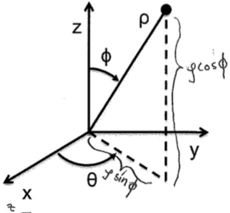
1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Spherical Coordinates

Provide a geometric interpretation the surfaces $\rho\sin\phi = 1$ and $\rho\cos\phi = 1$.

Solution: *Below is a screen capture of a previous year's handwritten notes.*

Fill in the blanks.



(1) $x = \rho\cos\theta \, \underline{\sin\phi}$

(2) $y = \rho\sin\theta \, \underline{\sin\phi}$

(3) $z = \rho \, \underline{\cos\phi}$

Cartesian in terms of spherical

There are equ's for expressing spherical in terms of Cartesian:

$\rho = \sqrt{x^2+y^2+z^2}$, $\tan\theta = y/x$, $\cos\phi = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

Provide a geometric interpretation of each expression.

a) $\rho\sin\phi = 1$, $x^2+y^2 = \rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta) = \rho^2\sin^2\phi$. But $\rho\sin\phi = 1$, so $x^2+y^2 = 1^2 \Rightarrow$ cylinder radius 1

b) $\rho\cos\phi = 1 \Rightarrow$ the plane $z=1$, from (3)

The xy-plane in spherical coord. is: $\phi = \pi/2$, from (3)
(because we need the value of ϕ that sets $z=0$)

1) A Triple Integral in Cylindrical Coordinates

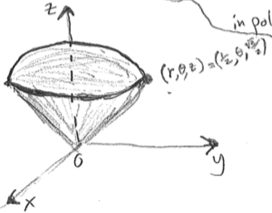
Use cylindrical coordinates to set-up an integral that represents the volume of the solid bounded by $x^2 + y^2 + z^2 = 1$, and $z^2 = 3(x^2 + y^2)$.

Solution: *Below is a screen capture of a previous year's handwritten notes.*

The sphere and cone intersect on: $z^2 = 1 - x^2 - y^2 = 3(x^2+y^2)$

in polar: $1 - r^2 = 3r^2$

$r = 1/2$,



$\Rightarrow$ surfaces intersect when $r = 1/2$, and when $z^2 = 3((1/2)^2) = 3/4$, or when $z = \sqrt{3}/2$.

(we don't really need z-coordinate)

USE $dz\,dr\,d\theta$

$\theta \in [0, 2\pi]$

$r \in [0, \frac{1}{2}]$

$z \in [\sqrt{3}\,r, \sqrt{1-r^2}]$

$$\boxed{V = \int_0^{2\pi} \int_0^{1/2} \int_{\sqrt{3}r}^{\sqrt{1-r^2}} r \text{ dz dr d(theta)}}$$

If interested, $V = \frac{\pi}{3}(2 - \sqrt{3})$

2) A Triple Integral in Spherical Coordinates

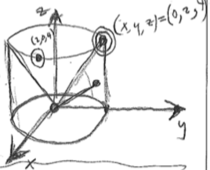
Use spherical coordinates to set-up an integral that represents the volume of the solid bounded by $z = 0$, $x^2 + y^2 = 4$, and $z = 2\sqrt{x^2 + y^2}$.

Solution: *Below is a screen capture of a previous year's handwritten notes.*

① Sketch solid: $x^2+y^2=4$ is a cylinder

$\frac{z}{2} = \sqrt{x^2+y^2}$ is a cone

bounded above by cone, below by plane.



② Integration limits

$\rho \in [0, 2\csc\phi]$, from ⊛

$\phi \in \left[\arctan(\tfrac{1}{2}), \tfrac{\pi}{2}\right]$, the xy-plane is $\phi = \pi/2$, and the "top" of the surface: $\phi = \arctan(1/2)$

$\theta \in [0, 2\pi]$ (symmetric about z-axis)

③ WRITE INTEGRAL

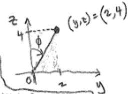
$$V = \int_0^{2\pi} \int_{\mathrm{atan}\,\frac{1}{2}}^{\pi/2} \int_0^{2\csc\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

Spherical Coordinates of ⊙:

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{4+16} = 2\sqrt{5}$

$\phi = \tan^{-1}\left(\frac{2}{4}\right)$

$\theta = 0$



⊛ OUTER SURFACE OF CYLINDER:

$\rho^2 \sin^2\phi = 4$, or $\rho \sin\phi = 2$

or $\rho = 2\csc\phi$.

NOTE

we almost always use $d\rho\, d\phi\, d\theta$, or $d\rho\, d\theta\, d\phi$

Recitation 25

Quiz 4 Topics
15.5 to 15.8 (I think)

Quiz 4 Learning Objectives

- Construct a triple integral that represents the area or volume of a region in Cartesian, polar, cylindrical, or spherical coordinates
- Change the order of integration, or coordinate system, for a triple integral
- Construct integrals that represent moments of inertia and centres of mass
- Identify a suitable transformation for a triple integral, and use that transform to find the area or volume of a given region

GRA4

1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Graded Recitation Activity 4

Instructions (same as before)

- Every student in your group needs to write their name or initials on the board.
- You have 15 minutes to answer both questions below.
- For full marks, show at least **two intermediate steps**.
- All students in the same group receive the same grade.
- Please do not share computers: every student should log in on their own computer.
- You do not need to simplify your answers.

Questions (5 points each, both questions are from old quizzes)

1. Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.
2. Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

GRA4.1

Set-up a triple integral that represents the volume of the ellipsoid $x^2 + (y/2)^2 + (z/9)^2 = 1$ in the 1st octant (x,y,z non-negative). Do not evaluate.

Solution: Let $u = x$, $2v = y$, $9w = z$, then $J = 18$, and we are integrating over the unit sphere in the 1st quadrant. From here, we can use Cartesian, cylindrical, or spherical coordinates. Using spherical coordinates, we have:

$$V = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^1 18\rho^2 \sin\phi d\rho d\phi d\theta$$

But there are other ways to set this integral up without using a uvw transformation. In Cartesian, we could use the following.

$$V = \int_0^1 \int_0^{2\sqrt{1-x^2}} \int_0^{9\sqrt{1-x^2-y^2/4}} dzdydx$$

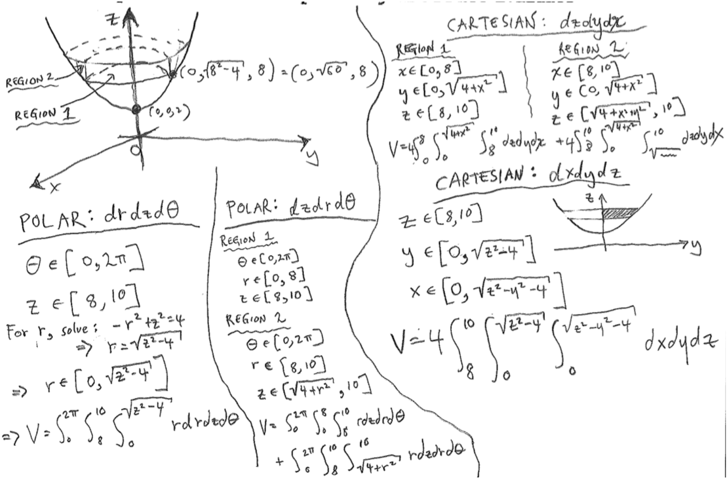
The value of the integral is 3π. WolframAlpha syntax for the above integrals are:

```
\int_0^{1} \int_0^{2\sqrt{1-x^2}}\int_0^{9\sqrt{1-x^2-y^2/4}} dz dy dx
\int_0^{pi/2} \int_0^{\pi/2} \int_0^1 18 r^2 sin(p) dr dp d\theta
```

GRA4.2

Set-up a triple integral that represents the volume of the solid bounded by the hyperboloid of two sheets $-x^2 - y^2 + z^2 = 4$, the plane $z = 8$ and the plane $z = 10$. Do not evaluate.

Solution: *Below is a screen capture of a previous year's handwritten notes.*



POLAR: $dr\,dz\,d\theta$

$\theta \in [0, 2\pi]$

$z \in [8, 10]$

For r, solve: $-r^2 + z^2 = 4$

$\Rightarrow r = \sqrt{z^2-4}$

$\Rightarrow r \in [0, \sqrt{z^2-4}]$

$\Rightarrow V = \int_0^{2\pi} \int_8^{10} \int_0^{\sqrt{z^2-4}} r\,dr\,dz\,d\theta$

POLAR: $dz\,dr\,d\theta$

REGION 1

$\theta \in [0, 2\pi]$

$r \in [0, 8]$

$z \in [8, 10]$

REGION 2

$\theta \in [0, 2\pi]$

$r \in [8, 10]$

$z \in [\sqrt{4+r^2}, 10]$

$V = \int_0^{2\pi} \int_0^{8} \int_8^{10} r\,dz\,dr\,d\theta + \int_0^{2\pi} \int_8^{10} \int_{\sqrt{4+r^2}}^{10} r\,dz\,dr\,d\theta$

CARTESIAN: $dz\,dy\,dx$

REGION 1

$x \in [0, 8]$

$y \in [0, \sqrt{4+x^2}]$

$z \in [8, 10]$

REGION 2

$x \in [8, 10]$

$y \in [0, \sqrt{4+x^2}]$

$z \in [\sqrt{4+x^2+y^2}, 10]$

$V = 4\int_0^8 \int_0^{\sqrt{4+x^2}} \int_8^{10} dz\,dy\,dx + 4\int_8^{10} \int_0^{\sqrt{4+x^2}} \int_{\sqrt{\ldots}}^{10} dz\,dy\,dx$

CARTESIAN: $dx\,dy\,dz$

$z \in [8, 10]$

$y \in [0, \sqrt{z^2-4}]$

$x \in [0, \sqrt{z^2-y^2-4}]$

$V = 4\int_8^{10} \int_0^{\sqrt{z^2-4}} \int_0^{\sqrt{z^2-y^2-4}} dx\,dy\,dz$

Change of Variables

- After completing HW 15.8, you might be familiar with computing an integral, if you are given a transform.
- But if we were given an integral over a complicated region, and were not given a suitable transform, how could we find one?
- The basic idea is to find a transform that converts a complicated region into a simple one, such as a square, or a circle

1) Change of Variables

Show that the area of the ellipse $(x/a)^2 + (y/b)^2 = 1$ is πab.

Solution: let $u = x/a$, and $v = y/b$, so that we are integrating over the unit circle, $u^2 + v^2 = 1$. We can show that $|J| = ab$, and the area, A, becomes

$$A = 4 \int_0^1 \int_0^{\sqrt{1-u^2}} abdvdu$$

Now let $u = r\cos\theta$ and $v = r\sin\theta$.

$$\begin{aligned}
A &= 4ab \int_0^{\pi/2} \int_0^1 rdrd\theta \\
&= 4ab \int_0^{\pi/2} \frac{r^2}{2}\Big|_0^1 d\theta \\
&= 2ab \int_0^{\pi/2} d\theta \\
&= 2ab\frac{\pi}{2} \\
&= \pi ab
\end{aligned}$$

2) Change of Variables

Set-up an integral that represents the area of a region bounded by $x+y=0$, $x+y=1$, $x-y=0$, $x-y=2$.
Solution: The appearance of the terms $(x+y)$ and $(x-y)$ in the integrand and in the lines that bound R suggests the transformation

$$u = x + y \tag{1}$$
$$v = x - y. \tag{2}$$

In order to compute the Jacobian, we need explicit expressions for u and v. If we add equations 1 and 2 we find that $x = \frac{u+v}{2}$. If we subtract equations 1 and 2 we find that $y = \frac{u-v}{2}$. The Jacobian becomes

$$J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}.$$

We also need to find the limits of integration in the transformed integral. Using equations 1 and 2 the four lines bounding R in the xy-plane become

$$u = 0, \quad u = 1, \quad v = 0, \quad v = 1.$$

The solution is continued on the next slide.

2) Change of Variables (continued)

The double integral therefore becomes

$$\iint\limits_R \left(x^2 - y^2\right) dxdy = \iint\limits_R (x-y)(x+y)dxdy$$
$$= \int_0^1 \int_0^1 uv \left| -\frac{1}{2} \right| dudv$$
$$= \frac{1}{2} \int_0^1 \int_0^1 (uv) dudv$$

We did not need to evaluate the integral, but this works out to be 1/8.

3) Triple Integrals

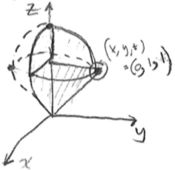
Set-up a triple integral that represents the volume of the solid bounded by $0 \leq x \leq 1$, $0 \leq y \leq \sqrt{1-x^2}$, and $\sqrt{x^2+y^2} \leq z \leq \sqrt{2-(x^2+y^2)}$.

Solution: *Below is a screen capture of a previous year's handwritten notes.*

Set-up an integral that represents the volume of the solid bounded by

$$\left.\begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq \sqrt{1-x^2} \end{array}\right\} \text{cylinder } x^2+y^2 \leq 1$$

$$\underbrace{\sqrt{x^2+y^2}}_{\text{cone}} \leq z \leq \underbrace{\sqrt{2-(x^2+y^2)}}_{\text{sphere}}$$

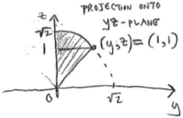


② INTEGRATION LIMITS

$\rho \in [0, \sqrt{2}]$

$\phi \in [0, \pi/4]$

$\theta \in [0, \pi/2]$



$$V = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \frac{\sqrt{2}}{6} \pi (2-\sqrt{2})$$

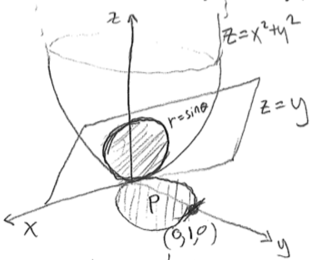
(NOTE: solid is a "quarter" of an ice-cream cone)

4) Cylindrical

Set-up a triple integral that represents the volume of the solid bounded by $z = x^2 + y^2$, and the plane $y = z$. Use cylindrical coordinates.

Solution: *Below is a screen capture of a previous year's handwritten notes.*

Set up an integral that represents the volume of solid bounded by $z = x^2 + y^2$, and $z = y$. Use cylindrical coordinates.



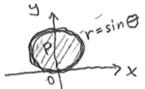
$z=x^2+y^2$ is an elliptic paraboloid, which intersects $z=y$ when:

$z=y=x^2+y^2$

In polar/cylindrical:

$r\sin\theta = r^2$

or: $r=\sin\theta$



P = projection of solid onto xy-plane on a given ray, r runs from 0 to $\sin\theta$.

$\theta \in [0, \pi]$

$r \in [0, \sin\theta]$

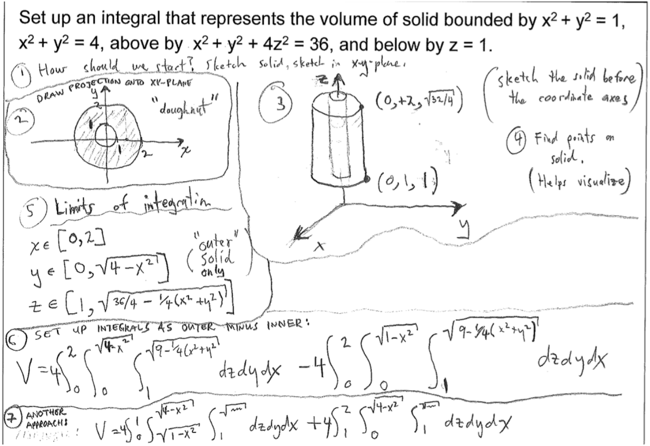
$z \in [x^2+y^2, y]$, or $z \in [r^2, r\sin\theta]$

$$V = \int_0^{\pi}\int_0^{\sin\theta}\int_{r^2}^{r\sin\theta} r\,dz\,dr\,d\theta$$

5) Triple Integral

Set-up a triple integral that represents the volume of the solid bounded by $1 = x^2 + y^2$, above $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

Solution: *Below is a screen capture of a previous year's handwritten notes.*



Set up an integral that represents the volume of solid bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, above by $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

1) How should we start? Sketch solid, sketch in xy-plane.

2) DRAW PROJECTION ONTO XY-PLANE

"doughnut"

3) $(0, 1, 1)$, $(0, +2, \sqrt{32/4})$

(sketch the solid before the coordinate axes)

4) Find points on solid. (Helps visualize)

5) Limits of integration

$x \in [0, 2]$

$y \in [0, \sqrt{4 - x^2}]$

$z \in [1, \sqrt{36/4 - \frac{1}{4}(x^2 + y^2)}]$

("outer" solid only)

6) SET UP INTEGRALS AS OUTER MINUS INNER:

$$V = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_1^{\sqrt{9 - \frac{1}{4}(x^2+y^2)}} dz\,dy\,dx - 4\int_0^2 \int_0^{\sqrt{1-x^2}} \int_1^{\sqrt{9 - \frac{1}{4}(x^2+y^2)}} dz\,dy\,dx$$

7) ANOTHER APPROACH:

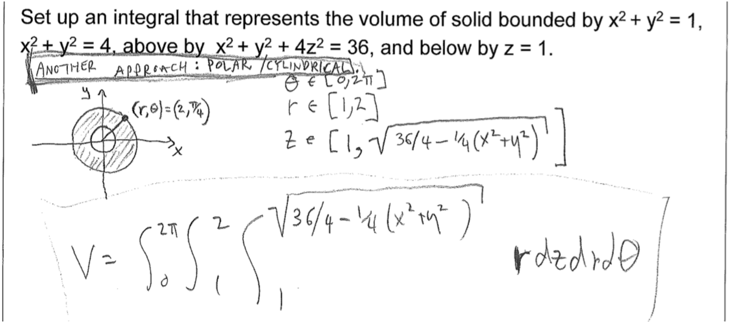
$$V = 4\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \int_1^{\sqrt{\cdots}} dz\,dy\,dx + 4\int_1^2 \int_0^{\sqrt{4-x^2}} \int_1^{\sqrt{\cdots}} dz\,dy\,dx$$

Let me know if you catch any typos in the above.

5) Triple Integral (Alternate Solution)

Set-up a triple integral that represents the volume of the solid bounded by $1 = x^2 + y^2$, above $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

Solution: *Below is a screen capture of a previous year's handwritten notes.*


Set up an integral that represents the volume of solid bounded by $x^2 + y^2 = 1$, $x^2 + y^2 = 4$, above by $x^2 + y^2 + 4z^2 = 36$, and below by $z = 1$.

ANOTHER APPROACH: POLAR / CYLINDRICAL.

$(r,\theta) = (2, \pi/4)$

$\theta \in [0, 2\pi]$

$r \in [1, 2]$

$z \in \left[1, \sqrt{36/4 - 1/4(x^2+y^2)}\right]$

$$V = \int_0^{2\pi}\int_1^2\int_1^{\sqrt{36/4 - 1/4(x^2+y^2)}} r\,dz\,dr\,d\theta$$

In the above, for the upper limit of the innermost integral, we should have used r^2, rather than $x^2 + y^2$.

R27
1

Recitation 27

Today's Topics

- 16.1 Line Integrals (brief review)

16.2 Vector Fields and Line Integrals, Work, Circulation, Flux

Learning Objectives

16.1 Set-up and evaluate a line integral to calculate the mass of a thin wire

16.2 Set-up and evaluate a line integral that represents total work

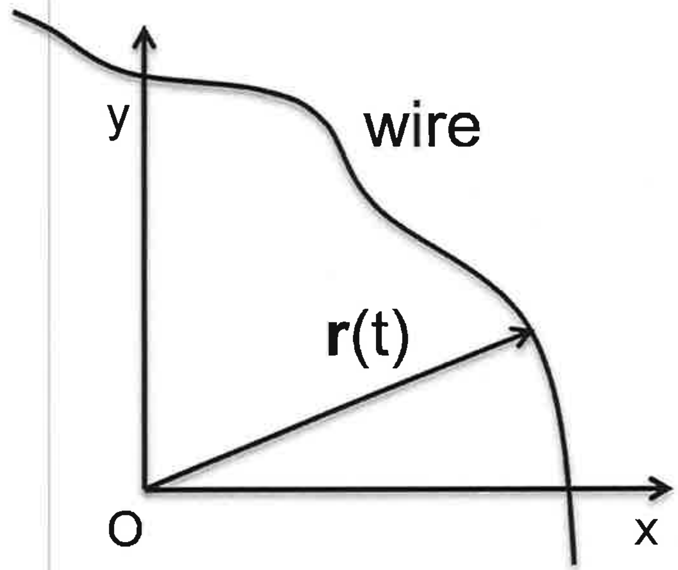
R27
2

16.1: Mass of a Thin Wire (a review of lecture material?)

How To Calculate Mass of a Wire

- position on wire given by parameterization, $\vec{\mathbf{r}}(t)$
- density of wire is $\delta = \delta(r(t))$
- length of a small piece of wire is $\Delta s(\mathbf{r}(t_k))$
- we can approximate the total mass with:

$$M \approx \sum \delta(\vec{r}(t)) \Delta s(\vec{r}(t_k))$$



In the limit as Δs tends to zero,

$$M = \int_C \delta \, ds$$

To compute total mass, we can show that:

$$M = \int_a^b \delta(\vec{r}) \, |\vec{r}\,'| \, dt$$

R27
3

16.1: Mass of a Thin Wire

Compute the total mass of a wire whose density is given by $\delta = 3x^2 - 2y$, and whose shape is given by the line segment from the origin to the point (2,4).

$$\vec{r} = t\hat{i} + 2t\hat{j} = \begin{bmatrix} t \\ 2t \end{bmatrix}, \quad t \in [0,2], \quad |\vec{r}'| = \sqrt{5}$$

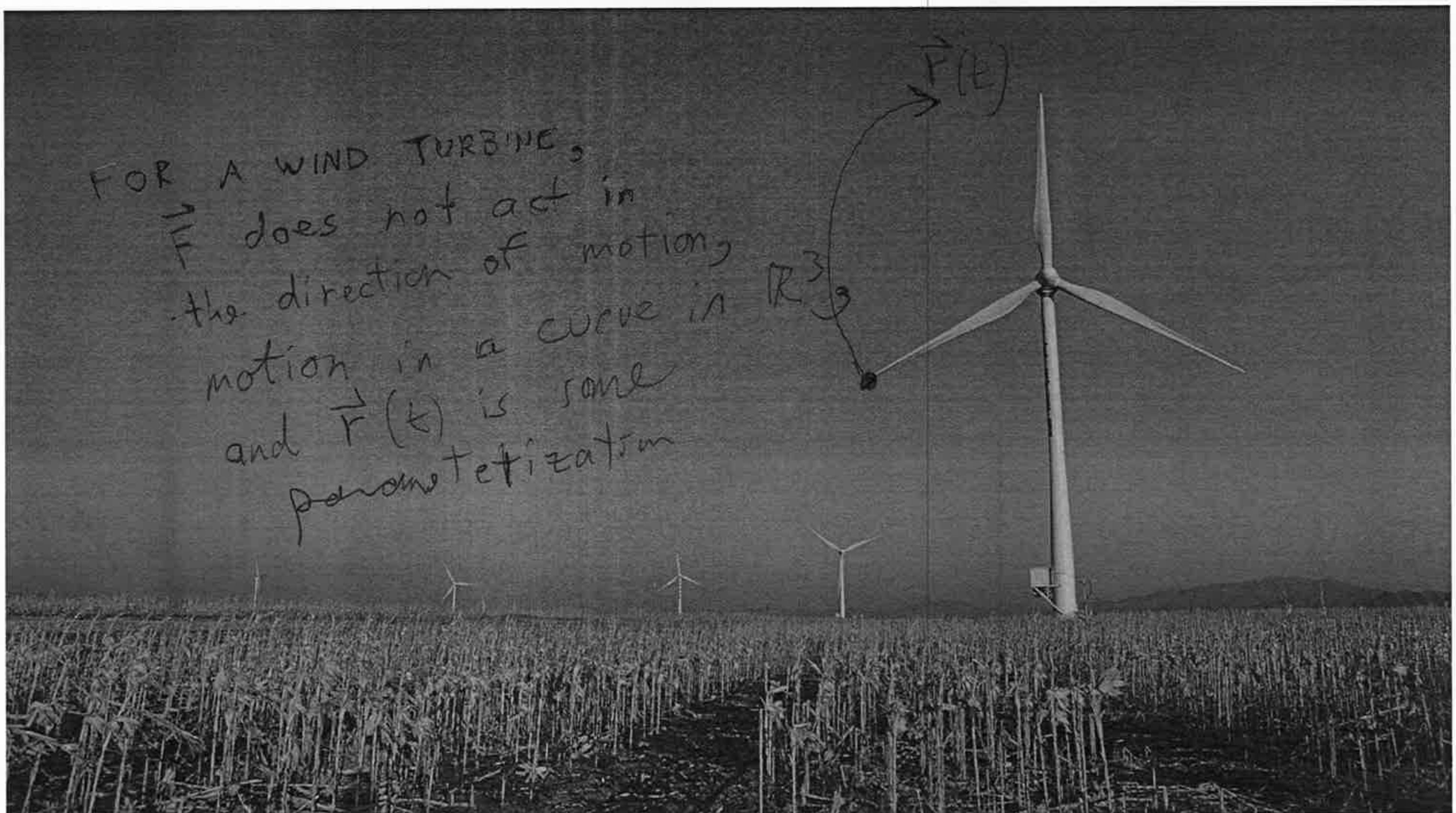
$$\delta = \delta(\vec{r}) = 3(x(t))^2 - 2y(t) = 3t^2 - 2t^2 = t^2$$

$$M = \int_0^2 (3t^2 - 2t^2)\sqrt{5}\, dt = \sqrt{5}\ 8/3$$

R27
4

16.2: Work (a review of lecture material?)

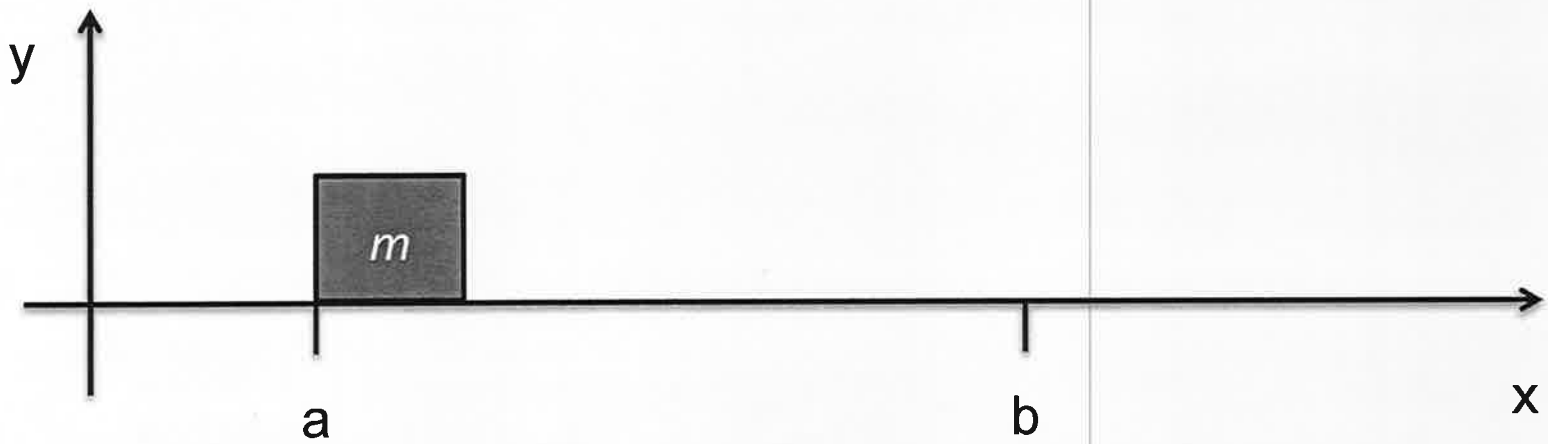
Work is the energy transferred to or from an object by means of a force acting on the object.



R27

16.2: Work Over a Straight Line Path

Force **F** is applied to an object as it moves from x = a to x = b along the x-axis.



	Applied Force	Work
Case 1	**F** = 4**i**	W = $\vec{F} \cdot \vec{r} = 4\hat{i} \cdot (b-a)\hat{i} = 4(b-a)$
Case 2	**F** = 4**i** – 2**j**	W = $\vec{F} \cdot \vec{r} = 4\hat{i} \cdot (b-a)\hat{i} = 4(b-a)$

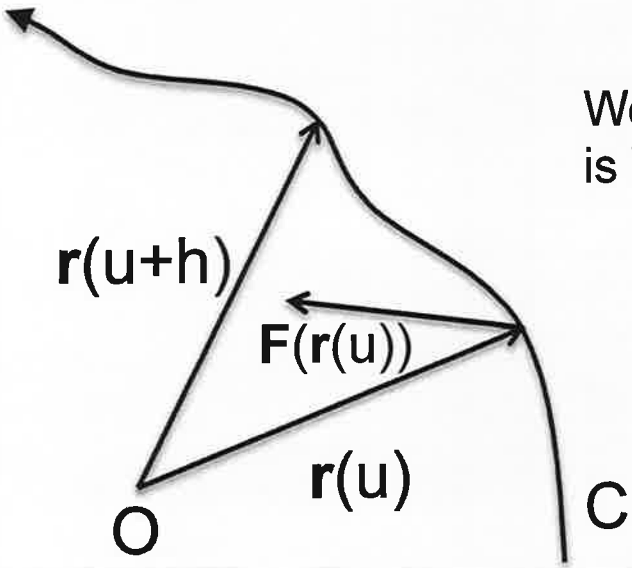
we need to extend this concept to curved paths in R^3

and work is a scalar, calculated with a dot product

R27
6

16.2: Force Over a Curved Path

Force **F** applied to an object as it moves from **r**(u) to **r**(u + h) along curve C.



Work done by force **F** from **r**(u) to **r**(u+h) is W(u + h) – W(u).

	Applied Force	Work
Case 3	**F = F(r(u))**	$\frac{W(u+h) - W(u)}{h} \approx F(\vec{r}(u)) \cdot \frac{\vec{r}(u+h) - \vec{r}(u)}{h}$

1) divide both sides by h

2) take limit as $h \to 0$: $w' = F(\vec{r}) \cdot \vec{r}\,'$

3) integrate: $w = \int F(\vec{r}) \cdot \vec{r}\,' \, du = \int \vec{F} \cdot d\vec{r}$

R27
7

16.2: Calculating Work

Set up an integral that represents the total work.

a) **F** = (x + 2y)**i** + (2x + y)**j**, path is $y = x^2$ from (0,0) to (2,4).

b) **F** = (x - y) **i** – xy **j**, along the line from (2,3) to (1,2).

c) **F** = xy **i** – 2 **j** + 4z**k**, along the circular helix **r** = cos(u)**i** + sin(u)**j** + u**k**, from u = 0 to u = 2π.

a) Find $\vec{r}$: let $x = u$, $y = u^2$, $\vec{F} = \begin{bmatrix} u+2u^2 \\ 2u+u^2 \end{bmatrix}$, $\vec{r} = \begin{bmatrix} u \\ u^2 \end{bmatrix}$, $\vec{r}\,' = \begin{bmatrix} 1 \\ 2u \end{bmatrix}$

$$W = \int_0^2 \begin{bmatrix} u+2u^2 \\ 2u+u^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2u \end{bmatrix} du = \int_0^2 (u+2u^2) + (2u+u^2+4u^2+2u^3)\, du$$

b) $\vec{F} = \begin{bmatrix} x-y \\ -xy \end{bmatrix} = \begin{bmatrix} -1 \\ -6-5u+u^2 \end{bmatrix}$ if $\vec{r} = \begin{bmatrix} 2-u \\ 3-u \end{bmatrix}$ and $u \in [0,1]$

$$W = \int_0^1 \begin{bmatrix} -1 \\ 6-5u+u^2 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \end{bmatrix} du = \int_0^1 1-6+5u-u^2\, du = -17/6$$

c) $$W = \int_0^{2\pi} \begin{bmatrix} cs \\ -2 \\ 4u \end{bmatrix} \cdot \begin{bmatrix} -s \\ c \\ 1 \end{bmatrix} du = \int_0^{2\pi} -cs^2 - 2c + 4u\, du = 0 + \int_0^{2\pi} 4u\,du = 8\pi^2$$

Recitation 28

Today's Topics

- 16.2 Vector Fields and Line Integrals, Work, Circulation, Flux
- 16.3 Path Independence

Learning Objectives

16.2 Set-up, evaluate, and interpret integrals to calculate **circulation** and **flux**

16.3 Determine whether a vector field is **conservative**

Circulation

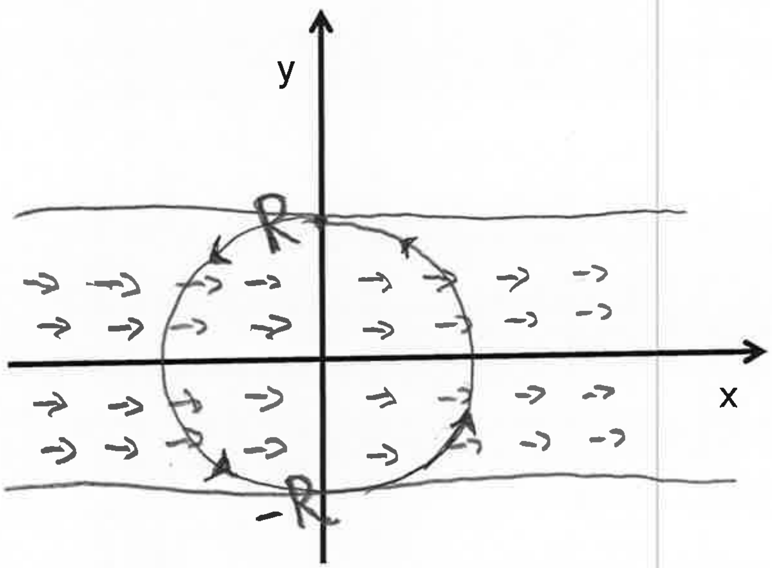
Cicrulation is a measure of the **flow** along a curve C, or **net velocity** along C.

$$\text{circulation} = \Gamma = \int_C \vec{v}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

16.2: Circulation

Sketch the velocity field for **v**, and calculate the **circulation** over curve C, where C is the circle of radius R.

$$\vec{v} = \begin{cases} 2\hat{\imath}, & R \le y \le R \\ 0, & \text{else} \end{cases}$$



$s = \sin t$, $c = \cos t$

$\vec{r} = \begin{bmatrix} c \\ s \end{bmatrix}$, $\vec{r}\,' = \begin{bmatrix} -s \\ c \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$$\Gamma = \int_C \vec{v} \cdot d\vec{r}$$

$$= \int_0^{2\pi} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -s \\ c \end{bmatrix} dt$$

$$= \int_0^{2\pi} -2 \sin t \, dt = 0$$

$$= \int_0^{\pi} -2s \, dt + \int_{\pi}^{2\pi} -2s \, dt$$

$$= -4 + 4 = 0$$

~~For part a)~~, the circulation is 0 because the circulation over $[0, \pi]$ cancels with the circulation over $[\pi, 2\pi]$.

~~For part b), the circulation is _____ because ______________.~~

Application of Circulation

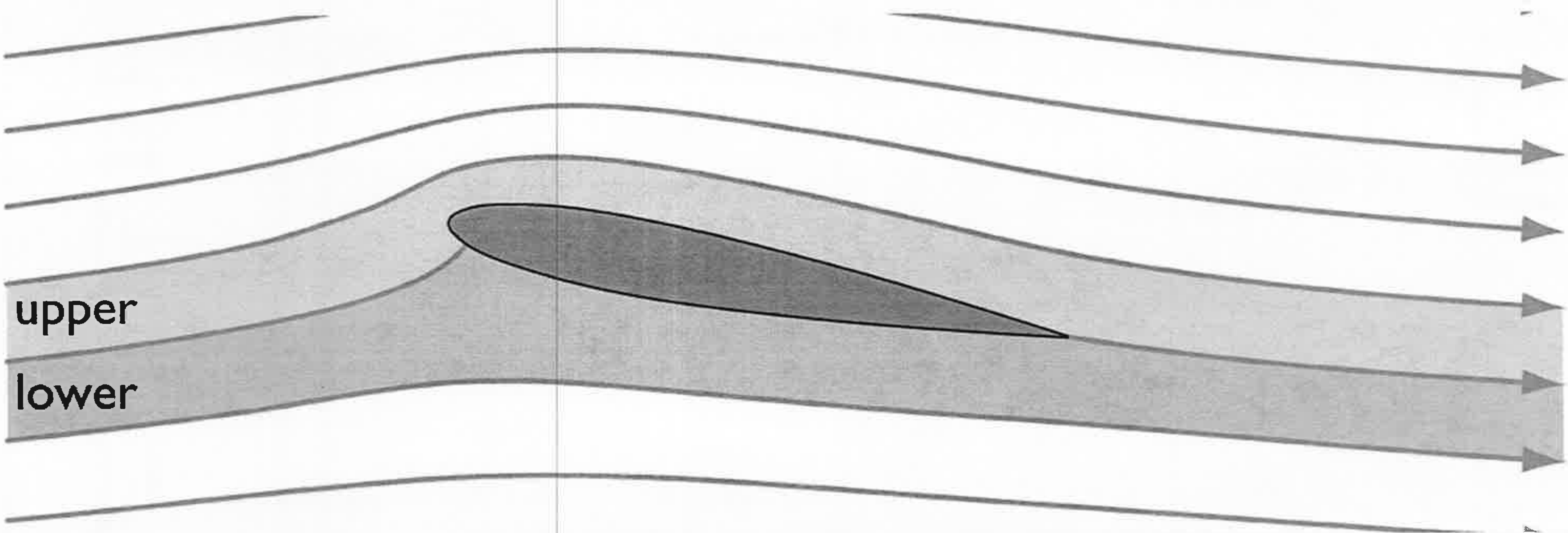
The circulation of a vector field **V** around a directed closed curve is

$$\text{circulation} = \Gamma = \int_C \vec{v}\left(\vec{r}\right)\cdot d\vec{r}$$

- Note the cross-sectional profile of the wing
- Take C to be a path around the wing, on its surface
- Upward lift force is proportional to circulation, Γ

16.2: An Application of Circulation

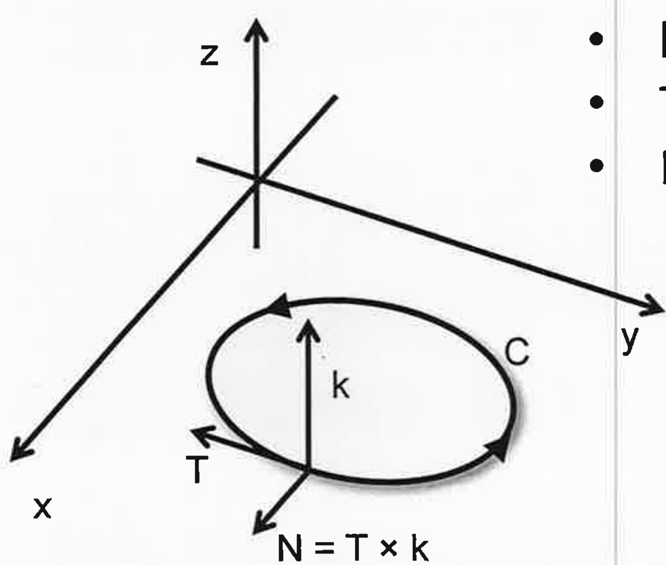
Take C to be a closed path around the wing on its surface



- Write Γ as $\Gamma = \Gamma_{upper} + \Gamma_{lower}$
- Γ_{upper} and Γ_{lower} have opposite signs
- the magnitude of **V** along the upper surface of the wing is greater than along the lower surface: net circulation is non-zero

16.2: Flux Across a Closed Plane Curve

Suppose we have a curve C in the xy plane, and a flow field **v** = M(x,y)**i** + N(x,y)**k**. We want to measure the **net flow through C**.



counterclockwise motion

- k is the unit vector parallel to the z-axis
- T is the tangent vector
- N is the outward pointing unit normal vector of C

$$\text{flux} = \oint_C \vec{v} \cdot \vec{N}\, dt = \oint_C M\, dy - N\, dx$$

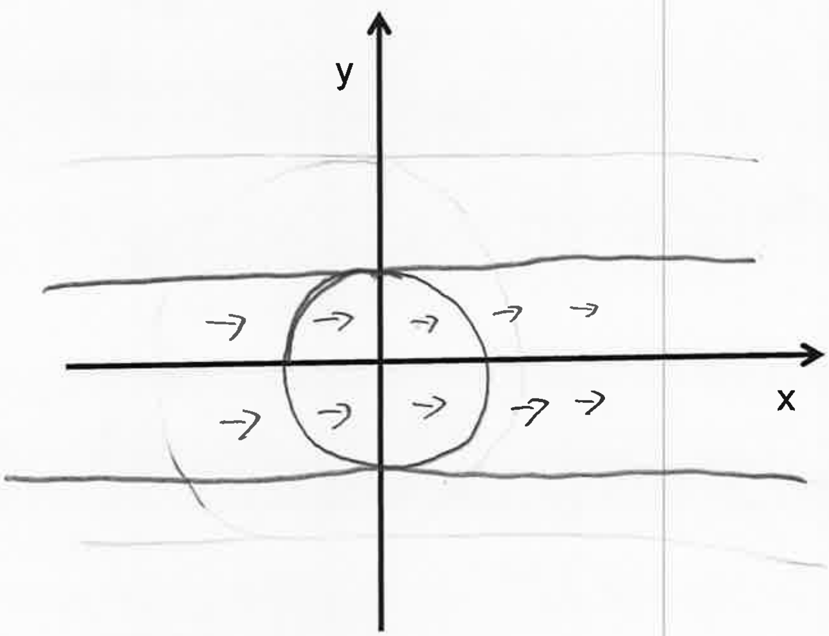
Note that:

- for a clockwise motion, we would instead use k × T
- later on, we will make a connection between flux and Green's theorem

16.2: Flux

Calculate the **flux** over curve C, where C is the circle of radius R.

$$\vec{v} = \begin{cases} 2\hat{\imath}, \; R \le y \le R \\ 0, \; \text{else} \end{cases}$$



$$\text{flux} = \oint_C M\,dy - N\,dx$$

$M=2,\; N=0$

$x = Rc, \quad dx = -Rs\,dt$
$y = Rs, \quad dy = Rc\,dt$ $\Big\}\; t \in [0, 2\pi]$

$$\text{flux} = \int_0^{2\pi} (2)(Rc\,dt) - (0)(-Rs\,dt)$$

$$= \int_0^{2\pi} 2Rc\,dt = 0$$

Therefore: the flux is 0 because in flow = out flow

R28
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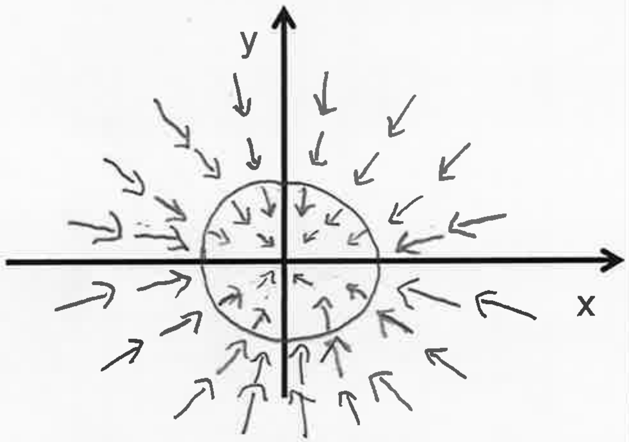
16.2: Circulation and Flux

1) Sketch the velocity field for **v = - xi - yj**, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.



$$\vec{r} = \begin{bmatrix} Rc \\ Rs \end{bmatrix}, \quad \vec{r}' = \begin{bmatrix} -Rs \\ +Rc \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -x \\ -y \end{bmatrix} = \begin{bmatrix} -Rc \\ -Rs \end{bmatrix}$$

$$\Gamma = \int_C \vec{v} \cdot d\vec{r} = \int_C \vec{v}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$

$$= \int_0^{2\pi} \begin{bmatrix} -Rc \\ -Rs \end{bmatrix} \cdot \begin{bmatrix} -Rs \\ Rc \end{bmatrix} dt$$

$$= \int_0^{2\pi} 0\, dt = 0$$

$M = -Rc,\ N = -Rs,$ $\quad x = Rc,\ dx = -Rs\,dt$, $\quad y = Rs,\ dy = Rc\,dt$

$$\text{flux} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} (-Rc)(Rc\,dt) - (-Rs)(-Rs\,dt)$$

$$= \int_0^{2\pi} -R^2(c^2+s^2)\,dt = -2\pi R^2$$

Therefore: the circulation is 0 because flow always perpendicular to C.

Therefore: the flux is $-2\pi R^2$ because all flow is inward

R28
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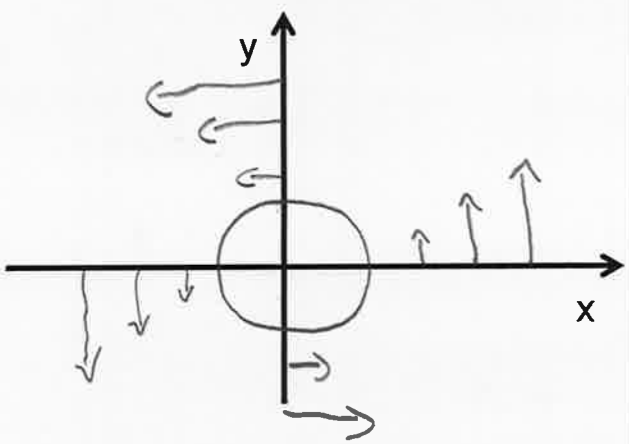
16.2: Circulation and Flux

2) Sketch the velocity field for **v = - yi + xj**, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.



$\vec{v}(1,0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v}(0,1) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

we solved this problem in the next recitation, R29.

16.3: Conservative Vector Fields

Recall the Pipe example.

a) Why was the circulation zero?

Γ on $[0,\pi]$ cancelled with Γ over $[\pi, 2\pi]$

b) For any path that starts and ends at point A, and stays inside "the pipe", the circulation is <u>the same</u>.

c) For all paths that starts at A and ends at point B, the integral $\int_C \vec{v} \cdot d\vec{r}$ is the same.

In general: if **v** is a conservative vector field (or is path independent), then there exists a scalar field, S, s.t. $\nabla S = \vec{v}$.

R28

1

Recitation 29

Today's Topics

16.2 Vector Fields and Line Integrals, Work, Circulation, Flux

16.3 Path Independence

~~16.4 Green's Theorem~~

Learning Objectives

16.2 Set-up, evaluate, and interpret integrals to calculate **circulation** and **flux**

16.3 Determine whether a vector field is **conservative** and apply the FTLI

~~16.4 Apply Green's theorem to calculate area and flux~~

Fundamental Theorem of Line Integrals

Circulation and Flux

Circulation is a measure of flow along a path.

Flux is a measure of flow through a region.

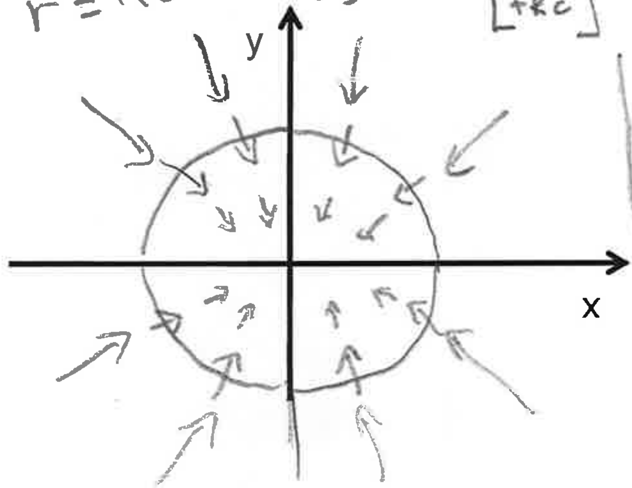
$$\text{circulation} = \Gamma = \int_C \vec{v}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{v}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$\text{flux} = \oint_C \vec{v} \cdot \vec{N}\,dt = \oint_C M\,dy - N\,dx$$

16.2: Circulation and Flux (review)

1) Sketch the velocity field for **v = - xi - yj**, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.

$\vec{r} = Rc\hat{i} + Rs\hat{j}$, $\vec{r}\,' = \begin{bmatrix} -Rs \\ +Rc \end{bmatrix}$.



NOTE: $c = \cos t$, $s = \sin t$

$$\Gamma = \int_C \vec{v} \cdot d\vec{r} = \int_0^{2\pi} \begin{bmatrix} -Rc \\ -Rs \end{bmatrix} \cdot \begin{bmatrix} -Rs \\ +Rc \end{bmatrix} dt$$

$$= \int_0^{2\pi} 0 \, dt = 0$$

$$\text{flux} = \int_0^{2\pi} M\,dy - N\,dx, \quad dy = \frac{dy}{dt}dt = +Rc\,dt, \quad dx = \frac{dx}{dt}dt = -Rs\,dt$$

$$= \int_0^{2\pi} (-Rc)(Rc\,dt) - (-Rs)(-Rs\,dt)$$

$$= \int_0^{2\pi} -R^2(c^2 + s^2)\,dt$$

$$= -2\pi R^2$$

Therefore: the circulation is $\underline{\ 0\ }$ because flow perpendicular to path

Therefore: the flux is $\underline{-2\pi R^2}$ because inward flow.

16.3: Conservative Vector Fields

In general: if **F** is a conservative vector field (or is path independent), then there exists a scalar field, f, s.t. $\nabla f = \vec{F}$, and

$$\int_C \vec{F} \cdot d\vec{r} = \int \nabla f \cdot d\vec{r} = f(b) - f(a)$$

Example: Calculate total work from the force **F** = (x^2-y)**i** + $(y^2 - x)$**j**, over the path **r** = a cos(t)**i** + b sin(t)**j**, where $0 \le t \le 2\pi$.

Is $\vec{F}$ conservative? Two ways to check.

① Check if derivatives match: $\frac{\partial}{\partial y}(x^2-y) = -1$, $\partial/\partial x(y^2-x) = -1$ $\Big\}$ equal, so $\vec{F}$ is conservative.

② Find potential:

$$\frac{\partial f}{\partial x} = x^2 - y \Rightarrow f = x^3/3 - yx + \phi(y) \Rightarrow \frac{\partial f}{\partial y} = -x + \phi',$$

and $\phi' = y^2$ by comparison. Thus f exists, and is

$$f = \tfrac{1}{3}(x^3 + y^3) - xy$$

$\vec{F}$ is conserv., so use FTLI: $\int_C \vec{F} \cdot d\vec{r} = f(2\pi) - f(0) = 0$

R28
4

16.3: Conservative Fields

Determine whether the following fields are conservative

1) v = – xi – yj

2) v = – yi + xj

1) $\frac{\partial}{\partial y}(-x) = 0 = \partial/\partial x(-y)$ $\Rightarrow$ conservative

2) $\partial/\partial y(-y) = -1 \neq \partial/\partial x(+x) = +1$ $\Rightarrow$ not conservative.

R28
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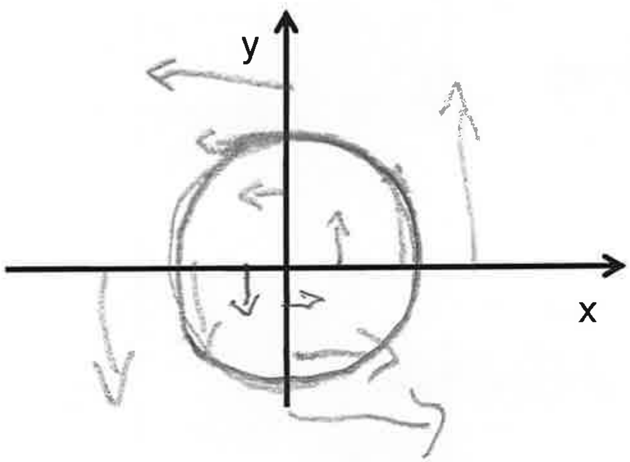
16.2: Circulation and Flux

2) Sketch the velocity field for **v = - yi + xj**, and calculate the **circulation** and **flux** over curve C, where C is the circle of radius R.



$$\Gamma = \int_C \vec{v}\cdot d\vec{r}, \quad \vec{r} = \begin{bmatrix} Rc \\ Rs \end{bmatrix}, \quad \vec{r}\,' = \begin{bmatrix} -Rs \\ +Rc \end{bmatrix}$$

$$= \int_0^{2\pi} \begin{bmatrix} -Rs \\ +Rc \end{bmatrix} \cdot \begin{bmatrix} -Rs \\ +Rc \end{bmatrix} dt$$

$$= R^2 \int_0^{2\pi} s^2 + c^2 \, dt = 2\pi R^2$$

$$\text{flux} = \int_0^{2\pi} (-Rs)(Rc\,dt) - (Rc)(-Rs\,dt)$$

$$= \int_0^{2\pi} 0\,dt = 0$$

FLUX IS ZERO BECAUSE THERE IS NO FLOW IN/OUT OF REGION, CIRCULATION, Γ, IS POSITIVE BECAUSE FLOW IS IN DIRECTION OF MOTION ALONG C.

R28
6

Conclusions

a) Circulation measures flow along path C.

b) Flux measures the flow through of ~~C~~ region

c) If a flow is conservative, the line integral $\int_C \vec{v}\cdot d\vec{r}$ is the same for any path C.

field name	velocity field equation	circulation	flux	is v conservative?
pipe	v = 2i for $-R \le y \le +R$, v = 0 otherwise	0	0	YES
"drain"	v = -xi - yj	0	$-2\pi R^2$	YES
"whirlpool" !	v = -yi + xj	$+2\pi R^2$	0	NO

R30

1

Recitation 30

Today's Topics

16.4 Green's Theorem

16.5 Surfaces and Areas

Learning Objectives

16.4 Apply Green's theorem to calculate area, flux, and circulation

16.5 Calculate the area of a surface given explicitly, implicitly, or parametrically

Green's Theorem

If R is a region that is closed, simple, and M and N are scalar fields that are differentiable on R, and C is the boundary of R, then:

$$\text{flux} = \oint_C M\,dy - N\,dx = \iint \underbrace{\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}_{\begin{bmatrix} M \\ N \end{bmatrix} \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix},\ \text{divergence } *} dx\,dy$$

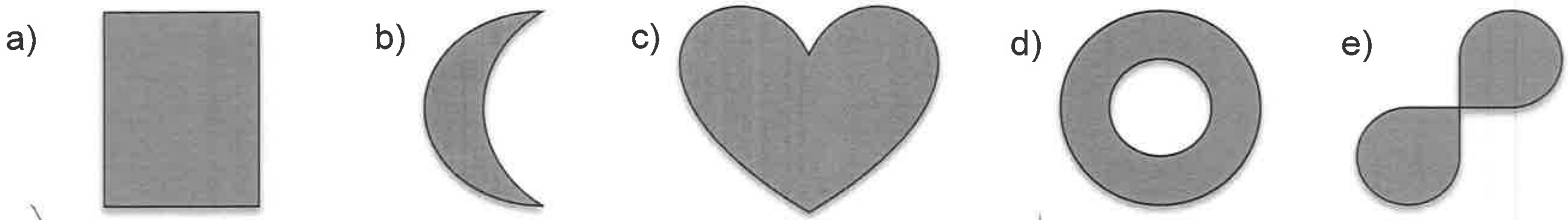
$$\text{circulation} = \oint_C M\,dx + N\,dy = \iint \underbrace{\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)}_{\text{curl} F \cdot \hat{k}\ *} dx\,dy$$

$$\left(= \oint_C \vec{F} \cdot d\vec{r}\right)$$

* we don't need div & curl at this point, but we will soon!

Green's Theorem Example (from an old quiz)

Below are five regions. For which regions can we apply Green's Theorem?



simple & closed
⇒ can apply GT

not simple.
⇒ can apply GT if we, eg - use two integrals

simple = no holes, and boundary is not "self-intersecting."

R30
3

Green's Theorem Example (from an old quiz)

Find the circulation AND flux for the field $\mathbf{F} = 3x^2y^2\,\mathbf{i} + 2x^3y\,\mathbf{j}$ around the rectangle $0 \le x \le 2$, $0 \le y \le 3$. Use Green's Theorem.

$$\text{flux} = \oint_C M\,dy - N\,dx = \oint_C \overbrace{3x^2y^2}^{M}\,dy - (\overbrace{2x^3y}^{N})\,dx = \iint \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy$$

$$= \int_0^3\!\!\int_0^2 (6xy^2 + 2x^3)\,dx\,dy$$

$$= 132$$

$$\Gamma = \iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy$$

$$= \iint \frac{\partial}{\partial x}(2x^3y) - \frac{\partial}{\partial y}(3x^2y^2)\,dx\,dy$$

$$= \int_0^3\!\!\int_0^2 6x^2y - 6x^2y\,dx\,dy$$

$$= 0$$

in the above, $M = 3x^2y^2$, $N = 2x^3y$

Green's Theorem Example (from an old quiz)

Let R be the region in the plane, inside the cardiod r = 1 + cos (θ), and C its boundary Consider the line integral

$$\int_C xy\,dx - xy^2\,dy.$$

Use Green's theorem to convert to an double integral, and express this as a double integral in polar coordinates with limits.

flux $= \oint M\,dy - N\,dx$

$M = -xy^2$, $\frac{\partial M}{\partial x} = -y^2$

$N = -xy$, $\frac{\partial N}{\partial y} = -x$

$$\text{flux } \cancel{\text{Area}} = \iint \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dx\,dy$$

$$= \iint \left(-y^2 - x\right) dx\,dy$$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} \left(-r^2\sin^2\theta - r\cos\theta\right) r\,dr\,d\theta$$

Alternate: $\Gamma = \oint M\,dx + N\,dy$

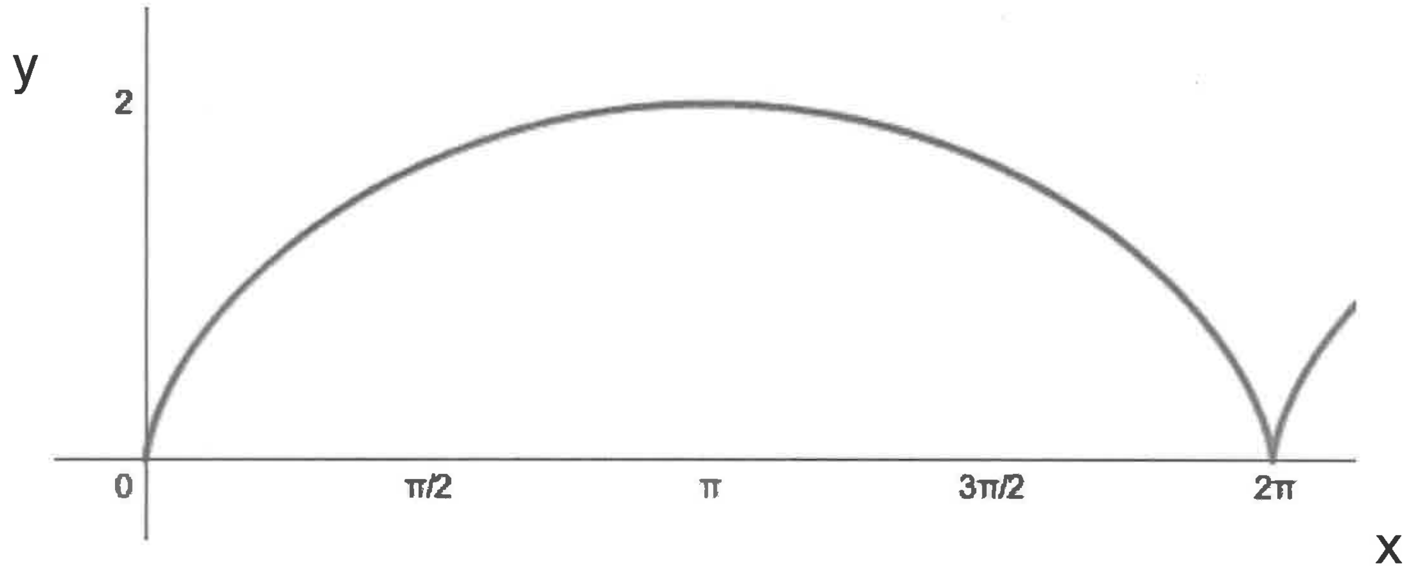
$M = xy$, $M_y = +x$

$N = -xy^2$, $N_x = -y^2$

$\Gamma = \iint N_x - M_y\,dx\,dy = \iint -y^2 - x\,dx\,dy$

$= \int_0^{2\pi}\int_0^{1+\cos\theta} \left(-r^2 s^2 - rc\right) r\,dr\,d\theta$

Additional Green's Theorem Example



The curve traced by a point on a rolling wheel is

$$x(t) = t - \sin(t)$$
$$y(t) = 1 - \cos(t)$$

R30
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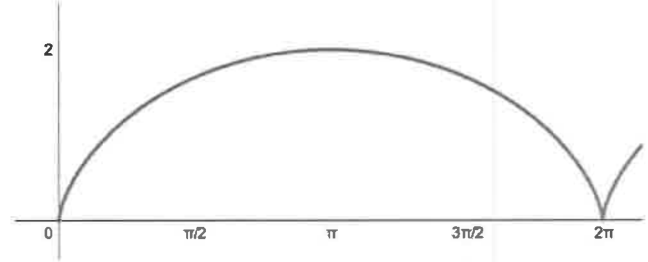
Additional Example: Green's Theorem

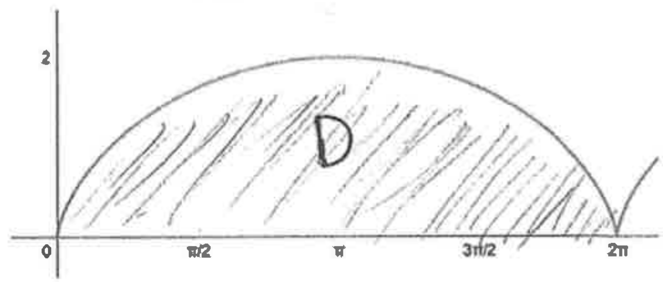
Find the area under one arch of the cycloid:
$x(t) = t - \sin(t)$, $y(t) = 1 - \cos(t)$



Find the area under one arch of the cycloid:
$x(t) = t - \sin(t)$, $y(t) = 1 - \cos(t)$

call the area D

$$A = \iint_D dx\,dy$$



We don't have $y = y(x)$ explicitly.
What can we do?

Introduce

$$\left.\begin{array}{ll} M = x, & \frac{\partial M}{\partial x} = 1 \\ N = 1, & \frac{\partial N}{\partial y} = 0 \end{array}\right\} A = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_D dx\,dy \text{, as needed.}$$

On C_1: $x = t$,
$y = 0$, $dy = 0\,dt$

On C_2: $x = t - \sin t$,
$y = 1 - \cos t$, $dy = -\sin t$

$$= \oint_C x\,dy - 0\,dx$$

$$= \int_{C_1} x\,dy + \int_{C_2} x\,dy$$

$$= \int_0^{2\pi} t \cdot 0\,dt + \int_0^{2\pi} \sin^2 t - t\sin t\,dt$$

$$= \dots = 3\pi$$

Additional Example: Green's Theorem

a) Evaluate $\oint_C y^2\,dx + 2xy\,dy$, C is one loop of $r = 2\sin 2\theta$

b) Change the integral so that it represents the area of one loop.

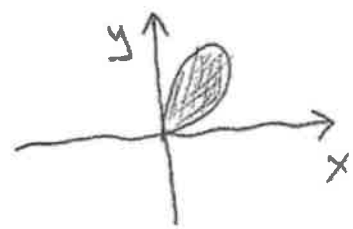
a) We need a formulation of Green's theorem. We can use

$$\text{flux} = \oint M\,dy - N\,dx = \iint_D \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\,dxdy \left(= \iint_D \underbrace{\begin{bmatrix} M \\ N \end{bmatrix} \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix}}_{\vec{V}\cdot\vec{\nabla}} \right)$$

$= \cancel{\text{AREA}}$

(some textbooks use a slightly different formula)

$$\Rightarrow \left.\begin{aligned} M &= 2xy, & \partial M/\partial x &= 2y \\ N &= -y^2, & \partial N/\partial y &= -2y \end{aligned}\right\}$$ integrand is zero, so answer is zero.



b) For area, we need $\frac{\partial M}{\partial x} + \partial N/\partial y = 1$.

We can choose:

$$\left.\begin{aligned} M &= +3xy \Rightarrow M_x = 3y \\ N &= -y^2 \Rightarrow N_y = -2y \end{aligned}\right\}$$ $\text{AREA} = \iint M_x + N_y\,dxdy = \int_0^{\frac{\pi}{2}}\int_0^{2\sin 2\theta} r\,dr\,d\theta$

16.5 Surfaces and Areas

Surface area for a parameterized surface:

$$S = \iint d\sigma, \quad d\sigma = \overbrace{|r_u \times r_v|}^{\text{Jacobian}} \, du\, dv$$

Your textbook has formulas for calcuatling the surface area for **implicit** and **explicit** surfaces, we probably won't have time to work on these in recitation.

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16.5 Surfaces and Areas

a) What properties does a parametric representation of a surface need to have?

① satisfy given surface

② one-to-one

③ continuous

b) Find a parametric representation for the part of the plane $z = x + 2$ in the first octant and inside the cylinder $x^2 + y^2 = 1$.

Parametric representation is:

$$\vec{r} = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

where

$x = u\cos v$

$y = u\sin v$

$z = u\cos v + 2$

which is ①, ②, ③, for $u \in [0,1]$, $v \in [0, \frac{\pi}{2}]$ first octant

R31

1

Recitation 31

Today's Topics

16.5 Surfaces and Areas

16.6 Surface Integrals

Learning Objectives

16.5 Calculate the area of a surface given explicitly, implicitly, or parametrically

16.6 Calculate outward flux through a surface

16.6 Calculate the total mass and centroid of a thin surface (if time permits)

Course Logistics

1. Has a final exemption cutoff been announced?
2. What is the cutoff?
3. When is your final exam?

16.5 Surfaces and Areas

Surface area for a parameterized surface:

$$\text{Surface area} = \iint_S d\sigma, \quad \sigma = |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

Your textbook has formulas for calcuatling the surface area for **implicit** and **explicit** surfaces, we probably won't have time to work on these in recitation.

$\vec{r}$ is a parameterization of our surface, and

$$\vec{r} = \vec{r}(u, v)$$

$$\vec{r}_u = \frac{\partial}{\partial u} \vec{r}$$

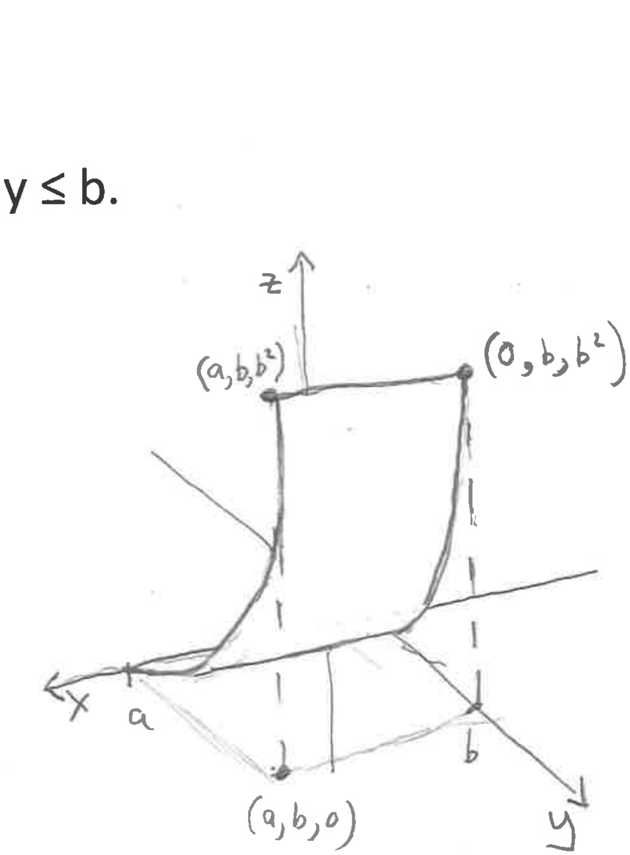
$$\vec{r}_v = \frac{\partial}{\partial v} \vec{r}$$

R31

3

16.5 Surface Area Example

Set-up an integral that represents ~~Calculate~~ the surface area of $z = y^2$, for $0 \leq x \leq a$, $0 \leq y \leq b$.



SURFACE AREA $= \iint |\vec{r}_u \times \vec{r}_v| \, du\, dv$

WE NEED PARAMETERIZATION, $\vec{r}(u,v)$.

$$\left.\begin{array}{ll} x = u, & u \in [0,a] \\ y = v, & v \in [0,b] \\ z = v^2 & \end{array}\right\} \quad \vec{r} = \begin{bmatrix} u \\ v \\ v^2 \end{bmatrix}$$

WE NEED $|\vec{r}_u \times \vec{r}_v|$

$$\vec{r}_u = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{r}_v = \begin{bmatrix} 0 \\ 1 \\ 2v \end{bmatrix}, \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 2v \end{vmatrix} = \begin{bmatrix} 0 \\ -2v \\ 1 \end{bmatrix}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{4v^2 + 1}$$

CALCULATE SURF. AREA

$$\iint |\vec{r}_u \times \vec{r}_v| \, du\, dv = \int_0^b \int_0^a \sqrt{1+4v^2} \, du\, dv$$

(if interested, can get area w/ table of integrals)

R31

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16.5 Surface Area Example

Set up an integral that represents the area of the part of the plane x + 2y + z = 4 that is inside the cylinder $x^2 + y^2 = 4$.

$$\text{surface area} = \iint_S d\sigma = \iint |\vec{r}_u \times \vec{r}_v| \, du\,dv$$

$$\vec{r} = \begin{bmatrix} u \\ v \\ 4-u-2v \end{bmatrix}, \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \end{vmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\iint_S d\sigma = \iint \sqrt{6} \, du\,dv, \quad u^2 + v^2 \le 4$$

$$= \int_0^{2\pi} \int_0^2 \sqrt{6} \, r\,dr\,d\theta$$

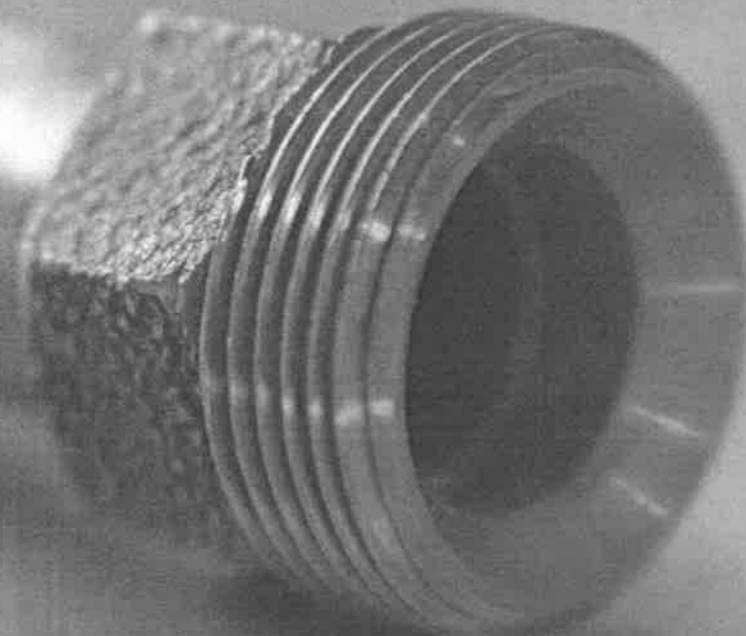
$$= 4\sqrt{6}\,\pi$$

16.6 Surface Integrals

Suppose we want to characterize 3D flow through a pipe.

To calculate 2D flux across a **curve**, we used: $\text{flux} = \int_C \bar{v}\cdot\bar{n}\,du = \int_C M\,dy - N\,dx$

If our flow field, **v**, is 3D, we calculate flux across a **surface.**



flux accross surface $\iint_S \vec{v}\cdot\vec{n}\,dS$

$\vec{n}$ = unit outward normal

$\vec{v}$ = velocity field

$$\iint_S \vec{v}\cdot\vec{n}\,dS = \iint \vec{v}\cdot\vec{n}\,|\vec{r}_u\times\vec{r}_v|\,du\,dv$$

$$= \iint \vec{v}\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv, \text{ because } \vec{n} = \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}$$

16.6 Flux Through a Surface

A fluid has velocity field **v** = y**i** + **j** + z**k**. Set up an integral that represents the flux through the paraboloid z = 9 – (x² + y²)/4, if x² + y² ≤ 36.

$$\text{flux} = \iint \vec{v}\cdot\vec{n}\,dS = \iint \left(\vec{v}\cdot(\vec{r}_u\times\vec{r}_v)\right)du\,dv$$

$$\vec{r} = \begin{bmatrix} u \\ v \\ 9-(u^2+v^2)/4 \end{bmatrix},\quad \vec{r}_u = \begin{bmatrix} 1 \\ 0 \\ -u/2 \end{bmatrix},\quad \vec{r}_v = \begin{bmatrix} 0 \\ 1 \\ -v/2 \end{bmatrix},\quad \vec{r}_u\times\vec{r}_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -u/2 \\ 0 & 1 & -v/2 \end{vmatrix} = \begin{bmatrix} u/2 \\ v/2 \\ 1 \end{bmatrix}$$

$x^2+y^2 \le 36$ implies $u^2+v^2 \le 36$

$$\text{flux} = \iint_{u^2+v^2\le 36} \vec{v}\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv = \iint \begin{bmatrix} v \\ 1 \\ 9-(u^2+v^2)/4 \end{bmatrix}\cdot\begin{bmatrix} u/2 \\ v/2 \\ 1 \end{bmatrix} du\,dv$$

$$= \iint uv/2 + v/2 + 9 - u^2/4 - v^2/4 \; du\,dv$$

For limits, use $u = r\cos\theta = rc$, $v = r\sin\theta = rs$:

$$\text{flux} = \int_0^{2\pi}\int_0^6 \tfrac{1}{2}(r^2cs + rs) + 9 - (r^2c^2 + r^2s^2)/4 \;\; r\,dr\,d\theta$$

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16.6 Surface Integrals (this was a 2014 pop quiz question)

Set up a double integral that represents the flux of flow **F** = x**i** + z**k** thorugh the surface $z(x,y) = x^2 - y^2$, where $0 \le x \le 1$, $-1 \le y \le 1$.

$$\vec{r} = \begin{bmatrix} u \\ v \\ u^2 - v^2 \end{bmatrix}, \quad \vec{r}_u = \begin{bmatrix} 1 \\ 0 \\ 2u \end{bmatrix}, \quad \vec{r}_v = \begin{bmatrix} 0 \\ 1 \\ -2v \end{bmatrix}, \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 1 & 0 & 2u \\ 0 & 1 & -2v \end{vmatrix} = \begin{bmatrix} -2u \\ +2v \\ 1 \end{bmatrix}$$

$$\text{flux} = \iint \vec{F} \cdot \vec{r}_u \times \vec{r}_v \, du\,dv$$

$$= \iint \begin{bmatrix} u \\ 0 \\ u^2 - v^2 \end{bmatrix} \cdot \begin{bmatrix} -2u \\ 2v \\ 1 \end{bmatrix} du\,dv$$

$$= \int_0^1 \int_{-1}^1 -u^2 - v^2 \, dv\,du$$

R31

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16.6 Centroid of a Thin Surface (if time permits)

The mass density at any point on a thin surface $z^2 = x^2 + y^2$, $0 \le z \le 1$, is proportional to its distance to the z-axis.

a) Find the total mass of the surface.

b) Find the centroid of the surface.

a) $$M = \iint_S \delta(x,y,z)\, d\sigma = \iint k(x^2+y^2)\, d\sigma, \quad k = \text{constant}, \quad d\sigma = |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$\vec{r} = \begin{bmatrix} u \\ v \\ \sqrt{u^2+v^2} \end{bmatrix}, \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} 1 & 0 & u/\sqrt{\ } \\ 0 & 1 & v/\sqrt{\ } \end{vmatrix} = \begin{bmatrix} u/\sqrt{\ } \\ -v/\sqrt{\ } \\ 1 \end{bmatrix}, \quad |\vec{r}_u \times \vec{r}_v| = \sqrt{2}$$

$$\Rightarrow M = \iint_S k(u^2+v^2)(\sqrt{2}\, du\, dv) = \sqrt{2}k \int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta = \frac{2}{3}\sqrt{2}\pi k$$

b) We want $(\bar{x}, \bar{y}, \bar{z})$, but $\bar{x} = \bar{y} = 0$ by symmetry.

$$\bar{z}M = \iint_S z\,\delta(x,y,z)\, d\sigma = \iint k(u^2+v^2)^2(\sqrt{2}\, du\, dv)$$

$$= \sqrt{2}k \int_0^{2\pi}\int_0^1 r^3\, dr\, d\theta$$

$$= \frac{\sqrt{2}}{2}\pi k$$

R31

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16.5 Surface Area Parameterization (additional example)

Find parametric representations for the following surfaces.

a) the upper half of $4x^2 + 9y^2 + z^2 = 36$

b) the part of the plane $z = x + 2$ inside the cylinder of $x^2 + y^2 = 1$

a) divide by 36 : $\frac{x^2}{3^2} + \frac{y^2}{2^2} + \frac{z^2}{6^2} = 1$, use modified spherical

$$\left.\begin{aligned} x &= 3\cos u \cos v \\ y &= 2 \sin u \cos v \\ z &= 6 \sin v \end{aligned}\right\} \quad \begin{aligned} u &\in [0, 2\pi] \\ v &\in [0, \pi/2] \end{aligned}$$

b)

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r \sin\theta \\ z &= r\cos\theta + 2 \end{aligned}\right\} \quad \begin{aligned} r &\in [0, 1] \\ \theta &\in [0, 2\pi] \end{aligned}$$

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Recitation 32

Today's Topics

Final Exam Review

16.7 Stokes Theorem

16.8 The Divergence Theorem

Learning Objectives

16.7 Use Stoke's theorem to calculate either work, or circulation over a curve

16.8 Calculate flux through a surface using the divergence theorem

Final Exam Logistics

Review session: information sent via email

Questions during final: information sent via email

Studying for the Final Exam

There are two prep-finals available on T^2. Each of them have five questions that focus on specific areas of our textbook.

	Chapter 13	Chapter 14	Chapter 15	Chapter 16
Prep-Final A	P1			P2, P3, P4, P5
Prep-Final B	P1	P2	P3	P4, P5

Ways you may want to study:

- solve prep final questions
- re-do quizzes 1 through 4
- re-do MML problems
- memorize formulas (especially from Chapters 13 and 16)

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PrepFinal Question A1

Find the speed, the tangential acceleration and the normal acceleration for the motion **r** = (t,t²,t²). Compute also the curvature of the corresponding curve as a function of t.

$$\text{speed} = |\vec{r}'| = \left|\frac{d}{dt}\begin{bmatrix} t \\ t^2 \\ t^2 \end{bmatrix}\right| = \left|\begin{bmatrix} 1 \\ 2t \\ 2t \end{bmatrix}\right| = \sqrt{1+4t^2+4t^2} = \sqrt{1+8t^2}$$

$$a_T = \frac{d}{dt}|\vec{v}| = \frac{d}{dt}\sqrt{1+8t^2} = \frac{1}{2}(1+8t^2)^{-1/2}\left(\frac{d}{dt}8t^2\right) = 8t/\sqrt{1+8t^2}$$

$$a_N = \sqrt{|\vec{r}''|^2 - |a_T|^2} = \left(\left|\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}\right|^2 - \frac{8^2t^2}{(\sqrt{1+8t^2})^2}\right)^{1/2} = \left(\left(\sqrt{0^2+2^2+2^2}\right)^2 - \frac{128t^2}{1+8t^2}\right)^{1/2} = \sqrt{8-\frac{128t^2}{1+8t^2}}$$

(we could have also used $a_N = \frac{|\vec{v}\times\vec{a}|}{|\vec{v}|}$)

$$K = \frac{|\vec{T}'|}{|\vec{v}|} = \frac{|\vec{r}'\times r''|}{|r'|^{3/2}} = \frac{1}{(1+8t^2)^{3/2}}\left\|\begin{matrix} 1 & 2t & 2t \\ 0 & 2 & 2 \end{matrix}\right\| = (1+8t^2)^{-3/2}\left|\begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}\right| = 8(1+8t^2)^{-3/2}$$

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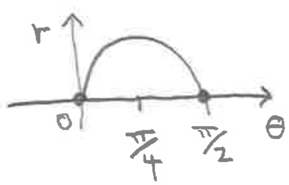
PrepFinal Question A2

Find the moment of inertia with respect to the x axis of a thin shell of mass δ that is in the first quadrant of the xy plane and bounded by the curve $r^2 = \sin 2\theta$.

1) PLOT CURVE



region is closed and simple

2) SET-UP INTEGRAL, INTEGRATE

$$I_x = \frac{-1}{3}\delta \oint_C y^3\,dx$$, did we do this in lecture? If not, how else would we do this?

$$= \frac{-\delta}{3}\iint -\frac{\partial}{\partial y}(y^3)\,dx\,dy$$, by Green's Thm

$$= \frac{\delta}{3}\,3\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\sin 2\theta}} (r^2 \sin^2\theta)\, r\,dr\,d\theta$$

$$= \delta \int_0^{\frac{\pi}{2}} \sin^2\theta \int_0^{\sqrt{\ }} r^3\,dr\,d\theta$$, $\sqrt{\ } = \sqrt{\sin 2\theta}$

$$= \delta \int \sin^2\theta \left.\tfrac{1}{4}r^4\right|_0^{\sqrt{\ }} d\theta$$

$$= \frac{\delta}{4}\int (\sin^2\theta)(\sin 2\theta)^2\,d\theta$$

$$= \delta \int_0^{\pi/2} \sin^4\theta \cos^2\theta\,d\theta$$, $\sin 2\theta = 2\sin\theta\cos\theta$

$$= \delta \int_0^{\pi/2} \sin^4\theta - \sin^6\theta\,d\theta$$, $\cos^2\theta = 1 - \sin^2\theta$

$$= \delta\left(\frac{3\pi}{2^4} - \frac{5\pi}{2^5}\right) = \delta\,\pi/2^5$$

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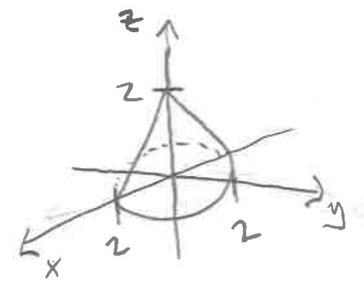
PrepFinal Question A3

Compute the center of mass of a thin shell that is formed by the cone $(z-2)^2 = x^2 + y^2$, $0 \le z \le 2$.



We want $(\bar{x}, \bar{y}, \bar{z})$, but $\bar{x} = \bar{y} = 0$ by symmetry.

$\bar{z} M = M_{xy}$, $M = \iint_S \delta \, d\sigma$, $M_{xy} = \iint_S z \, \delta \, d\sigma$.

Assume $\delta =$ constant.

$M = \delta \iint |r_u \times r_v| \, du\,dv$

$\vec{r} = \begin{bmatrix} u \\ v \\ 2 + \sqrt{u^2+v^2} \end{bmatrix}$, $\vec{r}_u \times \vec{r}_v = \begin{vmatrix} 1 & 0 & u/\sqrt{\ } \\ 0 & 1 & v/\sqrt{\ } \end{vmatrix} = \begin{bmatrix} -u/\sqrt{\ } \\ -v/\sqrt{\ } \\ 1 \end{bmatrix}$, $|\vec{r}_u \times \vec{r}_v| = \sqrt{\frac{u^2}{u^2+v^2} + \frac{v^2}{u^2+v^2} + 1} = \sqrt{2}$

$M = \delta \iint \sqrt{2} \, du\,dv = \sqrt{2}\,\delta \int_0^{2\pi} \int_0^2 r\,dr\,d\theta = \sqrt{2}\,\delta \cdot 2\pi \cdot \left(\frac{1}{2} r^2\right)\Big|_0^2 = +\sqrt{2}\,\delta\, 4\pi$

on the cone, $z = 2 - r$, and $r \in [0, 2]$.

$M_{xy} = \delta \iint \sqrt{2}\, z \, du\,dv = \sqrt{2}\,\delta \int_0^{2\pi}\int_0^2 (2 - r)\, r\,dr\,d\theta = \sqrt{2}\,\delta (2\pi)\left(\frac{2r^2}{2} - \frac{r^3}{3}\right)\Big|_0^2 = 2\pi\sqrt{2}\,\delta \cdot 4/3$

$\bar{z} = \frac{M_{xy}}{M} = \frac{2\pi(4/3)\sqrt{2}\,\delta}{\sqrt{2}\,\delta\, 4\pi} = \frac{2}{3} \Rightarrow (\bar{x}, \bar{y}, \bar{z}) = (0, 0, 2/3)$

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PrepFinal Question A4

Compute the line integral of the vector field **F** = (xyz + 1, x^2z, x^2y)e^{xyz} along the curve **r**(t) = (cost,sint,t), 0 ≤ t ≤ π.

We want $\int_C \vec{F}\cdot d\vec{r}$. We can't use Greens' Thm, or Stokes thm, because C is not closed.

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi} e^{cst}\begin{bmatrix} tcs+1 \\ tc^2 \\ c^2 s\end{bmatrix}\cdot\begin{bmatrix} -s \\ c \\ 1\end{bmatrix} dt, \quad c=\cos t,\ s=\sin t$$

$$= \int_0^{\pi} e^{cst}\left(\left(-tcs^2-s\right)+tc^3+c^2s\right) dt$$

Direct integration is very difficult. However, if $\vec{F}$ is a conservative field, we can use the FTLI. If $\nabla\times\vec{F}=\vec{0}$, then $\vec{F}$ is conservative; curl $\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (xyz+1)e^{xyz} & x^2ze^{xyz} & x^2ye^{xyz}\end{vmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$. Thus, $\vec{F}$ is conservative, and we can apply the FTLI. To do so, we need to find a $\phi(x,y,z)$, s.t. $\nabla\phi=\vec{F}$. By inspection, $\phi = xe^{xyz}$. Thus, $\int_C \vec{F}\cdot d\vec{r} = \phi(\cos\pi, \sin\pi, \pi) - \phi(\cos 0, \sin 0, 0)$

$=(-1)-(+1) = -2$

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7 # PrepFinal Question A5

Use the divergence theorem to compute the outward flux of the vector field **F** = (x^2 , y^2 , z^2) through the cylindrical can that is bounded on the side by $x^2 + y^2 = 4$, bounded above by $z = 1$ and below by $z = 0$.

outward flux $= \iint_S \vec{F}\cdot\hat{n}\, d\sigma = \iiint_V \nabla\cdot F\, dV$

$$= \iiint 2x + 2y + 2z\, dV$$

$$= 2\int_0^{2\pi}\int_0^2\int_0^1 (rc + rs + z)\, r\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2 \left(r^2c + r^2s + r/2\right) dr\, d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{r^3}{3}(c+s) + \frac{r^2}{4}\right)\Big|_0^2\, d\theta$$

$$= 2\int_0^{2\pi} \left(\frac{8}{3}(c+s) + 1\right) d\theta$$

$$= 2\left(\frac{8}{3}(s-c) + 1\right)\Big|_0^{2\pi} = 2\left(\left(\frac{8}{3}(0-1)+1\right) - \left(\frac{8}{3}(0-1)+1\right)\right)$$

$$= 0$$

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8 # PrepFinal Question B1

Find the parametric equations of the line that is tangent to the curve **r**(t) = (e^t, sin t, ln(1 − t)), at t = 0.

$$\vec{r}\,' = \begin{bmatrix} e^t \\ c \\ \frac{-1}{1-t} \end{bmatrix}, \quad c = \cos t, \quad \vec{r}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{r}\,'(0) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Line has equation $\vec{L} = \vec{r}(0) + \lambda \vec{r}\,'(0)$, $\lambda \in \mathbb{R}$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

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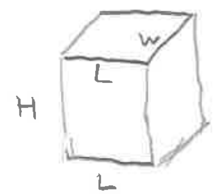
PrepFinal Question B2

Find the minimum cost ~~area~~ of a rectangular solid with volume 64 cubic inches, given that the top and sides cost 4 cents per square inch and the bottom costs 7 cents per square inch. Just set up the equations using Lagrange multipliers, you do not have to solve them.

$$V = 64 = LWH$$



$$C = 4LW + 4 \cdot 2LH + 4 \cdot 2WH + 7LW$$

$$= 11LW + 8LH + 8WH$$

$$\nabla C = \lambda \nabla V$$

$$\begin{bmatrix} 11W + 8H \\ 11L + 8H \\ 8L + 8W \end{bmatrix} = \lambda \begin{bmatrix} WH \\ LH \\ LW \end{bmatrix}, \quad \text{and} \quad LWH = 64$$

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PrepFinal Question B3

Compute the average of the function x^4 over the sphere centered at the origin whose radius is $R > 0$.

(I'm assuming that we want average over solid shere, not its surface)

In general, $\iiint_V f(x,y,z)\,dx\,dy\,dz = \begin{pmatrix}\text{average of } f\\ \text{over } V\end{pmatrix} \cdot (\text{volume of } V)$

So for this problem,

$$\left(\tfrac{4}{3}\pi R^3\right)(\text{average}) = \iiint x^4\,dx\,dy\,dz$$

$$= \int_0^{2\pi}\int_0^{R}\int_0^{\pi} x^4 \rho^2 \sin\phi \; d\phi\, d\rho\, d\theta, \quad x^4 = \rho^4 \sin^4\phi \cos^4\theta$$

$$= \int_0^{2\pi}\int_0^{R}\int_0^{\pi} \rho^6 \sin^5\phi \cos^4\theta \; d\phi\, d\rho\, d\theta$$

$$= \frac{R^7}{7}\int_0^{2\pi} \cos^4\theta\, d\theta \int_0^{\pi} \sin^5\phi\, d\phi = \ldots\ldots \text{(rest is straightforward)}$$

$$= R^7/7 \left(3\pi/4\right)\left(16/15\right) = 4\pi R^7/35$$

R32

11 # PrepFinal Question B4

Compute the flux $\int_S \mathbf{F}\cdot\mathbf{n}d\sigma$, S where S is the hemisphere $x^2 + y^2 + z^2 = 4$, $z \geq 0$, $\mathbf{n}$ points toward the origin and $\mathbf{F} = (x(z-y), y(x-z), z(y-x))$.

$$\iint_S \vec{F}\cdot\hat{n}\, d\sigma = -\iiint \nabla\cdot F\, dV,$$ use "−" because $\hat{n}$ is the inward normal

$$= -\iiint (z-y) + (x-z) + (y-x)\, dV$$

$$= \iiint 0\, dV$$

$$= 0$$

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PrepFinal Question B5

Compute the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the curve given by the intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = -y$, counterclockwise when viewed from above, and $\mathbf{F} = (x^2 + y, x + y, 4y^2 - z)$.

$r = \cancel{\ } \ u \ v$

$$\Gamma = \int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\,d\sigma, \text{ by stokes thm}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y & x+y & 4y^2-z \end{vmatrix} = \begin{bmatrix} 8y \\ 0 \\ 0 \end{bmatrix}, \quad \vec{n} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \nabla\times\vec{F}\cdot\hat{n} = 8xy$$

$x^2+y^2+(-y)^2 = 4$, so $x^2+2y^2=4$, so let $x = r\cos\theta = rc$

$y = \sqrt{2}\, r\sin\theta = \sqrt{2}\, r s$

$$\iint_S 8xy\,d\sigma = \int_0^{2\pi}\int_0^2 \sqrt{2}\,8 r^2 cs\,dr\,d\theta = \sqrt{2}\,8\int_0^{2\pi}\cos\theta\sin\theta\,d\theta\int_0^2 r^2\,dr$$

$$= \sqrt{2}\,8\,(\quad 0 \quad)\int_0^2 r^2\,dr$$

$$= 0$$

EASIER METHOD: $\vec{n} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, with S being surface $y+z=0$

$\Rightarrow \nabla\times F\cdot\vec{n} = \vec{0} \Rightarrow \Gamma = 0$,

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16.7 Stokes' Theorem

Curl describes the tendency a fluid has to turn at a specific point. Stokes' Theorem states that:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, d\sigma, \quad F = F(x,y,z)$$

Note that curve C must be closed

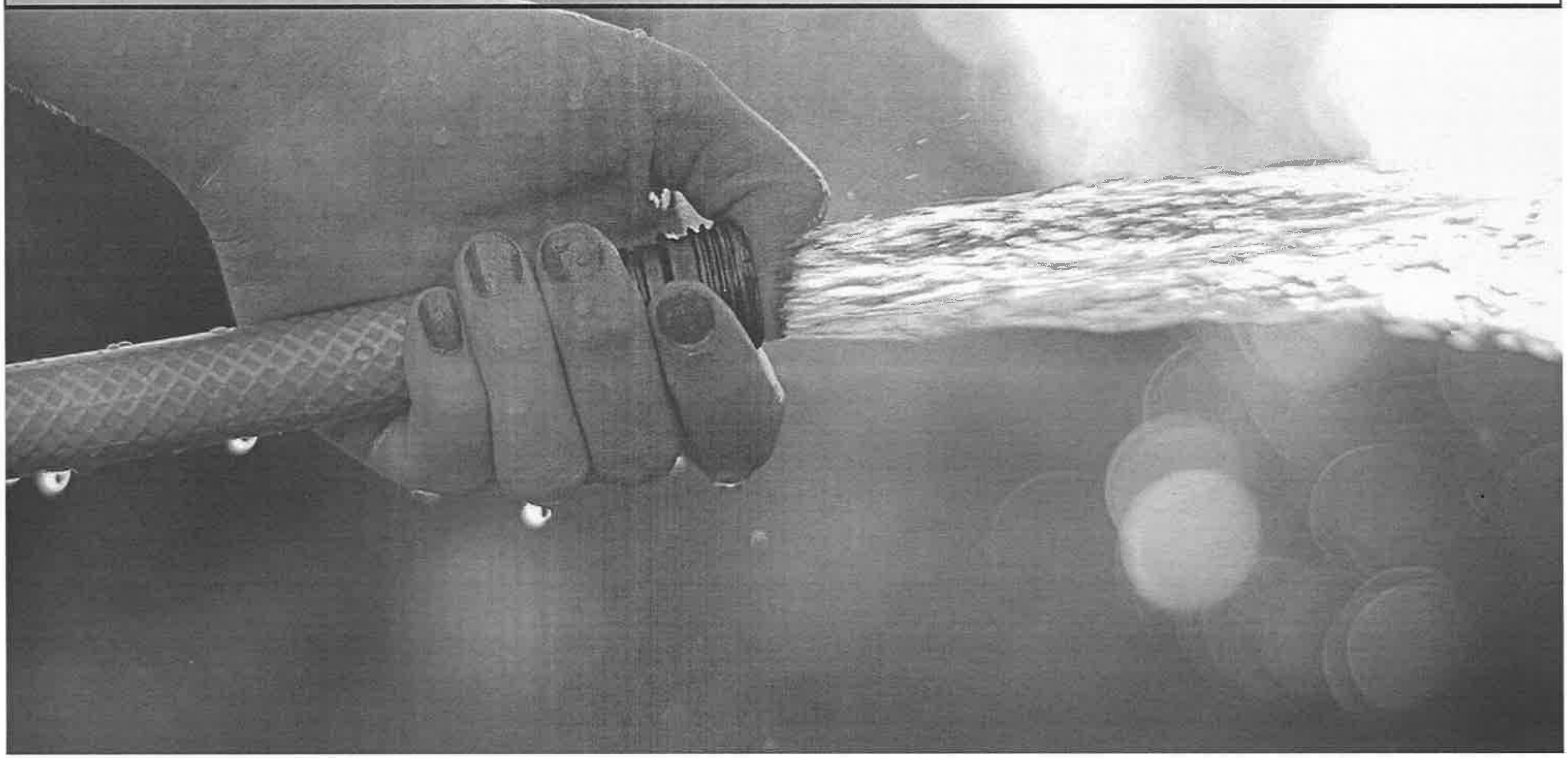
Stokes' theorem can be used to calculate work and flux.

Historical note: Stokes' theorem is named after Sir George Stokes, but was discovered by Sir William Thomson.

16.8 What is Divergence?

Divergence describes the tendency a fluid has to expand/compress.

Water is (approximately) an incompressible fluid. If you place your thumb at the end of a hose, the speed of the water increases, because incompressible, or because $\nabla \cdot \vec{v} = 0$.



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16.8 The Divergence Theorem

The divergence theorem states that

$$\text{flux} = \iint_S \vec{v} \cdot \vec{n} \, d\sigma = \iiint_V \nabla \cdot \vec{v} \, dV$$

n = unit outward normal

$\nabla \cdot \vec{v}$ = divergence

S = closed surface