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CONDITIONS ON FUNCTIONS, FUNCTIONAL CONVERGENCES AND SELF MAPS

A THESIS

Presented to

The Faculty of the Graduate Division

by

Richard Vernon Fuller

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy

in the School of Mathematics

Georgia Institute of Technology

June, 1967

CONDITIONS ON FUNCTIONS, FUNCTIONAL CONVERGENCES

AND SELF MAPS

Approved:

Chairman

Date Approved by Chairman: May 18, 1967

ACKNOWLEDGMENTS

I wish to express heart-felt thanks to my thesis advisor, Dr. R. H. Kasriel, whose advice and guidance were valuable and whose encouragement was invaluable in the pursuit of this study. Thanks are due also to Dr. G. L. Cain, Jr. and Dr. D. L. Finn for their reading and criticism of this thesis. Finally I am grateful to the National Aeronautics and Space Administration for a Traineeship which supported my studies from September, 1964 to June, 1967.

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CHAPTER I

INTRODUCTION

This thesis contains a study of three topics in point-set topology which are vaguely suggested by the title. These three topics, while to be sure related within topology, are not treated as being interrelated within this work and accordingly each topic occupies a separate chapter of the three chapters which follow.

In Chapter II, a number of conditions on a function from one topological space to another are considered. Among these conditions are those of a function or its inverse preserving openness, closedness, or compactness of sets. Other conditions are having a closed graph and a concept generalizing continuity, subcontinuity, which we introduce in Section 2.

Some interesting results uncovered in Chapter II are the following: (1) a function which is closed with closed point inverses, and a regular space for its domain has a closed graph. (2) If a function maps into a Hausdorff space, continuity of the function is equivalent to the requirement that the function be subcontinuous and have a closed graph. (3) The usual net characterization of continuity for a function with values in a Hausdorff space is still valid if it is required only that the image of a convergent net be convergent (not necessarily to the "right" value).

In Chapter III various modes of convergence of nets of functions will be considered. In particular three Ascoli type theorems and a Dini type theorem are proved.

The first two Ascoli theorems are closely related to the topological-uniform and purely topological Ascoli theorems found in Kelley [20, Chapter 7]. While the theorems in Kelley consider the question of when a family of functions with a certain topology is compact, we consider the question of when a net of functions has a subnet which converges in a certain mode. This approach enables us to consider in this connection modes of convergence which are not topological but are, from a certain point of view, very natural. In this respect our motivation is very much the same as Cook and Fischer [7] with their convergence and uniform convergence structures. Along these lines we should mention that Poppe [28] has generalized the topological Ascoli theorems in Kelley (due to Kelley and Morse) to the convergence space of Fischer.

The third Ascoli theorem is a uniform version. Another uniform version appears, for instance, in Isbell [17, p. 51].

Finally in Chapter III, we characterize uniform convergence and uniform convergence at a point (for nets of continuous functions with a continuous limit) with some theorems which use extensions of the notion of monotone convergence in the classical Dini theorem.

In Chapter IV we study various self maps of a uniform space which treat a base for the uniformity in a special way.

The contents of the first section of this chapter hopefully shed additional light on the question, "If a self map of a uniform space is nonexpansive relative to some base, is there a base (and if so of what sort) such that the map is invariant relative to the latter base?"

The second section of this chapter consists of a few simple theorems concerning when the pointwise limit of self maps is nonexpansive, invariant, or noncontractive relative to some base. A small application to real variables is given.

The last section of Chapter IV offers a Banach contraction principle for uniform space. Our main theorem of this section, in fact, generalizes two theorems for contraction maps in a metric space. One is, of course, Banach's contraction principle, and the other a sort of localized extension of Banach's principle due to Edelstein [11].

For any concepts which we do not define or elaborate upon the reader is referred to Kelley's book [20]. We will denote nets by a symbol such as x_a letting context distinguish between a net and a point of the range of the net and suppressing explicit mention of the directed set. A subnet of a net x_a will be denoted by a symbol such as x_{Nb} where b is a member of the domain of x_{Nb} and N is the appropriate function from the domain of x_{Nb} into the domain of x_a.

Additive notation will be used for the union of sets and multiplicative notation for intersection. Thus if A and B are two sets $A + B$ is their union and $A \cdot B$ their intersection. Moreover if $\mathcal{A}$ is a collection of sets $\Sigma\mathcal{A}$ is the union and $\Pi\mathcal{A}$ the intersection of these sets.

Finally, in general, parentheses will be dropped from functional notation, i.e., if f is a function and x a point in its domain, then fx is the value of f at x.

CHAPTER II

RELATIONS AMONG CONTINUOUS AND VARIOUS NON-CONTINUOUS FUNCTIONS

In this chapter a number of conditions on a function from one topological space to another are considered. Among these conditions are those of preserving closedness, openness, or compactness of a set. Other conditions are having a closed graph and a concept generalizing continuity, subcontinuity, which we introduce in Section 2.

Some interesting results which are uncovered are the following: (1) A function which is closed with closed point inverses and a regular space for its domain has a closed graph. (2) If a function maps into a Hausdorff space, continuity of the function is equivalent to the requirement that the function be subcontinuous and have a closed graph. (3) The usual net characterization of continuity for a function with values in a Hausdorff space is still valid if it is required only that the image of a convergent net be convergent (not necessarily to the "right" value).

Also several theorems of Halfar [15], [16] are extended including some sufficient conditions for continuity.

In general, spaces "X" and "Y" in this chapter are topological spaces and no function is assumed to be continuous unless explicitly stated to be so.

1. Functions with Closed Graphs

(1.1) Definition: A function $f : X \to Y$ has a closed graph (relative to $X \times Y$) if and only if $\{(x, fx) : x \in X\}$ is closed in the product topology of $X \times Y$.

Using a characterization of closed in terms of nets (See Kelley [20, Chapter 2]) a function $f : X \to Y$ has a closed graph if and only if the net (x_a, fx_a) converges to (p, q) in $X \times Y$ implies that $q = fp$. The following example shows that a function with a closed graph need not be continuous.

(1.2) Example: Let $D : C^1[0,1] \to C[0,1]$ be the differentiation operator, $C^1[0,1]$ all continuously differentiable functions on $[0,1]$, and $C[0,1]$ all continuous functions on $[0,1]$. Let $C^1[0,1]$ and $C[0,1]$ be given the sup-norm which in turn generates the topology of uniform convergence. Since D is a linear operator, if D were continuous it should map the bounded sequence $\{t^n, n \geq 1\}$ onto a bounded sequence. However

$$||Dt^n|| = ||nt^{n-1}|| = n .$$

Thus D is not continuous

But now suppose $(f_n, Df_n) \to (f, g)$ in $C^1[0,1] \times C[0,1]$. Then $f_n \to f$ uniformly and $Df_n \to g$ uniformly. It then follows by a well-known theorem (See Rudin [30, Theorem 7.17, p. 124]) that $Df = g$ and consequently D has a closed graph.

Let E_a be a net of sets in a topological space X. A point p in X belongs to $\lim\sup E_a$ ($\lim\inf E_a$) if and only if E_a frequently (resp., eventually) intersects each neighborhood of p. This generalization of the lim sup and lim inf of sets from sequences to nets has been studied by Mrówka [26, p. 237].

(1.3) Lemma: If $f : X \to Y$ is a function, y_a is a net in Y, and $p \in \limsup f^{-1}[y_a]$ then there is a net x_{Nb} in X such that x_{Nb} converges to p and fx_{Nb} is a subnet of y_a.

Proof: Assume y_a is a net in Y and $p \in \limsup f^{-1}[y_a]$. Then for each a and each neighborhood U of p there is an index $N(a,U) \geq a$ such that $f^{-1}[y_{N(a,U)}] \cap U \neq \emptyset$. Now direct the neighborhoods U downward by inclusion, give the pairs (a,U) the product direction, and choose $x_{N(a,U)}$ in $f^{-1}[y_{N(a,U)}] \cap U$, thus obtaining a net $x_{N(a,U)}$. Finally note that $fx_{N(a,U)} = y_{N(a,U)}$ is a subnet of y_a and $x_{N(a,U)}$ converges to p.

The following characterization of a function with a closed graph appears in Kuratowski [22, Definition, p. 32] in a considerably restricted form.

(1.4) Theorem: If $f : X \to Y$ is a function, then the following conditions are equivalent:

(a) f has a closed graph

(b) $y_a \to q$ in Y implies $\limsup f^{-1}[y_a] \subset f^{-1}[q]$

(c) $y_a \to q$ in Y implies $\liminf f^{-1}[y_a] \subset f^{-1}[q]$

Proof: Assume (a) holds. Let $y_a \to q$ in Y and $p \in \limsup f^{-1}[y_a]$. By the previous lemma there is a net x_{Nb} in X such that $x_{Nb} \to p$ and

fx_{Nb} is a subnet of y_a. Thus we have $(x_{Nb}, fx_{Nb}) \to (p,q)$. Since f has a closed graph $q = fp$ or $p \varepsilon f^{-1}[q]$. Consequently (b) holds.

If (b) holds then, since $\lim\inf f^{-1}[y_a] \subset \lim\sup f^{-1}[y_a]$, (c) evidently holds.

Assume then (c) holds. Suppose $(x_a, fx_a) \to (p,q)$. Then $y_a = fx_a \to q$ and $p \varepsilon \lim\inf f^{-1}[y_a]$. Consequently $p \varepsilon f^{-1}[q]$ or $q = fp$ and (a) holds.

2. Subcontinuous and Inversely Subcontinuous Functions

(2.1) <u>Definition</u>: The function $f : X \to Y$ is said to be <u>subcontinuous</u> if and only if $x_a \to p$ in X implies there is a subset x_{Nb} of x_a and a point q in Y such that $fx_{Nb} \to q$. The function f is said to be <u>inversely subcontinuous</u> if and only if $fx_a \to q$ in Y implies there is a subnet x_{Nb} of x_a and a p in X such that $x_{Nb} \to p$, that is, if and only if $y_a \to q$ in Y and $x_a \varepsilon f^{-1}[y_a]$ implies there is a subnet x_{Nb} of x_a and a p in X such that $x_{Nb} \to p$.

Our concept of a subcontinuous function is a generalization of a function whose range is compact, and similarly our concept of an inversely subcontinuous function is a generalization of a function whose domain is compact. In addition, a subcontinuous function is a generalization of a continuous function whence its name.

More generally, it is clear if $f : X \to Y$ is a function and each point p in X has a neighborhood U such that $f[U]$ is contained in a compact subset of Y, then f is subcontinuous. Likewise if $f : X \to Y$ is a function and each q in Y has a neighborhood V such that $f^{-1}[V]$ is contained in a compact subset of X, then f is inversely subcontinuous.

There are a couple more analogous pairs of facts about subcontinuous and inversely subcontinuous functions with analogous proofs of these facts. This seems to indicate that probably each pair of proofs could have been integrated into a single proof concerning a multiple-valued subcontinuous function. However such an approach did not seem of particular interest in the present investigation.

The following theorem says that subcontinuous functions which map into a completely regular space are very close to being compact preserving. Analogously inversely subcontinuous functions with a completely regular domain have an inverse which very nearly preserves compactness. Unfortunately, nearly is often not good enough. However, the subcontinuous functions (plain and inversely) have advantages over those which preserve compactness in one direction or another as will be pointed out later.

(2.2) <u>Theorem</u>: Let $f : X \to Y$ be a function.

(a) If f is inversely subcontinuous and X is completely regular then for each compact set $K \subset Y$, $f^{-1}[K]^-$ is compact ($^-$ denotes closure).

(b) If f is subcontinuous and Y is completely regular then for each compact set $K \subset X$, $f[K]^-$ is compact.

<u>Proof</u>: (a) Let K be a compact subset of Y. Let $\{z_a, a \in A\}$ be a net in $f^{-1}[K]^-$. Let B be a uniformity for X. Direct B downward by inclusion and let $A \times B$ have the product order. For each $(a,b) \in A{\times}B$ choose $x_{(a,b)}$ in $b[z_a] \circ f^{-1}[K]$. Since K is compact, a subnet of

$fx_{(a,b)}$ converges and thus a subnet $x_{(Nc,Mc)}$ of $x_{(a,b)}$ converges to say p. Clearly $p \in f^{-1}[K]^-$.

Now consider the net z_{Nc} which is a subnet of z_a. We proceed to show $z_{Nc} \to p$. Let $b \in B$. There is a symmetric $b_1 \in B$ such that $b_1 \circ b_1 \subset b$. By the choice of the $x_{(a,b)}$, it is clear that $(z_{Nc}, x_{(Nc,Mc)})$ is eventually in b_1. Since $x_{(Nc,Mc)} \to p$, $(x_{(Nc,Mc)}, p)$ is eventually in b_1. Thus (z_{Nc}, p) is eventually in b and $z_{Nc} \to p$. Consequently $f^{-1}[K]^-$ is compact.

(b) The proof is entirely analogous to that of (a).

3. Functions and Inverses Which Preserve Closedness and Compactness

Let $f : X \to Y$ be a function. If f maps compact sets of X onto compact sets of Y, f is said to be compact-preserving. If compact sets of Y go onto compact sets of X under f^{-1}, f is called compact. We point out that compact functions are usually required to be continuous and the modifier "strong" is often attached if f is not onto. When closed sets of X are mapped onto closed sets of Y, f is said to be closed. Again the modifier "strong" is often applied if f is not onto. Functions $f : X \to Y$ whose inverse carries closed sets of Y onto closed sets of X are sometimes called continuous.

Pursuing further the similarities noted in the previous section we characterize compact and compact preserving functions in terms of inverse subcontinuity and subcontinuity respectively.

(3.1) Theorem: Let $f : X \to Y$ be a function.

(a) f is compact if and only if $f|f^{-1}[K] : f^{-1}[K] \to K$ is inversely subcontinuous for each compact $K \subset Y$.

(b) f is compact preserving if and only if $f|K : K \to f[K]$ is subcontinuous for each compact $K \subset X$.

Proof: (a) Assume f is compact. Let K be a compact subset of Y and fx_a a net in K which converges to q in K. Then x_a, being in the compact set $f^{-1}[K]$, has a subnet x_{Nb} converging to some p in $f^{-1}[K]$. Thus $f|f^{-1}[K] : f^{-1}[K] \to K$ is inversely subcontinuous.

Now assume $f|f^{-1}[K]$ is inversely subcontinuous for each compact $K \subset Y$. Let K be a compact subset of Y and let x_a be a net in $f^{-1}[K]$. Then fx_a has a subnet fx_{Nb} converging to some q in K. We thus may conclude x_{Nb} has a subnet x_{NMc} converging to some p in $f^{-1}[K]$. Therefore $f^{-1}[K]$ is compact.

(b) The proof is entirely analogous to that of (a).

The following theorem giving sufficient conditions that a function be compact or compact preserving is deduced immediately from Theorem (3.1).

(3.2) Theorem: Let $f : X \to Y$ be a function.

(a) If f is inversely subcontinuous and $f^{-1}[K]$ is closed for each compact $K \subset Y$, then f is compact.

(b) If f is subcontinuous and $f[K]$ is closed for each compact $K \subset X$, then f is compact preserving.

Remark: From (3.6) it will then follow that f has a closed graph and is subcontinuous (inversely subcontinuous) implies f is compact preserving (resp., compact).

(3.3) <u>Corollary</u>: Let $f : X \to Y$ be a function.

(a) If Y is Hausdorff and f is both continuous and inversely subcontinuous, then f is compact.

(b) If X is Hausdorff and f is both closed and subcontinuous, then f is compact preserving.

We turn now to gathering some more facts about functions with closed graphs and their relation to other functions. In particular, the following two theorems show that the closed graph property complements the two subcontinuities in interesting ways.

(3.4) <u>Theorem</u>: Let $f : X \to Y$ be a function. A sufficient condition that f be continuous is that f have a closed graph and be subcontinuous. If Y is Hausdorff the condition is also necessary.

<u>Proof</u>: (Sufficiency) Let x_a be a net in X which converges to some point p. Suppose fx_a does not converge to fp. Then fx_a has a subnet, say fx_{Nb}, no subnet of which converges to fp (see Kelley [20, item (c), p. 74]). However by subcontinuity some subnet of fx_{Nb}, say fx_{NMc}, converges to some point q. Thus we have (x_{NMc}, fx_{NMc}) converges to (p,q). But by the closed graph property of f, $q = fp$ and we have a contradiction.

(Necessity) Assuming f is continuous then evidently f is subcontinuous. If $(x_a, fx_a) \to (p,q)$ then $x_a \to p$ and thus $fx_a \to fp$. Assuming Y is Hausdorff we conclude $q = fp$.

(3.5) <u>Theorem</u>: If the function $f : X \to Y$ has a closed graph and is inversely subcontinuous, then f is closed.

Proof: Let C be a closed subset of X. Suppose $f[C]$ is not closed. Then there is a q in $Y - f[C]$ and a net x_a in C such that $fx_a \to q$. Thus by inverse subcontinuity there is a subnet of x_a, say x_{Nb}, which converges to some p in X. Since C is closed, $p \in C$. Consequently we have (x_{Nb}, fx_{Nb}) converges to (p,q) but $q \neq fp$ since $q \notin f[C]$. This contradicts the closedness of the graph of f.

Theorem (3.4) tells us (among other things) that if $f : X \to Y$ is continuous with Y Hausdorff then f has a closed graph. The Hausdorff requirement cannot be dropped in general for if $i : X \to X$ is the identity on X, the graph of i is closed in $X \times X$ if and only if X is Hausdorff.

We can also see from (3.4) the dependency of the closed-graphness of a function $f : X \to Y$ upon the space Y in which $f[X]$ is imbedded. Let $f : X \to Y$ be a function which has a closed graph and is not continuous. Let Y^* be a compactification of Y. Then $f : X \to Y^*$ is subcontinuous (showing also the dependency of subcontinuity on the space in which the range is imbedded). Thus if $f : X \to Y^*$ had a closed graph then $f : X \to Y^*$ would be continuous. But continuity does not depend upon the space in which the range is imbedded and so $f : X \to Y$ would be continuous contradicting our assumption.

Assuming the range is embedded in a Hausdorff space, the next theorem says, roughly, that a function with a closed graph (in general) handles compact sets somewhat less successfully than a continuous function, but the inverse of a function of the former sort is just as successful as the inverse of a continuous function in this respect.

(3.6) Theorem: Let the function $f : X \to Y$ have a closed graph.

(a) If K is a compact subset of X, then $f[K]$ is a closed subset of Y.

(b) If K is a compact subset of Y, then $f^{-1}[K]$ is a closed subset of X.

Proof: We prove only (b), the proof of (a) being entirely analogous.

Let K be a compact subset of Y. Suppose $f^{-1}[K]$ is not closed. Then there is a p in $X - f^{-1}[K]$ and a net x_a in $f^{-1}[K]$ such that x_a converges to p. The net fx_a evidently has a subnet fx_{Nb} which converges to some q in K. Thus we have (x_{Nb}, fx_{Nb}) converges to (p, q) so that $p \in f^{-1}[q] \subset f^{-1}[K]$. But this contradicts the choice of p.

Remark: We note in passing that Theorem (3.6) also follows from an exercise in Kelley [20, Ex. A, p. 203].

Having already discovered in (3.4) conditions under which a continuous function has a closed graph, we now proceed to investigate conditions under which a closed function has a closed graph. The characteristic function of the interval $(0,1]$ mapping E_1 into $\{0,1\}$ shows us that a closed function does not always have a closed graph. In particular one should note that this function does not have closed point inverses (i.e., inverse image of each point in the range is closed).

Let us call a function $f : X \to Y$ locally closed if for every neighborhood U of each point p in X there is a neighborhood V of p such that $V \subset U$ and $f[V]$ is closed in Y. It is not clear that a closed function is always locally closed but if the domain of a closed function is regular then the function is locally closed. Also if a function $f : X \to Y$ is such that X is regular and locally compact and f maps compact sets onto closed sets, then f is locally closed.

A locally closed function need not be closed as the following example shows. Let X be the reals with the discrete topology, Y the reals with the usual topology, and $f : X \to Y$ the identity function. Then f is locally closed and in fact continuous but certainly not closed.

(3.7) <u>Theorem</u>: If $f : X \to Y$ is a locally closed function then $y_a \to q$ in Y implies $\lim\sup f^{-1}[y_a] \subset f^{-1}[q]^-$.

<u>Proof</u>: Let $y_a \to q$ in Y and $p \in \lim\sup f^{-1}[y_a]$. Suppose $p \notin f^{-1}[q]^-$. By Lemma (1.3) there is a net x_{Nb} in X such that $x_{Nb} \to p$ and fx_{Nb} is a subnet of y_a. Now $X - f^{-1}[q]^-$ is a neighborhood of p and thus there is a neighborhood of p, U, such that $U \subset X - f^{-1}[q]^-$ and $f[U]$ is closed in Y. But x_{Nb} is eventually in U, so that fx_{Nb} is eventually in $f[U]$. This means fx_{Nb} is eventually in the complement of a neighborhood of q, namely, $Y - f[U]$. We thus have a contradiction to the fact that fx_{Nb} must converge to q, being a subnet of y_a.

(3.8) <u>Corollary</u>: If $f : X \to Y$ is a locally closed function and has closed point inverses, then f has a closed graph.

<u>Proof</u>: The statement follows immediately from Theorem (1.4).

(3.9) <u>Corollary</u>: If the function $f : X \to Y$ is closed with closed point inverses and X is regular then f has a closed graph.

(3.10) Corollary: If the function $f : X \to Y$ is closed and subcontinuous with closed point inverses and X is regular, then f is continuous.

Proof: We have this result directly from (3.9) and Theorem (3.4).

The last corollary generalizes a theorem of Halfar's [16, Theorem 3] by replacing compactness of Y with subcontinuity of f. See also in this connection Rhoda Manning [24, Thm. 1.5].

(3.11) Theorem: If $f : X \to Y$ is a function where X is regular and locally compact, the following conditions are equivalent:

(a) f maps compact sets onto closed sets and has closed point inverses.

(b) f is locally closed and has closed point inverses.

(c) f has a closed graph.

Proof: We have already commented that (a) implies (b). By Corollary (3.8), (b) implies (c). Thus it remains to show (c) implies (a).

Assuming f has a closed graph, Theorem (3.6) gives us that f maps compact sets onto closed sets. Furthermore since points are compact, the same theorem yields that f has closed point inverses. Consequently (c) implies (a).

The following lemma which we prove in preparation for the next theorem is well known (see for example Dugundji [10, Problem 10, p. 96]) but we include it for completeness. This lemma supports further the feeling one gets that if one were going to define continuity of the inverse of a function for purposes of the present investigation, one would say that the inverse is continuous when the function is closed.

(3.12) Lemma: The function $f : X \to Y$ is closed if and only if for each q in Y and for each open set $U \supset f^{-1}[q]$ there is an open set $V \supset \{q\}$ such that $f^{-1}[V] \subset U$.

Proof: Assume the condition holds. Let C be a closed subset of X. Suppose $f[C]$ is not closed. Then there is a point q in $Y - f[C]$ which is an accumulation of $f[C]$. Since $X - C$ is an open set containing $f^{-1}[q]$, there is an open set $V \supset \{q\}$ such that $f^{-1}[V] \subset X - C$. But this is a contradiction since V must intersect $f[C]$.

Now assume f is closed. Let $q \in Y$ and U be an open set containing $f^{-1}[q]$. Then $f[X - U]$ is a closed set which does not contain q. Hence letting $V = Y - f[X - U]$, V is an open set containing q. Noting that

$$f^{-1}[V] = f^{-1}[Y] - f^{-1}[f[X - U]] \subset U$$

we have the desired condition.

The following theorem is essentially a special case of one to be found in Bergé [4, Theorem 3, p. 116]. However this fact is somewhat disguised in Bergé's terminology, and since the theorem is of interest in the present study, it is included (after all, isn't one of a mathematician's jobs to remove disguises?). The theorem extends a theorem of Halfar's [15, Theorem 1] by removing the requirement that f be continuous. (See also Michael [25, Lemma 5.18, p. 172].)

(3.13) <u>Theorem</u>: If $f : X \to Y$ is closed and has compact point inverses, then f is compact.

<u>Proof</u>: Let K be a compact subset of Y. Let $\mathcal{U}$ be an open cover of $f^{-1}[K]$. For each k in K, $f^{-1}[k]$ is compact and $\mathcal{U}$ is an open cover of $f^{-1}[k]$. Thus for each k in K there is a finite subcover of $\mathcal{U}$, say $\mathcal{U}(k)$, which covers $f^{-1}[k]$. Let $U(k) = \Sigma\mathcal{U}(k)$ (union). Then $U(k)$ is an open set containing $f^{-1}[k]$ so that there is an open set $V(k)$ containing k such that $f^{-1}[V(k)] \subset U(k)$.

Now $\{V(k) : k \in K\}$ is an open cover of K and thus there is a finite subcover $V(k_1),\ldots, V(k_n)$. Consequently $U(k_1),\ldots,U(k_n)$ is a cover of $f^{-1}[K]$. Finally $\Sigma\{\mathcal{U}(k_i) : i = 1,\ldots,n\}$ is a cover of $f^{-1}[K]$ which is a finite subcover of $\mathcal{U}$. Therefore $f^{-1}[K]$ is compact.

4. Compact Functions, Compact Preserving Functions, and k_i Spaces

It is intuitively clear that a mapping $f : X \to Y$ which is compact or compact preserving will have no particular tendency to treat other topological properties nicely unless the topologies of X or Y or both are to a considerable extent dictated by the compact sets. In this section we define some topological spaces which are, in a sense, determined by their compact sets, and prove a couple of theorems concerning these spaces and compact and compact preserving functions.

(4.1) <u>Definition</u>: Let X be a topological space and $p \in X$.

X is said to have <u>property k_1 at p</u> if and only if for each infinite subset A having p as an accumulation point, there is a compact subset, B, of $A + \{p\}$ such that $p \in B$ and p is an accumulation point of B. X is a <u>k_1 space</u> if it has property k_1 at each of its points.

X has <u>property k_2 at p</u> if and only if for each set A having p as an accumulation point, there is a subset B of A and a compact set $K \supset B + \{p\}$ such that p is an accumulation point of B. We call X a <u>k_2 space</u> if it has property k_2 at each of its points.

X is a <u>k_3 space</u> if and only if U is an open set in X precisely whenever $U \cdot K$ is open in K for each compact set K in X.

Halfar defines property k_1 at a point in one of his papers [16, Definition 2]. Property k_2 at a point is a slight variation of a definition I believe is due to S. B. Myers. A definition differing slightly from that of a k_3 space is discussed by Kelley in his book [20, p. 230]. If X is Hausdorff the k_2 and k_3 definitions agree with those of Myers and Kelley respectively.

It is immediate that X is k_1 at p implies X is k_2 at p. Also it is not difficult to show that a k_2 space is always a k_3 space, but I do not know whether k_2 space and k_3 space are equivalent concepts. It is easy to see that a locally compact or first countable space is a k_2 space. Finally the following example shows that X being k_2 at a point p does not necessitate X being k_1 at p.

(4.2) <u>Example</u>: This example provides a k_2 space which does not have property k_1 at any point.

Let F be the space of all functions mapping [0,1] into [0,1] with the topology of point-wise convergence. Let A be the collection of all finite subsets of $[0,1]$ and ω the set of positive integers. For each $a \in A$ and $n \in \omega$ let f_{na} be the function in F defined by $f_{na}x = 1/n$ for x in a, $f_{na}x = 1$ otherwise. Letting A be directed upward by inclusion, ω have the usual order, and $\omega \times A$ have the product order, $\{f_{na}, (n,a) \in w \times A\}$ is a net in F.

It is easy to see that f_{na} converges to the zero function, 0^*, and thus that 0^* is an accumulation point of $\{f_{na} : (n,a) \in \omega \times A\}$. We proceed to show that F lacks property k_1 at 0^* and it will then be clear that F has property k_1 at no point.

Let $P = \{f_{na} : (n,a) \in \omega \times A\}$ and let B be any subset of P which has 0^* as an accumulation point. Then $\{(n,a) : f_{na} \in B\}$ must be cofinal in $\omega \times A$ for if for any (n_o,a_o) there is no $(n_1,a_1) \geq (n_o,a_o)$ such that $f_{n_1a_1} \in B$ then there is no member of B in the neighborhood of 0^* given by $\{f \in F : |fx| < 1/n_o$ for $x \in a_o\}$.

Now let $f_{n_oa_o} \in B$. For every $k \in \omega$ choose $f_{n_ka_k} \in B$ such that $n_k > n_{k-1}$ and $a_k \supset a_{k-1}$. Consider $Q = \Sigma\{a_k : k \in \omega\}$. Let $f \in F$ be defined by $fx = 0$ for $x \in Q$, $fx = 1$ otherwise. The sequence $f_{n_ka_k}$ converges to f and $f \neq 0^*$ since Q is countable. Thus $B + 0^*$ is not closed and since F is Hausdorff it is consequently not compact.

Therefore F does not have property k_1 at 0^*, but on the other hand since by the Tychonoff Theorem F is compact, it is clearly a k_2 space.

The next theorem which we prove says that if a function $f : X \to Y$ is compact and has a closed graph with Y being a k_3 space then f is is closed. This theorem extends a theorem of Halfar [15, Theorem 2] by requiring f have only a closed graph instead of being continuous and by requiring that Y be only a k_3 space instead of locally compact Hausdorff.

(4.3) <u>Theorem</u>: Let $f : X \to Y$ be compact function and Y a k_3 space. A sufficient condition that f be closed is that f have a closed graph. If X is regular Hausdorff, the condition is also necessary.

<u>Proof</u>: Assume f has a closed graph. Let C be a closed subset of X. Since Y is a k_3 space it follows from the definition that to show $f[C]$ is closed we have only to show that the intersection of $f[C]$ with each compact set K in Y is closed in K.

Let K be a compact subset of Y. Then $f^{-1}[K]$ is compact, and it follows that $C \cdot f^{-1}[K]$ is compact. By Theorem (3.6) we have that $f[C \cdot f^{-1}[K]]$ is closed. Finally since

$$f[C \cdot f^{-1}[K]] = f[C] \cdot K$$

$f[C] \cdot K$ is closed in Y and thus in K.

Now assume that X is regular Hausdorff and f is closed. Then f being compact, point inverses are closed. Thus by Corollary (3.9), f has a closed graph.

Comparing Theorem (3.5) and the sufficiency portion of the one immediately preceding, (4.3), we see that in the former theorem "f is inversely subcontinuous" replaces "f is compact" and no requirements are put on the space Y. The example which follows show that the requirement that Y be a k_3 space cannot be dropped in Theorem (4.3). These observations illustrate that in some instances f being inversely subcontinuous is effectively a stronger requirement than f being compact.

(4.4) Example: We will display a function which is compact and has a closed graph but which is not closed.

Let X be an uncountable set. Let $\mathcal{U}_1$ be the topology on X consisting of all complements of countable sets (plus the empty set). Let $\mathcal{U}_2$ be the discrete topology on X.

First let us show that the only compact sets of $(X, \mathcal{U}_1)$ are the finite sets. Let $A = \{a_i : i = 1,2,...\}$ be a countable, non-finite set. Then $\{(X-A) + \{a_i\} : i = 1,2,...\}$ is an open cover of A which obviously has no finite subcover. Thus no countable non-finite set is compact. Suppose an uncountable set B were compact. Then since each countable set is closed, there would be countable, non-finite subsets of B which were compact. This contradiction leaves us with only finite subsets of $(X, \mathcal{U}_1)$ being compact.

Let A be a countable set. Then A is not open but $A \cdot K$ for any compact set K is open in K since K is finite and hence discrete ($(X, \mathcal{U}_1)$ being T_1, any accumulation point of a set B must have an infinite number of points of B in each neighborhood). Thus $(X, \mathcal{U}_1)$ is clearly not a k_3 space.

Now consider the identity map $i : (X, \mathcal{U}_2) \to (X, \mathcal{U}_1)$. The function i has a closed graph as we now show. Let $(x_a, ix_a) \to (p,q)$. Then x_a must eventually be the constant p in $(X, \mathcal{U}_1)$ and thus so must $ix_a = x_a$. If ix_a converged to $q \neq p$ then we would have contradiction to the evident fact that points are closed.

The inverse of i carries compact (finite) subsets of $(X, \mathcal{U}_1)$ onto compact (finite subsets of $(X, \mathcal{U}_2)$, and i is consequently compact. Finally, since $\mathcal{U}_2$ is a strictly larger topology than $\mathcal{U}_1$, i is not closed.

(4.5) Theorem: Let $f : X \to Y$ where X and Y are Hausdorff and X has property k_2 at a point p. If f is compact preserving and has closed point inverses, then f is continuous at p.

Proof: Suppose f is not continuous at p. Then there is a neighborhood V^* of fp such that for each neighborhood U of p there exists a point x_U in U with the property that $fx_U \notin V^*$. The collection $A = \{x_U : U$ is a neighborhood of $p\}$ has p as an accumulation point. Thus A has a subset B and a compact set $K \supset B + \{p\}$ such that p is an accumulation point of B.

Consider the function $f|K : K \to Y$. $f|K$ is strongly closed since Y is Hausdorff. As $(f|K)^{-1}[p] = f^{-1}[p] \cdot K$, $f|K$ has closed point inverses. The range of $f|K$ is compact and thus $f|K$ is subcontinuous. Finally, with the observation that K being compact Hausdorff is regular, we may conclude from Corollary (3.10) that $f|K$ is continuous.

However if we choose a net x_a in $B \subset K$ such that $x_a \to p$ then fx_a is never in the neighborhood V^* of fp. This contradiction proves the theorem.

The last theorem, (4.5), is a generalization of two theorems of Halfar [16, Theorems 2 and 5]. Halfar's Theorem 5 is the same as our theorem (4.5) except that Halfar requires X to have property k_1 at p instead of k_2 at p. Our example (4.2) shows then that Halfar's theorem 5 is not as strong as our (4.5) and in particular does not apply to all locally compact spaces.

5. Another Characterization of Continuity

In this section of the chapter we give a second characterization of continuity (The first occurred in Theorem (3.4)). This characterization was discovered while pursuing the question "How much must the usual net characterization of continuity (see Kelley [20, Theorem 1, p. 86]) be relaxed in order that something less than continuity be achieved?" One reasonable answer to this question is the subcontinuity condition (see Section 2). The principal theorem of this section seems to suggest this answer.

(5.1) Theorem: Let $f : X \to Y$ where Y is Hausdorff. The following conditions are equivalent:

(a) The function f is continuous.

(b) If $x_a \to p$ in X, then there is a q in Y such that $fx_a \to q$.

(c) If $x_a \to p$ in X, then there is a subnet x_{Nb} of x_a such that $fx_{Nb} \to fp$.

(d) For each p in X there is a q in Y such that $x_a \to p$ implies there is a subnet x_{Nb} of x_a such that $fx_{Nb} \to q$.

Note that (b), probably the most interesting equivalence, says that the usual net characterization of continuity is still valid even if it is not required that the image of a convergent net converge to the "right" value. In order to prove part (b) we will use the following lemmas which we state separately since they may be of interest in other applications of nets.

(5.2) <u>Lemma</u>: Let $(A, >_a)$ and $(B, >_b)$ be disjoint directed sets which are isomorphic (that is, there is a $1-1$ function h from A onto B such that $\alpha_1 >_a \alpha_2$ if and only if $h\alpha_1 >_b h\alpha_2$). Then there is a directed set $(C, >_c)$ such that A and B are cofinal subsets of C and $C = A + B$.

<u>Proof</u>: Let $C = A + B$ and define $>_c$ as follows: If $\gamma_1, \gamma_2 \in A$ then $\gamma_1 >_c \gamma_2$ if and only if $\gamma_1 >_a \gamma_2$, and similarly if $\gamma_1, \gamma_2 \in B$. If $\gamma_1 \in A$ and $\gamma_2 \in B$ then $\gamma_1 >_c \gamma_2$ if and only $h\gamma_1 >_b \gamma_2$ and $\gamma_2 >_c \gamma_1$ if and only if $\gamma_2 >_b h\gamma_1$.

In brief then, we order C by leaving the order on A and B the same and identifying points in A with their images in B.

The reflexivity of $>_c$ is inherited from $>_a$ and $>_b$. We check only the following three cases in the proof of the transitivity of $>_c$ (leaving the rest to the reader): given elements $\gamma_1 >_c \gamma_2$ and $\gamma_2 >_c \gamma_3$, (1) $\gamma_1, \gamma_2, \gamma_3 \in A$ (or B) (2) $\gamma_1, \gamma_3 \in A$ and $\gamma_2 \in B$, and (3) $\gamma_1, \gamma_2 \in A$ and $\gamma_3 \in B$. In the first case transitivity is inherited. In the second case $h\gamma_1 >_b \gamma_2 >_b h\gamma_3$ so that $\gamma_1 >_a \gamma_3$. In the third case $h\gamma_1 >_b h\gamma_2$ since h is an isomorphism and $h\gamma_2 >_b \gamma_3$ by definition. Thus $h\gamma_1 >_b \gamma_3$ so that $\gamma_1 >_c \gamma_3$.

Next we show that each two element subset of C has an upper bound. Let γ_1 and $\gamma_2 \varepsilon C$. If γ_1 and γ_2 both belong to A (or B). the desired result is obvious from inheritance. Hence assume $\gamma_1 \varepsilon A$, $\gamma_2 \varepsilon B$. Then there is a $\gamma_3 \varepsilon B$ such that $\gamma_3 >_b \gamma_2$, $\gamma_3 >_b h\gamma_1$. Thus $\gamma_3 >_c \gamma_2$ and $\gamma_3 >_c \gamma_1$ by definition.

We now have that $(C, >_c)$ is a directed set. Finally it is clear that A and B are cofinal in C.

(5.3) Lemma: Let X be a topological space and $\{x_a, a \varepsilon A\}$ and $\{z_b, b \varepsilon B\}$ nets in X with disjoint directed sets which are isomorphic. Then there is a net $\{\omega_c, c \varepsilon C\}$ in X such that x_a and z_b are subnets and $\omega_c \to p$ if and only if $x_a \to p$ and $z_b \to p$.

Proof: Let C be the directed set constructed in the previous lemma and define $\omega_c = x_c$ if $c \varepsilon A$, $\omega_c = z_c$ if $c \varepsilon B$. The assertions of the lemma are then clear.

Proof of Theorem (5.1): Since each of conditions (b), (c), (d) are clearly implied by continuity, we have only to show that each of these conditions implies continuity.

Assume condition (b) holds. Let $x_a \to p$. Then $fx_a \to q$ for some q in Y. Suppose $q \neq f(p)$. Let z_b be a net which is constantly p and whose directed set is disjoint and isomorphic to that of x_a. Then by Lemma (5.3) there is a net ω_c such that x_a and z_b are subnets and $\omega_c \to p$. However $fx_a \to q$ and $fz_b \to fp$. Since Y is Hausdorff, $f\omega_c$ cannot converge, and we have a contradiction.

Assume condition (d) holds. We will show condition (c) holds. Let $p \varepsilon X$ and q be the corresponding point assured by the condition.

We wish to show that $q = fp$. But this is evident for if x_a is a net constantly p then $x_a \to p$ and thus $fx_a \to fp$. Since Y is Hausdorff $q = fp$.

Finally we show condition (c) implies continuity. Suppose condition (c) holds and f is not continuous. Then there is a p in X and an open neighborhood V^* of fp such that for each neighborhood U of p there exists a point $x_U \in U$ such that $fx_U \notin V^*$. Now the function x_U defined on the set of neighborhoods of p directed downward by inclusion is a net converging to p. On the other hand, it is clear that for no subnet x_{Nb} of x_U is it true that $fx_{Nb} \to fp$. This contradiction shows that condition (c) implies continuity.

(5.4) Corollary to (5.1) (b) Let $f : X \to Y$ where $(Y, \mathcal{V})$ is a Hausdorff uniform space. If f maps convergent nets onto Cauchy nets then f is continuous.

Proof: Let $(h, Y^*, \mathcal{V}^*)$ be the Hausdorff completion of $(Y, \mathcal{V})$ where h is a uniform isomorphism of Y into Y^*. Note $h \circ f$ maps convergent nets onto Cauchy nets. Since Y^* is complete, Cauchy nets are convergent. Hence $h \circ f$ maps convergent nets onto convergent nets. Thus by (5.1), (b), $h \circ f$ is continuous. But then so is f since h is a homeomorphism.

The preceding corollary has been announced by Yu-Lee Lee [23].

6. Open Functions and Continuity of their Inverses

In a previous section we commented that saying that the many-valued inverse function of a closed function is continuous seemed quite appropriate.

On the other hand, one feels intuitively that openness of a function should be related to continuity of the inverse (in some sense). In this section we find a sense and a setting in which this is indeed the case for the set-valued inverse function.

(6.1) <u>Definition</u>: A function $f : X \to Y$ is <u>open</u> if and only if U is open in X implies $f[U]$ is open in Y.

(6.2) <u>Theorem</u>: If $f : X \to Y$ is a function, the following conditions are equivalent:

(a) The function f is open.

(b) $y_a \to q$ in Y implies $f^{-1}[q] \subset \lim\inf f^{-1}[y_a]$.

(c) $y_a \to q$ in Y implies $f^{-1}[q] \subset \lim\sup f^{-1}[y_a]$.

<u>Proof</u>: Assume (a) holds. Let $y_a \to q$ and p be in $f^{-1}[q]$. Suppose $p \notin \lim\inf f^{-1}[y_a]$. Then there is an open set U containing p such that frequently $f^{-1}[y_a] \cdot U = \phi$. Thus y_a is frequently outside $f[U]$. But this is absurd since $f[U]$ is a neighborhood of q. Thus (b) holds.

Clearly if (b) holds then (c) holds. Hence assume (c) holds. Suppose f is not open. Then there is an open set U in X and a net y_a in $Y - f[U]$ such that $y_a \to q$ for some q in $f[U]$. Thus $f^{-1}[q] \subset \lim\sup f^{-1}[y_a]$. Let $p \in f^{-1}[q] \cdot U$. Then U is a neighborhood of p and thus $f^{-1}[y_a] \cdot U$ is frequently nonempty. But this means y_a is frequently in $f[U]$ in contradiction to our choice. Consequently (a) holds.

If E_a is a net of sets in a topological space X such that $\lim\inf E_a = \lim\sup E_a = E$ (see paragraph preceding Lemma (1.3) for definition) then we say that the <u>limit of E_a</u> exists and write $\lim E_a = E$. Having made this definition the following theorem then follows from the previous theorem and Theorem (1.4).

(6.3) <u>Theorem</u>: The function $f : X \to Y$ is open and has a closed graph if and only if $y_a \to q$ in Y implies $\lim f^{-1}[y_a] = f^{-1}[q]$.

<u>Proof</u>: By the theorems cited in the paragraph preceding (6.3), f is open and has a closed graph if and only if $y_a \to q$ in Y implies

$$f^{-1}[q] \subset \liminf f^{-1}[y_a] \subset \limsup f^{-1}[y_a] \subset f^{-1}[q] .$$

Thus the theorem holds.

<u>Remark</u>: A theorem similar to the preceding one is given in Whyburn [31, Theorem (4.32), p. 130] for metric spaces.

Now let X be a locally bicompact space (i.e., a space which has a basis of open sets with compact closures). Let 2^X be the collection of all nonempty closed subsets of X. Consider all sets of the form $[U_1,\ldots,U_n;\ V_1,\ldots,V_m]$

$$= \{A \in 2^X : A \cdot U_i \neq \phi \text{ and } A \cdot V_j^- = \phi \text{ for } i = 1,\ldots,n \text{ and } j = 1,\ldots,m\}$$

where the U_i and V_j are open and have compact closure in X.

These sets form a base for a topology for 2^X which Mrówka calls the lbc topology. Mrówka proves in his paper [26, Theorem 4] that for a locally bicompact space X, convergence in 2^X relative to the lbc topology is the same as the convergence of sets previously described (i.e. when lim inf = lim sup). Since by Theorem (3.6), if $f : X \to Y$ has a closed graph then $f^{-1}[y]$ is closed for each y in Y, the following theorem follows from the preceding comments and Theorem (6.3).

(6.4) <u>Theorem</u>: Let $f : X \to Y$ be a function and X be locally bicompact. Let F be the set-valued function on Y defined by $Fy = f^{-1}[y]$. Let 2^X have the lbc topology. If f is open and has a closed graph then $F : Y \to 2^X$ is continuous. Conversely if f has closed point inverses and $F : Y \to 2^X$ is continuous then f is open and has a closed graph.

(6.5) <u>Theorem</u>: Let f be an open continuous function mapping a locally bicompact space X onto a Hausdorff space Y. Let

$$\mathcal{B} = \{f^{-1}[y] : y \in Y\}$$

and let $\mathcal{B}$ have the relativized l.b.c. topology. Then Y and $\mathcal{B}$ are topologically equivalent.

<u>Proof</u>: By Theorem (3.4), since f is continuous and Y is Hausdorff, f has a closed graph. Thus by the previous theorem $F : Y \to \mathcal{B}$ defined by $Fy = f^{-1}[y]$ is continuous. Thus it remains only to show that F^{-1} is continuous.

Let $f^{-1}[y_a]$ converge to $f^{-1}[q]$ in $\mathcal{B}$. Then $\limsup f^{-1}[y_a] = f^{-1}[q]$. By Lemma (1.3), if $p \in \limsup f^{-1}[y_a]$ there is a net x_{Nb} in X such that $x_{Nb} \to p$ in X and fx_{Nb} is a subnet of y_a. Now f is continuous and thus fx_{Nb} converges to $fp = q$. But $F^{-1}(f^{-1}[fx_{Nb}]) = fx_{Nb}$. Since $f^{-1}[fx_{Nb}]$ is a subnet of $f^{-1}[y_a]$, it follows by Theorem (5.1) that F^{-1} is continuous.

<u>Remark</u>: Theorems similar to the previous two theorems may be found in a paper of E. Michael [25] (see in particular Theorem 5.10.2).

CHAPTER III

CONCERNING CERTAIN FUNCTIONAL CONVERGENCES

In this chapter various modes of convergence of nets of functions will be investigated. In particular three Ascoli type theorems and a Dini type theorem are proved.

The first two Ascoli theorems are closely related to the topological-uniform and pure topological Ascoli theorems found in Kelley [20, Chapter 7]. While the theorems in Kelley consider the question of when a family of functions with a certain topology is compact, we consider the question of when a net of functions has a subnet which converges in a certain mode. This approach enables us to consider in this connection modes of convergence which are not topological but are, from a certain point of view, very natural. In this respect our motivation is very much the same as Cook and Fischer [7] with their convergence and uniform convergence structures. Also along these lines we should mention that Poppe [26] has generalized the topological Ascoli theorems in Kelley (due to Kelley and Morse) to the convergence space of Fischer.

The third Ascoli theorem is a uniform version. Another uniform version appears, for instance, in Isbell [17, p. 51].

Finally we characterize uniform convergence and uniform convergence at a point (for nets of continuous functions with a continuous limit) with some theorems which use extensions of the notion of monotone convergence in the classical Dini theorem.

7. Pseudo Even Continuity and Quite Continuous Convergence

In this section all spaces "X" and "Y" are topological spaces and "p" some point in X.

(7.1) Definition: Let F be a family of functions from X into Y and p belong to X. The family F is evenly continuous at p if and only if for each q in Y and each neighborhood V of q there are neighborhoods U of p and W of q such that f[U] is a subset of V whenever fp is in W and f is in F.

The family F is evenly continuous if it is evenly continuous at each point of X.

The preceding concept is used in the purely topological Ascoli theorem in Kelley [20, p. 234ff.]. We introduce the following concept for nets which is weaker than requiring a net to be evenly continuous (see Example (8.16)) but serves the same sort of purpose.

(7.2) Definition: Let $f_a : X \to Y$ be a net of functions. The net f_a is said to be pseudo evenly continuous at p if and only if for each net (f_{Nb}, x_b), where f_{Nb} is a subnet of f_a, x_b converges to p, and $f_{Nb}p$ converges to q, it is true that $f_{Nb}x_b$ converges to q.

The net f_a is pseudo evenly continuous if it is pseudo evenly continuous at each point of X.

(7.3) Theorem: A family of functions, F, is evenly continuous at p if and only if each net f_a in F is pseudo evenly continuous at p.

<u>Proof</u>: Assume F is evenly continuous and let f_a be a net in F. Let (f_{Nb}, x_b) be a net such that f_{Nb} is a subnet of f_a, $f_{Nb}p$ converges to q, and x_b converges to p. Let V be any neighborhood of q. By definition there is a neighborhood U of p and a neighborhood W of q such that $f[U]$ is a subset of V whenever fp is in W. Since x_b is eventually in U and $f_{Nb}p$ is eventually in W, it is true that eventually $f_{Nb}x_b$ is in V. Therefore $f_{Nb}x_b$ converges to q and f_a is pseudo evenly continuous at p.

Now assume each net in F is pseudo evenly continuous at p. Suppose F is not evenly continuous at p. Then there is a q in Y and a neighborhood V^* of q such that for every pair (U, W), where U is a neighborhood of p and W is a neighborhood of q, there is a function f_{UW} in F such that $f_{UW}p$ is in W and a point x_{UW} in U such that $f_{UW}x_{UW}$ is not in V^*.

Letting each of the neighborhoods U and the neighborhoods W be directed downward by inclusion and giving the pairs (U, W) the product order, f_{UW} is a net in F. Moreover $f_{UW}p$ converges to q and x_{UW} converges to p. Thus by definition $f_{UW}x_{UW}$ converges to q. But then $f_{UW}x_{UW}$ is eventually in V^* which is a contradiction. Therefore F is evenly continuous at p.

The preceding theorem and the definition of pseudo evenly continuous are suggested by a problem in Kelley [20, Problem L, p. 241]. Similarly the following definition and theorem (7.7) are closely related to another problem in Kelley [20, Problem M, p. 241].

(7.4) <u>Definition</u>: Let $f_a : X \to Y$ be a net of functions. We say that

f_a <u>converges continuously to f at p</u> if and only if whenever x_a is a net in X converging to p, $f_a x_a$ converges to fp.

The net f_a converges <u>quite continuously to f at p</u> if and only if each subnet of f_a converges continuously to f at p.

Finally f_a <u>converges continuously (quite continuously) to f</u> if and only if f_a converges continuously (quite continuously) to f at each point of X.

(7.5) <u>Lemma</u>: Let $f_a, f : X \to Y$ be functions. The net f_a converges to f quite continuously at p if and only if for each neighborhood V of fp there is a neighborhood U of p and an a_o such that $a \geq a_o$ implies $f_a[U] \subset V$.

<u>Proof</u>: Assume the condition holds. Let f_{Nb} be a subnet of f_a, $x_b \to p$, and V be a neighborhood of fp. There is then a neighborhood U of p such that $f_{Nb}[U]$ is eventually contained in V. Since x_b is eventually in U, it follows that $f_{Nb}x_b$ is eventually in V. Therefore f_a converges quite continuously to f at p.

Now assume f_a converges quite continuously to f at p. Suppose the condition does not hold. Then there is a neighborhood V^* of fp such that for every pair (a, U), where a is an index of f_a and U is a neighborhood of p, there is an index $N(a, U) \geq a$ and a point $x_{(a,U)} \in U$ such that $f_{N(a,U)}x_{(a,U)} \notin V^*$. Since $f_{N(a,U)}$ is a subnet of f_a and $x_{(a,U)} \to p$, it follows that $f_{N(a,U)}x_{(a,U)} \to fp$. Thus $f_{N(a,U)}x_{(a,U)}$ is eventually in V^* which is a contradiction. Therefore the condition holds.

(7.6) <u>Theorem</u>: Let f_a, $f : X \to Y$ where Y is regular. If f_a converges quite continuously to f at p then f is continuous at p.

<u>Proof</u>: Let V be a closed neighborhood of fp. By Lemma (7.5) there a neighborhood U of p such $f_a[U]$ is eventually contained in V. Thus for each x in U fx is in V since it is the limit of the net $f_a x$ which is eventually in the closed set V. Therefore f is continuous.

(7.7) <u>Definition</u>: Let F be a family of functions from X to Y and $\mathcal{J}$ be a topology for F. The topology $\mathcal{J}$ is said to be <u>jointly continuous</u> if and only if the function $P : F \times X \to Y$ defined by $P(f,x) = fx$ is continuous relative to the product topology on $F \times X$.

(7.8) <u>Theorem</u>: Let F be a family of functions from X to Y and $\mathcal{J}$ a topology for F. The topology $\mathcal{J}$ is jointly continuous if and only if f_a is a net in F which $\mathcal{J}$-converges to f implies f_a converges quite continuously to f.

<u>Proof</u>: Assume $\mathcal{J}$ is jointly continuous. Let f_a be a net in F which $\mathcal{J}$-converges to f. Let f_{Nb} be a subnet of f_a and x_b converge to p. Then (f_{Nb}, x_b) converges to (f, p) in the product topology of $F \times X$. Since by definition the map $P : F \times X \to Y$ defined by $P(g,x) = gx$ is continuous, it follows that $f_{Nb}x_b$ converges to fp in Y. Therefore f_a converges quite continuously to f.

Conversely assume $\mathcal{J}$-convergence of a net implies quite continuous convergence of the net. We wish to show the function P is continuous. Let (f_a, x_a) be a net in $F \times X$ converging to (f, p). Then

f_a $\mathcal{S}$-converges to f and hence f_a converges to f quite continuously. Thus $P(f_a, x_a) = f_a x_a$ converges to $fp = P(f, p)$. Therefore $\mathcal{S}$ is jointly continuous.

(7.9) Theorem: Let F be a family of functions from X to Y and $\mathcal{C}$ be the compact open topology for F. Let $f_a, f \in F$. If f_a converges to f quite continuously then f_a $\mathcal{C}$-converges to f. Conversely if X is regular, locally compact, f is continuous, and f_a $\mathcal{C}$-converges to f then f_a converges to f quite continuously.

Proof: Assume f_a converges to f quite continuously. Let $W(K,V) = \{g \in F : g[K] \subset V\}$ be a neighborhood of f where K is compact and V is open. If f_a is not eventually in $W(K,V)$, there exists for each a an index $Na \geq a$ and point x_a in K such that $f_{Na}x_a \notin V$. There is a subnet x_{Mb} of x_a which converges to some p in K. Thus $f_{NMb}x_{Mb} \to fp$ so that $f_{NMb}x_{Mb}$ is eventually in V. Consequently we have a contradiction.

Now assume X is regular locally compact, f is continuous, and f_a $\mathcal{C}$-converges to f. Let f_{Nb} be a subnet of f_a and $x_b \to p$. Let V be a neighborhood of fp. There is a compact neighborhood U of p such that $f[U] \subset V$. Thus f_{Nb} is eventually in $W(U, V)$, or $f_{Nb}[U]$ is eventually contained in V. Since x_b is eventually contained in U, $f_{Nb}x_b$ is eventually contained in V. Therefore f_a converges quite continuously to f.

As corollaries to Theorems (7.8) and (7.9) we have the following:

(7.10) <u>Corollary</u>: The compact open topology is smaller than each jointly continuous topology.

(7.11) <u>Corollary</u>: If F is a family of continuous functions from X to Y where X is locally compact regular, then the compact open topology for F is jointly continuous.

We also have from Theorem (7.9) that if F is a family of continuous functions from X into Y where X is regular locally compact, then quite continuous convergence of a net f_a in F to a member of F is a topological convergence. In fact, in this case the convergence is that of the compact-open topology. If X is not locally compact then Arens [2, section 5] shows that in general no smallest jointly continuous topology exists. Thus quite continuous convergence is in general not a topological convergence. This follows since by Theorem (7.8), if there were a topology $\mathcal{S}$ which induced quite continuous convergence, it would be jointly continuous and moreover smaller than any other jointly continuous topology.

The following theorem gives the simple relation which exists between quite continuous convergence and pseudo even continuity (Definition (7.2)) and provides the key to our Ascoli type theorem. The proof is so straight-forward it is omitted.

(7.12) <u>Theorem</u>: Let $f_a : X \to Y$ be a net of functions, and $f : X \to Y$ be a function. (a) If f_a converges to f point-wise at p and f_a is pseudo evenly continuous at p then f_a converges quite continuously to f at p. (b) Conversely, if Y is Hausdorff and f_a converges quite continuously to f at p, then f_a converges to f point-wise at p and f_a is pseudo evenly continuous.

(7.13) Definition: A net in a topological space is said to be subcompact if and only if each subnet has a cluster point.

Note in particular that a net which is contained in a compact set is subcompact.

(7.14) Theorem (Ascoli): Let $\{f_a, a \varepsilon A\}$ be a net of functions from X into Y where Y is Hausdorff. Then there exists a subnet of f_a which converges quite continuously to some function $f : X \to Y$ if and only if there is a subnet f_{Nb} of f_a which is pseudo evenly continuous and for each x in X, $f_{Nb}x$ is subcompact.

Proof: Assume f_{Nb} is pseudo evenly continuous and $f_{Nb}x$ is subcompact for each x in X. Let f_{NMc} be a universal subnet of f_{Nb} in Y^X (see Kelley [20, p. 81] or Gaal [14]). Since it is clear from definition that each subnet of a pseudo evenly continuous net is pseudo evenly continuous, f_{NMc} is pseudo evenly continuous. For each x, $f_{NMc}x$, being the image under the x^{th} projection of f_{NMc}, is a universal subnet. Thus since $f_{NMc}x$ has a cluster point, say fx, $f_{NMc}x$ converges to fx. Consequently we have that f_{NMc} converges pointwise to a function f. Therefore by (7.12) f_{NMc} converges quite continuously to f.

Conversely assume that a subnet f_{Nb} of f_a converges quite continuously to f. Then by (7.12) f_{Nb} is pseudo evenly continuous and for each x in X $f_{Nb}x$ converges. Thus $f_{Nb}x$ is subcompact for each

x in X.

(7.15) Corollary: Let $\{f_a, a \in A\}$ be a pseudo evenly continuous net of functions from X into a Hausdorff space Y and suppose for each x in X, $\{f_a x : a \in A\}$ has a compact closure. Then there is a subnet f_{Nb}, and a function $f : X \to Y$ such that f_{Nb} converges quite continuously to f.

8. Pseudo Equicontinuity and Uniform Convergence at a Point

Throughout this section X is a topological space, $(Y, \mathcal{V})$ a uniform space, and p some point in X.

(8.1) Definition: A net of functions $f_a : X \to (Y, \mathcal{V})$ is said to be pseudo equicontinuous at p if and only if for each V in $\mathcal{V}$ there is a neighborhood U of p and an a_o such that $x \in U$ and $a \geq a_o$ implies $(f_a x, f_a p) \in V$.

(8.2) Definition: A family of functions, F, from X to $(Y, \mathcal{V})$ is equicontinuous at p if and only if for each V in $\mathcal{V}$ there is a neighborhood U of p such that $x \in U$ implies $(fx, fp) \in V$ for all $f \in F$.

(8.3) Theorem: A family F of functions from X to $(Y, \mathcal{V})$ is equicontinuous at p if and only if each net in F is pseudo equicontinuous at p.

Proof: Clearly if F is equicontinuous at p then each net in F is pseudo equicontinuous at p.

Thus assume each net in F is pseudo equicontinuous at p. Suppose F is not equicontinuous at p. Then there is a V^* in $\mathcal{V}$ such that for every neighborhood U of p there is a function $f_U \varepsilon F$ and a point $x_U \varepsilon U$ such that $(f_U x_U, f_U p)$ does not belong to V^*.

But $\{f_U, U$ a neighborhood of $p\}$ is a net in F and consequently pseudo equicontinuous at p. Thus there is a neighborhood U_1 of p and a neighborhood U_2 of p such that $x \varepsilon U_1$, $U \subset U_2$ implies $(f_U x, f_U p) \varepsilon V^*$. In particular then $(f_{U_1 \cdot U_2} x_{U_1 \cdot U_2}, f_{U_1 \cdot U_2} p) \varepsilon V^*$ in contradiction to the choice of $f_{U_1 \cdot U_2}$ and $x_{U_1 \cdot U_2}$. Therefore F is equicontinuous at p.

(8.4) <u>Theorem</u>: If $\{f_n, n \geq 1\}$ is a sequence of functions from X to $(Y, \mathcal{V})$ which is pseudo equicontinuous at p and each of which is continuous at p, then $\{f_n : n \geq 1\}$ is equicontinuous at p.

<u>Proof</u>: Let $V \varepsilon \mathcal{V}$. Since f_n is pseudo equicontinuous there is an integer N and neighborhood U_N of p such that $x \varepsilon U_N$ and $n \geq N$ implies $(f_n x, f_n p) \varepsilon V$. Since each f_n is continuous, then there is a neighborhood U_n of p such that $x \varepsilon U_n$ implies $(f_n x, f_n p) \varepsilon V$ for $n = 1,\ldots,N-1$. Letting $U = \Pi \{U_n : n = 1,\ldots,N\}$ (intersection) then $x \varepsilon U$ implies $(f_n x, f_n p) \varepsilon V$ for all $n \geq 1$. Therefore $\{f_n : n \geq 1\}$ is equicontinuous at p.

Thus far we see that a net is pseudo equicontinuous at a point if, roughly, the final segments of the net come closer and closer to being equicontinuous at the point. Moreover under certain conditions appropriate pseudo equicontinuity at a point gives us equicontinuity at the point.

The next theorem tells us that when pseudo equicontinuity at point is defined the concepts of pseudo equicontinuity of a net at a point and pseudo even continuity of a net at a point coincide in the presence of pointwise convergence of the net.

(8.5) Theorem: Let $f_a : X \to (Y, \mathcal{V})$ be a net of functions and $f : X \to Y$.

(a) If f_a is pseudo equicontinuous at p then f_a is pseudo evenly continuous at p.

(b) If f_a is pseudo evenly continuous at p and f_a converges to f pointwise then f_a is pseudo equicontinuous at p.

Proof: (a) Assume f_a is pseudo equicontinuous at p. Let f_{Nb} be a subnet of f_a, $f_{Nb}p \to q$, and $x_b \to p$. Let $V[q]$ where $V \varepsilon \mathcal{V}$ be a neighborhood of q. There is a symmetric $V_1 \varepsilon \mathcal{V}$ such that $V_1 \circ V_1 \subset V$. By pseudo equicontinuity at p there is a neighborhood U of p such that for x in U, $(f_{Nb}x, f_{Nb}p)$ is eventually in V_1. Also clearly x_b is eventually in U and $(f_{Nb}p, q)$ is eventually in V_1. Thus $(f_{Nb}x_b, q)$ is eventually in V. Therefore f_a is pseudo evenly continuous at p.

(b) Assume f_a is pseudo evenly continuous at p and $f_ap \to fp$. Suppose f_a is not pseudo equicontinuous at p. Then there is a $V^* \varepsilon \mathcal{V}$ such that for each neighborhood U of p and each a there exists an index $N(a,U) \geq a$ and a point $x_{(a,U)}$ such that $(f_{N(a,U)}x_{(a,U)}, f_{N(a,U)}p) \notin V^*$. Now notice that $f_{N(a,U)}$ is a subnet of f_a, $f_{N(a,U)}p \to fp$, and $x_{(a,U)} \to p$. Hence $f_{N(a,U)}x_{(a,U)} \to fp$.

There is a symmetric $V_1 \varepsilon \mathcal{V}$ such that $V_1 \circ V_1 \subset V^*$. Since $(f_{N(a,U)}p, fp)$ and $(f_{N(a,U)}x_{(a,U)}, fp)$ are eventually in V_1, it follows that $(f_{N(a,U)}x_{(a,U)}, f_{N(a,U)}p)$ is eventually in V^*. This contradiction proves that f_a is pseudo equicontinuous at p.

(8.6) <u>Definition</u>: Let $f_a : X \to (Y, \mathcal{V})$ be a net of functions and $f : X \to Y$. The net <u>f_a converges to f uniformly at p</u> if and only if for each $V \in \mathcal{V}$ there is a neighborhood U of p and an index a_o such that $x \in U$ and $a \geq a_o$ implies $(f_a x, fx) \in V$.

(8.7) <u>Theorem</u>: Let $f_a : X \to (Y, \mathcal{V})$ be a net of functions and $f : X \to Y$. Then f_a converges quite continuously to f at p if and only if f_a converges to f uniformly at p and f is continuous at p.

<u>Proof</u>: Assume f_a converges to f quite continuously at p. By (7.6), since Y is completely regular f is continuous at p.

Let $V \in \mathcal{V}$. There is a symmetric $V_1 \in \mathcal{V}$ such that $V_1 \circ V_1 \subset V$. By (7.5) there is a neighborhood U_1 of p and an a_o such that $x \in U_1$ and $a \geq a_o$ implies $f_a x \in V_1[fp]$, that is, $(f_a x, fp) \in V_1$. Since f is continuous at p there is a neighborhood U_2 of p such that $x \in U_2$ implies $(fx, fp) \in V_1$. Thus if $x \in U_1 \cdot U_2$ and $a \geq a_o$ then $(f_a x, fx) \in V$. Therefore f_a converges to f uniformly at p.

Assume f_a converges to f uniformly at p and f is continuous at p. We show that the condition of Lemma (7.5) is satisfied and thus f_a converges to f quite continuously at p.

Let $V[fp]$ where $V \in \mathcal{V}$ is symmetric be a neighborhood of fp. There is a symmetric $V_1 \in \mathcal{V}$ such that $V_1 \circ V_1 \subset V$. There is a neighborhood U_1 of p and an index a_o such that $x \in U_1$ and $a \geq a_o$ implies $(f_a x, fx) \in V_1$. Finally there is a neighborhood U_2 of p such that $x \in U_2$ implies $(fx, fp) \in V_1$. Thus if $x \in U_1 \cdot U_2$ and $a \geq a_o$ then $(f_a x, fp) \in V$ or $f_a x \in V[fp]$. Therefore the condition of (7.5) is satisfied.

Remark: The concept of uniform convergence at a point for real-valued sequences of functions has been used to characterize pseudo-compact spaces. See for instance Bagley [3] or Iseki [18].

(8.8) Example: We now give an example which shows that a sequence of functions $f_n : X \to Y$ may converge uniformly at each point of X but still not converge uniformly on some neighborhood of each point of X.

Let X be the space of all real-valued sequences which converge to 0. For $\bar{x} = (x_1, x_2, \ldots)$ in X define $\ell(\bar{x}) = \sup \{|x_n| : n \geq 1\}$. The function ℓ is a norm on X.

Now define a sequence of functions $f_n : X \to X$ as follows: for each $n \geq 1$ let $f_n\bar{x} = (0,0,\ldots,x_n,x_{n+1},\ldots)$. Clearly f_n converges to $\bar{0} = (0,0,\ldots)$ pointwise on X.

First we show that f_n converges to $\bar{0}$ uniformly at each point of X. Let $e > 0$ and $\bar{x}_o \in X$ be given. Let $U = \{\bar{x} \in X : \ell(\bar{x} - \bar{x}_o) < e/2\}$. There is an N such that $n \geq N$ implies $\ell(f_n\bar{x}_o) < e/2$. Hence if $\bar{x} \in U$ and $n \geq N$ then

$$(f_n\bar{x}) \leq \ell(f_n\bar{x}_o) + \ell(f_n\bar{x} - f_n\bar{x}_o)$$

$$\leq \ell(f_n\bar{x}_o) + \ell(\bar{x} - \bar{x}_o) < e .$$

Therefore f_n converges to $\bar{0}$ uniformly at $\bar{x}_o$.

Now we show that it is false that f_n converges to $\bar{0}$ uniformly on a neighborhood of each point. Suppose it is true that f_n converges to $\bar{0}$ uniformly on a neighborhood of $\bar{0}$, say, the neighborhood

$$U = \{ \bar{x} \in X : \ell(\bar{x}) < d \}.$$

Then for each $e > 0$ there is a N_e such that $n \geq N_e$ implies $\ell(f_n \bar{x}) < e$ for $\bar{x} \in U$.

But consider $e = d/4$ and $\bar{x}^* = (x_1, x_2, \ldots)$ where $x_i = d/2$ for $i \leq N_{d/4} + 1$ and $x_i = 0$ for $i > N_{d/4} + 1$. Note that $\bar{x}^* \in U$. Note further that $\ell(f_{N_{d/4}} \bar{x}^*) = d/2 > d/4$. This contradiction shows that f_n does not converge to $\bar{0}$ uniformly on a neighborhood of $\bar{0}$.

(8.9) <u>Theorem</u>: Let $f_a : X \to (Y, \mathcal{V})$ be a net of functions and $f : X \to Y$. If f_a converges to f uniformly at each point of X then f_a converges to f uniformly on compacta.

<u>Proof</u>: Let K be a compact subset of X and $V \in \mathcal{V}$. For every x in K there is an open neighborhood U_x of x and an index a_x such that $z \in U_x$ and $a \geq a_x$ implies $(f_a z, fz) \in V$. Since $\{U_x : x \in K\}$ is an open cover of K there is a finite subcover $U_{x_1}, \ldots U_{x_n}$. There is an index a^* such that $a^* \geq a_{x_i}$ for $i = 1, \ldots, n$. Now let $a \geq a^*$ and $y \in K$. Then $y \in U_{x_i}$ for some i and thus $(f_a y, fy) \in V$. Therefore f_a converges to f uniformly on compacta.

(8.10) <u>Example</u>: This example shows that a net may converge uniformly on compacta but not converge uniformly at each point of the domain.

Let X be an uncountable set with the topology of countable complements as in example (4.4). Only finite sets are compact in X and thus to say a net of functions converges uniformly on compacta is the same as saying the net converges pointwise.

Since no finite sets are open, each point of X is an accumulation point of X. Let $p \varepsilon X$ and $\{x_a, a \varepsilon A\}$ be a net in $X - \{p\}$ which converges to p. We may assume $x_a \neq x_b$ for $a \neq b$ since X is a T_1 space. Define a net of functions f_a from X into $\{0, 1\}$ as follows: $f_a x = 1$ if $x \notin \{x_b : b \geq a\}$ and $f_a x = 0$ if $x \varepsilon \{x_b : b \geq a\}$.

Clearly f_a converges to 1 pointwise. On the other hand it is clear that f_a does not converge to f uniformly at p.

Having now shown that for a net $f_a : X \to (Y, \mathcal{V})$ the concepts of uniform convergence on a neighborhood of each point, uniform convergence at each point, and uniform convergence on compacta on occasion differ, we find in the next theorem that if X is locally compact, these concepts coincide.

(8.11) <u>Theorem</u>: Let $f_a : X \to (Y, \mathcal{V})$ be a net of functions and $f : X \to Y$. If X is locally compact the following assertions are equivalent:

(a) f_a converges to f uniformly on some neighborhood of each point of X.

(b) f_a converges to f uniformly at each point of X.

(c) f_a converges to f uniformly on compacta.

<u>Proof</u>: Clearly (a) implies (b). By Theorem (8.9) (b) implies (c). Since a neighborhood of each point of X is assumed to be compact, (c) implies (a).

(8.12) Theorem: If $f_a : X \to (Y, \mathcal{V})$ is a pseudo equicontinuous (pseudo evenly continuous) net of functions the following assertions are equivalent.

(a) f_a converges to f pointwise.

(b) f_a converges to f quite continuously.

(c) f_a converges to f uniformly at each point of X.

(d) f_a converges to f uniformly on compacta.

Proof: Since each of conditions (a) through (d) implies pointwise convergence, by Theorem (8.5) the overall hypotheses of pseudo evenly continuous or pseudo equicontinuous are equivalent. Condition (d) surely implies (a) and by (8.9) condition (c) implies (d). Further by (8.7) condition (b) implies (c). Finally by (7.12) condition (a) implies (b).

(8.13) Theorem: If $f_a : X \to (Y, \mathcal{V})$ converges to f uniformly at p, of the following conditions (a) and (d) are equivalent and (b) and (c) are equivalent. If in addition Y is Hausdorff all are equivalent.

(a) f is continuous at p.

(b) f_a is pseudo evenly continuous at p.

(c) f_a is pseudo equicontinuous at p.

(d) f_a converges to f quite continuously at p.

Proof: By remarks in (8.12) conditions (b) and (c) are equivalent. By (8.7) condition (a) holds if and only if (d) holds. Finally by (7.12) conditions (b) and (d) are equivalent if Y is Hausdorff.

(8.14) Theorem (Ascoli): Let $\{f_a, a \in A\}$ be a net of functions from X into $(Y, \mathcal{V})$. Then there exists a continuous function $f : X \to Y$ and a subnet of f_a which converges to f uniformly at each point of X if and only if there is a subnet of f_a, f_{Nb}, which is pseudo equicontinuous and for each x in X, $f_{Nb}x$ is subcompact.

Proof: Assume f_{Nb} is a subnet of f_a which is pseudo equicontinuous and for each x, $f_{Nb}x$ is subcompact. Let f_{NMc} be a universal subnet of f_{Nb}. Since for each x, $f_{NMc}x$ has a cluster point and is a universal net, f_{NMc} converges pointwise to some function $f : X \to Y$. But f_{NMc} is also pseudo equicontinuous so that by (8.12) and (8.13) f_{NMc} converges to f uniformly at each point of X and f is continuous.

Assume a subnet of f_a, f_{Nb}, converges to a continuous function f uniformly at each point of X. Evidently for each x, $f_{Nb}x$ is convergent and thus subcompact. By (8.13) f_a is pseudo equicontinuous.

(8.15) Corollary: Let $\{f_a, a \in A\}$ be a pseudo equicontinuous net of functions and suppose that for each x in X $\{f_a x : a \in A\}$ has a compact closure. Then there is a subnet f_{Nb} and a continuous function f such that f_{Nb} converges to f uniformly at each point of X.

Let us remark before closing this section that it is easily shown that if a net f_a converges to f uniformly at p and each f_a is continuous at p then f is continuous at p. Finally we give the following simple example.

(8.16) Example: This example provides a sequence which is pseudo equicontinuous (and thus pseudo evenly continuous) but is neither evenly continuous or equicontinuous.

Define $f_n x = 1/n$ for x in $[0, 1)$ and $f_n x = 2/n$ for x in $[1, 2]$. Then $\{f_n : n \geq 1\}$ is neither equicontinuous or evenly continuous since no f_n is continuous. On the other hand since f_n converges uniformly to the continuous function 0, by (8.13) f_n is pseudo equicontinuous.

9. Pseudo Uniform Equicontinuity and Uniform Convergence

In this section $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ will be uniform spaces, f_a a net of functions from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$, and f a function from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$.

(9.1) Definition: A family f of functions from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$ is uniformly equicontinuous if and only if for each $V \varepsilon \mathcal{V}$ there is a $U \varepsilon \mathcal{U}$ such that $(x, y) \varepsilon U$ implies $(fx, fy) \varepsilon V$ for every $f \varepsilon F$.

(9.2) Definition: A net f_a is pseudo uniformly equicontinuous if and only if for each $V \varepsilon \mathcal{V}$ there is a $U \varepsilon \mathcal{U}$ and an index a_o such that $(x, y) \varepsilon U$ and $a \geq a_o$ implies $(f_a x, f_a y) \varepsilon V$.

(9.3) Theorem: A family F of functions from $(X, \mathcal{U})$ to $(Y, \mathcal{V})$ is uniformly equicontinuous if and only if each net in F is pseudo uniformly equicontinuous.

Proof: If F is uniformly equicontinuous then surely each net in F is pseudo uniformly equicontinuous.

Hence assume each net in F is pseudo uniformly equicontinuous. Suppose F is not uniformly equicontinuous. Then there is a $V^* \varepsilon \mathcal{V}$ such that for each $U \varepsilon \mathcal{U}$ there a pair $(x_U, y_U) \varepsilon U$ and a function f_U such that $(f_U x_U, f_U y_U) \notin V^*$.

But f_U is a net in F with directed set $\mathcal{U}$ directed downward by inclusion. Thus there is a U_1, $U_2 \in \mathcal{U}$ such that $(x, y) \in U_1$ and $U \subset U_2$ implies $(f_U x, f_U y) \in V^*$. Consequently in particular $(f_{U_1 \cdot U_2} x_{U_1 \cdot U_2}, f_{U_1 \cdot U_2} y_{U_1 \cdot U_2}) \in V^*$ in contradiction to the choice of these objects. Therefore F is uniformly equicontinuous.

(9.4) <u>Theorem</u>: If f_a is pseudo uniformly equicontinuous and f_a converges to f point-wise then f is uniformly continuous.

<u>Proof</u>: Let V be any closed member of $\mathcal{V}$. There is a $U \in \mathcal{U}$ and an index a_o such that $(x, y) \in U$ and $a \geq a_o$ implies $(f_a x, f_a y) \in V$. Since $(f_a x, f_a y) \rightarrow (fx, fy)$ and V is closed, $(fx, fy) \in V$ for $(x, y) \in U$. Therefore f is uniformly continuous.

(9.5) <u>Theorem</u>: Let f_a converge uniformly to f. Then f is uniformly continuous if and only if f_a is pseudo uniformly equicontinuous.

<u>Proof</u>: If f_a is pseudo uniformly equicontinuous then by Theorem (9.4) f is uniformly continuous.

Hence assume f is uniformly continuous. Let $V \in \mathcal{V}$. There is a symmetric $V_1 \in \mathcal{V}$ such that $V_1 \circ V_1 \circ V_1 \subset V$. There is a $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $(fx, fy) \in V_1$. There is an index a_o such that $a \geq a_o$ implies $(f_a x, fx) \in V_1$ for all x in X.

Thus if $(x, y) \in U$ and $a \geq a_o$ then $(f_a x, fx)$, (fx, fy), and $(fy, f_a y) \in V_1$ so that $(f_a x, f_a y) \in V$. Therefore f_a is pseudo uniformly equicontinuous.

(9.6) <u>Definition</u>: A uniform space $(X, \mathcal{U})$ is said to be <u>totally bounded</u> if and only if for each $U \in \mathcal{U}$, there is a finite set of points of X, $x_1, \ldots, x_n$ such that

$$\Sigma \{U[x_i] : i = 1, \ldots, n\} = X .$$

(9.7) <u>Theorem</u>: Let f_a be a pseudo uniformly equicontinuous net and $(X, \mathcal{U})$ be totally bounded. Then f_a converges to f uniformly if and only if f_a converges to f pointwise.

<u>Proof</u>: Assume f_a converges to f pointwise. Let $V \in \mathcal{V}$. There is a symmetric $V_1 \in \mathcal{V}$ such that $V_1 \circ V_1 \circ V_1 \subset V$. There is a $U_1 \in \mathcal{U}$ and an index a_1 such that $(x, y) \in U_1$ and $a \geq a_1$ implies $(f_a x, f_a y) \in V_1$. Since by Theorem (9.4) f is uniformly continuous, there is a $U_2 \in \mathcal{U}$ such that $(x, y) \in U_2$ implies $(fx, fy) \in V_1$. Let U be a symmetric member of $\mathcal{U}$ such that $U \subset U_1 \cdot U_2$. Then as X is totally bounded there is a finite set of points of X say $x_1, \ldots, x_n$ such that $U[x_1], \ldots, U[x_n]$ covers X.

For a sufficiently large, say greater than a_0, $(f_a x_i, f x_i) \in V_1$ for $i = 1, \ldots, n$. Let a_2 be such that $a_2 \geq a_0$ and $a_2 \geq a_1$. Now let $x \in X$ and $a \geq a_2$. Then $x \in U[x_i]$ for some i or $(x, x_i) \in U$. Thus $(f_a x, f_a x_i)$, $(f_a x_i, f x_i)$, and $(f x_i, fx)$ all belong to V_1 so that $(f_a x, fx) \in V$. Therefore f_a converges to f uniformly.

(9.8) <u>Theorem (Ascoli)</u>: Let $f_a : (X, \mathcal{U}) \to (Y, \mathcal{V})$ be a net of functions and $(X, \mathcal{U})$ be totally bounded. Then there is a uniformly continuous function f and a subnet of f_a which converges uniformly to f if and only if there is a subnet of f_a, f_{Nb}, which is pseudo uniformly equicontinuous and for each x in X, $f_{Nb}x$ is subcompact.

Proof: Assume f_{Nb} is a pseudo uniformly equicontinuous subnet of f_a and for each x in X, $f_{Nb}x$ is subcompact. Let f_{NMc} be a universal subnet of f_{Nb}. Since by assumption $f_{NMc}x$ has a cluster point for each x in X, it follows that f_{NMc} converges point-wise to a function f. As f_{NMc} is evidently pseudo uniformly equicontinuous, by (9.4) we conclude f is uniformly continuous and by (9.7), f_{MNc} converges to f uniformly.

Conversely assume a subnet f_{Nb} of f converges uniformly to a uniformly continuous function f. Then $f_{Nb}x$ is evidently subcompact, and by (9.5) f_{Nb} is pseudo uniformly equicontinuous.

We close this section with the remark that it is well-known and easily shown that the limit of a uniformly convergent net of uniformly continuous functions is uniformly continuous.

10. Extensions of Monotone Convergence and Dini's Theorem

A more or less classical version of Dini's theorem would state that if f_n is a sequence of continuous functions from a closed, bounded subset of the reals to the reals which converges pointwise to a continuous function and is such that $f_{n+1}x \geq f_n x$ for each x, then f_n converges uniformly. An examination of the proof reveals that one does not need the sequence $f_n x$ increasing but the important point seems to be, calling the limit function f, that $f_{n+1}x$ be closer to fx than $f_n x$ for each n. We are led to the following generalizations of monotone convergence in a uniform space.

In this section X is a topological space, $(Y, \mathcal{V})$ a uniform space, f_a a net of functions from X to Y, f a function from X to Y, and $\mathcal{B}$ a base for $\mathcal{V}$.

(10.1) Definition: Let $\mathcal{B}$ be base for $\mathcal{V}$ and f_a converge to f pointwise.

(a) The net f_a converges to f monotonely relative to $\mathcal{B}$ ($f_a \to f\ (m, \mathcal{B})$) if and only if for each V in $\mathcal{B}$ and each x in X $(f_a x, fx) \varepsilon V$ and $a^* \geq a$ implies $(f_{a^*}x, fx) \varepsilon V$.

(b) The net f_a converges to f eventually monotonely relative to $\mathcal{B}$ ($f_a \to f(em, \mathcal{B})$) if and only if for each V in $\mathcal{B}$ there is an index $N(V)$ such that x in X, $a_1 \geq a_2 \geq N(V)$ and $(f_{a_2}x, fx) \varepsilon V$ implies $(f_{a_1}x, fx) \varepsilon V$.

(c) The net f_a converges to f eventually monotonely at each point relative to $\mathcal{B}$ ($f_a \to f\ (emp, \mathcal{B})$), if and only if for each V in $\mathcal{B}$ and each p in X there is an index $N(V, p)$ and a neighborhood U of p such that $a_1 \geq a_2 \geq N(V, p)$, $x \varepsilon U$ and $(f_{a_2}x, fx) \varepsilon V$ implies $(f_{a_1}x, fx) \varepsilon V$.

Note that if $V_e = \{(x, y) : |x - y| < e\}$ and $\mathcal{B} = \{V_e : e > 0\}$ then in the classical Dini Theorem f_n converges to f monotonely relative to $\mathcal{B}$.

(10.2) Theorem: If X is compact then $f_a \to f(em, \mathcal{B})$ if and only if $f_a \to f(emp, \mathcal{B})$.

Proof: Obviously $f_a \to f(em, \mathcal{B})$ implies $f_a \to f\ (emp, \mathcal{B})$.

Hence assume $f_a \to f\ (emp, \mathcal{B})$. Let $V \varepsilon \mathcal{B}$. Then for each p

in X there is an index $N(p)$ and an open neighborhood $U(p)$ of p such that $a_1 \geq a_2 \geq N(p)$, $x \in U(p)$, and $(f_{a_2}x, fx) \in V$ implies $(f_{a_1}x, fx) \in V$. Since $\{U(p) : p \in X\}$ is an open cover of X, there is a finite subcover $U(p_1),\ldots,U(p_n)$. Also there is a^* such that $a^* \geq N(p_i)$ for $i = 1,\ldots,n$.

Now suppose $x \in X$, $a_1 \geq a_2 \geq a^*$ and $(f_{a_2}x, fx) \in V$. Then $x \in U(p_i)$ for some i and thus $(f_{a_1}x, fx) \in V$. Therefore $f_a \to f(\text{em}, \mathcal{B})$.

(10.3) Theorem: Let f_a be a net of continuous functions and f a continuous function. The following conditions are equivalent:

(a) $f_a \to f$ uniformly at each point of X.

(b) $f_a \to f$ (emp, $\mathcal{V}$)

(c) $f_a \to f$ (emp, $\mathcal{B}$)

Proof: Condition (a) implies (b). Let $V \in \mathcal{V}$ and $p \in X$. There is an a_o and a neighborhood U of p such that $x \in U$ and $a \geq a_o$ implies $(f_a x, fx) \in V$. It is then clear that (b) follows.

Obviously (b) implies (c).

Assuming (c) holds we now show that (a) holds. Let $V \in \mathcal{B}$ and $p \in X$. There is a $V_1 \in \mathcal{B}$ and a symmetric $V_2 \in \mathcal{V}$ such that $V_1 \subset V_2$ and $V_2 \circ V_2 \circ V_2 \subset V$. There is a $N(V)$ and a neighborhood U_1 of p such that $x \in U_1$, $a_1 \geq a_2 \geq N(V)$ and $(f_{a_2}x, fx) \in V$ implies $(f_{a_1}x, fx) \in V_1$. There is an a_o such that $a \geq a_o$ implies $(f_a p, fp) \in V_1$. Now choose an index a^* such that $a^* \geq a_o$ and $a^* \geq N(V_1)$.

Then there is also a neighborhood U_2 of p such that $x \in U_2$ implies $(f_{a^*}x, f_{a^*}p) \in V_1$ and $(fx, fp) \in V_1$. Now if $x \in U_1 \cdot U_2$ then $(f_{a^*}x, f_{a^*}p)$, $(f_{a^*}p, fp)$, and $(fp, fx) \in V_2$ and thus

$(f_{a^*}x, fx) \in V$. But then if $a \geq a^*$ since $a^* \geq N(V)$, $(f_a x, fx) \in V$ for $x \in U_1 \cdot U_2$. Therefore $f_a \to f$ uniformly at each point of X.

(10.4) <u>Corollary</u>: If in addition to the hypothesis of (10.3), X is compact, then the following conditions are equivalent:

(a) $f_a \to f$ uniformly

(b) $f_a \to f$ (em, $\mathcal{V}$)

(c) $f_a \to f$ (em, $\mathcal{B}$).

<u>Proof</u>: By Theorem (8.11) condition (a) is equivalent to $f_a \to f$ uniformly at each point. Thus by Theorems (10.2) and (10.3) condition (a) is equivalent to (b) and (c).

CHAPTER IV

CONTRACTION AND RELATED SELF MAPS OF A UNIFORM SPACE

In this chapter various self maps of a uniform space which treat a base for the uniformity in a special way are studied.

It is hoped that its contents of section 11 shed additional light on the question, "If a self map of a uniform space is nonexpansive relative to some base, is there a base (and if so what sort) such that the map is invariant relative to the latter base."

Section 12 consists of a few simple theorems concerning when the pointwise limit of self maps is nonexpansive, invariant, or noncontractive relative to some base. A small application to real variables is given.

The last section of this chapter offers a Banach contraction principle for uniform space. Our main theorem of this section in fact generalizes two theorems for contraction maps in a metric space. One is, of course, Banach's contraction principle, and the other a sort of localized extension of Banach's principle due to Edelstein [11].

11. Nonexpansive Maps Which are Invariant

Our principal theorem of this section, (11.5), is very much along the lines of a theorem of Rhodes [29, Corollary 1, p. 402] and a theorem of Brown and Comfort [5, Theorem 2.1]. In addition to the fact that our theorem is valid in non-Hausdorff spaces, the main points of interest are the method of proof (reducing the uniform situation to a pseudometric situation and applying a similar theorem for pseudometrics) and the structure of the base relative to which the nonexpansive map is invariant (base induced by a family of pseudometrics).

(11.1) Definition: Let (X, d) be a pseudometric space and $f : X \to X$. (a) f is said to be {non-contractive} {invariant} {non-expansive} under d if and only if for x, y in X $\{d(fx, fy) \geq d(x,y)\}$ $\{d(fx, fy) = d(x, y)\}$ $\{d(fx, fy) \leq d(x,y)\}$. (b) f is said to be e-noncontractive e-invariant e-nonexpansive under d if and only if for $e > 0$ and x,y in X, $\{d(fx, fy) < e$ implies $d(fx,fy) \geq d(x,y)\}$ $\{d(x,y) < e$ implies $d(x,y) = d(fx, fy)\}$ $\{d(x,y) < e$ implies $d(fx, fy) \leq d(x, y)\}$.

Notes Freudenthal and Hurewicz [13] proved that a nonexpansive map of a totally bounded metric space onto itself is invariant. Edrei [12] supplemented this result with a localized version which states that an e-nonexpansive map of a totally bounded metric space onto itself is e-invariant.

(11.2) Definition: Let $(X, \mathcal{U})$ be a uniform space, $\mathcal{B}$ a base for $\mathcal{U}$, and $f : X \to X$. The function f is said to be {$\mathcal{B}$-noncontractive} {$\mathcal{B}$-invariant} {$\mathcal{B}$-nonexpansive} if and only if for (x, y) in $X \times X$ and U in $\mathcal{B}$ $\{(x,y) \in U$ if $(fx, fy) \in U\}$ $\{(x,y) \in U$ if and only if $(fx, fy) \in U\}$ $\{(x,y) \in U$ only if $(fx, fy) \in U\}$.

Notes: Rhodes proved that if f is $\mathcal{B}$-nonexpansive map of a totally bounded Hausdorff uniform space onto itself, there is a base $\mathcal{B}'$ such that f is $\mathcal{B}'$-invariant. Brown and Comfort by restricting the basis (open and ample) were able to prove that Rhodes' conclusion may be strengthened to "f is $\mathcal{B}$-invariant" (i.e., invariant relative to the original base $\mathcal{B}$).

First we extend the previously mentioned theorems for metric spaces to pseudometric spaces:

(11.3) <u>Theorem</u>: Let (X, d) be a totally bounded pseudometric space and f map X onto X.

(a) If f is nonexpansive then f is invariant

(b) If f is e-nonexpansive then f is e-invariant.

<u>Proof</u>: Consider the metric space (X', d') associated with (X, d) where $X' = \{\{x\}^- : x \in X\}$ and $d'(\{x\}^-, \{y\}^-) = d(x,y)$. Since X is totally bounded it is clear that X' is also. There is a function $f' : X' \to X'$ associated with f defined by $f'\{x\}^- = \{fx\}^-$. This function f' is well defined since if $d(x,y) = 0$ then $d(fx, fy) \leq d(x, y) = 0$ in either case (a) or (b). If (a) holds or if (b) holds and $d(x,y) < e$ then

$$d'(f'\{x\}^-, f'\{y\}^-) = d(fx, fy) \leq d(x, y) = d'(\{x\}^-, \{y\}^-).$$

Thus f' is nonexpansive or e-nonexpansive if (a) or (b) holds, respectively. Since it is clear that f' is onto, then, by Freudenthal's or Edrei's result we have f' is invariant or f' is e-invariant, respectively. But it then follows by the above inequality that f is invariant if (a) holds or f is e-invariant if (b) holds.

<u>Remark</u>: Edrei's theorem actually includes Freudenthal's: Let f be nonexpansive onto map on a totally bounded metric space of diameter less

than e. Then f is e-nonexpansive and thus by Edrei's theorem f is e-invariant. But in view of the choice of e this means f is invariant.

(11.4) <u>Theorem</u>: Let $(X, \mathcal{U})$ be a uniform space, $\mathcal{B}$ a base for $\mathcal{U}$, and f map X into X. If f is $\mathcal{B}$-nonexpansive then f is 1/2 nonexpansive under each member of a family of uniformly continuous pseudometrics which generates $\mathcal{U}$.

<u>Lemma</u>: If f is $\mathcal{B}$-nonexpansive then f is $\mathcal{B}'$-nonexpansive where $\mathcal{B}' = \{U \circ (U^{-1}) : U \in \mathcal{B} \text{ is symmetric.}\}$

<u>Proof of Lemma</u>: Let $U \in \mathcal{B}$. Then $(x, y) \in U^{-1}$ implies $(y,x) \in U$ which in turn implies $(fy, fx) \in U$ which finally implies $(fx,fy) \in U^{-1}$. Thus $(x,y) \in U \circ (U^{-1})$ implies $(fx, fy) \in U$ and $(fx, fy) \in U^{-1}$ or $(fx, fy) \in U \circ (U^{-1})$ completing the proof of our lemma.

<u>Proof of Theorem (11.4)</u>: We now assume without loss of generality that f is $\mathcal{B}$-nonexpansive where $\mathcal{B}$ is symmetric.

Let $U \in \mathcal{B}$. Let $U_0 = X \times X$, $U_1 = U$, and inductively choose $U_{n+1} \in \mathcal{B}$ such that $U^3_{n+1} \subset U_n$ for each positive integer n. (Recall $V^3 = V \circ V \circ V$, etc.). The sequence $\{U_n, n \geq 0\}$ satisfies the hypothesis of a lemma in Kelley [20, Lemma 12, p. 185], and thus there is a pseudometric d on X such that

$$U_n \subset \{(x, y) : d(x,y) < 2^{-n}\} \subset U_{n-1}$$

for $n \geq 1$. It then follows that d is uniformly continuous (see Kelley [20, Theorem 11, p. 183]) and $\{(x,y) : d(x,y) < 1/4\} \subset U$.

We now show that f is 1/2-nonexpansive under d. The pseudometric d is defined by

$$d(x, y) = \inf \left\{ \sum_{i=o}^{n} g(x_i, x_{i+1}) : x_o = x,\ x_{n+1} = y, \right.$$
$$\left. \text{and } x_i \in X \text{ for } 1 \leq i \leq n \right\}$$

where $g(x,y) = 2^{-n}$ if $(x,y) \in U_{n-1} - U_n$ and $g(x,y) = 0$ if $(x,y) \in U_n$ for every $n \geq 0$.

If $g(x,y) = 2^{-n}$ and $n \geq 2$, then $(x,y) \in U_{n-1}$. Hence $(fx, fy) \in U_{n-1}$ and thus $g(fx, fy) \leq 2^{-n}$. If $g(x,y) = 0$ then $(x,y) \in U_n$ for every $n \geq 0$. Hence $(fx, fy) \in U_n$ for every $n \geq 0$ and thus $g(fx, fy) = 0$. Therefore if $g(x,y) \leq 1/4$ then $g(fx,fy) \leq g(x,y)$.

Now suppose $d(x,y) < 1/2$. Let e be given such that $0 < e < (1/2) - d(x,y)$. Then there is a sequence $x_o, \ldots, x_{n+1}$ with $x_o = x$, $x_{n+1} = y$ such that

$$\sum_{i=o}^{n} g(x_i, x_{i+1}) < d(x,y) + e < 1/2$$

Thus $g(x_i, x_{i+1}) \leq 1/4$ for each i so that $g(fx_i, fx_{i+1}) \leq g(x_i, x_{i+1})$. Consequently

$$d(fx, fy) \leq \sum_{i=o}^{n} g(fx_i, fx_{i+1}) \leq \sum_{i=o}^{n} g(x_i, x_{i+1}) < d(x,y) + e$$

Therefore $d(fx, fy) \leq d(x, y)$ for $d(x, y) < 1/2$.

Taking one such pseudometric d for each U in $\mathcal{B}$, we obtain the desired family of pseudometrics.

(11.5) Theorem: Let $(X, \mathcal{U})$ be a totally bounded uniform space, $\mathcal{B}$ be a base for $\mathcal{U}$, and f map X onto X. If f is $\mathcal{B}$-nonexpansive, then f is 1/2-invariant under a family Q of uniformly continuous pseudometrics which generates $\mathcal{U}$. Consequently if $\mathcal{B}'$ consists of all finite intersections of sets of the form $V_{p,e}$ $(=\{(x,y) \in X \times X : p(x,y) < e\})$ where $0 < e \leq 1/2$ and $p \in Q$ then f is $\mathcal{B}'$-invariant.

Proof: By Theorem (11.4), f is 1/2-nonexpansive under a family Q of uniformly continuous pseudometrics which generates $\mathcal{U}$. Since $(X, \mathcal{U})$ is totally bounded, (X, d) is totally bounded for every d in Q. Thus by Theorem (11.3) it follows that f is 1/2-invariant under each d in Q.

12. Nets of Self Maps Whose Pointwise Limits Are Nonexpansive, Invariant, or Noncontractive

(12.1) Definition: Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{B}$ a base for $\mathcal{U}$.

$\mathcal{B}$ is ample if and only if $U \in \mathcal{B}$ and $(x,y) \in U$ implies there is a $W \in \mathcal{B}$ such that $(x, y) \in W$ and $W^{-} \subset U$.

$\mathcal{B}$ is coample if and only if $U \in \mathcal{B}$ and $(x, y) \notin U$ implies there is a $W \in \mathcal{B}$ such that $(x, y) \notin W$ and $W^{o} \supset U$.

Note: Ampleness of a base is a generalization of each member of the base being closed in the product topology of $X \times X$. Likewise coampleness of a base is a generalization of each member of the base being open.

(12.2) Definition: Let f_a be a net of functions from $(X, \mathcal{U})$ into $(X, \mathcal{U})$. The net f_a is said to be quasi $\{\mathcal{B}$-nonexpansive$\}$ $\{\mathcal{B}$-invariant$\}$ $\{\mathcal{B}$-noncontractive$\}$ if and only if for each U in $\mathcal{B}$ and x, y in X

there is an index a^* such that for $a \geq a^*$ $\{(x,y) \in U$ only if $(f_a x, f_a y) \in U\}$ $\{(x,y) \in U$ if and only if $(f_a x, f_a y) \in U\}$ $\{(x,y) \in U$ if $(f_a x, f_a y) \in U\}$.

(12.3) Theorem: Let f_a be a net of self maps of $(X, \mathcal{U})$ converging pointwise to f.

(a) If f_a is quasi $\mathcal{B}$-nonexpansive and $\mathcal{B}$ is ample then f is $\mathcal{B}$-nonexpansive.

(b) If $\mathcal{B}$ is open and f is $\mathcal{B}$-nonexpansive then f_a is quasi $\mathcal{B}$-nonexpansive.

(c) Thus if $\mathcal{B}$ is open and ample then f is $\mathcal{B}$-nonexpansive if and only if f_a is quasi $\mathcal{B}$-nonexpansive.

Proof: (a) Let $U \in \mathcal{B}$ and $(x,y) \in U$. There is a $W \in \mathcal{B}$ such that $(x,y) \in W$ and $W^- \subset U$. Also there is an index a^* such that $a \geq a^*$ implies $(f_a x, f_a y) \in W$. Thus since $(f_a x, f_a y) \to (fx, fy)$, $(fx, fy) \in W^-$. Therefore $(fx, fy) \in U$, and f is $\mathcal{B}$-nonexpansive.

(b) Let $U \in \mathcal{B}$ and $(x,y) \in U$. Then $(fx, fy) \in U$ and since $(f_a x, f_a y) \to (fx, fy)$ there is an a^* such that $a \geq a^*$ implies $(f_a x, f_a y) \in U$. Therefore f_a is quasi $\mathcal{B}$-nonexpansive.

(12.4) Theorem: Let f_a be a net of self maps of X converging pointwise to f.

(a) If f_a is quasi $\mathcal{B}$-noncontractive and $\mathcal{B}$ is coample, then f is $\mathcal{B}$-noncontractive.

(b) If $\mathcal{B}$ is closed and f is $\mathcal{B}$-noncontractive, then f_a is quasi $\mathcal{B}$-noncontractive.

(c) Thus if $\mathcal{B}$ is closed and coample, then f is $\mathcal{B}$-noncontractive if and only if f_a is quasi $\mathcal{B}$-noncontractive.

<u>Proof</u>: (a) Let $U \varepsilon \mathcal{B}$ and $(x,y) \notin U$. Then there is a $W \varepsilon \mathcal{B}$ such that $(x,y) \notin W$ and $W^o \supset U$. Also there is an index a^* such that $a \geq a^*$ implies $(f_a x, f_a y) \notin W$. Thus since $(f_a x, f_a y) \rightarrow (fx, fy)$, $(fx, fy) \notin W^o$. Therefore $(fx, fy) \notin U$ and f is $\mathcal{B}$-noncontractive.

(b) Let $U \varepsilon \mathcal{B}$ and $(x, y) \notin U$. Then $(fx, fy) \notin U$ and since $(f_a x, f_a y) \rightarrow (fx, fy)$ there is an a^* such that $a \geq a^*$ implies $(f_a x, f_a y) \notin U$. Therefore f_a is quasi $\mathcal{B}$-noncontractive.

(12.5) <u>Corollary</u>: Let f_a be a net of self maps of X converging pointwise to f. If f_a is quasi $\mathcal{B}$-invariant and $\mathcal{B}$ is ample and coample, then f is $\mathcal{B}$-invariant.

<u>Proof</u>: Since quasi $\mathcal{B}$-invariant implies quasi $\mathcal{B}$-nonexpansive, by (12.3) f is $\mathcal{B}$-nonexpansive. Similarly by (12.4) f is $\mathcal{B}$-noncontractive. Thus f is $\mathcal{B}$-nonexpansive.

(12.6) <u>Theorem</u>: Let f_a be a net of continuous self maps of X which converges pointwise to a function f which sends X onto X. If f_a is quasi $\mathcal{B}$-invariant and $\mathcal{B}$ is coample then f is $\mathcal{B}^-$-invariant.
$(\mathcal{B}^- = \{U^- : U \varepsilon \mathcal{B}\})$

<u>Proof</u>: Since each f_a is continuous, if $(x,y) \varepsilon U$ implies $(f_a x, f_a y) \varepsilon U$ then $(x,y) \varepsilon U^-$ implies $(f_a x, f_a y) \varepsilon U^-$. Thus by Theorem (12.3) since f_a is quasi $\mathcal{B}^-$nonexpansive, f is $\mathcal{B}^-$-nonexpansive.

By Theorem (11.4) f is $\mathcal{B}$-noncontractive and thus in particular

f is open. It then follows that the mapping (f, f) defined by $(f,f)(x,y) = (fx,fy)$ is open. Now suppose $(x,y) \notin U^-$ where $U \in \mathcal{B}$. Note $(f,f)[X \times X - U^-]$ is open and a subset of $X \times X - U$. But then a neighborhood of (fx, fy) misses U and thus $(fx, fy) \notin U^-$. Hence f is $\mathcal{B}^-$-noncontractive.

Therefore f is $\mathcal{B}^-$-invariant.

We close this section with an application of Theorem (12.3).

(12.7) <u>Theorem</u>: Let f_n be a sequence of functions with continuous first derivatives mapping an interval $[a, b]$ into itself. If f_n converges pointwise to a function f and

$$\lim \sup \left\{ \sup \left\{ |f_n' x| : x \in [a,b] \right\},\ n \geq 1 \right\} \leq 1$$

then $|fx - fy| \leq |x - y|$ for x, y in $[a, b]$.

<u>Proof</u>: For each $e > 0$ let $V_e = \{(x,y) : x, y \in [a,b] \text{ and } |x - y| < e\}$. Let $\mathcal{B} = \{V_e : e > 0\}$. It is easy to see that $\mathcal{B}$ is an ample base for the usual uniformity on $[a, b]$.

Let $V_e \in \mathcal{B}$ and $(x,y) \in V_e$ $(x \neq y)$. There is an N such that $n \geq N$ implies

$$\sup \left\{ |f_n' t| : t \in [a, b] \right\} < e/|x - y|$$

Thus by the mean value theorem

$$|f_n x - f_n y|/|x - y| = |f_n' z_{xy}^n| < e/|x - y|$$

for $n \geq N$. Consequently $(f_n x, f_n y) \in V_e$ for $n \geq N$. Hence f_n is quasi $\mathcal{B}$-nonexpansive and so by Theorem (12.3) f is $\mathcal{B}$-nonexpansive. But this means that $|fx - fy| \leq |x - y|$ for $x, y \in [a, b]$ which is what we were to prove.

13. A Banach's Contraction Principle for Uniform Space

Let us first mention some other extensions of Banach's principle to spaces more general than a metric space. In the ensuing discussion we will mean by a uniform space exactly that which we mean throughout this chapter, namely, a set X together with a collection of subsets of X x X which satisfy certain properties as are given in Kelley [20, p. 176]. By a gauge space we will mean a set X together with a collection of pseudometrics on X. Each uniform space has a gauge space associated with it, the collection of pseudometrics being all uniformly continuous pseudometrics on X (see Kelley [20, pp. 184-190]). Finally by a generalized metric space we will mean a set together with a function d which has the usual properties of a metric except that d takes on values in a certain partially ordered group instead of the reals (see Kalisch [19]). Kalisch associates a generalized metric space with each uniform space. This generalized metric space is itself constructed from the associated gauge space (or it could be constructed from a generating subfamily of pseudometrics).

Deleanu [9] has found that the classical Banach contraction principle in a metric space makes equally good sense in a generalized metric space and moreover can be proved in an analogous way. He applies his theorem to prove an existence and uniqueness theorem for locally-convex-vector-space-valued integral equations. Albrecht and Karrer [1] have theorems which assure a unique fixed point for a function in a generalized metric space under conditions somewhat more general than those of Deleanu. More recently Naimpally [27] has rediscovered the Banach contraction principle for generalized metric spaces.

Colojoară [6] gives a Banach contraction principle for a gauge space which except for a small innovation is a straightforward adaptation of the metric Banach theorem both in statement and in proof. Our theorem (13.13) is somewhat similar to Colojoară's.

Davis [8] in a space somewhat more general than a uniform space defines a contraction map (a rather more difficult trick than in the preceding cases) and proves a fixed point theorem for such a map on a well-chained, sequentially complete space. Davis' definition of a contraction map helped inspire ours.

Finally Knill [21] with a definition of contraction map different from Davis' proves a fixed point theorem for a sequentially complete, well-chained uniform space.

Now we come to the question of what our theorem has to offer that the theorems we have just mentioned do not. With regard to all but Davis' theorem and Knill's theorem, the answer is that their theorems apply in a uniform space only with those bases for the uniformity which are generated by generalized metrics or families of pseudometrics. In the cases of Davis and of Knill, Banach's contraction principle is, strictly speaking, not generalized since the space must be well-chained.

Moreover our proof gives insight into a connection which exists between a contraction map in uniform space in our sense and a contraction map in a gauge space (or generalized metric space. See comments at the last of the first paragraph of this section.). The sense in which we mean contraction map in a gauge space is the more or less evident one; namely, that for each member of the collection of pseudometrics or at least a generating subfamily the map in question is a contraction map in the sense of the classical Banach theorem.

Having, perhaps, justified the existence of our theorem, we proceed with its development.

(13.1) Definition: Let $(X, \mathcal{U})$ be a uniform space, $\mathcal{B}$ a base for $\mathcal{U}$, and $f : X \to X$. Then f is said to be r/s $\mathcal{B}$-contractive if and only if r and s are positive integers with $r < s$ and for each U_o in $\mathcal{B}$ there is a U_1 in $\mathcal{B}$ such that $U_o \supset U_1$, $U_o^r \supset U_1^s$, and $(f,f)[U_o] \subset U_1$.

Remark: An r/s $\mathcal{B}$-contractive map may be thought of as being somewhat like a map f in a metric space (X, d) which satisfies $d(fx, fy) \leq (r/s) \cdot d(x, y)$ for all x,y in X.

Another Remark: It is easy to show that a map f is an r/s map relative to some base $\mathcal{B}$ in the sense of Davis [8, p. 982] (i.e., if $(f,f)[V^s] \subset V^r$ for each V in $\mathcal{B}$) if f is an r/s $\mathcal{B}$-contractive map in our sense.

Before proceeding further let us outline our plan of attack for arriving at the main theorem of this section, Theorem (13.14). The lemmas and definitions (not the theorems) numbered (13.1) to (13.9) put us in a position to state and prove Lemma (13.10) which says that a certain (contracting) condition on a map f in a uniform space is equivalent to a condition on f in the associated gauge space. We then prove a gauge space Banach contraction theorem, Theorem (13.13), whose hypotheses are slightly weaker than the condition mentioned in the preceding sentence. Our main theorem then follows immediately.

(13.2) Lemma: Let f be an r/s $\mathcal{B}$-contractive map.

(a) If $r \leq r' < s' \leq s$, then f is r'/s' $\mathcal{B}$-contractive.

(b) If m is a positive integer then f is mr/ms $\mathcal{B}$-contractive

(c) f^n is r^n/s^n $\mathcal{B}$-contractive for each positive integer n

where f^n is the n^{th} iterate of f.

Proof: Let $U_o \in \mathcal{B}$. Then there is a $U_1 \in \mathcal{B}$ such that $U_o \supset U_1$, $U_o^r \supset U_1^s$ and $(f,f)[U_o] \subset U_1$. For part (a) we then note that $U_o^{r'} \supset U_o^r \supset U_1^s \supset U_1^{s'}$ and thus f is r'/s' $\mathcal{B}$-contractive. For part (b) we note that $(U_o^r)^m \supset (U_1^s)^m$ and thus f is mr/ms $\mathcal{B}$-contractive.

For part (c) we proceed by induction on n, noting the assertion is clear for $n = 1$. Assume it is true for $n \leq k$. Let $U_o \in \mathcal{B}$. There is a $U_1 \in \mathcal{B}$ such that $U_o \supset U_1$, $U_o^{r^k} \supset U_1^{s^k}$, and $(f^k, f^k)[U_o] \subset U_1$. There is a $U_2 \in \mathcal{B}$ such that $U_1 \supset U_2$, $U_1^r \supset U_2^s$ and $(f,f)[U_1] \subset U_2$. Thus $U_o \supset U_2$,

$$U_o^{r^{k+1}} = (U_o^{r^k})^r \supset (U_1^{s^k})^r = (U_1^r)^{s^k} \supset U_2^{s^{k+1}} ,$$

and

$$(f^{k+1}, f^{k+1})[U_o] \subset (f,f)[U_1] \subset U_2 .$$

Therefore f^{k+1} is r^{k+1}/s^{k+1} $\mathcal{B}$-contractive.

<u>Remark</u>: Davis [8] has lemmas which are similar to the preceding and the following lemma.

(13.3) <u>Lemma</u>: Let f be r/s $\mathcal{B}$-contractive and $t > 1$ be an integer. Then there is an $n \geq 1$ and a base $\mathcal{B}' = \{U^{r^n} : U \in \mathcal{B}\}$ such that f^n is $1/t$ $\mathcal{B}'$-contractive.

<u>Proof</u>: Choose n so that $tr^n < s^n$. Let $\mathcal{B}' = \{U^{r^n} : U \in \mathcal{B}\}$. By Lemma (13.2) f^n is r^n/s^n $\mathcal{B}$-contractive. Further since $r^n < tr^n \leq s^n$, f^n is r^n/tr^n $\mathcal{B}$-contractive. Now let $U_o^{r^n} \in \mathcal{B}'$. Then there is a $U_1 \in \mathcal{B}$ such that $U_o^{r^n} \supset U_1^{tr^n}$, $U_o \supset U_1$, and $(f^n, f^n)[U_o] \subset U_1$. (Note $U_o^{r^n} \supset U_1^{r^n}$).

Observe that $(f,f)[U] = G(f) \circ U \circ G(f)^{-1}$ and $G(f)^{-1} \circ G(f) \supset \Delta$ where $G(f)$ is the graph of f, $U \subset X \times X$ and $\Delta = \{(x, x) : x \in X\}$. Thus

$$U^{r^n} \supset \{(f^n, f^n)[U_o]\}^{r^n}$$

$$= [G(f^n) \circ U_o \circ G(f^n)^{-1}] \circ [G(f^n) \circ U_o \circ G(f^n)^{-1}] \circ \ldots \circ$$

$$[G(f^n) \circ U_o \circ G(f^n)^{-1}] \quad (r^n \text{ times})$$

$$\supset G(f^n) \circ U_o^{r^n} \circ G(f^n)^{-1} = (f^n, f^n)[U_o^{r^n}] .$$

Therefore f^n is $1/t$ $\mathcal{B}'$-contractive.

(13.4) <u>Definition</u>: Let $(X, \mathcal{U})$ be a uniform space, $\mathcal{B}$ a base for $\mathcal{U}$, and $f : X \to X$. $\mathcal{B}$ is said to be an <u>r/s inclusive base for f</u> if and only if for each $(x,y) \in X \times X$ and for each $U \in \mathcal{B}$ there is a $U_o \in \mathcal{B}$

such that $U_o \supset U$ and $(x,y) \in U_o$ and there is a sequence $U_o,\ldots,U_n$ in $\mathcal{B}$ such that $U_i^r \supset U_{i+1}^s$, $U_i \supset U_{i+1}$, and $(f,f)[U_i] \subset U_{i+1}$ for $i = 0,\ldots,n-1$ and $U_n \subset U$.

The following two theorems give situations in which a contractive map automatically has an inclusive base to go with it. Such situations are of interest since the hypotheses of our contraction principle require both a contractive map and an inclusive base.

(13.5) <u>Theorem</u>: Let $(X, \mathcal{U})$ be a uniform space and $\mathcal{P}$ a family of pseudometrics which generates $\mathcal{U}$. Suppose $d(fx, fy) \le (1/t)\, d(x,y)$ for (x, y) in $X \times X$ and d in $\mathcal{P}$ where $t \ge 2$ is an integer. Let $\mathcal{B}$ be the base for $\mathcal{U}$ generated by $\mathcal{P}$. Then f is $1/t$ $\mathcal{B}$-contractive and $\mathcal{B}$ is an $1/t$ inclusive base for f.

<u>Proof</u>: $\mathcal{B}$ consists of all finite intersections of sets of the form $V_{d,e} = \{(x,y) \in X \times X : d(x,y) < e\}$ where $d \in \mathcal{P}$ and $e > 0$. Let $U_o = \Pi \{V_{d_i,e_i} : i = 1,\ldots,n\} \in \mathcal{B}$. Consider $U_1 = \Pi \{V_{d_i,(1/t)e_i} : i = 1,\ldots,n\}$. If $(x,y) \in U_o$ then $d_i(fx, fy) < (1/t)\, e_i$ and thus $(fx, fy) \in U_1$. Since also clearly $U_o \supset U_1$ and $U_o \supset U_1^t$ it follows that f is $1/t$ $\mathcal{B}$-contractive.

To show that $\mathcal{B}$ is $1/t$ inclusive for f, let

$$U = \Pi \{V_{d_i,e_i} : i = 1,\ldots,n\}$$

belong to $\mathcal{B}$ and $(x,y) \in X \times X$. There is a positive number $e > 0$ such that $(x,y) \in \Pi \{V_{d_i,e} : i = 1,\ldots,n\}$ and a positive integer m such that $(1/t)^m e \le e_i$ for $i = 1,\ldots,n$. Define

$$U_k = \Pi \left\{ V_{d_i,(1/t)^k e} \ : \ i = 1,\dots,n \right\} \text{ for } k = 0,\dots,m .$$

Then $U_k \supset U_{k+1}$, $U_k \supset U_{k+1}^t$, and $(f,f)[U_k] \subset U_{k+1}$ for $k = 0,\dots,m-1$. Moreover $U_m \subset U$. Therefore $\mathcal{B}$ is $1/t$ inclusive for f.

(13.6) Theorem: If $(X, \mathcal{U})$ is well-chained (i.e., for each $(x,y) \in X \times X$ and for each $U \in \mathcal{U}$ there is a positive integer n such that $(x,y) \in U^n$) and f is $1/t$ $\mathcal{B}$-contractive then $\mathcal{B}' = \{U^n : n \geq 1, U \in \mathcal{B}\}$ is a $1/t$ inclusive base for f and f is $1/t$ $\mathcal{B}'$-contractive.

Proof: To show that f is $1/t$ $\mathcal{B}'$-contractive, let $U_o^n \in \mathcal{B}'$. There is a $U_1 \in \mathcal{B}$ such that $U_o \supset U_1$, $U_o \supset U_1^t$, and $(f,f)[U_o] \subset U_1$. Thus $U_o^n \supset U_1^n$, $U_o^n \supset (U_1^n)^t$ and (see Lemma (13.3))

$$(f,f)[U_o^n] = G(f) \circ U_o^n \circ G(f)^{-1}$$

$$\subset G(f) \circ U_o \circ G(f)^{-1} \circ G(f) \circ U_o \circ \dots \circ U_o \circ G(f)^{-1} \text{ (n times)}$$

$$= \{(f,f)[U_o]\}^n \subset U_1^n .$$

Now to see that $\mathcal{B}'$ is a $1/t$ inclusive base for f, let $U^n \in \mathcal{B}'$ and $(x,y) \in X \times X$. There is a positive integer m such that $(x,y) \in U^{nt^m}$. Let $V_{m-k} = U^{nt^k}$ for $k = 0,\dots,m$. Then $(x,y) \in V_o$, $V_p \supset V_{p+1}$, $V_p \supset (V_{p+1})^t$ and $(f,f)[V_p] \subset V_{p+1}$ for $p = 0,\dots,m$.

(13.7) Lemma: Let f be an r/s $\mathcal{B}$-contraction and $\mathcal{B}$ an r/s inclusive base for f.

(a) If $r \leq r' < s' \leq s$ then f is r'/s' $\mathcal{B}$-contractive and

$\mathcal{B}$ an r'/s' inclusive base for f.

(b) If m is a positive integer then f is mr/ms $\mathcal{B}$-contractive and $\mathcal{B}$ is an mr/ms inclusive base for f.

(c) If n is a positive integer f^n is r^n/s^n $\mathcal{B}$-contractive and $\mathcal{B}$ an r^n/s^n inclusive base for f^n.

Proof: In view of Lemma (13.2), (a) and (b) are clear.

We proceed then to prove (c). By Lemma (13.2), f^n is r^n/s^n $\mathcal{B}$-contractive. Let $(x,y) \in X \times X$ and $U \in \mathcal{B}$. Since $\mathcal{B}$ is an r/s inclusive base for f and f is r/s $\mathcal{B}$-contractive, there is a $U_o \in \mathcal{B}$ and a positive integer of the form kn such that $(x,y) \in U_o$, $U_o \supset U$, and there is a sequence $U_o,\ldots,U_{kn}$ in $\mathcal{B}$ such that $U_i^r \supset U_{i+1}^s$ $U_i \supset U_{i+1}$, $(f,f)[U_i] \subset U_{i+1}$, and $U_{kn} \subset U$.

Consider the sequence $\{V_i, i = 0,\ldots,k\}$ defined by $V_o = U_o$ and $V_i = U_{in}$ for $i = 1,\ldots,k$. One may see that

$$(f^n,f^n)[V_o] \subset (f^{n-1},f^{n-1})[U_1] \subset (f^{n-2},f^{n-2})[U_2]$$

$$\subset \ldots \subset (f,f)[U_{n-1}] \subset U_n = V_1 \ .$$

Similarly $(f^n,f^n)[V_i] \subset V_{i+1}$ for $i = 1,\ldots,k-1$. Now notice that

$$V_o^{r^n} = U_o^{r^n} \supset (U_1^s)^{r^{n-1}} = (U_1^r)^{sr^{n-2}} \supset (U_2^s)^{sr^{n-2}}$$

$$\supset \ldots \supset U_{n-1}^{s^{n-1}r} \supset U_n^{s^n} = V_1^{s^n} \ .$$

Likewise $V_i^{r^n} \supset V_{i+1}^{s^n}$ for $i = 1,\ldots,k-1$. Therefore $\mathcal{B}$ is an r^n/s^n inclusive base for f^n.

(13.8) Lemma: Let f be r/s $\mathcal{B}$-contractive and $\mathcal{B}$ an r/s inclusive base for f. Then there is a positive integer n and a base $\mathcal{B}' = \{U^{r^n} : U \in \mathcal{B}\}$ such that f^n is $1/3$ $\mathcal{B}'$-contractive and $\mathcal{B}'$ is a $1/3$ inclusive base for f^n.

Proof: By Lemma (13.3) there is a positive integer n such that f^n is $1/3$ $\mathcal{B}'$-contractive. Recall n is chosen so that $r^n < 3r^n \leq s^n$. Thus by Lemma (13.7) f^n is $r^n/3r^n$ $\mathcal{B}'$-contractive and $\mathcal{B}'$ is an $r^n/3r^n$ inclusive base for f^n.

Let $(x,y) \in X \times X$ and $U^{r^n} \in \mathcal{B}'$. There is U_0 in $\mathcal{B}$ such that $U_0 \supset U$, $(x,y) \in U_0$, and a sequence $U_0,\ldots,U_{kn}$ in $\mathcal{B}$ such that $U_i \supset U_{i+1}$, $U_i^r \supset U_{i+1}^s$, $(f,f)[U_i] \subset U_{i+1}$, and $U_{kn} \subset U$ for $i = 0,\ldots,kn-1$. Consider the sequence $\{U_{in}^{r^n},\ i = 0,\ldots,k\}$. It is true that $(x,y) \in U_0 \subset U_0^{r^n}$, $U_0^{r^n} \supset U^{r^n}$, and $U_{kn}^{r^n} \subset U^{r^n}$. Further

$$U_{in}^{r^n} = (U_{in}^r)^{r^{n-1}} \supset (U_{in+1}^s)^{r^{n-1}} \supset \ldots \supset U_{in+n}^{s^n} \supset U_{n(i+1)}^{3r^n} .$$

Moreover $(f^n, f^n)[U_{in}^{r^n}] \subset U_{n(i+1)}^{r^n}$ by an argument in Lemma (13.3). Therefore $\mathcal{B}'$ is a $1/3$ inclusive base for f^n.

(13.9) Definition: Let (X, d) be a pseudometric space and $f : X \to X$. Let e be a positive number and $\lambda < 1$. Then f is said to be (e, λ) locally uniformly contractive if and only if $d(p, q) < e$ implies $d(fp, fq) \leq \lambda d(p, q)$.

Remark: The preceding definition is due to Edelstein [11].

(13.10) Lemma: Let $(X, \mathcal{U})$ be a uniform space, $\mathcal{B}$ a base for $\mathcal{U}$, and f a self-map of X. If f is a $1/3$ $\mathcal{B}$-contractive map and $\mathcal{B}$ is a $1/3$ inclusive base for f, then there is a symmetric base $\mathcal{B}'$ for $\mathcal{U}$ such that f is a $1/3$ $\mathcal{B}'$-contractive map and $\mathcal{B}'$ is a $1/3$ inclusive base for f.

<u>Proof</u>: Let $\mathcal{B}' = \{U \circ (U^{-1}) : U \varepsilon \mathcal{B}\}$. Note that $\mathcal{B}'$ is a symmetric base for $\mathcal{U}$.

We first show that f is $1/3$ $\mathcal{B}'$-contractive. Let $U \cdot (U^{-1}) \varepsilon \mathcal{B}'$. There is a $U_1 \varepsilon \mathcal{B}$ such that $U \supset U_1^3$ and $(f,f)[U] \subset U_1$. Observe then that $U^{-1} \supset (U_1^{-1})^3$ and $(f,f)[U^{-1}] \subset U_1^{-1}$. Hence

$$(U_1 \circ U_1^{-1})^3 \subset U_1^3 \cdot (U_1^{-1})^3 \subset U \circ U^{-1}$$

and

$$(f,f)[U \circ U^{-1}] \subset (f,f)[U] \circ (f,f)[U^{-1}] \subset U_1 \cdot (U_1^{-1}).$$

Therefore f is $1/3$ $\mathcal{B}'$-contractive.

It remains now to show that $\mathcal{B}'$ is a $1/3$ inclusive base for f. Let $(x,y) \varepsilon X \times X$ and $U \circ (U^{-1}) \varepsilon \mathcal{B}'$. There is a $U_o \varepsilon \mathcal{B}$ such that $U_o \supset U$, $(x,y) \varepsilon U_o$, and there is a sequence $U_o, \ldots, U_n$ in $\mathcal{B}$ such that $U_i \supset U_{i+1}^3$ and $(f,f)[U_i] \subset U_{i+1}$ for $i = 0,\ldots,n-1$ with $U_n \subset U$. Also there is a V_o in $\mathcal{B}$ such that $V_o \supset U_o$, $(y,x) \varepsilon V_o$, and there is a sequence $V_o, \ldots, V_m$ in $\mathcal{B}$ such that $V_i \supset V_{i+1}^3$, and $(f,f)[V_i] \subset V_{i+1}$ with $V_m \subset U_o$.

As above note that the sequence $V_o \circ (V_o^{-1}), \ldots, V_m \circ (V_m^{-1})$ has the properties $V_i \circ (V_i^{-1}) \supset [V_{i+1} \circ (V_{i+1}^{-1})]^3$, $(f,f)[V_i \cdot (V_i)^{-1}] \subset V_{i+1} \cdot (V_{i+1}^{-1})$, and $V_m \circ (V_m^{-1}) \subset U_o \circ (U_o^{-1})$. Also note that

$(x,y) \varepsilon V_o \circ (V_o^{-1})$ since $(x,y) \varepsilon U_o \subset V_o$ and that $V_o \cdot (V_o^{-1}) \supset U_o \circ (U_o^{-1})$.

Since f is $1/3$ $\mathcal{B}'$-contractive there is a V_{m+1} in $\mathcal{B}'$ such that $(f,f)[V_m \circ (V_m^{-1})] \subset V_{m+1}$ and $V_m \cdot (V_m^{-1}) \supset V_{m+1}^3$. Since

$$(f,f)[V_m \cdot (V_m^{-1})] \subset (f,f)[U_o \cdot (U_o^{-1})] \subset U_1 \circ (U_1^{-1})$$

we may choose $V_{m+1} \subset U_1$. Similarly we may choose $V_{m+1},\ldots,V_{m+n}$ in $\mathcal{B}'$ such that $V_{m+i} \supset V_{m+i+1}^3$, $(f,f)[V_{m+i}] \subset V_{m+i+1}$, and $V_{m+i+1} \subset U_{i+1} \circ (U_{i+1}^{-1})$ for $i = 1,\ldots,n-1$. Thus

$$V_{m+n} \subset U_n \cdot (U_n^{-1}) \subset U \cdot (U^{-1}) .$$

Therefore $\mathcal{B}'$ is a $1/3$ inclusive base for f.

(13.11) Lemma: Let $(X, \mathcal{U})$ be a uniform space, P be the gauge for $\mathcal{U}$, and f be a self map of X. Then the following conditions are equivalent:

(a) There is a symmetric base $\mathcal{B}$ for $\mathcal{U}$ and a positive integer n such that f^n is $1/3$ $\mathcal{B}$-contractive and $\mathcal{B}$ is a $1/3$ inclusive base for f.

(b) There is a positive integer m and a positive number $\lambda < 1$ such that for every (x,y) in $X \times X$ the collection P_{xy} described as follows generates $\mathcal{U}$: P_{xy} consists of all members d of P such that for some positive number e_d, f^m is (e_d, λ) uniformly locally contractive relative to d and $d(x,y) < e_d$.

Proof: Assume (b) holds. Let $\mathcal{B}$ be the collection of all intersections of sets of the form

$$V_{d,e} = \left\{(x,y) \varepsilon X \times X : d(x,y) < e\right\}$$

where $d \varepsilon \Sigma \left\{P_{xy} : (x,y) \varepsilon X \times X\right\}$ and $e < e_d$. Then $\mathcal{B}$ is clearly a symmetric base for $\mathcal{U}$.

Choose a positive integer k such that $\lambda^k \leq 1/3$. Let $V_o \varepsilon \mathcal{B}$ where

$$V_o = \Pi \left\{V_{d_i,e_i} : i = 1,\ldots,n\right\}.$$

Consider $V_1 \varepsilon \mathcal{B}$ where

$$V_1 = \Pi \left\{V_{d_i,e_i/3} : i = 1,\ldots,n\right\}.$$

Then $V_o \supset V_1$,

$$V_o \supset \Pi \left\{V^3_{d_i,e_i/3} : i = 1,\ldots,n\right\} \supset V_1^3,$$

and

$$(f^{mk}, f^{mk})[V_o] \subset \Pi \left\{(f^{mk}, f^{mk})[V_{d_i,e_i}] : i = 1,\ldots,n\right\} \subset V_1.$$

Thus f^{mk} is $1/3$ $\mathcal{B}$-contractive.

It remains then to show $\mathcal{B}$ is a $1/3$ inclusive base for f. Let $(x,y) \varepsilon X \times X$ and $U \varepsilon \mathcal{B}$. There is an entourage V of the form

$$V = \Pi \left\{V_{d_i,e} : i = 1,\ldots,n\right\}$$

in $\mathcal{B}$ such that $V \subset U$ and $d_i \varepsilon P_{xy}$ for $i = 1,\ldots,n$. For $i = 1,\ldots,n$ there is an e_i such that $d_i(x,y) < e_i < e_{d_i}$. There is then a positive integer of the form ℓk such that $\lambda^{\ell k} e_i \leq e$ for

$i = 1,\ldots,n$. Let

$$U_j = \Pi\left\{V_{d_i,\lambda^{jk}e_i} : i = 1,\ldots,n\right\}$$

for $j = 0,\ldots,\ell$. Then by choice of k it follows that $U_j \supset U^3_{j+1}$, $U_j \supset U_{j+1}$ and $(f^{mk}, f^{mk})[U_j] \subset U_{j+1}$ for $j = 0,\ldots,\ell-1$.

Finally by choice of ℓ, $U_\ell \subset V \subset U$. Therefore $\mathcal{B}$ is a 1/3 inclusive base for f^{mk}. Hence condition (a) holds.

Now assume (a) holds. It suffices to show that given (x^*, y^*) in $X \times X$ and U in $\mathcal{B}$, there is a uniformly continuous pseudometric d such that f^n is (1/2, 1/2) uniformly locally contractive relative to d, $d(x^*, y^*) < 1/2$ and there is a positive number e such that $V_{d,e} \subset U$.

This will be shown. Using 1/3 inclusiveness of $\mathcal{B}$, there is a U_1 in $\mathcal{B}$ such that $(x^*, y^*) \in U_1$ and there is a sequence $U_1,\ldots,U_k$ in $\mathcal{B}$ such that $U_i \supset U^3_{i+1}$, $(f^n, f^n)[U_i] \subset U_{i+1}$ for $i = 1,\ldots,k-1$ with $U_k \subset U$.

Using 1/3 $\mathcal{B}$-contractiveness of f^n, there are sets U_i in $\mathcal{B}$ $i = k+1,\ldots$ such that for $i = k,\ldots$ $U_i \supset U_{i+1}$, $U_i \supset U^3_{i+1}$, and $(f^n, f^n)[U_i] \subset U_{i+1}$.

Letting $U_0 = X \times X$, $\{U_i, i \geq 0\}$ is a sequence of symmetric subsets of $X \times X$ satisfying a lemma in Kelley's book [20, Theorem 12, p. 185]. Moreover $(f^n, f^n)[U_i] \subset U_{i+1}$ for $i \geq 1$.

Thus by this lemma there exists a pseudometric d such that

$$U_i \subset V_{d,2^{-i}} \subset U_{i-1}$$

for $i \geq 1$. It then follows by a theorem in Kelley [20, Theorem 11, p. 183] that d is uniformly continuous. Furthermore

$$V_{d,2^{-(k+1)}} \subset U_k \subset U .$$

It remains then to show that f^n is $(1/2), 1/2)$ uniformly locally contractive relative to d and $d(x^*, y^*) < 1/2$. Recall from the lemma that d is defined by

$$d(x,y) = \inf \left\{ \sum_{i=o}^{n} g(x_i, x_{i+1}) : x_o = x, \quad x_{n+1} = y , \right.$$

$$\left. \text{and} \quad x_i \in X \quad \text{for} \quad 1 \leq i \leq n \right\}$$

where $g(x,y) = 2^{-i}$ if $(x,y) \in U_{i-1} - U_i$ and $g(x,y) = 0$ if $(x,y) \in U_i$ for every $i \geq 0$.

If $g(x,y) = 2^{-i}$ for some $i \geq 2$ then $(x,y) \in U_{i-1}$ which implies that $(f^n x, f^n y)$ is a member of U_i. This in turn implies that

$$g(f^n x, f^n y) \leq 2^{-(i+1)} = (1/2)\ g(x,y).$$

In the event $g(x,y) = 0$ then $(x,y) \in U_i$ for $i \geq 0$. As a result $(f^n x, f^n y) \in U_{i+1}$ for $i \geq 0$ so that

$$g(f^n x, f^n y) = 0 = (1/2)\ g(x, y) \quad .$$

Let $d(x,y)$ be less than $1/2$. Let $e > 0$ be given such that $e < (1/2) - d(x, y)$. Then there exist points $x_o, \ldots, x_{n+1}$ in X such that $x_o = x$, $x_{n+1} = y$, and

$$\sum_{i=o}^{n} g(x_i, x_{i+1}) < d(x,y) + e < 1/2 .$$

Thus $g(x_i, x_{i+1}) \leq 1/4$ for $i = 0,\ldots,n$ so that

$$g(f^n x_i, f^n x_{i+1}) \leq (1/2)\, g(x_i, x_{i+1}) .$$

for $i = 0,\ldots,n$. Consequently

$$d(f^n x, f^n y) \leq \sum_{i=o}^{n} g(f^n x_i, f^n x_{i+1})$$

$$\leq (1/2) \sum_{i=o}^{n} g(x_i, x_{i+1}) < (1/2) d(x,y) + (e/2) .$$

We then have $d(f^n x, f^n y) \leq (1/2)\, d(x,y)$ for $d(x,y) < 1/2$. Therefore f^n is $(1/2, 1/2)$ uniformly locally contractive.

Finally since $(x^*, y^*) \in U_1$ then $g(x^*, y^*) \leq 1/4$ and thus

$$d(x^*, y^*) \leq g(x^*, y^*) \leq 1/4 < 1/2 .$$

(13.12) Lemma: Let X be a set and f a self map of X. If for some positive integer k f^k has the unique fixed point z, then f also has the unique fixed point z.

Proof: Note that

$$f^k f z = f f^k z = f z .$$

Since z is unique and fz is a fixed point of f^k it follows that $fz = z$ and thus z is a fixed point of f. If w is a fixed point of f, then w is also a fixed point of f^k and hence $w = z$ so that z is the unique fixed point of f.

(13.13) <u>Lemma</u>: Let X be a topological space and f a self map of X. If for some positive integer k there is a z in X such that $\lim \{f^{nk}x, n \geq 1\} = z$ for every x in X, then $\lim \{f^n x, n \geq 1\} = z$ for every x in X.

<u>Proof</u>: For every integer i such that $1 \leq i \leq k - 1$ $\lim \{f^{nk+i}x, n \geq 1\} = z$. Hence if U is a neighborhood of z, there is an integer N_i such that $n \geq N_i$ implies $f^{nk+i}x \in U$. Let $N = \sup\{N_i : 1 \leq i \leq k - 1\}$ and $M = (N + 1)k$. For each positive integer m the representation $m = n_m k + i_m$ where $0 \leq i_m \leq k - 1$ is unique. Thus if $m \geq M$ then $n_m \geq N$ and hence $f^m x \in U$. Therefore $\lim \{f^m x, m \geq 1\} = z$.

(13.14) <u>Theorem</u>: Let $(X, \mathcal{U})$ be a sequentially complete Hausdorff uniform space, P be the gauge for $\mathcal{U}$, and f a self map of X. For each positive integer m and each $(x,y) \in X \times X$ let P^m_{xy} denote the collection consisting of all pseudometrics d in P such that for some positive number e_d and some positive number $\lambda_d < 1$ f^m is (e_d, λ_d) uniformly locally contractive relative to d and $d(x,y) < e_d$. If for some positive m, P^m_{xy} generates $\mathcal{U}$ for every (x,y) in $X \times X$ then f has a unique fixed point z and $\lim \{f^n x, n \geq 1\} = z$ for every x in X.

<u>Proof</u>: We will show that f^m has a unique fixed point z and

$\lim \{f^{nm}x, n \geq 1\} = z$ for every x in X, and the conclusion of the theorem will then follow by Lemmas (13.12) and (13.13).

Let $x \in X$, $g = f^m$, and $d \in P^m_{x,gx}$. Then $d(gx, g^2x) \leq \lambda_d d(x, gx)$ and in general for $n \geq 1$, $d(g^n x, g^{n+1}x) \leq \lambda_d^n\, d(x, gx)$. Hence for each pair of positive integers n and p

$$d(g^n x, g^{n+p}x) \leq \sum_{i=o}^{p-1} d(g^{n+i}x, g^{n+i+1}x)$$

$$\leq \left[\sum_{i=o}^{p-1} \lambda_d^{n+i}\right] d(x, gx) \leq [\lambda_d^n/(1 - \lambda_d)]\, d(x, gx) \ .$$

Since the last term in this inequality converges to zero as n tends to infinity, it follows that $\{g^n x, n \geq 1\}$ is a Cauchy sequence relative to d for every d in $P^m_{x,gx}$.

As $P^m_{x,gx}$ generates $\mathcal{U}$, we then have that $\{g^n x, n \geq 1\}$ is Cauchy relative to $\mathcal{U}$. Since $(X, \mathcal{U})$ is sequentially complete there is a point z in X such that $\{\lim\ g^n x, n \geq 1\} = z$. Moreover, $(X, \mathcal{U})$ being Hausdorff, z is the unique limit of this sequence. Thus

$$gz = g(\lim \{g^n x, n \geq 1\}) = \lim \{g^{n+1}x, n \geq 1\} = z$$

and z is a fixed point of g.

It remains to be shown that z is the only fixed point of g. Suppose $gw = w$ and $w \neq z$. Since $(X, \mathcal{U})$ is Hausdorff and P^m_{wz} generates $\mathcal{U}$, there is a d in P^m_{wz} such that $d(w, z) > 0$. Thus

$$d(w, z) = d(gw, gz) = \lambda_d d(w, z) < d(w, z)$$

which is a contradiction.

(13.15) <u>Theorem</u>: Let $(X, \mathcal{U})$ be a sequentially complete Hausdorff uniform space, $\mathcal{B}$ a base for $\mathcal{U}$, and f a self map of X. If there are positive integers m, r, and s such that f^m is an r/s $\mathcal{B}$-contraction map and $\mathcal{B}$ is an r/s inclusive base for f^m, then f has a unique fixed point z and $\lim \{f^n x, n \geq 1\} = z$ for every x in X.

<u>Proof</u>: By Lemma (13.8) there is a positive integer k and a base $\mathcal{B}'$ such that f^{mk} is $1/3$ $\mathcal{B}'$-contractive and $\mathcal{B}'$ is a $1/3$ inclusive base for f^{mk}. By Lemma (13.10) we may assume $\mathcal{B}'$ is symmetric. It then follows by Lemma (13.11) that the hypotheses of Theorem (13.14) are satisfied. Hence the conclusion holds.

(13.16) <u>Corollary (Banach)</u>: Let (X, d) be a complete metric space and f a self map of X. If there is a positive number $\lambda < 1$ such that $d(fx, fy) \leq \lambda\, d(x, y)$ for every x, y in X, then f has a unique fixed point z and $\lim \{f^n x, n \geq 1\} = z$ for every x in X.

<u>Proof</u>: There is a positive integer m such that $\lambda^m \leq 1/3$. Let $\mathcal{B} = \{V_e : e > 0\}$ where $V_e = \{(x, y) \in X \times X : d(x, y) < e\}$. Then f^m is a $1/3$ $\mathcal{B}$-contractive map and $\mathcal{B}$ is a $1/3$ inclusive base for f^m. The uniformity induced by d is clearly sequentially complete and Hausdorff. Hence by Theorem (13.15), the corollary is true.

(13.17) <u>Corollary (Edelstein)</u>: Let (X, d) be complete e_o-chainable metric space and f a mapping of X into X which is (e_o, λ)

uniformly locally contractive. Then there is a unique fixed point of f, z, and $\lim \{f^n x, \ n \geq 1\} = z$ for every x in X.

<u>Proof</u>: Let $\mathcal{B} = \{V_e^n : n$ is a positive integer and $e \leq e_o\}$. There exists a positive integer m such that $\lambda^m \leq 1/3$. Hence for $e \leq e_o$, $(f^m, f^m)[V_e] \subset V_{e/3}$ and of course $V_e \supset (V_{e/3})^3$. Thus

$$\begin{aligned}(V_{e/3})^n &\supset \{(f^m, f^m)[V_e]\}^n \\ &= G(f^m) \circ V_e \circ G(f^m)^{-1} \dots G(f^m) \circ V_e \circ G(f^m)^{-1} \\ &\supset G(f^m) \circ (V_e)^n \circ G(f^m)^{-1} = (f^m, f^m)[V_e^n]\end{aligned}$$

(Recall the argument in Lemma (13.3)) and clearly $V_e^n \supset (V_{e/3}^n)^3$. Therefore f^m is $1/3$ $\mathcal{B}$-contractive.

Let $(x,y) \in X \times X$ and $V_e^n \in \mathcal{B}$. Since (X, d) is e_o-chainable, there is a positive integer k such that $(x,y) \in V_{e_o}^k$. Also there is a positive integer ℓ such that $(1/3)^\ell \, e_o \leq e/k$. Define $U_i = V^k_{(1/3)^i e_o}$ for $i = 1, \dots, \ell$. Then $U_i \supset U_{i+1}^3$ and $(f^m, f^m)[U_i] \subset U_{i+1}$ as above. Finally note

$$U_\ell = V^k_{(1/3)^\ell e_o} \subset V^k_{e/k} \subset V_e \subset V_e^n \,.$$

Thus $\mathcal{B}$ is a $1/3$ inclusive base for f^m. Therefore by Theorem (13.15), the conclusion follows.

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