ECONOMIC DESIGN OF CONTROL CHARTS
FOR MULTIVARIATE, MULTISTATE PROCESSES

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CHAPTER I

INTRODUCTION

Statistical Quality Control

The beginning of statistical quality control, as reported by Duncan (3) in his historical introduction, came with the publication of Walter A. Shewhart's first work on control charts in 1924. Throughout the 1920's and into the 1930's work continued at the Bell Telephone Laboratories on sampling inspection, with H. Dodge and H. Romig, as well as Shewhart, making important contributions. It was during this period that Hotelling (8) developed the theory of the T^2 distribution.

American industry as a whole did not adopt the new statistical techniques as rapidly as one might have expected. It took the federal government, principally the interest of the armed services in statistical quality control generated during the second world war, to spread the knowledge and practice across the country. The government not only adopted statistical techniques in its procurement of war materiel, but sponsored training programs for personnel involved in production. In addition, further research into sampling techniques was conducted, the results of which were at times considered so important that they were classified for the duration of the war.

It became apparent during this time, with both government procurement agencies and industrial suppliers involved in statistical quality control, that two separate aspects were involved. One, which included the control chart techniques, involved the process control function of industry. The second included the tools of the purchaser, acceptance sampling and conformance analysis. The research conducted for this thesis was principally concerned with the first aspect, control chart analysis and process control.

In the last thirty years, the tools available have grown from the initial Shewhart control charts and wartime sampling plans to encompass design of experiments, analysis of variance, and even more recently response surface analysis.

In their discussion of economic aspects of quality decisions, Grant and Leavenworth (7) cite papers as early as 1950 by one of them, E. L. Grant (5) (6), tying engineering economy to statistical quality control. Duncan (2) and Cowden (1) in the mid-1950's began to deal with economic models to minimize the cost of sampling plans, whereas the traditional approach to designing a control chart involved using purely statistical considerations when choosing the control chart parameters of sample size, critical region, and interval between samples.

Knappenberger and Grandage (10) in 1969 developed a cost model similar to those of Duncan (2) and Cowden (1), and used it in developing a least cost sampling scheme for controlling the mean of a normal process with an $\bar{x}$ chart. The work of these earlier researchers produced greater interest in the economic design of control charts. Klatt (9) developed a technique, based on the Knappenberger and Grandage model (10), for designing minimum cost sampling schemes for two normal variates controlled by a T^2 chart. Montgomery, Heikes and Mance (14) used this model for the multistate fraction defective control chart. The current research was a direct outgrowth of these later papers.

Purpose and Scope of the Thesis

The purpose of this research was to develop a general cost model for the Hotelling T^2 control chart, and a method of solving this model for a least cost sampling plan given various input parameters. For complete generality it was assumed that $p \geq 2$ quality characteristics were necessary to describe the process output, and that there could be any number $s \geq 1$ out-of-control states in which the process could operate.

A computer program was developed which allowed studies of the sensitivity of the model to changes in the parameters and which will in addition be useable for determining least cost sampling plans for real world processes.

We assume that there are a set of costs which tell the decision maker the relative costs of sampling, of investigating and correcting the process, and of producing defective items. The program, using the model, will then design a control chart and sampling plan by choosing the sample size N, the interval between samples K, and the control limit parameter T^2_α, which minimize the cost per unit of quality control (defined through the model as a function of these three costs). The control chart design is relative to a set of specifications imposed on the process output which are expressed as a pair of limit vectors.

Survey of the Literature

The publication by "student" in 1908 of the "student's t" distribution, involving the ratio of a mean to a standard deviation where both are estimated from a sample taken from a population, was followed over twenty years later by Hotelling's (8) extension of t to the multivariate case. Hotelling developed both T and T^2 in his 1931 paper (8), but the statistic T^2, given by

$$T^2 = N(\underline{\bar{x}} - \underline{\mu}_0)' \underline{\underline{S}}^{-1} (\underline{\bar{x}} - \underline{\mu}_0)$$

is the statistic generally used as the measure of simultaneous deviation of the several variates x_i of a process.

The use of the T^2 statistic where several variables are to be considered requires considerable computation on the part of the personnel conducting the sampling, so it is not surprising that the T^2 statistic was not in widespread use before the general availability of digital computers.

Cost is not always considered in developing quality control charts and acceptance sampling plans. Two of the earliest models which incorporated cost into the design of a control chart were those of Cowden (1) and Duncan (2). Cowden's model was very similar to that which was used here, involving as it does costs of sampling, of correcting an out-of-control process, and of producing defectives. Cowden's assumptions about the process are rather different, however, principally the assumption that once corrected the process remains in control until the end of the day.

Duncan (2) employed a different cost model, although still very similar to Cowden's and to that used in this research. The principal contributions of Duncan to this research were the implications of the assumption that shifts out of control occur randomly and that the mean time to shift is exponentially distributed.

The cost model introduced by Knappenberger and Grandage (10), which was based on the models both of Cowden and Duncan, was used here, as it was in Klatt (9) and in Montgomery and Klatt (15), without modification. This model breaks the cost of sampling into a fixed cost per sample plus an incremental cost per unit of sample size.

Klatt (9) used the cost model of Knappenberger and Grandage (10) in developing an analysis of the two-variable two-state model for control of a mean vector. In his work he independently developed the non-central analogue to Paulson's (19) approximation to the central F distribution. This non-central approximation was earlier developed by Severo and Zelen (21) and by Laubscher (11) as mentioned in Mudholkar et al. (17). The results of Klatt's work were published in a paper by Montgomery and Klatt (15).

Knappenberger and Grandage (10) dealt with a one-variable multi-state case for control of a mean by an $\bar{X}$ chart. In addition to contributing to the cost model, their paper developed a method for assigning probabilities to the transitions into and among the higher states which has been used in this thesis.

Montgomery, Heikes and Mance (14) proposed a method of condensing several out-of-control states into one in their analysis of the multistate fraction defective control chart. This technique was analyzed herein, as certain of the results apply to the control of a multivariate normal process.

An important aspect of the model developed in this research was the use of a subroutine to evaluate certain probabilities associated with the multivariate normal integral. Much of this work was based on a publication by Milton (12). Milton based his work on the existence of a general FORTRAN subroutine for error-controlled multidimensional quadrature. He developed some simplifications of the expression for the iterated multidimensional normal integral, which speed solution in this special case. The internal integral is evaluated using a fast FORTRAN normal approximation developed by Milton and Hotchkiss (13).

CHAPTER II

MULTIVARIATE QUALITY CONTROL

Although statistical quality control has been accepted by modern industry, tools for implementing quality control in a multivariate situation are not yet as advanced as those for the univariate situation. It is not generally valid to apply univariate techniques to several variables simultaneously since in many cases the variables being measured are not independent. If we consider two independent normally distributed variables, the criteria of

$$l_1 \leq x_1 \leq u_1$$

$$l_2 \leq x_2 \leq u_2$$

where l = lower control limit
u = upper control limit

yields a rectangular control region as depicted in Figure 1.

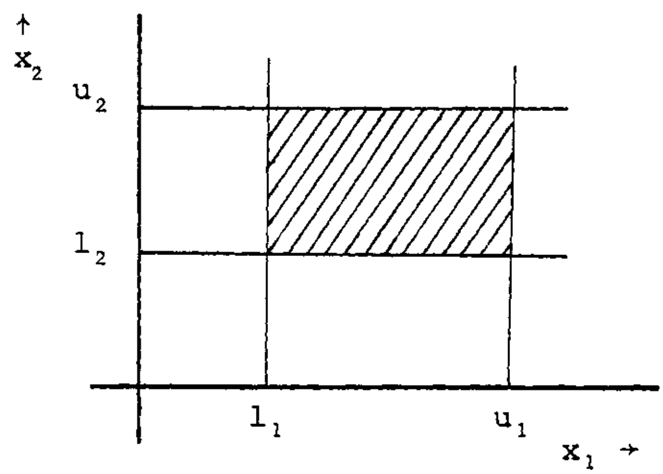

Figure 1: A Rectangular Control Region For Two Independently Distributed Variables

For two or more jointly distributed variables, the correct control technique as given by Hotelling (8) is the T^2 statistic, which defines a multidimensional ellipsoidal control region. In the bivariate case this might appear superimposed on the upper and lower independent limits as shown in Figure 2.

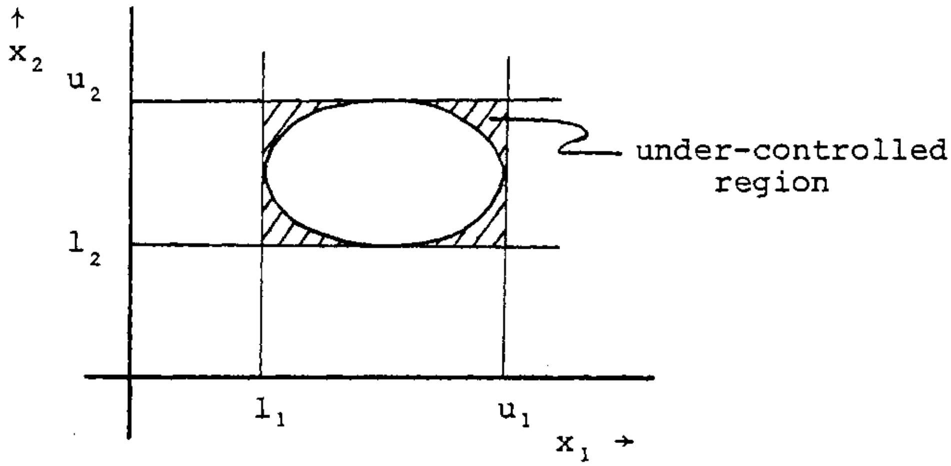

Figure 2. An Elliptical Control Region For Two Jointly Distributed Variables.

When using the independent limits as opposed to the correct elliptical region, there can obviously be areas of under-control.

If both variables are independently monitored by $\bar{x}$ charts based upon 3σ control limits, the true probability of rejecting the null hypothesis if $\underline{\mu} = \underline{\mu}_0$ is

$$1 - (1-\alpha)^2 = 1 - .9973^2 = .0054$$

and not the desired .0027 resulting from a univariate test using an $\bar{x}$ chart with the same 3σ control limits.

Costs associated with quality control include both those which penalize under-control, such as a customer's rejection of a shipment, and those costs which penalize over-control. Examples of these are the cost of items tested to destruction and other types of excessive sampling costs. In order to minimize the sum of these costs valid multivariate techniques must be employed.

The T^2 Control Chart

If we assume that the output of a process is described by a vector $\underline{x}$ of p variables, each representing a particular quality characteristic, and we assume that the distribution of $\underline{x}$ is given by the p-variate normal distribution, say

$$f(\underline{x}) = \frac{|\underline{\underline{\Sigma}}|^{-1/2}}{(2\pi)^{p/2}} e^{-1/2(\underline{x}-\underline{\mu})'\underline{\underline{\Sigma}}^{-1}(\underline{x}-\underline{\mu})}$$

where $\underline{\mu}$ is the mean vector of the p characteristics and $\underline{\underline{\Sigma}}$ is their variance - covariance matrix, then the correct control procedure is due to Hotelling (8). This states that the statistic

$$T^2 = n(\underline{\bar{x}} - \underline{\mu}_0)'\underline{\underline{S}}^{-1}(\underline{\bar{x}} - \underline{\mu}_0)$$

is distributed as Hotelling's T^2 with p and N_σ-p degrees of freedom, $\underline{\bar{x}}$ being the sample mean, $\underline{\mu}_0$ the vector of individual quality characteristic means corresponding to the in-control state, $\underline{\underline{S}}$ the estimate of $\underline{\underline{\Sigma}}$, N_σ the sample size used in computing $\underline{\underline{S}}$, and n the sample size associated with $\underline{\bar{x}}$. The T^2 variate with p and N_σ-p degrees of freedom is denoted by $T^2_{p,N_\sigma-p}$.

It is obvious in the bivariate case that the quadratic form

$$(\underline{\bar{x}} - \underline{\mu}_0)' \, \underline{\underline{S}}^{-1} (\underline{\bar{x}} - \underline{\mu}_0)$$

can be arranged into the conic equation of an ellipse, and, in fact, the T^2 statistic describes ellipsoidal control regions in any p-space. Thus a p-dimensional ellipsoid is the correct general shape of the control region.

If we now define a number $T^2_{\alpha;p,N_\sigma-p}$ as the upper tail α percentage point of the T^2 distribution, where

$$P\,\{T^2 \geq T^2_{\alpha;p,N_\sigma-p}\} = \alpha$$

then if $T^2 > T^2_{\alpha;p,N_\sigma-p}$ we conclude that the process mean has shifted. Implementation of this control procedure is fairly simple once the percentage point $T^2_{\alpha;p,N_\sigma-p}$ has been determined. A T^2 chart similar in appearance to an $\bar{x}$ chart for controlling only an increase in $\underline{\mu}$ is shown in Figure 3. Note that the chart has only an upper control limit. Furthermore, for p > 2, this form of the chart is easier to implement in practice than the control ellipse.

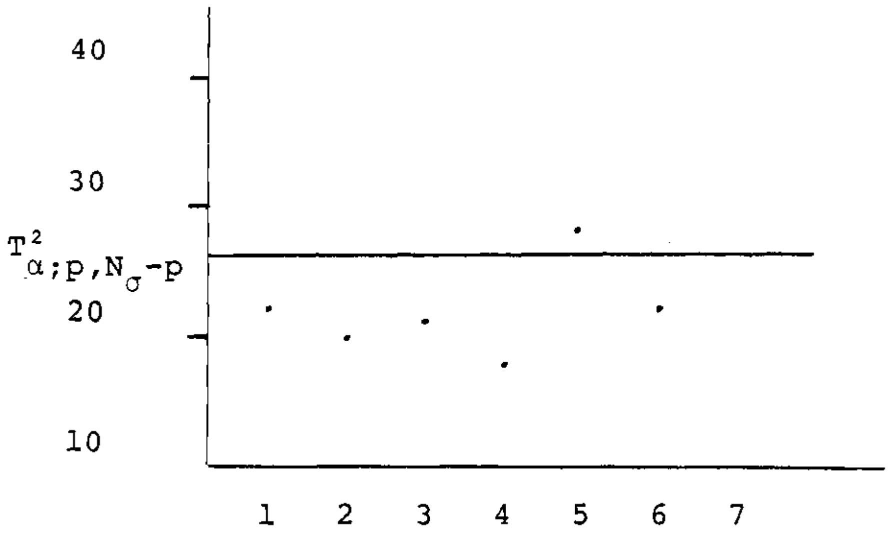

Figure 3. An Example T^2 Chart

The mean difference vector $(\underline{\bar{X}}-\underline{\mu}_0)$ is computed from current sample data. (Typically, the mean vector $\underline{\mu}_0$ and covariance estimate $\underline{\underline{S}}$ are known from historical data.) The T^2 statistic is computed, which for p above 2 or 3 is best done mechanically, and the statistic is plotted on the T^2 chart. If $T^2 > T^2_{\alpha;p,N_\sigma-p}$ the process is assumed to be out of control. It is not possible to tell which individual quality characteristics are out of control from this chart, and this may not even be immediately apparent from analysis of the individual data points. The T^2 chart merely provides an indication as to whether or not to interrupt the process. In Figure 3 above, the process would be halted after the fifth sample for investigation and possible correction.

It is possible to obtain the percentage points of $T^2_{\alpha;p,N_\sigma-p}$ from the tables of the F distribution, since

$$T^2_{\alpha;p,N_\sigma-p} = \frac{p(N_\sigma-1)}{N_\sigma-p} F_{\alpha;p,N_\sigma-p}$$

Therefore, it is not necessary to provide special tables of the cumulative distribution of $T^2_{p,N_\sigma-p}$.

The Cost Model

The general form of the economic model was developed by Knappenberger and Grandage (10). It has been used by several authors, including Montgomery and Klatt (15) in their previous study of the T^2 chart, and Montgomery, Heikes and Mance (14) in an analysis of the fraction defective or p chart. This model has the dual advantages of simplicity and intuitive appeal. It is a general multi-state formulation and totally independent of the number of quality characteristics considered. For this reason, although Knappenberger and Grandage (10) considered a multi-state univariate case, and Montgomery and Klatt (15) considered a two-variable two-state case, the models used in those two papers and in this research are identical.

The expected value of the total cost of the quality control program per unit of product is represented as the sum of three terms:

$$E(C) = E(C_1) + E(C_2) + E(C_3)$$

Here, C_1 is the cost per unit of testing, C_2 the cost per unit of investigation and possibly correcting the process when an out-of-control situation is detected, and C_3 is the cost per unit of producing defective product.

The cost C_1 was first represented by Cowden (1) and Duncan (2) as consisting of two terms, one representing the cost of taking a sample regardless of the sample size, and the second representing the incremental cost per unit times the sample size. Thus,

$$E(C_1) = \frac{A_1}{K} + \frac{A_2 N}{K} \tag{2.1}$$

where A_1 is the fixed cost per sample

A_2 is the cost per unit sampled

N is the sample size

K is the number of units produced between samples, including the sample itself.

Knappenberger and Grandage (10), in analyzing the cost C_2 of investigating and correcting the process when it appears out of control, concluded that because of inaccurate information and offsetting effects it was often unfeasible to consider information more detailed than the long term values for the frequency with which the process goes out of control, how long the process remains inoperative while being corrected, and the hourly cost of having the process inoperative. Then if we define the cost A_3 as the expected value of the total cost of investigating and correcting an out of control process, and consider the probability of failure to accept the null hypothesis (where the cost of correction is zero when H_0 is accepted) the expected cost per unit of investigation and correction can be written

$$E(C_2) = \frac{A_3 P\{\text{Fail to accept } H_0 | \text{Sample}\}}{K}$$

If we let s be the number of possible out-of-control states, $\underline{\rho}'$ be the row vector of probabilities $\rho_{i,i=0,\ldots,s}$ such that

$$\rho_i = P\{\text{Fail to accept } H_0 | \underline{\mu} = \underline{\mu}_i\}$$

and we let $\underline{\beta}$ be the column vector of steady-state probabilities β_i that the process is in state "i", i.e.,

$$\beta_i = P\{\underline{\mu} = \underline{\mu}_i\}$$

then

$$E(C_2) = \frac{A_3}{K} \sum_{i=0}^{s} P\{\text{Fail to accept } H_0 | \underline{\mu} = \underline{\mu}_i\} P\{\underline{\mu} = \underline{\mu}_i\}$$

$$= \frac{A_3}{K} \sum_{i=0}^{s} \rho_i \beta_i = \frac{A_3}{K} \underline{\rho}' \underline{\beta} \qquad (2.2)$$

Now consider the cost A_4 of producing a defective item. Let $\underline{\phi}'$ be the row vector of probabilities ϕ_i, where ϕ_i is the conditional probability of producing a defective item given that the process is in state "i", and let $\underline{\gamma}$ be the vector of probabilities γ_i that the process is in state "i" when a sample is taken. Then the expected cost per unit of erroneously accepting the null hypothesis becomes

$$E(C_3) = A_4 P\{\text{produce a defective} \mid \underline{\mu}=\underline{\mu}_i\} P\{\underline{\mu}=\underline{\mu}_i \mid \text{sample}\}$$

$$= A_4 \sum_{i=0}^{s} \phi_i \gamma_i$$

$$= A_4 \underline{\phi}' \underline{\gamma} \qquad (2.3)$$

Thus the total expected cost is represented as the sum of the components in (2.1), (2.2), and (2.3) or

$$E(C) = \frac{A_1 + A_2 N}{K} + \frac{A_3}{K} \underline{\rho}' \underline{\beta} + A_4 \underline{\phi}' \underline{\gamma}$$

Note that A_4 is really an expected cost, weighted over all possible types of defective units. In general, some types of defective units could produce costs much greater than others, with costs of product recall and of suits for damages among the highest.

Mathematical Development

The four vectors of the cost model, $\underline{\rho}$, $\underline{\beta}$, $\underline{\phi}$, and $\underline{\gamma}$ are all dependent on the three control chart parameters, that is the sample size N, the sampling interval K, and the critical region for the chart α. The vectors $\underline{\rho}$, $\underline{\beta}$, and $\underline{\gamma}$ depend on the interval between samples K, the sample size N, and the critical region of the test. The vector $\underline{\phi}$ depends upon the upper and lower specification limits imposed on product quality characteristics.

The assumption is made that the quality characteristics x_i which form the vector $\underline{x}$ are jointly distributed as the p-variate normal. This assumption of normality is intuitively appealing, and leads to the T^2 distribution with p and N_σ-p degrees of freedom as the basis of the control procedure. In computing the values of the integral of $T^2_{p,N_\sigma-p}$ an approximation due to Paulson (19) involving $F^{1/3}$ is used. As Paulson notes, we must have $N_\sigma - p \geq 3$ to insure that $P\{F<0\} \cong 0$. In the current research, N_σ was always assumed sufficiently large so that this was not a problem.

The importance of this restriction was brought home clearly in re-running the four examples from Klatt (9) using his program where the sample size N and N_σ are always equal. There is a trap in his program to detect non-convergent cases which, if invoked, still prints the results of the computations. This trap showed non-convergence in the optimization routine only in those instances where N-p was equal to 1 or 2. A similar trap has been incorporated into the new program as situations causing model non-convergence due to an attempt to detect too small a shift in the mean have also been found.

The region of acceptable process output is defined by two specification vectors $\underline{l}$ and $\underline{u}$, such that each quality characteristic $x_{j,j=1,2,\ldots,p}$ obeys the rule $l_j \leq x_j \leq u_j$. Trial values of the type I error α are input to the economic model. These are mapped into appropriate values of $T^2_{\alpha;p,N_\sigma-p}$ and the optimal value of α is printed along with the optimal value of $T_{\alpha;p,N_\sigma-p}$ for the control chart, to give the prospective user of the program a fuller understanding of the test criteria.

The elements $\phi_{i,i=0,1,\ldots,s}$ of the probability vector $\underline{\phi}$ are properly

$$\phi_i = 1-\int\left|\int_{\underline{l}}^{\underline{u}}(2\pi)^{-p/2}|\underline{\underline{\Sigma}}^{-1}|e^{-1/2(\underline{x}-\underline{\mu}_i)'\underline{\underline{\Sigma}}^{-1}(\underline{x}-\underline{\mu}_i)}dx_1dx_2\ldots dx_p\right.$$

Since the values of the elements of $\underline{\underline{\Sigma}}$ are not known, $\underline{\underline{\Sigma}}$ is approximated by its unbiased estimator $\underline{\underline{S}}$, which is assumed based on a previous sample of size N_σ. Then the $\phi_{i,i=0,1,\ldots,s}$ are approximated by

$$\phi_i = 1-\int \Big| \int_{\underline{l}}^{\underline{u}} (2\pi)^{-p/2} |\underline{\underline{S}}^{-1}| e^{-1/2(\underline{x}-\underline{\mu}_i)'\underline{\underline{S}}^{-1}(\underline{x}-\underline{\mu}_i)} dx_1 dx_2 \ldots dx_p$$

The value of the multivariate normal integral is determined by use of a program developed by Milton (12) and incorporated into the program utilized in this study with minor modifications. This is discussed more fully in the section on Milton's Normal Integral Evaluation in Chapter III.

The vector $\underline{\rho}$ is defined such that

$$\rho_0 = P\{T^2 > T^2_{\alpha;p,N_\sigma-p}\} = \int_{T^2_{\alpha;p,N_\sigma-p}}^{\infty} f(T^2)dT^2$$

and the $\rho_{i,i=1,2,\ldots,s}$

$$\rho_i = P\{T^{2'} > T^2_{\alpha;p,N_\sigma-p}\} = \int_{T^2_{\alpha;p,N_\sigma-p}}^{\infty} f(T_i^{2'})dT_i^{2'}$$

where $f(T^2)$ is the distribution function of T^2 with p and N_σ-p degrees of freedom given in Hotelling, and $f(T^{2'})$is the distribution function of the non-central T^2 with p and N_σ-p degrees of freedom and non-centrality parameter

$$\tau_i = N(\underline{\mu}_i-\underline{\mu}_0)'\underline{\underline{\Sigma}}^{-1}(\underline{\mu}_i-\underline{\mu}_0), i=0,1,2,\ldots,s$$

where N is the sample size associated with $\underline{\bar{x}}$. The matrix $\underline{\underline{\Sigma}}$ is not known. Consequently, it must be approximated by its unbiased estimator $\underline{\underline{S}}$, and the τ_i's we must work with are estimates of the true process τ_i's. The τ_i form a vector $\underline{\tau}$ of dimension s+1, where τ_0 is obviously zero, allowing the non-central T^2 distribution to degenerate to the central T^2 when $\underline{\mu} = \underline{\mu}_0$. The integral $\int f(T^2)dT^2$ is estimated as mentioned above, the manner of estimation being explained in the section on conduct of the validation steps in Chapter III. Note that ρ_0 is α, the probability of type I error for the control chart while ρ_i, for $i \geq 1$ is the power of the test, since ρ_i is the probability of rejecting the hypothesis of statistical control when the process is in state "i".

In order to compute the vector $\underline{\beta}$ of probabilities β_j that the process is in state "j" when a sample is taken, the transition probability matrix $\underline{\underline{B}}$ is required. Its elements represent the probabilities $b_{j,k}$ of the process shifting from state "j" to state "k" during the interval between samples. The mathematical development of equations for computing the elements of the $\underline{\underline{B}}$ matrix requires several additional assumptions about the behavior of the process.

It must first be assumed that the process is neither self-correcting nor self-improving. This implies that all transitions from a higher state to a lower state are forbidden. The process never gets better, but it may get worse. It is also assumed that the average interval between shifts is the same for transitions between higher states as it is for transitions out of state 0.

There are considerations involving definitions of states which were not introduced into the current model. In particular, when one characteristic x_i can go out of control either above or below the mean, it is unrealistic to permit transitions from a state where $x_i < \mu_i$ to one where $x_i > \mu_i$. Also, since a vector field is not well-ordered, there is no natural hierarchy of states. For instance, of the four states

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1.3 \end{bmatrix}, \quad \begin{bmatrix} 1.6 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1.6 \\ 1.3 \end{bmatrix}$$

which state is state 2? For the current development, this problem is passed over, but it is discussed again under "Reccommendations."

As a first step in developing $\underline{\beta}$, we define the vector $\underline{q}$ as the a priori vector of probabilities $q_j, j=1,2,\ldots,s$ that the process will shift from the in-control state 0 with mean vector $\underline{\mu}_0$, to an out-of-control state "j" with mean vector $\underline{\mu}_j$. If we assume that the time the process remains in control, given that it is now in control, is exponentially distributed, then a known mean-time-to-shift λ^{-1} gives a probability of remaining in state 0 for a time of h hours of

$$1 - \int_0^h \lambda e^{-\lambda t} \, dt = e^{-\lambda h}$$

If we also assume a known production rate of R units per hour, then the probability of remaining in control while producing K units of product is

$$q_0 = e^{-\lambda K/R}$$

The remaining probabilities must be assigned such that

$$\sum_{j=1}^{s} q_j = 1 - e^{-\lambda K/R} \tag{2.4}$$

Knappenberger and Grandage (10) have proposed a binomial weighting scheme for assigning these probabilities. If we assume a binomial distribution among the states, then the weights w_j are

$$w_j = \frac{s!}{j!(s-j)!} \pi^j (1 - \pi)^{s-j}, \quad 0 < \pi < 1$$

If w_0 is known then

$$\sum_{j=1}^{s} w_j = 1 - w_0 = 1 - (1 - \pi)^s. \tag{2.5}$$

Combining (2.4) and (2.5) so that

$$q_0 + \sum_{j=1}^{s} q_j = 1$$

we can set

$$q_j = \frac{(1 - e^{-\lambda K/R})}{1 - (1 - \pi)^s} \cdot \frac{s!}{j!(s-j)!} \pi^j (1 - \pi)^{s-j}$$

There is no real-world justification for this assignment, but it is a reasonably simple solution, and choice of $\pi \cong j/s$ can give a distribution favoring a known most likely out of control state "j". (For a most probable state "s", π must equal some number close to but less than 1, e.g., $\pi = 1 - 1/s^2$.) This technique in fact gives an entire family of distributions indexed by the parameter π.

Having defined the q_j, we proceed to develop the matrix $\underline{\underline{B}}$. The elements $b_{j,k}$ of $\underline{\underline{B}}$ are composed, in general, of terms representing transfers into and out of state 0, transfers from lower to higher states, and non-transfers when the process remains in the same state. The matrix $\underline{\underline{P}}$ of probabilities $P_{j,k}$ of transfers among states above state 0 must be defined. Knappenberger and Grandage (10) make these probabilities $P_{j,k}$ proportional to the binomially weighted probabilities q_k of transfer into state "k" from state 0. Again, there is no real-world justification for this, but it seems a reasonable assignment in the absence of knowledge to the contrary. According to this technique, the $P_{j,k}$ are

$$P_{j,k} = 0 \qquad , \; j > k$$

$$P_{j,k} = \sum_{m=1}^{k} \frac{q_m}{1-q_0} \qquad , \; j = k$$

$$P_{j,k} = \frac{q_k}{1-q_0} \qquad , \; j < k$$

The elements $b_{j,k}$ of $\underline{\underline{B}}$ can now be written as follows:

(1) For $j = 0$, $b_{j,k} = q_k$

(2) For $j \neq 0$, each element contains a term

P{Reject H_0 at time t}*P{Shift $\underline{\mu}_0 \rightarrow \underline{\mu}_k$ during production of K units}

$= \rho_j q_k$

This term represents the probabilities of a shift out of state 0 in the time between two successive samples, following the correction of a previous shift.

(a) For $j > k$, this is the only term

(b) For $j = k$, $k > 0$ there is an additional term of the form

P{Fail to reject H_0 at time t}*P{Remain in state "j" during production of the next K units}

$$= (1-\rho_j)P_{j,j} = (1-\rho_j)\,\frac{\sum_{m=1}^{j} q_m}{1 - q_0}$$

which is the probability of the process's remaining undetected for an entire sampling period in the same out-of-control state.

(c) For $0 < j < k$ there is an additional term of the form

P{Fail to reject H_0 at time t}*P{Shift $\underline{\mu}_j \to \underline{\mu}_k$ during production of the next K units}

$$= (1-\rho_j)P_{j,k} = (1-\rho_j)\,\frac{q_j}{1-q_0}$$

which is the probability of an out-of-control process's remaining undetected and going further out of control before the next sample.

As an example, the matrix $\underline{\underline{B}}$ for s=2 is as follows:

$$\begin{bmatrix} q_0 & q_1 & q_2 \\ \rho_1 q_0 & \rho_1 q_1 + (1-\rho_1)\,\dfrac{\sum_{m=1}^{1} q_m}{1-q_0} & \rho_1 q_2 + (1-\rho_1)\,\dfrac{q_2}{1-q_0} \\ \rho_2 q_0 & \rho_2 q_1 & \rho_2 q_2 + (1-\rho_2)\,\dfrac{\sum_{m=1}^{2} q_m}{1-q_0} \end{bmatrix}$$

It is easy to demonstrate that each row of such a matrix $\underline{\underline{B}}$ sums to 1. This, together with the assumption that the process is never self-correcting, is given in many references (4) (18) as defining the transition matrix of an irreducible aperiodic positive recurrent Markov chain. Therefore, it can be shown that there exists a vector $\underline{\beta}$ of elements β_j such that

$$\underline{\beta}'\underline{\underline{B}} = \underline{\beta}', \quad \text{where} \sum_{j=0}^{s} \beta_j = 1$$

and that the vector $\underline{\beta}$ is the vector of steady-state probabilities β_j of being in state "j".

It can be shown that, since $\sum_{m=0}^{s} \beta_m = 1$

$$\underline{\beta}' = (\underline{0}',1)\underline{\underline{B}}^{*-1}$$

where $(\underline{0}',1)$ is an s + 1 dimensional row vector of s zeros and a 1, and $\underline{\underline{B}}^*$ is the s + 1 dimensional square sub-matrix of the matrix

$$\left[\underline{\underline{B}}-\underline{\underline{I}}\,|\,\underline{1}\right]$$

formed by eliminating any column of $[\underline{\underline{B}}-\underline{\underline{I}}]$. In this development, and the example given later, column s was arbitrarily eliminated since when coded into the computer program this involves merely a replacement of column s by ones.

The vector $\underline{\gamma}$ is the vector of probabilities γ_j of being in the state "j" given the previous state of the process. (The vector $\underline{\beta}$ was defined irrespective of the previous state of the process). Duncan (2) has shown that given a shift between any two samples from $\underline{\mu}_0$ to some other $\underline{\mu}_j$, the average fraction of the interval that elapses before the shift occurs is

$$F = \frac{1 - (1+\lambda K/R)e^{-\lambda K/R}}{\lambda K/R(1-e^{-\lambda K/R})}$$

Knappenberger and Grandage (10) make the assumption that the fraction F is valid for shifts from $\underline{\mu}_j$ to $\underline{\mu}_{k,j\neq 0}$, as well as for those from $\underline{\mu}_0$ to $\underline{\mu}_k$

Then the value of γ_0, as given in Knappenberger and Grandage (10) is

$$\gamma_0 = \beta_0 q_0 + F\beta_0(1 - q_0),$$

the two terms representing the probability of being in and remaining in state $\underline{\mu}_0$, and of being in but shifting out of state $\underline{\mu}_0$.

The $\gamma_{j,j\neq 0}$ include four terms -- the probability that the process is in state $j > 0$ at time t and remains there for the production of K units, the probability that the process is in state 0 at time t and shifts to state "j", the probability that the process is in state $k < j$ at time t and shifts to "j" (this probability is zero when $j = 1$) and the probability that the process is in state "j" and shifts to $k > j$ (zero when $j = s$). Thus we have for $j = 1,2,\ldots,s$

$$\gamma_j = \beta_j \frac{\sum_{m=1}^{j} q_m}{1-q_0} + \beta_0(1-F)q_j + \sum_{m=1}^{j-1} \beta_m \, (q_j/1-q_0)(1-F)$$

$$+ \frac{\beta_j F}{1-q_0} \sum_{m=j+1}^{s} q_m$$

CHAPTER III

DEVELOPMENT AND VALIDATION OF THE COMPUTER PROGRAM

Precursor Programs

Klatt's Bivariate Bistate Solution

The program presented in this thesis is based on other programs presented in Klatt (9) and Milton (12)(13). Klatt's work, which was also reported in a subsequent paper (15), developed a computer program which solved the same cost equation as used here for the two-variable, two-state (i.e., one out-of-control state) case. From the point of view of the potential user, however, this program is inconvenient in that it requires a great deal of preparatory work. This is largely attributable to a direct conversion of the mathematical operations into FORTRAN without any attempt at using efficient programming techniques or simplifying input. A second cause was the need to use bivariate normal tables for the probabilities of producing defectives, since an approximation or numerical evaluation of the bivariate normal integral was not available.

As mentioned briefly in the Introduction, if the program presented here is to be of use to professional practitioners, it must be both correct and easy to use. Therefore, the required input has been changed in many cases. A comparison table of the inputs required by Klatt's program and the program presented here is to be found in Appendix 7. Internally, Klatt's program in the parameter optimization phase steps through various values of N, K and T^2. This is perfectly valid, but for a better intuitive meaning the variables in the new program have been changed to N, K and α, with the value of T^2 computed from α. The value of the optimal α is displayed in each case with the optimal T^2.

Milton's Normal Integral Evaluation

Milton has published several papers (12) (13) on normal integral evaluation. In his paper on the multivariate integral (12),he presented a self-contained program which would evaluate the multivariate integral with distribution parameters as input. For the present purpose this was insufficient. Once again, ease of implementation to the potential user had to be considered, and this consideration ruled out taking output from one free-standing program and using it as input to another. Even communication via files was unreasonably complicated since file input and output in FORTRAN varies from one compiler to another. The evaluation program was therefore changed to give it the format of a FORTRAN function subprogram. The multi-dimensional volume within a set of limits can, via this technique, be determined by one use of the subprogram, with arguments of limit vectors, covariance matrix, mean vector, and other items as necessary to define the problem. A driving function was written which replaces Milton's main program and first level subroutine and which fragments the outermost level of the iterated integral into the desired number of parts.

In the course of modifying Milton's program, it was necessary to change from a form where the entire program was coded for a particular number of variables to one in which the number of variables could be determined at run time. This was done by deleting internally defined matrices (the covariance matrix and its inverse) and passing both them and their dimension, the number of variables, through the subroutine or function argument list. To use this technique throughout the program, the main program is very short, serving merely to define the matrices in their largest form (12 x 12) and inquire of the user what the problem dimensions are. The matrices and dimensioning information are then passed to a first level subroutine where all further inputs are processed and where all the primary computation is done. Control never returns to the main program.

A General Matrix Inversion Routine

A subroutine called INVERT was included among the routines received from Dr. Milton at the National Institute of Health. This routine inverts symmetric matrices and provides their determinants, and is sufficient when dealing with covariance matrices. The matrix $\underline{\underline{B}}$ used in developing the $\underline{\beta}$ vector is not symmetric, however, and consequently another routine, MTINV, was incorporated into the program. MTINV, from the Honeywell Level 66 mathematics library (23), inverts any non-singular matrix but returns no determinant. Because of the need for the covariance matrix determinant, both routines were retained and used where appropriate.

Program Validation

One of the most serious problems in the development of programs for iterative solutions is their validation, since if there were resources to check all the program's steps thoroughly, the need for the program would be greatly lessened. The validation therefore had to be less than complete, and proceeded piecemeal.

Outline of Validation Steps

Validation of the program presented herein fell into several phases:

(1) Checking of steps in the mathematical development and the adherence of the programs and subroutines to those steps.

(2) Hand checking of the development of Klatt's (9) inputs (requiring some extended calculation as mentioned above).

(3) Re-generation of Klatt's (9) output using his program converted to run on the Honeywell 66/40.

(4) Comparing output from Milton's (13) univariate normal routine against normal tables.

(5) Re-running of the problems used by Milton (12) to validate his multivariate normal integration program.

(6) Comparing output from Milton's (12) program against NBS bivariate normal tables (22).

(7) Comparing output from the multivariate $T^2(\alpha)$ integral evaluation (using successive approximations of the $\int T^2(\alpha)$ inverse function) against the bivariate evaluation (a closed-form calculation).

(8) Checking intermediate results against hand computation.

(9) Iterating to a solution using the example problem from Montgomery and Klatt (15).

(10) Running the new program to compute values at the tabulated points in Montgomery and Klatt (15).

Conduct of Validation Steps

Mathematical Development. This was checked internally in Klatt (9) and Milton (12). In following Klatt's development of the non-central F approximation and in modifying the program to compute the multivariate T^2 integral, calculations were checked back to the original papers of Paulsen (19) and Hotelling (8). Klatt's (9) development of the non-central F was a duplication of the work of Laubscher (11) and Severo and Zelen (21) as noted in Mudholkar et al. (17). The accuracy of this approximation is studied in Mudholkar et al. (17), who note that it is the least accurate of the four approximations studied. However, at its worst case it is accurate to 1.6×10^{-2}, or about 10% of the true value at that point. It is seldom worse than 1% of the true value, and with carefully chosen values of N_{σ}, it can be held within this range. When the uncertainty of some of the estimated parameters is considered, this seems to be reasonable accuracy, and the approximation was not changed to one of those shown in Mudholkar et al. (17), nor to the approximation in Searle (20).

Klatt's coding was checked line by line against his development and was correct except for typographical errors. Milton's coding was checked similarly up to the point where the multidimensional quadrature routine was entered. Beyond this, the agreement of numerical results with known values seemed to justify acceptance. As well, the fact that Milton's routines were loaded from a punched card deck, a reliable medium, argues for their correctness.

Re-computation of Klatt's Input. Klatt's program was confirmed in regenerating the output values published in Montgomery and Klatt (15). There were errors in preparation of input parameters in Klatt (9), however, which are traceable to an incorrect definition of the domain of integration of the bivariate normal distribution. Instead of integrating over domain I, shown in Figure 4, integration was performed over domain II. These errors in Klatt's original work were deleted prior to publication in journal form, and consequently the tables reported in Montgomery and Klatt

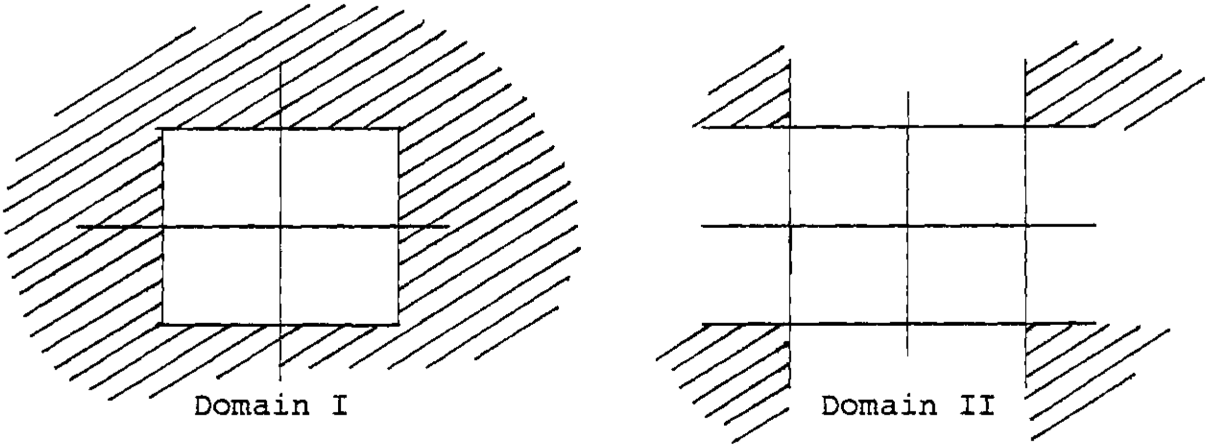

Figure 4. Error In Definition Of Domain Of Integration Of Bivariate Normal Distribution

(15) are valid. Ultimately, good agreement with this paper was achieved.

Regeneration of Klatt's Output. Given the published input, the program converted to the Honeywell machine gave good agreement. In this phase of validation, the divergent solutions showed up corresponding to $N = N_{\sigma} < p+3$.

Comparing Values of the Univariate Normal Integral. The values produced by Milton's (12) subroutine were tabulated and compared to published values. Results were good to $\pm 5 \times 10^{-5}$. Detailed results are given in Appendix 2.

Re-running Milton's Checks. Values for the volumes of infinite rectangles and hyper-rectangles and of the n-dimensional orthants gave good agreement with known values.

Bivariate Integral Comparison. The bivariate tables published by NBS (22) were re-generated for correlation coefficients of 0, .45, and .5 corresponding to Klatt's three trial covariance matrices. Results were good to approximately 5 x 10^{-5}. Detailed results are shown in Appendix 3.

Closed Form for T^2 Compared to Approximation. In the bivariate case, the formula from page 375 of Hotelling (8)

$$\alpha = \frac{1}{\left[1+\frac{T^2}{n}\right]^{\frac{n-1}{2}}}$$

where n is the degrees of freedom, N_σ-1, gives a closed form expression for T^2, that is,

$$T^2 = n\left[\left[\frac{1}{\alpha}\right]^{\frac{2}{n-1}} -1\right]$$

This formula was used by Klatt to convert the initial α parameters into T^2, but no such formula exists for problems where the number of variables, p, is odd. To derive the value of T^2, successive approximations to the desired α were made using Paulson's (19) approximation to the cumulative F distribution function. The results yielded by this technique agree very well with the closed form, except near the limits of small N_σ, p and F. Detailed results are shown in Appendix 4. As is shown in Mudholkar et al. (17), so long as N_σ is adjusted to keep $F \geq 3$, the error remains $\leq 2\%$.

Checking Intermediate Results

Manually checking the results of a full iterative solution would involve considerable effort. Therefore, the cost at only one point was manually computed and compared to the program's derived cost. For this purpose, since the subroutines had previously been checked, their results were accepted as input to the manual computations whenever subroutines were encountered. Thus the manual computations checked only the main program's results and did not further validate the subroutines.

The central point in the table of Optimal Test Parameters from Montgomery and Klatt (15) (their Table 1) was selected as the point to use. Machine differences, the publishing of the value of K only within ±50, and the use of the subroutine for evaluating the multivariate normal cumulative distribution function instead of interpolating from bivariate tables, prevented complete agreement with the results of Montgomery and Klatt (15). But having their cost as an approximation of the desired result kept errors in the mathematical development from remaining uncorrected through this step. In this way it was discovered that in the computation of $\underline{\beta}$, where

$$\underline{\beta}' = (\underline{0}', 1)\underline{\underline{B}}^{*-1}$$

and where $\underline{\underline{B}}^{*}$ is the s+1 dimensional square sub-matrix of the matrix

$$[\ \underline{\underline{B}}-\underline{\underline{I}}\,|\,\underline{1}\ ],$$

the subtraction $B_{0,0}-I_{0,0}$ was not being made, resulting in negative probabilities in $\underline{\beta}$. This was corrected, and results agreed well with Montgomery and Klatt (15). Their estimated cost was \$.08711, while the cost produced here was \$.0834.

In order to compare results against those of Montgomery and Klatt (15), where T^2 but not α is printed, the ability to define T^2 rather than α and to disable the iteration was incorporated into the program. This ability was also important in the last validation step.

Comparing Minimal Solutions. Once it was established that the costs derived by Montgomery and Klatt could be reproduced, the program was allowed to solve for the minimal cost. Surprisingly, while the minimal cost was close to that of Montgomery and Klatt, the operating point was quite different. In fact, the minimal operating point was always found at the minimal N (N=5) and α(α=.001). This seemed unlikely, so all total costs were printed - 27 per iteration - which showed that the minimum was being correctly determined. A recheck of the program reproduced in Klatt (9) showed that his search scheme involved only one reduction in step size. The final search was conducted with a step of 1/10 K. This meant, for this problem, a size for K in the neighborhood of 70. To this degree of refinement the surface seemed convex. The program used in this study will reduce step size to 1 in both N and K in all cases, at which time non-convexity in the surface is revealed. The iterative scheme was thus proven, though not by its agreement with prior work.

Comparison with Tabulated Values. Comparison with the results in Montgomery and Klatt (15) was, for the reasons given above, made using the non-iterative technique. Agreement was good, except that for $A_2=A_3=.001$, all results given in Montgomery and Klatt (15) are forbidden by Paulsen's rule that N be $\geq p+3$. A comparison of estimated costs derived by the two programs is given in Appendix 5. Interestingly, in those situations where N was equal to three or four in Montgomery and Klatt, for an N of 5 the value of α is obviously divergent, as shown in Appendix 5.

Language Standardization

If any computer program is intended to be run on a number of machines, standardization of language must be a consideration. Most major computer manufacturers provide a close approximation to ANSI standard FORTRAN, or an extended standard FORTRAN. To insure compatibility among machines, it is necessary to understand what constitutes such extensions and where compatibility problems will arise in non-standard areas.

In writing this program, any dependence on the Honeywell Level 66 computer or its software was avoided. In particular, no executive functions such as a system clock interface are invoked, and, while a line-numbered program is used, the format of each line is such that it is left in standard fixed-field FORTRAN format if the line numbers are removed. No line is longer than 72 characters, to permit standard teletype listing. Also, since different systems have different input and output device numbers, all input and output statements reference variables containing logical unit numbers, rather than referencing constants.

Avoidance of extensions amounts to limiting the FORTRAN repertoire to a fairly common subset. In this program, no ENTRY statements were used and no CHARACTER variables were defined, the one alphabetic input being read into an integer word. All DIMENSION and TYPE statements are located at the head of programs and subroutines. No null arguments have been passed. No logical variables have been used, although some logical IF statements exist. In short, to move from the Honeywell Level 66 to another machine which supports six character variable names and logical IF statements, the only consideration should be the format or existence of free-field input. Free-field input is such a convenience to the user that it was incorporated despite this compatibility drawback. (A listing of the program can be found in Appendix 7.)

Search Procedure

The search procedure is a multi-step grid search patterned directly after Klatt (9). However, where Klatt used a two-stage grid search, the program presented here uses as many step reductions as necessary for convergence. The grid in question is defined by the limits on the parameters N, K, and α. In all, 27 points are evaluated at each iteration, corresponding to the center point (N_0, K_0, α_0) and all combinations of $(N_0 \pm \Delta N, K_0 \pm \Delta K, \alpha_0 \pm \Delta\alpha)$. These points lie at the center, corners, centers of edges and centers of faces of a solid rectangle in N, K, α space. This rectangle is loosely called a cube here for convenience.

Klatt's sole convergence criterion was that the minimum of the 27 costs computed within the N, K, α cube occur at the cube's center point. Here, this criterion is sufficient to declare convergence only if all step sizes are also minimal. Before this condition is met, Klatt's convergence criterion merely triggers a reduction in step size. The step size is never increased.

To optimize the search, some experimentation with the step reduction factor was done. A factor of 1/8 was settled upon, since it was optimal in one case and held the number of iterations (27 points of evaluation per iteration) to about 12 in most others. As a comparison, a step reduction factor of 1/2 seems to cause convergence in about twice this number of iterations.

The user is required to enter search limits for N, K, and α. These are in reality the boundaries of the first cube to be searched, not true limits. Whenever a minimum is found on the cube's surface, the limits are adjusted to place that point at the center of a new cube of the same size. Thus after each iteration, either the step size is reduced or the cube is shifted toward the minimum.

There are certain limits imposed on the N, K, and α parameters as listed in Table 1.

Table 1. Limits on N, K, and α

1. $N \geq 5$
2. $N \leq K$
3. $\alpha \geq .001$
4. $\Delta N \geq 1$
5. $\Delta K \geq 1$
6. $\Delta\alpha \geq .001$

The first of these, for reasonably well-behaved distributions, assures that the distribution of the sample mean is close to normal via the central limit theorem. The .001 limit on α and on $\Delta\alpha$ is arbitrary. The other limits are obvious conditions, the important point being that all these limits are enforced by the program. When a limit is violated, the particular parameter is set to the value of the limit. The 27 point cube in such case will have one or more sides collapsed onto the center, and the cube can be moved along the limiting value while seeking a minimum on another axis. In most cases of interest, the lower limits in both N and α were reached and a minimum was found by moving in the K direction.

Occasional instabilities were encountered, generally where the shift to be detected was too small. An arbitrary limit of fifty shifts without a step size reduction was imposed to detect these points. Typical values of K when the limit was reached were between 10,000 and 80,000. Type I error α also typically would grow to large figures near or above 50%.

CHAPTER IV

A NUMERICAL STUDY OF THE COST MODEL

General Approach

The number of parameters in the model, from the point of view of a researcher, is too great to permit any attempt at an exhaustive study of all possible combinations within even a small region. Several operating points, defined by fixing all 13 parameters (N_σ, $\underline{\underline{S}}$, p, s, π, λ, R, $\underline{1}$, $\underline{u}$, A_1, A_2, A_3, A_4), were therefore chosen and tests on the sensitivity of the model to changes in the values of individual parameters were made in the vicinity of these points. In all, five separate operating points were chosen. Three of them use the $\underline{\underline{S}}$ matrix from the example problem in Montgomery and Klatt (15), that is

$$\underline{\underline{S}} = \begin{bmatrix} 2 & 1 \\ 1 & 2.5 \end{bmatrix}$$

The other two use the three-dimensional identity matrix. Results in the vicinity of the two sets of points were not directly compared, but the behavior of the model was generally the same. The switch to the identity matrix was made because of the greater ease in redefining the process control limits and the out-of-control mean vectors in terms of the variances.

In all, over 225 examples were run manually, and a set of runs was automatically initiated to generate the 625 points of the cost tables found in Table 11.

The Effect of Number of Variables p and Allowed Error on Compute Time

The multi-dimensional quadrature subroutine used in evaluating the multivariate normal integral controls the allowable error in each integration step. A total allowable error is provided this routine as an argument, and this error is apportioned to the intervals of integration, ultimately defining the step size for the inner integrations. Increasing the allowed error, therefore, increases the step size and allows faster evaluation at a cost in accuracy.

While the main program has provisions for up to 12 quality characteristics, no test cases involving more than four variables were studied, as the multivariate normal CDF routine used a great deal of computer time when the number of variables p was greater than three. This computing time could be lessened somewhat by reducing the allowed error in the quadrature routine as mentioned previously, but to reduce the allowed error below 10^{-3} renders the probabilities in the tail regions meaningless. Table 2 gives the times spent in this routine for various combinations of p and allowed error.

Table 2. Effect of p and Error on Time (Sec)

Error \ P	3	4	5
.001	5	69	1200*
.0009		73	
.0008		77	
.0007		83	
.0006		98	
.0005		110	
.0004		140	
.0003		182	
.0002		265	
.0001	10	365	
.00001	28	360*	

*Run terminated without solution

Effects of π and State Condensation on Cost

The effect of the binomial weighting parameter π on cost can be seen from runs made about operating point 2. The operating points are listed in Appendix 10 for comparison, but are set out in the text as well. Operating point 2 is defined as the point where

$p = 2$

$s = 6$

$N_\sigma = 13$

$$\underline{\underline{S}} = \begin{bmatrix} 2 & 1 \\ 1 & 2.5 \end{bmatrix}$$

π = not applicable

$\lambda = 1$

$R = 10000$

$$\underline{l}' = (-3.5\sigma_1,\ -3.5\sigma_2)$$

$$\underline{u}' = (3.5\sigma_1,\ 3.5\sigma_2)$$

$$\underline{\mu}_{i,i=0,1,\ldots,6} = \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}2\sigma_1\\2\sigma_2\end{bmatrix}, \begin{bmatrix}2.2\sigma_1\\2.2\sigma_2\end{bmatrix}, \begin{bmatrix}2.4\sigma_1\\2.4\sigma_2\end{bmatrix}, \begin{bmatrix}2.6\sigma_1\\2.6\sigma_2\end{bmatrix},$$

$$\begin{bmatrix}2.8\sigma_1\\2.8\sigma_2\end{bmatrix}, \begin{bmatrix}3\sigma_1\\3\sigma_2\end{bmatrix}$$

$$A_1 = 1$$

$$A_2 = .1$$

$$A_3 = 100$$

$$A_4 = 1$$

Table 3 shows the relationship of π to cost. Pi obviously has a significant effect, with the cost increasing almost as fast as the numerator in π.

Table 3. Effect of π on Cost

π	1/6	2/6	3/6	4/6	5/6	.97
C	.0585	.0973	.137	.176	.216	.251

At the same time that these runs were made, the suggestion in Montgomery, Heikes and Mance (14), that the several states might be condensed into one, was tested. In their study of the fraction defective control chart, they suggest that a condensed fraction defective state be defined as

$$p^* = \frac{\sum_{i=1}^{s} p_{o,i}p_i}{\sum_{i=1}^{s} p_{o,i}}$$

where $P_{0,i}$ are the same probabilities p_i defined by Knappenberger and Grandage (10), q_i in this development. This formula should be directly transferrable to mean vectors, so that one could set

$$\mu_j^* = \frac{\sum_{i=1}^{s} q_i \mu_{i,j}}{\sum_{i=1}^{s} q_i}$$

where $\mu_{i,j}$ is the j^{th} element of the mean vector $\underline{\mu}_i$. If this were to be done before the optimal K was determined, it would imply some a priori knowledge of the probabilities q_i involving the parameter K. The factor $(1-e^{-\lambda K/R})$ can be taken outside the summation in the numerator, however, whereupon it cancels with the denominator, and we are left with an expression in only the binomial weights w_i,

$$\mu_j^* = \sum_{i=1}^{s} w_i \mu_{i,j} = \sum_{i=1}^{s} \binom{s}{j} \frac{\pi^i (1-\pi)^{s-i}}{(1-(1-\pi)^s)} \mu_{i,j}$$

These probabilities can be assigned, since the mean vectors and π value are known, and the program was made to compute the vector $\underline{\mu}^*$ using this formula in all runs where $s>2$. Table 4 shows the value of $\underline{\mu}^*$ in each case tried. The costs given are as follows:

C_0 - The same C as in Table 3, the minimum C with six out-of-control states.

C_1 - The value of the cost function with all three variables frozen, letting s=1 and using $\underline{\mu}^*$.

C_2 - The minimal cost with one out-of-control state and using $\underline{\mu}^*$.

C_3 - The value of the cost function used to define C_2, with all three variables frozen, letting s=6 and using the original $\underline{\mu}_{i,i=1,\ldots,6}$.

Table 4. Effect of State Condensation on Cost

π	1/6	2/6	3/6	4/6	5/6	.97
$\underline{\mu}^*$	$\begin{bmatrix}2.97\\3.32\end{bmatrix}$	$\begin{bmatrix}3.17\\3.54\end{bmatrix}$	$\begin{bmatrix}3.41\\3.81\end{bmatrix}$	$\begin{bmatrix}3.68\\4.11\end{bmatrix}$	$\begin{bmatrix}3.96\\4.43\end{bmatrix}$	$\begin{bmatrix}4.19\\4.69\end{bmatrix}$
C_0	.0585	.0973	.137	.176	.216	.251
C_1	.0184	.0191	.0200	.0211	.0223	.0233
C_2	.0166	.0176	.0188	.0201	.0214	.0225
C_3	.0626	.0999	.138	.177	.217	.253

The match between C_0 and C_3 is almost exact for $\pi \geq 1/2$, and within 10% at its worst. The interpretation of this is that if a sampling scheme is selected using $\underline{\mu}^*$, the cost computed will be incorrect, but the values of the other parameters, N, α and K, when used to evaluate the cost function on the surface defined by the true situation of six out-of-control states, will give a value very close to the true minimum. Thus the state-condensation idea of Montgomery, Heikes and Mance seems to be valid in the case of the T^2 chart model also.

Effects of N and α on Cost

The effect of all model parameters and problem values on the optimal size of N was negligible. All test problems produced cost minima at the minimum allowable N, the value p+3, except for the six out-of-control state case with π=1/6 and π=2/6, where N was p+5 and p+4 respectively. Even divergent cases gave a minimal N. The effect of N on cost near operating point 5 can be seen in Table 5 below. Operating point 5 is defined as the point where

$p = 3$

$s = 3$

$N_\sigma = 25$

$\underline{\underline{S}} = \underline{\underline{I}}_3$

$\pi = 1/3$

$\lambda = 1$

$R = 10000$

$\underline{l}' = (-3.5,-3.5,-3.5)$

$\underline{u}' = (3.5,3.5,3.5)$

$$\underline{\mu}_{i, i=0,1,2,3} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2.5 \\ 2.5 \\ 2.5 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

$A_1 = 1$

$A_2 = .1$

$A_3 = 100$

$A_4 = 1$

Table 5. Effect of N on Cost

N	6	8	10	15	25	50	100
C	.108	.109	.110	.112	.116	.127	.149

With R, the rate of production, set at 10,000 items per hour, an increase of N from 6 to 10 costs an additional \$20. per hour, or about a 2% increase. An increase to 15, still a reasonable value, doubles this increase. The behavior that leads to a minimal N in almost all cases can be seen, but the effect of sampling using a slightly larger value for N would probably be tolerable.

The optimal value of α was always minimal (i.e., .001) except in a few divergent cases. Again working from operating point 5, the effect of α on cost, given by non-iterative runs, can be seen in Table 6.

Table 6. Effect of α on Cost

α	.001	.005	.01	.05
C	.108	.109	.112	.129

Selection of too large an α merely because it is an often-used figure, such as .05, can be costly. Selecting α=.05 adds 20% to the cost.

Effect of K on Cost

The effect of the interval between samples K on cost was studied around the previously defined operating point 5, where p=3, s=3, $\underline{\underline{S}}=\underline{\underline{I}}_3$, and all other parameters are as previously specified. The results of the study can be seen in Table 7. K, unlike N and α, gave a minimal cost at some value between the smallest and largest allowable values in all convergent cases. As is obvious from the table, the cost surface is very nearly flat in the K direction.

Table 7. Effect of K on Cost

K	50	229	300	1000	5000
C	.129	.108	.108	.125	.203

Effect of N_σ on Cost and on T^2

The effect of N_σ was checked at various values around that used at operating point 2 for the studies of π, which was N_σ=13, and also around that used at operating point 5 for all later studies, which was N_σ=25. The value of N_σ, the preliminary sample size used to derive $\underline{\underline{S}}$, the estimator of $\underline{\underline{\Sigma}}$, had almost no effect on cost. The principal effect of increasing N_σ was to lower the T^2 tolerance as shown in Table 8. This says, in effect, that the more we know about the variance, the smaller a change in the sample mean need be to be significant. Again, the table was prepared by searching in the neighborhood of operating point 5, where p=3, s=3, and $\underline{\underline{S}}=\underline{\underline{I}}_3$.

Table 8. Effect of N_σ on Cost and on T^2

N_σ	13	20	25	30
C	.108	.108	.108	.108
α	.001	.001	.001	.001
T^2	46.6	29.6	25.8	23.7

Effect of $\underline{\underline{S}}$ on Cost

Like A_4, the cost of producing a defective, $\underline{\underline{S}}$ is very difficult to estimate. Even if the variances are well known, the covariances often are not. An indication that the model is insensitive to changes in $\underline{\underline{S}}$ would be very important. As it turns out, the cost is almost totally insensitive to variations in $\underline{\underline{S}}$ because of the flatness of the cost surface in the K direction. The four examples shown in Table 9 were run with p=3 about operating point 5 (p=3, s=3, $\underline{\underline{S}}=\underline{\underline{I}}_3$) and with covariance matrices of $\underline{\underline{S}}_1=\underline{\underline{I}}_3$, $\underline{\underline{S}}_2=.5\underline{\underline{I}}_3$, $\underline{\underline{S}}_3=2\underline{\underline{I}}_3$, and

$$\underline{\underline{S}}_4 = \begin{bmatrix} 1 & .5 & .5 \\ .5 & 1 & 0 \\ .5 & 0 & 1 \end{bmatrix}$$

The values shown are the minimal K and cost using the various matrices as if they were incorrect estimates (C_1), and the true cost using the minimal K and the $\underline{\underline{I}}_3$ matrix in a single point evaluation (C_2).

Table 9. Effect of $\underline{\underline{S}}$ on Optimal Cost

$\underline{\underline{S}}$	$\underline{\underline{S}}_1$	$\underline{\underline{S}}_2$	$\underline{\underline{S}}_3$	$\underline{\underline{S}}_4$
K	229	347	184	247
C_1	.108	.0697	.176	.0949
C_2		.109	.108	.108

Effect of the Model Parameters on Cost

The model parameters A_1, A_2, A_3 and A_4 were varied individually by ±10%, and again by ±50%, giving, with the central point, a total of 17 points on four axes through the central point of $A_1=1$, $A_2=.1$, $A_3=100$ and $A_4=1$. The optimal costs for the values of the A's were evaluated at operating point 5, where p=3, s=3, $\underline{\underline{S}}=\underline{\underline{I}}_3$, and other parameters as well remain unchanged. The resulting Table 10 gives a brief idea of the behavior of the cost surface in response to changes in the A's. A much more comprehensive study was conducted involving the computation of the optimal costs at all 625 points on the four dimensional grid defined by successively multiplying all model parameters by the five factors .5, 1., 1.5, 2. and 5. The results appear in Table 11.

The abbreviated data in Table 10 show the cost to be fairly robust to changes in A_1, A_2 and A_3, but changes in cost are almost equal in proportion to changes in A_4. This is unfortunate, since A_4 is probably the hardest of

Table 10. Effect of Model Parameters on Cost

Vary/By	-50%	-10%	0	+10%	+50%
A_1	.105	.107	.108	.108	.110
A_2	.106	.108	.108	.108	.109
A_3	.103	.107	.108	.109	.113
A_4	.0618	.0987	.108	.117	.153

the four costs to estimate in many cases. Good values could probably be derived in other situations, however, such as that of a customer who performs sampling inspection of received items, with specified contractual penalties for lot nonconformance to standard. In any case, though, the estimation of A_4 is an important matter.

All tests were run with the costs defined as at operating point 5, i.e.,

$$A_1 = 1$$
$$A_2 = .1$$
$$A_3 = 1$$
$$A_4 = 100$$

In varying the A_i to test the model's sensitivity, these values were varied by no more than a factor of five. To see if the model was still well-behaved in more distant regions of the domain, one run was made at operating point 5, but with the costs as follows:

$$A_1 = 500$$
$$A_2 = 1$$
$$A_3 = 25$$
$$A_4 = 2500$$

Table 11. Effect of Estimated Process Costs on Total Cost and Sample Spacing

COST A1 = 0.5

COSTS	A4:	0.5		1.0		1.5		2.0		5.0	
A2	A3	K	COST	K	COST	K	COST	K	COST	K	COST
	50	229	.053	162	.098	135	.142	113	.186	70	.443
	100	238	.059	168	.104	135	.148	117	.191	70	.449
.05	150	247	.064	175	.109	141	.153	120	.197	76	.455
	200	258	.069	178	.114	141	.158	125	.202	76	.460
	500	309	.099	206	.145	168	.190	141	.234	85	.494
	50	270	.055	191	.100	153	.144	135	.188	85	.447
	100	276	.060	194	.105	156	.150	135	.194	85	.453
.10	150	282	.065	198	.111	159	.155	137	.199	85	.458
	200	293	.070	202	.116	162	.160	141	.204	85	.464
	500	344	.101	229	.147	189	.192	159	.236	100	.497
	50	303	.056	212	.102	175	.146	150	.190	94	.451
	100	309	.061	216	.107	178	.152	153	.196	94	.456
.15	150	321	.066	223	.112	181	.157	153	.201	98	.462
	200	327	.071	225	.117	184	.162	156	.206	100	.467
	500	377	.102	250	.148	202	.193	175	.238	110	.500
	50	334	.057	233	.103	191	.148	162	.192	103	.454
	100	340	.062	238	.108	194	.153	168	.198	103	.459
.20	150	351	.067	242	.113	196	.158	168	.203	103	.465
	200	356	.072	246	.118	199	.164	175	.208	110	.470
	500	409	.102	272	.149	216	.195	184	.239	115	.503
	50	482	.061	334	.109	272	.156	236	.201	147	.468
	100	491	.066	337	.114	273	.161	238	.207	150	.473
.50	150	497	.071	344	.120	276	.166	238	.212	150	.479
	200	505	.076	347	.125	282	.171	242	.217	150	.484
	500	555	.106	370	.155	296	.202	254	.248	156	.516

Table 11. (Continued)

COST A1 = 1.0

COSTS A4:		0.5		1.0		1.5		2.0		5.0	
A2	A3	K	COST	K	COST	K	COST	K	COST	K	COST
.05	50	291	.055	202	.101	168	.146	141	.190	92	.450
	100	299	.060	210	.106	168	.151	147	.195	94	.455
	150	308	.066	212	.112	175	.156	150	.200	94	.461
	200	315	.071	216	.117	178	.161	153	.206	94	.466
	500	368	.101	246	.148	196	.193	168	.237	103	.499
.10	50	323	.056	225	.103	184	.147	159	.192	102	.453
	100	330	.061	229	.108	184	.153	162	.197	103	.458
	150	340	.066	236	.113	191	.158	162	.202	103	.464
	200	347	.072	238	.118	194	.163	168	.208	103	.469
	500	398	.102	264	.149	212	.194	184	.239	113	.502
.15	50	353	.057	247	.104	199	.149	175	.194	110	.456
	100	360	.062	250	.109	202	.154	178	.199	113	.461
	150	368	.067	254	.114	206	.159	178	.204	113	.466
	200	377	.072	258	.119	210	.165	181	.209	113	.472
	500	426	.103	282	.150	229	.196	196	.241	121	.504
.20	50	380	.058	264	.105	216	.150	189	.195	117	.458
	100	388	.063	270	.110	216	.156	191	.200	118	.464
	150	395	.068	273	.115	223	.161	191	.206	118	.469
	200	402	.073	276	.120	223	.166	194	.211	120	.474
	500	454	.104	303	.151	242	.197	206	.242	128	.507
.50	50	516	.062	358	.111	291	.158	250	.203	156	.471
	100	523	.067	360	.116	293	.163	254	.209	159	.477
	150	533	.072	368	.121	296	.168	254	.214	159	.482
	200	541	.077	370	.126	298	.173	258	.219	159	.487
	500	591	.107	391	.156	315	.203	270	.250	168	.519

Table 11. (Continued)

COST A1 = 1.5

COSTS A4:		0.5		1.0		1.5		2.0		5.0	
A2	A3	K	COST	K	COST	K	COST	K	COST	K	COST
	50	344	.057	238	.103	194	.148	168	.193	103	.455
	100	351	.062	244	.109	198	.154	168	.198	103	.460
.05	150	360	.067	247	.114	201	.159	175	.203	110	.466
	200	368	.072	252	.119	206	.164	178	.209	113	.471
	500	416	.103	276	.150	223	.195	191	.240	118	.503
	50	372	.058	258	.105	210	.150	181	.195	113	.457
	100	379	.063	264	.110	212	.155	184	.200	115	.463
.10	150	388	.068	268	.115	216	.160	184	.205	117	.468
	200	395	.073	272	.120	223	.166	191	.210	118	.474
	500	444	.103	296	.151	238	.196	202	.242	128	.506
	50	400	.059	276	.106	225	.151	194	.196	121	.460
	100	405	.064	282	.111	229	.157	196	.201	125	.465
.15	150	412	.069	282	.116	229	.162	199	.207	125	.471
	200	420	.074	291	.121	233	.167	201	.212	125	.476
	500	468	.104	315	.152	250	.198	216	.243	135	.508
	50	422	.059	293	.107	238	.153	206	.198	128	.462
	100	430	.064	298	.112	242	.158	210	.203	128	.468
.20	150	436	.069	303	.117	244	.163	210	.208	135	.473
	200	444	.074	306	.122	247	.168	212	.213	135	.473
	500	494	.105	330	.153	264	.199	225	.244	141	.510
	50	547	.063	382	.112	308	.159	264	.205	168	.475
	100	555	.068	385	.117	309	.164	268	.211	168	.480
.50	150	564	.073	388	.122	315	.169	270	.216	168	.485
	200	574	.078	391	.127	315	.174	272	.221	168	.490
	500	627	.108	416	.157	334	.205	282	.251	178	.522

Table 11. (Continued)

COST A1 = 2.0

COSTS A2	A4: A3	0.5 K	0.5 COST	1.0 K	1.0 COST	1.5 K	1.5 COST	2.0 K	2.0 COST	5.0 K	5.0 COST
	50	388	.058	270	.105	223	.151	191	.196	118	.459
	100	398	.063	275	.111	223	.156	194	.201	120	.464
.05	150	405	.068	280	.116	225	.161	194	.206	121	.470
	200	412	.073	282	.121	229	.166	198	.211	125	.475
	500	461	.104	308	.151	247	.197	210	.242	135	.507
	50	412	.059	291	.107	234	.152	202	.197	128	.462
	100	422	.064	291	.112	238	.157	202	.202	125	.467
.10	150	430	.069	296	.117	238	.163	206	.208	128	.472
	200	436	.074	299	.122	242	.168	210	.213	128	.478
	500	487	.104	323	.152	261	.198	223	.244	137	.510
	50	436	.060	305	.108	247	.153	212	.199	135	.464
	100	444	.065	308	.113	250	.159	216	.204	135	.469
.15	150	454	.070	313	.118	254	.164	216	.209	135	.475
	200	461	.075	315	.123	254	.169	223	.214	137	.480
	500	511	.105	340	.153	272	.200	233	.245	147	.512
	50	461	.060	321	.108	261	.155	223	.200	141	.466
	100	468	.065	323	.114	261	.160	225	.205	141	.471
.20	150	476	.070	327	.119	264	.165	229	.210	141	.477
	200	482	.075	330	.124	268	.170	229	.216	143	.482
	500	533	.106	355	.154	282	.201	244	.246	150	.514
	50	578	.064	400	.113	323	.161	282	.207	178	.477
	100	589	.069	405	.118	327	.166	282	.212	178	.483
.50	150	597	.074	409	.124	330	.171	282	.217	178	.488
	200	604	.079	412	.129	334	.176	287	.223	181	.493
	500	658	.109	436	.159	347	.206	298	.253	184	.524

Table 11. (Continued)

COST A1 = 5.0

COSTS A2	A4: A3	0.5 K	0.5 COST	1.0 K	1.0 COST	1.5 K	1.5 COST	2.0 K	2.0 COST	5.0 K	5.0 COST
	50	597	.064	416	.114	334	.162	291	.208	181	.479
	100	604	.069	418	.119	337	.167	291	.213	181	.484
.05	150	615	.074	422	.124	340	.172	293	.218	184	.490
	200	622	.079	426	.129	344	.177	295	.224	184	.495
	500	678	.109	447	.159	358	.207	306	.254	191	.526
	50	616	.065	426	.115	344	.163	298	.209	189	.481
	100	624	.070	430	.120	347	.168	299	.214	184	.486
.10	150	631	.075	433	.125	351	.173	301	.219	191	.491
	200	639	.080	436	.130	353	.178	303	.225	191	.496
	500	698	.110	461	.160	368	.208	315	.255	194	.528
	50	631	.065	436	.116	353	.163	306	.210	191	.482
	100	639	.070	441	.121	356	.169	306	.215	194	.488
.15	150	651	.075	444	.126	360	.174	309	.220	194	.493
	200	658	.080	447	.131	360	.179	309	.226	194	.498
	500	714	.110	472	.161	377	.209	323	.256	199	.529
	50	651	.066	451	.116	364	.164	315	.211	196	.484
	100	658	.071	454	.121	368	.169	315	.216	196	.489
.20	150	666	.076	458	.126	368	.174	315	.221	198	.494
	200	674	.081	461	.131	372	.179	321	.226	199	.499
	500	733	.110	482	.161	385	.210	330	.257	204	.531
	50	742	.068	513	.120	416	.169	356	.217	223	.492
	100	751	.073	516	.125	416	.174	358	.222	223	.498
.50	150	760	.078	521	.130	418	.179	360	.227	225	.503
	200	769	.083	523	.135	422	.184	360	.232	225	.508
	500	833	.113	547	.165	436	.214	373	.262	229	.539

This resulted in a minimal cost of $220.00 per unit, an associated N of 6 (still minimal), and an α of .006 which, while finally non-minimal in a convergent case, still is very small.

CHAPTER V

SUMMARY OF RESULTS AND RECOMMENDATIONS

Summary of Results

A computer program was developed which can serve both as a tool to study the multivariate multistate model and its behavior, and as a means for practitioners to find a least-cost solution to their real world problems in controlling the mean vector of a multivariate normal process. The program has been used to make numerical studies of the objective function and of the sensitivity of its value to most of the parameters of the system.

These numerical studies have revealed an objective function of very gentle curvature and slope within the region studied. The value of the cost function is not strongly sensitive to anything except the cost A_4, the cost of producing a defective.

Recommendations

Recommendations for further work have generally been mentioned as they occurred in the text. They are repeated here with some amplification.

State Definition

The problem of defining an ordered succession of states, particularly when variations both above and below one or more of the mean values can be expected, should be studied more fully. The development of the $\underline{\beta}$ vector depends upon the assumption of no spontaneous correction, and one approach might be to study the sensitivity of $\underline{\beta}$ to this assumption. Further study might also define a condensed state formula that would yield the same minimum point, or a point close to the same minimal point, as would be given by the full state complement. Alternatively, a rigorous definition of state and a technique of state definition or state grouping which can be explained to users or, better, incorporated into the program, would be helpful.

Dependence on N and α

An analysis of the mathematical model ought to be made to see if it is indeed monotonically increasing with N and α within a portion of the domain of convergence. Such a study might point up errors in the model, or ways to improve it. The idea of a minimal N in all cases does not have much intuitive appeal.

Using the current model, tests with larger expected costs might be run to observe the behavior of the minimum cost α. It would be of value to determine what sorts of costs give rise to more reasonable values of α, in the neighborhood of α=.01 to α=.10

Search Technique

It is quite likely that in a real-world implementation of this model it will be used in slower machines than the Honeywell 66/40. The step size algorithm might be a candidate for improvement, particularly with regard to increases in step size under certain conditions. The search algorithm itself might be altered, but gradient search techniques will probably not prove too useful because of the impossibility of normalizing the units of measure of N, K, and α.

Program Size

Program size depends on the machine being used. On the 66/40 this program used about 16K words, or 64K bytes. The best possiblity of reducing storage requirements is to study the MDQUAD routine supplied with Milton's program (12) to determine whether the size of the 5000-word work vector defined within it can be reduced or made dependent on p.

Further Analysis of the Cost A_4

Further studies of the sensitivity of the model to changes in the ratio of A_4 to the other costs might prove fruitful. Modifications of the model to provide for different A_4 type costs for different out-of-control variables might also be reasonable. As an instance, in testing packaged foods, the mean weight and mean Salmonella content surely would involve A_4 costs out of all proportion to each other.

Real-World Examples

The strongest recommendation to be made is that this model be utilized and studied with real-world examples.

Choosing operating points and costs in this study had no basis in experience. They were defined arbitrarily, simply be selecting points which it was felt were reasonable, and which produced convergent results. A real-world application would lend great authority to this analysis and sensitivity study.

APPENDICES

Appendix 1. Example Run

```
*RUN

ENTER NUMBER OF VARIABLES AND NUMBER OF OUT-OF-CONTROL STATES
=2,1

ENTER THE LIMITING VALUES OF
    SAMPLE SIZE 'N'
=5,25
    NUMBER BETWEEN SAMPLES 'K'
=5,500
    TYPE 1 ERROR 'ALPHA'
=.001,.1

ENTER  2 DIMENSIONAL L AND U SPECIFICATION VECTORS
=-4,-4
=4,4

ENTER  2 DIMENSIONAL IN-CONTROL MEAN VECTOR
=0,0

FOR THE  1 STATES, ENTER THE  2 DIMENSIONAL MEAN VECTORS
STATE  1
=5,6

ARE YOU ENTERING THE S (S) OR S INVERSE (SI) MATRIX??
=S

ENTER THE SAMPLE SIZE USED IN ESTIMATING
    THE COVARIANCE MATRIX
=13

ENTER THE S MATRIX
=2,1
=1,2.5
```

Appendix 1. (Continued)

```
ENTER THE TRIAL COST VALUES
=.001,.0001,.01,1

ENTER THE MEAN TIME TO SHIFT (IN HOURS),
    AND UNITS PER HOUR PRODUCED
=1,10000


ESTIMATED COSTS AFTER    7 ITERATIONS:
 FIXED COST PER SAMPLE        $     0.001
 INSPECTION COST PER UNIT     $     0.000
 AVERAGE COST OF REPAIR       $     0.010
 COST PER UNIT OF DEFECTIVES $     1.000

MINIMA:
   FOR TESTING                $ 0.250E-03
   FOR CORRECTING             $ 0.267E-05
   FOR BAD PRODUCT            $ 0.159E-01
 COST PER UNIT                $ 0.161E-01
 SAMPLE SIZE                        5
 SAMPLE INTERVAL                    6
 TYPE I ERROR                       0.100 %
 T-SQUARED VALUE                  30.804

*
```

Appendix 2. Comparison of Univariate Normal CDF Values

Z	0	1	2	3	4	5	6	7	8	9
-3.0	0013	0010	0007	0005	0003	0002	0002	0001	0001	0000
-2.9	0019	0018	0018	0017	0016	0016	0015	0015	0014	0014
-2.8	0026	0025	0024	0023	0023	0022	0021	0021	0020	0019
-2.7	0035	0034	0033	0032	0031	0030	0029	0028	0027	0026
-2.6	0047	0045	0044	0043	0041	0040	0039	0038	0037	0036
-2.5	0062	0060	0059	0057	0055	0054	0052	0051	0049	0048
-2.4	0082	0080	0078	0075	0073	0071	0069	0068	0066	0064
-2.3	0107	0104	0102	0099	0096	0094	0091	0089	0087	0084
-2.2	0139	0136	0132	0129	0125-	0122	0119	0116	0113	0110
-2.1	0179	0174	0170	0166	0162	0158	0154	0150	0146	0143
-2.0	0228	0222	0217	0212	0207	0202	0197	0192	0188	0183
-1.9	0287	0281	0274	0268	0262	0256	0250	0244	0239+	0233
-1.8	0359	0351-	0344	0336	0329	0322	0314	0307	0301+	0294
-1.7	0446	0436	0427	0418	0409	0401	0392	0384	0375	0367
-1.6	0548	0537	0526	0516	0505	0495	0485	0475	0465	0455
-1.5	0668	0655	0643	0630	0618	0606	0594	0582	0571+	0559
-1.4	0808	0793	0778	0764	0749	0735	0721-	0708	0694	0681
-1.3	0968	0951	0934	0918	0901	0885	0869	0853	0838	0823
-1.2	1151	1131	1112	1093	1075	1056	1038	1020	1003	0985
-1.1	1357	1335	1314	1292	1271	1251	1230	1210	1190	1170
-1.0	1587	1562	1539	1515	1492	1469	1446	1423	1401	1379
-0.9	1841	1814	1788	1762	1736	1711	1685	1660	1635	1611
-0.8	2119	2090	2061	2033	2005	1977	1949	1922	1894	1867
-0.7	2420	2389	2358	2327	2296-	2266	2236	2206	2177	2148
-0.6	2743	2709	2676	2643	2611	2578	2546	2514	2483	2451
-0.5	3085	3050	3015	2981	2946	2912	2877	2843	2810	2776
-0.4	3446	3409	3372	3336	3300	3264	3228	3192	3156	3121
-0.3	3821	3783	3745	3707	3669	3632	3594	3557	3520	3483
-0.2	4207	4168	4129	4090	4052	4013	3974	3936	3897	3859
-0.1	4602	4562	4522	4483	4443	4404	4364	4325	4286	4247
0.	5000	4960	4920	4880	4840	4801	4761	4721	4681	4641

Note: 1. Computed value is shown. This, where shown, is greater than published values by 10^{-4} (+) or lesser by 10^{-4} (-).

2. $z = -3.9(.1)-3.0(.01)+3.0(.1)+3.9$

Appendix 2. (Continued)

Z	0	1	2	3	4	5	6	7	8	9
0.	5000	5040	5080	5120	5160	5199	5239	5279	5319	5359
0.1	5398	5438	5478	5517	5557	5596	5636	5675	5714	5753
0.2	5793	5832	5871	5910	5948	5987	6026	6064	6103	6141
0.3	6179	6217	6255	6293	6331	6368	6406	6443	6480	6517
0.4	6554	6591	6628	6664	6700	6736	6772	6808	6844	6879
0.5	6915	6950	6985	7019	7054	7088	7123	7157	7190	7224
0.6	7257	7291	7324	7357	7389	7422	7454	7486	7517	7549
0.7	7580	7611	7642	7673	7704+	7734	7764	7794	7823	7852
0.8	7881	7910	7939	7967	7995	8023	8051	8078	8106	8133
0.9	8159	8186	8212	8238	8264	8289	8315	8340	8365	8389
1.0	8413	8438	8461	8485	8508	8531	8554	8577	8599	8621
1.1	8643	8665	8686	8708	8729	8749	8770	8790	8810	8830
1.2	8849	8869	8888	8907	8925	8944	8962	8980	8997	9015
1.3	9032	9049	9066	9082	9099	9115	9131	9147	9162	9177
1.4	9192	9207	9222	9236	9251	9265	9279+	9292	9306	9319
1.5	9332	9345	9357	9370	9382	9394	9406	9418	9429-	9441
1.6	9452	9463	9474	9484	9495	9505	9515	9525	9535	9545
1.7	9554	9564	9573	9582	9591	9599	9608	9616	9625	9633
1.8	9641	9649+	9656	9664	9671	9678	9686	9693	9699-	9706
1.9	9713	9719	9726	9732	9738	9744	9750	9756	9761-	9767
2.0	9772	9778	9783	9788	9793	9798	9803	9808	9812	9817
2.1	9821	9826	9830	9834	9838	9842	9846	9850	9854	9857
2.2	9861	9864	9868	9871	9875+	9878	9881	9884	9887	9890
2.3	9893	9896	9898	9901	9904	9906	9909	9911	9913	9916
2.4	9918	9920	9922	9925	9927	9929	9931	9932	9934	9936
2.5	9938	9940	9941	9943	9945	9946	9948	9949	9951	9952
2.6	9953	9955	9956	9957	9959	9960	9961	9962	9963	9964
2.7	9965	9966	9967	9968	9969	9970	9971	9972	9973	9974
2.8	9974	9975	9976	9977	9977	9978	9979	9979	9980	9981
2.9	9981	9982	9982	9983	9984	9984	9985	9985	9986	9986
3.0	9987	9990	9993	9995	9997	9998	9998	9999	9999	10000

Appendix 3. Comparison of Bivariate Normal CDF Values

NBS Table

r =0.00

h / k	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.0	250000	230086	210370	191044	172289	154269	137127	120982	105928
0.1	230086	211758	193613	175827	158565	141980	126204	111345	097490
0.2	210370	193613	177022	160760	144978	129814	115389	101804	089136
0.3	191044	175827	160760	145992	131659	117889	104789	092452	080948
0.4	172289	158565	144978	131659	118734	106315	094502	083375	073001
0.5	154269	141980	129814	117889	106315	095195	084617	074655	065365
0.6	137127	126204	115389	104789	094502	084617	075215	066359	058102
0.7	120982	111345	101804	092452	083375	074655	066359	058546	051261
0.8	105928	097490	089136	080948	073001	065365	058102	051261	044883
0.9	092030	084699	077442	070327	063423	056789	050479	044536	038994
1.0	079328	073009	066753	060620	054669	048951	043512	038389	033612
1.1	067833	062430	057080	051836	046748	041858	037207	032826	028742
1.2	057535	052952	048414	043967	039650	035503	031558	027843	024378
1.3	048400	044545	040728	036986	033355	029867	026548	023422	020508
1.4	040378	037162	033978	030856	027827	024916	022148	019540	017109
1.5	033404	030743	028108	025526	023020	020613	018322	016165	014153
1.6	027400	025217	023056	020938	018883	016908	015029	013259	011610
1.7	022283	020508	018750	017028	015356	013750	012222	010783	009441
1.8	017965	016534	015117	013729	012381	011086	009854	008694	007612
1.9	014358	013215	012082	010972	009895	008860	007876	006948	006084
2.0	011375	010469	009572	008693	007839	007019	006239	005505	004820
2.1	008932	008221	007516	006826	006156	005512	004899	004323	003785
2.2	006952	006398	005850	005312	004791	004290	003813	003364	002946
2.3	005362	004935	004512	004098	003695	003309	002941	002595	002272
2.4	004099	003772	003449	003132	002825	002529	002248	001984	001737
2.5	003105	002858	002613	002373	002140	001916	001703	001503	001316
2.6	002331	002145	001961	001781	001606	001438	001278	001128	000987
2.7	001733	001595	001459	001325	001195	001070	000951	000839	000734
2.8	001278	001176	001075	000976	000880	000788	000701	000618	000541
2.9	000933	000859	000785	000713	000643	000576	000512	000451	000395
3.0	000675	000621	000568	000516	000465	000416	000370	000327	000286
3.1	000484	000445	000407	000370	000333	000299	000265	000234	000205
3.2	000344	000316	000289	000263	000237	000212	000188	000166	000146
3.3	000242	000222	000203	000185	000167	000149	000133	000117	000102
3.4	000168	000155	000142	000129	000116	000104	000092	000082	000071
3.5	000116	000107	000098	000089	000080	000072	000064	000056	000049
3.6	000080	000073	000067	000061	000055	000049	000044	000038	000034
3.7	000054	000050	000045	000041	000037	000033	000030	000026	000023
3.8	000036	000033	000030	000028	000025	000022	000020	000018	000015
3.9	000024	000022	000020	000018	000017	000015	000013	000012	000010
4.0	000016	000015	000013	000012	000011	000010	000009	000008	000007

Computed

R = 0.

\H K\	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.0	249911	229997	210281	190956	172200	154180	137038	120893	105839
0.1	230004	211677	193531	175745	158484	141899	126122	111263	097408
0.2	210295	193538	176948	160685	144903	129739	115315	101729	089061
0.3	190976	175759	160692	145924	131592	117821	104721	092384	080880
0.4	172228	158504	144917	131598	118673	106254	094440	083314	072940
0.5	154214	141925	129759	117834	106260	095141	084563	074600	065310
0.6	137078	126155	115340	104740	094453	084569	075166	066310	058053
0.7	120939	111302	101761	092408	083332	074612	066316	058503	051218
0.8	105890	097452	089098	080910	072963	065328	058064	051223	044845
0.9	091997	084666	077409	070294	063390	056757	050446	044503	038961
1.0	079299	072980	066724	060592	054641	048923	043483	038360	033584
1.1	067809	062405	057056	051812	046723	041834	037182	032802	028717
1.2	057514	052931	048394	043946	039630	035483	031537	027822	024357
1.3	048383	044527	040710	036969	033339	029849	026530	023405	020490
1.4	040364	037147	033963	030841	027812	024902	022133	019525	017094
1.5	033391	030730	028096	025514	023008	020600	018310	016153	014141
1.6	027389	025207	023046	020928	018872	016897	015019	013249	011599
1.7	022274	020499	018742	017020	015348	013742	012214	010775	009433
1.8	017958	016527	015110	013722	012374	011079	009847	008687	007605
1.9	014353	013209	012077	010967	009890	008855	007870	006943	006078
2.0	011371	010464	009567	008688	007835	007015	006235	005500	004815
2.1	008929	008217	007513	006822	006152	005508	004896	004319	003781
2.2	006949	006395	005847	005309	004788	004287	003810	003361	002943
2.3	005360	004933	004510	004095	003693	003306	002939	002592	002270
2.4	004097	003770	003447	003130	002823	002527	002246	001982	001735
2.5	003103	002856	002611	002371	002138	001914	001701	001501	001314
2.6	002329	002144	001960	001780	001605	001437	001277	001127	000986
2.7	001732	001594	001458	001324	001194	001069	000950	000838	000733
2.8	001277	001175	001074	000975	000879	000787	000700	000617	000540
2.9	000932	000858	000784	000712	000642	000575	000511	000451	000394
3.0	000674	000620	000567	000515	000464	000416	000369	000326	000285
3.1	000483	000445	000406	000369	000333	000298	000265	000233	000204
3.2	000343	000316	000288	000262	000236	000211	000188	000166	000145
3.3	000241	000222	000203	000184	000166	000149	000132	000116	000102
3.4	000168	000154	000141	000128	000116	000103	000092	000081	000071
3.5	000116	000106	000097	000088	000080	000071	000063	000056	000049
3.6	000079	000073	000066	000060	000054	000049	000043	000038	000033
3.7	000053	000049	000045	000041	000037	000033	000029	000026	000022
3.8	000036	000033	000030	000027	000024	000022	000019	000017	000015
3.9	000023	000022	000020	000018	000016	000014	000013	000011	000010
4.0	000015	000014	000013	000012	000010	000009	000008	000007	000006

Appendix 3. (Continued)

NBS Table

r = 0.00

k \ h	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	092030	079328	067833	057535	048400	040378	033404	027400	022283
0.1	084699	073009	062430	052952	044545	037162	030743	025217	020508
0.2	077442	066753	057080	048414	040728	033978	028108	023056	018750
0.3	070327	060620	051836	043967	036986	030856	025526	020938	017028
0.4	063423	054669	046748	039650	033355	027827	023020	018883	015356
0.5	056789	048951	041858	035503	029867	024916	020613	016908	013750
0.6	050479	043512	037207	031558	026548	022148	018322	015029	012222
0.7	044536	038389	032826	027843	023422	019540	016165	013259	010783
0.8	038994	033612	028742	024378	020508	017109	014153	011610	009441
0.9	033878	029202	024971	021180	017817	014864	012297	010086	008203
1.0	029202	025171	021524	018256	015358	012812	010599	008694	007071
1.1	024971	021524	018405	015611	013133	010956	009063	007434	006046
1.2	021180	018256	015611	013241	011139	009293	007688	006306	005178
1.3	017817	015358	013133	011139	009370	007817	006467	005305	004314
1.4	014864	012812	010956	009293	007817	006522	005395	004425	003599
1.5	012297	010599	009063	007688	006467	005395	004463	003661	002977
1.6	010086	008694	007434	006306	005305	004425	003661	003003	002442
1.7	008203	007071	006046	005128	004314	003599	002977	002442	001986
1.8	006613	005700	004875	004134	003478	002902	002400	001969	001601
1.9	005286	004556	003896	003304	002780	002319	001918	001574	001280
2.0	004187	003609	003086	002618	002202	001837	001520	001247	001014
2.1	003288	002834	002424	002056	001729	001443	001194	000979	000796
2.2	002559	002206	001886	001600	001346	001123	000929	000762	000620
2.3	001974	001701	001455	001234	001038	000866	000716	000588	000478
2.4	001509	001301	001112	000943	000794	000662	000548	000449	000365
2.5	001143	000985	000842	000715	000601	000502	000415	000340	000277
2.6	000858	000740	000632	000536	000451	000376	000311	000255	000208
2.7	000638	000550	000470	000399	000336	000280	000232	000190	000155
2.8	000470	000405	000347	000294	000247	000206	000171	000140	000114
2.9	000343	000296	000253	000215	000181	000151	000125	000102	000083
3.0	000248	000214	000183	000155	000131	000109	000090	000074	000060
3.1	000178	000154	000131	000111	000094	000078	000065	000053	000043
3.2	000126	000109	000093	000079	000067	000055	000046	000038	000031
3.3	000089	000077	000066	000056	000047	000039	000032	000026	000022
3.4	000062	000053	000046	000039	000033	000027	000023	000018	000015
3.5	000043	000037	000032	000027	000023	000019	000016	000013	000010
3.6	000029	000025	000022	000018	000015	000013	000011	000009	000007
3.7	000020	000017	000015	000012	000010	000009	000007	000006	000005
3.8	000013	000011	000010	000008	000007	000006	000005	000004	000003
3.9	000009	000008	000007	000006	000005	000004	000003	000003	000002
4.0	000006	000005	000004	000004	000003	000003	000002	000002	000001

Computed

R = 0.

\H K\	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	091941	079239	067744	057446	048312	040290	033315	027311	022194
0.1	084518	072927	062348	052870	044463	037080	030661	025136	020426
0.2	077357	066678	057006	048340	040653	033903	028034	022982	018676
0.3	070259	060553	051769	043899	036919	030788	025458	020870	016960
0.4	063362	054608	046686	039589	033294	027766	022959	018821	015295
0.5	056735	048896	041803	035448	029812	024862	020558	016853	013695
0.6	050430	043463	037158	031509	026499	022099	018273	014980	012173
0.7	044493	038346	032783	027800	023379	019497	016122	013216	010740
0.8	038956	033574	028704	024340	020470	017071	014116	011572	009404
0.9	033845	029169	024938	021147	017784	014831	012264	010053	008170
1.0	029174	025143	021496	018228	015329	012784	010571	008666	007042
1.1	024946	021500	018381	015587	013108	010932	009039	007410	006022
1.2	021159	018236	015590	013220	011118	009272	007667	006285	005107
1.3	017800	015340	013115	011121	009353	007800	006449	005287	004296
1.4	014849	012798	010941	009278	007803	006507	005380	004411	003584
1.5	012284	010587	009051	007675	006455	005383	004451	003649	002965
1.6	010076	008684	007424	006296	005295	004415	003651	002993	002432
1.7	008194	007062	006038	005120	004306	003591	002969	002434	001978
1.8	006607	005694	004868	004128	003471	002895	002394	001962	001595
1.9	005280	004551	003890	003299	002774	002314	001913	001568	001274
2.0	004183	003605	003082	002613	002198	001833	001515	001242	001009
2.1	003285	002831	002420	002052	001726	001439	001190	000975	000793
2.2	002556	002203	001883	001597	001343	001120	000926	000759	000617
2.3	001972	001699	001453	001232	001036	000864	000714	000585	000476
2.4	001507	001299	001110	000941	000792	000660	000546	000447	000363
2.5	001141	000984	000841	000713	000600	000500	000413	000339	000275
2.6	000857	000738	000631	000535	000450	000375	000310	000254	000206
2.7	000637	000549	000469	000398	000335	000279	000231	000189	000153
2.8	000469	000404	000346	000293	000246	000205	000170	000139	000113
2.9	000343	000295	000252	000214	000180	000150	000124	000102	000082
3.0	000248	000213	000182	000155	000130	000108	000090	000073	000060
3.1	000177	000153	000131	000111	000093	000078	000064	000052	000043
3.2	000126	000108	000093	000079	000066	000055	000045	000037	000030
3.3	000088	000076	000065	000055	000046	000039	000032	000026	000021
3.4	000062	000053	000045	000038	000032	000027	000022	000018	000015
3.5	000042	000036	000031	000026	000022	000018	000015	000012	000010
3.6	000029	000025	000021	000018	000015	000012	000010	000008	000007
3.7	000019	000017	000014	000012	000010	000008	000007	000005	000004
3.8	000013	000011	000009	000008	000007	000005	000004	000004	000003
3.9	000008	000007	000006	000005	000004	000003	000003	000002	000002
4.0	000005	000005	000004	000003	000003	000002	000002	000001	000001

Appendix 3. (Continued)

NBS Table

r = 0.00

k \ h	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
0.0	017965	014358	011375	008932	006952	005362	004099	003105	002331
0.1	016534	013215	010469	008221	006398	004935	003772	002858	002145
0.2	015117	012082	009572	007516	005850	004512	003449	002613	001961
0.3	013729	010972	008693	006826	005312	004098	003132	002373	001781
0.4	012381	009895	007839	006156	004791	003695	002825	002140	001606
0.5	011086	008860	007019	005512	004290	003309	002529	001916	001438
0.6	009854	007876	006239	004899	003813	002941	002248	001703	001278
0.7	008694	006948	005505	004323	003364	002595	001984	001503	001128
0.8	007612	006084	004820	003785	002946	002272	001737	001316	000987
0.9	006613	005286	004187	003288	002559	001974	001509	001143	000858
1.0	005700	004556	003609	002834	002206	001701	001301	000985	000740
1.1	004875	003896	003086	002424	001886	001455	001112	000842	000632
1.2	004134	003304	002618	002056	001600	001234	000943	000715	000536
1.3	003478	002780	002202	001729	001346	001038	000794	000601	000451
1.4	002902	002319	001837	001443	001123	000866	000662	-000502	000376
1.5	002400	001918	001520	001194	000929	000716	000548	000415	000311
1.6	001969	001574	001247	000979	000762	000588	000449	000340	000255
1.7	001601	001280	001014	000796	000620	000478	000365	000277	000208
1.8	001291	001032	000817	000642	000500	000385	000295	000223	000168
1.9	001032	000825	000653	000513	000399	000308	000235	000178	000134
2.0	000817	000653	000518	000406	000316	000244	000186	000141	000106
2.1	000642	000513	000406	000319	000248	000192	000146	000111	000083
2.2	000500	000399	000316	000248	000193	000149	000114	000086	000065
2.3	000385	000308	000244	000192	000149	000115	000088	000067	000050
2.4	000295	000235	000186	000146	000114	000088	000067	000051	000038
2.5	000223	000178	000141	000111	000086	000067	000051	000039	000029
2.6	000168	000134	000106	000083	000065	000050	000038	000029	000022
2.7	000125	000100	000079	000062	000048	000037	000028	000022	000016
2.8	000092	000073	000058	000046	000036	000027	000021	000016	000012
2.9	000067	000054	000042	000033	000026	000020	000015	000012	000009
3.0	000049	000039	000031	000024	000019	000014	000011	000008	000006
3.1	000035	000028	000022	000017	000013	000010	000008	000006	000005
3.2	000025	000020	000016	000012	000010	000007	000006	000004	000003
3.3	000017	000014	000011	000009	000007	000005	000004	000003	000002
3.4	000012	000010	000008	000006	000005	000004	000003	000002	000002
3.5	000008	000007	000005	000004	000003	000002	000002	000001	000001
3.6	000006	000005	000004	000003	000002	000002	000001	000001	000001
3.7	000004	000003	000002	000002	000001	000001	000001	000001	000001
3.8	000003	000002	000002	000001	000001	000001	000001	000000	000000
3.9	000002	000001	000001	000001	000001	000001	000000	000000	000000
4.0	000001	000001	000001	000001	000000	000000	000000	000000	000000

Computed

R = 0.

\H K\	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
0.0	017877	014270	011286	008844	006863	005273	004010	003016	002242
0.1	016453	013133	010387	008139	006316	004853	003691	002776	002063
0.2	015043	012008	009497	007442	005775	004437	003374	002538	001887
0.3	013661	010904	008625	006758	005245	004030	003064	002305	001713
0.4	012320	009834	007778	006095	004730	003634	002764	002079	001545
0.5	011031	008805	006964	005457	004235	003254	002474	001861	001383
0.6	009805	007827	006191	004851	003764	002892	002199	001654	001230
0.7	008651	006905	005462	004279	003321	002552	001940	001459	001085
0.8	007574	006046	004782	003747	002908	002234	001699	001278	000950
0.9	006580	005253	004155	003255	002526	001941	001476	001110	000825
1.0	005672	004528	003581	002806	002177	001673	001272	000957	000711
1.1	004850	003872	003062	002399	001862	001431	001088	000818	000608
1.2	004114	003284	002597	002035	001579	001213	000923	000694	000516
1.3	003461	002762	002185	001712	001328	001021	000776	000584	000434
1.4	002887	002304	001823	001428	001108	000851	000647	000487	000362
1.5	002388	001906	001508	001181	000917	000704	000535	000403	000299
1.6	001959	001564	001237	000969	000752	000578	000439	000330	000245
1.7	001593	001271	001006	000788	000611	000470	000357	000268	000199
1.8	001284	001025	000811	000635	000493	000379	000288	000216	000161
1.9	001026	000819	000648	000508	000394	000302	000230	000173	000128
2.0	000813	000649	000513	000402	000312	000240	000182	000137	000102
2.1	000638	000509	000403	000316	000245	000188	000143	000107	000080
2.2	000497	000396	000313	000246	000190	000146	000111	000083	000062
2.3	000383	000306	000242	000189	000147	000113	000086	000064	000048
2.4	000293	000234	000185	000145	000112	000086	000065	000049	000036
2.5	000222	000177	000140	000109	000085	000065	000049	000037	000027
2.6	000166	000133	000105	000082	000064	000049	000037	000028	000021
2.7	000124	000099	000078	000061	000047	000036	000027	000021	000015
2.8	000091	000073	000057	000045	000035	000027	000020	000015	000011
2.9	000066	000053	000042	000033	000025	000019	000015	000011	000008
3.0	000048	000038	000030	000023	000018	000014	000010	000008	000006
3.1	000034	000027	000021	000017	000013	000010	000007	000005	000004
3.2	000024	000019	000015	000012	000009	000007	000005	000004	000003
3.3	000017	000013	000011	000008	000006	000005	000003	000003	000002
3.4	000012	000009	000007	000006	000004	000003	000002	000002	000001
3.5	000008	000006	000005	000004	000003	000002	000001	000001	000001
3.6	000005	000004	000003	000002	000002	000001	000001	000001	000000
3.7	000003	000003	000002	000002	000001	000001	000000	000000	000000
3.8	000002	000002	000001	000001	000001	000000	000000	000000	000000
3.9	000001	000001	000001	000000	000000	000000	000000	000000	000000
4.0	000001	000000	000000	000000	000000	000000	000000	000000	000000

Appendix 3. (Continued)

NBS Table

r = 0.00

k \ h	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
0.0	001733	001278	000933	000675	000484	000344	000242	000168	000116
0.1	001595	001176	000859	000621	000445	000316	000222	000155	000107
0.2	001459	001075	000785	000568	000407	000289	000203	000142	000098
0.3	001325	000976	000713	000516	000370	000263	000185	000129	000089
0.4	001195	000880	000643	000465	000333	000237	000167	000116	000080
0.5	001070	000788	000576	000416	000299	000212	000149	000104	000072
0.6	000951	000701	000512	000370	000265	000188	000133	000092	000064
0.7	000939	000618	000451	000327	000234	000166	000117	000082	000056
0.8	000734	000541	000395	000286	000205	000146	000102	000071	000049
0.9	000638	000470	000343	000248	000178	000126	000089	000062	000043
1.0	000550	000405	000296	000214	000154	000109	000077	000053	000037
1.1	000470	000347	000253	000183	000131	000093	000066	000046	000032
1.2	000399	000294	000215	000155	000111	000079	000056	000039	000027
1.3	000336	000247	000181	000131	000094	000067	000047	000033	000023
1.4	000280	000206	000151	000109	000078	000055	000039	000027	000019
1.5	000232	000171	000125	000090	000065	000046	000032	000023	000016
1.6	000190	000140	000102	000074	000053	000038	000026	000018	000013
1.7	000155	000114	000083	000060	000043	000031	000022	000015	000010
1.8	000125	000092	000067	000049	000035	000025	000017	000012	000008
1.9	000100	000073	000054	000039	000028	000020	000014	000010	000007
2.0	000079	000058	000042	000031	000022	000016	000011	000008	000005
2.1	000062	000046	000033	000024	000017	000012	000009	000006	000004
2.2	000048	000036	000026	000019	000013	000010	000007	000005	000003
2.3	000037	000027	000020	000014	000010	000007	000005	000004	000002
2.4	000028	000021	000015	000011	000008	000006	000004	000003	000002
2.5	000022	000016	000012	000008	000006	000004	000003	000002	000001
2.6	000016	000012	000009	000006	000005	000003	000002	000002	000001
2.7	000012	000009	000006	000005	000003	000002	000002	000001	000001
2.8	000009	000007	000005	000003	000002	000002	000001	000001	000001
2.9	000006	000005	000003	000003	000002	000001	000001	000001	000000
3.0	000005	000003	000003	000002	000001	000001	000001	000000	000000
3.1	000003	000002	000002	000001	000001	000001	000000	000000	000000
3.2	000002	000002	000001	000001	000001	000000	000000	000000	000000
3.3	000002	000001	000001	000001	000000	000000	000000	000000	000000
3.4	000001	000001	000001	000000	000000	000000	000000	000000	000000
3.5	000001	000001	000000	000000	000000	000000	000000	000000	000000
3.6	000001	000000	000000	000000	000000	000000	000000	000000	000000
3.7	000000	000000	000000	000000	000000	000000	000000	000000	000000
3.8	000000	000000	000000	000000	000000	000000	000000	000000	000000
3.9	000000	000000	000000	000000	000000	000000	000000	000000	000000
4.0	000000	000000	000000	000000	000000	000000	000000	000000	000000

Computed

H = 0.

\H K\	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
0.0	001645	001189	000844	000586	000395	000255	000153	000080	000036
0.1	001514	001094	000777	000540	000364	000235	000141	000073	000033
0.2	001384	001000	000710	000493	000332	000214	000129	000067	000030
0.3	001257	000908	000645	000448	000302	000195	000117	000061	000027
0.4	001133	000819	000582	000404	000272	000176	000105	000055	000024
0.5	001015	000734	000521	000362	000244	000157	000094	000049	000022
0.6	000902	000652	000463	000321	000217	000140	000084	000044	000019
0.7	000796	000575	000408	000284	000191	000123	000074	000038	000017
0.8	000697	000504	000358	000248	000167	000108	000065	000034	000015
0.9	000605	000437	000311	000216	000145	000094	000056	000029	000013
1.0	000522	000377	000268	000186	000125	000081	000048	000025	000011
1.1	000446	000322	000229	000159	000107	000069	000041	000021	000009
1.2	000378	000273	000194	000135	000091	000058	000035	000018	000008
1.3	000318	000230	000163	000113	000076	000049	000029	000015	000007
1.4	000265	000192	000136	000094	000063	000041	000024	000013	000005
1.5	000219	000159	000112	000078	000052	000034	000020	000010	000004
1.6	000180	000130	000092	000064	000043	000028	000016	000008	000004
1.7	000146	000106	000075	000052	000035	000022	000013	000007	000003
1.8	000118	000085	000060	000042	000028	000018	000011	000005	000002
1.9	000094	000068	000048	000033	000022	000014	000008	000004	000002
2.0	000074	000054	000038	000026	000018	000011	000007	000003	000001
2.1	000059	000042	000030	000021	000014	000009	000005	000002	000001
2.2	000045	000033	000023	000016	000011	000007	000004	000002	000001
2.3	000035	000025	000018	000012	000008	000005	000003	000001	000000
2.4	000027	000019	000013	000009	000006	000004	000002	000001	000000
2.5	000020	000014	000010	000007	000005	000003	000002	000001	000000
2.6	000015	000011	000007	000005	000003	000002	000001	000000	000000
2.7	000011	000008	000005	000004	000002	000001	000001	000000	000000
2.8	000008	000006	000004	000003	000002	000001	000000	000000	000000
2.9	000006	000004	000003	000002	000001	000001	000000	000000	000000
3.0	000004	000003	000002	000001	000001	000000	000000	000000	000000
3.1	000003	000002	000001	000001	000000	000000	000000	000000	000000
3.2	000002	000001	000001	000000	000000	000000	000000	000000	000000
3.3	000001	000001	000000	000000	000000	000000	000000	000000	000000
3.4	000001	000000	000000	000000	000000	000000	000000	000000	000000

Appendix 3. (Continued)

NBS Table

Computed

$r = 0.00$

k \ h	3.6	3.7	3.8	3.9	4.0
0.0	000080	000054	000036	000024	000016
0.1	000073	000050	000033	000022	000015
0.2	000067	000045	000030	000020	000013
0.3	000061	000041	000028	000018	000012
0.4	000055	000037	000025	000017	000011
0.5	000049	000033	000022	000015	000010
0.6	000044	000030	000020	000013	000009
0.7	000038	000026	000018	000012	000008
0.8	000034	000023	000015	000010	000007
0.9	000029	000020	000013	000009	000006
1.0	000025	000017	000011	000008	000005
1.1	000022	000015	000010	000007	000004
1.2	000018	000012	000008	000006	000004
1.3	000015	000010	000007	000005	000003
1.4	000013	000009	000006	000004	000003
1.5	000011	000007	000005	000003	000002
1.6	000009	000006	000004	000003	000002
1.7	000007	000005	000003	000002	000001
1.8	000006	000004	000003	000002	000001
1.9	000005	000003	000002	000001	000001
2.0	000004	000002	000002	000001	000001
2.1	000003	000002	000001	000001	000001
2.2	000002	000001	000001	000001	000000
2.3	000002	000001	000001	000001	000000
2.4	000001	000001	000001	000000	000000
2.5	000001	000001	000000	000000	000000
2.6	000001	000001	000000	000000	000000
2.7	000001	000000	000000	000000	000000

Appendix 3. (Continued)

NBS Table

r = 0.45

k \ h	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.0	324288	303974	283071	261810	240432	219182	198300	178014	158530
0.1	303974	285438	266295	246755	227039	207375	187987	169091	150884
0.2	283071	266295	248905	231089	213048	194990	177124	159651	142759
0.3	261810	246755	231089	214979	198603	182152	165817	149783	134229
0.4	240432	227039	213048	198603	183864	169000	154185	139590	125380
0.5	219182	207375	194990	182152	169000	155684	142361	129186	116309
0.6	198300	187987	177124	165817	154185	142361	130483	118691	107120
0.7	178014	169091	159651	149783	139590	129186	118691	108229	097923
0.8	158530	150884	142759	134229	125380	116309	107120	097923	088825
0.9	140029	133540	126615	119312	111704	103870	095901	087889	079932
1.0	122658	117207	111362	105172	098693	091993	085148	078236	071341
1.1	106532	101999	097116	091921	086459	080785	074962	069056	063139
1.2	091729	087998	083961	079645	075087	070330	065426	060429	055401
1.3	078294	075255	071951	068403	064637	060690	056601	052416	048186
1.4	066235	063786	061110	058223	055145	051903	048529	045060	041538
1.5	055531	053579	051435	049111	046621	043986	041230	038384	035480
1.6	046136	044596	042897	041046	039053	036934	034708	032397	030027
1.7	037981	036779	035447	033989	032411	030725	028945	027088	025176
1.8	030979	030052	029019	027882	026646	025320	023912	022436	020909
1.9	025034	024326	023534	022658	021701	020667	019566	018406	017200
2.0	020041	019507	018906	018238	017505	016709	015857	014955	014012
2.1	015893	015494	015044	014540	013985	013379	012727	012033	011304
2.2	012485	012191	011856	011481	011065	010609	010115	009588	009031
2.3	009714	009500	009255	008978	008670	008330	007961	007564	007143
2.4	007487	007332	007155	006953	006727	006477	006204	005909	005594
2.5	005715	005605	005478	005332	005169	004987	004787	004570	004338
2.6	004321	004243	004153	004050	003933	003802	003657	003500	003330
2.7	003235	003181	003118	003045	002963	002870	002766	002653	002530
2.8	002390	002362	002318	002268	002210	002144	002071	001991	001903
2.9	001762	001737	001707	001672	001632	001587	001536	001479	001417
3.0	001282	001265	001245	001221	001194	001162	001127	001088	001045
3.1	000923	000912	000899	000883	000864	000843	000819	000792	000762
3.2	000659	000651	000642	000632	000619	000605	000589	000571	000550
3.3	000465	000461	000455	000448	000440	000430	000419	000407	000393
3.4	000326	000323	000319	000314	000309	000303	000296	000288	000278
3.5	000226	000224	000221	000218	000215	000211	000206	000201	000195
3.6	000155	000154	000152	000150	000148	000146	000143	000139	000135
3.7	000105	000104	000103	000102	000101	000099	000097	000095	000093
3.8	000071	000070	000070	000069	000068	000067	000066	000065	000063
3.9	000047	000047	000047	000046	000046	000045	000044	000043	000042
4.0	000031	000031	000031	000030	000030	000030	000029	000029	000028

Computed

R = 0.45

K \ H	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.0	324270	303956	283053	261792	240414	219164	198282	177996	158512
0.1	303963	285427	266284	246744	227028	207364	187976	169080	150872
0.2	283058	266282	248892	231076	213035	194977	177111	159638	142745
0.3	261797	246742	231076	214966	198590	182139	165804	149770	134216
0.4	240415	227023	213032	198587	183848	168984	154169	139574	125364
0.5	219171	207364	194979	182141	168989	155673	142350	129175	116298
0.6	198287	187974	177111	165803	154172	142348	130470	118677	107107
0.7	178002	169078	159638	149771	139578	129173	118678	108216	097910
0.8	158514	150868	142743	134213	125364	116293	107104	097907	088809
0.9	140019	133530	126605	119302	111693	103860	095890	087879	079922
1.0	122645	117194	111350	105160	098681	091981	085135	078224	071328
1.1	106521	101988	097105	091910	086448	080774	074951	069045	063128
1.2	091725	087994	083957	079641	075083	070326	065421	060425	055397
1.3	078292	075253	071949	068400	064635	060687	056599	052414	048183
1.4	066233	063784	061109	058222	055143	051901	048527	045058	041535
1.5	055530	053577	051434	049110	046620	043984	041229	038383	035478
1.6	046135	044595	042896	041045	039052	036933	034707	032396	030026
1.7	037980	036778	035446	033988	032410	030724	028944	027088	025175
1.8	030978	030051	029018	027881	026646	025319	023911	022436	020908
1.9	025033	024325	023533	022657	021700	020667	019565	018406	017199
2.0	020040	019506	018905	018237	017504	016708	015856	014954	014011
2.1	015892	015494	015043	014539	013984	013378	012726	012032	011304
2.2	012484	012190	011856	011480	011064	010608	010115	009587	009030
2.3	009714	009499	009254	008977	008669	008330	007960	007564	007142
2.4	007486	007331	007154	006952	006726	006477	006203	005908	005594
2.5	005714	005604	005477	005332	005168	004986	004786	004569	004337
2.6	004320	004242	004152	004049	003932	003801	003656	003499	003329
2.7	003235	003181	003117	003045	002962	002869	002765	002652	002529
2.8	002399	002361	002318	002267	002209	002144	002071	001990	001902
2.9	001762	001736	001706	001671	001631	001586	001535	001478	001416
3.0	001281	001264	001244	001220	001193	001162	001126	001087	001044
3.1	000923	000911	000898	000882	000863	000842	000818	000791	000761
3.2	000658	000651	000642	000631	000619	000605	000588	000570	000550
3.3	000465	000460	000454	000447	000439	000430	000419	000406	000393
3.4	000325	000322	000318	000314	000308	000302	000295	000287	000278
3.5	000225	000223	000221	000218	000214	000210	000206	000200	000194
3.6	000154	000153	000151	000150	000147	000145	000142	000138	000134
3.7	000104	000104	000103	000102	000100	000099	000097	000095	000092
3.8	000070	000070	000069	000068	000067	000066	000065	000064	000062
3.9	000046	000046	000046	000045	000045	000044	000044	000043	000042
4.0	000030	000030	000030	000030	000029	000029	000029	000028	000028

Appendix 3. (Continued)

NBS Table

r = 0.45

k \ h	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	140029	122658	106532	091729	078294	066235	055531	046136	037981
0.1	133540	117207	101999	087998	075255	063786	053579	044596	036779
0.2	126615	111362	097116	083961	071951	061110	051435	042897	035447
0.3	119312	105172	091921	079645	068403	058223	049111	041046	033989
0.4	111704	098693	086459	075087	064637	055145	046621	039053	032411
0.5	103870	091993	080785	070330	060690	051903	043986	036934	030725
0.6	095901	085148	074962	065426	056601	048529	041230	034708	028945
0.7	087889	078236	069056	060429	052416	045060	038384	032397	027088
0.8	079932	071341	063139	055401	048186	041536	035480	030027	025176
0.9	072120	064543	057280	050400	043959	038000	032551	027627	023229
1.0	064543	057922	051549	045486	039787	034494	029634	025225	021272
1.1	057280	051549	046008	040715	035719	031059	026763	022850	019327
1.2	050400	045486	040715	036138	031798	027734	023971	020529	017418
1.3	043959	039787	035719	031798	028066	024554	021289	018290	015568
1.4	038000	034494	031059	027734	024554	021549	018743	016154	013795
1.5	032551	029634	026763	023971	021289	018743	016355	014142	012116
1.6	027627	025225	022850	020529	018290	016154	014142	012269	010547
1.7	023229	021272	019327	017418	015568	013795	012116	010547	009097
1.8	019347	017769	016193	014639	013126	011669	010283	008981	007773
1.9	015960	014700	013438	012186	010961	009776	008644	007576	006579
2.0	013038	012045	011043	010046	009065	008112	007197	006329	005515
2.1	010548	009772	008987	008201	007424	006665	005933	005235	004578
2.2	008450	007851	007241	006628	006019	005422	004843	004288	003763
2.3	006702	006245	005777	005304	004832	004367	003913	003477	003062
2.4	005263	004917	004562	004202	003840	003482	003131	002791	002467
2.5	004091	003833	003567	003295	003021	002748	002479	002218	001967
2.6	003149	002958	002761	002558	002352	002147	001943	001744	001553
2.7	002399	002260	002115	001965	001813	001660	001507	001358	001213
2.8	001809	001709	001604	001495	001383	001270	001157	001046	000938
2.9	001350	001279	001203	001125	001044	000962	000879	000797	000717
3.0	000998	000947	000894	000838	000780	000721	000661	000602	000543
3.1	000730	000695	000657	000618	000577	000535	000492	000449	000407
3.2	000528	000504	000478	000451	000422	000392	000362	000332	000301
3.3	000378	000362	000344	000325	000305	000285	000264	000242	000221
3.4	000268	000257	000245	000232	000219	000205	000190	000175	000160
3.5	000188	000181	000173	000164	000155	000146	000136	000125	000115
3.6	000131	000126	000121	000115	000109	000102	000096	000089	000082
3.7	000090	000087	000083	000080	000076	000071	000067	000062	000057
3.8	000061	000059	000057	000055	000052	000049	000046	000043	000040
3.9	000041	000040	000039	000037	000035	000033	000031	000029	000027
4.0	000027	000027	000026	000025	000024	000023	000021	000020	000019

Computed

R = 0.45

K \ H	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	140011	122640	106514	091711	078276	066217	055513	046118	037963
0.1	133529	117195	101987	087987	075244	063775	053568	044585	036768
0.2	126602	111349	097103	083948	071938	061097	051422	042884	035434
0.3	119299	105159	091908	079632	068390	058210	049098	041033	033976
0.4	111687	098677	086443	075071	064621	055129	046605	039037	032395
0.5	103859	091982	080774	070319	060678	051892	043975	036923	030714
0.6	095887	085134	074948	065412	056587	048515	041217	034694	028931
0.7	087877	073223	069044	060417	052404	045047	038372	032384	027076
0.8	079916	071325	063123	055385	048170	041520	035464	030011	025160
0.9	072110	064533	057270	050390	043949	037990	032541	027617	023219
1.0	064531	057910	051537	045474	039775	034481	029622	025213	021260
1.1	057269	051538	045997	040704	035708	031048	026752	022839	019316
1.2	050396	045482	040711	036134	031794	027730	023967	020525	017414
1.3	043957	039785	035717	031796	028064	024552	021287	018288	015566
1.4	037998	034492	031057	027732	024552	021547	018742	016153	013793
1.5	032550	029633	026762	023970	021288	018742	016354	014141	012115
1.6	027626	025224	022849	020528	018289	016153	014141	012269	010546
1.7	023228	021271	019326	017417	015567	013794	012115	010546	009096
1.8	019346	017768	016192	014638	013125	011668	010282	008981	007772
1.9	015959	014700	013437	012185	010960	009775	008643	007575	006579
2.0	013037	012044	011042	010045	009064	008111	007196	006328	005515
2.1	010547	009772	008986	008200	007423	006664	005932	005234	004578
2.2	008449	007850	007240	006627	006019	005421	004842	004287	003762
2.3	006701	006244	005776	005303	004831	004366	003913	003476	003062
2.4	005262	004917	004562	004201	003839	003481	003130	002791	002466
2.5	004090	003833	003566	003294	003020	002747	002478	002217	001966
2.6	003148	002958	002760	002557	002352	002146	001942	001744	001552
2.7	002398	002259	002114	001965	001812	001659	001507	001357	001212
2.8	001808	001708	001603	001494	001382	001269	001157	001045	000937
2.9	001350	001278	001203	001124	001043	000961	000879	000797	000717
3.0	000997	000947	000893	000837	000779	000720	000660	000601	000542
3.1	000729	000694	000656	000617	000576	000534	000491	000448	000406
3.2	000527	000503	000477	000450	000421	000392	000361	000331	000301
3.3	000378	000361	000343	000325	000305	000284	000263	000242	000220
3.4	000268	000257	000245	000232	000218	000204	000189	000175	000160
3.5	000188	000180	000172	000164	000154	000145	000135	000125	000114
3.6	000130	000125	000120	000114	000108	000102	000095	000088	000081
3.7	000089	000086	000083	000079	000075	000071	000066	000061	000057
3.8	000061	000058	000056	000054	000051	000048	000045	000042	000039
3.9	000041	000039	000038	000036	000035	000033	000031	000029	000027
4.0	000027	000026	000025	000024	000023	000022	000021	000019	000018

Appendix 3. (Continued)

NBS Table

r = 0.45

k \ h	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
0.0	030979	025034	020041	015893	012485	009714	007487	005715	004321
0.1	030052	024326	019507	015494	012191	009500	007332	005605	004243
0.2	029019	023534	018906	015044	011856	009255	007155	005478	004153
0.3	027882	022658	018238	014540	011481	008978	006953	005332	004050
0.4	026646	021701	017505	013985	011065	008670	006727	005169	003933
0.5	025320	020667	016709	013379	010609	008330	006477	004987	003802
0.6	023912	019566	015857	012727	010115	007961	006204	004787	003657
0.7	022436	018406	014955	012033	009588	007564	005909	004570	003500
0.8	020909	017200	014012	011304	009031	007143	005594	004338	003330
0.9	019347	015960	013038	010548	008450	006702	005263	004091	003149
1.0	017769	014700	012045	009772	007851	006245	004917	003833	002958
1.1	016193	013438	011043	008987	007241	005777	004562	003567	002761
1.2	014639	012186	010046	008201	006628	005304	004202	003295	002558
1.3	013126	010961	009065	007424	006010	004832	003840	003021	002352
1.4	011669	009776	008112	006665	005422	004367	003482	002748	002147
1.5	010283	008644	007197	005933	004843	003913	003131	002479	001943
1.6	008981	007576	006329	005235	004288	003477	002791	002218	001744
1.7	007773	006579	005515	004578	003763	003062	002467	001967	001553
1.8	006665	005661	004763	003968	003273	002673	002161	001729	001370
1.9	005661	004826	004074	003406	002820	002312	001876	001507	001198
2.0	004763	004074	003452	002897	002407	001980	001613	001300	001038
2.1	003968	003406	002897	002440	002035	001680	001374	001112	000890
2.2	003273	002820	002407	002035	001703	001412	001159	000941	000757
2.3	002673	002312	001980	001680	001412	001175	000968	000789	000637
2.4	002161	001876	001613	001374	001159	000968	000800	000655	000531
2.5	001729	001507	001300	001112	000941	000789	000655	000539	000438
2.6	001370	001198	001038	000890	000757	000637	000531	000438	000358
2.7	001074	000942	000819	000706	000602	000509	000426	000353	000289
2.8	000833	000734	000640	000554	000474	000402	000338	000281	000232
2.9	000640	000565	000495	000430	000370	000315	000266	000222	000183
3.0	000486	000431	000379	000330	000285	000244	000206	000173	000144
3.1	000365	000325	000287	000251	000218	000187	000159	000134	000111
3.2	000272	000243	000215	000189	000164	000141	000121	000102	000085
3.3	000200	000179	000159	000140	000123	000106	000091	000077	000065
3.4	000145	000131	000117	000103	000091	000079	000068	000058	000049
3.5	000105	000095	000085	000075	000066	000058	000050	000043	000036
3.6	000075	000068	000061	000054	000048	000042	000036	000031	000027
3.7	000053	000048	000043	000039	000034	000030	000026	000023	000019
3.8	000037	000033	000030	000027	000024	000021	000019	000016	000014
3.9	000025	000023	000021	000019	000017	000015	000013	000011	000010
4.0	000017	000016	000014	000013	000012	000010	000009	000008	000007

Computed

R = 0.45

K \ H	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
0.0	030961	025016	020023	015875	012467	009696	007469	005697	004303
0.1	030041	024315	019496	015483	012180	009489	007321	005594	004232
0.2	029006	023521	018893	015030	011843	009242	007141	005464	004140
0.3	027869	022645	018225	014527	011468	008965	006940	005320	004037
0.4	026630	021684	017488	013969	011049	008654	006711	005153	003917
0.5	025308	020656	016698	013368	010598	008319	006466	004976	003791
0.6	023898	019553	015844	012714	010102	007948	006191	004774	003644
0.7	022424	018394	014942	012021	009575	007552	005897	004559	003487
0.8	020893	017184	013996	011289	009015	007127	005578	004322	003314
0.9	019337	015950	013028	010538	008440	006692	005253	004081	003139
1.0	017756	014688	012033	009760	007839	006232	004905	003821	002946
1.1	016182	013427	011032	008976	007230	005766	004552	003556	002750
1.2	014635	012182	010042	008197	006624	005300	004198	003291	002554
1.3	013124	010959	009063	007422	006017	004830	003838	003019	002350
1.4	011667	009775	008110	006663	005421	004365	003480	002747	002145
1.5	010282	008643	007195	005932	004842	003912	003130	002478	001942
1.6	008981	007575	006328	005234	004287	003476	002790	002217	001744
1.7	007772	006579	005515	004578	003762	003062	002466	001966	001552
1.8	006664	005660	004762	003967	003272	002672	002160	001729	001369
1.9	005661	004825	004073	003406	002819	002311	001875	001506	001197
2.0	004762	004073	003451	002896	002406	001979	001612	001300	001037
2.1	003967	003406	002896	002439	002034	001680	001373	001111	000890
2.2	003272	002819	002406	002034	001703	001411	001158	000941	000756
2.3	002672	002311	001979	001680	001411	001174	000967	000789	000636
2.4	002160	001875	001612	001373	001158	000967	000800	000655	000530
2.5	001729	001506	001300	001111	000941	000789	000655	000538	000438
2.6	001369	001197	001037	000890	000756	000636	000530	000438	000357
2.7	001073	000942	000819	000705	000602	000508	000425	000352	000289
2.8	000832	000733	000640	000553	000474	000402	000337	000281	000231
2.9	000639	000565	000495	000429	000369	000314	000265	000221	000183
3.0	000485	000430	000378	000330	000284	000243	000206	000172	000143
3.1	000365	000325	000286	000250	000217	000186	000158	000133	000111
3.2	000271	000242	000214	000188	000164	000141	000120	000102	000085
3.3	000199	000179	000159	000140	000122	000105	000090	000077	000064
3.4	000145	000130	000116	000103	000090	000078	000067	000057	000048
3.5	000104	000094	000084	000075	000066	000057	000049	000042	000036
3.6	000074	000067	000060	000054	000047	000041	000036	000031	000026
3.7	000052	000047	000043	000038	000034	000029	000026	000022	000019
3.8	000036	000033	000030	000027	000024	000021	000018	000016	000013
3.9	000025	000023	000020	000018	000016	000014	000013	000011	000009
4.0	000017	000015	000014	000012	000011	000010	000009	000007	000006

NBS Table

r = 0.45

k \ h	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
0.0	003235	002399	001762	001282	000923	000659	000465	000326	000226
0.1	003181	002362	001737	001265	000912	000651	000461	000323	000224
0.2	003118	002318	001707	001245	000899	000642	000455	000319	000221
0.3	003045	002268	001672	001221	000883	000632	000448	000314	000218
0.4	002963	002210	001632	001194	000864	000619	000440	000309	000215
0.5	002870	002144	001587	001162	000843	000605	000430	000303	000211
0.6	002766	002071	001536	001127	000819	000589	000419	000296	000206
0.7	002653	001991	001479	001088	000792	000571	000407	000288	000201
0.8	002530	001903	001417	001045	000762	000550	000393	000278	000195
0.9	002399	001809	001350	000998	000730	000528	000378	000268	000188
1.0	002260	001709	001279	000947	000695	000504	000362	000257	000181
1.1	002115	001604	001203	000894	000657	000478	000344	000245	000173
1.2	001965	001495	001125	000838	000618	000451	000325	000232	000164
1.3	001813	001383	001044	000780	000577	000422	000305	000219	000155
1.4	001660	001270	000962	000721	000535	000392	000285	000205	000146
1.5	001507	001157	000879	000661	000492	000362	000264	000190	000136
1.6	001358	001046	000797	000602	000449	000332	000242	000175	000125
1.7	001213	000938	000717	000543	000407	000301	000221	000160	000115
1.8	001074	000833	000640	000486	000365	000272	000200	000145	000105
1.9	000942	000734	000565	000431	000325	000243	000179	000131	000095
2.0	000819	000640	000495	000379	000287	000215	000159	000117	000085
2.1	000706	000554	000430	000330	000251	000189	000140	000103	000075
2.2	000602	000474	000370	000285	000218	000164	000123	000091	000066
2.3	000509	000402	000315	000244	000187	000141	000106	000079	000058
2.4	000426	000338	000266	000206	000159	000121	000091	000068	000050
2.5	000353	000281	000222	000173	000134	000102	000077	000058	000043
2.6	000289	000232	000183	000144	000111	000085	000065	000049	000036
2.7	000235	000189	000150	000118	000092	000071	000054	000041	000030
2.8	000189	000152	000122	000096	000075	000058	000044	000034	000025
2.9	000150	000122	000097	000077	000061	000047	000036	000028	000021
3.0	000118	000096	000077	000062	000049	000038	000029	000022	000017
3.1	000092	000075	000061	000049	000038	000030	000023	000018	000014
3.2	000071	000058	000047	000038	000030	000024	000018	000014	000011
3.3	000054	000044	000036	000029	000023	000018	000014	000011	000009
3.4	000041	000034	000028	000022	000018	000014	000011	000009	000007
3.5	000030	000025	000021	000017	000014	000011	000009	000007	000005
3.6	000022	000019	000015	000013	000010	000008	000006	000005	000004
3.7	000016	000014	000011	000009	000008	000006	000005	000004	000003
3.8	000012	000010	000008	000007	000006	000004	000004	000003	000002
3.9	000008	000007	000006	000005	000004	000003	000003	000002	000002
4.0	000006	000005	000004	000004	000003	000002	000002	000002	000001

Computed

R = 0.45

\H K\	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
0.0	003218	002382	001744	001264	000906	000643	000448	000309	000210
0.1	003170	002351	001726	001254	000901	000641	000451	000313	000214
0.2	003105	002305	001694	001231	000885	000630	000443	000306	000209
0.3	003033	002255	001659	001208	000870	000619	000435	000303	000207
0.4	002947	002194	001616	001178	000848	000603	000424	000294	000201
0.5	002859	002134	001576	001151	000832	000594	000419	000294	000201
0.6	002753	002058	001522	001114	000806	000576	000406	000283	000194
0.7	002640	001978	001467	001075	000779	000558	000395	000275	000190
0.8	002514	001887	001401	001029	000746	000535	000378	000263	000179
0.9	002389	001799	001340	000988	000720	000518	000368	000258	000179
1.0	002248	001697	001267	000935	000683	000492	000350	000245	000169
1.1	002104	001593	001193	000883	000646	000467	000333	000234	000162
1.2	001961	001491	001121	000834	000614	000447	000321	000229	000160
1.3	001811	001381	001042	000779	000575	000420	000304	000217	000153
1.4	001658	001269	000961	000720	000533	000391	000284	000203	000144
1.5	001506	001156	000878	000660	000491	000361	000263	000189	000135
1.6	001357	001045	000797	000601	000448	000331	000242	000174	000125
1.7	001212	000937	000716	000542	000406	000301	000220	000160	000114
1.8	001073	000832	000639	000485	000365	000271	000199	000145	000104
1.9	000942	000733	000565	000430	000325	000242	000179	000130	000094
2.0	000819	000640	000495	000378	000286	000214	000159	000116	000084
2.1	000705	000553	000429	000330	000250	000188	000140	000103	000075
2.2	000602	000474	000369	000284	000217	000164	000122	000090	000066
2.3	000508	000402	000314	000243	000186	000141	000105	000078	000057
2.4	000425	000337	000265	000206	000158	000120	000090	000067	000049
2.5	000352	000281	000221	000172	000133	000102	000077	000057	000042
2.6	000289	000231	000183	000143	000111	000085	000064	000048	000036
2.7	000234	000188	000150	000118	000091	000070	000053	000040	000030
2.8	000188	000152	000121	000095	000075	000058	000044	000033	000025
2.9	000150	000121	000097	000077	000060	000047	000036	000027	000020
3.0	000118	000095	000077	000061	000048	000037	000029	000022	000016
3.1	000091	000075	000060	000048	000038	000030	000023	000017	000013
3.2	000070	000058	000047	000037	000030	000023	000018	000014	000010
3.3	000053	000044	000036	000029	000023	000018	000014	000011	000008
3.4	000040	000033	000027	000022	000017	000014	000011	000008	000006
3.5	000030	000025	000020	000016	000013	000010	000008	000006	000005
3.6	000022	000018	000015	000012	000010	000008	000006	000005	000003
3.7	000016	000013	000011	000009	000007	000006	000004	000003	000003
3.8	000011	000009	000008	000006	000005	000004	000003	000002	000002
3.9	000008	000007	000005	000004	000004	000003	000002	000002	000001
4.0	000005	000005	000004	000003	000002	000002	000001	000001	000001

Appendix 3. (Continued)

NBS Table

$r = 0.45$

k \ h	3.6	3.7	3.8	3.9	4.0
0.0	000155	000105	000071	000047	000031
0.1	000154	000104	000070	000047	000031
0.2	000152	000103	000070	000047	000031
0.3	000150	000102	000069	000046	000030
0.4	000148	000101	000068	000046	000030
0.5	000146	000099	000067	000045	000030
0.6	000143	000097	000066	000044	000029
0.7	000139	000095	000065	000043	000029
0.8	000135	000093	000063	000042	000028
0.9	000131	000090	000061	000041	000027
1.0	000126	000087	000059	000040	000027
1.1	000121	000083	000057	000039	000026
1.2	000115	000080	000055	000037	000025
1.3	000109	000076	000052	000035	000024
1.4	000102	000071	000049	000033	000023
1.5	000096	000067	000046	000031	000021
1.6	000089	000062	000043	000029	000020
1.7	000082	000057	000040	000027	000019
1.8	000075	000053	000037	000025	000017
1.9	000068	000048	000033	000023	000016
2.0	000061	000043	000030	000021	000014
2.1	000054	000039	000027	000019	000013
2.2	000048	000034	000024	000017	000012
2.3	000042	000030	000021	000015	000010
2.4	000036	000026	000019	000013	000009
2.5	000031	000023	000016	000011	000008
2.6	000027	000019	000014	000010	000007
2.7	000022	000016	000012	000008	000006
2.8	000019	000014	000010	000007	000005
2.9	000015	000011	000008	000006	000004
3.0	000013	000009	000007	000005	000004
3.1	000010	000008	000006	000004	000003
3.2	000008	000006	000004	000003	000002
3.3	000006	000005	000004	000003	000002
3.4	000005	000004	000003	000002	000002
3.5	000004	000003	000002	000002	000001
3.6	000003	000002	000002	000001	000001
3.7	000002	000002	000001	000001	000001
3.8	000002	000001	000001	000001	000001
3.9	000001	000001	000001	000001	000000
4.0	000001	000001	000001	000000	000000

Computed

R = 0.45

\H K\	3.6	3.7	3.8	3.9	4.0
0.0	000138	000088	000053	000030	000014
0.1	000143	000095	000060	000036	000020
0.2	000140	000091	000057	000034	000018
0.3	000138	000090	000057	000034	000018
0.4	000133	000086	000053	000030	000014
0.5	000135	000089	000057	000035	000019
0.6	000131	000085	000053	000032	000016
0.7	000127	000084	000053	000031	000017
0.8	000120	000078	000048	000027	000013
0.9	000122	000081	000052	000032	000018
1.0	000114	000075	000048	000028	000015
1.1	000110	000073	000047	000028	000015
1.2	000111	000077	000051	000033	000021
1.3	000107	000074	000051	000034	000022
1.4	000101	000070	000048	000033	000022
1.5	000095	000066	000045	000031	000021
1.6	000088	000061	000042	000029	000020
1.7	000081	000057	000039	000027	000018
1.8	000074	000052	000036	000025	000017
1.9	000067	000047	000033	000023	000015
2.0	000060	000043	000030	000020	000014
2.1	000054	000038	000027	000018	000012
2.2	000047	000034	000024	000016	000011
2.3	000041	000029	000021	000014	000010
2.4	000036	000026	000018	000013	000009
2.5	000031	000022	000016	000011	000007
2.6	000026	000019	000013	000009	000006
2.7	000022	000016	000011	000008	000005
2.8	000018	000013	000009	000007	000005
2.9	000015	000011	000008	000005	000004
3.0	000012	000009	000006	000004	000003
3.1	000010	000007	000005	000004	000002
3.2	000008	000006	000004	000003	000002
3.3	000006	000004	000003	000002	000001
3.4	000005	000003	000002	000002	000001
3.5	000003	000003	000002	000001	000001
3.6	000003	000002	000001	000001	000000
3.7	000002	000001	000001	000001	000000
3.8	000001	000001	000001	000000	000000
3.9	000001	000001	000000	000000	000000

NBS Table

r = 0.50

k \ h	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.0	333333	312961	291886	270344	248589	226878	205468	184605	164512
0.1	312961	294422	275161	255392	235345	215260	195377	175927	157126
0.2	291886	275161	257709	239718	221397	202965	184644	166650	149190
0.3	270344	255392	239718	223488	206888	190114	173370	156858	140769
0.4	248589	235345	221397	206888	191979	176847	161676	146649	131946
0.5	226878	215260	202965	190114	176847	163320	149694	136139	122816
0.6	205468	195377	184644	173370	161676	149694	137570	125451	113486
0.7	184605	175927	166650	156858	146649	136139	125451	114718	104071
0.8	164512	157126	149190	140769	131946	122816	113486	104071	094686
0.9	145388	139168	132448	125281	117733	109882	101819	093640	085448
1.0	127398	122215	116586	110550	104159	097477	090578	083546	076465
1.1	110671	106398	101733	096704	091350	085722	079882	073896	067839
1.2	095297	091814	087990	083844	079407	074718	069825	064784	059656
1.3	081329	078522	075421	072042	068404	064539	060485	056284	051988
1.4	068785	066546	064061	061337	058388	055237	051913	048451	044891
1.5	057646	055882	053912	051740	049377	046836	044142	041320	038401
1.6	047867	046493	044950	043238	041365	039340	037180	034905	032539
1.7	039379	038321	037126	035793	034325	032729	031018	029204	027307
1.8	032095	031290	030375	029348	028211	026968	025627	024198	022695
1.9	025912	025307	024615	023834	022963	022006	020968	019854	018677
2.0	020724	020274	019756	019169	018510	017782	016987	016130	015218
2.1	016417	016087	015704	015268	014776	014228	013627	012974	012276
2.2	012882	012642	012363	012043	011679	011272	010823	010332	009804
2.3	010011	009840	009638	009406	009141	008842	008510	008145	007750
2.4	007706	007585	007441	007275	007083	006867	006624	006357	006065
2.5	005875	005790	005689	005571	005435	005280	005105	004911	004698
2.6	004436	004377	004307	004225	004129	004019	003894	003755	003602
2.7	003317	003277	003229	003172	003105	003029	002941	002842	002733
2.8	002457	002430	002397	002358	002312	002259	002198	002130	002053
2.9	001802	001784	001762	001736	001705	001669	001627	001579	001526
3.0	001309	001297	001283	001265	001244	001220	001192	001159	001123
3.1	000942	000934	000925	000913	000899	000883	000864	000842	000817
3.2	000671	000666	000660	000652	000644	000633	000620	000606	000589
3.3	000473	000470	000466	000462	000456	000449	000441	000431	000420
3.4	000331	000329	000326	000323	000320	000315	000310	000304	000297
3.5	000229	000228	000226	000224	000222	000219	000216	000212	000207
3.6	000157	000156	000155	000154	000153	000151	000149	000146	000143
3.7	000107	000106	000106	000105	000104	000103	000102	000100	000098
3.8	000072	000071	000071	000071	000070	000069	000069	000068	000066
3.9	000048	000048	000047	000047	000047	000046	000046	000045	000045
4.0	000031	000031	000031	000031	000031	000031	000030	000030	000030

Computed

R = 0.50

\H K\	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
0.0	333324	312952	291877	270335	248580	226869	205459	184596	164503
0.1	312950	294411	275150	255381	235334	215249	195366	175916	157115
0.2	291872	275147	257695	239704	221383	202951	184630	166636	149176
0.3	270333	255384	239710	223480	206880	190106	173362	156850	140761
0.4	248579	235335	221387	206878	191969	176837	161666	146639	131936
0.5	226868	215250	202955	190104	176837	163309	149684	136128	122806
0.6	205455	195363	184631	173357	161663	149681	137556	125438	113473
0.7	184596	175918	166642	156849	146641	136130	125442	114709	104062
0.8	164501	157115	149179	140758	131935	122805	113476	104060	094675
0.9	145378	139157	132438	125271	117722	109872	101808	093630	085437
1.0	127391	122208	116578	110542	104151	097470	090571	083538	076458
1.1	110662	106390	101725	096695	091341	085714	079873	073888	067830
1.2	095289	091806	087982	083836	079399	074710	069817	064776	059648
1.3	081326	078518	075418	072038	068401	064536	060481	056280	051984
1.4	068781	066543	064058	061333	058385	055234	051910	048448	044887
1.5	057644	055880	053911	051739	049375	046835	044140	041318	038400
1.6	047866	046492	044949	043237	041364	039339	037179	034904	032538
1.7	039378	038320	037125	035792	034324	032729	031017	029203	027306
1.8	032094	031289	030374	029347	028210	026967	025626	024197	022694
1.9	025911	025306	024614	023833	022962	022005	020967	019853	018676
2.0	020723	020273	019755	019168	018510	017781	016986	016129	015217
2.1	016416	016086	015704	015267	014775	014227	013626	012973	012275
2.2	012881	012641	012362	012042	011678	011271	010822	010331	009803
2.3	010010	009839	009638	009405	009140	008841	008509	008144	007750
2.4	007705	007584	007440	007274	007083	006866	006623	006356	006064
2.5	005874	005789	005688	005570	005434	005279	005104	004910	004697
2.6	004435	004376	004306	004224	004128	004018	003894	003754	003601
2.7	003316	003276	003228	003171	003105	003028	002940	002842	002732
2.8	002456	002429	002396	002357	002312	002259	002198	002129	002052
2.9	001801	001783	001761	001735	001704	001668	001626	001579	001525
3.0	001308	001296	001282	001264	001244	001219	001191	001159	001122
3.1	000941	000933	000924	000912	000899	000882	000863	000842	000817
3.2	000670	000665	000659	000652	000643	000632	000620	000605	000588
3.3	000473	000470	000466	000461	000455	000448	000440	000430	000419
3.4	000330	000328	000326	000323	000319	000315	000309	000303	000296
3.5	000228	000227	000226	000224	000221	000218	000215	000211	000207
3.6	000156	000155	000155	000153	000152	000150	000148	000146	000143
3.7	000106	000105	000105	000104	000103	000102	000101	000099	000097
3.8	000071	000071	000070	000070	000069	000069	000068	000067	000066
3.9	000047	000047	000047	000046	000046	000046	000045	000045	000044
4.0	000031	000031	000031	000030	000030	000030	000030	000029	000029

Appendix 3. (Continued)

NBS Table

r = 0.50

k \ h	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	145388	127398	110671	095297	081329	068785	057646	047867	039379
0.1	139168	122215	106398	091814	078522	066546	055882	046493	038321
0.2	132448	116586	101733	087990	075421	064061	053912	044950	037126
0.3	125281	110550	096704	083844	072042	061337	051740	043238	035793
0.4	117733	104159	091350	079407	068404	058388	049377	041365	034325
0.5	109882	097477	085722	074718	064539	055237	046836	039340	032729
0.6	101819	090578	079882	069825	060485	051913	044142	037180	031018
0.7	093640	083546	073896	064784	056284	048451	041320	034905	029204
0.8	085448	076465	067839	059656	051988	044891	038401	032539	027307
0.9	077344	069426	061786	054504	047649	041275	035421	030110	025349
1.0	069426	062514	055812	049394	043323	037651	032418	027648	023353
1.1	061786	055812	049991	044388	039062	034063	029428	025184	021345
1.2	054504	049394	044388	039545	034920	030556	026490	022749	019349
1.3	047649	043323	039062	034920	030942	027170	023639	020373	017391
1.4	041275	037651	034063	030556	027170	023944	020907	018085	015494
1.5	035421	032418	029428	026490	023639	020907	018323	015909	013681
1.6	030110	027648	025184	022749	020373	018085	015909	013864	011969
1.7	025349	023353	021345	019349	017391	015494	013681	011969	010372
1.8	021134	019534	017915	016297	014701	013146	011652	010233	008902
1.9	017447	016178	014888	013591	012304	011044	009826	008663	007566
2.0	014260	013266	012249	011221	010196	009186	008204	007261	006367
2.1	011539	010769	009977	009171	008363	007563	006780	006024	005304
2.2	009242	008654	008043	007420	006790	006163	005546	004947	004373
2.3	007328	006883	006419	005941	005457	004971	004490	004021	003568
2.4	005751	005418	005069	004708	004339	003968	003598	003234	002882
2.5	004468	004222	003962	003692	003415	003134	002852	002574	002303
2.6	003435	003255	003065	002865	002659	002449	002237	002027	001820
2.7	002613	002484	002346	002200	002049	001894	001736	001579	001424
2.8	001968	001876	001777	001672	001562	001449	001333	001217	001102
2.9	001467	001402	001332	001257	001178	001097	001013	000928	000843
3.0	001082	001037	000988	000935	000879	000821	000761	000700	000639
3.1	000789	000759	000725	000688	000649	000608	000566	000522	000478
3.2	000570	000549	000526	000501	000474	000446	000416	000385	000354
3.3	000408	000394	000378	000361	000343	000323	000303	000281	000260
3.4	000288	000279	000269	000257	000245	000232	000218	000203	000188
3.5	000202	000196	000189	000181	000173	000164	000155	000145	000135
3.6	000140	000136	000132	000127	000121	000115	000109	000102	000096
3.7	000096	000093	000091	000087	000084	000080	000076	000072	000067
3.8	000065	000064	000062	000060	000057	000055	000052	000049	000046
3.9	000044	000043	000042	000040	000039	000037	000036	000034	000032
4.0	000029	000028	000028	000027	000026	000025	000024	000023	000022

Computed

R = 0.50

\H K\	0.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7
0.0	145379	127389	110662	095288	081321	068776	057637	047858	039371
0.1	139157	122204	106387	091803	078511	066535	055871	046482	038310
0.2	132434	116571	101719	087976	075407	064047	053898	044936	037112
0.3	125273	110541	096695	083836	072033	061328	051732	043230	035785
0.4	117723	104148	091339	079397	068394	058377	049366	041354	034315
0.5	109872	097467	085712	074708	064529	055227	046826	039330	032719
0.6	101805	090565	079868	069812	060471	051900	044129	037167	031004
0.7	093631	083537	073887	064775	056275	048442	041311	034896	029195
0.8	085437	076454	067828	059645	051977	044880	038390	032528	027296
0.9	077333	069415	061775	054493	047639	041265	035411	030099	025339
1.0	069419	062507	055805	049386	043315	037644	032410	027640	023346
1.1	061777	055804	049982	044379	039054	034055	029420	025175	021336
1.2	054496	049386	044380	039538	034912	030548	026482	022741	019341
1.3	047646	043319	039059	034916	030938	027167	023635	020370	017387
1.4	041272	037643	034060	030553	027167	023941	020904	018082	015491
1.5	035420	032416	029427	026489	023638	020906	018322	015907	013680
1.6	030109	027647	025182	022748	020372	018084	015907	013863	011968
1.7	025348	023352	021344	019348	017390	015494	013680	011968	010371
1.8	021133	019533	017914	016296	014700	013145	011651	010232	008901
1.9	017446	016178	014887	013590	012303	011043	009825	008662	007565
2.0	014259	013265	012248	011220	010195	009185	008203	007260	006366
2.1	011538	010768	009976	009171	008362	007562	006779	006023	005303
2.2	009242	008653	008043	007419	006789	006162	005545	004946	004372
2.3	007327	006882	006418	005940	005456	004970	004489	004020	003568
2.4	005751	005417	005068	004707	004339	003967	003597	003233	002881
2.5	004467	004221	003961	003691	003414	003133	002851	002573	002302
2.6	003434	003254	003064	002865	002658	002448	002237	002026	001820
2.7	002613	002483	002345	002200	002048	001893	001736	001578	001423
2.8	001967	001875	001776	001671	001561	001448	001333	001216	001101
2.9	001466	001401	001331	001256	001178	001096	001012	000927	000843
3.0	001081	001036	000987	000934	000879	000820	000760	000699	000638
3.1	000789	000758	000724	000687	000648	000608	000565	000522	000477
3.2	000570	000549	000525	000500	000473	000445	000415	000385	000354
3.3	000407	000393	000377	000360	000342	000322	000302	000281	000259
3.4	000288	000278	000268	000257	000244	000231	000217	000202	000187
3.5	000201	000195	000188	000181	000173	000164	000154	000144	000134
3.6	000139	000135	000131	000126	000121	000115	000108	000102	000095
3.7	000095	000093	000090	000087	000083	000079	000075	000071	000066
3.8	000064	000063	000061	000059	000057	000054	000052	000049	000046
3.9	000043	000042	000041	000040	000038	000037	000035	000033	000031
4.0	000028	000028	000027	000026	000025	000024	000023	000022	000021

Appendix 3. (Continued)

NBS Table

r = 0.50

k \ h	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
0.0	032095	025912	020724	016417	012882	010011	007706	005875	004436
0.1	031290	025307	020274	016087	012642	009840	007585	005790	004377
0.2	030375	024615	019756	015704	012363	009638	007441	005689	004307
0.3	029348	023834	019169	015268	012043	009406	007275	005571	004225
0.4	028211	022963	018510	014776	011679	009141	007083	005435	004129
0.5	026968	022006	017782	014228	011272	008842	006867	005280	004019
0.6	025627	020968	016987	013627	010823	008510	006624	005105	003894
0.7	024198	019854	016130	012974	010332	008145	006357	004911	003755
0.8	022695	018677	015218	012276	009804	007750	006065	004698	003602
0.9	021134	017447	014260	011539	009242	007328	005751	004468	003435
1.0	019534	016178	013266	010769	008654	006883	005418	004222	003255
1.1	017915	014888	012249	009977	008043	006419	005069	003962	003065
1.2	016297	013591	011221	009171	007420	005941	004708	003692	002865
1.3	014701	012304	010196	008363	006790	005457	004339	003415	002659
1.4	013146	011044	009186	007563	006163	004971	003968	003134	002449
1.5	011652	009826	008204	006780	005546	004490	003598	002852	002237
1.6	010233	008663	007261	006024	004947	004021	003234	002574	002027
1.7	008902	007566	006367	005304	004373	003568	002882	002303	001820
1.8	007671	006546	005531	004626	003830	003138	002544	002041	001620
1.9	006546	005608	004758	003996	003322	002733	002225	001793	001429
2.0	005531	004758	004053	003418	002853	002358	001928	001560	001249
2.1	004626	003996	003418	002895	002427	002014	001654	001344	001081
2.2	003830	003322	002853	002427	002043	001703	001405	001146	000926
2.3	003138	002733	002358	002014	001703	001426	001181	000968	000785
2.4	002544	002225	001928	001654	001405	001181	000983	000809	000659
2.5	002041	001793	001560	001344	001146	000968	000809	000669	000548
2.6	001620	001429	001249	001081	000926	000785	000659	000548	000451
2.7	001273	001127	000989	000859	000740	000630	000532	000444	000367
2.8	000989	000879	000775	000676	000584	000500	000424	000355	000295
2.9	000760	000678	000600	000526	000457	000393	000334	000282	000235
3.0	000578	000518	000460	000405	000353	000305	000261	000221	000185
3.1	000434	000391	000349	000308	000270	000234	000201	000171	000144
3.2	000323	000292	000262	000232	000204	000178	000154	000131	000111
3.3	000238	000216	000194	000173	000153	000134	000116	000099	000084
3.4	000173	000157	000142	000127	000113	000099	000086	000075	000064
3.5	000124	000114	000103	000093	000083	000073	000064	000055	000047
3.6	000088	000081	000074	000067	000060	000053	000047	000040	000035
3.7	000062	000057	000052	000048	000043	000038	000034	000029	000025
3.8	000043	000040	000037	000033	000030	000027	000024	000021	000018
3.9	000030	000028	000025	000023	000021	000019	000017	000015	000013
4.0	000020	000019	000017	000016	000015	000013	000012	000010	000009

Computed

R = 0.50

\H K\	1.8	1.9	2.0	2.1	2.2	2.3	2.4	2.5	2.6
0.0	032086	025904	020715	016408	012873	010003	007697	005866	004427
0.1	031279	025296	020263	016076	012631	009829	007574	005779	004366
0.2	030361	024601	019742	015690	012349	009624	007427	005675	004293
0.3	029340	023825	019161	015260	012034	009398	007266	005563	004216
0.4	028201	022953	018500	014766	011669	009130	007073	005425	004119
0.5	026958	021996	017772	014218	011262	008832	006857	005270	004009
0.6	025614	020954	016974	013614	010809	008497	006611	005092	003881
0.7	024189	019846	016121	012966	010323	008137	006348	004902	003747
0.8	022684	018666	015207	012265	009793	007740	006054	004687	003591
0.9	021124	017436	014249	011528	009232	007318	005741	004457	003424
1.0	019527	016171	013259	010762	008646	006876	005411	004214	003248
1.1	017906	014879	012241	009968	008035	006410	005061	003954	003056
1.2	016289	013583	011213	009164	007412	005933	004700	003685	002858
1.3	014697	012301	010192	008360	006787	005453	004336	003411	002656
1.4	013143	011041	009183	007560	006160	004968	003965	003131	002446
1.5	011650	009824	008202	006779	005545	004489	003596	002851	002236
1.6	010231	008662	007259	006023	004946	004020	003233	002573	002026
1.7	008901	007565	006366	005303	004372	003568	002881	002302	001820
1.8	007670	006545	005530	004625	003829	003137	002543	002040	001620
1.9	006545	005608	004757	003995	003321	002732	002225	001792	001428
2.0	005530	004757	004052	003417	002853	002357	001927	001559	001248
2.1	004625	003995	003417	002894	002426	002013	001653	001343	001080
2.2	003829	003321	002853	002426	002043	001702	001404	001146	000925
2.3	003137	002732	002357	002013	001702	001425	001180	000967	000785
2.4	002543	002225	001927	001653	001404	001180	000982	000808	000659
2.5	002041	001792	001559	001343	001146	000967	000808	000669	000547
2.6	001620	001428	001248	001080	000925	000785	000659	000547	000450
2.7	001272	001126	000988	000859	000739	000629	000531	000443	000366
2.8	000988	000879	000774	000675	000584	000500	000423	000355	000294
2.9	000759	000678	000600	000526	000456	000392	000334	000281	000234
3.0	000577	000517	000459	000404	000353	000304	000260	000220	000184
3.1	000434	000390	000348	000308	000269	000234	000201	000170	000143
3.2	000322	000291	000261	000232	000204	000177	000153	000131	000110
3.3	000237	000215	000193	000172	000152	000133	000115	000099	000084
3.4	000172	000157	000142	000127	000112	000099	000086	000074	000063
3.5	000124	000113	000102	000092	000082	000072	000063	000055	000047
3.6	000088	000081	000073	000066	000059	000052	000046	000040	000034
3.7	000062	000057	000052	000047	000042	000037	000033	000029	000025
3.8	000043	000039	000036	000033	000030	000026	000023	000020	000018
3.9	000029	000027	000025	000023	000020	000018	000016	000014	000012
4.0	000020	000018	000017	000015	000014	000013	000011	000010	000009

Appendix 3. (Continued)

NBS Table

r = 0.50

k \ h	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
0.0	003317	002457	001802	001309	000942	000671	000473	000331	000229
0.1	003277	002430	001784	001297	000934	000666	000470	000329	000228
0.2	003229	002397	001762	001283	000925	000660	000466	000326	000226
0.3	003172	002358	001736	001265	000913	000652	000462	000323	000224
0.4	003105	002312	001705	001244	000899	000644	000456	000320	000222
0.5	003029	002259	001669	001220	000883	000633	000449	000315	000219
0.6	002941	002198	001627	001192	000864	000620	000441	000310	000216
0.7	002842	002130	001579	001159	000842	000606	000431	000304	000212
0.8	002733	002053	001526	001123	000817	000589	000420	000297	000207
0.9	002613	001968	001467	001082	000789	000570	000408	000288	000202
1.0	002484	001876	001402	001037	000759	000549	000394	000279	000196
1.1	002346	001777	001332	000988	000725	000526	000378	000269	000189
1.2	002200	001672	001257	000935	000688	000501	000361	000257	000181
1.3	002049	001562	001178	000879	000649	000474	000343	000245	000173
1.4	001894	001449	001097	000821	000608	000446	000323	000232	000164
1.5	001736	001333	001013	000761	000566	000416	000303	000218	000155
1.6	001579	001217	000928	000700	000522	000385	000281	000203	000145
1.7	001424	001102	000843	000639	000478	000354	000260	000188	000135
1.8	001273	000989	000760	000578	000434	000323	000238	000173	000124
1.9	001127	000879	000678	000518	000391	000292	000216	000157	000114
2.0	000989	000775	000600	000460	000349	000262	000194	000142	000103
2.1	000859	000676	000526	000405	000308	000232	000173	000127	000093
2.2	000740	000584	000457	000353	000270	000204	000153	000113	000083
2.3	000630	000500	000393	000305	000234	000178	000134	000099	000073
2.4	000532	000424	000334	000261	000201	000154	000116	000086	000064
2.5	000444	000355	000282	000221	000171	000131	000099	000075	000055
2.6	000367	000295	000235	000185	000144	000111	000084	000064	000047
2.7	000300	000242	000194	000153	000120	000093	000071	000054	000040
2.8	000242	000197	000158	000126	000099	000077	000059	000045	000034
2.9	000194	000158	000128	000102	000081	000063	000049	000037	000028
3.0	000153	000126	000102	000082	000065	000051	000040	000030	000023
3.1	000120	000099	000081	000065	000052	000041	000032	000025	000019
3.2	000093	000077	000063	000051	000041	000032	000025	000020	000015
3.3	000071	000059	000049	000040	000032	000025	000020	000016	000012
3.4	000054	000045	000037	000030	000025	000020	000016	000012	000009
3.5	000040	000034	000028	000023	000019	000015	000012	000009	000007
3.6	000030	000025	000021	000017	000014	000011	000009	000007	000006
3.7	000022	000018	000015	000013	000011	000009	000007	000005	000004
3.8	000016	000013	000011	000009	000008	000006	000005	000004	000003
3.9	000011	000010	000008	000007	000006	000005	000004	000003	000002
4.0	000008	000007	000006	000005	000004	000003	000003	000002	000002

Computed

R = 0.50

\H K\	2.7	2.8	2.9	3.0	3.1	3.2	3.3	3.4	3.5
0.0	003308	002448	001793	001300	000935	000663	000466	000323	000221
0.1	003266	002419	001773	001286	000923	000656	000460	000319	000219
0.2	003215	002383	001748	001269	000911	000648	000453	000314	000213
0.3	003164	002350	001728	001257	000905	000645	000454	000316	000217
0.4	003095	002302	001695	001234	000889	000633	000447	000310	000213
0.5	003019	002249	001659	001210	000873	000623	000439	000306	000211
0.6	002923	002186	001614	001179	000851	000607	000428	000298	000204
0.7	002834	002121	001571	001151	000834	000597	000423	000296	000204
0.8	002722	002042	001515	001112	000807	000578	000409	000286	000198
0.9	002603	001958	001456	001071	000779	000560	000397	000278	000192
1.0	002477	001869	001395	001030	000752	000542	000387	000272	000189
1.1	002333	001763	001323	000979	000716	000518	000370	000260	000181
1.2	002193	001664	001249	000927	000680	000493	000353	000250	000174
1.3	002046	001559	001175	000876	000646	000471	000339	000242	000170
1.4	001891	001446	001094	000818	000605	000443	000320	000229	000161
1.5	001735	001332	001012	000760	000565	000415	000301	000217	000154
1.6	001578	001216	000927	000699	000521	000384	000280	000202	000144
1.7	001423	001101	000843	000638	000477	000353	000259	000187	000134
1.8	001272	000988	000759	000577	000434	000322	000237	000172	000124
1.9	001126	000879	000678	000517	000390	000291	000215	000157	000113
2.0	000986	000774	000600	000459	000348	000261	000193	000142	000102
2.1	000859	000675	000526	000404	000308	000232	000172	000127	000092
2.2	000739	000584	000456	000353	000269	000204	000152	000112	000082
2.3	000629	000500	000392	000304	000234	000177	000133	000099	000072
2.4	000531	000423	000334	000260	000201	000153	000115	000086	000063
2.5	000443	000355	000281	000220	000170	000131	000099	000074	000055
2.6	000366	000294	000234	000184	000143	000110	000084	000063	000047
2.7	000299	000242	000193	000153	000119	000092	000070	000053	000040
2.8	000242	000196	000158	000125	000098	000076	000059	000044	000033
2.9	000193	000158	000127	000101	000080	000062	000048	000037	000028
3.0	000153	000125	000101	000081	000064	000050	000039	000030	000023
3.1	000119	000098	000080	000064	000051	000040	000031	000024	000018
3.2	000092	000076	000062	000050	000040	000032	000025	000019	000015
3.3	000070	000059	000048	000039	000031	000025	000020	000015	000012
3.4	000053	000044	000037	000030	000024	000019	000015	000012	000009
3.5	000040	000033	000028	000023	000018	000015	000012	000009	000007
3.6	000029	000025	000020	000017	000014	000011	000009	000007	000005
3.7	000021	000018	000015	000012	000010	000008	000006	000005	000004
3.8	000015	000013	000011	000009	000007	000006	000005	000004	000003
3.9	000011	000009	000008	000006	000005	000004	000003	000003	000002
4.0	000007	000006	000005	000004	000004	000003	000002	000002	000001

Appendix 3. (Continued)

NBS Table

r = 0.50

k \ h	3.6	3.7	3.8	3.9	4.0
0.0	000157	000107	000072	000048	000031
0.1	000156	000106	000071	000048	000031
0.2	000155	000106	000071	000047	000031
0.3	000154	000105	000071	000047	000031
0.4	000153	000104	000070	000047	000031
0.5	000151	000103	000069	000046	000031
0.6	000149	000102	000069	000046	000030
0.7	000146	000100	000068	000045	000030
0.8	000143	000098	000066	000045	000030
0.9	000140	000096	000065	000044	000029
1.0	000136	000093	000064	000043	000028
1.1	000132	000091	000062	000042	000028
1.2	000127	000087	000060	000040	000027
1.3	000121	000084	000057	000039	000026
1.4	000115	000080	000055	000037	000025
1.5	000109	000076	000052	000036	000024
1.6	000102	000072	000049	000034	000023
1.7	000096	000067	000046	000032	000022
1.8	000088	000062	000043	000030	000020
1.9	000081	000057	000040	000028	000019
2.0	000074	000052	000037	000025	000017
2.1	000067	000048	000033	000023	000016
2.2	000060	000043	000030	000021	000015
2.3	000053	000038	000027	000019	000013
2.4	000047	000034	000024	000017	000012
2.5	000040	000029	000021	000015	000010
2.6	000035	000025	000018	000013	000009
2.7	000030	000022	000016	000011	000008
2.8	000025	000018	000013	000010	000007
2.9	000021	000015	000011	000008	000006
3.0	000017	000013	000009	000007	000005
3.1	000014	000011	000008	000006	000004
3.2	000011	000009	000006	000005	000003
3.3	000009	000007	000005	000004	000003
3.4	000007	000005	000004	000003	000002
3.5	000006	000004	000003	000002	000002
3.6	000004	000003	000003	000002	000001
3.7	000003	000003	000002	000001	000001
3.8	000003	000002	000002	000001	000001
3.9	000002	000001	000001	000001	000001
4.0	000001	000001	000001	000001	000000

Computed

R = 0.50

K \ H	3.6	3.7	3.8	3.9	4.0
0.0	000149	000096	000063	000039	000023
0.1	000146	000096	000061	000037	000021
0.2	000142	000092	000058	000034	000018
0.3	000147	000097	000063	000039	000023
0.4	000143	000094	000061	000037	000021
0.5	000142	000094	000060	000037	000021
0.6	000136	000090	000056	000033	000018
0.7	000139	000092	000060	000037	000022
0.8	000133	000088	000056	000034	000019
0.9	000130	000086	000055	000034	000019
1.0	000130	000087	000057	000036	000022
1.1	000123	000083	000054	000034	000020
1.2	000119	000080	000052	000033	000020
1.3	000118	000081	000054	000036	000023
1.4	000112	000077	000052	000035	000022
1.5	000108	000075	000051	000035	000023
1.6	000102	000071	000049	000033	000022
1.7	000095	000066	000046	000031	000021
1.8	000088	000062	000043	000029	000020
1.9	000081	000057	000039	000027	000018
2.0	000073	000052	000036	000025	000017
2.1	000066	000047	000033	000023	000015
2.2	000059	000042	000030	000020	000014
2.3	000052	000037	000026	000018	000013
2.4	000046	000033	000023	000016	000011
2.5	000040	000029	000020	000014	000010
2.6	000034	000025	000018	000012	000009
2.7	000029	000021	000015	000011	000007
2.8	000025	000018	000013	000009	000006
2.9	000020	000015	000011	000008	000005
3.0	000017	000012	000009	000006	000004
3.1	000014	000010	000007	000005	000004
3.2	000011	000008	000006	000004	000003
3.3	000009	000006	000005	000003	000002
3.4	000007	000005	000004	000003	000002
3.5	000005	000004	000003	000002	000001
3.6	000004	000003	000002	000001	000001
3.7	000003	000002	000002	000001	000001
3.8	000002	000002	000001	000001	000000
3.9	000001	000001	000001	000000	000000
4.0	000001	000001	000000	000000	000000

Appendix 4

Comparison of Bivariate $T^2_{\alpha;\ 2,N-2}$ Values

N	α	0.0010	0.0025	0.0050	0.0100	0.0250	0.0500
5	T^2	396.00	213.15	132.80	82.18	42.78	25.47
	T^{2*}	3685.17	595.25	237.66	113.13	48.67	26.90
6	T^2	153.11	95.00	65.71	45.00	26.62	17.36
	T^{2*}	276.72	131.47	80.11	50.43	27.91	17.69
7	T^2	89.09	59.91	43.95	31.86	20.24	13.89
	T^{2*}	114.71	69.26	48.10	33.56	20.66	13.99
8	T^2	63.00	44.58	33.94	25.49	16.94	12.00
	T^{2*}	71.90	48.15	35.61	26.20	17.10	12.02
9	T^2	49.57	36.31	28.35	21.82	14.95	10.83
	T^{2*}	53.62	38.03	29.17	22.16	15.01	10.82
10	T^2	41.61	31.25	24.85	19.46	13.63	10.03
	T^{2*}	43.80	32.21	25.30	19.64	13.65	10.00
11	T^2	36.42	27.86	22.46	17.83	12.70	9.46
	T^{2*}	37.76	28.46	22.74	17.93	12.69	9.42
12	T^2	32.79	25.46	20.74	16.63	12.00	9.03
	T^{2*}	33.70	25.87	20.93	16.69	11.98	8.98
13	T^2	30.13	23.67	19.44	15.72	11.47	8.69
	T^{2*}	30.80	23.97	19.58	15.75	11.43	8.64
14	T^2	28.11	22.29	18.44	15.01	11.04	8.42
	T^{2*}	28.64	22.52	18.54	15.02	11.00	8.37
15	T^2	26.52	21.19	17.63	14.43	10.69	8.20
	T^{2*}	26.96	21.39	17.71	14.44	10.65	8.15
16	T^2	25.24	20.30	16.97	13.96	10.41	8.01
	T^{2*}	25.62	20.48	17.04	13.96	10.36	7.96
17	T^2	24.19	19.57	16.43	13.57	10.17	7.86
	T^{2*}	24.53	19.72	16.49	13.56	10.12	7.80
18	T^2	23.31	18.95	15.97	13.23	9.96	7.72
	T^{2*}	23.63	19.10	16.02	13.23	9.91	7.67
19	T^2	22.57	18.42	15.57	12.94	9.78	7.61
	T^{2*}	22.87	18.56	15.62	12.94	9.73	7.55

Appendix 4 (Continued)

N	α	0.0010	0.0025	0.0050	0.0100	0.0250	0.0500
20	T^2	21.93	17.97	15.23	12.69	9.63	7.50
	T^{2*}	22.22	18.10	15.28	12.69	9.58	7.45

Note: T^2 = Exact Closed Form Value
T^{2*} = Approximate value

Appendix 5
Comparison of Derived Costs Using Problems From Table 1, Montgomery & Klatt (15)

Upper Figure is M&K, Lower Figure is Harris.

$A_4 = 1$				
A_2	A_3	A_1		
		.0001	.001	.01
.00001	.001	.0356 .0357	.0611 .0598	.1500 .138
	.010	.0459 .0458	.0709 .0691	.1600 .146
	.100	.1381 .136	.1637 .157	.2571 .231
.0001	.001	.0558 .0546	.0729 .0707	.1543 .142
	.010	.0715 .0689	.0871 .0834[1]	.1660 .152
	.100	.1691 .162	.1840 .178	.2655 .237
.001	.001	.1076 .111[2]	.1091 .119[3]	.1761 .165[4]
	.010	.1429 .131	.1498 .138	.2022 .208
	.100	.2609 .232	.2671 .238	.3173 .298

[1] Example Case at Operating Point 1

[2] Divergent, $N<p+3$, $\alpha = 44\%$

[3] Divergent, $N<p+3$, $\alpha = 69\%$

[4] Divergent, $N<p+3$, $\alpha = 50\%$

Appendix 6

Comparison of Required Program Input - Harris vs. Klatt

	Harris	Klatt	Description
1.	p	$(p\equiv 2)$	number of variables
2.	s	$(s\equiv 1)$	number of out-of-control states
3.	π	not applicable	binomial weighting factor
4.	N	N	sample size
5.	N_σ	$(N_\sigma\equiv N)$	sample size for $\underline{\underline{\Sigma}}$ estimate
6.	K		Sample spacing
7.	λ^{-1}		mean-time-to-shift
8.	R		Units/hour produced
9.		$\lambda K/R$	Function of 6-8 above
10.	α	$q_0=\alpha$	Probability of type I error
11.	A_1		Fixed cost/sample
12.	A_2		Cost/unit of sampling
13.	A_3		Cost of checking and correcting
14.	A_4		Cost of producing a defective
15.		$a_1=A_1/A_4$	Function of 11-14 above
16.		$a_2=A_2/A_4$	Function of 11-14 above
17.		$a_3=A_3/A_4$	Function of 11-14 above

Appendix 6 (Continued)

	Harris	Klatt	Description
18.	$\underline{\mu}_0$		In-control mean vector
19.	$\underline{\mu}_1,\ldots,\underline{\mu}_s$		Out-of-control mean vectors
20.	$\underline{\underline{S}}$		Variance-covariance matrix
21.	$\underline{u}$		Upper specification limit vector
22.	$\underline{l}$		Lower specification limit vector
23.		f_0	Probability of producing a defective when in state 0
24.		f_1	Probability of producing a defective when in state 1

where f_0, f_1, functions of 18-22 above, are defined as

$$f_0 = \int_{l_1}^{u_1}\int_{l_2}^{u_2} (2\pi)^{-1/4}\left|\underline{\underline{S}}^{-1}\right|^{1/2} e^{-1/2(\underline{x}-\underline{\mu}_0)\underline{\underline{S}}^{-1}(\underline{x}-\underline{\mu}_0)} dx_1 dx_2$$

$$f_1 = \int_{l_1}^{u_1}\int_{l_2}^{u_2} (2\pi)^{-1/4}\left|\underline{\underline{S}}^{-1}\right|^{1/2} e^{-1/2(\underline{x}-\underline{\mu}_1)\underline{\underline{S}}^{-1}(\underline{x}-\underline{\mu}_1)} dx_1 dx_2$$

and are derived using bivariate normal tables.

Appendix 7. Program Listing

```
*#RUN *=(NWARN)
C     THE MAIN PROGRAM DIMENSIONS THE MATRICES FOR
C       THE PROBLEM.
C
      DIMENSION SS(12,12), SSINV(12,12), BB(13,13),
     &          PP(13,13), DELTA(12,13), MU(12,13)
C
      INTEGER S,P,SP1
      DATA MAXP/10/, MAXS/12/, LDI/5/, LDO/6/
C
      WRITE(LDO,1)
    1 FORMAT (/," ENTER NUMBER OF VARIABLES AND NUMBER",
     &" OF OUT-OF-CONTROL STATES")
      READ(LDI,2)P,S
    2 FORMAT (V)
      IF (P.LE.MAXP.AND.S.LE.MAXS) GO TO 3
      WRITE(LDO,4)
    4 FORMAT (/," VALUE TOO LARGE")
      STOP 180
C
C     NOTE --
C       BECAUSE FORTRAN SUBSCRIPTS START AT 1, ALL STATE
C       NUMBERS USED AS SUBSCRIPTS MUST BE PROMOTED BY 1.
C
    3 SP1=S+1
      CALL MAIN(LDI,LDO,P,SP1,SS,SSINV,BB,PP,DELTA,MU)
      STOP 250
      END
CMAIN   MOST OF THE MAIN PROGRAM RUNS AS A SUBROUTINE
C
      SUBROUTINE MAIN(LDI,LDO,IP,ISP1,SS,SSINV,BB,PP,
     &                DELTA,MU)
C
C     VARIABLES INDEXED BY STATE
      DIMENSION BB(ISP1,ISP1), PP(ISP1,ISP1)
      DIMENSION PHI(13), GAMMA(13), RHO(13), BETA(13),
     & FACTI(13), SUMP(13), PR(13), TAU(13), SUMB(13),
     & WEGHT(13)
C
C     VARIABLES INDEXED BY QUALITY CHARACTERISTIC
      DIMENSION SS(IP,IP), SSINV(IP,IP)
      DIMENSION LSL(12), USL(12), WORK(12), WMU(12)
C
C
C      VARIABLES INDEXED BOTH WAYS
      DIMENSION DELTA(IP,ISP1), MU(IP,ISP1)
C
C     VARIABLES DIMENSIONED OTHERWISE
      DIMENSION NVEC(3), KVEC(3), ALPHA(3)
      DIMENSION CSAV(3,3,3)
C
      REAL MU, LAMDA, LKR, LSL
      INTEGER S,ANS,ESS,SP1, P
      DATA EPSS/5.E-4/, NDIV/6/
      DATA ESS/"S "/
C     FACTORIALS OF 0 THRU 12
      DATA FACTI/1.,1.,2.,6.,24.,120.,720.,5040.,40320.,
     &           362880.,3628800.,39916800.,4790016.E02/
C
      SP1=ISP1
      S=SP1-1
      P=IP
      V1=IP
      P=V1
      SFACT=FACTI(SP1)
      TWO9=2./9.
```

Appendix 7. (Continued)

```
C         ITERATION STEP REDUCTION FACTOR
          STEP=8.
C
        2 FORMAT (V)
C
C         TO ENTER TSQ INSTEAD OF ALPHA AS A DEBUG TOOL,
C              ENTER A NEGATIVE ALPHA.
C
C         TO FORCE SOLUTION FOR ONE POINT, SET NSOLV = 1.
          NSOLV=2
C
      124 WRITE(LDO,3)
        3 FORMAT (/," ENTER THE LIMITING VALUES OF",/,
         &  "       SAMPLE SIZE 'N'")
          READ(LDI,2)(NVEC(I),I=1,NSOLV)
          DO 120 I=1,NSOLV
          IF (NVEC(I)-(P+3)) 121,120,120
      120 CONTINUE
          GOTO 123
      121 WRITE(6,122)
      122 FORMAT (/,"      SAMPLE SIZE 'N' MUST BE AT",
         & " LEAST THREE",/,"  GREATER THAN THE NUMBER",
         & " OF VARIABLES.")
          GOTO 124
      123 WRITE(LDO,31)
       31 FORMAT ("      NUMBER BETWEEN SAMPLES 'K'")
          READ(LDI,2)(KVEC(I),I=1,NSOLV)
          DO 125 I=1,NSOLV
          IF (KVEC(I)-(P+3)) 126,125,125
      125 CONTINUE
          GOTO 128
      126 WRITE(6,127)
      127 FORMAT (/," SAMPLE INTERVAL 'K' MUST BE AT LEAST",
          & " THREE",/,"  GREATER THAN THE NUMBER OF",
          & " VARIABLES")
           GOTO 123
       128 WRITE(LDO,4)
         4 FORMAT ("      TYPE 1 ERROR 'ALPHA'")
           READ(LDI,2)(ALPHA(I),I=1,NSOLV)
           IF (NSOLV.EQ.1) GOTO 5000
C
C          SORT VARIABLES IN ASCENDING ORDER
           IF (NVEC(1).LT.NVEC(2)) GOTO 130
           I=NVEC(1)
           NVEC(1)=NVEC(2)
           NVEC(2)=I
       130 IF (KVEC(1).LT.KVEC(2)) GOTO 131
           I=KVEC(1)
           KVEC(1)=KVEC(2)
           KVEC(2)=I
       131 IF (ALPHA(1).LT.ALPHA(2)) GOTO 132
           A=ALPHA(1)
           ALPHA(1)=ALPHA(2)
           ALPHA(2)=A
       132 CONTINUE
           NVEC(3)=NVEC(2)
           KVEC(3)=KVEC(2)
           ALPHA(3)=ALPHA(2)
C
C          SET INITIAL STEP SIZE AND CENTER POINT
           KSTEP=(KVEC(3)-KVEC(1))/2
           KSTEP=MAX0(KSTEP,1)
           NSTEP=(NVEC(3)-NVEC(1))/2
           NSTEP=MAX0(NSTEP,1)
           ASTEP=(ALPHA(3)-ALPHA(1))/2.
           ASTEP=AMAX1(ASTEP,.005)
```

Appendix 7. (Continued)

```
      NVEC(2)=NVEC(1)+NSTEP
      KVEC(2)=KVEC(1)+KSTEP
      ALPHA(2)=ALPHA(1)+ASTEP
C
 5000 CONTINUE
      WRITE(LDO,5)P
    5 FORMAT (/," ENTER ",I2," DIMENSIONAL L AND U",
     & " SPECIFICATION VECTORS")
      READ(LDI,2)(LSL(I),I=1,P)
      READ(LDI,2)(USL(I),I=1,P)
C
      WRITE(LDO,6)P
    6 FORMAT (/," ENTER ",I2," DIMENSIONAL IN-CONTROL",
     & " MEAN VECTOR")
      READ(LDI,2)(MU(J,1),J=1,P)
C
      WRITE(LDO,19)S,P
   19 FORMAT (/," FOR THE ",I2," STATES, ENTER THE ",I2,
     & " DIMENSIONAL MEAN VECTORS")
      DO 20 I=2,SP1
      WRITE (LDO,29)I-1
   29 FORMAT (" STATE ",I2)
      READ(LDI,2)(MU(J,I),J=1,P)
   20 CONTINUE
C
   40 WRITE(LDO,7)
    7 FORMAT (/," ARE YOU ENTERING THE S (S) OR S",
     & " INVERSE (SI) MATRIX??")
      READ(LDI,8)ANS
    8 FORMAT (A2)
C
      WRITE(LDO,32)
   32 FORMAT (/," ENTER THE SAMPLE SIZE USED IN",/,
     & "           ESTIMATING THE COVARIANCE MATRIX")
      READ(LDI,2)SGMAN
C
      IF (ANS-ESS) 9,11,9
C
    9 WRITE(LDO,23)
   23 FORMAT (/," ENTER THE S INVERSE MATRIX")
      GO TO 24
C
   11 WRITE(LDO,13)
   13 FORMAT (/," ENTER THE S MATRIX")
   24 DO 14 I=1,P
      READ(LDI,2)(SS(J,I),J=1,P)
      DO 14 J=1,P
   14 SSINV(J,I) = SS(J,I)
C
C     CHECK FOR SYMMETRY
      DO 35 I=1,P-1
      DO 35 J=I+1,P
      IF (SS(I,J).NE.SS(J,I)) GOTO 33
   35 CONTINUE
      GOTO 15
C
   33 WRITE(6,34)
   34 FORMAT (/," MATRIX NOT SYMMETRIC")
      STOP 1171
C
   15 WRITE (LDO,21)
   21 FORMAT (/," ENTER THE TRIAL COST VALUES")
      READ(LDI,2)A1,A2,A3,A4
C
      WRITE(LDO,28)
   28 FORMAT (/," ENTER THE MEAN TIME TO SHIFT (IN"
```

Appendix 7. (Continued)

```
     & ,/,"      HOURS), AND UNITS PER HOUR PRODUCED")
      READ(LDI,2)LAMDA,R
      LAMDA=1./LAMDA
C
      PI=1.
      PI1=0.
      IF (S.EQ.1) GO TO 26
   39 WRITE(LDO,25)
   25 FORMAT (/," ENTER PI, THE BINOMIAL WEIGHTING",
     & " FACTOR")
      READ(LDI,2)PI
      IF (PI)  41,41,36
   36 IF (PI-1) 37,41,41
   41 WRITE(6,38)
   38 FORMAT (/," ERROR - YOU MUST USE 0<PI<1")
      GOTO 39
   37 PI1=1.-PI
   26 CONTINUE
C
C     END OF INPUT PHASE
C     -------------------------------------
      LDO=6
C
C     INVERT THE INPUT MATRIX
      IF (ANS-ESS) 110,112,110
  110 CALL INVERT (SS,P,1,P,NSING,DET)
      IF (NSING.EQ.0) GO TO 114
  115 WRITE(6,113)
  113 FORMAT (/," S OR S-INVERSE INVERSION DIFFICULTY")
      STOP 1530
C
  112 CALL INVERT (SSINV,P,1,P,NSING,DET)
      IF (NSING.NE.0) GO TO 115
C     THE DETERMINANT DESIRED IS THAT OF THE S-INVERSE
C       MATRIX
      DET=1./DET
  114 CONTINUE
C
C     COMPUTE THE MEAN DIFFERENCE VECTORS 'DELTA'
      DO 101 I=1,SP1
      DO 101 J=1,P
  101 DELTA(J,I)=MU(J,I)-MU(J,1)
C
C     COMPUTE TAU, THE NON-CENTRALITY VECTOR, EACH
C       ELEMENT A QUADRATIC FORM
C         SGRAN*(DELTA*SSINV*DELTA)
      TAU(1)=0.
      DO 102 I=2,SP1
      TAU(I)=0.
      DO 103 J=1,P
      WORK(J)=0.
      DO 104 K=1,P
  104 WORK(J)=WORK(J)+SSINV(J,K)*DELTA(K,I)
  103 TAU(I)=TAU(I)+DELTA(J,I)*WORK(J)
  102 CONTINUE
C
C     COMPUTE THE PHI VECTOR OF PROBABILITIES OF
C       PRODUCING A DEFECTIVE
C
      DO 109 I=1,SP1
  109 PHI(I)=1.-
     & CDMVN(P,LSL,USL,MU(1,I),SSINV,DET,NDIV,NSEC,EPSS)
C
C     EXECUTE THE SEARCH
C     -------------------------------------
C
```

Appendix 7. (Continued)

```
C       CMIN=1.E35
        LIM=NSOLV*3/2
        KONT2=0
C
C       WITHIN THE K,N,ALPHA LOOPS, COMPUTE  PR,RHO,
C         BETA,GAMMA
C
 3000   DO 1012 I4=1,LIM
        K=KVEC(I4)
        IF (K.LT.P+3) K=P+3
        AK=K
C
C       GENERATE THE BINOMIALLY WEIGHTED P VECTOR (PR)
C         AND ITS PARTIAL SUMS.
        LKR=LAMDA*AK/R
        ELKR=EXP(-LKR)
        F=(1.-(1.+LKR)*ELKR)/(LKR*(1.-ELKR))
        PR(1)=ELKR
        DENOM=1.-PR(1)
        PR(2)=DENOM
        SUMP(1)=0.
        SUMP(2)=DENOM
        IF (S-1) 1044,1044,1014
 1014   CONST=DENOM/(1.-PI1**S)*SFACT
        SUMP(1)=0.
        DO 1015 I=1,S
        II=I+1
        PR(II) = CONST*PI**I*
     &  PI1**(S-I)/(FACTI(II)*FACTI(S-I+1))
        IF (IFRST.GT.0) GOTO 1015
        WEGHT(II)=PR(II)/DENOM
 1015   SUMP(II)=SUMP(I)+PR(II)
C
C       NOW COMPUTE PP(I,J), THE PROBABILITIES OF
C         TRANSITIONS AMONG THE HIGHER STATES.
        DO 1040 I=1,SP1
        DO 1040 J=1,SP1
        IF (I-J) 1041,1042,1043
 1041   PP(J,I)=PR(J)/DENOM
        GO TO 1040
 1042   PP(I,I) = SUMP(I)/DENOM
        GO TO 1040
 1043   PP(J,I) = 0.
 1040   CONTINUE
        IF (IFRST.GT.0) GOTO 1044
        DO 1045 I=2,SP1
        DO 1045 J=1,P
        WMU(J)=WMU(J)+MU(J,I)*WEGHT(I)
 1045   CONTINUE
 1044   CONTINUE
C
        DO 1011 I5=1,LIM
        N=NVEC(I5)
        IF (N.LT.P+3) N=P+3
        IF (N.GT.K) N=K
        AN=N
        V2=AN-V1
C
        DO 1010 I6=1,LIM
        ALPH1=ALPHA(I6)
C
C       SEARCH FOR T-SQUARED VALUE CORRESPONDING TO ALPHA
        IF (ALPH1) 1031,1032,1032
 1031   T=-ALPH1
        ALPH1=CDFT2(T,P,INT(SGMAN),0.)
        ALPH2=ALPH1
```

Appendix 7. (Continued)

```
        GOTO 1024
 1032   T1=0.
        IF (ALPH1.LT..001) ALPH1=.001
        TMAX=100000
        T2=TMAX
        T=T2/3.
 1020   ALPH2=CDFT2(T,P,INT(SGMAN),0.)
        IF (ABS(ALPH2/ALPH1-1.)-.1*ALPH1) 1024,1024,1021
 1021   IF (ALPH1-ALPH2) 1022,1022,1023
 1022   T1=T
        GO TO 1025
 1023   T2=T
 1025   T=T1+(T2-T1)/3.
        IF (TMAX-T-1.) 1026,1026,1020
 1026   PRINT,"NO CONVERGENCE IN TSQ FOR ",T,P,SGMAN
        STOP 2550
 1024   TSQ=T
C
C       NOW APPROXIMATE RHO (RHO(1) IS ALPHA)
        RHO(1)=ALPH1
        DO 1030 I=2,SP1
 1030   RHO(I)=CDFT2(TSQ,P,INT(SGMAN),AV*TAU(I))
C
C       NOW CREATE THE B MATRIX
        DO 1050 I=1,SP1
        DO 1050 J=1,S
        IF (I-1) 1053,1053,1054
 1053   BB(J,I) = PR(J)
        GOTO 1050
 1054    IF (J-I) 1055,1056,1057
 1055   BB(J,I) = RHO(I)*PR(J)
        GOTO 1050
 1056   BB(J,I) = RHO(I)*PR(J)+(1.-RHO(I))*PP(I,I)-1.
        GOTO 1050
 1057   BB(J,I) = RHO(I)*PR(J)+(1.-RHO(I))*PR(J)/DENOM
 1050   CONTINUE
        BB(1,1) = BB(1,1) - 1.
C
C       REPLACE COLUMN SP1 OF BB WITH A COLUMN OF 1'S
        DO 1058 I=1,SP1
 1058   BB(SP1,I) = 1.
C
C       INVERT THE ASYMMETRIC MATRIX 'BB'
C
        CALL MTINV (BB,SP1)
C
C       BETA IS THE LAST ROW OF THE B INVERSE MATRIX
        DO 1060 I=1,SP1
 1060   BETA(I)=BB(I,SP1)
C
C       ACCUMULATE PARTIAL SUMS OF BETA
        SUMB(1)=0.
        DO 1065 I=2,S
 1065   SUMB(I)=SUMB(I-1)+BETA(I-1)
        SUMB(SP1)=0.
C
C       NOW THE GAMMA VECTOR
        GAMMA(1)=BETA(1)*PR(1)+F*BETA(1)*DENOM
        DO 1070 I=2,SP1
        GAMMA(I)=BETA(I)*PP(I,I)+BETA(1)*(1.-F)*PR(I)
       & +SUMB(I-1)*PR(I)/DENOM*(1.-F)
        IF (I-SP1) 1071,1070,1070
 1071   GAMMA(I)=
       & GAMMA(I)+BETA(I)*F/DENOM*(1.-SUMB(I)-PR(1))
 1070   CONTINUE
C
```

Appendix 7. (Continued)

```
C      NOW GENERATE THIS COST
       C1=(A1+A2*AN)/AK
       SUM=0.
       IFRST=1
       DO 1080 J=1,SP1
  1080 SUM=SUM+RHO(J)*BETA(J)
       C2=A3*SUM/AK
       SUM=0.
       DO 1090 J=1,SP1
  1090 SUM=SUM+PHI(J)*GAMMA(J)
       C3=SUM*A4
       COST=C1+C2+C3
       CSAV(I4,I5,I6)=COST
       KONT2=KONT2+1
       IF (COST .GE. CMIN) GO TO 1010
       CMIN1=C1
       CMIN2=C2
       CMIN3=C3
       CMIN=COST
       NMIN=AN
       KMIN=AK
       TSUMM=TSUM
       ALPHM=ALPH1
  1010 CONTINUE
  1011 CONTINUE
  1012 CONTINUE
C
C      MOVE OR CHANGE STEP SIZE
C      -----------------------------------
C
  2002 FORMAT (1X,F9.3,1X,I4,1X,I4,1X,F9.3)
C
C      IF THE MIN COST IS AT THE CENTER OF UN-CONSTRAINED
C           VARIABLES, REDUCE THE STEP.
       IF (NSOLV.EQ.1) GOTO 1000
       KONT1=KONT1+1
       IF (KONT1.GT.50) GOTO 4000
       IF (CSAV(2,2,2).NE.CMIN) GOTO 2000
       IF (FLOAT(KSTEP)*FLOAT(NSTEP)*ASTEP.LT..0051)
      & GOTO 1000
C
C      REDUCE STEP
       KSTEP=(KVEC(3)-KVEC(1))/INT(STEP)
       KSTEP=MAX0(KSTEP,1)
       NSTEP=(NVEC(3)-NVEC(1))/INT(STEP)
       NSTEP=MAX0(NSTEP,1)
       ASTEP=(ALPHA(3)-ALPHA(1))/STEP
       ASTEP=AMAX1(ASTEP,.005)
       KONT1=0
C
C      DEFINE NEW REGION
  2000 DO 2001 I=1,3
       ISIGN=I-2
       KVEC(I)=KMIN+KSTEP*ISIGN
       NVEC(I)=NMIN+NSTEP*ISIGN
  2001 ALPHA(I)=ALPHM+ASTEP*ISIGN
       GOTO 3000
C
C      NO CONVERGENCE
  4000 WRITE(LDO,4001)KONT1
  4001 FORMAT (/," AFTER ",I3," STEPS, NO CONVERGENCE.")
C
C      PRINT THE RESULT
C      -----------------------------------
  1000 CONTINUE
       WRITE(LDO,4013)KONT2/27,A1,A2,A3,A4,
```

Appendix 7. (Continued)

```
     &      CMIN1,CMIN2,CMIN3,CMIN,NMIN,KMIN,ALPHM,TSQMN
C
        IF (S.EQ.1) GOTO 4014
        WRITE(LOO,4015)(WMU(I),I=1,P)
   4015 FORMAT(/," WEIGHTED MEAN WOULD BE:",/,6(1X,G10.5))
   4014 CONTINUE
   4013 FORMAT (1H1,/,
     & " ESTIMATED COSTS AFTER ",I4," ITERATIONS:",/,
     & "   FIXED COST PER SAMPLE",7X,"$ ",F9.3,/,
     & "   INSPECTION COST PER UNIT",4X,"$ ",F9.3,/,
     & "   AVERAGE COST OF REPAIR",6X,"$ ",F9.3,/,
     & "   COST PER UNIT OF DEFECTIVES $ ",F9.3,//,
     & " MINIMA:",/,
     & "      FOR TESTING",15X,"$ ",G9.3,/,
     & "      FOR CORRECTING",12X,"$ ",G9.3,/,
     & "      FOR BAD PRODUCT",11X,"$ ",G9.3,/,
     & "   COST PER UNIT",15X,"$ ",G9.3,/,
     & "   SAMPLE SIZE",19X,I5,/,
     & "   SAMPLE INTERVAL",15X,I5,/,
     & "   TYPE I ERROR",13X,2PF9.3," %",/,
     & "   T-SQUARED VALUE",14X,0PF9.3)
C
        STOP
        END
CMTINV    INVERT ANY NON-SINGULAR MATRIX
C
        SUBROUTINE MTINV (A,IDIM)
        DIMENSION A(IDIM,IDIM), L(13)
      1 NR=IDIM
        DO 21 J1=1,NR
     21 L(J1)=J1
        DO 291 J1=1,NR
C
        FIND REMAINING ROW CONTAINING LARGEST ABSOLUTE
C          VALUE IN PIVOTAL COLUMN
    101 TEMP=0.
        DO 121 J2=J1,NR
        IF(ABS(A(J2,J1)).LT.TEMP) GO TO 121
        TEMP=ABS(A(J2,J1))
        IBIG=J2
    121 CONTINUE
        IF(IBIG.EQ.J1) GO TO 201
C
C   REARRANGE ROWS TO PLACE LARGEST ABSOLUTE VALUE
C     IN PIVOTAL POSITION
        DO 141 J2=1,NR
        TEMP=A(J1,J2)
        A(J1,J2)=A(IBIG,J2)
    141 A(IBIG,J2)=TEMP
        I=L(J1)
        L(J1)=L(IBIG)
        L(IBIG)=I
C
C       COMPUTE COEFFICIENTS IN PIVOTAL ROW
    201 TEMP=A(J1,J1)
        A(J1,J1)=1.
        DO 221 J2=1,NR
    221 A(J1,J2)=A(J1,J2)/TEMP
C
C       COMPUTE COEFFICIENTS IN OTHER ROWS
        DO 251 J2=1,NR
        IF(J2.EQ.J1) GO TO 251
        TEMP=A(J2,J1)
        A(J2,J1)=0.
        DO 241 J3=1,NR
    241 A(J2,J3)=A(J2,J3)-TEMP*A(J1,J3)
```

Appendix 7. (Continued)

```
  281 CONTINUE
  291 CONTINUE
C
C     INTERCHANGE COLUMNS ACCORDING TO INTERCHANGES OF
C       ROWS OF ORIGINAL MATRIX
  301 N1=NR-1
      DO 391 J1=1,N1
      DO 321 J2=J1,NR
      IF(L(J2).NE.J1) GO TO 321
      IF(J2.EQ.J1) GO TO 391
      GO TO 341
  321 CONTINUE
  341 DO 361 J3=1,NR
      TEMP=A(J3,J1)
      A(J3,J1)=A(J3,J2)
  361 A(J3,J2)=TEMP
      L(J2)=L(J1)
  391 CONTINUE
 5001 RETURN
      END
CFNU      UN-NORMALIZED UNIVARIATE INTEGRAL, FROM LSL(N)
C         TO USL(N), TIMES THE LUMPED MULTIPLIER
C             DET(SSINV)/(2*PI)**(N/2)
C
      FUNCTION FNU(Z,F,G,N,D)
      DIMENSION VECMN(12)
      DIMENSION Z(10), D(N,N), X(10)
      COMMON/BLK2/CONST,RTANN
      COMMON/BLK3/KNT1
      COMMON /BLK5/ VECMN
      EXTERNAL F,G
      COMMON/BLK7/XSAVE,KFIRST
      KNT1 = KNT1 + 1
      N1 = N-1
      N2 = N-2
      IF(KFIRST) 4,20,4
    4   SUMB = 0.
      SUMC1 = 0.
      SUMC2 = 0.
      DO  5 I = 1,N1
      X(I) = Z(I) - VECMN(I)
      SUMB = SUMB + X(I)*D(I,N)
    5   SUMC1 = SUMC1 + X(I)*X(I)*D(I,I)
      IF(N2) 6,10,6
    6   DO  8 I = 1,N2
      IP1 = I + 1
      DO  8 J = IP1,N1
    8   SUMC2 = SUMC2 + X(I)*X(J)*D(I,J)
      GO TO 12
   10   SUMC2 = 0.
   12   B = SUMB
      C = SUMC1 + 2.*SUMC2
      E = B/RTANN
      XP = EXP(-.5*(C-B*B/D(N,N)))/RTANN
   13   YU = G(N1,Z) - VECMN(N)
      YL = F(N1,Z) - VECMN(N)
      FNU = XP*(CDFN(YU*RTANN+E)-CDFN(YL*RTANN+E))*CONST
      RETURN
   20   X(N1) = Z(N1) - VECMN(N1)
      DELTA1 = Z(N1) - XSAVE
      DELTA2 = Z(N1) + XSAVE - 2.*VECMN(N1)
      SUMB = SUMB + D(N1,N)*DELTA1
      SUMC1 = SUMC1 + D(N1,N1)*(DELTA1*DELTA2)
      IF(N2) 26,10,26
   26   SUMC3 = 0.
      DO  27 I = 1,N2
```

Appendix 7. (Continued)

```
      SUMC3 = SUMC3 + D(N1,I)*X(I)
   27   CONTINUE
      SUMC2 = SUMC2 + SUMC3*DELTA1
      GO TO 12
      END
CMDQUAD   MULTI-DIMENSIONAL QUADRATURE
C
      SUBROUTINE MDQUAD(N,D,A,B,F,G,U,EPS,NSEC,S,NS,V,
     & IFLAG)
C ***********************************************************
C *                                                         *
C *                                                         *
C *                                                         *
C *        UNIVERSITY OF WISCONSIN COMPUTING CENTER         *
C *                                                         *
C *                                                         *
C *           MULTIDIMENSIONAL QUADRATURE PROGRAM           *
C *                                                         *
C *                                                         *
C ***********************************************************
      DIMENSION S(NS)
C *********** ASSIGN WORKSPACE
      NT = NS/5
      NC=NT/N
C *********** TEST PARAMETERS
      IFLAG=0
      IF(N.LE.10 .AND. N.GE.1)GO TO 10
      IFLAG=4
      WRITE(6,101)
   10   IF(B.GT.A)GO TO 20
      IFLAG=4
      WRITE(6,102)
   20   IF(EPS.GT.0)GO TO 30
      IFLAG=4
      WRITE(6,103)
   30   IF(NSEC.GE.0)GO TO 40
      IFLAG=4
      WRITE(6,104)
   40   IF(NC.GE.1)GO TO 50
      IFLAG=4
      WRITE(6,105)
   50   IF(IFLAG.GT.0)RETURN
      CALL WORK(U,F,G,N,D,A,B,EPS,V,NSEC,IFLAG,NC,S(1),
     &     S(NT+1),S(2*NT+1),S(3*NT+1),S(4*NT+1))
  101  FORMAT(6H0*****' N .GT. 10 OR N .LT. 1'    )
  102  FORMAT(6H0*****' B .LE. A')
  103  FORMAT(6H0*****' EPS .LE. 0')
  104  FORMAT(6H0*****' NSEC .LT. 0')
  105  FORMAT(6H0*****' INSUFFICIENT WORKSPACE')
      RETURN
      END
CWORK     MAIN PROCESSING SUBROUTINE FOR COMVN
      SUBROUTINE WORK(U,F,G,N,D,AEP,BEP,EPSMAX,VALUE,
     &         NSEC,IFLAG,NC,A,I,F1,F2,F3)
C ********* VARIABLE WORKSPACE
      DIMENSION A(N,NC),I(N,NC),F1(N,NC),F2(N,NC),
     &          F3(N,NC)
C *********** FIXED WORKSPACE
      DIMENSION V(13),H(13),X(13),KK(13),LENGTH(13),
     &          SUM1(13),SUM2(13),SUM3(13),JTAB(13),
     &          EPS(13),J(13),M(13),MSTART(13),JEND(13),
     &                    K(13),P(13),LL(13),FA(13),FB(13)
      EQUIVALENCE (FA(1),SUM2(1)),(FB(1),SUM3(1))
      REAL I,LENGTH,I1,I2
      COMMON/BLK7/XSAVE,KFIRST
      EXTERNAL F,G,U
```

Appendix 7. (Continued)

```
      KFIRST = -1
C ********** INITIALIZE
*       TIMLST=URTIMG(0)
*         TIMEND=TIMLST+NSEC
      MAXCNT=10
*         NCOUNT=10
      EDVN15=EPSMAX/N*15.
      A(1,1)=AEP
      LENGTH(1)=BEP - AEP
      L=1
      GO TO 60
C ********** GO DOWN A LEVEL
   50   L=L+1
      KFIRST = -1
C C********  KFIRST = -1 MEANS AT LEAST 2 ELEMENTS IN
C C********      ARRAY X HAVE BEEN CHANGED SINCE LAST
C C********      CALL TO FNU(X,F,G)
      A(L,1)=F(L-1,X)
      LENGTH(L)=G(L-1,X) - A(L,1)
   60   V(L)=0
      IF(LENGTH(L))62,2000,65
   62   IF(IFLAG.EQ.0 .OR. IFLAG.EQ.1)IFLAG=IFLAG+2
      GO TO 2000
   65     JTAB(L)=1
      H(L)=.5*LENGTH(L)
      EPS(L)=EDVN15
      LM1 = L - 1
      IF(LM1 .EQ. 0) GO TO 75
      DO 70 INDEX = 1,LM1
   70   EPS(L)=EPS(L)/LENGTH(INDEX)
   75   X(L)=A(L,1)
      K(L)=-1
      J(L)=1
      M(L)=NC
      MSTART(L)=NC
      JEND(L)=0
  100  IF(L.NE.N)GO TO 50
C
C ********** BEGIN -- TIME CHECK SECTION
*       IF(NSEC.EQ.0)GO TO 200
*        NCOUNT = NCOUNT-1
*        IF(NCOUNT .NE. 0) GO TO 200
*        TIME=TIMLST
*           TIMLST=URTIMG(0)
*        TIME=TIMLST-TIME
*        IF(TIME)140,200,140
* 140    IF(TIMLST.GT.TIMEND) GO TO 5000
*       NCOUNT=MAXO(INT(MAXCNT/TIME+.5),1)
C ********** END -- TIME CHECK SECTION
C
C 200  V(L+1)=FNU(X)
  200  XSAVE = XSAVET
      XSAVET = X(L)
      V(L+1) = FNU(X,F,G,N+1,0)
      KFIRST = 0
      NGO=JTAB(L)
      GO TO (500,600,700,1100,1200,3100),NGO
  500  F1(L,1)=V(L+1)
      X(L)=A(L,1) + H(L)
      JTAB(L)=2
      GO TO 100
  600  F2(L,1) = V(L+1)
      X(L)=A(L,1) + LENGTH(L)
      JTAB(L)=3
      GO TO 100
  700  F3(L,1)=V(L+1)
```

Appendix 7. (Continued)

```
            I(L,1)=LENGTH(L)/6.*(F1(L,1)+4.*F2(L,1)+F3(L,1))
            GO TO 1050
C ********** VARIABLE STEP
      1000  EPS(L) = .5*EPS(L)
            H(L) = .5*H(L)
            IF(H(L) .LE. 0.0)GO TO 5010
      1050  IF(J(L).EQ.JEND(L))GO TO 1800
            NN=J(L)
            X(L)=A(L,NN) + .5*H(L)
            JTAB(L)=4
            GO TO 100
      1100  FA(L)=V(L+1)
            NN=J(L)
            X(L)=A(L,NN) + 1.5*H(L)
            JTAB(L)=5
            GO TO 100
      1200  FB(L)=V(L+1)
            J1=J(L)
            H06=H(L)/6.
            I1=H06*(F1(L,J1) + 4.*FA(L) + F2(L,J1))
            I2=H06*(F2(L,J1) + 4.*FB(L) + F3(L,J1))
            ERR = I1+I2-I(L,J1)
            IF(ABS (ERR).LT.EPS(L))GO TO 1500
            IF(M(L).EQ.J1)GO TO 3000
            M1=M(L)
            M2=M(L) + K(L)
            A(L,M1)=A(L,J1)
            A(L,M2) =A(L,J1) + H(L)
            I(L,M1)=I1
            I(L,M2)=I2
            F1(L,M1)=F1(L,J1)
            F2(L,M1)=FA(L)
            F3(L,M1)=F2(L,J1)
            F1(L,M2)=F2(L,J1)
            F2(L,M2)=FB(L)
            F3(L,M2)=F3(L,J1)
            M(L) = M2 + K(L)
            J(L)=J(L) + K(L)
            GO TO 1050
      1500  V(L) = ERR/15. + I1 + I2 + V(L)
            J(L)=J(L) + K(L)
            GO TO 1050
      1800  IF(M(L).EQ.MSTART(L))GO TO 2000
            IF(K(L))1900,1900,1950
      1900  K(L)=1
            J(L)=M(L) + 1
            JEND(L)=NC + 1
            M(L)=1
            MSTART(L)=1
            GO TO 1000
      1950  K(L)=-1
            J(L)=M(L) - 1
            JEND(L)=0
            MSTART(L)=NC
            M(L)=NC
            GO TO 1000
C ********** GO UP A LEVEL
      2000  L=L - 1
            IF(L.EQ.0)GO TO 2100
            NGO=JTAB(L)
            GO TO (500,600,700,1100,1200,3100),NGO
      2100  VALUE=V(1)
            RETURN
C ********** UNIFORM STEP
      3000  IF(IFLAG.EQ.0 .OR. IFLAG.EQ.2)IFLAG=IFLAG+1
      3025  SUM1(L) = F1(L,J1) + F3(L,J1)
```

Appendix 7. (Continued)

```
             SUM2(L)=FA(L) + F2(L,J1)+ F5(L)
C     ***    SAVE H(L)
             F1(L,J1) = H(L)
             H(L)=.5*H(L)
             IF(H(L).LE.0)GO TO 5010
             KK(L)=4
             P(L)=I1 + I2
      3030   SUM3(L)=0
             X(L)=A(L,J1) + .5*H(L)
             LL(L)=1
      3050   IF(LL(L).GT.KK(L))GO TO 3500
             JTAB(L)=6
             GO TO 100
      3100   SUM3(L)=SUM3(L) + V(L+1)
             LL(L)=LL(L) + 1
             X(L)=X(L) + H(L)
             GO TO 3050
      3500   I2=H(L)/6.*(SUM1(L) + 2.*SUM2(L) + 4.*SUM3(L))
             ERR = I2 - P(L)
             IF(ABS (ERR).LT.EPS(L))GO TO 3600
             SUM2(L)=SUM2(L) + SUM3(L)
             KK(L)=2*KK(L)
             H(L)=.5*H(L)
             P(L)=I2
             GO TO 3030
      3600   V(L) = ERR/15. + I2 + V(L)
C     ***    RESTORE H(L)
             NN=J(L)
             H(L)=F1(L,NN)
             J(L)=J(L)+K(L)
             GO TO 1050
C     *********** ERROR RETURN
*     5000 WRITE(6,5001)
*     5001 FORMAT(6HC*****' ESTIMATED MODQUAD COMPUTATION'
*         & ' TIME HAS ELAPSED.')
*          IFLAG=4
*          RETURN
      5010  WRITE(6,5011)
      5011  FORMAT(5HO*****' INTEGRATION STEP UNDERFLOW,',
          & ' POSSIBLY CAUSED BY ILL-CONDITIONING,',
          & ' PROGRAMMING ERROR, OR EPS TOO SMALL.')
             IFLAG=4
             RETURN
             END
CINVERT   INVERT POS DEF SYMM MATRIX
C
             SUBROUTINE INVERT(TT,N,KK,KKK,NSING,DET)
C  C          ALGORITHM 55 (CACM 4(1961)322) FOR INVERSE OF
C  C             POS. DEF. SYMMETRIC MATRIX STORED IN ONE-
C  C             DIMENSIONAL ARRAY BY ROWS OF UPPER TRIANGLE,
C  C             INCLUDING DIAGONAL ELEMENT.
C  C          MODIFIED TO ACCEPT INPUT FROM TWO-DIMENSIONAL
C             ARRAY.  THE DETERMINANT RETURNED IS THAT OF THE
C             INPUT MATRIX PRIOR TO INVERSION
             DIMENSION T(55), TT(N,N), V(10)
             EPS = 10.E-10
C  CC         2-DIMENSIONAL SYMMETRIC MATRIX INTO 1-DIMENSIONAL
C  CC            ARRAY, BY ROWS OF UPPER TRIANGLE
             JJ = 1
             DO  5 I = 1,N
             DO  5 J = I,N
             T(JJ) = TT(I,J)
        5       JJ = JJ + 1
             LIMT = N*(N+1)/2
C             IF T(1) IS NON-POSITIVE, T IS NOT POSITIVE
C               DEFINITE.
```

Appendix 7. (Continued)

```
      IF(1.LT.KK)  GO TO 11
   10   DET = 1.0
   11   DO  40 K = KK,KKK
      IF(T(1).LE.EPS)   GO TO 50
C      IF T(1) IS ZERO OR NEGATIVE: SINGULAR
   16   PIVOT = 1.0/T(1)
      DET = DET*T(1)
      DO  18 I = 2,N
   18   V(I-1) = T(I)
      NN = N-1
      L = N
      JJ = 1
      M = N
      DO  30 I = 1,NN
      T(L) = -V(I)*PIVOT
      W = T(L)
      L = L+N-I
      DO  20 J = I,NN
      INDX1 = JJ+M
      T(JJ) = T(INDX1) + V(J)*W
   20   JJ = JJ+1
      M = M-1
   30   JJ = JJ+1
   40   T(LIMT) = -PIVOT
C      AT THIS POINT WE HAVE -(T INVERSE)
      NSING = 0
C CC   1-DIMENSIONAL UPPER TRIANGLE (BY ROWS) INTO
C CC      2-DIMENSIONAL SYMMETRIC MATRIX  (WITH SIGN
C CC      CHANGE)
      JJ = 1
      DO  45 I = 1,N
      DO  45 J = I,N
      TT(I,J) = -T(JJ)
      TT(J,I) = TT(I,J)
   45   JJ = JJ+1
      RETURN
   50   NSING = K
      RETURN
      END
CURTIMG  PROC TIME SUBROUTINE
C
      FUNCTION URTIMG(J)
      GO TO 1
      ENTRY URTIMS(J)
    1 URTIMG=0.
      RETURN
      END
CCDFN     UNI-VARIATE CUM DIST FUNC NORMAL
C
      FUNCTION CDFN(X)
C
C      INTEGRAL OF NORMAL VARIATE, FROM -INFINITY TO X
C
      DIMENSION A(6),B(6)
      DOUBLE PRECISION C(6,10)
C
      DATA A/ .625, 1.250, 2.000, 2.450, 3.50, 4.62/
      DATA B/ .300,  .925, 1.625, 2.225, 2.95, 4.15/
C
      DATA C/6.79824032910-04, -1.27097535980-03,
     &         6.79645257970-04, -8.35003142970-05,
     &         1.47915541520-07,  5.48790726730-07,
     &         5.10286403680-05, -1.48226497440-03,
     &        -3.45896837320-04,  5.68250709910-04,
     &        -6.19650131250-03, -8.06561253370-03,
     &        -6.01603623790-03,  8.77189656040-03,
```

Appendix 7. (Continued)

```
     &          -3.0634010430D-03, -4.5015531032D-04,
     &           2.2265710841D-04,  6.6877769441D-07,
     &          -2.2725611373D-02, -2.0200415601D-03,
     &           7.2141082947D-03, -1.3872389725D-03,
     &          -4.4390013966D-04, -6.4034653798D-07,
     &           3.3555772127D-02, -3.4661674433D-02,
     &          -1.0677039440D-03,  5.5422135916D-03,
     &           5.2973750343D-04,  6.2471732476D-07,
     &           7.2703311325D-02,  4.7642403207D-02,
     &          -2.4359750304D-02, -1.0221129236D-02,
     &          -5.6352630662D-04, -4.2105004094D-07,
     &          -1.4093895654D-01,  5.6350293056D-02,
     &           5.7420315319D-02,  1.1350195831D-02,
     &           3.1255376121D-04,  2.0742133302D-07,
     &          -1.5463912770D-01, -2.2130615136D-01,
     &          -6.5753705170D-02, -6.8664030973D-03,
     &          -2.7657162553D-04, -7.6355040218D-08,
     &           5.1563045450D-01,  2.3979043073D-01,
     &           4.0236129479D-02,  3.9933335701D-03,
     &           9.3751413487D-05,  1.5722020822D-08,
     &           1.6431337971D-01,  4.0453365155D-01,
     &           4.3922166659D-01,  4.9917416725D-01,
     &           4.9993489349D-01,  4.9999997779D-01/
C
      Y=X*0.70710678119D0
      SGNY=1.
      IF (Y) 2,1,3
    1 CDFN=.5
      RETURN
    2 SGNY=-1.
      Y=-Y
    3 DO 4 I=1,5
      IF (Y-A(I)) 5,5,4
    4 CONTINUE
      Z=.5
      GOTO 7
    5 Y=Y-B(I)
      Z=C(I,1)
      DO 6 J=2,10
    6 Z=Z*Y+C(I,J)
    7 CDFN=.5+SGNY*Z
      RETURN
      END
CFNF      FNF AND FNG FOR INTEGRATION LIMITS
C
      FUNCTION FNF(I,X)
C
      COMMON /LIMIT/ LSL(12), USL(12)
      REAL LSL
C
      FNF=LSL(I+1)
      RETURN
C
      END
      FUNCTION FNG(I,X)
C
      COMMON /LIMIT/ LSL(12), USL(12)
      REAL LSL
C
      FNG=USL(I+1)
      RETURN
C
      END
CCDMVN    CUM DIST FUNC MULTI-VARIATE NORMAL
C
      FUNCTION CDMVN(P,LSL,USL,MU,SSINV,DET,NDIV,NSEC,
```

Appendix 7. (Continued)

```
     & EPS)
C
      INTEGER P
      REAL LSLA, LSL(P), USL(P), MU(P), SSINV(P,P)
      EXTERNAL FNF, FNG, FNU
      DIMENSION WORK(5000)
      DATA TWOPI/6.2831853/, NS/5000/
C
      COMMON /LIMIT/ LSLA(12), USLA(12)
      COMMON /BLK2/ CONST,RTANN
      COMMON /BLK3/ KNT1
      COMMON /BLK4/ A,B
      COMMON /BLK5/ VECMN(12)
C
C     THIS ROUTINE REPLACES THE MAIN PROGRAM FROM NIH,
C       AND DRIVES THE MVNORMAL COMPUTATION ROUTINES.
C
      DO 1 I=1,P
      LSLA(I)=LSL(I)
      USLA(I)=LSL(I)
    1 VECMN(I)=MU(I)
C
      A=LSL(1)
      B=USL(1)
      K=P-1
      FN=K
      CONST=SQRT(DET)/(TWOPI**(FN*.5))
      RTANN=SQRT(SSINV(P,P))
      SUMANS=0.
      DEL=(B-A)/NDIV
C
      DO 75 JINT=1,NDIV
      V=A+(JINT-1)*DEL
      W=V+DEL
      IF (JINT.EQ.NDIV) W=B
      EPSS=EPS/NDIV
      CALL MDQUAD(K,SSINV,V,W,FNF,FNG,FNU,EPSS,NSEC,
     &            WORK,NS,ANS,IFLAG)
   75 SUMANS=SUMANS+ANS
      CDMVN=SUMANS
      RETURN
      END
CCDFT2    CUM DIST FUNCTION OF T-SQUARED
C
      FUNCTION CDFT2(TSQ,IP,N,TAU)
      DATA TWO9/.22222222/
C
C     INTEGRAL OF T-SQUARED, FROM TSQ TO INFINITY
C
      AN=N
      V1=IP
      V2=AN-IP
      V1T=V1+TAU
      FVAR=(V2*TSQ/(V1T*(AN-1)))**(1./3.)
      X=FVAR*(1.-TWO9/V2)-(1.-TWO9*(V1T+TAU)/V1T**2)
      X=X/SQRT(TWO9*(V1T+TAU)/V1T**2+TWO9/V2*FVAR*FVAR)
      CDFT2=1.-CDFN(X)
      RETURN
      END
```

Appendix 8
Symbol Cross-Reference List

Source code: Harris (H), Harris' program (HP), Klatt (K), Klatt's program (KP), Montgomery and Klatt (MK), Knappenberger and Grandage (KG)

	Source					
Description	H	HP	K	KP	MK	KG
No. variables	p	P	p		p	
No. out-of-control states	s	S				s
Covariance matrix	$\underline{\underline{S}}$	SS	$\underline{\underline{V}}$		$\underline{\underline{S}}$	
Covariance^{-1} matrix	$\underline{\underline{S}}^{-1}$	SSINV	$\underline{\underline{V}}^{-1}$			
mean vector	$\underline{\mu}$	MU	$\underline{\mu}$		$\underline{\mu}$	$\underline{\mu}$
Transition matrix	$\underline{\underline{B}}$	BB	$\underline{\underline{B}}$		$\underline{\underline{G}}$	$\underline{\underline{B}}$
P{higher state transition}	$\underline{\underline{P}}$	PP				$\underline{\underline{P}}$
mean difference vector	$\underline{\delta}$	DELTA	$\underline{\delta}$		$\underline{\delta}$	
lower specification limit	$\underline{l}$	LSL	$\underline{\rho}$		$\underline{l}$	
upper specification limit	$\underline{u}$	USL	$\underline{\rho}$		$\underline{u}$	
cost	A_1	A1	a_1		a_1	a_1
cost	A_2	A2	a_2		a_2	a_2
cost	A_3	A3	a_3		a_3	a_3
cost	A_4	A4	a_4		a_4	a_4
1/(mean time to shift)	λ	LAMDA	λ	LAMBDA	λ	λ
degrees of freedom #1	ν_1	V1	v_1	V1		

Appendix 8 (Continued)

Description	H	HP	Source K	KP	MK	KG
degrees of freedom #2	ν_2	V2	v_2	V2		
Sample size	N	NVEC	N	SN	n	N
No. between samples	K	KVEC	k	SK	k	k
Type I error	α	ALPHA				
Sample size for $\underline{\underline{\Sigma}}$ estimation	N_σ	SGMAN				
Units/hour produced	R	R	R		R	R
Binomial weighting factor	π	PI				π
Non-centrality parameter	$\underline{\tau}$	TAU	λ	$\pi=f(\lambda)$	$\underline{\tau}$	
P{producing defective}	$\underline{\phi}$	PHI	$\underline{f}$	F0, 1	$\underline{\phi}$	$\underline{f}$
P{remain in state}	$\underline{q}$	PR	$\underline{p}$	P0,1		$\underline{p}$
$T^2_{p,n-p}$	T^2	TSQ		TSQ		
$P\{T^2>T^2_{\alpha;p,n-p}\}$	$\underline{\rho}$	RHO	$\underline{q}$	Q0,1	$\underline{\rho}$	$\underline{q}$
P{in state i (steady state)}	$\underline{\beta}$	BETA	$\underline{\alpha}$	ALPHA0,1	$\underline{\beta}$	$\underline{\alpha}$
P{in state i|sample}	$\underline{\gamma}$	GAMMA	$\underline{\gamma}$	GAMA0,1	$\underline{\gamma}$	$\underline{\gamma}$

Appendix 9
Listing of the Five Operating Points

Parameter \ O.P.	1	2	3	4	5
P	2	2	2	4	3
s	1	6	1	3	3
N_σ	13	13	13	13	25
$\underline{\underline{S}}$	$\begin{bmatrix} 2 & 1 \\ 1 & 2.5 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 2.5 \end{bmatrix}$	$\begin{bmatrix} 2 & 1 \\ 1 & 2.5 \end{bmatrix}$	$\underline{\underline{I}}_3$	$\underline{\underline{I}}_3$
π	-	1/2	-	1/4	1/3
λ	1	1	1	1	1
R	10000	10000	10000	10000	10000
$\underline{l}$	$\begin{bmatrix} -4 \\ -4 \end{bmatrix}$	$\begin{bmatrix} -3.5\sigma_1 \\ -3.5\sigma_2 \end{bmatrix}$	$\begin{bmatrix} -3.5\sigma_1 \\ -3.5\sigma_2 \end{bmatrix}$	$\begin{bmatrix} -3.5 \\ -3.5 \\ -3.5 \\ -3.5 \end{bmatrix}$	$\begin{bmatrix} -3.5 \\ -3.5 \\ -3.5 \end{bmatrix}$
$\underline{u}$	$\begin{bmatrix} 4 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 3.5\sigma_1 \\ 3.5\sigma_2 \end{bmatrix}$	$\begin{bmatrix} 3.5\sigma_1 \\ 3.5\sigma_2 \end{bmatrix}$	$\begin{bmatrix} 3.5 \\ 3.5 \\ 3.5 \\ 3.5 \end{bmatrix}$	$\begin{bmatrix} 3.5 \\ 3.5 \\ 3.5 \end{bmatrix}$

Appendix 9 (Continued)

Parameter O.P.	1	2	3[1]	4	5
$\underline{\mu}_0$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
$\underline{\mu}_1$	$\begin{bmatrix} 5 \\ 6 \end{bmatrix}$	$\begin{bmatrix} 2\sigma_1 \\ 2\sigma_2 \end{bmatrix}$	$\begin{bmatrix} 3.41 \\ 3.81 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$
$\underline{\mu}_2$	-	$\begin{bmatrix} 2.2\sigma_1 \\ 2.2\sigma_2 \end{bmatrix}$	-	$\begin{bmatrix} 2.5 \\ 2.5 \\ 2.5 \\ 2.5 \end{bmatrix}$	$\begin{bmatrix} 2.5 \\ 2.5 \\ 2.5 \end{bmatrix}$
$\underline{\mu}_3$	-	$\begin{bmatrix} 2.4\sigma_1 \\ 2.4\sigma_2 \end{bmatrix}$	-	$\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$	$\begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$
$\underline{\mu}_4$	-	$\begin{bmatrix} 2.6\sigma_1 \\ 2.6\sigma_2 \end{bmatrix}$	-	-	-
$\underline{\mu}_5$	-	$\begin{bmatrix} 2.8\sigma_1 \\ 2.8\sigma_2 \end{bmatrix}$	-	-	-
$\underline{\mu}^6$	-	$\begin{bmatrix} 3.0\sigma_1 \\ 3.0\sigma_2 \end{bmatrix}$	-	-	-

[1]In O.P. 3, $\underline{\mu}_1$ is $\underline{\mu}^*$ from O.P. 2.

Appendix 9 (Continued)

Parameter	O.P. 1	2	3	4	5
A_1	10	1	1	1	1
A_2	1	.1	.1	.1	.1
A_3	100	100	100	100	100
A_4	1	1	1	1	1

BIBLIOGRAPHY

BIBLIOGRAPHY

1. Cowden, D.J., Statistical Methods in Quality Control, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1957.

2. Duncan, A.J., "The Economic Design of $\bar{X}$ Charts Used to Maintain Current Control of a Process," Journal of the American Statistical Association, v. 51, 1956.

3. Duncan, A.J., Quality Control and Industrial Statistics, Richard D. Irwin, Inc., Homewood, Ill., 1965.

4. Feller, W., An Introduction to Probability Theory and Its Applications, v. 1, 2nd Ed., John Wiley & Sons, Inc., New York, 1957.

5. Grant, E.L., "The Economic Relationship Between Design and Acceptance Specifications," Special Technical Publication No. 103, Symposium on Application of Statistics, American Society for Testing and Materials, Philadelphia, Pa., 1950.

6. Grant, E.L., "Some Possible Contributions of Statistical Quality Control to Engineering Economy," Paper No. 1, Fourth Annual Convention ASQC, ASQC, New York, 1950.

7. Grant, E.L., and Leavenworth, R.S., Statistical Quality Control, McGraw-Hill, Inc., 1972.

8. Hotelling, Harold, "The Generalization of Students' Ratio," Annals of Mathematical Statistics, v. 2, 1931.

9. Klatt, Phillip J., Design of Control Charts for the Mean Vector of a Multivariate Normal Process, M.S. Thesis, Georgia Institute of Technology, 1971.

10. Knappenberger, H.A., and Grandage, A.H., "Minimum Cost Quality Control Tests," AIIE Transactions, v. 1, No. 1, 1969.

11. Laubscher, N.H., "Normalizing the Noncentral t and F Distributions," Annals of Mathematical Statistics, v. 31, 1960, 1105-1112.

12. Milton, R.C., "Computer Evaluation of the Multivariate Normal Integral," Technometrics, v. 14, No. 4, 1972.

BIBLIOGRAPHY (Continued)

13. Milton, R.C., and Hotchkiss, R., "Computer Evaluation of the Normal and Inverse Normal Distribution Functions," Technometrics, v. 11, No. 4, 1969.

14. Montgomery, D.C., Heikes, R.G., and Mance, J.F., "Economic Design of Fraction Defective Control Charts," Management Science, v. 21, No. 11, 1975.

15. Montgomery, D.C., and Klatt, P.J., "Economic Design of T^2 Control Charts to Maintain Current Control of a Process," Management Science, v. 19, No. 1, 1972.

16. Morrison, D.F., Multivariate Statistical Methods, McGraw-Hill Inc., New York, 1967.

17. Mudholkar, G.S., Chaubey, Y.P., and Lin, Ching-Chuong, "Some Approximations for the Noncentral-F Distribution," Technometrics, v. 18, No. 3, 1976.

18. Parzen, S., Stochastic Processes, Holden-Day, Inc., San Francisco, Cal., 1962.

19. Paulson, E., "An Approximate Normalization of the Analysis of Variance Distribution," Annals of Mathematical Statistics, No. 13, 1942.

20. Searle, S.R., Linear Models, Wiley, New York, 1971.

21. Severo, N.C., and Zelen, M., "Normal Approximations to the Chi-Square and Non-Central F Probability Functions," Biometrika, v. 47, 411-416.

22. Tables of the Bivariate Normal Distribution Function and Related Functions, Compiled and Edited by the National Bureau of Standards - Applied Mathematics Series 50, Issued June 1959.

23. Time Sharing Applications Library Guide: Volume 1 - Mathematics, Honeywell Information Systems, Inc.