Graphs of small *rank-width* are pivot-minors of graphs of small *tree-width*

Sang-il Oum (KAIST)
joint work with O-Joung Kwon.

Graph Theory at Georgia Tech
2012.5.9

Rank-width

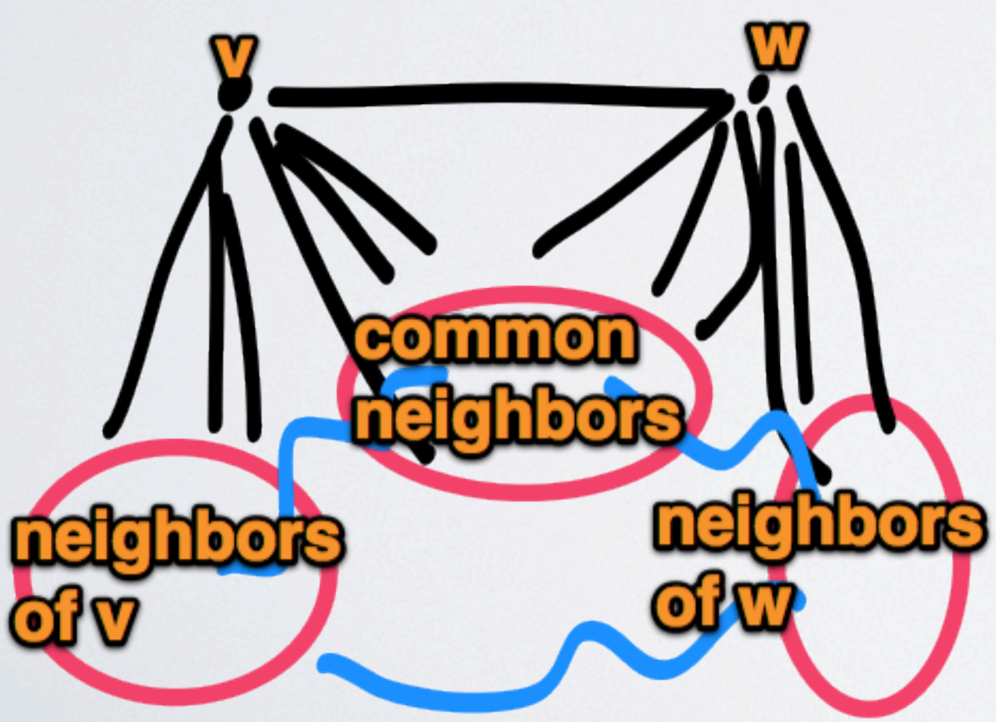
(O. and Seymour 2005)

- Width parameter of graphs
- Generalizing tree-width (Dense graphs can have small rank-width)
- Small rank-width implies many algorithmic problems to be solvable in polynomial time
  - Any problem expressible in MSOL can be solved in $O(n^3)$ time for graphs of rank-width at most k (for fixed k) (Courcelle Makowsky Rotics, 2000)

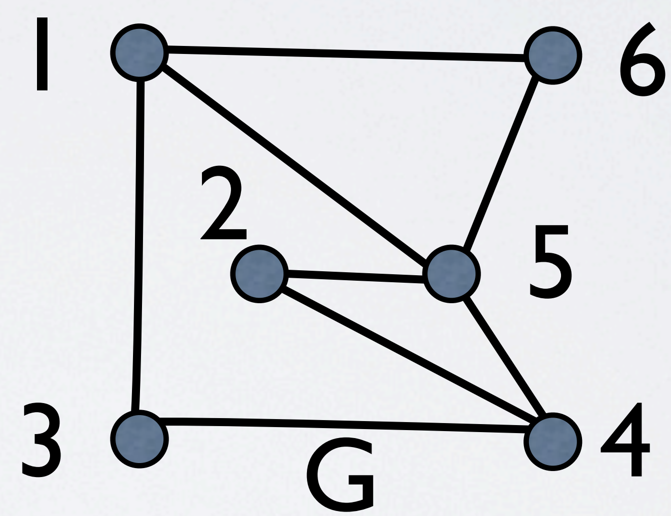
Pivot-minors

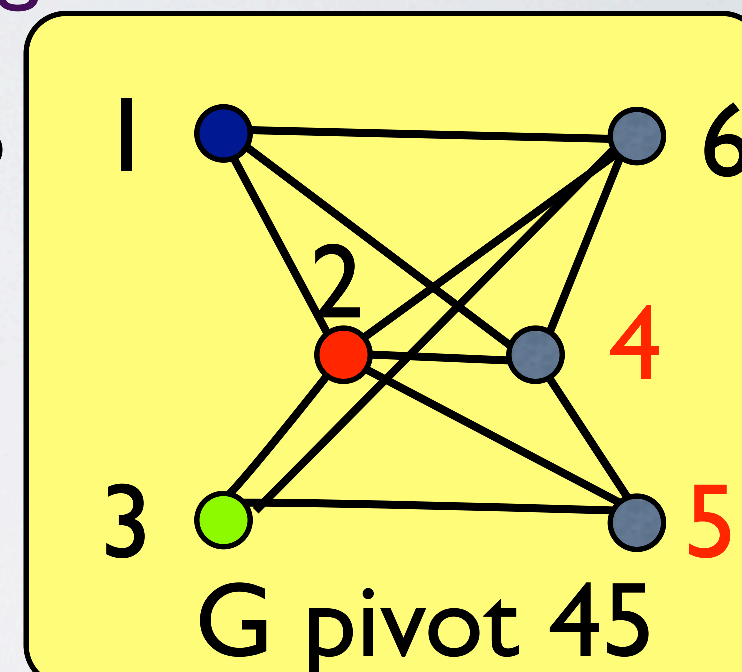
- Containment relation, suitable for the study on rank-width

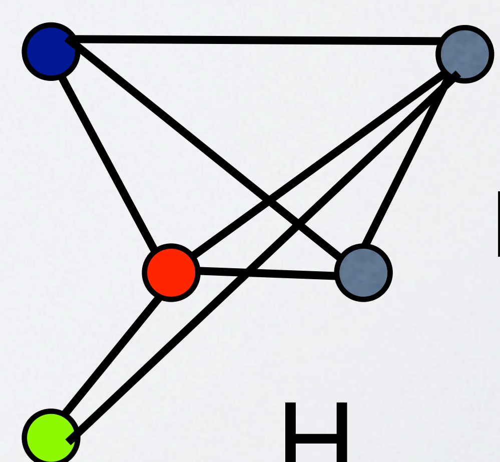
Pivot: Flip adjacencies between "blue" pairs and swap v and w



- H is a ***pivot-minor*** of G if H is obtained from G by applying a sequence of ***pivots*** and deleting vertices.







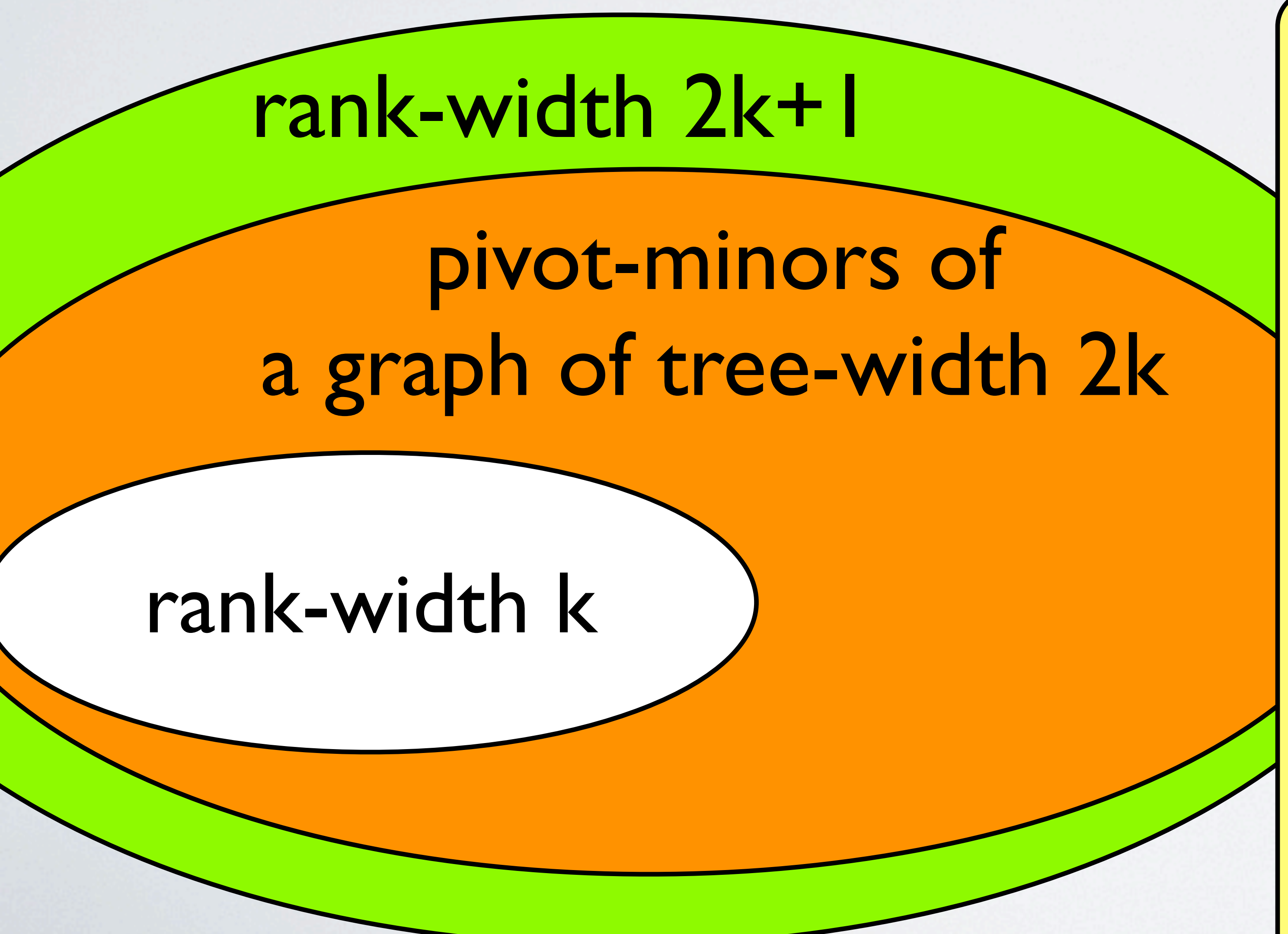
H is a pivot-minor of G

Theorem (Kwon-O.)

If rank-width(G)=k,
then there is a graph H such that
(i) tree-width(H)=2k
(ii) G is a pivot-minor of H.

Known results

- (O. 08) If tree-width(H)=k, then rank-width(H)≤k+1.
- (O. 05) If rank-width(H)=k, then rank-width(pivot-minor of H)≤k.



Corollary of Known ... :
Let *I* be a minor ideal.
The minimal pivot-minor ideal containing *I* is {all graphs}
iff
***I* contains all planar graphs.**

Theorem (Kwon-O.)

If rank-width(G)=k,
then there is a graph H such that

(i) tree-width(H)=2k

(ii) G is a pivot-minor of H.

Known results

- (O. 08) If tree-width(H)=k, then rank-width(H)≤k+1.
- (O. 05) If rank-width(H)=k, then rank-width(pivot-minor of H)≤k.

Corollary:
Let *I* be a minor ideal.
The minimal pivot-minor ideal *J* containing *I* is {all graphs}
iff
***J* has bounded rank-width.**

Corollary of Known ... :
Let *I* be a minor ideal.
The minimal pivot-minor ideal containing *I* is {all graphs}
iff
***I* contains all planar graphs.**

Rank-width (Oum, Seymour 2005)

Cut-Rank

Rank-decomposition (T,μ)

Rank-width (Oum, Seymour 2005)

Cut-Rank

$$\rho_G(X)=\text{rank}\ \begin{array}{cc} & X^c \\ & y \\ X & \\ x & 1 \end{array}$$

rk over GF(2)

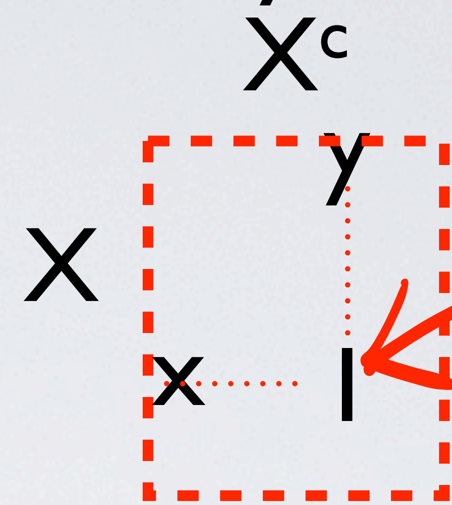
$$\begin{cases} 1 & \text{if } x\sim y \\ 0 & \text{otherwise} \end{cases}$$

Rank-decomposition (T,μ)

Rank-width (Oum, Seymour 2005)

Cut-Rank

$\rho_G(X)$=rank

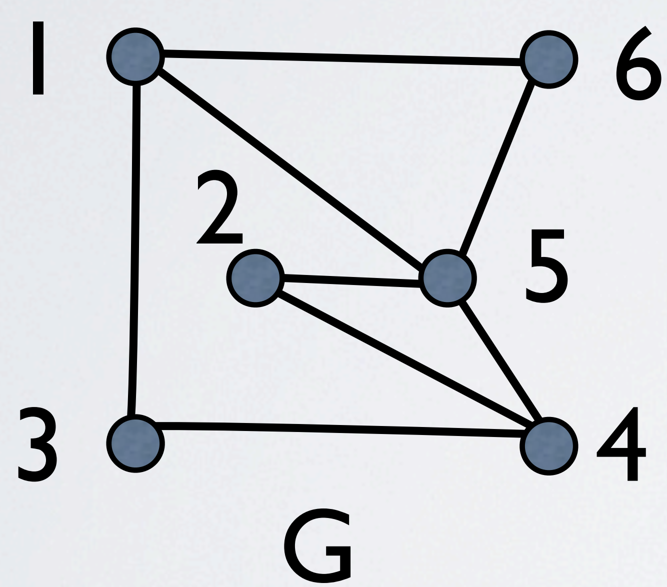


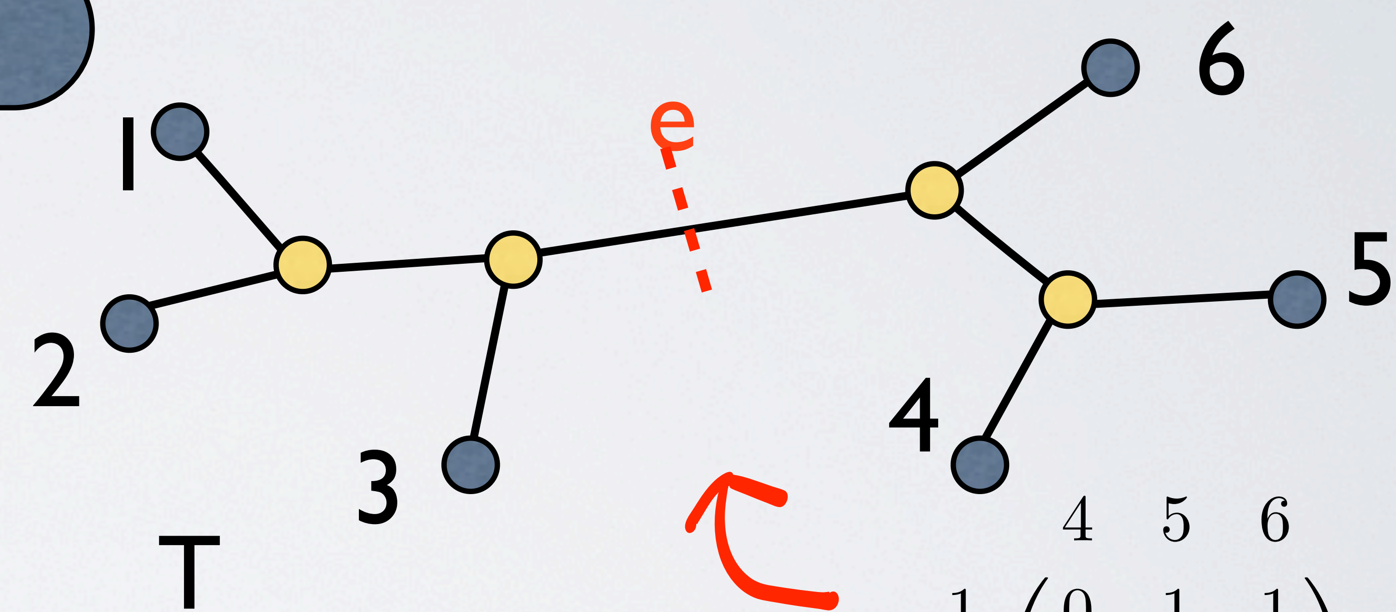
rk over GF(2)

1 if x~y

0 otherwise

Rank-decomposition (T,μ)



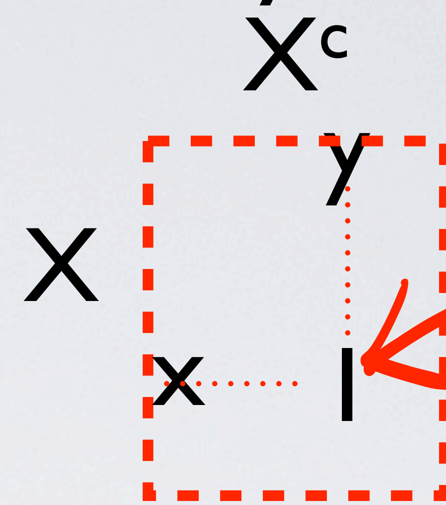


Width(e)= $\mathrm{rank}\ \begin{array}{c c} & \begin{matrix} 4 & 5 & 6 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} = 3$

Rank-width (Oum, Seymour 2005)

Cut-Rank

$\rho_G(X)$=rank

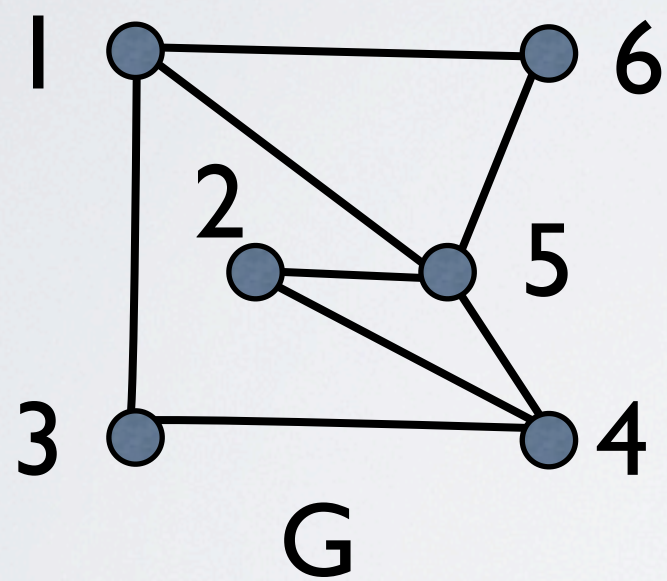


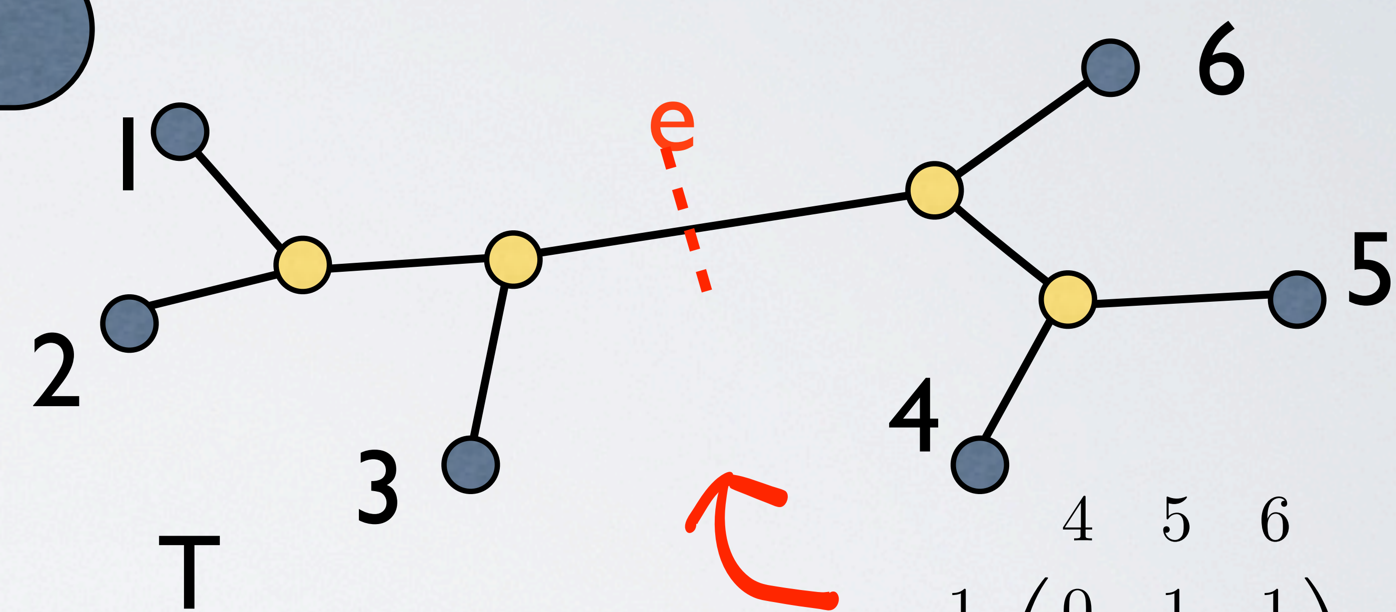
rk over GF(2)

1 if x~y

0 otherwise

Rank-decomposition (T,μ)





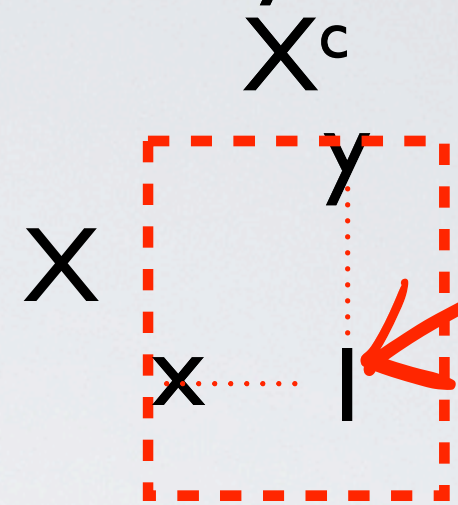
$$\text{Width}(e) = \text{rank}\ \begin{array}{c|c} & \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} = 3$$

Width of (T,μ)= max (Width(e): e is an edge of T)

Rank-width (Oum, Seymour 2005)

Cut-Rank

$\rho_G(X)$=rank

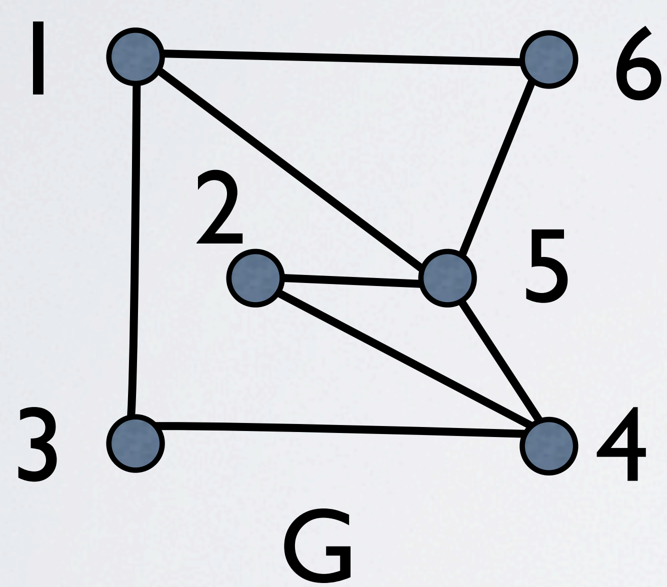


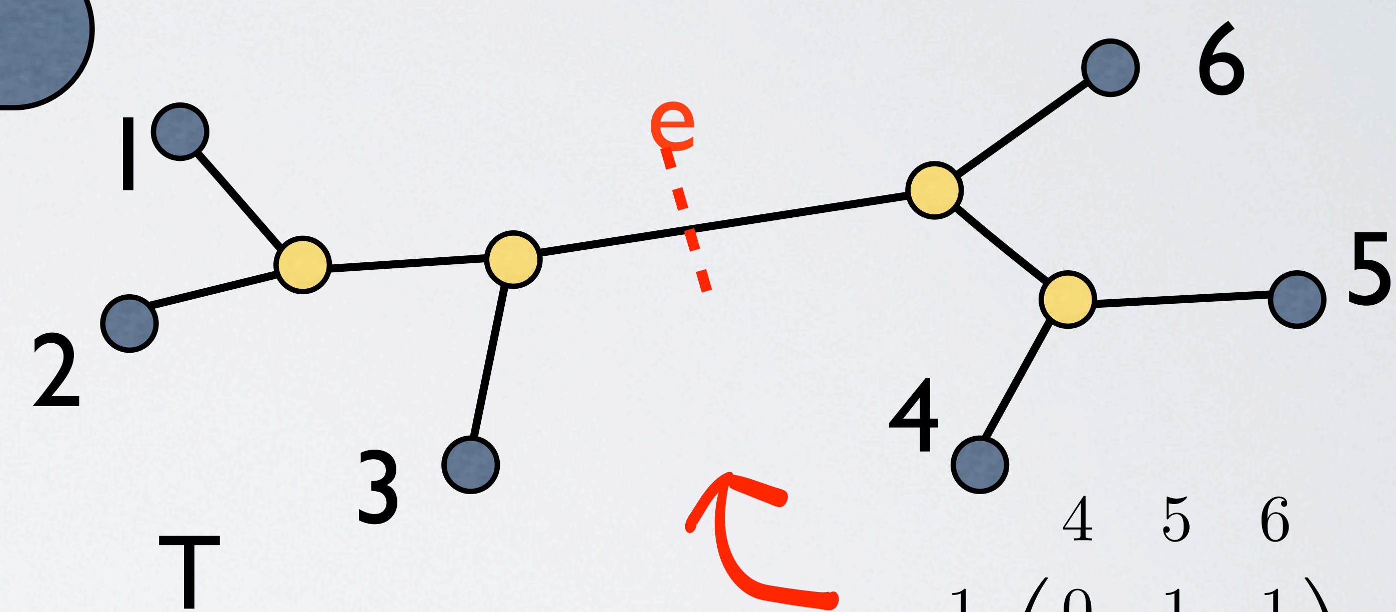
rk over GF(2)

1 if x~y

0 otherwise

Rank-decomposition (T,μ)





Width(e)= $\text{rank} \begin{array}{c c} & \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} = 3$

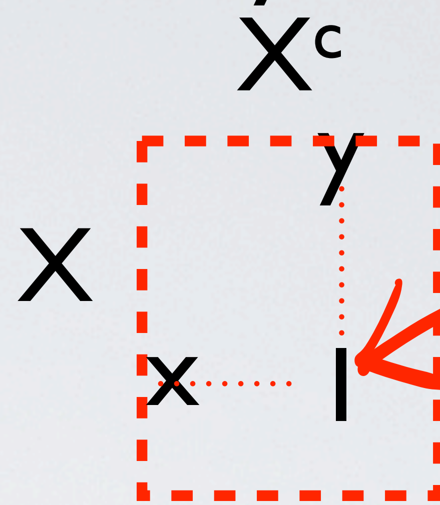
Width of (T,μ)= max (Width(e): e is an edge of T)

Rank-width: Min Width of All Rank-Decompositions

Rank-width (Oum, Seymour 2005)

Cut-Rank

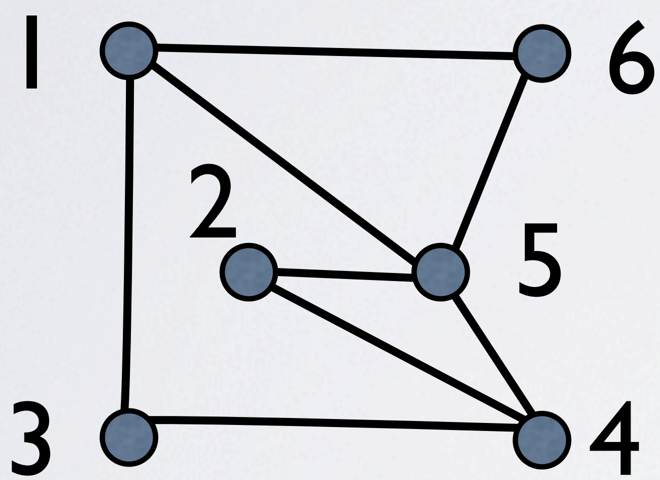
$\rho_G(X)$=rank



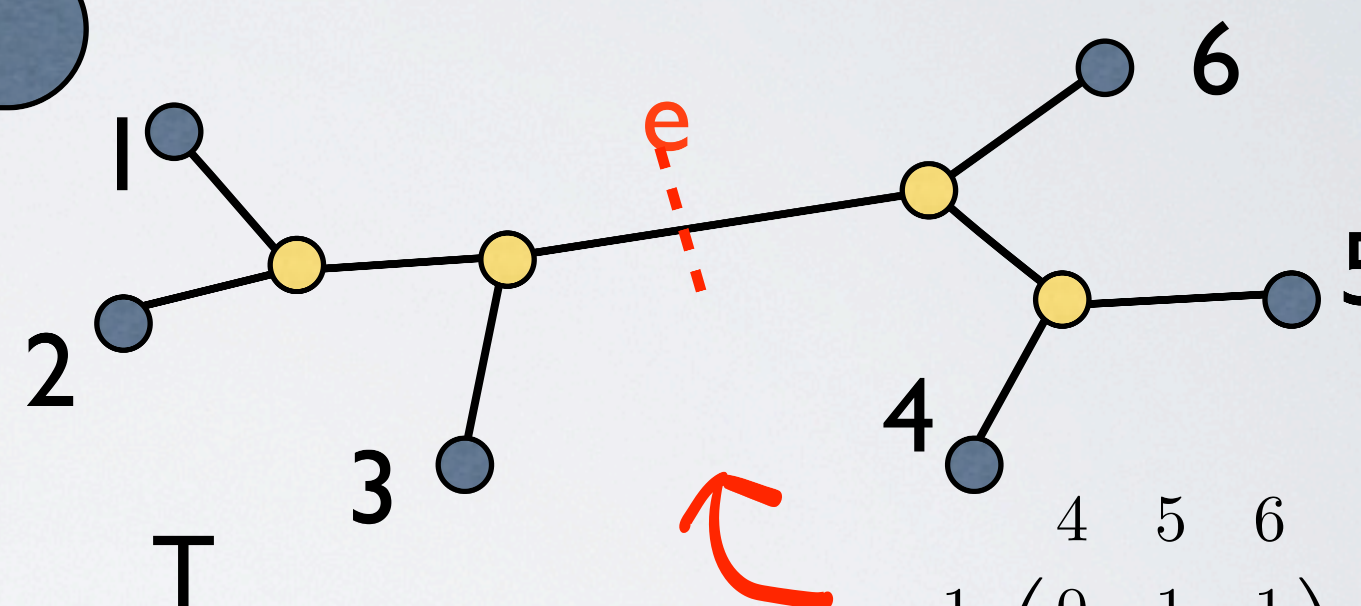
rk over GF(2)

$\begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{otherwise} \end{cases}$

Rank-decomposition (T,μ)



G



Width(e)= $\text{rank} \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} = 3$

Width of (T,μ)= max (Width(e): e is an edge of T)

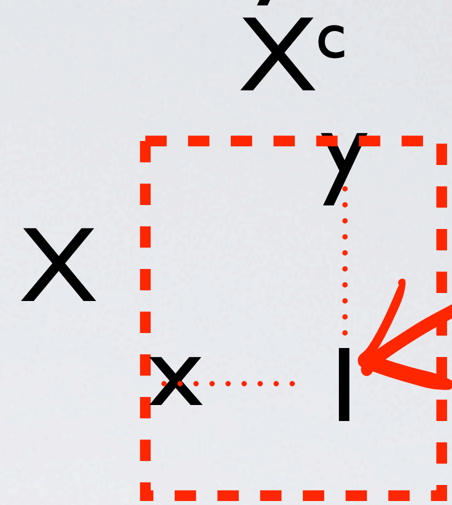
Rank-width: Min Width of All Rank-Decompositions

Fact: Rank-width $\leq$ Clique-width $\leq 2^{1+\text{Rank-width}}-1$

Rank-width (Oum, Seymour 2005)

Cut-Rank

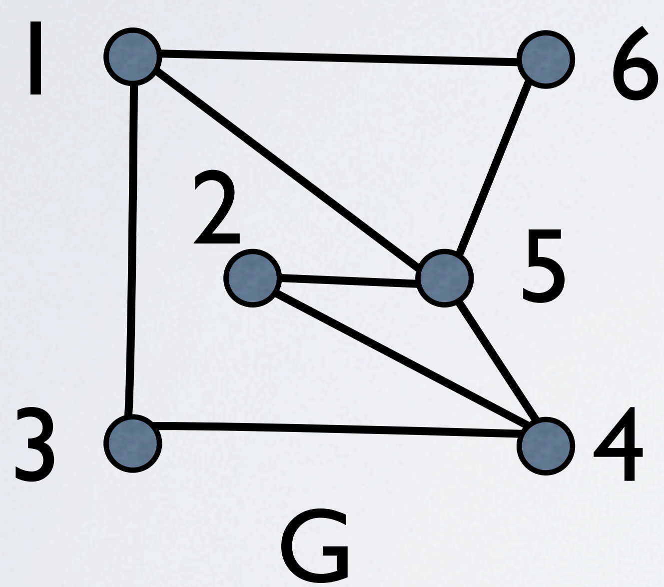
$\rho_G(X)$=rank

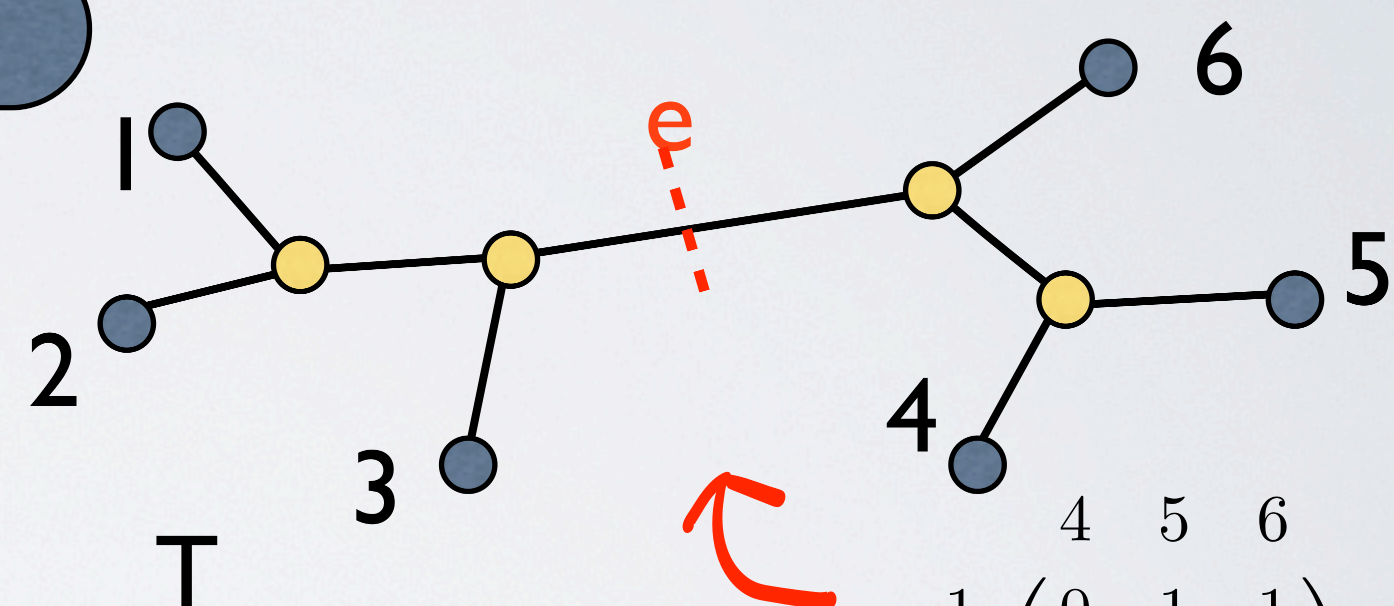


rk over GF(2)

$\begin{cases} 1 \text{ if } x\sim y \\ 0 \text{ otherwise} \end{cases}$

Rank-decomposition (T,μ)





Width(e)= $\text{rank} \begin{array}{c c} & \begin{array}{ccc} 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array} = 3$

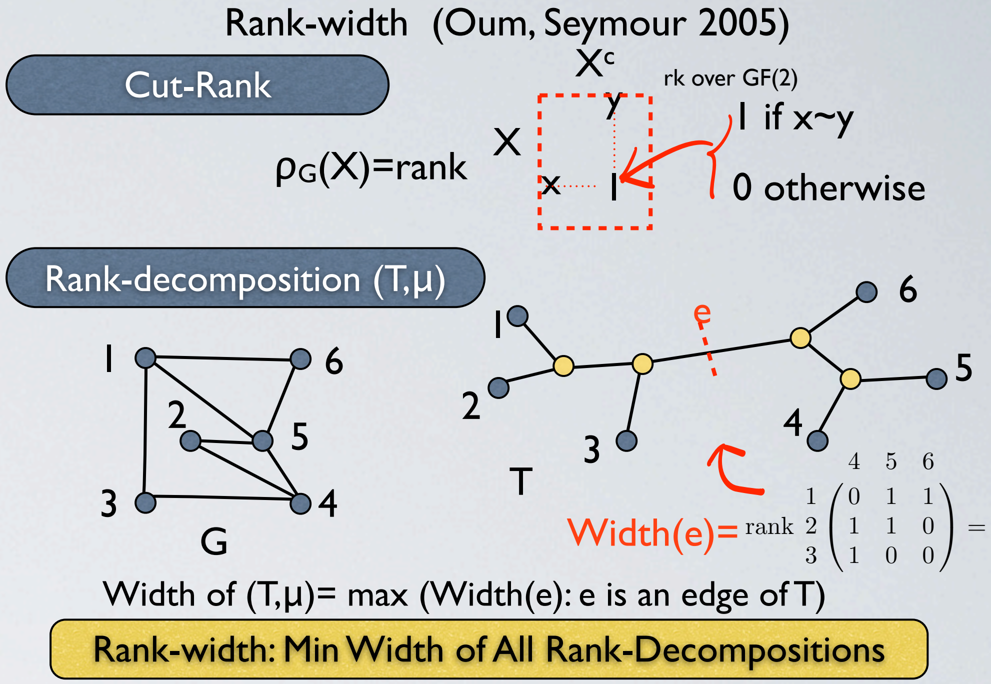
Width of (T,μ)= max (Width(e): e is an edge of T)

Rank-width: Min Width of All Rank-Decompositions

Fact: Rank-width ≤ Clique-width ≤ $2^{1+\text{Rank-width}}$-1

Examples

Graph	Rank-width	Tree-width
Trees	1	1
K_n	1	n-1
nxn grid	n-1	n



Fact: Rank-width ≤ Clique-width ≤ $2^{1+\text{Rank-width}}-1$

Properties

- If rank-width(G)=k, then rank-width($\bar{G}$)≤k+1.
- For **fixed** k, it is possible to decide rank-width≤k in **O(n^3) time**, and if yes, output a rank-decomposition of width≤k. (Hlineny, Oum '08)
- NP-complete to decide rank-width≤k for an input k: implied by *Seymour and **Thomas** (1994), "Call routing and the ratcatcher"*
- Graphs of rank-width≤k are wqo by pivot-minors (O. '08)
- If H is a pivot-minor of G, then rank-width(H)≤rank-width(G).

KAIST

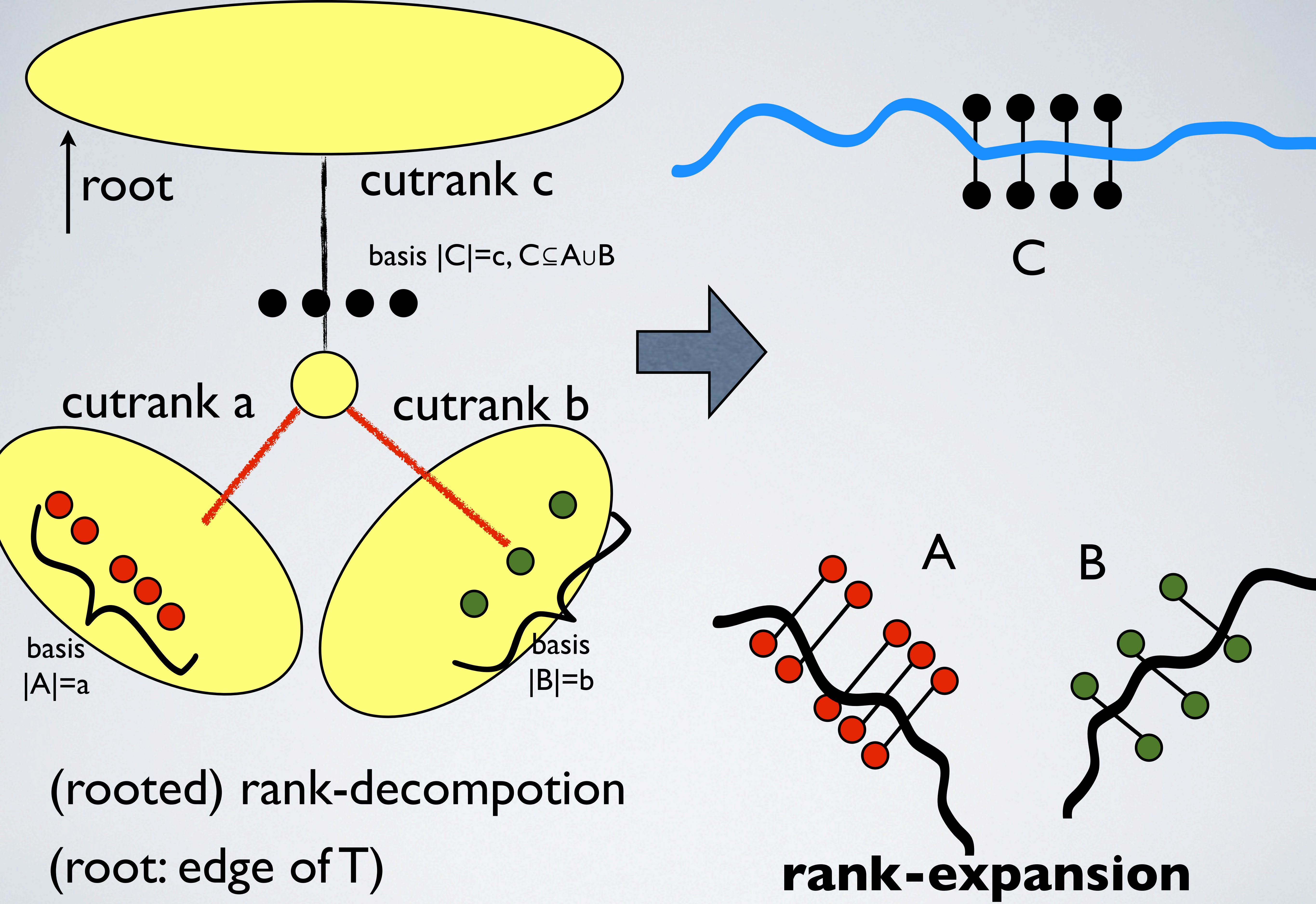
Proof

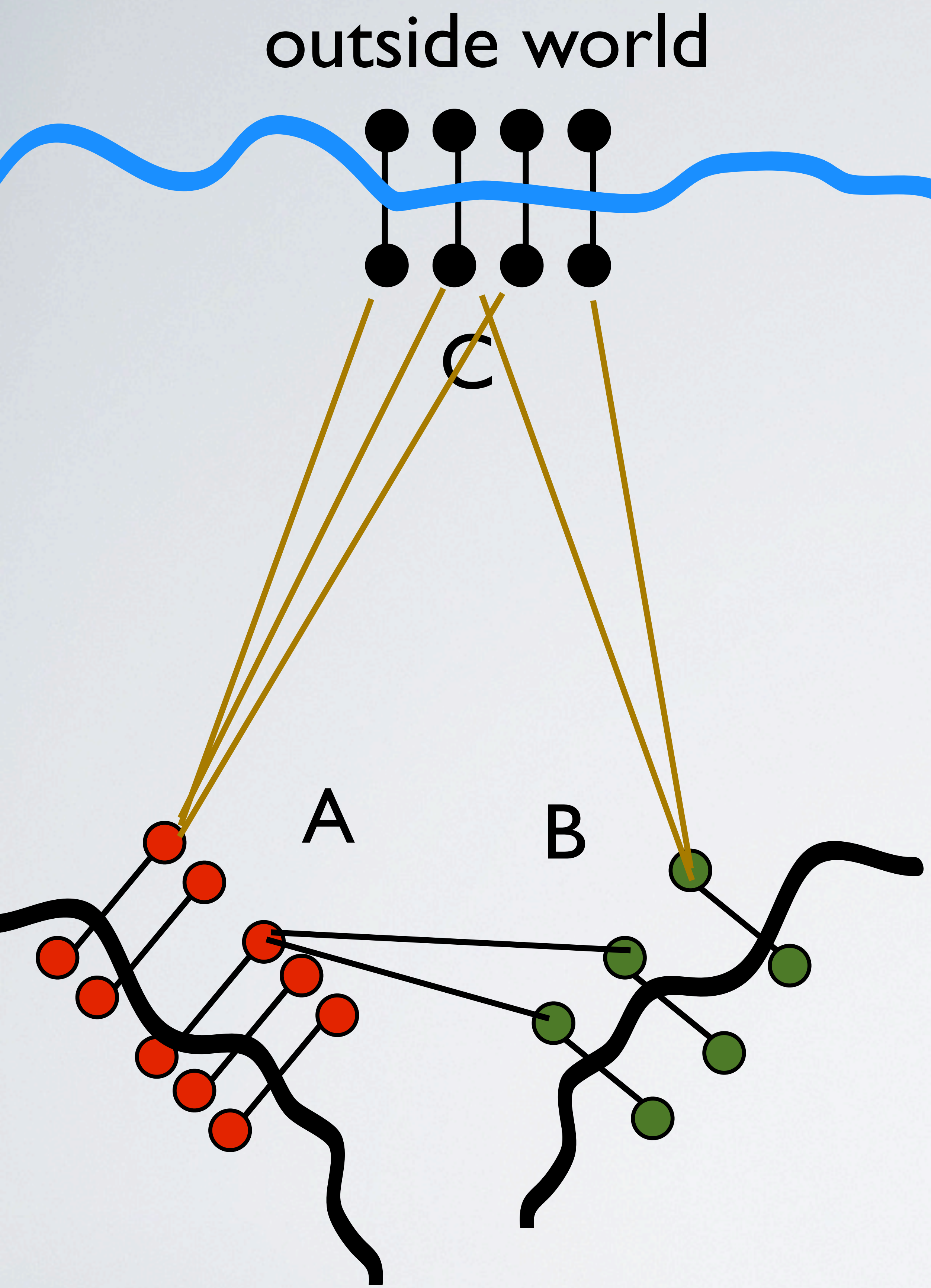
Theorem (Kwon-O.)

If rank-width(G)=k,
then there is a graph H such that
(i) tree-width(H)=2k
(ii) G is a pivot-minor of H.

Given a graph G
and a rank-decomposition (T, μ) of width k,
we explicitly construct
a graph H, called a ***rank-expansion***, such that
(i) tree-width(H)≤2k
(ii) G is a pivot-minor of H.

Our proof: $|V(H)|\leq(2k+1)n-6k$
(We assume $n\geq 3$ and G is connected)


root
cutrank c
basis |C|=c, C⊆A∪B
C
cutrank a
cutrank b
A
B
basis
|A|=a
basis
|B|=b
(rooted) rank-decompotion
(root: edge of T)
rank-expansion



For each vertex v in A or B,
find a subset C_v of C
such that
(neighbors of v outside)=
sum (neighbors of vertices in x outside: x in C_v)
as a 0-1 vector over GF(2)

Since C is a basis,
Cv is uniquely determined

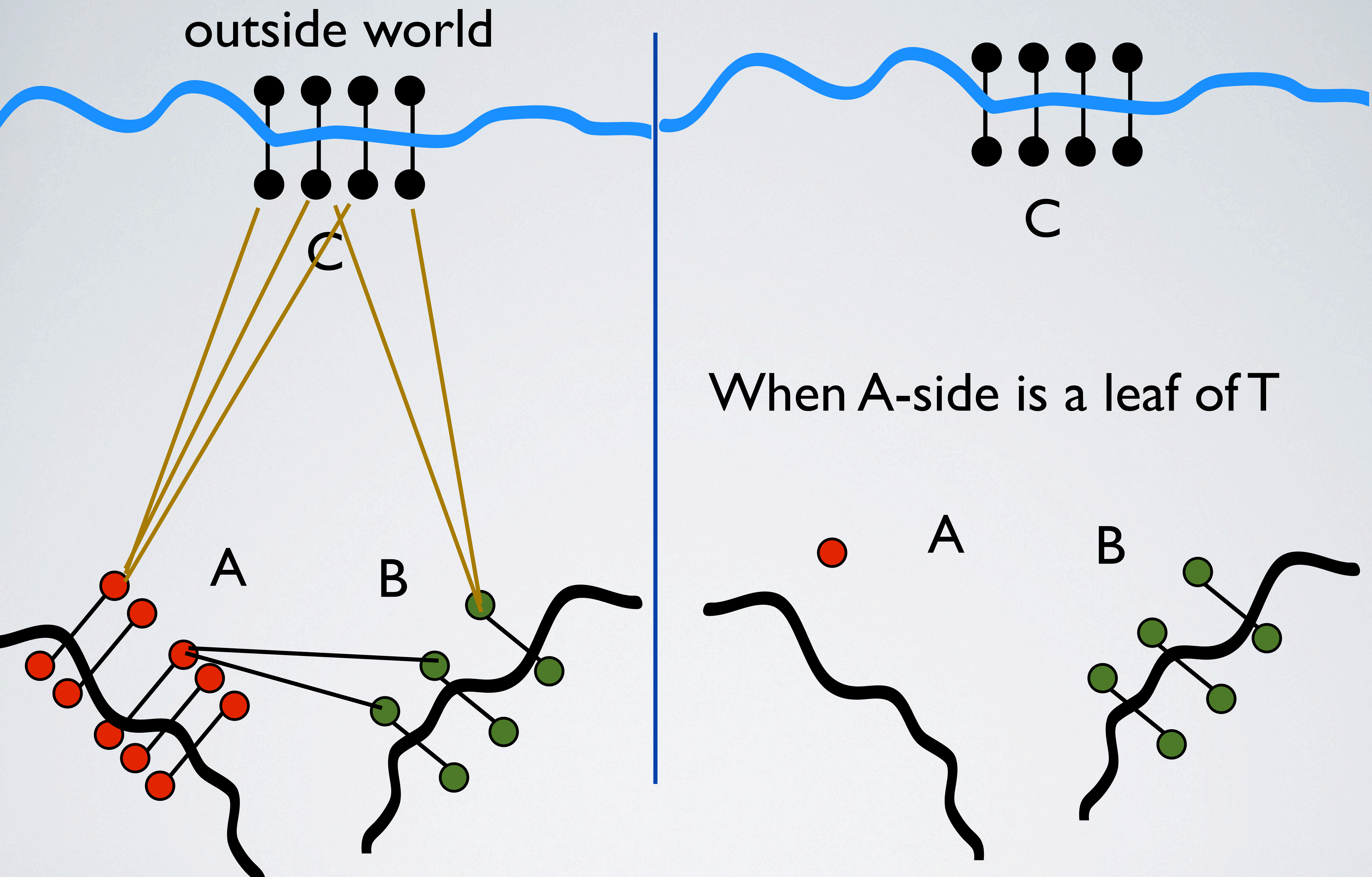
Edges between A∪B and C

Add edges from v to C_v

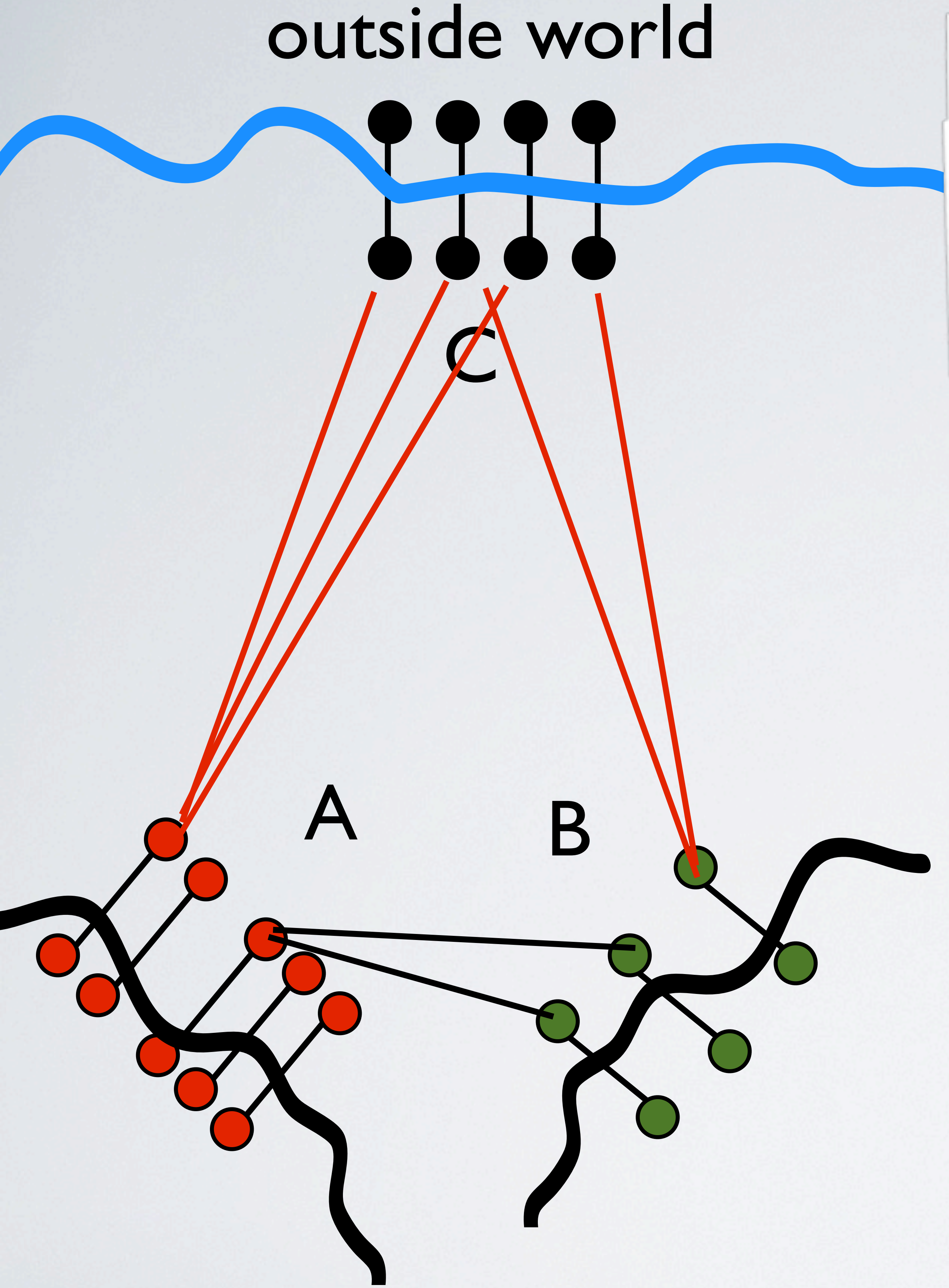
Edges between A and B

Join them if they are adjacent in G

KAIST



Theorem 1:
G= H pivot (all black matching edges) - (vertices on matching edges)



Theorem 1:
G= H pivot (all black matching edges) - (vertices on matching edges)

Sketch: Proved directly or by some
linear algebraic lemma (matrix multiplication)

Theorem 2:
tree-width (H)≤ 2k

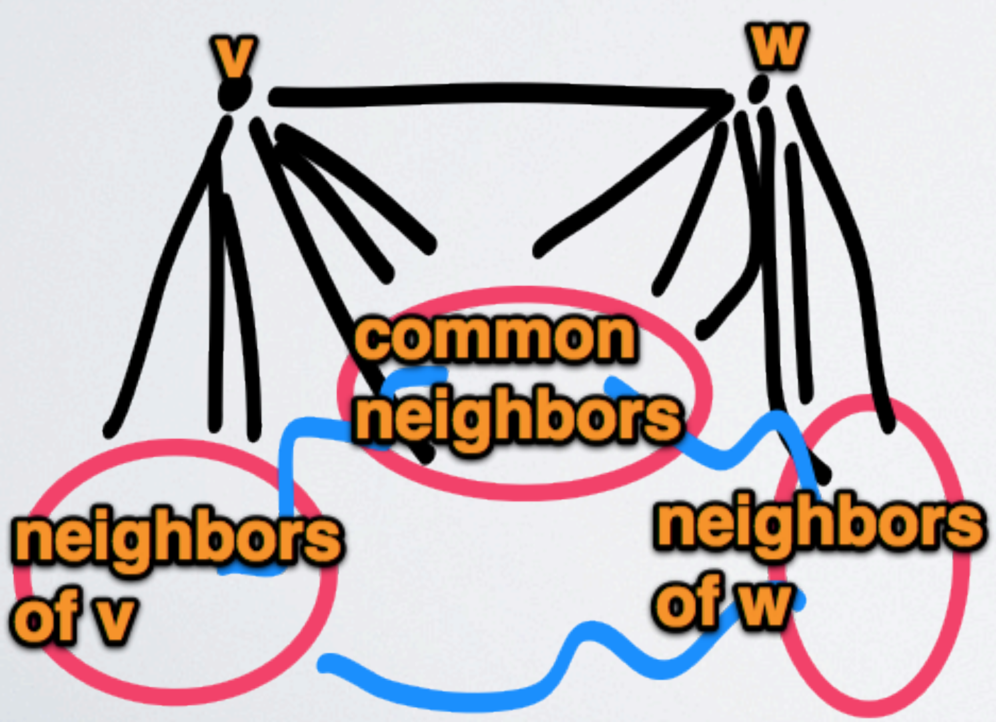
- a, b, c≤k
- there is a matching covering C in the red edges
(we choose C so that C⊆A∪B)

Observations

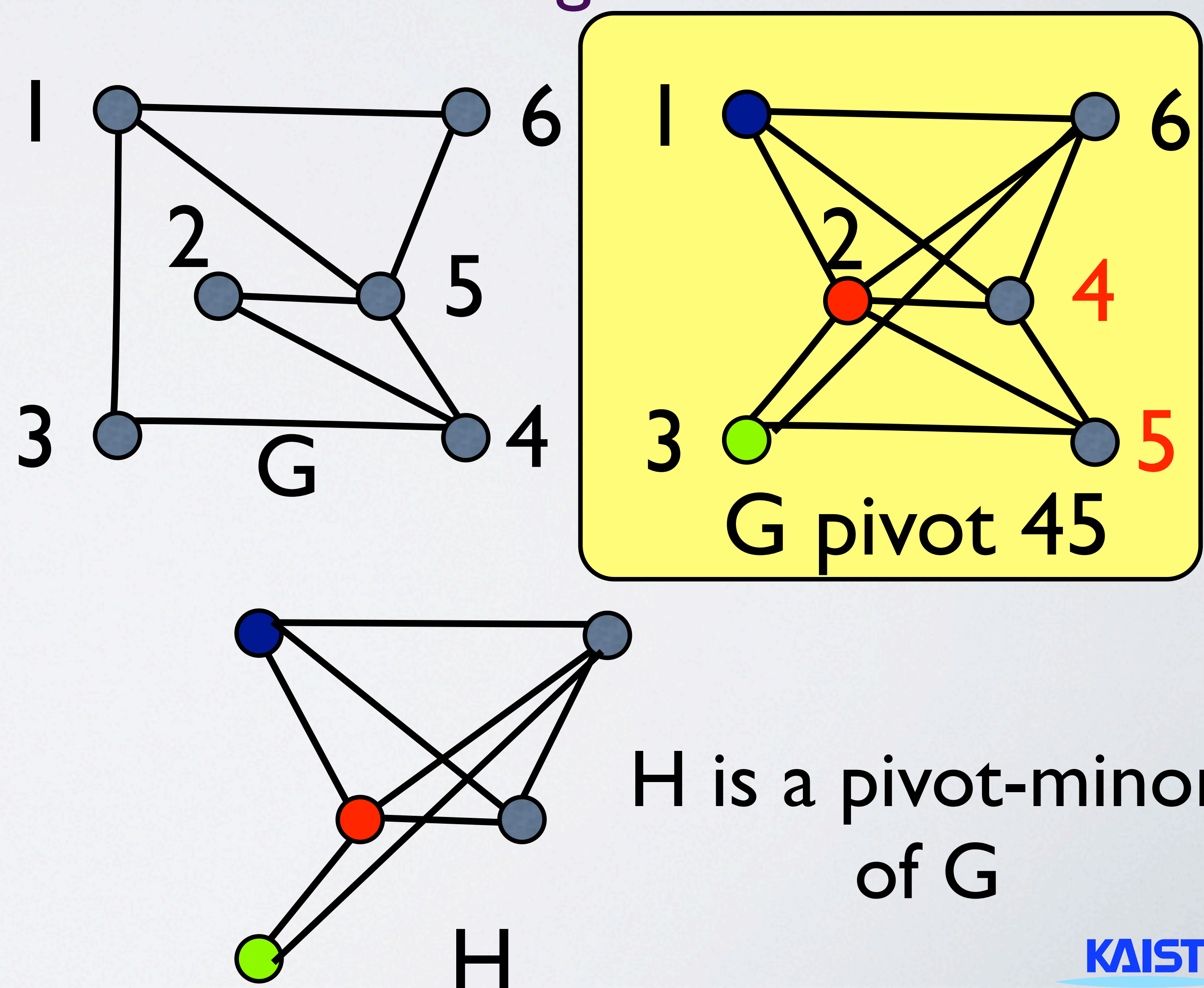
Pivot-minors

- Containment relation, suitable for the study on rank-width

Pivot: Flip adjacencies between "blue" pairs and swap v and w



- H is a ***pivot-minor*** of G if H is obtained from G by applying a sequence of ***pivots*** and deleting vertices.



H is a pivot-minor of G

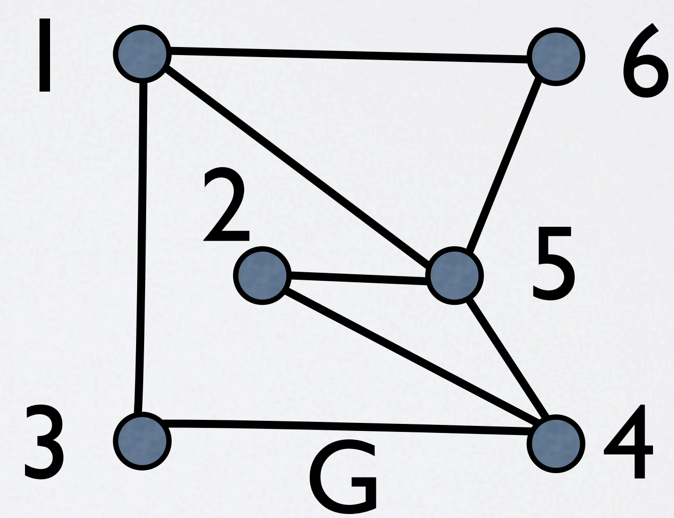
Vertex-minors

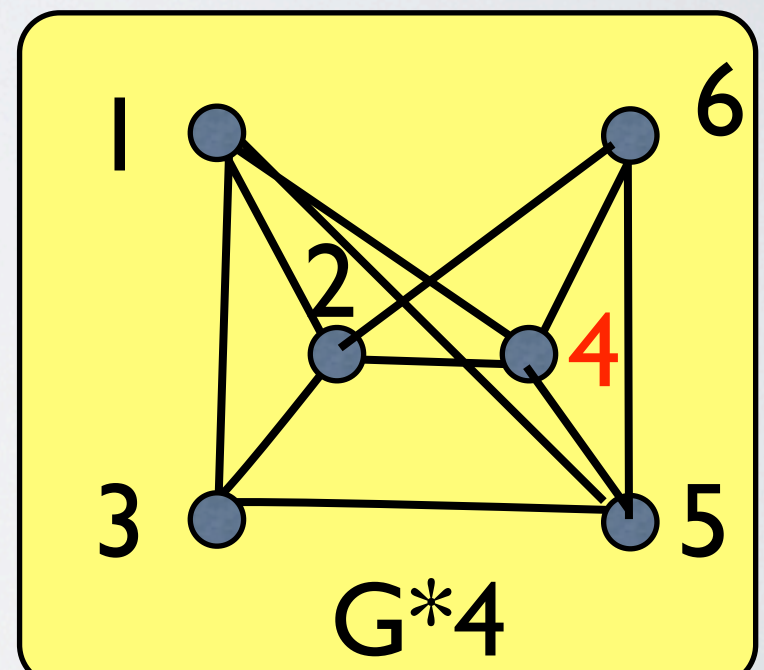
- Another Containment relation, suitable for rank-width

Local Complementation: Flip adjacencies between neighbors of v

flip "blue" pairs

- H is a ***vertex-minor*** of G if H is obtained from G by applying a sequence of ***local complementations*** and deleting vertices.

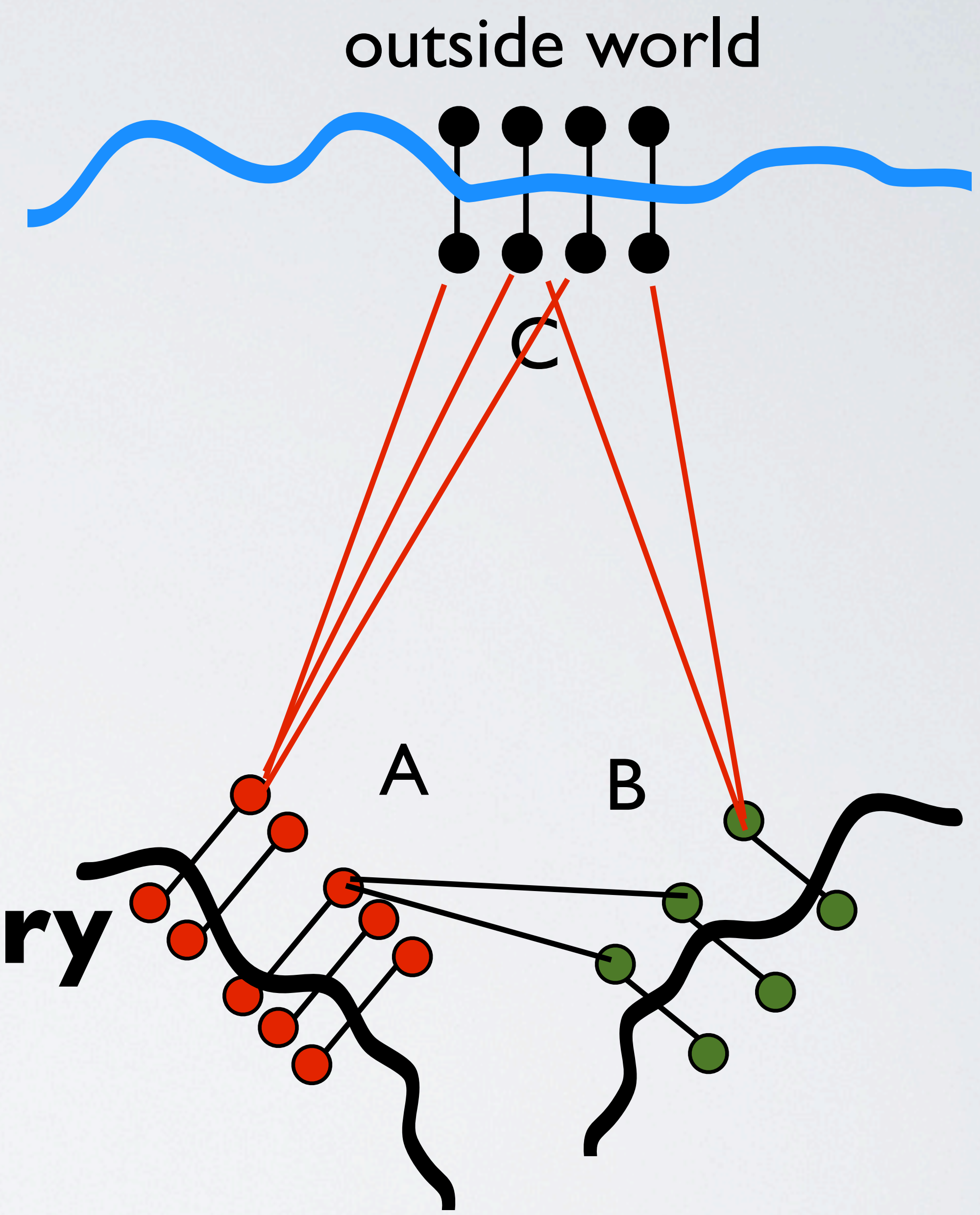




G*3*4-3 is a vertex-minor of G

Rank-width 1

- Graphs of rank-width 1 are exactly distance-hereditary graphs (O. 06)
- In our proof, when k=1, we only create a tree + disjoint triangles.
- If we replace a triangle by a claw (Δ-Y operation) from H, we obtain a new graph H'.



Corollary:
A graph is **distance-hereditary**
iff
it is a vertex-minor of a tree

More corollaries

- A graph is distance-hereditary (rank-width 1) iff it is a **vertex-minor** of a **tree**.
- A graph is **bipartite** distance-hereditary (rank-width 1) iff it is a **pivot-minor** of a **tree**.
- If a graph has **linear rank-width** k, then it is a **pivot-minor** of a graph of **path-width** k+1.
- A graph has **linear rank-width** 1 iff it is a **vertex-minor** of a **path**.
- A graph is **bipartite** and **linear rank-width** 1 iff it is a **pivot-minor** of a **path**.

One more thing

Happy birthday Robin!