ON CLOSEST-POINT MAPS IN BANACH SPACES

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CHAPTER 0

INTRODUCTION

This thesis is an investigation of closest-point maps in Banach spaces. Our primary intent here is to relate the properties of the closest-point maps to the geometrical structure of the unit ball of the space, and in so doing we place particular emphasis on the relationship between the structure of the unit ball and the behavior of the product of two of these closest-point maps. The following is a very brief abstract of some of the results contained in the text.

In Chapter I, we discuss briefly some characterizations of inner-product spaces which prove useful in the sequel. From some well known characterizations, we derive others which apparently are not so well known. In particular, we show that, for $n \geq 3$, an n-dimensional normed linear space admits an inner product if and only if every k-dimensional subspace admits a projection of unit norm for some k, $2 \leq k \leq n-1$. This seems to be a non-trivial extension of Kakutani's well known theorem which states that this fact is true for $k = 2$. We use this result to show that if for some k, $0 < k < n$, the closest-point map shrinks distance for every k-dimensional subspace in an n-dimensional strictly convex normed linear space with $n \geq 3$, then the space is an inner-product space.

In Chapter II, we let P_A denote the closest-point map on the subspace A, and we consider the sequence of iterates $\left\{(P_A P_B)^n(x)\right\}$

between two subspaces, A and B. We show that E is a complete inner-product space if and only if these mappings are defined and the sequence converges to $P_{A \cap B}(x)$ for every x in E. We thereby prove the converse of a theorem due to von Neumann. We also investigate the behavior of this sequence in finite-dimensional strictly convex normed linear spaces, and show that every such sequence converges to a point in $A \cap B$ if and only if the space is smooth. We further show that the iterates always converge in two- or three-dimensional spaces, and we give an example to show that they do not always converge in infinite-dimensional spaces.

In Chapter III, we continue our investigation of the iterates of the product of two closest-point maps; however, here we do not restrict ourselves to subspaces. We consider the closest-point maps defined for arbitrary convex sets, and we show that if A and B are two closed convex sets, with at most one point in common, in a finite-dimensional normed linear space and P_A and P_B their respective closest-point maps, then the sequence of iterates $\{(P_A P_B)^n(x)\}$ converges to a point in A whose distance from B is equal to the distance between A and B if either A or B is compact and either is strictly convex. We also show that if K_j, $j = 1,\ldots,N$, are closed convex sets in a complete inner-product space and at least one of these sets is compact, then the sequence of iterates $\{(P_{K_1} \cdots P_{K_N})^n(x)\}$ converges for all x. These results extend some ideas introduced by Cheney and Goldstein in their short note [3].

In Chapter IV, we prove, for any strictly convex, smooth, finite-dimensional normed linear space E, that

$$\lim (I - P_A)(I - P_B) \ . \ . \ . \ (I - P_A)(I - P_B)(x) = (I - P_{A+B})(x)$$

for every $x \in E$ and for every pair of subspaces A and B. This answer negatively a question, raised by Hirschfeld eight years ago, which has been the subject of some recent investigation by Klee [13].

In Chapter V, we investigate an unsolved problem posed by Cheney: Let A be a subspace of n-dimensional Euclidean space E_n, and let x be any element of E_n. If x_p is the point in A closest to x in the l_p norm, then Cheney has asked what can be said about the convergence of the sequence $\{x_p\}$. We show that when A is either one-dimensional or a hyperplane, the sequence converges, and thus answers his question in the three-dimensional case.

Throughout this thesis, we tacitly assume that all spaces are taken over the real field; however, in many cases the corresponding result follows easily when the field is complex by considering the real restriction of the space.

CHAPTER I

CHARACTERIZATIONS OF INNER-PRODUCT SPACES

In this chapter, we mention briefly some of the better known and more useful methods of characterizing inner-product spaces, and we give two characterizations which apparently are not quite so well known. Our intent here is, for the most part, to familiarize the reader with some basic techniques for analyzing the structure of Banach spaces. These techniques will allow him to more fully appreciate later developments. The first of our characterizations is due to Jordan and von Neumann [11]. It states that a normed linear space admits an inner product if and only if the norm satisfies the equality $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$ for all x and y. This of course implies that a normed linear space admits an inner product if and only if every two-dimensional subspace does. Several refinements of this idea have been investigated (see Day [4]).

Definition. A normed linear space E is smooth if and only if each point on the surface of the unit ball S has a unique supporting hyperplane.

Definition. A vector x is orthogonal to a vector y (written $x \perp y$) if and only if $||x|| \leq ||x + \alpha y||$ for all scalars α. A vector is orthogonal to a subset if and only if it is orthogonal to each element of the subset.

This definition specializes to the usual definition of orthogonality if the space is an inner-product space.

Birkhoff [2] showed that a normed linear space which is smooth admits an inner product if and only if orthogonality is symmetric, i.e., x orthogonal to y implies y orthogonal to x. James [9] removed the restriction of smoothness and gave several other useful and related characterizations of inner-product spaces. This concept of orthogonality was investigated by Day [5] in two-dimensional spaces.

The third and final characterization that we mention is the well-known theorem of Kakutani (see [12]) which states that if every two-dimensional subspace of a normed linear space E, whose dimension n is greater than or equal to three, admits a projection of norm one, then E is an inner product space. We discuss this characterization more fully than the others and prove the following:

Theorem 1.1. If for some k, $2 \leq k \leq n-1$, every k-dimensional subspace admits a projection of unit norm, then E is an inner-product space when $n \geq 3$.

Proof. According to the Jordan-von Neumann representation just mentioned, we need only show that every $(k+1)$-dimensional subspace is an inner-product space. Let E_1 be any $(k+1)$-dimensional subspace and let H by any hyperplane in E_1. H is k-dimensional and by hypotheses there must be a projection P of unit norm mapping E onto H. Let P_1 be the restriction of P to E_1. Then P_1 is a projection of norm one of E_1 onto H. Let x be any element of E_1-H, and let $z = x - P(x)$. If h is any element of H, we will show that h is orthogonal to z, i.e., $||h + \alpha z|| \geq ||h||$ for all scalars α. Write

$h = (h + \alpha z) - \alpha z$ and apply P to both sides of this equation. Noting that $||P|| = 1$, we have $||h|| = ||P(h)|| = ||P(h + \alpha z)|| \leq ||h + \alpha z||$ as required. Therefore, every hyperplane in E_1 is orthogonal to some non-zero element of E_1. James has shown ([9, Theorem 4]) that if every hyperplane is orthogonal to some non-zero element in a space of at least three dimensions, then the space is an inner-product space. Hence E_1 admits an inner product, and, since E_1 is an arbitrary $(k+1)$-dimensional subspace of E, E must be an inner-product space.

The preceding proof also shows that if for some k, $1 \leq k \leq n-2$, every subspace of deficiency k admits a projection of unit norm, then E is an inner-product space. This of course is useful in the infinite-dimensional case.

Definition. A closed convex set in a normed linear space is strictly convex if and only if its boundary contains no line segment, and a normed linear space is strictly convex if and only if its unit ball $S = \{x: ||x|| \leq 1\}$ is strictly convex.

Definition. If K is a subset of E, let $P_K(x) = \{y \in K: ||x - y|| = \inf_{k \in K} ||x - k||\}$. If P_K is single valued for each x, we will call it the closest-point map on K.

Noting that the closest-point map is well-defined for each subspace of a strictly convex finite-dimensional space and using Kakutani's characterization of inner-product spaces, one can show that if for some k, $0 < k < n-1$, every k-dimensional subspace has a linear closest-point map, then the space is an inner-product space. This was shown directly by Rudin and Smith in [19].

<u>Definition</u>. A mapping P <u>shrinks distance</u> if and only if $||P(x) - P(y)|| \leq ||x - y||$ for all x and y.

Phelps [18] has shown that if the closest-point map, P, shrinks distance for all one-dimensional subspaces of a space which is at least three-dimensional, the space is an inner-product space. The proof is evident because this hypothesis implies, immediately, that orthogonality is symmetric -- a condition which is sufficient according to the Birkhoff-James characterization just mentioned. Hirschfeld [8] has shown that the same conclusion can be drawn if the closest-point map on all two-dimensional subspaces shrinks distance. However, these are only special cases of the following:

<u>Theorem 1.2</u>. Let E be strictly convex and n-dimensional with $n \geq 3$. If for some k, $0 < k < n$, the closest-point map on every k-dimensional subspace shrinks distance, then the space E is an inner-product space.

<u>Proof</u>. If $k \geq 2$, let E_1 be a $(k + 1)$-dimensional subspace of E and H a k-dimensional subspace of E_1. Let x be any element in $E_1 - H$, and let $z = x - P_H(x)$. Then all vectors in E_1 orthogonal to H are scalar multiples of z by Lemma 2.2.2. Hence P_H is a linear mapping, and, therefore, a projection of E_1 onto H. Since the closest-point map shrinks distance, $||P_H|| \leq 1$. Hence every k-dimensional subspace of E_1 admits a projection of unit norm, and, by Theorem 1.1, E_1 must be an inner-product space. Since every $(k+1)$-dimensional subspace of E is an inner-product space, E must be too. For $k = 1$, the result follows from the previous discussion.

The interested reader should consult Klee's review of [19] for additional information.

CHAPTER II

CLOSEST-POINT MAPS AND THEIR PRODUCTS

2.0 Introduction. Let A and B be subspaces of a normed linear space E and let P_A and P_B be their respective closest-point maps. In this chapter, we attempt to show a relation between the structure of the unit ball of E and the convergence of the sequence $\{(P_AP_B)^n(x)\}$. In Section 2.1 we give the definitions and background basic to our development. In Section 2.2 we show that the closest-point map is well defined for every subspace and $\lim (P_AP_B)^n(x) = P_{A\cap B}(x)$ for every $A,B,$ and x if and only if E is a complete inner-product space. In Section 2.3 we assume that E is strictly convex and finite-dimensional and show the following: (1) If A and B are subspaces of E and E is smooth, there exists a constant k, $0 \leq k < 1$, such that $||P_AP_B(x) - P_B(x)|| \leq k||P_B(x) - x||$ for every $x \in A$. (2) The sequence $\{(P_AP_B)^n(x)\}$ converges to a point in $A \cap B$ for all A, B, and x if and only if E is smooth. (3) If the dimension of E is less than or equal to three, the iterations $(P_AP_B)^n(x)$ either converge to a point in the intersection of A and B or repeat after two steps. In Section 2.4 we show that there is at least one case where the iterates $(P_AP_B)^n(x)$ do not converge. This is in the infinite-dimensional space l_1.

2.1 General. In the following, we let E denote a normed linear space, and we use the notation introduced in the previous chapter.

Using the weak compactness of S in a reflexive space, we see by a direct proof that $P_A(x)$ is non-empty for all subspaces A. Phelps [17, p. 253] has shown the converse: If $P_A(x)$ is non-empty for all x and for all subspaces, then the space is reflexive. If E is both reflexive and strictly convex, we see immediately, from the definition of strictly convex, that $P_A(x)$ must consist of exactly one point. We will assume the above facts in the sequel, and also assume, unless otherwise stated, that A is a subspace whenever we write P_A.

If E is a complete inner-product space, then P_A is well defined, shrinks distance, and is linear for every subspace A. This is true because P_A is the orthogonal projection of E on A. Unfortunately P_A is not so well-behaved in general. As we mentioned in Chapter I, Phelps [18, Theorem 5.4] has shown that if P_A shrinks distance for all one-dimensional subspaces A of E, and E is at least three-dimensional, then E is an inner-product space; and Hirschfeld [8, Theorem 2] has shown the following:

Lemma 2.1.1. If E is strictly convex, at least three-dimensional, and $P_L(x + y) = P_L(x) + P_L(y)$ for every one-dimensional subspace L, then E is an inner-product space.

2.2 Inner-product spaces. The main result of this section is a converse to the following theorem of von Neuman [16, p. 475]. (Theorem 2.2.1 has also been proved by Weiner [24, p. 101] and generalized by Halperin in [7].) A proof of this theorem appears in Chapter III.

Theorem 2.2.1. (von Neumann). If E is a complete inner-product space, then P_A is single-valued for every subspace A and

$$\lim (P_A P_B \ . \ . \ . \ P_A P_B)(x) = P_{A \cap B}(x)$$

for all x in E.

The reader, at this point, should recall the definition of orthogonality made on page 4.

Lemma 2.2.1 (James [10, Theorem 2.1]). Let A be a subspace of E. Then $x \perp A$ if and only if there is a continuous linear functional x' such that $|x'(x)| = ||x'|| \cdot ||x||$ and $x'[A] = 0$.

Lemma 2.2.2. If H is a hyperplane containing the null vector and E is strictly convex, there is at most one linearly independent vector orthogonal to H.

Proof. Suppose $x, y \perp H$ and $||x|| = ||y|| = 1$. By Lemma 2.2.1, there are unit vectors x', y' in the conjugate space such that $x'(x) = y'(y) = 1$ and $x'[H] = y'[H] = 0$. $x'[H] = y'[H] = 0$ and $||x'|| = ||y'|| = 1$, together, imply that $x' = \beta y'$ for some scalar β with $|\beta| = 1$; so $x'(\beta^{-1}y) = \beta y'(\beta^{-1}y) = 1$. But this means that $||x'(\alpha x + (1 - \alpha)\beta^{-1}y)|| = 1$, and so $||\alpha x + (1 - \alpha)\beta^{-1}y|| = 1$ for all α, $0 < \alpha < 1$. Since E is strictly convex, this implies that $x = \beta^{-1}y$.

Theorem 2.2.2. If E is a strictly convex space of dimension greater than two such that $\lim(P_A P_B \ . \ . \ . \ P_A P_B)(x) = P_{A \cap B}(x)$ for every $X \varepsilon E$ and for every pair A, B of two-dimensional subspaces, then E is an inner-product space.

Proof. According to Lemma 2.1.1, we need only show that $P_L(x + y) = P_L(x) + P_L(y)$ for every one-dimensional subspace L. Let x, y and L be given. Suppose the span M of x, y, L is a three-dimensional subspace. Then the span of x, L and the span of y, L, denoted by A and B respectively, are each two-dimensional. Because the distance in M is inherited from E, we see that $\lim (P_A P_B \ldots P_A P_B)(x) = P_{A \cap B}(x) = P_L(x)$, and that $\lim (P_A P_B \ldots P_A P_B)(y) = P_{A \cap B}(y) = P_L(y)$. Moreover, since A and B are hyperplanes in M and M is strictly convex, Lemma 2.2.2 implies that there is at most one linearly independent vector b, in M, orthogonal to B. Therefore, M can be written as the direct sum of A and scalar multiples of a, and P_A is linear; and a similar argument shows that P_B is linear. Hence $(P_A P_B \ldots P_A P_B)(x + y) = (P_A P_B \ldots P_A P_B)(x) + (P_A P_B \ldots P_A P_B)(y)$. Taking limits and using the above relations, we get $P_L(x + y) = P_L(x) + P_L(y)$.

Suppose the dimension of M is less than three. If M is one-dimensional, then $P_L(x + y) = P_L(x) + P_L(y)$ holds trivially. If M is two-dimensional, we need only select a third linearly independent vector z and repeat the previous argument with A the span of x, y, L and B the span of z, L to show that P_L is additive.

The analogue of Theorem 2.2.2 does not necessarily hold if E is two-dimensional. In fact, Theorem 2.3.2 implies that every strictly convex, smooth, two-dimensional space satisfies the hypotheses of this theorem.

It is of some interest to note the behavior of the iterates in spaces other than inner-product spaces. Generally speaking, this is a very difficult task; however, we can examine the iterations in three-dimensional l_p, $p > 1$, spaces when both subspaces are two-dimensional.

Let A and B be two two-dimensional subspaces in a three-dimensional space, and let A and B be given by their outward normals $a = (a_1,a_2,a_3)$ and $b = (b_1,b_2,b_3)$. According to Fortet [6], any element (x,y,z) orthogonal to A in the l_p sense must satisfy

$$|x|^{p-2}\ xa_1^1 + |y|^{p-2}\ ya_2^1 + |z|^{p-2}\ za_3^1 = 0$$

and

$$|x|^{p-2}\ xa_1^2 + |y|^{p-2}\ ya_2^2 + |z|^{p-2}\ za_3^2 = 0$$

where (a_1^j, a_2^j, a_3^j), $j = 1,2,$ are two linearly independent vectors in A. This implies that

$$(|x|^{p-2}x, |y|^{p-2}y, |z|^{p-2}z)$$

is a scalar multiple of the vector (a_1,a_2,a_3), i.e., for some $k \neq 0$,

$$|x|^{p-2}x = ka_1$$
$$|y|^{p-2}y = ka_2$$
$$|z|^{p-2}z = ka_3\ .$$

Assuming, without loss of generality, that $k = 1$, we find that

$$x = (\text{sign } a_1)|a_1|^{1/(p-1)}$$
$$y = (\text{sign } a_2)|a_2|^{1/(p-1)}$$
$$z = (\text{sign } a_3)|a_3|^{1/(p-1)}$$

with analogous formulas holding for elements orthogonal to B. As we saw in the proof of Theorem 2.2.2, all iterates between the spaces A and B remain in a plane. By our previous discussion, we see that the outward normal of that plane must be

$$\begin{aligned}(&\operatorname{sign}(a_2b_3)|a_2b_3|^{1/(p-1)} - \operatorname{sign}(a_3b_2)|a_3b_2|^{1/(p-1)},\\ &\operatorname{sign}(a_3b_1)|a_3b_1|^{1/(p-1)} - \operatorname{sign}(a_1b_3)|a_1b_3|^{1/(p-1)},\\ &\operatorname{sign}(a_1b_2)|a_1b_2|^{1/(p-1)} - \operatorname{sign}(a_2b_1)|a_2b_1|^{1/(p-1)}).\end{aligned}$$

From this, it is clear from Theorem 2.3.2 that whenever $a_jb_k = 0$ or 1 for all j,k, the iterates converge to the same point in $A \cap B$ for all $p > 1$.

<u>Theorem 2.2.3</u>. Suppose E is at least three-dimensional. If P_A is single-valued for every subspace A, and $\lim (P_AP_B \ldots P_AP_B)(x) = P_{A\cap B}(x)$ for all A, B, and x, then E is a complete inner-product space.

<u>Proof</u>. If P_A is single-valued for every one-dimensional subspace, then E is strictly convex. Therefore, by Theorem 2.2.2, it is only necessary to show that E is complete. Let F be the completion of E and x any element of F. Let $H = \{y : y \in F,\ (y,x) = 0\}$ and let $H_o = H \cap E$. Then H and H_o are hyperplanes in F and E respectively. Let $z \in (E - H_o)$. By hypotheses, there exists an element $P_{H_o}(z)$ in H_o such that $||z - P_{H_o}(z)|| = \inf_{h \in H_o} ||z - h||$. Clearly $(z - P_{H_o}(z)) \perp H_o$ and $(z - P_{H_o}(z), h_o) = 0$ for every $h_o \in H_o$. By continuity of the inner product and by the fact that E is dense in F,

$(z - P_{H_o}(z), y) = 0$ for every $y \varepsilon H$. Because F is strictly convex, there is at most one linearly independent element orthogonal to H. Thus $x = \alpha(z - P_{H_o}(z))$ for some scalar α. Hence $x \varepsilon E$ and, therefore, E is complete.

We would like to mention that Hirschfeld [8] has posed a problem which, on account of the relation of orthogonal complements in a complete inner-product space, can also be considered a converse of Theorem 2.2.1: If P_A exists and is single-valued for every subspace A and

$$\lim (I - P_A)(I - P_B) \ . \ . \ . \ (I - P_A)(I - P_B)(x) = (I - P_{A+B})(x)$$

for all A, B and all $x \varepsilon E$, is E necessarily an inner-product space? Klee [13] showed that this conjecture is not true in two-dimensional spaces, and we show in Chapter IV that it is not true in any finite-dimensional space.

2.3 Smooth spaces. This section will be devoted to showing that if E is finite-dimensional, strictly convex, and smooth, the iterates $(P_A P_B)^n(x)$ always converge to a point in the intersection of A and B.

If x is a non-zero vector in a smooth space, Lemma 2.2.1 implies that there is at most one hyperplane H containing the zero vector with $x \perp H$. James [10] has shown that E is smooth if and only if $x \perp (y + z)$ whenever $x \perp y$ and $x \perp z$ -- a fact which is geometrically evident. Other characterizations of a smooth space are given by Day [4].

Definition. A normed linear space E is uniformly convex if and only if given any number $\varepsilon > 0$, there exists a number $\delta(\varepsilon) > 0$ such that $||x + y|| \leq 2(1 - \delta(\varepsilon))$ whenever $||x|| = ||y|| = 1$ and $||x - y|| \geq \varepsilon$.

The continuity of the norm and compactness of the unit ball imply immediately that any finite-dimensional strictly convex space is uniformly convex. Of course any uniformly convex space is strictly convex.

The following lemma is stated without proof by Klee in [14].

Lemma 2.3.1. If K is any closed convex set of a uniformly convex Banach space E, then P_K is a continuous function.

Proof. Let K be a convex set which does not contain the zero vector, and suppose $P_K(0) = x$ with $||x|| = 1$. Let $\{x_n\}$ be a sequence of vectors with limit 0, and let $\{P_K(x_n)\}$ be their closest points in K. There exists a hyperplane H supporting K and S at x, and we assume that x_n and 0 are on the same side of H. By the continuity of the distance function, $\lim ||x_n - P_K(x_n)|| = 1$, and so there is a sequence $\{\varepsilon_n\}$ with $\lim \varepsilon_n = 0$ such that $||x - P_K(x_n)|| = 1 + \varepsilon_n$. Each $P_K(x_n)$ has norm greater than one, and, since $||P_K(x_n)|| \leq 1 + \varepsilon_n + ||x_n||$ and $\lim x_n = 0$, $\lim||P_K(x_n)|| = 1$. If $\{P_K(x_n)\}$ does not converge to x, we can assume there is some $\varepsilon > 0$ such that $||P_K(x_n) - x|| > \varepsilon$. Noting that $(1/2)(P_K(x_n) + x)$ lies on the side of H opposite 0, and that $\lim||P_K(x_n) - [1/||P_K(x_n)||](P_K(x_n))|| = 0$, we see that $\lim || 1/2(P_K(x_n)/||P_K(x_n)|| + x)|| = 1$; and this contradicts the uniform convexity of E.

Lemma 2.3.2. Let $\{x_n\}$ be a sequence of unit vectors in a uniformly convex space E, and let $\{H_n\}$ be a sequence of hyperplanes such that H_n supports the unit ball S at x_n. If $y_n \varepsilon H_n$ and $\lim ||y_n|| = 1$, then $\lim (x_n - y_n) = 0$.

Proof. The essential ideas needed for this proof are contained in the proof of Lemma 2.3.1, and, therefore, will be omitted here.

Lemma 2.3.3. Let E be a normed linear space and $\{x_n\}$ a sequence of vectors convergent to x. If each x_n is orthogonal to a subset A of E, then x is orthogonal to A.

Proof. Since $x_n \perp A$, $||x_n|| \leq ||x_n + \alpha y||$ for all α and for all $y \varepsilon A$. By continuity of the norm, $||x|| \leq ||x + \alpha y||$ also holds for all α and for all $y \varepsilon A$.

Lemma 2.3.4. If A is any subspace of E, then $\alpha P_A(x) = P_A(\alpha x)$ for all scalars α.

Proof. $P_A(x)$ satisfies $||x - P_A(x)|| \leq ||x - y||$ for every $y \varepsilon A$. By multiplying both sides of this inequality by $|\alpha|$, we see that $||\alpha x - \alpha P_A(x)|| \leq ||\alpha x - \alpha y||$ holds for every $y \varepsilon A$. If $a = 0$, the result is trivial; if $\alpha \neq 0$, the last inequality is equivalent to $||\alpha x - \alpha P_A(x)|| \leq ||\alpha x - z||$ for all $z \varepsilon A$. Hence $P_A(\alpha x) = \alpha P_A(x)$.

Theorem 2.3.1. Let E be finite-dimensional, strictly convex, and smooth. Given any two subspaces, A and B, there exists a number k $0 \leq k < 1$, such that $||P_A P_B(x) - P_B(x)|| \leq k||P_B(x) - x||$ for all $x \varepsilon A$.

Proof. Assume, without loss of generality, that A and B span E. Since $x \varepsilon A$ and $P_A P_B(x)$ is the point in A closest to $P_B(x)$, $||x - P_B(x)|| \geq ||P_A P_B(x) - P_B(x)||$. Furthermore, since the space is strictly convex, either $x = P_A P_B(x)$ or $||x - P_B(x)|| > ||P_A P_B(x) - P_B(x)||$.

If $x = P_A P_B(x)$, there are two distinct possibilities. Either $x = P_B(x)$ or $x \neq P_B(x)$. If $x = P_B(x)$, $P_A P_B(x) = P_B(x)$ and we may choose k arbitrarily. If $x \neq P_B(x)$, $x - P_B(x)$ is a non-zero vector orthogonal to both A and B. Since E is smooth, Lemma 2.2.1 implies there is a unique hyperplane H containing the zero vector such that $(x - P_B(x)) \perp H$. Since $x - P_B(x)$ is orthogonal to A and to B, both of these subspaces must be contained in H. Hence E, being the span of A and B, must be contained in H and this is impossible.

Therefore, if the theorem is not true, there exists a sequence $\{x_n\}$ of elements in A such that

$$\lim \frac{||P_B(x_n) - P_A P_B(x_n)||}{||x_n - P_B(x_n)||} = 1.$$

Let

$$\alpha_n = \frac{1}{||P_B(x_n) - P_A P_B(x_n)||}$$

then

$$\lim \frac{||P_B(x_n) - P_A P_B(x_n)||}{||x_n - P_B(x_n)||} = \lim \frac{||\alpha_n P_A P_B(x_n) - \alpha_n P_B(x_n)||}{||\alpha_n x_n - \alpha_n P_B(x_n)||} = 1 .$$

Because the closest-point map is homogeneous (Lemma 2.3.4), this is equivalent to

$$\lim \frac{||P_A P_B(\alpha_n x_n) - P_B(\alpha_n x_n)||}{||\alpha_n x_n - P_B(\alpha_n x_n)||} = 1 .$$

It follows from Lemma 2.3.2 that

$$\lim \left\{[P_A P_B(\alpha_n x_n) - P_B(\alpha_n x_n)] - [\alpha_n x_n - P_B(\alpha_n x_n)]\right\} = 0 .$$

Moreover, each point in the sequence

$$\left\{P_A P_B(\alpha_n x_n) - P_B(\alpha_n x_n)\right\}$$

is orthogonal to A, and each point in the sequence

$$\left\{\alpha_n x_n - P_B(\alpha_n x_n)\right\}$$

is orthogonal to B. Therefore, since we may assume without loss of generality that each of these sequences converges because of the finite dimensionality of E, Lemma 2.3.2 and Lemma 2.3.3, together, imply that their common limit is a non-zero vector orthogonal to both A and B. As before, this is impossible.

If Theorem 2.3.1 were true in infinite-dimensional spaces, we could handle the convergence problem there. This is not the case however. To see this, we give the following simple example in the complete inner-product space l_2.

Let A be the subspace spanned by $(1, 1/2, 0,\ldots)$, $(0,0,1/3^3,1/4,0,\ldots),\ldots,(0,\ldots,0,1/(2n+1)^{(2n+1)}, 1/(2n+2),0,\ldots)$, and let B be the subspace spanned by $(0,1/2,0,\ldots),(0,0,0,1/4,0,\ldots),\ldots$. Selecting an element $(0,\ldots,0,1/n^n,1/(n+1),0,\ldots)$ from A and projecting orthogonally to B and then back to A, we observe that we come back to a scalar multiple of $(0,\ldots,0,1/n^n,1/(n+1),0,\ldots)$. Thus, by normalizing these vectors and performing this operation for large integers, one can see, from the following argument that Theorem 2.3.1 cannot apply in l_2. Let x and y be any two unit vectors situated at the origin in E_2, and let ζ denote the smaller angle between them. If we project orthogonally from x to y and then back to the line generated by x, we find that the ratio of $P_y(x) - x$ to $P_xP_y(x) - P_y(x)$ is $1/\cos \zeta$ whose limit as ζ approaches zero is one.

To simplify notation, we will use the following in the sequel. For $x \in E$, let

$$x_1 = P_B(x)$$
$$x_2 = P_AP_B(x)$$
$$\cdots\cdots\cdots$$
$$x_{2n-1} = P_B(x_{2n-2})$$
$$x_{2n} = P_A(x_{2n-1}) \;.$$

<u>Theorem 2.3.2</u>. If E is strictly convex and finite-dimensional, the sequence of iterates $\{x_n\}$ converges to a point in $A \cap B$ for every $x \in E$ and for every pair of subspaces A and B if and only if E is smooth.

<u>Proof</u>. Suppose E is smooth. According to Theorem 2.3.1, there exists a k, $0 \leq k < 1$, such that $k||x_{n-1} - x_n|| \geq ||x_{n+1} - x_n||$ for $n > 1$. If $m > n$,

$$\begin{aligned} ||x_n - x_m|| &\leq ||x_n - x_{n+1}|| + \ldots + ||x_{n+1} - x_m|| \\ &\leq k^{n-1}||x_2 - x_1|| + \ldots + k^{m-2}||x_2 - x_1|| \\ &\leq k^{n-1}||x_2 - x_1|| \frac{1}{1-k} . \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence whose limit must be in both A and B.

Conversely, if E is not smooth, there exists some unit vector x and two hyperplanes, H_1, H_2, each supporting S at x. It will be shown that there is a vector z whose iterates do not converge to a point in the intersection of the subspaces A and B, where $A = H_1 - x$ and $B = H_2 - x$ are the translates of H_1 and H_2, respectively. Since E is strictly convex and A and B are hyperplanes, any vector orthogonal to either A or B is a scalar multiple of x by Lemma 2.2.2. Thus, if z is any point in A but not in B, both $P_B(z) - z$ and $P_A P_B(z) - P_B(z)$ are scalar multiples of x. This implies that $P_A P_B(z) = z$.

At this point, we note that many questions arise naturally; e.g., are there necessary or sufficient conditions which, when imposed upon a space of arbitrary dimension, will insure convergence? In particular, will the iterates always converge when the space is both uniformly convex and smooth? Will they always converge in a finite-dimensional strictly convex space? We are unable to answer these questions, but the following discussion suggests that we might expect convergence if the space is finite-dimensional and $A \cap B = \{0\}$.

<u>Theorem 2.3.3</u>. Suppose E is finite-dimensional and A and B are subspaces satisfying $A \cap B = \{0\}$. For every positive number m, there exists some number M such that $||x||, ||y|| \leq M$ whenever $x \in A$, $y \in B$, and $||x - y|| \leq m$.

<u>Proof</u>. Let $R_1 = \{x \in A : ||x - y|| \leq m \text{ for some } y \in B\}$ and $R_2 = \{x \in B : ||x - y|| \leq m \text{ for some } y \in A\}$. It suffices (see Banach [1, p. 80]) to show that for any linear functional f,

$\sup_{x \in R_1} |f(x)| < \infty$. Let f be in A^o, the annihilator of A. Then if $y \in R_2$, $x \in A$, and $||x - y|| \leq m$, $|f(y)| = |f(x - y)| \leq ||f||m$ and $\sup_{y \in R_2} |f(y)| < \infty$. A similar argument shows that $\sup_{x \in R_1} |f(x)| < \infty$ if $f \in B^o$, the annihilator of B. Since $A \cap B = \{0\}$, the span of $A^o \cup B^o$ is E^*, the conjugate of E. Thus, there exists a finite number of linear functionals f_k, $k = 1, \ldots, r$, with $f_k \in A^o$ or $f_k \in B^o$ such that any linear functional f may be written

$$f = \sum_{k=1}^{r} a_k f_k .$$

Let $x \in R_1$, $y \in B$, and $||x - y|| \leq m$. Then

$$f(x) = \sum_{k=1}^{r} a_k f_k(x)$$

$$= \sum_{j=1}^{s} a_{k_j} f_{k_j}(x - y)$$

where $f_{k_j} \in B^o$ for $j = 1, \ldots, s$. Hence

$$|f(x)| \leq \sum_{j=1}^{s} |a_{k_j}| \; ||f_{k_j}|| \cdot ||x - y||$$

$$\leq m \sum_{k=1}^{r} |a_k| \; ||f_k||$$

and, therefore, $\sup_{x \in R_1} |f(x)| < \infty$. Similarly, $\sup_{y \in R_2} |f(y)| < \infty$.

With our usual notation for the sequence of iterates $\{x_n\}$, $||x_n - x_{n+1}|| \leq ||x_2 - x_1||$ for all $n > 1$. Thus, Theorem 2.3.3 implies that the sequence $\{x_n\}$ is bounded whenever $A \cap B = \{0\}$. Hence, if the iterations do not converge when E is strictly convex, finite-dimensional and $A \cap B = \{0\}$, a direct proof, using Lemma 2.3.1, shows that there must be at least two linearly independent vectors orthogonal to both A and B. However, much more than this can be said. In fact, if the sequence $\{x_n\}$ does not converge, we can show that the set C of limit points of the sequence $\{x_{2n}\}$ must be a continuum, and hence, that there are uncountably many vectors, any two of which are linearly independent, orthogonal to both A and B. The set C is closed and bounded. If C is not connected, it is the disjoint union of two non-empty compact sets C_1 and C_2, respectively. Since $D_1 \cup D_2$ contains all limit points of $\{x_{2n}\}$, $D_1 \cup D_2$ contains all but a finite number of points from the sequence $\{x_{2n}\}$. But this contradicts the fact that an infinite number of these points have to lie in both D_1 and D_2 and that $\lim ||x_{2n} - x_{2n+2}|| = 0$. Hence C must be a continuum.

Since each point of C is a fixed point of $P_A P_B$ and since $||c - P_B(c)||$ is constant for all $c \in C$, in the case under discussion there must be an uncountable number of unit vectors orthogonal to both A and B.

Definition. Let x be a point on the surface of the unit ball S. If there is exactly one supporting hyperplane for S at x, then x will be called a smooth point. If the intersection of all hyperplanes containing x and supporting S is x, then x will be called a <u>vertex</u> of S.

<u>Theorem 2.3.4</u>. Let E be strictly convex and finite-dimensional and suppose $A \cap B = \{0\}$. If each point x on the surface of the unit ball S is either a smooth point or a vertex, then the sequence of iterates $\{(P_A P_B)^n(x)\}$ converges.

<u>Proof</u>. The remarks following Theorem 2.3.3 imply that there are an uncountable number of points on S which are not smooth points. Since S can have at most a countable number of vertices (see Valentine [23, Theorem 11.2]), the sequence must converge.

If E is a normed linear space and B a subspace of E, let E/C represent the quotient space modulo B, i.e., the normed linear space consisting of all equivalence classes $[x]$ with $||[x]|| = \inf_{c \in C} ||x + c||$.

<u>Theorem 2.3.5</u>. Let B and C be subspaces of a strictly convex Banach space E, and let B contain C. Then B/C is a subspace of E/C, and, if $x \in E$ and $P_B(x) = y$, then $P_{B/C}([x]) = [y]$ where $[\cdot]$ indicates an element of E/C.

<u>Proof</u>. It is clear that B/C is a linear manifold of E/C, and by Lemma II.1.1 of [4], a quotient space of a complete space is complete. This implies that B/C is closed in E/C and, hence, a subspace.

Suppose $x \in E$ and $P_B(x) = y$. Then $||[x] - [y]|| = ||[x - y]|| = \inf_{c \in C} ||(x - y) + c||$. Since $x - y$ is orthogonal to C, $||[x] - [y]||$ $||[x - y]||$. If z is any element of B, then $||[x] - [z]|| = ||[x - y|| = \inf_{c \in C} ||(x - z) + c|| = \inf_{c \in C} ||x + (c - z)|| \geq \inf_{b \in B} ||x - b|| =$

$||x - y||$, and $\inf_{c \varepsilon C} ||x + (c - z)|| = ||x - y||$ if and only if $z - c = y$ for some $c \varepsilon C$ since E is strictly convex. This proves the theorem.

Using ideas similar to those used in Theorem 2.3.5, we can easily prove the following:

Theorem 2.3.6. If every two-dimensional quotient space is strictly convex, then E is strictly convex. If E is reflexive and strictly convex, then E/C is strictly convex for all subspaces C of E.

Proof. If E is not strictly convex, there are elements x and y in E such that $0, y \varepsilon P_{Ry}(x)$ (R denotes the set of real numbers). Since $x \perp y$, there exists a hyperplane H_1 containing 0 and y such that $x \perp H_1$. Let H_2 be any hyperplane such that $y \perp H_2$, and let $C = H_1 \cap H_2$. Then C has deficiency two, and this implies that the quotient space E/C is two-dimensional. Since $x \perp C$ and $x \perp H_1$, $||[x] - [0]|| = \inf_{c \varepsilon C} ||x + c|| = ||x||$, $||[x] - [y]|| = \inf_{c \varepsilon C} ||(x - y) + c|| = ||x - y|| = ||x||$, and $||[x] - \alpha[y]|| = \inf_{c \varepsilon C} ||(x - \alpha y) + c|| \geq \inf_{h \varepsilon H_1} ||x - h|| = ||x||$ for all scalars α. These relations show that E/C is not strictly convex.

Suppose that E/C is not strictly convex for some subspace E. Then there are two vectors $[x]$ and $[y]$ in E/C such that $||[x]|| = 1$, $||x|| = 1$, $[x] \perp R[y]$, and $||[x] - \alpha[y]|| = 1$ for all α, $0 \leq \alpha \leq 1$. Since E is reflexive, there exists a point c_o in C such that $||x + (c_o - y)|| = \inf_{c \varepsilon C} ||x + (c - y)|| = 1$. If α is any number between zero and one, then $||x + \alpha(c_o - y)|| \geq 1$ because $\alpha(y - c_o) \varepsilon \alpha[y]$, and, since S is convex, $||x + \alpha(c_0 - y)|| \leq 1$, for all such α; therefore, $|| x + \alpha(c_0 - y)|| = 1$ if $0 < \alpha \leq 1$ and E is not strictly convex.

We now prove the dual of Theorem 2.3.6.

<u>Theorem 2.3.7</u>. If A is a subspace of a smooth reflexive space E, then E/A is smooth. If the quotient space E/A is smooth for every subspace of deficiency two, then E is smooth. Thus, if every two-dimensional quotient space is smooth, E is smooth.

<u>Proof</u>. It is well known that smoothness and strict convexity of the conjugate space E^* imply the dual property in the space E, and that these two properties are actually dual properties in a reflexive space ([4, p. 112]). Furthermore, the conjugate space $(E/A)^*$ is linearly isometric to $E^* \cap A^o$ ([4, p. 25]). Therefore, if E is smooth, E^* is strictly convex. This implies that $E^* \cap A^o$ is strictly convex. Because $(E/A)^*$ is linearly isometric to a strictly convex space, it too must be strictly convex. Therefore E/A is smooth.

Let A^o be any two-dimensional subspace of E^*, and let A be the null space of A^o, i.e., $A = \{x \in E : x'(x) = 0$ for every $x' \in A^o\}$. Then A has deficiency two and A^o is the annihilator of A. Since $(E/A)^*$ is linearly isometric to A^o and E/A is reflexive, A^o must be strictly convex. Since every two-dimensional subspace of E^* is strictly convex, E^* is strictly convex. This implies that E is smooth.

<u>Theorem 2.3.8</u>. Let E be a uniformly convex Banach space, let A be a subspace of E, and let $A_{\perp} = \{x \in E : ||x|| = 1, \ x \perp A\}$.

Then $A_\perp$ is homeomorphic to the surface of the unit ball of the quotient space E/A.

Proof. If $[x] \varepsilon E/A$ and $||[x]|| = 1$, there exists a unique element a_x, of A, such that $||x - a_x|| = 1$. Since $||x - a|| \geq 1$ for all $a \varepsilon A$, $(x - a_x)$ is orthogonal to A. Let h be the mapping of the surface of the unit ball of E/A into $A_\perp$ such that $h([x]) = x - a_x$ for each $[x] \varepsilon E/A$. Clearly h is onto; for if p is any unit vector orthogonal to A, then $[p]$ is an element of E/A with unit norm and $h([p]) = p$. Suppose $h([y]) = h([x])$. Then $y - a_y = x - a_x$, $y - x = a_y - a$, and so $[y] = [x]$. Hence h is a one-to-one mapping of the unit ball of E/A onto $A_\perp$. It is immediately evident that h^{-1} is continuous. To see that h is continuous, and hence complete the proof, let $\{[x_n]\}$, $||x_n|| = 1$, be a sequence on the surface of the unit ball of E/A such that $\lim [x_n] = [x]$ and $||x|| = 1$. Let y_n be the unique element in $[x_n]$ such that $||x - y_n||$ is the distance from the point x to the linear variety $[x_n]$. Then $\lim \ ||x - y_n|| = 0$, and so $\lim || y_n || = 1$. Since E is uniformly convex, Lemma 2.3.2 implies that $\lim ||x_n - y_n|| = 0$. This of course implies that $\lim \ x_n = x$. It follows immediately that h is continuous.

If E is strictly convex, finite-dimensional, and $C = A \cap B$, we see by applying Theorem 2.3.3 to the quotient space E/C that there exists some number M such that $\inf ||x - x_n|| \leq M$ for all n.

Theorem 2.3.9. Let E be a strictly convex finite-dimensional normed linear space and let either A or B be a hyperplane. With the previous notation, the sequence of iterates $\{x_n\}$ either converges to a point in $A \cap B$ or repeats after two steps.

Proof. Suppose A is a hyperplane. Then by Lemma 2.2.2, there is exactly one linearly independent vector, p, orthogonal to A, and if $x_1 \in B$, $x_2 - x_1 = \alpha p$ for some scalar α. Consider the ratio

$$\nabla_n = \frac{||x_{n+1} - x_n||}{||x_n - x_{n-1}||} .$$

If there exists a k, $0 \leq k < 1$, such that $\nabla_n \leq k$ for all n, we see from the proof of Theorem 2.3.2 that $\{x_n\}$ must converge to a point in $A \cap B$. Otherwise, there exists a sequence $\{n_k\}$ of integers such that $\lim \nabla_{n_k} = 1$. By the method used in the proof of Theorem 2.3.1, we can find a vector which is orthogonal to both A and B. Since all vectors orthogonal to A are scalar multiples of p, $p \perp B$. This means that $x_3 = x_1$.

An immediate consequence of Theorem 2.3.9 is the following:

Corollary 2.3.10. If E is strictly convex and of dimension less than or equal to three, the iterates $\{x_n\}$ either converge to a point in $A \cap B$ or repeat after two steps.

2.4. A divergent iteration. The following example shows that the prescribed iterations do not always remain bounded -- even when the subspaces have only the zero vector in common and the mapping is single valued at each step.

Let l_1 be the space of sequences $x = (x_i)$ with

$$||x|| = \sum_{i=1}^{\infty} |x_i| < \infty ,$$

and let A and B be the smallest subspaces containing the sets $\{e_1, e_3, e_5, \ldots\}$ and $\{e_2, e_4, e_6, \ldots\}$, respectively, where $e_n = (1, 1/2, 1/3, \ldots, 1/n, 0, \ldots)$. Under these conditions, we will show that $P_B(e_{2n-1}) = e_{2n}$ and $P_A(e_{2n}) = e_{2n+1}$ for all n.

Let M be the set of all sequences $m = (m_i)$ with $||m|| = \sup_i |m_i|$. It is well known that the conjugate of l_1 is isometric to M and that m' is a continuous linear functional on l_1 if and only if there is an $m \in M$ such that

$$m'(x) = \sum_{i=1}^{\infty} m_i x_i$$

for all $x \in l_1$. For any bounded sequence m, we shall denote by m' the corresponding linear functional on l_1. Thus $m = (m_j) \in M$, $||m|| = 1$, and $m'[A] = 0$ if and only if m has the form $m = (0, -(2/3)m_3, m_3, -(4/5)m_5, m_5, \ldots)$, with $\sup_j |m_j| = 1$. Using this fact in conjunction with Lemma 2.2.1, we see that a vector a is orthogonal to A if and only if it has the form $a = (0, 0, a_3, 0, a_5, 0, \ldots)$. Similarly, a vector b is orthogonal to B if and only if it has the form $b = (0, b_2, 0, b_4, \ldots)$. Thus, because $e_{2n-1} + (0, \ldots, 0, 1/(2n), 0, \ldots) = e_{2n}$, $e_{2n} \in P_B(e_{2n-1})$. By the same reasoning, $e_{2n+1} \in P_A(e_{2n})$. Since any point in A is the limit of a sequence of finite linear combinations of e_n, with n odd, any point in A must be of the form $(a_1, a_2, (2/3)a_2, a_4, (4/5)a_4, \ldots)$, and similarly any point in B of the form $(b_1,(1/2)b_1,b_3(3/4)b_3,\ldots)$. If $z \varepsilon P_B(e_{2n-1})$, then $z \varepsilon B$ and $z = (z_1,(1/2)z_1,z_3,(3/4)z_3,\ldots)$. Also $z = e_{2n-1} + (0,b_2 0,b_4,0,\ldots)$ because all elements orthogonal to B must be of the form $(0,b_2,0,b_4,$ $0,\ldots)$. By comparing the coordinates in this last equation, we see that $z = e_{2n}$. This means that $e_{2n} = P_B(e_{2n-1})$; a similar argument shows that $e_{2n+1} = P_A(e_{2n})$.

It only remains to show that $A \cap B = 0$. Suppose $w \varepsilon A \cap B$, Then by our previous characterization of elements of A and B, $w = (w_1,w_2,(2/3)w_2,w_4,(4/5)w_4,\ldots) = (w_1,(1/2)w_1,w_3,(3/4)w_3,\ldots)$. Let w_k be the first non-zero term of the sequence $\{w_j\}$. Equating like terms, and assuming $k > 1$, we see that $w_k = ((k-1)/k)w_{k-1}$. Then $w_{k-1} = 0$ which is contrary to our assumption that w_k is the first non-zero term. If $k = 1$, $w_2 = (1/2)w_1$, $w_3 = (2/3)(1/2)w_1,\ldots,$ $w_n = ((n-1)/n)\ldots(2/3)(1/2)w_1$, and $w = w_1(1,1/2,1/3,\ldots)$. But the sum $\sum_{n=1}^{\infty} 1/n$ diverges, and so w is not in l_1 unless $w_k = 0$ for all k.

CHAPTER III

CLOSEST-POINT MAPS AND THEIR PRODUCTS, II

Let A and B be closed convex sets in a complete inner-product space E, and let P_A and P_B be their respective closest-point maps, i.e., for each $x \varepsilon E$ let $P_K(x)$ be the unique point in K such that $||x - P_K(x)|| = \inf_{k \varepsilon K} ||x - k||$ for $K = A, B$. It is well known that P_K is a well defined function in E. In [3], Cheney and Goldstein showed that the sequence of iterates $\{(P_A P_B)^n(x)\}$ converges to a point in A which is nearest to B, in the sense that the distance from this point to B is the same as the distance between A and B, whenever one of these sets is compact or one set is finite-dimensional and the distance between the two sets is attained at some point in one of the sets. Their arguments were based on properties of closest-point maps in inner-product spaces that are even stronger than the well known distance shrinking property:

$$||P_K(x) - P_K(y)|| \leq ||x - y||.$$

It can be shown (see e.g. [17]) that the closest-point map exists and is single-valued for every closed convex set in a strictly convex normed linear space if and only if the space is reflexive. Phelps [18] has shown that if the closest-point map shrinks distance for every one-dimensional subspace of a space which is at least three-dimensional, the space is an inner-product space (see our remarks in Chapter I).

Therefore, any serious generalization of the Cheney-Goldstein results to more general spaces cannot be based on the distance shrinking property of the closest-point map. In this chapter, we generalize their results and show that, for any pair, A and B, of closed convex sets in a normed linear space, the sequence of iterates $\{(P_A P_B)^n(x)\}$ always converges when the space is strictly convex and smooth, P_A and P_B are continuous, one of the sets is compact, one is strictly convex, and there is at most one point in their intersection. We also show that their requirement of an inner product could easily be replaced by the distance shrinking property. We then show that, in a complete inner-product space, the iterations converge, not only for a pair of convex sets, but for any finite number of them when at least one is compact. We conclude by giving a short proof that these iterations always converge when the convex sets are subspaces.

In the following, E will denote a reflexive normed linear space, A and B closed convex sets in E, and P_A and P_B the closest-point mapping of E onto A and B.

<u>Theorem 3.1</u>. If either A or B is strictly convex and A and B are disjoint, the distance between A and B is attained at most at one point. If C and D are closed convex sets in a reflexive space, the distance between them is attained if at least one of these sets is bounded.

<u>Proof</u>. Suppose that B is strictly convex, $x,y \in A$, $x \neq y$, and $m = ||x - P_B(x)|| = ||y - P_B(y)|| = \inf\limits_{\substack{a\in A\\ b\in B}} ||a - b||$. For any α, $0 < \alpha < 1$, $||(\alpha P_B(x) + (1 - \alpha)P_B(y)) - (\alpha x + (1 - \alpha)y)|| \leq \alpha||P_B(x)-x|| +$

$+ (1 - \alpha)||P_B(y) - y|| = m$. Since the distance from A to B is m, the equality sign must hold throughout this last inequality. Therefore, the line segment $[P_B(x), P_B(y)]$ must lie on the surface of B. This contradicts the assumption that B is strictly convex and, thus, proves the first part of the theorem.

Suppose that C and D are disjoint and that C is bounded. Let $\{c_n\}$ and $\{d_n\}$ be sequences from C and D, respectively, such that $\lim ||c_n - d_n|| = m$, the distance between C and D. Since the sequences $\{c_n\}$ and $\{c_n - d_n\}$ are both bounded and E is reflexive, we may assume that they both converge weakly. Since the difference of two convergent sequences must converge, the sequence $\{d_n\}$ must also converge. Let $\lim c_n \overset{w}{=} c$ and $\lim d_n \overset{w}{=} d$. By well known properties of weak limits, $||c - d|| \leq \lim ||c_n - d_n|| = m$. Since C and D are closed and convex, they are weakly closed. Hence $c \in C$ and $d \in D$, and, therefore, $m = ||c - d||$.

<u>Theorem 3.2</u>. Let E be smooth and strictly convex. If A or B is strictly convex and A and B have at most one point in common, then the sequence of iterates $\{(P_A P_B)^n(x)\}$ converges to a point x_o in A if any one of the following conditions hold:

i) A is compact and P_B is continuous.

ii) A and B are compact.

iii) A is compact and E is finite-dimensional.

Furthermore, $||x_o - P_B(x_o)|| = \inf\limits_{\substack{a \in A \\ b \in B}} ||a - b||$.

<u>Proof</u>. The hypotheses of either i, ii, or iii is sufficient to insure continuity of P_A and P_B. The straightforward proof will be

omitted. Thus, we need only prove the assertion when these mappings are continuous and when A is compact.

Since A is compact, the sequence $\{(P_A P_B)^n(x)\}$ has a convergent subsequence, say $\{x_k\}$. Let $x_o = \lim x_k$. Then by the continuity of P_A and P_B, $\lim P_B(x_k) = P_B(x_o)$ and $\lim P_A P_B(x_k) = P_A P_B(x_o)$. Because P_A and P_B are closest-point maps, $\lim ||x_k - P_B(x_k)|| = \lim ||P_B(x_k) - P_A P_B(x_k))||$, and, therefore, by the continuity of the norm, $||P_B(x_o) - P_A P_B(x_o)|| = ||x_o - P_B(x_o)||$. Since E is strictly convex, $P_A P_B(x_o) = x_o$.

Suppose that A and B are disjoint. Then $P_B(x_o) \neq x_o$. Let $d = ||x_o - P_B(x_o)||$, and let $C_1 = d(S) + x_o$ be a sphere of radius d centered at x_o. Then $C_1 \cap B = P_B(x_o)$. Hence, there is a hyperplane, H_2, containing the point $P_B(x_o)$ and supporting both B and $d(S) + x_o$ and, furthermore, separating these two sets. Using a similar argument, we see that there is a corresponding hyperplane H_1 containing x_o and separating $d(S) + P_B(x_o)$ from K_1. Because of the symmetry and smoothness of S, H_1 and H_2 must be translates of each other. This implies, immediately, that $||x_o - P_B(x_o)|| = \inf\limits_{\substack{a \varepsilon A \\ b \varepsilon B}} ||a - b||$. By Theorem 3.1, there is at most one point in A at which the distance between A and B is attained. Therefore, the above argument proves that the sequence $\{(P_A P_B)^n(x)\}$ must converge to x_o.

Suppose A and B have exactly one point in common. Let $\{x_n\}$ be a convergent sequence of elements from A whose limit is x_o. By using an argument similar to the one we used in the first part of the proof, we see that $P_A P_B(x_o) = x_o$. If $x_o \notin A \cap B$, unique parallel hyperplanes H_1 and H_2 can be found such that H_1 contains x_o and supports A, and H_2 contains $P_B(x_o)$ and supports B; but this is impossible because A and B have an element in common. Hence, the sequence $\{(P_A P_B)^n(x)\}$ must converge to the point contained in both A and B.

Whether all geometric conditions imposed on the unit ball, the separation of the sets A and B, or the strict convexity of at least one of the sets is necessary in the hypotheses of Theorem 3.2 is unknown to the author.

In the two-dimensional case, we can prove that the iterations behave as expected. We do this in the following theorem.

<u>Theorem 3.3</u>. Let A and B be any two closed convex sets in a two-dimensional strictly convex normed linear space E. If x is any point in E, let $x_1 = P_B(x)$, $x_2 = P_A P_B(x)$, and $x_n = P_A(x_{n-1})$ if n-1 is odd or $x_n = P_B(x_{n-1})$ if n-1 is even. If either A or B is compact, then each of the sequences $\{x_n\colon n \text{ even}\}$ and $\{x_n\colon n \text{ odd}\}$ converges.

<u>Proof</u>. Assume that A is compact. Since the sequence $\{||x_{n+1} - x_n||\}$ is decreasing, it has a limit, say d. We consider two cases depending upon the value of d.

Case 1. $d > 0$. Since A is compact, every subsequence of the sequence $\{x_{2n}\}$ has a convergent subsequence. If all of these subsequences converge to the same point, a simple argument using the continuity of P_B will complete the proof. Therefore, suppose $\{x_{2n}\}$ contains two convergent subsequences $\{q_n\}$ and $\{r_n\}$, whose limits, q

and r, are not the same. By the continuity of P_B, $\lim P_B(q_n) = P_B(q)$, $\lim P_B(r_n) = P_B(r)$, and $||q - P_B(q)|| = ||r - P_B(r)|| = d$. We now show that the set of limit points of $\{x_{2n}\}$ form a continuum. To see this, we consider the following modification of Lemma 2.3.2. Given any n, there exists a hyperplane H_n containing x_{2n+2} and supporting both A and the set $x_{2n+1} + ||x_{2n+2} - x_{2n+1}||(S)$ where S is the unit ball of E. Since $x_{2n} \varepsilon A$, the line segment $L[x_{2n}, x_{2n+1}]$ must intersect H_n. Let h_n denote the point of intersection. Since $\lim ||x_n - x_{n+1}|| = d$, $\lim ||h_n - x_{2n}|| = 0$. Hence, according to Lemma 2.3.2, $\lim ||x_{2n+2} - h_n|| = 0$. Applying the triangle inequality, we have $||x_{2n+2} - x_{2n}|| \leq ||x_{2n} - h_n|| + ||h_n - x_{2n+2}||$. Thus, $\lim ||x_{2n+2} - x_{2n}|| = 0$. The set C of limit points of the sequence $\{x_{2n}\}$ must be closed and bounded. If C is not connected, it is the disjoint union of two non-empty compact sets C_1 and C_2, and there are disjoint closed neighborhoods D_1 and D_2 containing C_1 and C_2, respectively. Since $D_1 \cup D_2$ contains all limit points of $\{x_{2n}\}$, $D_1 \cup D_2$ contains all but a finite number of points from the sequence $\{x_{2n}\}$. But this contradicts the fact that an infinite number of these points have to lie in both D_1 and D_2 and that $\lim ||x_{2n} - x_{2n+2}|| = 0$. Hence, C must be a continuum. Now, both the boundary of A and the boundary of B are Jordan Curves (or line segments), and, therefore, the limit points of the sequence $\{x_{2n}\}$ must form a non-degenerate Jordan arc T. By the continuity of the distance function, the distance from each point of T to B is d. Furthermore, since the elements of the sequence $\{x_{2n}\}$ are dense in T, there must be some element, say x_{2k}, which lies in the relative interior of this arc. Therefore

$||x_{2k} - x_{2k+1}|| = d$, and the iterations terminate which is impossible.

Case 2: $d = 0$. Again let $\{x_{2n}\}$ be the sequence of iterates in A, and suppose that this sequence has at least two limit points. Since $\lim ||x_{2n} - x_{2n+2}|| = 0$, an argument similar to the one used in Case 1 shows that the set of all limit points of the sequence $\{x_{2n}\}$ must be a non-degenerate Jordan arc on the boundary of A. As before, some x_{2k} must lie in this arc, and the iterations must terminate at this point. This completes the proof.

Because inner-product spaces are the only spaces where closest-point maps shrink distance for all convex sets, the following theorem is only a mild generalization of the corresponding result of Cheney and Goldstein.

Theorem 3.4. Suppose that E is strictly convex and that P_A and P_B shrink distance. If A or B is compact or if E is finite-dimensional and the distance between A and B is attained, the sequence of iterates $\{(P_A P_B)^n(x)\}$ converges to a point x_o such that $||x_o - P_B(x_o)|| = \inf_{\substack{a \in A \\ b \in B}} ||a - b||$.

Proof. Suppose E is finite-dimensional and the distance between A and B is attained at $a \in A$. Using the properties of the closest-point map, we see that $P_A P_B(a) = a$. Since P_A and P_B shrink distance, $||(P_A P_B)^n(x) - a|| \leq ||P_A P_B(x) - a||$ for all n. This implies that the sequence $\{(P_A P_B)^n(x)\}$ has a convergent subsequence whose limit must be in A because A is closed. This last statement is also true if A is compact. In either case, let $\{x_n\}$ be the convergent subsequence and let $x_o = \lim x_n$. As in the proof of Theorem 3.2,

$P_AP_B(x_o) = x_o$. If the sequence $\{(P_AP_B)^n(x)\}$ does not converge to x_o, it contains a second convergent subsequence whose limit y is not equal to x_o. As was the case with x_o, $P_AP_B(y) = y$. This leads us to a contradiction. For the distance shrinking property of each of P_A and P_B implies that we can find disjoint spheres about x_o and y such that the sequence $\{(P_AP_B)^n(x)\}$ is eventually in each of these spheres.

We now turn our attention to inner-product spaces, and we start by proving the following known result.

Theorem 3.5. If K is a closed convex set in a complete inner-product space E, P_K is well defined, shrinks distance, and $||P_K(x) - P_K(y)|| = ||x - y||$ only if $||x - P_K(x)|| = ||y - P_K(y)||$.

Proof. Since it is known that a complete inner-product space is smooth, strictly convex, and reflexive, our previous discussion of Phelp's work implies that the closest-point map exists and is unique for each closed convex set. Let x and y be any points in E, and let $P_K(x)$ and $P_K(y)$ be their respective closest-point maps on K. Since E is smooth, there are two unique hyperplanes, H_1 and H_2, H_1 supporting K at $P_K(x)$ and H_2 supporting K at $P_K(y)$, whose translates through the origin are orthogonal to $P_K(x) - x$ and $P_K(y) - y$, respectively. Since K is convex, the line segment $L[P_K(x), P_K(y)]$ must be on the side of H_1 opposite x and on the side of H_2 opposite y. By using the symmetry of the inner product in a real inner-product space, we see that x must lie on the side of H_x, the hyperplane containing the point $P_K(x)$ whose translate through the origin is orthogonal to the vector $P_K(x) - P_K(y)$, opposite $P_K(y)$. Similarly, y must lie on the side of H_y, the translate of H_x containing $P_K(y)$, opposite $P_K(x)$. This implies that $||x - y|| \geq ||P_K(x) - P_K(y)||$ with equality holding if and only if $x \in H_x$ and $y \in H_y$. It is now possible to verify that $||P_K(x) - x|| = ||P_K(y) - y||$ when the equality holds (an easy method for showing this will be used in the proof of Theorem 3.6, and, therefore, will be omitted here).

Theorem 3.6. Let $K_1, K_2, \ldots, K_N$ be closed convex sets in a complete inner-product space E, and let $P = P_{K_1} P_{K_2} \ldots P_{K_N}$. If for some j, K_j is compact, then $\lim P^n(x)$ exists and is a fixed point of P. If any K_j is strictly convex or if, for some j, $||P_{K_j}(x) - x|| \neq ||P_{K_j}(y) - y||$ for all $x, y, \in K_{j-1}$, $x \neq y$, then the limit is unique.

Proof. Suppose, without loss of generality, that K_1 is compact. It follows directly from Theorem 3.5 that P shrinks distance, and also that $||P(x) - P(y)|| = ||x - y||$ only if $||P_{K_j} \ldots P_{K_N}(x) - P_{K_j} \ldots P_{K_N}(y)|| = ||x - y||$ and $||P_{K_j} \ldots P_{K_N}(x) - P_{K_{j+1}} \ldots P_{K_N}(x)|| = ||P_{K_j} \ldots P_{K_N}(y) - P_{K_{j+1}} \ldots P_{K_N}(y)||$ for all j. Suppose that for some $x \in K_1$, $||P(x) - P^2(x)|| = ||x - P(x)||$. The preceding equalities imply that

$$||P_{K_j} \ldots P_{K_N}(x) - P_{K_j} \ldots P_{K_N} P(x)|| = ||x - P(x)|| ,$$

and that

$$||P_{K_j} \ldots P_{K_N}(x) - P_{K_{j+1}} \ldots P_{K_N}(x)|| = ||P_{K_j} \ldots P_{K_N} P(x) - P_{K_{j+1}} \ldots P_{K_N} P(x)||$$

for all j.

Consider the quadrilateral whose vertices are x, $P_{K_N}(x)$, $P(x)$, and $P_{K_N}P(x)$. The last equalities above imply that $||x - P(x)|| = ||P_{K_N}(x) - P_{K_N}P(x)||$ and that $||x - P_{K_N}(x)|| = ||P(x) - P_{K_N}P(x)||$. We will show that $x - P(x) = P_{K_N}(x) - P_{K_N}P(x)$, $x - P_{K_N}(x) = P(x) - P_{K_N}P(x)$ and that $x - P(x)$ is orthogonal to $x - P_{K_N}(x)$. Assuming that none of these quantities is the zero vector, we note that $P_{K_N}(x)$ is the point in K_N closest to x. Hence there exists a hyperplane H_1, whose translate at the origin is orthogonal to $x - P_{K_N}(x)$, which supports K_N at $P_{K_N}(x)$. Clearly x and $P_{K_N}P(x)$ lie on opposite sides of H_1, and, by the symmetry of the real inner product, x and $P_{K_N}P(x)$ lie on opposite sides of the hyperplane H containing $P_{K_N}(x)$, whose translate at the origin is orthogonal to $P_{K_N}(x) - P_{K_N}P(x)$. Similarly, $P(x)$ and $P_{K_N}(x)$ lie on opposite sides of the hyperplane $H + (P_{K_N}P(x) - P_{K_N}(x))$, the translate of H containing $P_{K_N}P(x)$. Since $||x - P(x)|| = ||P_{K_N}(x) - P_{K_N}P(x)||$, x and $P(x)$ must lie in H and $H + (P_{K_N}P(x) - P_{K_N}(x))$ respectively, and this implies $x - P(x) = P_{K_N}(x) - P_{K_N}P(x)$. It is also evident from this construction that $x - P(x)$ is orthogonal to $x - P_{K_N}(x)$.

Continuing this process inductively, we see that, for all j, $x - P(x) = P_{K_j}\ldots P_{K_N}(x) - P_{K_j}\ldots P_{K_N}P(x)$, and that $x - P(x)$ is orthogonal to $P_{K_{j+1}}\ldots P_{K_N}(x) - P_{K_j}\ldots P_{K_N}(x)$.

This implies that

$$\begin{aligned}||x - P(x)||^2 &= (x - P(x),\ x - P(x)) \\ &= (x - P(x),\ (x - P_{K_N}(x)) + (P_{K_N}(x) \\ &\quad - P_{K_{N-1}}P_{K_N}(x)) + \dots + (P_2 \dots P_{K_N}(x) \\ &\quad - P(x))) \\ &= 0 .\end{aligned}$$

Therefore, either $x = P(x)$ or $||P(x) - P^2(x)|| < ||x - P(x)||$.

Since K_1 is compact, the sequence $\{P^n(x)\}$ has a convergent subsequence, say $\{x_n\}$. Let $x = \lim x_n$. Then $||P(x) - P^2(x)|| \leq ||x - P(x)|| = \lim ||x_n - P(x_n)|| \leq \lim ||P(x_{n-1}) - P^2(x_{n-1})|| = ||P(x) - P^2(x)||$, and, therefore, $x = P(x)$. Thus, any convergent subsequence of $\{P^n(x)\}$ converges to a fixed point of P. As we saw in the proof of Theorem 3.4, this implies that the sequence $\{P^n(x)\}$ converges.

If K_j is strictly convex, then $||P_{K_j}(x) - P_{K_j}(y)|| < ||x - y||$ for all x and y. Hence, the same strict inequality holds for P. Since the sequences $\{P^n(x)\}$ and $\{P^n(y)\}$ must converge to a fixed point of P, their limit must be unique. The other assertions of the theorem follow immediately from the proof just presented.

<u>Definition</u>. A set R is <u>boundedly compact</u> if and only if $R \cap nS$ is compact for every positive number n.

<u>Corollary 3.7</u>. Let $K_1,\dots,K_N$ be closed convex sets in a complete inner-product space E, and suppose that $\bigcap_{j=1}^{N} K_j$ is non-empty and that at least one K_j is boundedly compact. If $P = P_{K_1} \dots P_{K_N}$,

$\lim P^n(x)$ exists for all x in E and, furthermore, the limit point is in $\bigcap_{j=1}^{N} K_j$.

Proof. Let $k \in \bigcap_{j=1}^{N} K_j$. Then k is a fixed point of P and $||P^n(x) - k|| \leq ||P(x) - k||$ holds for all n and all x. This implies that the sequence $\{P^n(x)\}$ is bounded. Since at least one K_j is boundedly compact, Theorem 3.6 implies that $\lim P^n(x)$ exists.

To see that the limit of the sequence $\{P^n(x)\}$ is in every K_j, we let $x_o = \lim P^n(x)$ and we let $k \in \bigcap_{j=1}^{N} K_j$. Suppose $x_o \notin K_m$. Then, by what has been established, in Theorem 3.5,

$$\begin{aligned} ||x_o - k|| &= ||P(x_o) - k|| \\ &= ||P_{K_1} \cdots P_{K_N}(x_o) - k|| \\ &< ||P_{K_{m+1}} \cdots P_{K_N}(x_o) - k|| \\ &\leq ||x_o - k|| , \end{aligned}$$

and this is impossible.

A simple, but interesting, result of Theorem 3.6, is that inside any acute triangle $[a,b,c]$ we may inscribe a unique triangle $[d,e,f]$ such that $[d,c]$ is orthogonal to $[b,c]$, $[e,f]$ is orthogonal to $[a,c]$, and $[f,d]$ is orthogonal to $[a,b]$. The points d, e, and f may be obtained by the aforementioned iterations.

If each K_j is a subspace and E is finite-dimensional, Corollary 3.7 implies convergence of the iterates to a point in their interaction. However, the restriction of finite dimensionality is not required in this case. This was first shown by von Neumann [16] for two subspaces and later shown by Halperin [7] for any finite number of subspaces.

We finish this chapter by giving a short proof (the material we cite from Fortet's paper is quite brief) of Halperin's theorem.

Let P be any linear operator of norm one in a strictly convex reflexive Banach space E. Fortet has shown, by using a very clever argument, that every element x of E can be written $x = y + z$ where $y \in A = \{x \in E : x = P(x)\}$ and $z \in B = \mathrm{Cl}\{x : x = y - P(y)$ for some $y \in E\}$, and that the set of elements orthogonal to B is precisely A. It follows directly from these facts that the sequence $\{P^n(x)\}$ converges for every x in E if and only if $\lim\ (P^n(y) - P^{n+1}(y)) = 0$ for every $y \in E$. Using these facts, we can easily generalize von Neumann's theorem and arrive at Halperin's theorem.

<u>Theorem 3.8</u>. Let $A_1,\ldots A_N$ be N subspaces of a complete inner-product space E, and let $P_1,\ldots,P_N$, be their respective closest-point maps. If $P = P_1\ldots P_N$, then the sequence $\{P^n(x)\}$ converges to $P_{A_1 \cap \ldots \cap A_N}(x)$ for every $x \in E$.

<u>Proof</u>. According to the previous discussion, it is only necessary to show that $\lim\ (P^n(y) - P^{n+1}(y)) = 0$ for every $y \in E$. Let $y_1 = P_N(y)$, $y_2 = P_{N-1}P_N(y),\ldots,y_N = P_1P_2\ldots P_N(y)$, and let y_n be defined for all positive integers, inductively, in the obvious manner. Noting that P_j is the orthogonal projection on A_j, and using the Pythagorean law, we see that $||y_n - y_{n+1}||^2 + ||y_{n+1}||^2 = ||y_n||^2$ for all $n \geq 1$. Summing, we have

$$\sum_{n=1}^{\infty} ||y_n - y_{n+1}||^2 = ||y_1|| - \lim\ ||y_n|| .$$

Since $||P|| \leq 1$, the sequence $\{||y_n||\}$ is monotonically decreasing, and, therefore, has a limit. Thus, $\lim ||y_n - y_{n+1}|| = 0$. Since $||P^n(y) - P^{n+1}(y)|| \leq ||y_{nN+1} - y_{nN+2}|| + \ldots + ||y_{(n+1)N+1} - y_{(n+1)N}||$, $\lim (P^n(y) - P^{n+1}(y)) = 0$ as required.

We now show that $\lim P^n(x) = P_{A_1 \cap \ldots \cap A_N}(x)$. For each x in E, let $R(x) = \lim P^n(x)$. It is clear that R is a linear operator and that $||R|| \leq 1$. Now for each positive integer k, $R(x) = \lim P^n(x) = P^k \lim P^n(x) = P^k R(x)$; therefore, taking limits, we see that $R(x) = R^2(x)$ and that R is a projection. Suppose $R(x) = x$. Then $\lim P^n(x) = x$ and so $P^2(x) = x$. Hence $x \varepsilon A_1 \cap \ldots \cap A_N$. Conversely, if $P^2(x) = x$, then $R^2(x) = x$. So R is a projection on the subspace $A_1 \cap \ldots \cap A_N$. Since R has norm one, the results of Fortet mentioned earlier imply that the subspace $A_1 \cap \ldots \cap A_N$ is orthogonal to $R(x) - x$ for every $x \varepsilon E$. Since orthogonality is symmetric in an inner-product space, R must correspond to $P_{A_1 \cap \ldots \cap A_N}$.

CHAPTER IV

A SOLUTION TO HIRSCHFELD'S PROBLEM

This chapter gives a negative answer, for all finite-dimensional spaces, to a question raised by Hirschfeld in [8]. Let A be a subset of a real Banach space E, and let $P_A(x) = \{y \in A : ||x - y|| = \inf_{a \in A} ||x - a||\}$. If A is a subspace and E is strictly convex and reflexive, then it is well known that $P_A(x)$ consists of exactly one point (see Chapter II). It follows directly from [16, Lemma 22] that if E is a complete inner-product space and A and B are any two subspaces of E, then $\lim (I - P_A)(I - P_B) \ldots (I - P_A)(I - P_B)(x) = (I - P_{A+B})(x)$ for all x in E. Hirschfeld [8] asked the following question: If E is a strictly convex reflexive Banach space and $\lim (I - P_A)(I - P_B) \ldots (I - P_A)(I - P_B)(x) = (I - P_{A+B})(x)$ for all subspaces A and B and for all $x \in E$, is E necessarily an inner-product space? Klee [13] proved that this is not necessarily true if E has only two dimensions. He did this by displaying a rather extensive class of non inner-product spaces which satisfied Hirschfeld's conditions. As Klee mentioned, the situation in higher dimensional spaces could be markedly different. However, we will show that it is not by giving a large class of finite-dimensional normed linear spaces which are not inner product spaces but which satisfy Hirschfeld's conditions. The solution of this problem in the finite-dimensional case is probably indicative of the solution in the infinite-dimensional case, but we do not show this.

Lemma 4.1. Let E be strictly convex and reflexive, and let A be a subspace of E. Then $P_A(a + x) = a + P_A(x)$ for every $a \varepsilon A$ and for every $x \varepsilon E$.

Proof. Since $||(a + x) - P_A(a + x)|| = ||x - (P_A(a + x) - a)|| = \inf_{y \varepsilon A} ||(a + x) - y)|| = \inf_{y \varepsilon A} ||x - (y - a)|| = \inf_{y \varepsilon A} ||x - y||$, $P_a(x) = P_A(a + x) - a$.

Lemma 4.2. Any subspace of a strictly convex smooth space is strictly convex and smooth.

Proof. The strict convexity is immediate and the smoothness follows easily from the Hahn-Banach theorem.

Theorem 4.1. If E is any strictly convex, smooth, finite-dimensional space and A and B are any two subspaces of E, then $\lim (I - P_A)(I - P_B) \ldots (I - P_A)(I - P_B)(x) = (I - P_{A+B})(x)$ for every $x \varepsilon E$.

Proof. The proof will be carried out in two parts.

Case 1: x not contained in the span of A and B. We may write $x = (x - P_{A+B}(x)) + a + b$ where $a \varepsilon A$, $b \varepsilon B$, and we may assume, without loss of generality, that the span of A, B, and z is E where $z = x - P_{A+B}(x)$. Note that this makes the span of A and B a hyperplane in E. By our choice of notation, it is necessary and sufficient to show that $\lim (I - P_A)(I - P_B) \ldots (I - P_A)(I - P_B)(x) = z$. First of all, we show that the left hand limit of this last equality exists and is a scalar multiple of z. To simplify notation, let

$x_1 = (I - P_B)(x)$, $x_2 = (I - P_A)(x_1)$, and let x_n be defined inductively in the obvious manner for the remaining positive integers. The sequence $\{||x_n||\}$ is monotonically decreasing and, therefore, has a limit. Call this limit d. Let $\{x_{n_k}\}$ be any subsequence of the sequence $\{x_n\}$. Since the sequence $\{x_{n_k}\}$ is bounded, it contains a convergent subsequence $\{y_n\}$. We can select the elements y_n so that either they are all orthogonal to A or they are all orthogonal to B. Suppose, for definiteness, that each y_n is orthogonal to A. If $y = \lim y_n$, it is an easy matter (see Lemma 2.3.3) to show that y is orthogonal to A. Letting $D(r,T)$ denote the distance from the vector r to the set T, we have $d \leq D(y_{n+1},A) \leq D(y_n,B) \leq D(y_n,A)$ for all n. Taking limits and using the continuity of D, we have $D(y,B) = D(y,A)$. Therefore, y is orthogonal to B as well as to A. Since E is strictly convex and the span of A and B is a hyperplane, y must be a scalar multiple of z (see Lemma 2.2.2). Calling this scalar β, we have $y = \beta z$ where, evidentially, β is non-negative. The above argument shows that any subsequence of $\{x_n\}$ contains a subsequence which converges to a non-negative multiple of z. Since the sequence $\{||x_n||\}$ is monotonically decreasing, each of these convergent subsequences must converge to βz. This, of course, implies that $\lim x_n = \beta z$. It only remains to show that $\beta = 1$.

Using Lemma 4.1 and the notation introduced above, we see that

$$x_1 = z + a - P_B(z + a)$$

$$x_2 = z - P_B(z + a) - P_A(z - P_B(z + a))$$

$$x_3 = z - P_A(z - P_B(z + a)) - P_B(z - P_A(z - P_B(z + a)))$$

$$x_4 = z - P_B(z - P_A(z - P_B(z + a))) - P_A(z - P_B(z - P_A(z - P_B(z + a)))).$$

Continuing this process inductively, we note that $x_n = z - b_n - a_n$ where $a_n \varepsilon A$, and $b_n \varepsilon B$. Since the sequence $\{x_n\}$ is bounded, the sequence $\{a_n + b_n\}$ is bounded. Thus, if $\lim (a_n + b_n) \neq 0$, we can find a convergent subsequence $\{a_{n_k} + b_{n_k}\}$ of the sequence $\{a_n + b_n\}$ such that $\lim (a_{n_k} + b_{n_k}) = c \neq 0$. Then $\lim x_{n_k} = z - c$. Recalling that $\lim x_{n_k} = \beta z$, we see that $\beta z = z - c$ or $c = z(1 - \beta)$. Because c is in the hyperplane spanned by A and B, and z is orthogonal to this hyperplane, the last equality implies that $\beta = 1$ and $c = 0$. This is a contradiction; therefore, $\lim (a_n + b_n) = 0$ and $\lim x_n = z$.

Case 2: x is contained in the span of A and B. This assumption, of course, implies that all x_n are contained in the span of A and B. We assume that E is the span of A and B. Since $(I - P_{A+B})(x) = 0$, we need only show that $\lim x_n = 0$.

Let $A_\perp = \{y \varepsilon E : y \perp A \text{ and } ||y|| = 1\}$ and let $B_\perp = \{y \varepsilon E : y \perp B \text{ and } ||y|| = 1\}$. Since E is smooth (by Lemma 4.2) and x is contained in the span of A and B, $A_\perp \cap B_\perp$ is the empty set. Furthermore, both $A_\perp$ and $B_\perp$ are compact-both of these sets being closed by Lemma 2.3.3. Hence there exists a number k, $0 \leq k < 1$, such that $D(y,B) \leq k$ for all $y \varepsilon A_\perp$ and $D(y,A) \leq k$ for all $y \varepsilon B_\perp$. By Lemma 2.3.4, P_A and P_B are homogeneous. Therefore, for any scalar α, $||(I - P_B)(\alpha y)|| \leq k||\alpha y||$ for every $y \varepsilon A_\perp$ and $||(I - P_A))\alpha y)|| \leq k||\alpha y||$ for every $y \varepsilon B$. This implies that

$||x_{n+2}|| \leq k^2||x_n||$. Since $0 \leq k < 1$, $\lim x_n = 0$ as required.

The above method is not directly applicable to the analogous infinite-dimensional problem. The main obstacle seems to be the lack of weak continuity of the distance function D. However, it seems likely that one can circumvent this difficulty and show that all uniformly convex smooth Banach spaces satisfy Hirschfeld's conditions.

CHAPTER V

ON CHENEY'S PROBLEM

In [15], Cheney asked the following question: If A is a subspace of finite-dimensional real Euclidean space E_n (the space of all real n-tuples) and x_p is the best approximation of some vector x in A in the l_p norm, i.e., $||x - x_p||_p = \inf_{a \in A} ||x - a||_p$, what can be said about the sequence $\{x_p\}$? In the following, we show that this sequence always converges when A is either a hyperplane or a line as well as when the l_∞ approximation is unique. The fact that the sequence always converges for hyperplanes and lines implies, immediately, that all of these sequences converge in E_3.

Theorem 5.1. Let A be any subspace in E_n and x any element in E_n. If x_p is the best approximation of x in A, then x_p is a continuous function of p for $1 < p < \infty$.

Proof. Let p_1 be any real number greater than one, and let $\{p_k\}$ be any increasing sequence with limit equal to p_o. Since $p_k \leq p_o$, $||x - x_{p_o}||_{p_k} \geq ||x - x_{p_k}||_{p_k} \geq ||x - x_{p_k}||_{p_o} \geq ||x - x_{p_o}||_{p_o}$. From the definition of the l_p norm, it is clear that $\lim ||x - x_{p_o}||_{p_k} = ||x - x_{p_o}||_{p_o}$. Hence, these inequalities imply that $\lim ||x - x_{p_k}||_{p_o} = ||x - x_{p_o}||_{p_o}$. Noting that the unit ball in l_{p_o} is strictly convex, we see by Lemma 2.3.2 that $\lim x_{p_k} = x_{p_o}$. Now suppose that $\{p_k\}$ is a decreasing sequence with limit p_o, and that $\{x_{p_k}\}$ is the corresponding sequence of best p-th approximations. Since

$||x - x_p||_\infty \le ||x - x_p||_p \le ||x - x_1||_1$, $||x_p||_\infty \le ||x - x_1||_1 + ||x||_\infty$, and, therefore, the sequence $\{x_{p_k}\}$ is bounded. Thus suppose, without loss of generality, that the sequence $\{x_{p_k}\}$ converges. Noting that $||x - x_{p_k}||_{p_o} \ge ||x - x_{p_o}||_{p_o} \ge ||x - x_{p_o}||_{p_k} \ge ||x - x_{p_k}||_{p_k} \ge ||x - x_{p_k}||_r$ for $r \ge p_k$, we see, by taking limits and letting $\lim x_{p_k} = y$, that $||x - y||_{p_o} \ge ||x - x_{p_o}||_{p_o} \ge ||x - y||_r$. Letting r approach p_o, we have $||x - y||_{p_o} = ||x - x_{p_o}||_{p_o}$. Again the strict convexity of l_{p_o} implies that $y = x_{p_o}$. The above discussion, combined with the fact that the set of best approximations is bounded, implies that the mapping is continuous.

<u>Theorem 5.2</u>. If H is a hyperplane in E_n, then $\lim_{p\to\infty} x_p$ exists.

<u>Proof</u>. Let $a = (a_1,\ldots,a_n)$ be the outward normal of H. Fortet [6] showed that any vector $y = (y_1,\ldots,y_n)$ orthogonal to H, in the l_p sense, must satisfy

$$\sum_{k=1}^{n} |y_k|^{p-2} y_k a_k^j = 0 \quad \text{for} \quad j = 1,\ldots,n-1$$

where $a^j = (a_1^j,\ldots,a_n^j)$, $j = 1,\ldots,n-1$, are any $n-1$ linearly independent vectors spanning H. Since any solution of the linear homogeneous system

$$\sum_{k=1}^{n} z_k a_k^j = 0, \quad j = 1,\ldots,n-1,$$

must be a scalar multiple of the vector a,

$$|y_k|^{p-2} y_k = m a_k \qquad \text{for} \quad k = 1, \ldots, n$$

for some constant m. Assuming that $m = 1$, and solving, we see that

$$y_k = (\text{sign } a_k) |a_k|^{1/(p-1)} .$$

From this, it is clear that $\lim_{p \to \infty} y_k = (\text{sign } a_k) \lim_{p \to \infty} |a_k|^{1/(p-1)}$, and that $\lim_{p \to \infty} y_k$ exists. The theorem follows directly from this and the fact that the lengths of the vectors $x - x_p$ approach a common limit in the l_1 norm.

We remarked earlier that we would prove that the sequence $\{x_p\}$ always converges when the l_∞ approximation, x_∞, is unique. Since we have indicated in Theorem 3.1 that the sequence $\{x_p\}$ is bounded, the convergence of $\{x_p\}$ follows, immediately, from the fact that $||x - x_\infty||_\infty \leq ||x - x_p||_\infty \leq ||x - x_p||_p \leq ||x - x_\infty||_p$ and that $\lim_{p \to \infty} ||x - x_\infty||_p = ||x - x_\infty||_\infty$.

Theorem 5.3. If A is a one-dimensional subspace of E_n, then $\lim_{p \to \infty} x_p$ exists.

Proof. Instead of the stated problem, we consider the equivalent problem of approximating the zero vector in a linear variety B which is a translate of A, and we assume, without loss of generality, that $\min_{b \in B} ||b||_\infty = 1$. The proof will be by induction. For $n = 1$, the result is trivial. In E_2, the problem is easily handled; for either the set $\{x_p\}$ consists of a single point or x_∞ is unique, and, if x_∞ is unique, the remarks preceding this theorem imply that $\lim_{p \to \infty} x_p$ exists.

Suppose the theorem holds in E_k, $k = 1,\ldots,n-1$. Let $S_p^k, 1 \leq k \leq n$ and $1 \leq p \leq \infty$, denote the unit l_p ball in E_k, and let $d_p = \inf_{b \in B} \|b\|_p$. If $B \cap S_\infty^n$ consists of a single point, $\lim_{p \to \infty} x_p$ exists. If not, B must lie in a linear variety, C, generated by one of the faces of S_∞^n. C must be of the form $C = \{(\alpha_1,\ldots,\alpha_n)$: for each k, $1 \leq k \leq n$, α_k is either identically one, identically minus one, or takes on all real values$\}$. Let $I_1 = \{k\colon\ \alpha_k = 1\}$ and $I_2 = \{1,\ldots,n\} - I_1$, and let the dimension of C be c. Then it is easy to verify that $(d_p S_p^n) \cap C$ is a multiple of the unit ball S_p^c centered about the n-tuple $\beta = (\beta_1,\ldots,\beta_n)$, $\beta_j = 1$ if $j \in I_1$ and $\beta_j = 0$ if $j \in I_2$, and tangent to B. Conversely, if dS_p^c is any multiple of the unit l_p ball in C which is centered at β and tangent to B, dS_p^c is the intersection of $(d_p S_p^n)$ with C. Therefore, the sequence $\{x_p\}$ of best approximations of zero, in B, in the n-dimensional space E_n, is the same as the sequence of best approximations of β, in B, in the c-dimensional space C. Since c is less than or equal to $n-1$, our induction hypothesis implies that this latter sequence converges, thereby, completing the proof.

Combining Theorem 5.2 and Theorem 5.3, we have the following:

<u>Theorem 5.4</u>. In two- or three-dimensional real Euclidean space, $\lim_{p \to \infty} x_p$ exists.

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*Abbreviations here used follow the form used by Mathematical Reviews.

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VITA

Wilbur Janes Stiles was born in Suffern, New York on January 12, 1932 to Wilbur Stiles and Elizabeth Janes Stiles. He spent most of his early life in Suffern and was educated in the Suffern public schools. He graduated from high school in June 1950 and entered Lehigh University in Bethlehem, Pennsylvania the following fall. He graduated from Lehigh in 1954 with a bachelor's degree in civil engineering.

Having taken ROTC training at Lehigh, he was called to active duty as a second lieutenant in the United States Air Force in October 1954. He was promptly sent to flight training in Tucson, Arizona and later to flight training in Laredo, Texas where he won his wings as a single-engine jet fighter pilot in December 1955. He then proceeded to basic single-engine pilot instructor school in Selma, Alabama for training as an instructor pilot, and subsequently returned to Laredo where he stayed as an instructor for the remainder of his tour of duty.

While in Laredo, Wilbur was married to Miss Evalyn A. Long in 1956, and their first child, Wilbur Janes Stiles II, was born in May 1959 - just one month prior to Wilbur's entrance to the Georgia Institute of Technology as a student in mathematics. Wilbur received a B.S. in Applied Mathematics in June 1960, and decided to remain at Georgia Tech for a master's degree in mathematics. He received the M.S. in Applied Mathematics in June 1962 and elected to study toward the Ph.D. Before receiving the Ph.D. in June 1965, his second child, John Edmund Stiles, was born in October 1962.

At present, Wilbur's principal interest is research in mathematics, and he has accepted a teaching and research position at the Florida State University with work beginning in September 1965.