5-LIST-COLORING GRAPHS ON SURFACES

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Luke Jamison Postle

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Approved by:

Dr. Robin Thomas, Advisor
School of Mathematics
Georgia Institute of Technology

Dr. Xingxing Yu
School of Mathematics
Georgia Institute of Technology

Dr. William T. Trotter
School of Mathematics
Georgia Institute of Technology

Dr. Zdenek Dvorak
Department of Computer Science
Charles University

Dr. William Cook
School of Industrial and Systems Engineering
Georgia Institute of Technology

Date Approved: August 15, 2012

To my parents, for educating me in what truly matters.

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SUMMARY

Thomassen proved that there are only finitely many 6-critical graphs embeddable on a fixed surface. He also showed that planar graphs are 5-list-colorable. This thesis develops new techniques to prove general theorems for 5-list-coloring graphs embedded in a fixed surface. Indeed, a general paradigm is established which improves a number of previous results while resolving several open conjectures. In addition, the proofs are almost entirely self-contained.

In what follows, let Σ be a fixed surface, G be a graph embedded in Σ and L a list assignment such that, for every vertex v of G, $L(v)$ has size at least five. First, the thesis provides an independent proof while also improving the bound obtained by DeVos, Kawarabayashi and Mohar that says that if G has large edge-width, then G is 5-list-colorable. The bound for edge-width is improved from exponential to logarithmic in Euler genus, which is best possible up to a multiplicative constant. Second, the thesis proves that there exist only finitely many 6-list-critical graphs embeddable in Σ, solving a conjecture of Thomassen from 1994. Indeed, it is shown that the number of vertices in a 6-list-critical graph is at most linear in genus, which is best possible up to a multiplicative constant. As a corollary, there exists a linear-time algorithm for deciding 5-list-colorability of graphs embeddable in Σ.

Furthermore, we prove that the number of L-colorings of an L-colorable graph embedded in Σ is exponential in the number of vertices of G, with a constant depending only on the Euler genus g of Σ. This resolves yet another conjecture of Thomassen from 2007. The thesis also proves that if X is a subset of the vertices of G that are pairwise distance $\Omega(\log g)$ apart and the edge-width of G is $\Omega(\log g)$, then any L-coloring of X extends to an L-coloring of G. For planar graphs, this was conjectured by Albertson and recently proved by Dvorak, Lidicky, Mohar, and Postle. For regular coloring, this was proved by Albertson and Hutchinson. Other related generalizations are examined.

CHAPTER I

INTRODUCTION

In this chapter, we will provide the graph theoretic context of the results to follow. In Section 1.1, we give descriptions of the basic terminology and structures used for our results. In Section 1.2, we explain how graphs can be embedded on surfaces other than the plane. In Section 1.3, we present an overview of the history of coloring graphs on surfaces, especially in regards to 5-coloring. In Section 1.4, we introduce list-coloring and begin to examine the history of 5-list-coloring graphs on surfaces. In Section 1.5, 1.6 and 1.7 we review results about extending colorings of precolored subgraphs, 5-list-coloring graphs with crossings, and proving the existence of exponentially many 5-list-colorings. In Section 1.8, we state the main results of this thesis. In Section 1.9, we provide an outline of the proof of the main results.

1.1 Graph Theoretic Preliminaries

We follow the exposition of Diestel in [16]. A *graph* is an ordered pair $(V(G), E(G))$ consisting of a nonempty set $V(G)$ of vertices and a set $E(G)$ of edges, which are two elements subsets of $V(G)$. Thus, we do not allow loops or multiple edges; that is, all graphs in this thesis are assumed to be *simple*.

If $e = \{u, v\}$ is an edge where $u, v \in V(G)$, then we write $e = uv$ and say that u and v are the *ends* of e. If u is an end of e then we say that e is *incident* with u and vice versa. If $u, v \in V(G)$ such that there exists $e \in E(G)$ with $e = uv$, then we say that u and v are *adjacent* and we denote this by writing $u \sim v$.

Two graphs G and H are *isomorphic* if there exists a bijection f between $V(G)$ and $V(H)$ such that any two vertices u and v in G are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in H. If $v \in V(G)$, the *neighborhood* of v, denoted by $N(v)$, is the set of all vertices in G adjacent to v. The *degree* of a vertex v, denoted by $d(v)$ is the size of its neighborhood.

Graphs are usually represented in a pictorial manner with vertices appearing as points and edges represented by lines connecting the two vertices associated with the edge. One class of graphs is the class of *complete graphs* which consist of graphs with vertex set V and an edge joining every pair of distinct vertices in V.

For a graph $G = (V, E)$, if $V' \subseteq V$, $E' \subseteq E$ and for every edge $e' \in E'$ both ends of e' are in V', then $G' = (V', E')$ is a *subgraph* of G. Given a graph $G = (V, E)$, if X is a subset of vertices, we denote by $G[X]$ the subgraph with vertex set X and edge set containing every edge of G with both ends contained in X. Then $G[X]$ is the graph *induced* by X.

A graph G is *connected* if there exists a path between any two vertices of G, and *disconnected* otherwise. A subgraph H of G is a *connected component* of G if H is connected and there does not exist an edge $e \notin E(H)$ with an end in $V(H)$. A vertex v of a connected graph G is a *cutvertex* if $G - v$ is disconnected.

We say that (G_1, G_2) is a *separation* of G if G_1, G_2 are edge-disjoint subgraphs of G whose union is G. We say $|V(G_1) \cap V(G_2)|$ is the *order* of a separation (G_1, G_2) of G. We say that a cutvertex v of G *divides* into two graphs G_1, G_2 if (G_1, G_2) is a separation of G such that $V(G_1) \cap V(G_2) = \{v\}$.

The *distance* between two vertices u and v of G, denoted by $d(u, v)$, is the the length of the shortest path between them. We let $N_k(v) = \{u \in V(G) | d(u, v) = k\}$, that is the vertices at distance k from v. We say that $B_k(v) = \{u \in V(G) | d(u, v) \leq k\}$, the set of vertices distance at most k from v, is the *ball of radius* k centered at v.

1.2 Graphs on Surfaces

A *surface* is defined to be a connected, compact, 2-dimensional manifold with empty boundary. We follow the exposition of Mohar and Thomassen [41] to describe how we view graphs on surfaces and ask the reader to refer to this text for further details. Two surfaces are *homeomorphic* if there exists a bijective continuous mapping between them such that the inverse is also continuous. Let X be a topological space. An *arc* in X is the image of a continuous one-to-one function $f : [0, 1] \to X$. We say a graph G is *embedded* in a topological space X if the vertices of G are distinct elements of X and every edge of G is an arc connecting in X the two vertices it joins in G, such that its interior is disjoint from other edges and vertices. An *embedding* of a graph G in topological space X is an isomorphism of G with a graph G' embedded in X.

A topological space X is *arcwise connected* if any two elements of X are connected by an arc in X. The existence of an arc between two points of X determines an equivalence relation whose equivalence classes are called the arcwise connected components, or the regions of X. A *face* of $C \subseteq X$is an arcwise connected component of $X \setminus C$. A *2-cell embedding* is an embedding where every face is homeomorphic to an open disk.

If G is a graph embedded in the plane, then we say that G is a *plane* graph. In that case, there exists an infinite face of G. If G is connected, we say that boundary walk of the infinite face of G is the *outer walk* of G. We say an edge e of G is a *chord* of the outer walk of G if the edge does not lie on the boundary of the infinite face but both its ends do.

A *curve* in X is the image of a continuous one-to-one function $f : S_1 \to X$ where S_1 is the unit circle. A curve is *two-sided* if traversing along it preserves orientation and *one-sided* otherwise. A surface is *nonorientable* if there exists a one-sided curve in the surface. A surface is *orientable* if all curves are two-sided.

A useful method for constructing a surface is as follows. Let $\mathcal{P}$ be a collection of pairwise disjoint regular polygons in the plane such that the sum of the number of edges in the collection of polygons is even, every edge has the same length and is oriented from one of its end, called the *tail*, to the other, called the *head*. Now identify pairs of edges so that heads are identified with heads and tails with tails. Consequently, all points in the union of these polygons have open neighborhoods homeomorphic to the plane and hence their union is a surface. It can be shown - see [41] - that every surface is homeomorphic to a surface constructed from such a set $\mathcal{P}$ where all the polygons are triangles.

All surfaces have been characterized by the classification theorem of surfaces. Before we state the theorem, some more definitions are in order. The most basic surface that is considered in the classification theorem of surfaces is the sphere, denoted S_0. The sphere can be constructed by letting $\mathcal{P}$ be a collection of four equilateral triangles and identifying them to yield a regular tetrahedron.

Given a surface Σ, there exists a set of operations to yield a different, and in a sense we will define later, a more complicated surface. In particular, these operations are *adding a handle*, *adding a twisted handle* or *adding a crosscap.*

First let us define adding a handle. Let T_1 and T_2 be two disjoint triangles in S all of whose side lengths are the same. If Σ is orientable, then orient the edges of T_1 and T_2 so that the directions of T_1's edges are the opposite of T_2's when each is viewed in a clockwise direction. Then if we remove the interiors of T_1 and T_2 and identify the edges of T_1 to the edges of T_2, this creates a new surface Σ'. We say that S' is obtained by adding a handle to Σ. Notice that we can only add a handle to an orientable surface.

Suppose that S is orientable and the clockwise orientations of T_1 and T_2 are the same. If we remove the interior of T_1 and T_2 and we identify the edges of T_1 to the edges of T_2 then the resulting surface, call it Σ'', is the result of adding a twisted handle to Σ. If S is nonorientable, then any handle added is a twisted handle. Finally, suppose that we have a simple closed disk, call it T. Suppose that we delete the interior of T from S and identify diametrically opposite points of T. This adds a crosscap to Σ. It can also be shown that adding two crosscaps is equivalent to adding a twisted handle. We can now state the classification theorem of surfaces. It states that every surface is homeomorphic to either S_g, the surface obtained from the sphere by adding g handles, or N_k, the surface obtained from the sphere by adding k cross-caps.

Using this terminology, $S_0 = N_0$ is the sphere, S_1 is the torus, N_1 is the projective plane and N_2 is the Klein bottle. Define the *Euler characteristic* of a surface Σ to be $\chi(\Sigma) = 2 - 2h$ if $\Sigma = S_h$ and $\chi(\Sigma) = 2 - k$ if $\Sigma = N_k$. Also, define the *Euler genus* of surface S, denoted by $g(S)$, to be $g(\Sigma) = 2 - \chi(\Sigma)$. In this whenever we refer to the genus of a surface, we shall mean the Euler genus. We can now state *Euler's formula* for surfaces.

Theorem 1.2.1 ([41])**.** *Let G be a graph which is 2-cell embedded in a surface S. If G has n vertices, q edges and f faces in Σ, then*

$$n - q + f = \chi(\Sigma).$$

Another important property of graphs embedded in surfaces is that curves in the surface may have different properties. A *homotopy* between two functions f and g from a space X to a space Y is a continuous map G from $X \times [0,1] \to Y$ such that $G(x, 0) = f(x)$ and $G(x, 1) = g(x)$. Two functions are *homotopic* if there is a homotopy between them. A *contractible cycle* of a graph embedded in a surface is a cycle in the graph which is the image of a closed curve homotopic to a a constant map. We call it contractible because it can be contracted to a point. However, on some surfaces there also exist cycles which are noncontractible.

One metric useful in the study of embedded graphs is the property of edge-width. The *edge-width* of a graph G embedded in a surface S, denoted by $ew(G)$, is the length of the smallest noncontractible cycle in G.

1.3 Coloring Graphs on Surfaces

Graph coloring is an area of study in graph theory that has received much attention. Indeed, mathematicians have long been interested in coloring maps. A natural question is to ask what is the fewest number of colors so that the regions or countries of a map that touch one another have different colors. For planar maps, it was long conjectured that four colors suffices. The Four-Color Theorem [7, 8, 44], proved in the 1970s, settled this conjecture in the affirmative.

Definition. Let X be a nonempty set. We say that a function $\phi : V(G) \to X$ is a *coloring* of G if for all $e = uv \in E(G)$, $\phi(u) \neq \phi(v)$. We say that a coloring $\phi : V(G) \to X$ is a *k-coloring* if $|X| = k$. We say that a graph G is *k-colorable* if there exists a k-coloring of G. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum k such that G is k-colorable.

Mathematicians have wondered what generalizations of the Four-Color Theorem might be true. A natural class of graphs to determine the coloring properties for is graphs embedded in a surface. A fundamental question in topological graph theory is as follows: Given a surface Σ and an integer $t > 0$, which graphs embedded in Σ are t-colorable? Heawood proved that if Σ is not the sphere, then every graph in Σ is t-colorable as long as $t \geq H(\Sigma) := \lfloor (7 + \sqrt{24g + 1})/2 \rfloor$, where g is the Euler genus of Σ.

Ringel and Youngs [43] proved that the bound is best possible for all surfaces except the Klein bottle. In 1934, Franklin [31] proved that every graph embeddable in the Klein bottle requires only six colors, but Heawood's bound gives only seven. Dirac [17] and Albertson and Hutchinson [2] improved Heawood's result by showing that every graph in Σ is actually $(H(\Sigma) - 1)$-colorable, unless it has a subgraph isomorphic to the complete graph on $H(\Sigma)$ vertices.

Thus the maximum chromatic number for graphs embeddable in a surface has been found for every surface. Yet the modern view argues that most graphs embeddable in a surface have small chromatic number. To formalize this notion, we need a definition. We say that a graph G is *t-critical* if it is not $(t-1)$-colorable, but every proper subgraph of G is $(t-1)$-colorable. Using Euler's formula, Dirac [18] proved that for every $t \geq 8$ and every surface Σ there are only finitely many t-critical graphs that embed in Σ. By a result of Gallai [33], this can be extended to $t = 7$. Indeed, we will see in a moment that this extends to $t = 6$ by a deep result of Thomassen.

First however, let us mention a different approach used by Thomassen to formalize the notion that most graphs on a surface are 5-colorable. He was able to show that graphs with large edge-width, that is graphs in which local neighborhood of every vertex is planar, are 5-colorable. Thomassen proved the following.

Theorem 1.3.1. *If G is a graph embedded in a surface Σ such that $ew(G) \geq 2^{\Omega(g(\Sigma))}$, then G is 5-colorable.*

Yet we note that Theorem 1.3.1 is implied by a corresponding bound on the size of 6-critical graphs embedded in a surface since k-colorability and having large edge-with are properties preserved by subgraphs. That is, if $H \subseteq G$, then $\chi(H) \leq \chi(G)$ and $ew(H) \geq ew(G)$. Nevertheless, we will improve the required lower bound on edge-width in Theorem 1.3.1 to $\Omega(\log g(\Sigma))$ in Chapter 5. Moreover such a logarithmic bound is best possible up to a multiplicative constant as we will demonstrate in Chapter 5 using Ramanujan graphs. Using deep and powerful new techniques, Thomassen was able to prove the following.

Theorem 1.3.2. *For every surface Σ, there are finitely many 6-critical graphs that embed in Σ.*

Furthermore, Theorem 1.3.2 yields an algorithm for deciding whether a graph on a fixed surface is 5-colorable.

Corollary 1.3.3. *There exists a linear-time algorithm for deciding* 5*-colorability of graphs on a fixed surface.*

This follows from a result of Eppstein [28, 29] which gives a linear-time algorithm for testing subgraph isomorphism on a fixed surface. Hence if the list of 6-critical graphs embeddable on a surface is known, one need merely test whether a graph contains one of the graphs on the list. The list is known only for the projective plane [2], torus [45], and Klein bottle [13, 37].

Theorem 1.3.2 is best possible as it does not extend to $t \leq 5$ for surfaces other than the plane and $t \leq 4$ for the plane. Indeed, Thomassen [50], using a construction of Fisk [32], constructed infinitely many 5-critical graphs that embed in the torus. One may also ask how large the 6-critical graphs on a fixed surface can be. Theorem 1.3.2 implies an implicit bound on the number of vertices in a 6-critical graph embeddable in Σ in terms of the genus of Σ. However, Thomassen did not prove an explicit bound. Postle and Thomas [42] gave a new proof of Theorem 1.3.2 that also provides an explicit bound. They proved the following.

Theorem 1.3.4. *The number of vertices of a* 6*-critical graph embedded in a surface* Σ *is* $O(g(\Sigma))$.

Their bound is best possible up to a multiplicative constant as demonstrated by Hajos' construction on copies of K_6.

1.4 List-Coloring Graphs on Surfaces

There exists a generalization of coloring where the vertices do not have to be colored from the same palette of colors.

Definition. We say that L is a *list-assignment* for a graph G if $L(v)$ is a set of colors for every vertex v. We say L is a *k-list-assignment* if $|L(v)| = k$ for all $v \in V(G)$. We say that a graph G has an *L-coloring* if there exists a coloring ϕ such that $\phi(v) \in L(v)$

for all $v \in V(G)$. We say that a graph G is *k-choosable*, also called *k-list-colorable*, if for every k-list-assignment L for G, G has an L-coloring. The *list chromatic number* of G, denoted by $ch(G)$, is the minimum k such that G is k-list-colorable.

Note that $\chi(G) \leq ch(G)$ as a k-coloring is a k-list-coloring where all the lists are the same. In fact, Dirac's Theorem[17] has been generalized to list-coloring by Bohme, Mohar and Stiebitz [12] for most surfaces; the missing case, $g(\Sigma) = 3$, was completed by Kral and Skrekovski [39].

Nevertheless, list-coloring differs from regular coloring. One notable example of this is that the Four Color Theorem does not generalize to list-coloring. Indeed Voigt [53] constructed a planar graph that is not 4-choosable.

Yet the list chromatic number of planar graphs is now well understood, thanks to Thomassen [46]. He was able to prove the following remarkable theorem with an outstandingly short proof.

Theorem 1.4.1. *Every planar graph is 5-choosable.*

Actually, Thomassen [46] proved a stronger theorem.

Theorem 1.4.2 (Thomassen)**.** *If G is a plane graph with outer cycle C and $P = p_1p_2$ is a path of length one in C and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(P)$, and $|L(p_1)| = |L(p_2)| = 1$ with $L(p_1) \neq L(p_2)$, then G is L-colorable.*

Indeed, this theorem will be the starting point for this research. In Chapter 2, we will generalize this result in a number of different ways. To understand questions about list-coloring, it is helpful to define a similarly useful notion of being list-critical.

Definition. If L is list assignment for a graph G, then we say that G is *L-critical* if G does not have an L-coloring but every proper subgraph of G does. Similarly, we say that G is *k-list-critical* if G is not $(k-1)$-list-colorable but every proper subgraph of G is.

We should mention the following nice theorem of Gallai [33].

Theorem 1.4.3. *Let G be an L-critical graph where L is a list assignment for G. Let H be the graph induced by the vertices v of G such that $d(v) = |L(v)|$. Then each block of H is a complete graph or an odd cycle.*

Theorem 1.4.3 is the key trick to proving there are only finitely many 7-critical graphs embedded in a fixed surface. In fact using Theorem 1.4.3, Thomassen [50] gave a simple proof that there are only finitely many 7-list-critical graphs on a fixed surface. Indeed, Thomassen proved the following stronger theorem.

Theorem 1.4.4. *Let G be a graph embedded in a surface Σ. Let L be a list assignment of G and let S be a set of vertices in G such that $|L(v)| \geq 6$ for each $v \in V(G) \setminus S$. If G is L-critical, then $|V(G)| \leq 150(g(\Sigma) + |S|)$.*

Naturally then, Thomassen conjectured (see Problem 5 of [50]) that Theorem 1.3.2 generalizes to list-coloring.

Conjecture 1.4.5. For every surface Σ, there are finitely many 6-list-critical graphs that embed in Σ.

Note that Kawarabayashi and Mohar [38] announced without proof a resolution of Conjecture 1.4.5. Indeed, they claim a strengthening of Conjecture 1.4.5 when there are precolored vertices as in Theorem 1.4.4, though not with a linear bound. We will nevertheless provide an independent proof of Conjecture 1.4.5 in Chapter 5. Our proof also gives a new proof of Theorem 1.3.2 as his techniques do not apply for list-coloring. In fact, we will also generalize the linear bound of Postle and Thomas to list-coloring.

Meanwhile, DeVos, Kawarabayashi, and Mohar [15] generalized Theorem 1.3.1 to list-coloring.

Theorem 1.4.6. *If G is a graph embedded in a surface Σ such that $ew(G) \geq 2^{\Omega(g(\Sigma))}$, then G is 5-list-colorable.*

Indeed in Chapter 5, we give an independent proof of Theorem 1.4.6 and improve the required lower bound to $\Omega(\log g(\Sigma))$ with a completely different proof. Moreover it should be noted that while a linear bound is implied by a linear bound for 6-list-critical graphs, a logarithmic bound requires some additional ideas.

1.5 Extending Precolored Subgraphs

An important technique in Thomassen's proofs is to ask what colorings of a graph are possible when a certain subgraph has already been precolored. To that end if H is a subgraph of G and ϕ is a coloring of H and ϕ' is a coloring of G, we say that ϕ *extends* to ϕ' if $\phi'(v) = \phi(v)$ for all $v \in V(H)$. Thomassen proved the following.

Theorem 1.5.1. *Let G be a 2-connected plane graph with no separating triangle and with outer cycle C. Let ϕ be a 5-coloring of $G[V(C)]$. Then G contains a connected subgraph H with at most $5^{|C|^3}$ vertices such that either*

(i) ϕ cannot be extended to a 5-coloring of H, or,

(ii) ϕ can be extended to a 5-coloring of H such that each vertex of $G \setminus H$ which sees more than two colors of H either has degree at most 4, or, has degree 5 and is joined to two distinct vertices of H of the same color.

The coloring of H in (ii) can be extended to a 5-coloring of G.

Yerger [54] was able to improve Theorem 1.5.1 by showing that there exists such an H with $|V(H)| \leq O(|C|^3)$. Postle and Thomas further improved Theorem 1.5.1 by proving that there exists such an H with $|V(H)| \leq O(|C|)$, which is best possible up to a multiplicative constant.

As for list-coloring, here is a very useful little theorem of Bohme et al. [12], originally shown by Thomassen for regular coloring [45], that characterizes when precolorings of cycles of length at most six do not extend. First let us say that if ϕ is a coloring of a subgraph of H of a graph G, then a vertex $u \in V(G) \setminus V(H)$ *sees* a color c if there exists $v \in V(H) \cap N(u)$ such that $\phi(v) = c$. With this in mind, we let $S(u) = L(u) \setminus \{\phi(v)|v \in N(u) \cap V(H)\}$, the set of *available colors* of u.

Theorem 1.5.2. *Let G be a plane graph, $C = c_1c_2 \ldots c_k$ be a facial cycle of G such that $k \leq 6$, and L be a 5-list-assignment. Then every proper precoloring ϕ of $G[V(C)]$ extends to an L-coloring of G unless one of the following conditions holds:*

(i) *$k \geq 5$ and there is a vertex $u \in V(G) \setminus V(G)$ such that v is adjacent to at least five vertices in C and u has no available colors, or,*

(ii) *$k = 6$ and there is an edge u_1u_2 in $E(G \setminus H)$ such that u_1, u_2 each have one available color and it is the same for both,*

(iii) *$k = 6$ and there is a triangle $u_1u_2u_3$ in $G \setminus H$ such that u_1, u_2, u_3 have the same set of available colors and that set has size two.*

Meanwhile Dvorak, Lidicky, Mohar and Postle [27] generalized Theorem 1.5.1 to list-coloring with a quadratic bound. They also conjectured the existence of a linear bound. In Chapter 3, we prove just such a linear bound for list-coloring.

Thomassen then extended Theorem 1.5.1 to the case when the precolored subgraph has more than one component. He proved the following stronger version of Theorem 1.3.2.

Theorem 1.5.3. *For all $g, q \geq 0$, there exists a function $f(g, q)$ such the following holds: Let G be a graph embedded in a surface Σ of Euler genus g and let S be a set of at most q vertices in G. If ϕ is a 5-coloring of S, then ϕ extends to a 5-coloring of G unless there is a graph H with at most $f(g, q)$ vertices such that $S \subseteq H \subseteq G$ and the 5-coloring of S does not be extend to a 5-coloring of H.*

In fact, Postle and Thomas [42] proved that f is linear. In Chapters 4 and 5, we generalize Theorem 1.5.3 to list-coloring. Indeed, we prove that f is linear, which is best possible up to a multiplicative constant.

Furthermore, Thomassen wondered though whether the dependence of f on the number of components in Theorem 1.5.3 could be dropped if certain conditions were satisfied. Specifically, Thomassen conjectured [50] that if all the components of S were just isolated vertices whose pairwise distance in the graph was large, then any pre-coloring of S always extends. Albertson [1] proved this in 1997. He then conjectured that this generalizes to list-coloring.

Conjecture 1.5.4. There exists D such that the following holds: If G is a plane graph with a 5-list assignment L and $X \subset V(G)$ such that $d(u,v) \geq D$ for all $u \neq v \in X$, then any L-coloring of X extends to an L-coloring of G.

Dvorak, Lidicky, Mohar, and Postle [27] recently announced a proof of Albertson's conjecture. In Chapter 5, we will give a different proof of Albertson's conjecture more in line with the results of Axenovich, Hutchinson, and Lastrina [10].

Indeed, Thomassen [50] conjectured something more.

Problem 1.5.5. Let G be a planar graph and $W \subset V(G)$ such that $G[W]$ is bipartite and any two components of $G[W]$ have distance at least d from each other. Can any coloring of $G[W]$ such that each component is 2-colored be extended to a 5-coloring of G if d is large enough?

Thomassen proved Problem 1.5.5 when W consists of two components (see Theorem 7.3 of [50]). Albertson and Hutchinson [4] proved Problem 1.5.5. As for list coloring, Theorem 1.4.2 proves Problem 1.5.5 when W has one component and the question asks whether the coloring can be extended to an L-coloring of G where L is a 5-list-assignment. In Chapter 5, we prove the list-coloring version when W has two components. We believe the results of Chapters 3 and 5 will also yield a proof when W has any number of components but for now this remains open. Note that a proof of the list-coloring vertsion of Problem 1.5.5 was announced without proof by

Kawarabayashi and Mohar in [38], where the distance d grows as a function of the number of components of W.

In addition, Albertson and Hutchinson [3] have generalized Albertson's result to other surfaces. They proved that if the graph is locally planar, then any precoloring of vertices far apart extends.

Theorem 1.5.6. *Let G be a graph embedded in a surface $\Sigma \neq S_0$ such that $ew(G) \geq 208(2^{g(\Sigma} - 1)$. If $X \subseteq V(G)$ such that $d(u,v) \geq 18$ for all $u \neq v \in X$, then any 5-coloring of X extends to an 5-coloring of G.*

In Chapter 5, we prove a similar generalization for list-coloring for surfaces when $ew(G) \geq \Omega(\log g)$ but the distance between vertices in X is also at least $\Omega(\log g)$. Note that such a generalization for surfaces when $ew(G) \geq 2^{\Omega(g)}$ and the distance between vertices in X grows as a function of X was announced by Kawarabayshi and Mohar in [38].

Meanwhile, Dean and Hutchinson [14] have proven that if G is a graph embedded in a surface Σ, L is a $H(\Sigma)$-list-assignment for $V(G)$ and $X \subset V(G)$ such that all vertices $u \neq v \in X$ have pairwise distance at least four, then any L-coloring of X extends to an L-coloring of G. They also asked the following:

Question 1.5.7. For which $k \geq 5$ does there exist $d_k > 0$ such that their result holds when $H(\Sigma)$ is replaced by k?

Of course, some additional proviso is necessary in Question 1.5.7 as there exists graphs that are not k-list-colorable for $k \leq H(\Sigma)$. Hence either G being L-colorable or the stronger assumption of large edge-width seems to be required.

In Chapter 5, we generalize Theorem 1.5.6 to list-coloring. We also improve the necessary lower bound on the edge-width to be $\Omega(\log g(\Sigma))$, which is best possible up to a multiplicative constant. This also answers Question 1.5.7 in the affirmative for all $k \geq 5$ with the proviso that G has edge-width $\Omega(\log g(\Sigma))$.

Albertson and Hutchinson [5] also prove a similar version of Problem 1.5.5 for surfaces. We believe our techniques can also generalize that result to list-coloring while improving their bound, but for now this remains open.

1.6 5-List-Coloring with Crossings Far Apart

Definition. We say a graph G is *drawn* in a surface Σ if G is embedded in Σ except that there are allowed to exist points in Σ where two — but only two — edges *cross*. We call such a point of Σ and the subsequent pair of edges of G, a *crossing*.

Dvorak, Lidicky and Mohar [26] proved that crossings far apart instead of precolored vertices also leads to 5-list-colorability.

Theorem 1.6.1. *If G can be drawn in the plane with crossings pairwise at distance at least 15, then G is 5-list-colorable.*

In Chapter 5, we provide an independent proof of Theorem 1.6.1. Indeed, we generalize Theorem 1.6.1 to other surfaces. Of course, Theorem 1.6.1 does not generalize verbatim as some condition is necessary to even guarantee that a graph drawn on a surface without crossings is 5-list-colorable. A lower bound on the edge-width seems to be a natural condition that guarantees that a graph is 5-list-colorable. Thus we will generalize Theorem 1.6.1 to other surfaces with addition requirement of having large edge-width. Indeed we will prove that edge-width logarithmic in the genus of the surface suffices, which is best possible.

1.7 Exponentially Many Colorings

Thomassen wondered whether a planar graph has many 5-list-colorings. He [51] proved the following.

Theorem 1.7.1. *If G is a planar graph and L is a 5-list assignment for G, then G has $2^{|V(G)|/9}$ L-colorings.*

Thomassen [49, 51] then conjectured that Theorem 1.7.1 may be generalized to other surfaces. Of course not every graph on other surfaces is 5-list-colorable. Hence, Thomassen conjectured the following.

Conjecture 1.7.2. Let G be a graph embedded in a surface Σ and L is a 5-list-assignment for G. If G is L-colorable, then G has $2^{c|V(G)|}$ L-colorings where c is a constant depending only on g, the genus of Σ.

Note that a proof of Conjecture 1.7.2 was announced without proof by Kawarabayashi and Mohar in [38]. We provide an independent proof of Conjecture 1.7.2 in Chapter 5. Indeed, we will show that precoloring a subset of the vertices still allows exponentially many 5-list-colorings where the constant depends only on the genus and the number of precolored vertices. In fact, we show that the dependence on genus and the number of precolored vertices can be removed from the exponent.

1.8 Main Results

Let us now state the main results of this thesis. First let us note the following theorem about extending the coloring of a precolored cycle to a list coloring of the whole graph.

Theorem 1.8.1. *Let G be a 2-connected plane graph with outer cycle C and L a 5-list-assignment for G. Then G contains a connected subgraph H with at most $29|C|$ vertices such that for every L-coloring ϕ of C either*

(i) ϕ cannot be extended to an L-coloring of H, or,

(ii) ϕ can be extended to an L-coloring of G.

This settles a conjecture of Dvorak et al. [27] in the affirmative. The fact that the bound is linear is crucial to proving many of the main results. Indeed the main results of Chapter 5 are first proved in a general setting about families of graphs satisfying a more abstract version of Theorem 1.8.1.

Another key ingredient of the proof is to extend Theorem 1.8.1 to two cycles. However, before this could be done, we found it necessary to prove a number of generalizations of Theorem 1.4.2. Here is one of the more elegant generalizations of Theorem 1.4.2, which we prove in Section 2.5.

Theorem 1.8.2. *If G is a plane graph with outer cycle C and $p_1, p_2 \in V(G)$ and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus \{p_1, p_2\}$, and $|L(p_1)| = |L(p_2)| = 2$, then G is L-colorable.*

This settles a conjecture of Hutchinson [36] in the affirmative.

Applying the general theory developed in Chapter 5, we prove a number of new results. In Section 5.8.1, we settle Conjecture 1.4.5 in the affirmative. Indeed, mirroring Theorem 1.3.4, we prove a linear bound in terms of genus of the number of vertices of a 6-list-critical graph.

Theorem 1.8.3. *If G is a 6-list-critical graph embedded in a surface Σ, then $|V(G)| = O(g(\Sigma))$.*

Indeed, Theorem 1.8.3 is best possible up to a multiplicative constant. Also it should be noted that Theorem 1.8.3 provides independent proofs of Theorems 1.3.2 and 1.3.4, though the constant in Theorem 1.3.4 may be better. As a corollary of Theorem 1.8.3, we can approximately determine the number of 6-list-critical graphs embeddable in a fixed surface.

Theorem 1.8.4. *Let Σ be a surface. There exist only finitely many 6-list-critical graphs embeddable in Σ.*

An immediate corollary of Theorem 1.8.3 is that we are now able to decide 5-list-colorablity on a fixed surface in linear-time:

Theorem 1.8.5. *There exists a linear-time algorithm to decide 5-list-colorability on a fixed surface.*

In Section 5.8, we actually prove a stronger version of Theorem 1.8.3 that allows the precoloring of a set of vertices.

Theorem 1.8.6. *Let G be a connected graph 2-cell embedded in a surface Σ, $S \subseteq V(G)$ and L a 5-list-assignment of G. Then there exists a subgraph H with $|V(H)| = O(|S| + g(\Sigma))$ such that for every L-coloring ϕ of S either*

(1) *ϕ does not extend to an L-coloring of H, or*

(2) *ϕ extends to an L-coloring of G.*

In addition, in Section 5.8.2, we use Theorem 1.8.6 to give an independent proof of Theorem 1.4.6 while improving the bound on the necessary edge-width from exponential in genus to logarithmic in genus. This also improves the best known lower bound for regular coloring which was linear in genus.

Theorem 1.8.7. *If G is 2-cell embedded in a surface Σ and $ew(G) \geq \Omega(\log g(\Sigma))$, then G is 5-list-colorable.*

Indeed, Theorem 1.8.7 is best possible given the existence of Ramanujan graphs [40] which have girth k, $2^{\Theta(k)}$ vertices and large fixed chromatic number (e.g. 6). Moreover in Section 5.8.2, using Theorem 1.8.6 we are also able to prove the following theorem about extending a precoloring of vertices pairwise far apart.

Theorem 1.8.8. *Let G be 2-cell embedded in a surface Σ, $ew(G) \geq \Omega(\log g)$ and L be a 5-list-assignment for G. If $X \subset V(G)$ such that $d(u, v) \geq \Omega(\log g(\Sigma))$ for all $u \neq v \in X$, then every L-coloring of X extends to an L-coloring of G.*

When Σ is the sphere, Theorem 1.8.8 reduces to Conjecture 1.5.4 and therefore provides an independent proof of that conjecture. Indeed, this is the generalization of that conjecture as developed by Albertson and Hutchinson, except that we have improved the necessary lower bound from exponential in genus to logarithmic in genus, which is best possible up to a multiplicative constant. In fact, in Section 5.8.2, we are able to prove a stronger version of Theorem 1.8.8. Namely, we generalize to the case of precoloring cycles far apart.

Theorem 1.8.9. *Let G be 2-cell embedded in a surface Σ, $ew(G) \geq \Omega(\log g)$ and L be a 5-list-assignment for G. Let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be a collection of disjoint cycles of G such that $d(C_i, C_j) \geq \Omega(\log(|C_i| + |C_j| + g(\Sigma)))$ for all $C_i \neq C_j \in \mathcal{C}$ and the inherited embedding of $G_i = B_{\Omega(\log(|C_i|+g(\Sigma)))}(C_i)$ is plane for all $C_i \in \mathcal{C}$. If ϕ is an L-coloring of the cycles in $\mathcal{C}$ such that $\phi \restriction C_i$ can be extended to an L-coloring of G_i for all $C_i \in \mathcal{C}$, then ϕ extends to an L-coloring of G.*

Note that we do not require a strict upper bound on the size of the cycles, rather just that the pairwise distance between the cycles as well as the locally planar neighborhood of the cycles reflect their size. In Section 5.8.2, we provide an application of Theorem 1.8.9 for cycles of size four. We prove the following theorem.

Theorem 1.8.10. *Let G be drawn in a surface Σ with a set of crossings X and L be a 5-list-assignment for G. Let G_X be the graph obtained by adding a vertex v_x at every crossing $x \in X$. If $ew(G_X) \geq \Omega(\log g(\Sigma))$ and $d(v_x, v_{x'}) \geq \Omega(\log g(\Sigma))$ for all $v_x \neq v_{x'} \in V(G_X) \setminus V(G)$, then G is L-colorable.*

When Σ is the sphere, Theorem 1.8.10 reduces to Theorem 1.6.1 and hence provides an independent proof of that result. Finally in Section 5.9, we settle Conjecture 1.7.2 in the affirmative.

Theorem 1.8.11. *For every surface Σ there exists a constant $c > 0$ such that following holds: Let G be a graph embedded in Σ and L a 5-list-assignment for G. If G has an L-coloring, then G has at least $2^{c|V(G)|}$ L-colorings of G.*

Indeed, we prove a stronger version of Theorem 1.8.11 about extending a precoloring of a subset of the vertices.

Theorem 1.8.12. *There exist constants $c, c' > 0$ such that following holds: Let G be a graph embedded in a surface Σ, $X \subseteq V(G)$ and L a 5-list-assignment for G. If ϕ is an L-coloring of $G[X]$ such that ϕ extends to an L-coloring of G, then ϕ extends to at least $2^{c|V(G)|-c'(g(\Sigma)+|X|)}$ L-colorings of G.*

1.9 Outline of the Proof

In Chapter 2, we generalize Theorem 1.4.2 to the case where the two precolored vertices are not adjacent but have lists of size two. This resolves a conjecture of Hutchinson [36]. We then proceed to characterize the critical graphs when two non adjacent precolored vertices have lists of size one and two and then lists of size one. We then characterize the critical graphs for two precolored edges that are not incident. Thomassen [51] characterized these when the edges are incident with the same vertex, that is for paths of length two. Indeed, we show that if the two edges are far apart, then there is a proportionally long segment of the graph which has a particularly nice structure, called a bottleneck, that is one of two types, called accordions and harmonicas.

In Chapter 3, we prove a linear bound for Theorem 1.5.1 for list-coloring. We then generalize this in the manner of Theorem 1.4.2 to prove a linear bound in terms of the precolored path. Furthermore, we show that if there are many precolored paths, as opposed to just two precolored edges, then either a linear bound is obtained or there a long bottleneck as in Chapter 2. Furthermore, we expand the usefulness of such linear bounds by showing that these critical graphs have other nice properties. Namely, we show that all vertices have logarithmic distance from the precolored vertices and the balls around vertices grow exponentially in their radius. In addition, we prove that precolorings of a cycle have exponentially many extensions to the interior, less a linear factor in the size of the cycle.

In Chapter 4, we use the bottleneck theorem to prove the any coloring of two precolored triangles that are far apart extends to the whole graph. Our strategy is as follows. We show there exists a long chain of separating triangles where the graph between any two consecutive triangles in the chain is one of three types, called tetrahedral, octahedral and hexadecahedral after their number of faces. We then develop a theory somewhat akin to that in Chapter 2, to show that if the chain is long enough then any coloring of the outer and inner triangle extends to the whole graph.

In Chapter 5, we generalize the main result of Chapter 4 to the case of two precolored cycles. We then extend the linear bound, logarithmic distance and exponential growth results from Chapter 3 to the case of two cycles. Next we proceed to develop an abstract theory for families of graphs satisfying such linear isoperimetric inequalities for the disc and cylinder, which we call hyperbolic families. We prove all of our main theorem hold in the setting of hyperbolic families. The theory of hyperbolic families has wider applications beyond 5-list-coloring as there exist other hyperbolic families of interest. The families of critical graphs of a number of other coloring problems are hyperbolic. Indeed, it follows that any coloring problem satisfying Theorem 1.8.1 leads to similar theorems as developed in this thesis. After developing the general theory of hyperbolic families, we then apply the theory to the family of 6-list-critical graphs to derive the main results for 5-list-coloring. Finally, we apply the theory for a slightly different family to obtain the exponentially many 5-list-colorings result.

CHAPTER II

TWO PRECOLORED VERTICES

2.1 Introduction

In this chapter, we prove generalizations of Theorem 1.4.2 of Thomassen, restated here for convenience.

Theorem 2.1.1 (Thomassen). *If G is a plane graph with outer cycle C and $P = p_1p_2$ is a path of length one in C and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(P)$, and $|L(p_1)| = |L(p_2)| = 1$ with $L(p_1) \neq L(p_2)$, then G is L-colorable.*

In Section 2.5, we will resolve a conjecture of Hutchinson [36] in the affirmative that Theorem 1.4.2 can be extended to the case where p_1 and p_2 are not required to be adjacent and both p_1 and p_2 have lists of size two. This is Theorem 1.8.2, which we restate here for convenience.

Theorem 2.1.2. *If G is a plane graph with outer cycle C and $P = \{p_1, p_2\}$ and L is a list assignment with $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$, $|L(v)| \geq 3$ for all $v \in V(C) \setminus V(P)$, and $|L(p_1)| = |L(p_2)| = 2$, then G is L-colorable.*

Note that Theorem 1.8.2 is a strict generalization of Theorem 1.4.2 as any minimum counterexample to that theorem yields a counterexample to Theorem 2.2.2. Indeed as we will in Chapter 3, Theorem 1.8.2 will imply a very useful characterization of when colorings of paths P with $|V(P)| > 2$ as in Theorem 1.4.2 do not extend. Furthermore, we also generalize Hutchinson's results [36] about the case when one or both of p_1, p_2 have a list of size one and G is an outerplane graph to case when G is plane while also providing independent proofs of said results.

In Section 2.6, we shall begin characterizing how the colorings of P in Theorem 1.4.2 extend to other paths of length one in the outer cycle. In Section 2.7, we will characterize the minimal non-colorable graphs when one of p_1, p_2 is allowed to have a list of size one. In Section 2.8, we will characterize the minimal non-colorable graphs when both p_1, p_2 are allowed to have lists of size one. Finally in Section 2.11, we will show that minimal non-colorable graphs when there are two precolored paths of length one far apart then there exists a a special structure of one of two types whose length is proportional to the distance between the edges.

2.2 Two with Lists of Size Two Theorem

In this section, we prove a generalization of this theorem. Let us define the graphs we will be investigating.

Definition (Canvas)**.** We say that (G, S, L) is a *canvas* if G is a connected plane graph, S is a subgraph of the boundary of the infinite face of G, and L is a list assignment for the vertices of G such that $|L(v)| \geq 5$ for all $v \in V(G) \setminus V(C)$ where C is the boundary of the infinite face of G, $|L(v)| \geq 3$ for all $v \in V(G) \setminus V(S)$, and there exists a proper L-coloring of S.

If S is a path that is also a subwalk of the the outer walk of G, then we say that (G, S, L) is a *path-canvas.* If the outer walk of G is a cycle C and $S = C$, then we say that (G, S, L) is a *cycle-canvas.*

We say an L-coloring ϕ of S is *non-extendable* if there does not exist an L-coloring ϕ' of G such that $\phi'(v) = \phi(v)$ for all $v \in V(S)$. We say the canvas (G, S, L) is L-*critical* if there does not an exist an L-coloring of G but for every edge $e \notin E(S)$ where both ends of e not in S, there exists an L-coloring of $G \setminus e$.

Hence, Thomassen's theorem restated in these terms is as follows.

Theorem 2.2.1 (Thomassen)**.** *If (G, P, L) is a path-canvas and $|V(P)| \leq 2$, then G is L-colorable.*

We can also restate Theorem 1.8.2 in these terms.

Theorem 2.2.2 (Two with List of Size Two Theorem)**.** *If (G, S, L) is a canvas with $V(S) = \{v_1, v_2\}$ and $|L(v_1)|, |L(v_2)| \geq 2$, then G is L-colorable.*

It should be noted that this theorem is not true when one allows three vertices with list of size two (e.g. an even fan). We actually prove a stronger statement. But first we need some preliminaries.

2.3 Fans and Bellows

A useful reduction is that found by Thomassen in his proof of 5-choosability.

Definition (Thomassen Reduction)**.** Let $T = (G, S, L)$ be a canvas. Let C be the outer walk of G. Suppose that there exists $v \in V(C) \setminus V(S)$ such that $uv \in E(C)$ where $u \in V(S)$, u is the only neighbor of v belonging to S, v is not a cutvertex of G and v does not belong to a chord of C. Given a coloring ϕ of S, we may define a *Thomassen reduction with respect to ϕ and v of T*, denoted by $T(\phi, v) = (G', S, L')$, as follows. Let $G' = G \setminus v$. Define a list assignment L' as follows. Let $S(v)$ be a subset of size two of $L(v) \setminus \phi(u)$. Let $L'(w) = L(w) \setminus S(v)$ for all w such that w is not in $V(C)$ and w is adjacent to v and let $L'(w) = L(w)$ otherwise.

Proposition 2.3.1. *Let $T(\phi, v) = (G', S, L')$ be a Thomassen reduction of $T = (G, S, L)$ with respect to ϕ and v. The following holds:*

(1) *$T(\phi, v)$ is a canvas.*

(2) *If G' has an L'-coloring extending ϕ, then G has an L-coloring extending ϕ.*

(3) *If T is L-critical, then G' contains an L'-critical subgraph G'' and hence there exists an L''-critical canvas (G'', S, L'') where $L''(v) = \{\phi(v)\}$ for $v \in S$ and $L''(v) = L'(v)$ otherwise.*

Proof. If $x \in V(G')$ such that $|L'(x)| < 5$, then either $x \in C$ or $x \sim v$. In either case, x is in the outer walk of G'. Note that if $x \in V(G')$ such that $L'(x) \neq L(x)$, then $x \notin C$ and hence $|L(x)| = 5$, $|L'(x)| \geq 3$. Thus, if $x \in V(G')$ such that $|L'(x)| < 3$, then $x \in S$. This proves that (1).

Let ϕ be an L'-coloring of G'. Let $\phi(v) = S(v) \setminus \phi(w)$ where $w \notin S$ and $vw \in E(C)$. Now ϕ is an L-coloring of G. This proves (2).

If T is L-critical, then there does not exist an L-coloring of G. Hence by (2), there does not exist an L'-coloring of G' extending ϕ. Hence, there exists an L''-critical subcanvas and (3) follows. □

Definition (Fans)**.** We say a graph G is a *fan* if G consists of a cycle $C = v_1v_2 \ldots v_k$ and edges v_1v_i for all $i, 3 \leq i \leq k-1$. We say that v_1 is the *hinge* of the fan and that the path $v_2v_1v_k$ is the *base* of the fan. We define the *length* of the fan to be $k-2$. We say a fan G is *even* if its length is even and *odd* if its length is odd.

Proposition 2.3.2. *Let G be a fan with cycle $v_1v_2 \ldots v_k$ where v_1 is the hinge of the fan and let L be a list assignment for G such that $|L(v)| \geq 3$ for all $v \in V(G \setminus \{v_1, v_2, v_k\})$. If ϕ is an L-coloring of P, then ϕ extends to an L-coloring of G unless there exist colors $c_3 \ldots, c_{k-2}$ such that $c_1 \neq c_i$ and $c_i \neq c_{i+1}$ for all i, $2 \leq i \leq k-2$ and $L(v_i) = \{c_1, c_{i-1}, c_i\}$ for all i, $3 \leq i \leq k-1$, where $c_1 = \phi(v_1), c_2 = \phi(v_2), \phi(v_k) = c_{k-1}$.*

Definition. Let $T = (G, P, L)$ be a path-canvas with $|V(P)| = 3$. Suppose that G is a fan with cycle $C = v_1v_2 \ldots v_k$ and hinge v_1 where $P = v_2v_1v_k$. We say that T is a *fan* if there exists a non-extendable L-coloring of P.

We say a fan T is *even* if G is an even fan and *odd* if G is odd fan. We say an odd fan T is *exceptional* if there exist two non-extendable L-colorings of P which differ only in the color of the hinge. We say an even fan T is *exceptional* if there exist two non-extendable L-colorings of P which interchange the colors of the hinge and another vertex in the base.

Definition (Wheel)**.** We say a graph G is a *wheel* if G is a cycle C and a vertex $v \notin V(C)$ such that v adjacent to every vertex of C. We say v is the *center* of the wheel.

Proposition 2.3.3. *Let G be a wheel with cycle $v_1v_2\dots v_k$ and center v and let L be a list assignment for G such that $|L(v)| \geq 3$ for all $v \in V(G \setminus \{v_1, v_2, v_3\})$. If ϕ is an L-coloring of $P = v_1v_2v_3$, then ϕ extends to an L-coloring of G unless $\phi(v_1) \neq \phi(v_3)$, $\{\phi(v_1), \phi(v_2), \phi(v_3)\} \subset L(v)$ and $(G \setminus v_2, P', L)$ is an exceptional odd fan where $P' = v_1vv_3$ with two non-extendable colorings, ϕ_1, ϕ_2 extending ϕ, where $\{\phi_1(v), \phi_2(v)\} = L(v) \setminus \{\phi(v_1), \phi(v_2), \phi(v_3)\}$.*

Definition (Turbofans)**.** Let $T = (G, P, L)$ be a path-canvas with $|V(P)| = 3$. Let $P = p_1p_2p_3$. Suppose that G is a wheel with center $v \notin V(P)$. We say that T is a *turbofan* if there exists a non-extendable L-coloring of P. We say that P is the *base* of T.

Definition (1-Sum)**.** Let $T = (G, S, L)$ be a canvas. Let C be the facial walk of the infinite face of G. Suppose that $v \in C$ is a cutvertex of C. Thus v divides G into two graphs G_1, G_2. Let $V(S_1) = V(S \cap G_1) \cup \{v\}$ and $E(S_1) = E(S) \cap E(G_1)$ and similarly for S_2. Let $L_1(v), L_2(v) \subset L(v)$ such that $L_1(v) \cup L_2(v) = L(v)$ and let $L_1(x) = L(x)$, $L_2(x) = L(x)$ otherwise. Let $T_1 = (G_1, S_1, L_1)$ and $T_2 = (G_2, S_2, L_2)$. We say that T is the *1-sum* of T_1 and T_2 along the vertex v.

Definition (2-Sum)**.** Let $T = (G, S, L)$ be a canvas. Let C be the facial walk of the infinite face of G. Suppose that uv is a chord of C. Thus uv divides G into two graphs G_1, G_2. Let $V(S_1) = V(S \cap G_1) \cup \{u, v\}$ and $E(S_1) = (E(S) \cap E(G_1)) \cup \{uv\}$ and similarly for S_2. Let $T_1 = (G_1, S_1, L)$ and $T_2 = (G_2, S_2, L)$. We say that T is the *2-sum* of T_1 and T_2 along the edge uv.

Definition (Bellows)**.** Let $T = (G, P, L)$ be a canvas with $|V(P)| = 3$. Let $P = p_1p_2p_3$ and C be the outer walk of G. We say T is a *bellows* if either T is a fan, a turbofan, or T is the 2-sum of two smaller bellows along the edge p_2x for some vertex $x \in C \setminus P$ such that there exists a non-extendable L-coloring of P. We say that P is the *base* of T.

Thomassen [51, Theorem 3] proved the following.

Theorem 2.3.4 (Thomassen)**.** *If $T = (G, P, L)$ is a path-canvas with $|V(P)| = 3$, then G has an L-coloring unless there exists a subgraph $G' \subseteq G$, $P \subset G'$, such that $T' = (G', P, L)$ is a bellows.*

We will also need two lemmas of Thomassen. The following can be found in [51, Lemma 1].

Lemma 2.3.5. *If $T = (G, P, L)$ is a bellows that is not a fan, then there exists at most one proper L-coloring of P that does not extend to an L-coloring of $G \setminus P$.*

The following can be found in [51, Lemma 4].

Lemma 2.3.6. *If T is a bellows with $|L(p_1)| = 1$ and $|L(p_3)| = 3$, then there exists a color c in $L(p_3)$ such that any proper L-coloring ϕ of P with $\phi(p_3) = c$ can be extended to an L-coloring of G. If T is not an exceptional even fan (where p_2 and p_3's colors are interchanged), then there exist at least two such colors.*

We will need a very similar lemma later on, which we state here for convenience.

Lemma 2.3.7. *If T is a bellows with $|L(p_3)| = 2$, then there exist at most two colors c in $L(p_1)$ such that there exists a proper L-coloring ϕ of $\{p_1, p_2\}$ with $\phi(p_1) = c$ that cannot be extended to an L-coloring of G. Moreover there exists at most one such color, unless T is an odd fan, $L(p_3) \subset L(p_1)$ and the two such colors are $L(p_3)$ (and the non-extendable colorings are from $L(p_3)$).*

Proof. Let ϕ be a non-extendable L-coloring of $\{p_1, p_2\}$. By Theorem 1.4.2, it follows that $\phi(p_2) \in L(p_3)$. Let $L(p_3) = \{c_1, c_2\}$. Let $\phi_1(p_3) = c_1$ and $\phi_1(p_2) = c_2$; let

$\phi_2(p_3) = c_2$ and $\phi_2(p_2) = c_1$. By Theorem 1.4.2, there exists at most one color b_i in $L(p_1)$ such that ϕ_i extends to a non-extendable coloring of P. Thus there exist at most two such colors in $L(p_1)$, namely b_1 and b_2 satisfying the conclusion as desired.

Suppose b_1 and b_2 exist. We may assume without loss of generality that $L(p_1) = \{b_1, b_2\}$. Let p_3' such that $p_3'p_3p_2$ is a triangle in G. If $p_3' = p_1$, it follows that $L(p_3) = \{b_1, b_2\}$ and the lemma follows. So we may suppose that $p_3' \neq p_1$. Consider $T' = (G \setminus \{p_3\}, P', L')$ where $P' = p_1p_2p_3'$, $L'(p_3') = L(p_3') \setminus L(p_3)$, $L'(p_1) \supseteq L(p_1)$ be a set of size three and $L'(v) = L(v)$ for all $v \in G \setminus \{p_3, p_3'\}$. Now T' is a bellows with $|L'(p_3)| = 1$ and $|L'(p_1)| = 3$. By Lemma 2.3.6, there exist at least two colors c in $L'(p_1)$ such that any proper L'-coloring ϕ' of P' with $\phi'(p_3) = c$ can be extended to an L'-coloring of $G \setminus \{p_1\}$ unless T' is an exceptional even fan. If T' is not an exceptional fan, this implies that there exists a color c in $L(p_1)$ such that any proper L-coloring ϕ of $\{p_1, p_2\}$ with $\phi(p_1) = c$ can be extended to an L-coloring of G. Yet $c \in \{b_1, b_2\}$, a contradiction.

So T' is an exceptional even fan. Indeed the colors of p_1 and p_2 must be interchanged its non-extendable L'-colorings. Thus $L(p_1) = \{c_1, c_2\} = L(p_3)$ and the lemma follows. $\square$

Another useful lemma we will need, which has the same spirit as the above, is the following.

Lemma 2.3.8. *If T is a bellows with $|L(p_1)| = 1$, $|L(p_3)| = 1$, then there exist at most two colors c in $L(p_2) \setminus (L(p_1) \cup L(p_3))$ such that the proper L-coloring ϕ of P with $\phi(p_1) = L(p_1)$, $\phi(p_3) = L(p_3)$ and $\phi(p_2) = c$ can not be extended to an L-coloring of G. If T is not an exceptional odd fan, then there exists at most one color.*

Proof. We proceed by induction on the number of vertices of G. By Lemma 2.3.5, it follows that T is a fan. Let the outer cycle of G be labeled as $p_1, v_1, \ldots, v_k, p_3, p_2$. Clearly $k \geq 1$. Let $P' = v_1p_2p_3$. Consider the bellows $T' = (G \setminus \{p_1\}, P', L)$. By

Lemma 2.3.6, there exists a color c in $L(v_1)$ such that any proper L-coloring ϕ of P' with $\phi(v_1) = c$ can be extended to an L-coloring of $G \setminus \{p_1\}$. Yet it follows that such a color c must be in $L(p_1)$. Hence there does not exist at least two such colors and by Lemma 2.3.6, T' is an exceptional even fan whose non-extendable colorings interchange the colors of v_1 and p_2. It follows that T is an exceptional odd fan and the two colors are colors in the non-extendable colorings of T'. □

We may now characterize the non-extendable L-colorings of a bellows.

Proposition 2.3.9. *Let $T = (G, P, L)$ be a path-canvas with $|V(P)| = 3$. Let $P = p_1p_2p_3$ and C be the outer walk of G. Suppose that T is the sum of two bellows $T_1 = (G_1, P_1, L)$ and (G_2, P_2, L) along the edge p_2x where $x \in V(C \setminus P)$. If ϕ is an L-coloring of P, then ϕ is non-extendable if and only if $L(x)$ has size three and can be denoted $\{\phi(p_2), c_1, c_2\}$ such that the coloring ϕ_1 with $\phi_1(p_1) = \phi(p_1), \phi_1(p_2) = \phi(p_2)$ and $\phi_1(x) = c_1$ does not extend to G_1 and the coloring ϕ_2 with $\phi_2(p_3) = \phi(p_3), \phi_2(p_2) = \phi(p_2)$ and $\phi_2(x) = c_2$ does not extend to G_2.*

Proof. Suppose ϕ is non-extendable. We may assume without loss of generality that $L(p_i) = \phi(p_i)$ for all $i \in \{1, 2, 3\}$. First suppose that $\phi(p_2) \notin L(x)$ or $|L(x)| \geq 4$. By Theorem 1.4.2, there exists an L-coloring ϕ_1 of G_1 that extends $\phi \upharpoonright P_1 \cap P$. Let $L_1(x) = L(x) \setminus \phi_1(x)$ and $L_1(v) = L(v)$ for all $v \in G_1 \setminus \{x\}$. By Theorem 1.4.2, there exists an L_1-coloring ϕ_2 of G_1 that extends $\phi \upharpoonright P_1 \cap P$. Let $L_2(x) = \{\phi_1(x), \phi_2(x), \phi(p_2)\}$ and $L_2(v) = L_2(v)$ for all $v \in G_2 \setminus \{x\}$. By Theorem 1.4.2, there exists an L_2-coloring ϕ_3 of G_2. Let $\phi' = \phi_3 \cup \phi_j$ where $j \in \{1, 2\}$ and $\phi_3(x) = \phi_j(x)$. Now ϕ' is an L-coloring of G that extends ϕ, a contradiction.

So we may suppose that $\phi(p_2) \in L(x)$ and $|L(x)| = 3$. Let $L(x) = \{\phi(p_2), c_1, c_2\}$. By Theorem 1.4.2, there exist L-colorings ϕ_1 of G_2 and ϕ_2 of G_1. Yet $\phi_1(x) \neq \phi_2(x)$ as otherwise $\phi_1 \cup \phi_2$ is an L-coloring of G, a contradiction. We may suppose without loss of generality that $\phi_1(x) = c_1$ and $\phi_2(x) = c_2$. But then $\phi_1 \upharpoonright P_1$ does not extend to G_1 and $\phi_2 \restriction P_2$ does not extend to G_2 and the proposition holds.

Similarly the converse holds as x must receive a color in any L-coloring of G and yet that color must either be c_1, which does not extend to G_1, or c_2, which does not extend to G_2. $\square$

2.4 Critical Lemmas

Definition. Let $T = (G, S, L)$ be a canvas and let C be the outer walk of G. We say a cutvertex v of G is *essential* if whenever v divides G into graphs $G_1, G_2 \neq G$, where $V(G_1) \cap V(G_2) = \{v\}$, $G_1 \cup G_2 = G$, then $S \cap (V(G_i) \setminus \{v\}) \neq \emptyset$ for all $i \in \{1, 2\}$. We say a chord U of C is *essential* if for every division of G into graphs $G_1, G_2 \neq G$, such that $V(G_1) \cap V(G_2) = V(U)$ and $G_1 \cup G_2 = G$, then $S \cap (V(G_i) \setminus V(U)) \neq \emptyset$ for all $i \in \{1, 2\}$

Lemma 2.4.1. *If $T = (G, S, L)$ is an L-critical canvas, then*

(1) *every cutvertex of G is essential, and*

(2) *every chord of the outer walk of G is essential, and*

(3) *there does not exist a vertex in the interior of a cycle of size at most four, and*

(4) *there exists at most one vertex in the interior of a cycle of size five.*

Proof. Suppose v is a cutvertex of G that is not essential. Hence there exist graphs G_1, G_2 such that $V(G_1) \cap V(G_2) = \{v\}$ and $V(S) \cap V(G_2) \subseteq \{v\}$. As T is L-critical, there exists an L-coloring ϕ of G_1. By Theorem 1.4.2, ϕ can be extended to G_2. Thus G has an L-coloring, a contradiction. This proves (1).

Suppose $U = u_1u_2$ is a chord of the outer walk of G that is not essential. Hence there exist graphs G_1, G_2 such that $V(G_1) \cap V(G_2) = \{u_1, u_2\}$ and $V(S) \cap V(G_2) \subseteq \{u_1, u_2\}$. As T is L-critical, there exists an L-coloring ϕ of G_1. By Theorem 1.4.2, ϕ can be extended to G_2. Thus G has an L-coloring, a contradiction. This proves (2).

Let C be a cycle of size at most five in G. Let $G_1 = G \setminus (Int(C) \setminus C)$ and $G_2 = Int(C)$. Suppose $Int(C) \setminus C \neq \emptyset$. As T is L-critical, there exists an L-coloring ϕ of G_1. By Theorem 1.5.2, ϕ can be extended to an L-coloring of G_2, a contradiction, unless $|C| = 5$ and there exists a vertex $v \in Int(C) \setminus C$ adjacent to all of the vertices of C. (3) has thus been proved. In addition, by (3), there cannot be vertices in the interior of the triangles containing v in $C + v$ and hence there is at most one vertex in the interior of C. This proves (4). □

2.5 *Proof of the Two with Lists of Size Two Theorem*

In this section, we prove Theorem 2.2.2. But first a definition.

Definition (Democratic Reduction)**.** Let $T = (G, S, L)$ be a canvas and L_0 be a set of two colors. Suppose that $P = p_1 \dots p_k$ is a path in C such that, for every vertex v in P, v is not the end of a chord of C or a cutvertex of C and $L_0 \subset L(v)$. Let x be the vertex of C adjacent to p_1 and y be the vertex of C adjacent to p_k. We define the *democratic reduction* of P with respect to L_0 and centered at x, denoted as $T(P, L_0, x)$, as $(G \setminus P, S, L')$ where $L'(w) = L(w) \setminus L_0$ if $w \in (N(P) \setminus \{y\}) \cup \{x\}$ and $L'(w) = L(w)$ otherwise.

Indeed, we now prove a stronger version of Theorem 2.2.2.

Theorem 2.5.1. *Let $T = (G, S, L)$ be a canvas where S has two components: a path P and an isolated vertex u with $|L(u)| \geq 2$. If $|V(P)| \geq 2$, suppose that G is 2-connected, that u is not adjacent to an internal vertex of P and that there does not exist a chord of the outer cycle of G with an end in P which separates a different vertex of P from u.*

If $L(v) = L_0$ for all $v \in V(P)$ where $|L_0| = 2$, then G has an L-coloring unless $L(u) = L_0$ and $G[V(P) \cup \{u\}]$ is an odd cycle.

Proof. Let $T = (G, S, L)$ be a counterexample such that $|V(G)|$ is minimized and subject to that $|V(P)|$ is maximized. Let C be the outer walk of G. Hence T is

L-critical. By Claim 2.4.1(2), all chords of C are essential. By Claim 2.4.1(1), all cutvertices of G are essential.

So we may assume there is no chord of C with an end in P. Let v_1 and v_2 be the two vertices (not necessarily distinct) of C adjacent to P.

Claim 2.5.2. *G is 2-connected.*

Proof. Suppose there is a cutvertex v of G. By assumption then, $|V(P)| = 1$. Let $V(P) = \{u'\}$. Now v divides G into two graphs G_1 and G_2. As v is an essential cutvertex of G, we may suppose without loss of generality that $u \in V(G_2) \setminus V(G_1)$ and $u' \in V(G_1) \setminus V(G_2)$.

Consider the canvas (G_1, S_1, L) where $S_1 = P + v$. As $|V(G_1)| < |V(G)|$, there exists an L-coloring ϕ_1 of G_1 as T is a minimum counterexample. Let $L_1(v) = L(v) \setminus \{\phi_1(v)\}$ and $L_1(x) = L(x)$ for all $x \in G_1 \setminus \{v\}$. As $|V(G_1)| < |V(G)|$, there exists an L'-coloring ϕ_2 of G_1 by the minimality of G. Note that $\phi_1(v) \neq \phi_2(v)$. Let $L_2(v) = \{\phi_1(v), \phi_2(v)\}$ and $L_2(x) = L(x)$ for all $x \in G_2 \setminus \{v\}$. Consider the canvas (G_2, S_2, L_2) where $S_2 = P' + u$ and P' is a path with sole vertex v_2. As $|V(G_2)| < |V(G)|$, there exists an L_2-coloring ϕ of G_2. Let i be such that $\phi_i(v) = \phi(v)$. Therefore, $\phi \cup \phi_i$ is an L-coloring of G, contrary to the fact that T is a counterexample.

□

Claim 2.5.3. *Either $v_1 \neq u$ or $v_2 \neq u$.*

Proof. Suppose not. That is, $v_1 = v_2 = u$. If $L(u) \setminus L_0 \neq \emptyset$, let $\phi(u) \in L(u) \setminus L_0$ and extend ϕ to a coloring of $G[V(P) \cup \{u\}]$. Let $L'(v) = L(v) \setminus L_0$ for all $v \in V(G) \setminus (V(P) \cup \{u\})$ and $L'(u) = \{\phi(u)\}$. By Theorem 1.4.2, $G \setminus P$ has an L'-coloring and thus G has an L-coloring, contrary to the fact that T is a counterexample. So we may suppose that $L(u) = L_0$. Thus by assumption $G[V(P) \cup \{u\}]$ is an even cycle. Let ϕ be an L-coloring of $G[V(P) \cup \{u\}]$. Let $L'(v) = L(v) \setminus L_0$ for all

$v \in V(G) \setminus (V(P) \cup \{u\})$. By Theorem 1.4.2, $G \setminus (P + u)$ has an L'-coloring and thus G has an L-coloring, contrary to the fact that T is a counterexample. □

Thus $v_1 \neq v_2$ as there is no cutvertex of G.

Claim 2.5.4. *For all $i \in \{1, 2\}$, if $v_i \neq u$, then v_i is the end of an essential chord of C.*

Proof. As v_1 and v_2 are symmetric, it suffices to prove the claim for v_1. So suppose $v_1 \neq u$ and v_1 is not the end of an essential chord of C. First suppose that $|L(v_1) \setminus L_0| \geq 2$. Let $G' = G \setminus V(P)$, $S' = P' + u$ where P' is a path with sole vertex v_1. Furthermore, let $L'(v_1)$ be a subset of size two of $L(v_1) \setminus L_0$, $L'(x) = L(x) \setminus L_0$ for all $x \in N(P) \setminus \{v_1, v_2\}$ and $L'(x) = L(x)$ otherwise. As $|V(G')| < |V(G)|$, G' has an L'-coloring as T is a minimum counterexample. Thus G has an L-coloring, contrary to the fact that T is a counterexample.

So we may assume that $L(p_1) \subseteq L(v_1)$ and $|L(v_1)| = 3$. Let P' be the path obtained from P by adding v_1. Let $S' = P' + u$,$L'(v_1) = L_0$ and $L'(x) = L(x)$ for all $x \in V(G) \setminus \{v_1\}$. Consider the canvas (G, S', L'). As v_1 is not the end of an essential chord of C and (G, S, L) was chosen so that $|V(P)|$ was maximized, we find that $G[V(P) \cup \{v_1, u\}]$ is an odd cycle and $L(u) = L_0$.

Now color G as follows. Let $\phi(v_1) \in L(v_1) \setminus L_0$. Extend ϕ to a coloring of $V(P) \cup \{v_1, u\}$. Let $L'(x) = L(x) \setminus L_0$ for all $x \in V(G) \setminus (V(P) \cup \{v_1, u\})$ and $L'(v_1) = \phi(v_1)$. By Theorem 1.4.2, there exists an L'-coloring of $G \setminus (P + u)$ and hence G has an L-coloring, contrary to the fact that T is a counterexample. □

By Claim 2.5.3, we may assume without loss of generality that $v_1 \neq u$. By Claim 2.5.4, v_1 is the end of an essential chord of C. But this implies that $v_2 \neq u$. By Claim 2.5.4, v_2 is the end of an essential chord of C. As G is planar, it follows that $v_1 v_2$ is a chord of C.

Claim 2.5.5. $|V(P)| = 1$

Proof. Suppose not. Now v_1v_2 divides G into two graphs G_1, G_2 where without loss of generality $V(P) \subset V(G_1)$ and $u \in V(G_2)$. Construct a new graph G' with $V(G') = V(G_2) \cup \{v\}$ and $E(G') = E(G_2) \cup \{vv_1, vv_2\}$. Let $L(v) = L_0$. Consider the canvas (G', S, L) where $S = P' + u$ and P' is a path with sole vertex v. As $|V(P)| \geq 2$, $|V(G')| < |V(G)|$. So there exists an L-coloring ϕ of G' as T is a minimum counterexample. Hence there exists an L-coloring ϕ of G_2 where $\{\phi(v_1), \phi(v_2)\} \neq L(p_1)$. We extend ϕ to an L-coloring of $P \cup G_2$. Let $L'(x) = L(x) \setminus L(p_1)$ for all $x \in V(G_1) \setminus (V(P) \cup \{v_1, v_2\})$, $L'(v_1) = \phi(v_1)$ and $L'(v_2) = \phi(v_2)$. By Theorem 1.4.2, there exists an L'-coloring of $G_1 \setminus P$. Thus G has an L-coloring, contrary to the fact that T is a counterexample. □

So we may assume that $P = \{v\}$. Let $L_0 = L(v) = \{c_1, c_2\}$.

Claim 2.5.6. *For $i \in \{1, 2\}$, $L_0 \subset L(v_i)$ and $|L(v_i)| = 3$*

Proof. By symmetry it suffices to prove the claim for v_1. If $|L(v_i)| = 3$, then $L_0 \setminus L(v_1) \neq \emptyset$ and so let $c \in L_0 \setminus L(v_1)$. Otherwise we may suppose that $|L(v_1)| \geq 4$. In this case let $c \in L(v)$.

In either case, let $L'(v_1) = L(v_1) \setminus \{c\}$, $L'(v_2) = L(v_2) \setminus \{c\}$ and $L'(x) = L(x)$ otherwise. Consider the canvas (G', S', L') where $G' = G \setminus \{v\}$, $S' = P' + u$ and P' is a path with sole vertex v_2. As $|V(G')| < |V(G)|$, there exists an L'-coloring of G' as T is a minimum counterexample. Thus G has an L-coloring, contrary to the fact that T is a counterexample. □

Claim 2.5.7. $L(v_1) = L(v_2)$

Proof. Suppose not. As G is planar, either v_1 is not the end of a chord of C separating v_2 from u or v_2 is not the end of a chord separating v_1 from u. Assume without loss of generality that v_1 is not in a chord of C separating v_2 from u. This implies that v_1 is not the end of a chord in C. Let v' be the vertex in C distinct from v_2 that is adjacent to v_1.

Let $c = L(v_1) \setminus L_0$. Let $G' = G \setminus \{v, v_1\}$, $L'(x) = L(x) \setminus \{c\}$ if $x \sim v_1$ and $L'(x) = L(x)$ otherwise. Note that $|L'(v_2)| \geq 3$ as $L(v_1) \neq L(v_2)$. First suppose that $u \neq v'$. Hence $|L(u)|, |L(v')| \geq 2$. Let $S' = P' + u$ where P' is a path with sole vertex v'. Hence (G', S', L') is a canvas. As $|V(G')| < |V(G)|$, there exists an L'-coloring of G' as T is a minimum counterexample. Thus there exists an L-coloring of G, contrary to the fact that T is a counterexample.

So we may suppose that $u = v'$. Hence $|L(u)| \geq 1$. Now (G', S', L') is a canvas with $V(S') = \{u\}$. By Theorem 1.4.2, there exists an L'-coloring of G. Thus there exists an L-coloring of G, contrary to the fact that T is a counterexample. □

Let $c_3 = c_1' = c_2'$. Let $P' = v_1v_2$. Let $L_1(v_1) = L_1(v_2) = \{c_1, c_3\}$ and $L_1(x) = L(x)$ for all $x \in V(G) \setminus \{v, v_1, v_2\}$. Similarly let $L_2(v_1) = L_2(v_2) = \{c_2, c_3\}$ and $L_2(x) = L(x)$ for all $x \in V(G) \setminus \{v, v_1, v_2\}$.

Claim 2.5.8. *One of v_1, v_2 is the end of an essential chord of C distinct from v_1v_2.*

Proof. Suppose not. Consider the canvases $(G \setminus \{v\}, P'+u, L_1)$ and $(G \setminus \{v\}, P'+u, L_2)$ which satisfy the hypotheses of theorem. As $|V(G')| < |V(G)|$, these canvases must satisfy the conclusion as T is a counterexample with a minimum number of vertices. Now either $L_1(v_1)$ or $L_2(v_1)$ is not equal to $L(u)$. So assume without loss of generality that $L_1(v_1) \neq L(u) = L_1(u)$. Thus there exists an L_1-coloring of $G \setminus v$. Hence there exists an L-coloring of G, contrary to the fact that T is a counterexample. □

Suppose without loss of generality that v_2 is the end of an essential chord of C distinct from v_1v_2. Choose such a chord v_2u_1 such that u_1 is closest to v_1 measured by the distance in $C \setminus \{v_2\}$. Now v_2u_1 divides G into two graphs G_1 and G_2 where we suppose without loss of generality that $v \in V(G_1)$ and $u \in V(G_2)$.

First suppose v_1 is adjacent to u_1. Let $L'(u_1) = L(u_1) \setminus \{c_3\}$ and $L'(x) = L(x)$ otherwise. Consider the canvas (G_2, S', L') where $S' = P' + u$ and P' is a path with sole vertex u_1. As $|V(G_2)| < |V(G)|$, there exists an L'-coloring ϕ of G_2 as T is a minimum counterexample. But then we may extend ϕ to v, v_1 to obtain an L-coloring of G, contrary to the fact that T is a counterexample.

So we may suppose that v_1 is not adjacent to u_1. Consider the canvas $T_1 = (G_1, P', L)$ where $P' = vv_2u_1$. As u_1 is not adjacent to v_1, T_1 is not a fan. By Lemma 2.3.5, there is at most one coloring ϕ of P' which does not extend to G_1. Let $L'(u_1) = L(u_1) \setminus \{\phi(u_1)\}$ and $L'(x) = L(x)$ otherwise. Consider the canvas (G_2, S', L') where $S' = P' + u$ and P' is a path with sole vertex u_1. As $|V(G_2)| < |V(G)|$, there exists an L'-coloring ϕ' of G' as T is a minimum counterexample. Then we extend ϕ' to v as $|L(v)| \geq 2$ and then to G_1 as $\phi'(u_1) \neq \phi(u_1)$. Thus we obtain an L-coloring of G, contrary to the fact that T is a counterexample. □

2.6 Accordions

In this section, we will begin to characterize how the coloring of P in Theorem 1.4.2 extend to colorings of other paths of length one on the boundary of the outer walk. Indeed we will show that any L-coloring of P extends to at least two L-colorings of any other path P' of length one unless a very specific structure occurs.

2.6.1 Coloring Extensions

Definition. Suppose $T = (G, P, L)$ is a path-canvas where $P = p_1p_2$ is a path of length one in C. Suppose we are given a collection $\mathcal{C}$ of L-colorings of P. Let P' be an edge of G with both ends in C. We let $\Phi_G(P', \mathcal{C})$ denote the collection of proper colorings of P' that can be extended to a proper coloring ϕ of G such that ϕ restricted to P is a coloring in $\mathcal{C}$. We will drop the subscript G when the graph is clear from context.

We may now restate Theorem 1.4.2 in these terms.

Theorem 2.6.1 (Thomassen)**.** *Let $T = (G, P, L)$ be a path-canvas with $|V(P)| = 2$, $\mathcal{C}$ a collection of proper L-colorings of P and P' is an edge of G with both ends in C.*

$\Phi(P', \mathcal{C})$ *is nonempty.*

Note the following easy proposition.

Proposition 2.6.2. *Let T, P, P' be as in Theorem 2.6.1. If $U = u_1u_2$ is a chord of C separating P from P', then*

$$\Phi(P', \Phi(U, \mathcal{C})) = \Phi(P', \mathcal{C})$$

2.6.2 Governments

To explain the structure of extending larger sets of coloring, we focus on two special sets of colorings, defined as follows.

Definition (Government)**.** Let $\mathcal{C} = \{\phi_1, \phi_2, \ldots, \phi_k\}$, $k \geq 2$, be a collection of disjoint proper colorings of a path $P = p_1p_2$ of length one. For $p \in P$, let $\mathcal{C}(p)$ denote the set $\{\phi(p) | \phi \in \mathcal{C}\}$.

We say $\mathcal{C}$ is *dictatorship* if there exists $i \in \{1, 2\}$ such that $\phi_j(p_i)$ is the same for all $1 \leq j \leq k$, in which case, we say p_i is the dictator of $\mathcal{C}$. We say $\mathcal{C}$ is *democracy* if $k = 2$ and $\phi_1(p_1) = \phi_2(p_2)$ and $\phi_2(p_1) = \phi_1(p_2)$. We say $\mathcal{C}$ is a *government* if $\mathcal{C}$ is a dictatorship or a democracy.

Here is a useful lemma about non-extendable colorings of bellows.

Lemma 2.6.3. *Let $T = (G, P, L)$ be a bellows with base $P = p_1p_2p_3$. Suppose that $p_1 \not\sim p_3$. Let $\phi(p_1) = c$. Then there exist at most two colorings ϕ_1, ϕ_2 of p_2, p_3 extending ϕ that do not extend to an L-coloring of $G \setminus P$. Furthermore, $\mathcal{C} = \{\phi_1, \phi_2\}$ is a government. In addition, if $\mathcal{C}$ is a democracy then T is an exceptional even fan. If $\mathcal{C}$ is a dictatorship, then p_3 is its dictator and T is an exceptional odd fan.*

Proof. Follows from Proposition 2.3.9. □

2.6.3 Accordions

Definition. We say a graph G is an *accordion* with *ends* P_1, P_2, which are distinct paths of length one, if G is a bellows with base $P_1 \cup P_2$ or there exists a chord U of G that divides G into two accordions: G_1 with ends P_1, U and G_2 with ends P_2, U.

Definition (Accordion)**.** Let $T = (G, S, L)$ be a canvas such that $S = P_1 \cup P_2$ where P_1, P_2 are distinct paths of length one. We say that T is an *accordion* with *ends* P_1, P_2 if T is a bellows with base $P_1 \cup P_2$, or T is the 2-sum of two smaller accordions $T_1 = (G_1, P_1 \cup U, L)$ with ends P_1, U and $T_2 = (G_2, P_2 \cup U, L)$ with ends U, P_2 along an edge $U = u_1u_2$ such that $|L(u_1)|, |L(u_2)| \leq 3$.

Definition (1-accordion)**.** Let $T = (G, P, L)$ be a path-canvas with $|V(P)| = 2$ and $|L(v)| = 1$ for all $v \in V(P)$. Let $P' = p_1p_2$ be an edge of the outer walk of G. We say T is a 1-*accordion* from P to P' if G is an accordion whose ends are P and P' and there exists exactly one L-coloring of G.

Proposition 2.6.4. *If $T = (G, P, L)$ is a 1-accordion from P to P' where P' is an edge of C, then $|L(v)| = 3$ for all $v \in V(C) \setminus V(P)$ where C is the outer walk of G.*

Proof. Suppose not. Then there exists $v \in V(C) \setminus V(P)$ such that $|L(v)| \geq 4$. As T is a 1-accordion, then G has exactly one L-coloring ϕ of G by definition. Let $T' = (G, P, L')$ where $L'(v) = L(v) \setminus \{\phi(v)\}$ and $L'(z) = L(z)$ otherwise. Now T' is a canvas. By Theorem 1.4.2, there exists an L'-coloring ϕ' of G. Yet ϕ' is an L-coloring of G and $\phi'(v) \neq \phi(v)$, which contradicts the fact that ϕ was the only L-coloring of G. $\square$

Hence a 1-accordion is also an accordion.

Theorem 2.6.5 (Accordion)**.** *Let $T = (G, P, L)$ be a canvas, where P is a path of length one, and P' be a path of length one distinct from P. If $\mathcal{C}$ is a non-empty set of proper L-colorings of P such that if $|\mathcal{C}| \geq 2$ then $\mathcal{C}$ contains a government, then*

$\Phi(P', \mathcal{C})$ does not contain a government if and only if T contains a subcanvas T' such that T' is a 1-accordion from P to P' and $\mathcal{C} = \{\phi\}$, where ϕ is the unique proper coloring of P in T'.

Proof. We proceed by induction on the number of vertices of G. As a 1-accordion has a unique L-coloring, one implication is clear. So let us prove the other. Suppose $\Phi(P', \mathcal{C})$ does not contain a government. Following the proof of Lemma 2.4.1, we may assume that the canvas $(G, P \cup P', L)$ does not have non-essential chords, non-essential cutvertices or a vertex in the interior of a triangle or 4-cycle, as otherwise theorem follows by induction. Let C be the outer walk of G.

First suppose there is a cutvertex v of G. If v does not separate P from P', then we may delete a block of G not containing P, P' and the theorem follows by induction and Theorem 1.4.2. So we may suppose v divides G into graphs G_1, G_2 where $V(P) \subset V(G_1)$ and $V(P') \subset V(G_2)$. By Theorem 2.6.1, $\Phi(v, \mathcal{C})$ is nonempty. Let $c \in \Phi(v, \mathcal{C})$ and uv be an edge of G_2 incident with the infinite face of G. Let $\mathcal{C}' = \{(c, c') : c' \in L(u) \setminus \{c\}\}$ be a set of colorings of $P'' = uv$. If $P' = P''$, the the theorem follows. Otherwise, apply induction to (G_2, P'', L) with $\mathcal{C}'$ to find that $\Phi_{G_2}(P', \mathcal{C}')$ contains a government. But it follows that $\Phi_{G_2}(P', \mathcal{C}) \subseteq \Phi_G(P', \mathcal{C})$ and hence $\Phi(P', \mathcal{C})$ contains a government, a contradiction.

Next suppose there is a chord $U = u_1u_2$ of C, the outer walk of G, separating G into G_1, G_2 where $V(P) \subset V(G_1)$ and $V(P') \subset V(G_2)$. Let $T_1 = (G_1, P, L)$ and $T_2 = (G_2, U, L)$. By Theorem 2.6.1, $\Phi(U, \mathcal{C})$ is nonempty. By induction for T_2, $|\Phi(P', \Phi(U, \mathcal{C}))|$ does not contain a government if and only if T_2 contains a 1-accordion T_2' from U to P', $|\Phi(U, \mathcal{C})| = 1$ and $\Phi(U, \mathcal{C}) = \{\phi'\}$, where ϕ' is the unique coloring of U in T_2'. By induction for T_1 then, T_1 contains a 1-accordion T_1' from P to U, $\mathcal{C} = \{\phi\}$ and $\Phi(U, \mathcal{C}) = \{\phi'\}$, where ϕ is the unique coloring of P in T_1'. Thus T contains a 1-accordion T' from P to P', the 1-sum of T_1' and T_2' and ϕ is the unique coloring of P in T' as desired. So we may assume there is no chord of G separating P from P'.

Suppose $P \cap P' \neq \emptyset$. We may assume by induction that T is a bellows. Let $P = p_1p_2$ and $P' = p_2p_3$. As there is no chord of C, G is either a triangle or a turbofan. Suppose G is a triangle. If $\mathcal{C}$ contains a democracy, then $\Phi(P', \mathcal{C})$ contains a dictatorship as desired. Suppose $\mathcal{C}$ contains a dictatorship $\mathcal{C}'$. If p_1 is the dictator of $\mathcal{C}'$, then $\Phi(P', \mathcal{C})$ contains a dictatorship as desired unless $L(p_3)$ has size three and consists of the color of p_1 in $\mathcal{C}$ and the colors of p_2 in $\mathcal{C}$. But then $\Phi(P', \mathcal{C})$ contains a democracy as desired. If p_2 is the dictator of $\mathcal{C}'$, then $\Phi(P', \mathcal{C})$ contains a dictatorship with dictator p_2 as desired.

So we may suppose that $\mathcal{C}$ does not contain a government and hence $|\mathcal{C}| = 1$ by assumption. Let $\mathcal{C} = \{\phi\}$. If $|L(v) \setminus \{\phi(p_1), \phi(p_2)\}| \geq 2$, then $\mathcal{C}$ contains a dictatorship with dictator p_2 as desired. So we may suppose that $|L(v)| = 3$ and $\phi(p_1), \phi(p_2) \in L(v)$. But then T contains a 1-accordion, a contradiction.

So we may suppose that G is a turbofan. By Lemma 2.3.5, there exists exactly one coloring ϕ of P that does not extend to an L-coloring of G. Let $\phi' \in \mathcal{C}$. Now ϕ' extends to at least two colorings of P. If more than one of these extends, then $\Phi(P', \mathcal{C})$ contains a dictatorship with dictator p_2 as desired. It follows that $|L(v)| = 3$, $\phi'(p_1) = \phi(p_1)$, $\phi'(p_2) = \phi(p_2)$ and $\phi(p_2), \phi(p_3) \in L(v)$. Thus $|\mathcal{C}| = 1$ and T contains a 1-accordion whose unique coloring restricts on P to the unique coloring in $\mathcal{C}$.

So we may suppose that $P \cap P' = \emptyset$. Let $P = p_1p_2$ and $P' = p'_1p'_2$. Let v_1, v_2 be the vertices of the infinite face not in P adjacent to p_1, p_2 respectively. We claim that that either v_1 or v_2 is not in $V(P')$. Suppose not. Then G is precisely a four-cycle $p_1p_2v_2v_1$ and $V(P') = \{v_1, v_2\}$. We fix a coloring ϕ of P, remove $\phi(p_1)$ from $L(v_1)$ and $\phi(p_2)$ from $L(v_2)$. Thus we obtain all L'-colorings of P', where $L'(v_1) = L(v_1) \setminus \{\phi(p_1)\}$ and $L'(v_2) = L(v_2) \setminus \{\phi(p_2)\}$. But this set contains a government, a contradiction. This proves the claim. So we may suppose without loss of generality that $v_1 \notin V(P')$.

Now consider the Thomassen reduction of v_1, $T_1 = T(\Phi, v_1)$. If $\Phi_{G \setminus v_1}(P', \mathcal{C})$ contains a government, then so does $\Phi_G(P', \mathcal{C})$ as desired. Thus by induction T_1

contains a 1-accordion T_1' from P to P', $|\mathcal{C}| = 1$ and $\mathcal{C} = \{\phi\}$ is the unique proper L-coloring of P in T_1'. Similarly if $v_2 \notin V(P')$, the Thomassen reduction of v_2, $T_2 = T(\Phi, v_2)$, contains a 1-accordion T_2' from P to P' and ϕ is the unique proper L-coloring of P in T_2'.

But now we may assume without loss of generality that p_1, p_1', p_2', p_2 appear in that order in the outer walk of G. We claim that there exists $x \sim p_1, p_2, p_1', p_2'$ or the theorem follows. First suppose $v_2 \in V(P')$. Hence $v_2 = p_2'$. As T_1' is a 1-accordion and there does not exist a chord of the outer walk of G, we find that there exists $x_1 \sim v_1, p_1, p_2, p_2'$. Let $G' = G \setminus V(P)$. Let $L'(x_1) = L(x_1) \setminus \{\phi(p_1), \phi(p_2)\}$, $L'(v_1) = L(v_1) \setminus \{\phi(p_1)\}$, $L'(p_2') = L(p_2') \setminus \{\phi(p_2)\}$ and $L' = L$ otherwise.

If $|L'(p_2')| = 2$, let $\phi(x_1) \in L'(x_1) \setminus L'(p_2')$; otherwise, let $\phi(x_1) \in L'(x_1)$. Then let $\phi(v_1) \in L'(v_1) \setminus \{\phi(x_1)\}$. Let $L''(p_2') = L'(p_2') \cup \{\phi(x_1)\}$ and $L'' = L'$ otherwise. Let $C' = \{\phi\}$ be a set of L''-colorings for $P'' = x_1v_1$. By induction on $T' = (G \setminus P, P'', L'')$, we find that either $\Phi(P', \mathcal{C}')$ contains a government or T' contains a 1-accordion from P'' to P'. If $\Phi(P', \mathcal{C})$ contains a government, then so does $\Phi(P', \mathcal{C})$ and the theorem follows. So we may assume that T' contains a 1-accordion from P'' to P'. As there is no chord of the outer walk of G, it follows that x_1 is adjacent to p_1' and the claim follows with $x = x_1$.

So we may suppose that $v_2 \notin V(P')$. By considering T_1 and T_2 we find that either the claim holds or there exists $x_1, x_2 \in V(G)$ such that $x_1 \sim v_1, p_1, p_2, v_2, x_2$ and $x_2 \sim v_1, v_2, p_1', p_2'$. Without loss of generality we may suppose that $\mathcal{C}(p_1) = \{1\}$, $\mathcal{C}(p_2) = \{2\}$, $L(v_1) = \{1, 3, 4, \}$, $L(v_2) = \{2, 3, 5\}$ and $L(x_1) = \{1, 2, 3, 4, 5\}$. Thus in the unique coloring ϕ_1 of T_1', we find that $\phi_1(x_1) = 5$ and $\phi_1(v_2) = 3$. But now $|L(x_2) \setminus \{3, 4, 5\}| \geq 2$ and hence $|L_1(x_2) \setminus \{3, 5\}| \geq 2$. It follows that there exist at least two L_1-colorings of T_1', contradicting that T_1' is a 1-accordion.

As $x \sim p_1, p_2, p_1', p_2'$, we find that p_1xp_1' and p_2xp_2' are bellows B_1 and B_2 respectively as otherwise we may delete $B_1 \setminus \{p_1xp_1'\}$ or $B_2 \setminus \{p_2xp_2'\}$ and apply induction.

Suppose without loss of generality that $\phi(p_1) = 1$ and $\phi(p_2) = 2$. If $|L(v_1) \setminus \{1\}| \geq 3$, then consider $T' = (G \setminus \{p_1 v_1\}, P, L')$ where $L'(v_1) = L(v_1) \setminus \{1\}$ and $L' = L$. By induction, T' contains a 1-accordion T'' from P to P'. But then x is in T'' and yet $|L'(x)| = 5$, contradicting Proposition 2.6.4.

We claim that that B_1 and B_2 are fans. Suppose not. Suppose without loss of generality that B_1 is not a fan. By Lemma 2.3.5, there exists a unique non-extendable coloring ϕ' of B_1. Consider $T' = (G \backslash (B_1 \backslash \{p_1 x p_1'\}), P, L')$ where $L'(x) = L(x) \backslash \{\phi'(x)\}$ and $L' = L$. By induction, T' contains a 1-accordion T'' from P to P'. But then x is in T'' and yet $|L'(x)| = 4$, contradicting Proposition 2.6.4. This proves the claim.

By Lemma 2.3.6, for $i \in \{1, 2\}$ there exists at most two colors in $L(p_i')$ that extend to a coloring of $p_i x p_i'$ that does not extend to a coloring of B_i. Suppose there is at most one such color c_1 for $i = 1$ and at most one such color c_2 for $i = 2$. Now let $L'(p_1') = L(p_1) \setminus \{c_1\}$ and $L'(p_2') = L(p_2) \setminus \{c_2\}$. But then every L'-coloring of P' is in $\Phi(P', \mathcal{C})$ and yet the set of L'-colorings of P' contains a government as desired. This also shows that $|L(p_1)| = |L(p_2)| = 3$.

So we may suppose without loss of generality that p_1' has two colors c_1, c_2 in $L(p_1')$ that do not extend to an L-coloring of B_1. By Lemma 2.3.6, it follows that $L(v_1) \setminus \phi(p_1) = \{c_1, c_2\}$, $c_1, c_2 \in L(x)$ and the non-extendable colorings of p_1, x, p_1' are $\phi(p_1), c_1, c_2$ and $\phi(p_1), c_2, c_1$.

We may assume without loss of generality that $L(x) = \{\phi(p_1), \phi(p_2), c_1, c_2, c_3\}$. Now let c_3' be a color in $L(p_2')$ such that every L-coloring of p_2', x with p_2' colored c_3 extends to an L-coloring of B_2. Suppose $c_3 \neq c_3'$. But then $\Phi(P', \mathcal{C})$ contains the colorings (with p_1' color first): (c_1, c_3') and (c_2, c_3'), a dictatorship as desired. So we may assume that $c_3 = c_3'$ and hence there exist two colors c_1', c_2' in $L(p_2')$ that do not extend to an L-coloring of B_2. It follows then that $c_1', c_2' \in L(x) \setminus \{\phi(p_1), \phi(p_2)\}$. Yet $c_1', c_2' \neq c_3$. So we may assume without loss of generality that $c_1' = c_1$ and $c_2' = c_2$.

Now color x with c_3. It follows that the colorings c_1, c_2 and c_2, c_1 are in $\Phi(P', \mathcal{C})$

and hence $\Phi(P', \mathcal{C})$ contains a democracy as desired. $\square$

2.7 Harmonicas

In this section, we will characterize how governments extend. Specifically, we will show that a government extends to two governments unless a very specific structure occurs. We will then show that this structure is the only obstruction to generalizing Theorem 2.2.2 to the case of one vertex with a list of size one and one with a list of size two.

Definition. Let $\mathcal{C}$ be a collection of disjoint proper colorings of a path $P = p_1p_2$ of length one. We say $\mathcal{C}$ is a *confederacy* if $\mathcal{C}$ is not a government and yet $\mathcal{C}$ is the union of two governments.

Definition (Harmonica)**.** Let $T = (G, S, L)$ be a canvas such that $S = P \cup P'$ where P, P' are paths of length one. Let $\mathcal{C}$ be a government for P. We say T is a *harmonica* from P to P' with government $\mathcal{C}$ if

- $G = P = P'$, or
- $\mathcal{C}$ is a dictatorship, $G = P \cup P'$, $P \cap P' = z$ where z is the dictator of $\mathcal{C}$, or
- $\mathcal{C}$ is a dictatorship and there exists a triangle zu_1u_2 where $z \in V(P)$ is the dictator of $\mathcal{C}$ in color c, $L(u_1) = L(u_2) = c \cup L_0$ where $|L_0| = 2$ and the canvas $(G \setminus (P \setminus U), U \cup P', L)$ is a harmonica from $U = u_1u_2$ to P' with democracy $\mathcal{C}'$ whose colors are L_0, or
- $\mathcal{C}$ is a democracy, there exists $z \sim p_1, p_2$ such that $L(z) = L_0 \cup \{c\}$ where L_0 are the colors of $\mathcal{C}$ and there exists $i \in \{1, 2\}$ such that the canvas $(G \setminus p_i, U \cup P', L')$ is a harmonica with dictatorship $\mathcal{C}' = \{\phi_1, \phi_2\}$ where $U = zp_{3-i}$ and $\phi_1(z) = \phi_2(z) = c$ and $\{\phi_1(p_{3-i}), \phi_2(p_{3-i})\} = L_0$.

Note that $\Phi_T(P', \mathcal{C})$ is a government $\mathcal{C}'$. We say a harmonica is *even* if $\mathcal{C}$ and $\mathcal{C}'$ are both dictatorships or both democracies and *odd* otherwise.

Lemma 2.7.1. *Let $T = (G, S, L)$ be a canvas. Suppose that $S = P \cup P'$ where $P = p_1p_2$ is a path of length one and $P \neq P'$. If T contains a harmonica $T_1 = (G_1, P + P', L)$ with dictatorship $\mathcal{C}$ where p_1 is the dictator of $\mathcal{C}'$, then T does not contain*

(1) *an harmonica $T_2 = (G_2, P \cup P', L)$ with democracy $\mathcal{C}'$, or*

(2) *a harmonica $T_2 = (G_2, P \cup P', L)$ with government $\mathcal{C}'$ such that p_2 is the dictator of $\mathcal{C}'$.*

Proof. Let T_1, T_2 be a counterexample such that $|V(G_1)| + |V(G_2)|$ is minimized.

First suppose that (i) holds. As T_2 is a harmonica with a democracy, there exists $p \sim p_1, p_2$ such that $L_0 \subset L(p)$ where $L_0 = L_2(p_1) = L_2(p_2)$ and $|L_0| = 2$. First suppose p_1p is a chord of the outer walk of G. Then $T_2' = (G_2 \backslash p_1, pp_1 + u, L_2')$ is an even harmonica, where $L_2'(p) = L(p) \setminus L_0$ and $L_2'(v) = L(v)$ otherwise. Thus $p_2 \notin G_1$ and we may consider the canvases $T' = (G \backslash p_2, pp_1 + u, L), T_1, T_2'$. Yet T_2' has smaller length than T_2 and satisfies (ii), contrary to the fact that T_1, T_2 were chosen to minimize the sum of the sizes of the harmonicas. So we may assume that p_2p is a chord of the outer walk of G. But then $T_1' = (G_1 \setminus p_1, pp_2 + u, L_1)$ is a harmonica with democracy. We may then consider the canvases $T' = (G \backslash p_1, pp_2 + u, L), T_1', T_2'' = (G_2 \backslash p_1, pp_2 + u, L_2')$. Yet T_2' has smaller length than T_2 and T_1' has smaller length than T_1. Moreover, T_1' satisfies (i), and hence T_2'', T_1' is a counterexample that contradicts the fact that T_1, T_2 were chosen to minimize the sum of the sizes of the harmonicas.

Finally suppose that (ii) holds. As G is planar there exists $p \sim p_1, p_2$ and p in at least one of G_1 or G_2. Suppose without loss of generality that $p \in V(G_1)$. Let $L_1'(p_2) = L_1'(p) = L_1(p) \setminus L_1(p_1)$. Thus $T_1' = (G_1 \setminus p_1, pp_2 + u, L_1')$ is a harmonica with a democracy. Moreover, T_1' has smaller length than T_1. But then T_2, T_1' is a counterexample that contradicts the fact that T_1, T_2 were chosen to minimize the sum of the sizes of the harmonicas.

□

Theorem 2.7.2. *Let $T = (G, P, L)$ be a canvas and P, P' be paths of length one in C. Given a collection $\mathcal{C}$ of proper colorings of P such that $\mathcal{C}$ is a government or a confederacy, then $\Phi(P', \mathcal{C})$ contains a confederacy unless $\mathcal{C}$ is a government and there exists a subgraph G' of G such that $(G', P \cup P', L)$ is a harmonica from P to P' with government $\mathcal{C}$.*

Proof. Suppose that $T = (G, S, L)$ is a counterexample with $|V(G)|$ minimized and subject to the condition that $\mathcal{C}$ is a government if possible. Following the proof of Lemma 2.4.1, we may suppose that T does not have non-essential chords, non-essential cutvertices or a vertex in the interior of a triangle or 4-cycle. Let C be the outer walk of G.

Claim 2.7.3. *G is 2-connected.*

Proof. Suppose not. Then there exists a curvertex v of G. We may assume by Lemma 2.4.1, that v is essential, which implies that v separates P from P'. If $v \in P \cup P'$, then theorem follows from Bellows Coloring Lemmas?

So suppose v divides G into two graphs G_1, G_2 such that without loss of generality $V(P') \subset V(G_2)$ and $V(P) \subset V(G_1)$. Consider the canvases $T_1 = (G_1, S_1, L)$ and $T_2 = (G_2, S_2, L)$ where $S_1 = P \cup U$ and $S_2 = U' \cup P'$ where U is an edge of the outer walk of G_1 containing v and U' is an edge of the outer walk of G_2 containing v. If there exist two colorings ϕ_1, ϕ_2 of T_1 such $\phi_1(v) \neq \phi_2(v)$. But then there exists a confederacy $\mathcal{C}''$ for U' such that every coloring in it extends back to T_1. As T is a minimum counterexample, it follows that $\Phi_{T_2}(P', \mathcal{C}'')$ has a confederacy, a contradiction.

Now as T is a minimum counterexample, either $\Phi_{T_1}(U, \mathcal{C})$ is a government and contains a harmonica T_1' from P to U or $\Phi_{T_1}(U, \mathcal{C})$ has a confederacy $\mathcal{C}''$. The latter is a contradiction as then there would exist two colorings ϕ_1, ϕ_2 of T_1 such that $\phi_1(v) \neq \phi_2(v)$. It follows similarly then that T_1' is an even harmonica if $\mathcal{C}$ is a dictatorship and odd if $\mathcal{C}$ is a democracy. But then as T is a minimum counterexample, it follows by considering $T_2' = (G_2 \cup U, U + P', L)$ that $\Phi_{T_2'}(P', \mathcal{C}'')$ has a confederacy, a contradiction. $\square$

Claim 2.7.4. *$\mathcal{C}$ is a government.*

Proof. Suppose not. Then $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy. As T is a minimum counterexample, there exists a harmonica T_1 from P to P' for $\mathcal{C}_1$ and a harmonica T_2 from P to P' for $\mathcal{C}_2$. But this contradicts Lemma 2.7.1 unless $\mathcal{C}_1, \mathcal{C}_2$ are both dictatorship with the same dictator. It is not hard to see though that $\Phi(P', \mathcal{C}_1) \neq \Phi(P', \mathcal{C}_2)$ and hence that $\Phi(P', \mathcal{C})$ has a confederacy, a contradiction. $\square$

Claim 2.7.5. *There does not exist a chord of C.*

Proof. Suppose there exists a chord U of C. We may assume U is essential, separating a vertex of P from a vertex of P'. Now U divides G into graphs G_1, G_2 where we may assume without loss of generality that $P \subseteq G_1$ and $P' \subseteq G_2$. Consider the canvases $T_1 = (G_1, P \cup U, L)$ and $T_2 = (G_2, U \cup P', L)$. As T is a minimum counterexample, either $\Phi_{T_1}(U, \mathcal{C})$ contains a confederacy $\mathcal{C}'$ or there exists harmonica T_1' from P to U with government $\mathcal{C}$. Suppose the former. But then as T is a minimum counterexample, $\Phi_{T_2}(P', \mathcal{C}')$ contains a confederacy and hence so does $\Phi(P', \mathcal{C})$ a contradiction.

So we may suppose the latter. But then $\Phi_{T_1}(U, \mathcal{C})$ is a government $\mathcal{C}'$. As T is a minimum counterexample, $\Phi_{T_2}(P', \mathcal{C}')$ contains or a confederacy or there exists a harmonica T_2' from U to P' with government $\mathcal{C}'$. If the former holds, then $\Phi(P', \mathcal{C})$ contains a confederacy, a contradiction. So suppose the latter. But then the 2-sum of T_1' and T_2' with respect to U is harmonica from P to P' with government $\mathcal{C}$, a contradiction. $\square$

Claim 2.7.6. $P \cap P' = \emptyset$.

Proof. Suppose not. If $P = P'$, then $(G, P \cup P', L)$ is a harmonica, a contradiction. So we may suppose that $P \neq P'$. Let $z = P \cap P'$. If there do not exist $\phi_1, \phi_2 \in \mathcal{C}$ such that $\phi_1(z) \neq \phi_2(z)$, then $\mathcal{C}$ and $\Phi(P', \mathcal{C})$ are dictatorships with dictator z. But then $(P \cup P', P \cup P', L)$ is a harmonica, a contradiction.

So we may suppose there exists $\phi_1, \phi_2 \in \mathcal{C}$ such that $\phi_1(z) \neq \phi_2(z)$. First suppose T is not a bellows with base $P \cup P'$. We may assume by criticality that $G = P \cup P'$. But then ϕ_1 and ϕ_2 extend to distinct governments of P' and hence $\Phi(P', \mathcal{C})$ contains a confederacy, a contradiction.

So we may assume that T is a bellows with base $P \cup P'$. By Claim 2.7.5, it follows that either G is a triangle or T is a turbofan. In the former case, it is not hard to see that either $\Phi(P', \mathcal{C})$ contains a confederacy or T is a harmonica, a contradiction. So suppose the latter.

By Lemma 2.3.5, there exists a unique L-coloring ϕ_0 of $P \cup P'$ that does not extend to an L-coloring of G. Let $i \in \{1, 2\}$ such that $\phi_i(z) \neq \phi_0(z)$. Hence there is a dictatorship $\mathcal{C}_1 \subseteq \Phi(P', C)$ such that $\phi(z) = \phi_i(z)$ for all $\phi \in \mathcal{C}_1$. Let $P' = zz'$. Suppose $L(z') \setminus \{\phi_1(z), \phi_2(z), \phi_0(z')\} \neq \emptyset$. Let c be a color in $L(z') \setminus \{\phi_1(z), \phi_2(z), \phi_0(z')\}$. Hence there exists a dictatorship $\mathcal{C}_2 \subseteq \Phi(P', \mathcal{C})$ such that $\phi(z') = c$ for all $\phi \in \mathcal{C}_2$. But then $\Phi(P', \mathcal{C})$ contains the confederacy $\mathcal{C}_1 \cup \mathcal{C}_2$, a contradiction.

So we may assume that $L(z') = \{\phi_0(z'), \phi_1(z), \phi_2(z)\}$ as $|L(z')| \geq 3$. Hence, $\phi_0(z') \neq \phi_1(z), \phi_2(z)$. Hence the democracy $\mathcal{C}_2$ in colors $\phi_1(z), \phi_2(z)$ is in $\Phi(P', \mathcal{C})$. But then $\Phi(P', \mathcal{C})$ contains the confederacy $\mathcal{C}_1 \cup \mathcal{C}_2$, a contradiction. $\square$

Claim 2.7.7. *$\mathcal{C}$ is a dictatorship.*

Proof. Suppose not. Hence $\mathcal{C}$ is a democracy. Let L_0 be the colors of $\mathcal{C}$. Let $Q = q_1 \dots q_k$ be a maximal path in C such that $E(Q) \cap E(P') = \emptyset$, $V(P) \subseteq V(Q)$, $L_0 \subset L(v)$ for all $v \in V(Q)$. Suppose $q_1 \notin V(P')$. Let $q_1 v_1 \in E(C) \setminus E(Q)$. Let $T' = (G \setminus Q, v_1 + P', L')$ be the democratic reduction of Q with democracy L_0 centered around v_1. As Q is maximal and $q_1 \notin V(Q)$, L_0 is not a subset of $L(v_1)$ as otherwise $Q + v_1$ would also be path satisfying the above conditions, contradicting that Q is maximal. Hence $|L'(v_1)| \geq 2$. Let P'' be a path of length one of C containing v_1. Thus the set of L'-coloring of P'' contains a confederacy $\mathcal{C}'$. As T is a minimum counterexample, it follows from considering $T'' = (G \setminus Q, P'' \cup P', L')$ that $\Phi_{T''}(P', \mathcal{C}')$ contains a confederacy and hence $\Phi_T(P', \mathcal{C})$ contains a confederacy, a contradiction. So we may suppose that $q_1 \in V(P')$. By symmetry, it follows that $q_k \in V(P')$.

Yet $P \cap P' = \emptyset$ by Claim 2.7.6. So $q_1, q_k \notin V(P)$. Let $c_1 \in L(q_1) \setminus L_0$ and $c_2 \in L(q_2) \setminus L_0$. Let $\mathcal{C}_1 = \{\phi_1, \phi_1'\}$ where $\phi_1(q_1) = \phi_1'(q_1) = c_1$ and $\{\phi_1(q_k), \phi_1'(q_k)\} = L_0$. Similarly, let $\mathcal{C}_2 = \{\phi_2, \phi_2'\}$ where $\phi_2(q_k) = \phi_2'(q_k) = c_2$ and $\{\phi_2(q_1), \phi_2'(q_1)\} = L_0$. Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are distinct governments of P' and $\mathcal{C}' = \mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy. Moreover, for all $\phi \in \mathcal{C}'$, $\phi \in \Phi(P', \mathcal{C})$. To see this, simply extend ϕ to Q using the colors of L_0. Then if $\phi \in \mathcal{C}_1$, consider the democratic reduction T' of $Q \setminus q_1$ centered around q_1. There exists a coloring of T' by Theorem 1.4.2. Hence $\phi \in \Phi(P', \mathcal{C})$. Similarly if $\phi \in \mathcal{C}_2$, consider the democratic reduction T' of $Q \setminus q_k$ centered around q_k. There exists a coloring of T' again by Theorem 1.4.2. Hence $\phi \in \Phi(P', \mathcal{C})$.

□

Suppose without loss of generality that p_1 is the dictator of $\mathcal{C}$ in color c. Let v_1, v_2 be the vertices of C adjacent to p_1. Now let $T_1 = (G \setminus \{v_1\}, S, L_1), T_2 = (G \setminus \{v_2\}, S, L_2)$ be the Thomassen reductions of v_1, v_2 respectively. If $\Phi_{T_1}(P', \mathcal{C})$ contains a confederacy, then so does $\Phi_T(P', \mathcal{C})$, a contradiction. So we may suppose, as T is a minimum counterexample, that T_1 contains a harmonica from P to P' with government $\mathcal{C}$. Similarly for T_2.

Thus there exists $v \notin C$ such that $v \sim p_1, v_1, v_2$. As T_1 contains a harmonica, $L_1(v) = L_1(v_2)$. So it follows though that $c \in L(v_2)$, that $|L(v_2)| = 3$. Similarly $c \in L(v_1)$ and $|L(v_1)| = 3$. In addition, we now find that $L(v_1) \cap L(v_2) = \{c\}$ and $L(v) = L(v_1) \cup L(v_2)$. Consider the furthest chords $U_1 = vu_1$ and $U_2 = vu_2$ with one end in C and the other end v in the paths from v_1 to P' and v_2 to P', respectively, avoiding p_1. We find by planarity that u_1 is adjacent to u_2 given the edges of the harmonicas. Hence $P' = u_1u_2$.

On the other hand we claim that $L(v_1) = L(v_2)$. Let $\mathcal{C}_i = \Phi_{T_i}(P', \mathcal{C})$. As $\mathcal{C}_i$ is not a confederacy, $\mathcal{C}_i$ is government as T is a minimal counterexample. Yet $\mathcal{C}_1$ must be a dictatorship, as $\Phi_{T_1}(vu_2, \mathcal{C})$ is either a democracy or a dictatorship with dictator u_2. Similarly $\mathcal{C}_2$ must be a dictatorship. But then $\mathcal{C}_1$, $\mathcal{C}_2$ must have the same dictator in the same color as otherwise $\mathcal{C}_1 \cup \mathcal{C}_2$ is a confederacy in $\Phi(P', \mathcal{C})$, a contradiction. Suppose without loss of generality that u_1 is the dictator of $\mathcal{C}_1$ and $\mathcal{C}_2$. But now it follows that $L(v_2) \setminus \{c\} \subset L(u_1)$ given the democracy on vu_2 in T_1. However, $L(v_1) \setminus \{c\} \subset L(u_1)$ given the democracy on v_1v in T_2. Yet $|L(u_1)| = 3$ as T_1 and T_2 are harmonicas, a contradiction to the fact that $L(v_1) \cap L(v_2) = \{c\}$.

□

Definition (Harmonica)**.** Let $T = (G, S, L)$ be a canvas such that $S = P + u$ where P is a path of length at most one and $|L(u)| = 2$. We say T is a *harmonica* from P to u if there exists a color c such that either

- $V(P) = \{p\}$, $|L(p)| = 1$ and there exists $p' \sim p$ and $u' \sim u$ such that $(G, P \cup P', L')$ is a harmonica with dictatorship $\mathcal{C}$ where $P = pp''$, $\mathcal{C}$ is the set of L-colorings of P, $P' = uu'$, $L'(u) = L(u) \cup \{c\}$ and $L'(w) = (w)$ otherwise, and $\Phi(P', \mathcal{C})$ is a dictatorship with dictator u in color c.

- $P = p_1p_2$, $L(p_1) = L(p_2)$, $|L(p_1)| = 2$, and there exists $u' \sim u$ such that $(G, P \cup P', L')$ is a harmonica with democracy $\mathcal{C}$ where $\mathcal{C}$ is the set of L-colorings of P, $P' = uu'$, $L'(u) = L(u) \cup \{c\}$ and $L'(w) = (w)$ otherwise, and $\Phi(P', \mathcal{C})$ is a dictatorship with dictator u in color c.

Following our earlier definition, we say such a harmonica is *odd* if $|V(P)| = 2$ and *even* if $|V(P)| = 1$.

Theorem 2.7.8. *Let $T = (G, S, L)$ be a canvas, where $S = P + u$ and $P = p_1p_2$ is a path of length one and $|L(u)| \geq 2$. Let $\mathcal{C}$ be the set of L-colorings of P. If $|\mathcal{C}| \geq 2$, then G is L-colorable unless there exists a canvas $T' = (G', P + u, L)$ with $G' \subseteq G$, $S' = P + P'$, unless $\mathcal{C}$ is a government, $|L(u)| = 2$ and T is a harmonica from P to u.*

Proof. Follows from Theorem 2.7.2. □

2.8 Orchestras

In this section, we will characterize when governments on two distinct paths P, P' of length one on the outer walk of a canvas do not extend to an L-coloring of the whole graph. In addition, this characterizes the obstructions to generalizing Theorem 2.2.2 to the case where both vertices have lists of size one. First we prove a useful coloring lemma which will be required for the proof.

Lemma 2.8.1. *Let G be a plane graph as follows: $x \sim x'$, $x \sim v_1, v_2$, $x' \sim w_1, w_2$, there exists $u_1, u_2 \sim x, x'$ and $u_1xv_1, u_2xv_2, u_1x'w_1, u_21x'w_2$ are the bases of bellows, or are edges because $u_i = v_i$ or $u_i = w_i$. Let $T = (G, S, L)$ be a canvas where $S = \{v_1, v_2, w_1, w_2\}$. If $|L(x)|, |L(x')| \geq 4$ and $|L(v)| \geq 2$ for all $v \in S$, then G has an L-coloring unless there exists $z \in \{x, x'\}$ such that z is adjacent to all of S, v_1zw_1 and v_2zw_2 are the bases of exceptional odd fans, $L(v_1) = L(w_1)$ has size two, $L(v_2) = L(w_2)$ has size two, and $L(z) = L(v_1) \cup L(v_2)$.*

Proof. Suppose not. Let T be a counterexample with a minimum number of vertices.

Claim 2.8.2. *For all $i \in \{1, 2\}$, $u_i \neq v_i, w_i$.*

Proof. Suppose not. We may assume without loss of generality that $u_1 = w_1$ and hence $x \sim w_1$. Consider the bellows T_1 with base v_1xw_1. If there is only one color of w_1 in $L(w_1)$ that extends to an L'-coloring of $\{w_1, x\}$ that does not then extend to an L-coloring of T_1, color w_1 with a different color, delete $T_1 \setminus x$, and remover the color of w_1 from $L(x)$ and $L(x')$. Now x, x' have lists of size three, so we may find a coloring by Theorem 2.2.2.

So suppose there are two such colors. By Lemma 2.3.7, $L(w_1) = L(v_1)$. Now remove $L(w_1)$ from $L(x)$. Thus x, v_2, w_2 have lists of size two and x' has a list of size four. This has a coloring by Theorem 2.7.8 unless xv_2w_2 is an exceptional odd fan and $L(w_2) = L(v_2)$. Applying the same argument symmetrically shows that xw_1v_1 is an exceptional odd fan and hence x is adjacent to all of S. Furthermore $L(x) = L(v_1) \cup L(v_2)$ where these have size two. So $|L(x)| = 4$. But then the conclusion of the lemma holds, a contradiction. This proves the claim. □

So we may suppose that $u_1 \neq v_1, w_1$ and $u_2 \neq v_2, w_2$. Thus v_1xu_1 is the base of a bellows T_1 and $u_1x'w_1$ is the base of a bellows T_2. Suppose there exists only one color of u_1 in $L(u_1)$ that extends to an L'-coloring of $\{u_1, x\}$ that does not then extend to an L-coloring of T_1. Let $L'(u_1) = L(u_1)$ and $L' = L$ otherwise. Consider the canvas $T' = (G', S', L')$ where $G' = G \setminus (T_1 \setminus \{u_1, x\})$ and $S' = S \setminus \{v_1\} \cup \{u_1\}$. Now T' satisfies the hypotheses of the lemma. As T is a minimum counterexample, it follows that either G' has an L'-coloring, a contradiction as then G has an L-coloring or there exists $z \in \{x, x'\}$ such that z is adjacent to all of S, contradicting Claim 2.8.2 as then $u_2 = w_2$ or $u_2 = v_2$.

So we may assume there are two such non-extendable colors of u_1 in $L(u_1)$. Thus by Lemma 2.3.7, $L(v_1) \subseteq L(u_1)$ where the two non-extendable colors are $L(v_1)$. Similarly, we may assume that $L(w_1) \subseteq L(u_1)$ where the two non-extendable colors are $L(w_1)$.

Suppose $L(v_1) = L(w_1)$. Now color u_1 from $L(u_1) \setminus L(v_1)$, delete $T_1 \cup T_2 \setminus \{x, x'\}$

and remove the color of u_1 from $L(x)$ and $L(x')$. Then x, x' have lists of size three and there exists a coloring by Theorem 2.2.2. But this extends to an L-coloring of G, a contradiction. So we may assume that $L(v_1) \neq L(w_1)$.

By symmetry of u_1 and u_2, we may assume that $L(v_2), L(w_2) \subset L(u_2)$ and $L(v_2) \neq L(w_2)$. Now color v_1, w_1 with the same color from $L(w_1) \cap L(v_1)$ and remove that color from $L(x), L(x'), L(u_1)$. This leaves x, x', u_2 with lists L' of size three and u_1, v_2, w_2 with lists L' of size two.

We claim this has a coloring as $L'(v_2) = L(v_2) \neq L(w_2) = L'(w_2)$. Color w_2, v_2 with the same color from $L'(w_2) \cap L'(v_2)$; this must generate four lists of size two, as three two's and a three has a coloring. But this will color unless x, x', u_1 or x, x', u_2 have the same lists. Suppose the latter case. Color x and w_2 with the same color not in $L'(u_1)$. Then color v_2,u_2,x' and finally u_1. So suppose the former case. Color x and w_2 with the same color from $L'(w_2) \setminus L'(v_2)$, then color u_1, x', u_2, v_2. $\square$

Definition (Orchestra)**.** Let $T = (G, P \cup P', L)$ be a canvas, where P, P' are distinct paths of length at most one in C and they are disjoint if either is a path of length zero. We say T is a *double bellows* with *sides* P, P' if there exists a vertex v adjacent to all vertices in $V(P) \cup V(P')$ and the inlets of $G[P \cup P' \cup \{v\}]$ are the bases of bellows. We say that a double bellows T is a *wheel bellows* if G is a wheel. We say T if a *defective double bellows* if T is a wheel bellows less an edge from the center of the wheel v to a vertex in $P \cup P'$.

We say T is an *instrument* with sides P, P' if T is a bellows with base $P \cup P'$, or T is double bellows or defective double bellows with sides P, P'.

We say T is an *instrumental orchestra* with sides P, P' if T is an instrument with sides P, P', or T is the 1-sum or 2-sum of two smaller instrumental orchestras $T_1 = (G_1, P + U, L)$ and $T_2 = (G_2, P' + U, L)$, along the vertex or edge U, respectively where $|L(v)| \leq 4$ for all $v \in U$.

We say T is a *special orchestra* with sides P, P' if there exists an edge uu' such that T consists of an harmonica (possibly null) from P to u, the edge uu', and a harmonica (possibly null) from u' to P', where $|L(u)|, |L(u')| \leq 3$.

We say T is an *orchestra* if T is an instrumental orchestra or a special orchestra.

Theorem 2.8.3. *Let $T = (G, P_1 \cup P_2, L)$ be a canvas, where P_1, P_2 are disjoint edges of C. Let $\mathcal{C}_1$ be a government for P_1 and $\mathcal{C}_2$ be a government for P_2. If there do not exist colorings $\phi_1 \in \mathcal{C}_1, \phi_2 \in \mathcal{C}_2$ such that $\phi_1 \cup \phi_2$ extends to an L-coloring of G, then there exists an orchestra $T' = (G', P_1' \cup P_2', L)$ with sides P_1', P_2' where G' is a subgraph of G, and for all $i \in \{1, 2\}$, $P_i' \subseteq P_i$, and $P_i = P_i'$ if $\mathcal{C}_1$ is a democracy.*

Moreover, if T' cannot be found such that T' is instrumental, then T' is a special orchestra with cut-edge uu', the harmonica from P to u is even if $\mathcal{C}_1$ is a dictatorship and odd otherwise and similarly the harmonica from P' to u' is even if $\mathcal{C}_2$ is a dictatorship and odd otherwise.

Proof. Let $T = (G, P \cup P', L)$ be a counterexample with a minimum number of vertices. By proofs similar to that of Lemma 2.4.1, we may assume no vertices of G are in the interior of a triangle or 4-cycle and there is at most one vertex in the interior of a 5-cycle. Let C be the outer walk of G.

Claim 2.8.4. *There does not exist a cutvertex v of G.*

Proof. Suppose there exists a cutvertex v of G. Then v divides G into two graphs G_1, G_2. Suppose $P \cup P' \subseteq G_1$. We now apply induction to $(G_1, P \cup P', L)$. If there exists an orchestra with sides P, P', the theorem follows. So we may assume that there exist $\phi_1 \in \mathcal{C}_1, \phi_2 \in C_2$ such that $\phi_1 \cup \phi_2$ extends to an L-coloring of G_1. But then by Theorem 1.4.2, this extends to an L-coloring of G, a contradiction.

So we may assume that v separates P from P'. Let $\phi_1 \in \mathcal{C}$. By Theorem 1.4.2, ϕ_1 extends to an L-coloring of G_1. Let $L_1(v) = L(v) \setminus \{\phi_1(v)\}$ and $L_1(x) = L_1(x)$ otherwise. First suppose that $T_1 = (G_1, P + v, L_1)$ has an L-coloring ϕ_2. Then let $L_2(v) = \{\phi_1(v), \phi_2(v)\}$ and $L_2(x) = L(x)$ otherwise. But then $T_2 = (G_2, P' + v, L_2)$

does not have an L-coloring as T is a counterexample. By Theorem 2.7.8, there exists a harmonica $T_2' = (G_2', P' + v, L_2)$ from P' to v. It follows from Theorem 1.4.2, however that for all $c \in L(v) \setminus L_2(v)$, there exists an L-coloring ϕ_c of G_2 such that ϕ_c restricted to P' is in $\mathcal{C}'$ and $\phi_c(v) = c$.

Let $T_1' = (G_1, P + v, L_1')$ where $L_1'(v) = L(v) \setminus L_2(v)$. Suppose that $|L(v)| = 3$. Then as T is a minimum counterexample, T_1' contains an orchestra $T_1'' = (G_1', P + v, L_1')$. If T_1'' is a special orchestra, then it follows that $(G_1' \cup G_2', P \cup P', L)$ is a special orchestra, a contradiction. If T_1'' is an instrumental orchestra, then $(G_1' \cup G_2', P \cup P', L)$ is an instrumental orchestra, a contradiction. So we may suppose that $|L(v)| = 4$. By Theorem 2.7.8, T_1' contains a harmonica $T_1'' = (G_1', P + v, L_1')$ from P to v. Hence, $(G_1' \cup G_2', P \cup P', L)$ is an instrumental orchestra, a contradiction. Finally suppose $|L(v)| = 5$. By Theorem 1.4.2, G_1 has an L_1'-coloring bu then G has an L-coloring, contrary to the fact that T is a counterexample.

So we may suppose that T_1 does not have an L-coloring. By Theorem 2.7.8, T_1 contains a harmonica $T_1' = (G_1', P + v, L_1)$. Note then that $|L_1(v)| = 2$ and hence $|L(v)| = 3$. Let $T_2 = (G_2, v + P', L_2)$ where $L_2(v) = L(v) \setminus L_1(v)$ and $L_2(x) = L(x)$ otherwise. Then as T is a minimum counterexample, T_2 contains an orchestra $T_2' = (G_2', v + P', L_2)$. If T_2' is a special orchestra, then $(G_1' \cup G_2', P \cup P', L)$ is a special orchestra, a contradiction. If T_2' is an instrumental orchestra, then $(G_1' \cup G_2', P \cup P', L)$ is an instrumental orchestra, a contradiction. $\square$

Claim 2.8.5. *There does not exist a chord U of C with both ends having lists of size less than five.*

Proof. Suppose there is such a chord U. Thus U divides G into two graphs G_1, G_2. First suppose that $P \cup P' \subset V(G_1)$. If there exists an orchestra with sides P, P' in G_1, the theorem follows. So we may assume that there exist $\phi_1 \in \mathcal{C}_1, \phi_2 \in C_2$ such that $\phi_1 \cup \phi_2$ extends to an L-coloring of G_1. But then by Theorem 1.4.2, this extends to an L-coloring of G, a contradiction.

So we may assume without loss of generality that $V(P) \setminus V(G_2) \neq \emptyset$ and $V(P') \setminus V(G_1) \neq \emptyset$. By Theorem 2.6.5, $\Phi_{G_1}(U, \mathcal{C}_1)$ has a government. Similarly $\Phi_{G_2}(U, \mathcal{C}_2)$ has a government. Consider $T_1 = (G_1, P + U, L)$ with governments $\mathcal{C}_1$ for P and $\Phi_{G_2}(U, \mathcal{C}_2)$ for U. As T is a minimum counterexample, T_1 must contain an orchestra $T_1' = (G_1', P + U, L)$. Similarly, $T_2 = (G_2, U + P', L)$ must contain an orchestra $T_2' = (G_2', P' + U, L)$. If T_1' and T_2' are instrumental then T contains an instrumental orchestra $T = (G_1' \cup G_2', P \cup P', L)$, a contradiction.

So we may assume without loss of generality that T_1' is a special orchestra. Let T_P be the harmonica of T_1' from P. If T_P is not empty, then as there is no cutvertex of G by Claim 2.8.4, it follows that there exists a chord U' of C distinct from U separating P from U. But there is an instrumental orchestra T_1'' from U' to P, namely that given by the harmonica. Applying the argument above to U', we obtain a contradiction unless the orchestra found between U' and P' is a special orchestra T_2''. But then the 2-sum of T_1'' and T_2'' is a special orchestra. So we may suppose that T_P is empty.

It follows that $\mathcal{C}$ is a dictatorship and the dictator x of $\mathcal{C}$ is adjacent to a vertex u in the the harmonica in T_1' from U. Let U' be the chord from said harmonica separating P_1 from P_2 and incident with u. But then $x \cup U'$ is the base of a bellows as T is critical and u is not a cutvertex of G by Claim 2.8.4. Therefore T_1 contains an instrumental orchestra, a contradiction. $\square$

Case 1: At least one of $\mathcal{C}_1, \mathcal{C}_2$ is a democracy.

Suppose that at least one of $\mathcal{C}_1, \mathcal{C}_2$ is a democracy. Without loss of generality, suppose that $\mathcal{C}_1$ is a democracy. Let L_0 be the two colors of the democracy. Choose a path $Q \supseteq P$ in $C \setminus P'$ if $\mathcal{C}'$ is a democracy, and in $C \setminus \{u\}$ if $\mathcal{C}$ is a dictatorship with dictator u, such that $L_0 \subseteq L(x)$ and $|L(x)| \leq 4$ for all $x \in Q$, and subject to that $|V(Q)|$ is maximum. Let $Q = q_1 \dots q_k$.

Let $v_1, v_2 \in V(C)$ such that $p_1 v_1, p_2 v_2 \in E(C)$ and $w_1, w_2 \in V(C)$ such that

$q_1w_1, q_kw_2 \in E(C)$.

First suppose that $w_1 \in V(P')$ if $\mathcal{C}_2$ is a democracy or w_1 is the dictator if $\mathcal{C}_2$ is a dictatorship. Suppose $\mathcal{C}_2$ is a dictatorship. So $P' = u$. Thus $L(u) = \{c\}$. If $c \in L(p_1)$, color u with c and then color P' using $\mathcal{C}$. Let $G' = G \setminus (P' \cup \{u\})$ and remove the colors in $L(p_1)$ from vertices in $N(P' \cup \{u\})$. The resulting graph is a canvas that has two vertices with list of size two on the boundary. By the Theorem 2.2.2, there is an L'-coloring of G', a contradiction.

So suppose that $\mathcal{C}_2$ is a democracy. Let L_0' be the colors of $\mathcal{C}_2$. If $L_0' \cap L_0 = \emptyset$, we may color the democratic reduction of Q centered around w_1 by Theorem 1.4.2, a contradiction. If $|L_0' \cap L_0| = 1$, color w_1 from $L_0' \cap L_0$ and extend to Q and $P' \setminus w_1$. We may then extend this coloring to a coloring of $G \setminus (Q \cup P')$ by Theorem 1.4.2. If $L_0 = L_0'$, we may color the democratic reduction of $Q \cup P'$ by Theorem 1.4.2, a contradiction. So we may assume that $w_1 \notin V(P')$ and by symmetry that $w_2 \notin V(P')$.

Now we may add w_1 to Q and get a larger path, which contradicts the choice of Q, unless $|L(w_1) \cap L_0| \leq 1$ or $|L(w_1)| = 5$. Similarly we may add w_2 to Q and get a larger path, which contradicts the choice of Q, unless $|L(w_2) \cap L_0| \leq 1$ or $|L(w_2)| = 5$.

Consider the democratic reduction, $T_1 = (G \setminus Q, w_1 + P', L_1)$, of Q centered around w_1, and the democratic reduction, $T_2 = (G \setminus Q, w_1 + P', L_2)$, of Q centered around w_2. If $|L_1(w_1)| \geq 3$, then $G \setminus Q$ has an L_1-coloring by Theorem 1.4.2, a contradiction. Thus $|L_1(w_1)| \leq 4$ and hence $L(w_1) \cap L_0| \leq 1$. So we may assume that $|L(w_1)| = 3$ and $|L_1(w_1)| = |L(w_1) \setminus L_0| = 2$. Similarly, $|L(w_2)| = 3$ and $|L_2(w_2)| = |L(w_2) \setminus L_0| = 2$. Note that $|L(v)| \leq 4$ for all $v \in Q$ by the choice of Q.

By Theorem 2.7.8, there exists a harmonica $T_1' = (G_1, w_1 + P', L_1)$ from P' to w_1. Let $x_1', u_1' \in V(G_1)$ such that $w_1x_1'u_1'$ is a triangle. Note that $|L_1(x_1')|, |L_1(u_1')| = 3$. By Claim 2.8.5, either $|L(x_1')| = 5$ or $|L(u_1')| = 5$. So assume without loss of generality that $|L(x_1')| = 5$. But then $x_1' \in N(Q)$ as $|L_1(x_1')| = |L(x_1') \setminus L_0| = 3$. Suppose $u_1' \in N(Q)$. But then either u_1' or x_1' is a cutvertex of the harmonica T_1' and yet is adjacent to w_1, which is impossible. So $u_1' \notin N(Q)$ and hence $u_1' \in V(C)$. As $u_1' \notin N(Q)$, $L(u_1') = L_1(u_1')$. Yet as T_1' is a harmonica, $|L_1(u_1')| = 3$. Furthermore, $L_1(u_1') = L_1(x_1')$. Yet, $L_0 \cup L_1(x_1') = L(x_1')$ and so $L_0 \cap L_1(x_1') = \emptyset$. It follows then that $L_0 \cap L(u_1') = \emptyset$.

Let P_1 be the path from v_1 to w_1 in C avoiding P and let P_2 be the path from v_2 to w_2 in C avoiding P. Note that P_1 and P_2 are subpaths of Q and thus $|L(v)| \leq 4$ for all $v \in P_1 \cup P_2$. Now consider the coloring ϕ with $\phi(w_1) \in L(w_1) \cap L_0$, where we note that this is nonempty as $|L(w_1) \setminus L_0| = 2$ and $|L(w_1)| = 3$. Extend ϕ to $Q \setminus V(P_2)$ using one of the colorings in $\mathcal{C}$. Now let $T_2 = (G \setminus (P \cup P_1), v_2 + P', L_2)$ where $L_2(x) = L(x) \setminus \{\phi(p) : p \in P \cup P_1, p \sim x\}$. As $L_0 \cap L(u_1') = \emptyset$, there is only one vertex not in P' that has a list of size less than three and that is v_2, which has a list of size two. As G is not L-colorable, Theorem 2.7.8 implies that there exists a harmonica $T_2' = (G_2, v_2 + P', L_2)$ from P' to v_2. Let $x_2, u_2 \in G_2$ such that $v_2x_2u_2$ is a triangle. Using an identical argument as above, we find $u_2 \in V(C) \setminus N(P \cup P_1)$, $x_2 \in N(P \cup P_1) \setminus V(C)$. We find then that $L_2(x_2) \cup L_0 = L(x_2)$ and hence $L_2(x_2) \cap L_0 = \emptyset$. Yet $L_2(v_2) \subset L_2(x_2)$ and hence $L_2(v_2) \cap L_0 = \emptyset$. As C has no chord whose ends have lists of size less than five by Claim 2.8.5, v_2 has at most one neighbor in $P \cup P_1$, namely p_2. Thus, L_0 is not a subset of $L(v_2)$, but this implies that $v_2 = w_2$. Note that x_2 has at least two neighbors in $P \cup P_1$.

Now we let $\phi'(v_2) \in L(v_2) \cap L(p_1)$ and extend ϕ' to P using $\mathcal{C}$. By Theorem 2.7.8, there exists a harmonica $T_1'' = (G \setminus (P \cup \{v_2\}), v_1 + P', L_1')$ from P' to v_1 where $L_1'(x) = L(x) \setminus \{\phi'(p) : p \in P \cup \{v_2\}, p \sim x\}$. Let $x_1, u_1 \in G_1$ such that $v_1x_1u_1$ is a triangle. Using an identical argument as above, we find $u_1 \in V(C) \setminus N(P \cup \{v_2\})$, $x_2 \in N(P \cup \{v_2\}) \setminus V(C)$. But then we also find as above that $v_1 = w_1$. Thus $P = Q$. As $v_1 = w_1$, then x_2 has two neighbors in $P \cup \{v_1\}$. Similarly, x_1 has at least two neighbors in $P \cup \{v_2\}$. As G is planar, we find that $x_1 = x_2 = x$.

Thus x is adjacent to v_1, v_2, u_1, u_2. Moreover, x is adjacent to at least one of p_1, p_2.

But notice that x_2 must have two neighbors with different colors in $P \cup \{v_1\}$ and yet p_2 receives the same color as v_1; hence, $x = x_2$ is adjacent to p_1. Similarly x_1 must have two neighbors with different colors in $P \cup \{v_2\}$ and yet p_1 receives the same color as v_2; hence, $x = x_1$ is adjacent to p_2. Let u_1' be the neighbor of x in the path from v_1 to P' closest to P', Similarly let u_2' be the neighbor of x in the path from v_2 to P' closest to P'. Given the harmonicas T_1'', we find that either u_1' is in a chord of C whose ends have lists of size less than five (indeed, lists of size three), contradicting Claim 2.8.5, or $u_1' \in V(P')$. Similarly we find that $u_2' \in V(P')$. But then T is a double bellows with sides P, P' as desired.

Case 2: $\mathcal{C}_1$ and $\mathcal{C}_2$ are dictatorships.

So we may assume that $\mathcal{C}_1$ and $\mathcal{C}_2$ are dictatorships. Let u_1, u_2 be their dictators. Let v_1, v_2 be the neighbors of u_1 in C and w_1, w_2 be the neighbors of u_2 in C, where we may assume without loss of generality that v_1 and w_1 (and similarly v_2 and w_2) are on the subwalk from u_1 to u_2 of C.

Claim 2.8.6. *$d(u_1, u_2) \geq 3$.*

Proof. Suppose u_1 is adjacent to u_2. If $L(u_1) = L(u_2)$ is allowed, then the edge u_1u_2 is an orchestra. So we may suppose $L(u_1) \neq L(u_2)$. But then there exists an L-coloring of G by Theorem 1.4.2. So we may suppose that $u_1 \not\sim u_2$. Similarly suppose $d(u_1, u_2) = 2$. Thus there exists a vertex v adjacent to u_1 and u_2. If $v \in V(C)$, then T is a bellows with base u_1vu_2. So suppose $v \notin V(C)$. Then u_1vu_2 is the base of two bellows and thus T is a double bellows. Either way, T is an orchestra, a contradiction. □

Claim 2.8.7. *v_1, v_2, w_1, w_2 have lists of size less than five.*

Proof. Suppose not. So without loss of generality $|L(v_1)| = 5$. Let G' be obtained from G by deleting the edge u_1v_1. Let $L'(v_1) = L(v_1) \setminus L(u_1)$ and $L'(v) = L(v)$ for all $v \in G \setminus v_1$. As T is a minimum counterexample, $T' = (G', S, L')$ contains an orchestra

$T'' = (G'', S, L')$. Show that $v_1 \notin G''$ and hence that T contains an orchestra, a contradiction... $\square$

It also follows from Claim 2.8.5 that the Thomassen reductions T_1, T_2 on v_1 or v_2 respectively are canvases. We now consider these in detail. Let P_1 be the path from v_1 to w_1 in $C \setminus \{u_1, u_2\}$ and P_2 be the path from v_2 to w_2 in $C \setminus \{u_1, u_2\}$. As G is 2-connected by Claim 2.8.4, $P_1 \cap P_2 = \emptyset$.

Claim 2.8.8. *There exists a neighbor z_1 of v_1, $|L(z_1)| = 5$, such that z_1 is adjacent to a vertex with a list of size less than five in P_2.*

Proof. As T is a minimum counterexample, $T_1 = (G \setminus \{v_1\}, \{u_1, u_2\}, L_1)$ contains an orchestra $T_1' = (G', \{u_1, u_2\}, L_1)$. Suppose T_1' is a special orchestra. If the cut-edge of T_1' is not incident with u_1, then u_1 is in a triangle $u_1 v_2 z_1$ of T_1'. Now z_1 is not in $V(C)$ and hence $|L(z_1))| = 5$. Yet $|L_1(z_1)| = 3$; so z_1 is adjacent to v_1 and the claim follows. So we may suppose that T_1' is an instrumental orchestra. Let F be the instrument of T_1' that contains u_1. If F is a bellows, then let $z_1 z_2$ be the edge in the base of F such that $z_1, z_2 \neq u$. If F is a double bellows or a defective double bellows, let $z_1 z_2$ be the side of F not containing u_1.

Suppose $u_2 \notin \{z_1, z_2\}$. Now $|L_1(z_1)|, |L_1(z_2)| < 5$ and yet v_1 is not adjacent to both z_1 and z_2. So we may assume that $z_2 \in V(C)$. As there is no chord of C with lists of size less than five by Claim 2.8.5, it follows that $z_2 \in P_2$ and $z_1 \notin C$. Thus $z_1 \sim v_1$ and the lemma follows. So we may suppose that $u_2 \in \{z_1, z_2\}$. But now it follows that $d(u_1, u_2) \leq 2$, contradicting Claim 2.8.6. $\square$

Claim 2.8.9. *There exists x_1, $|L(x_1)| = 5$ such that x_1 is adjacent to v_1, v_2.*

Proof. By symmetry, there exists a neighbor z_2 of v_2, $|L(z_2)| = 5$, such that z_2 is adjacent to a vertex with a list of size less than five in P_1. As G is planar, we choose z_1, z_2 such that $z_1 = z_2$. Call this vertex x_1. Now x_1 is adjacent to v_1 and v_2, and $|L(x_1)| = 5$. $\square$

Note that $x_1 \notin V(C)$ as otherwise $u_1v_1x_1$ is the base of a bellows F as v_1 is a not a cutvertex by Claim 2.8.4. If F is fan, it must be that $u_1 \sim x_1$ as there is no chord of C whose ends have lits of size less than five by Claim 2.8.5. But then we can delete u_1 and remove its color from $L(v_1)$ and $L(x_1)$ and find a harmonica by Theorem 2.7.8. So T contains a special orchestra, a contradiction. So we may suppose that F is not a fan. But then we delete u_1, remove its color from $L(v_1)$ and remove the color c of x_1 in the non-extendable coloring of F from $L(x_1)$ and find a harmonica by Theorem 2.7.8. So T contains a special orchestra, a contradiction.

By symmetry there exists $x_2 \notin V(C)$, $|L(x_2)| = 5$ such that x_2 is adjacent to w_1, w_2.

Claim 2.8.10. (1) *Either there exists $i \in \{1, 2\}$ such that the only edge uv with $u \in N(v_i)$, $|L(u)| = 5$ and $v \in P_{3-i}$, $|L(v)| < 5$, is x_1v_{3-i}, or,*

(2) *$N(x_1) \cap (P_i \setminus \{v_i\}) \neq \emptyset$ for all $i \in \{1, 2\}$.*

Proof. Suppose not. As (2) does not hold, we may assume without loss of generality that $N(x_1) \cap (P_1 \setminus \{v_1\}) = \emptyset$. As (1) does not hold for $i = 2$, there exists uv, $u \sim v_2$, $|L(u)| = 5$ and $v \in P_1$, $|L(v)| < 5$ such that either $u \neq x_1$ or $v \neq v_1$. As $N(x_1) \cap P = \{v_1\}$, it follows that $u \neq x_1$. As (1) does not hold for $i = 1$, there exists $u'v'$, $u' \sim v_1$, $|L(u')| = 5$ and $v' \in P_2$, $|L(v')| < 5$ such that either $u' \neq x_1$ or $v' \neq v_2$. As G is planar, it follows that either $u = u'$ or $v' = v_2$. In either case, $u \sim v_1, v_2$ and hence x_1 is in the interior of the 4-cycle $u_1v_1uv_2$, a contradiction. □

A symmetric claim holds for x_2, w_1, w_2.

Claim 2.8.11. $x_1 \sim u_1$.

Proof. Suppose not. It follows that T_1 contains a special orchestra T_1' where the cut-edge of T_1' is u_1v_2, and similarly T_2 contains a special orchestra T_2' where the cut-edge of T_2' is u_1v_1.

Suppose Claim 2.8.10(1) holds with $i = 2$. It follows that T_2 contains a special orchestra T'_2 where the cut-edge of T'_2 is u_1v_1. Let $v_1z_1z_2$ be the triangle in the harmonica of T'_2 which contains v_1. But then z_1z_2 is an edge such that - without loss of generality - $z_1 \in N(v_2)$, $|L(z_1)| = 5$ and $z_2 \in P_1$, $|L(z_2)| = 3$, contradicting that Claim 2.8.10(1) holds with $i = 2$. So we may suppose that Claim 2.8.10(1) does not hold with $i = 2$. By symmetry, Claim 2.8.10(1) does not hold with $i = 1$ and hence Claim 2.8.10(1) does not hold.

So Claim 2.8.10(2) holds. As Claim 2.8.10(2) holds and there is no chord of C whose ends have lists of size less than five, it follows that $x_1 \sim u_2$ and $v_1x_1u_2$ is the base of an even fan. Similarly, $v_2x_1u_2$ is the base of an even fan. Hence T contains a defective double bellows, a contradiction. □

By symmetry, $x_2 \sim u_2$. As $d(u_1, u_2) \geq 3$ by Claim 2.8.6, $\{v_1, x_1, v_2\} \cap \{w_1, x_2, w_2\} = \emptyset$.

Claim 2.8.12. *For all $i \in \{1, 2\}$, if Claim 2.8.10(2) holds or Claim 2.8.10(1) holds with i, then either there exists a vertex in P_{3-i} adjacent to both x_1 and x_2, or, there exists adjacent vertices z_1, z_2 in P_{3-i} such that $z_1z_2x_2x_1$ is a 4-cycle.*

Proof. By symmetry, it suffices to prove the claim for $i = 2$. Let z_1 be the neighbor of x_1 in P_1 closest to w_1 in P_1. Now T_2 contains an orchestra T'_2. Suppose T'_2 is a special orchestra. It follows from an argument similar to that given in the proof of Claim 2.8.11 that w_1u_2 is the cut-edge of T'_2. Let $w_1z_1z_2$ be the triangle in the harmonica of T'_2 which contains w_1. But then z_1z_2 is an edge such that - without loss of generality - $z_1 \in N(v_2)$, $|L(z_1)| = 5$ and $z_2 \in P_1$, $|L(z_2)| = 3$, and hence Claim 2.8.10(1) does not hold with $i = 2$. So Claim 2.8.10(2) holds and it follows that $z_1 = x_1$ and hence $w_1 \sim x_1$. Therefore w_1 is adjacent to both x_1 and x_2 as desired.

So we may suppose that T'_2 is an instrumental orchestra. Note that there does not a cutvertex of T_2' as then there would exists an edge z_1z_2 such that - without loss of generality - $z_1 \in N(v_2)$, $|L(z_1)| = 5$ and $z_2 \in P_1$, $|L(z_2)| = 3$, where $z_1, z_2 \notin \{v_1, x_1\}$ contradicting that either Claim 2.8.10(2) holds or Claim 2.8.10(1) holds with $i = 2$. Let F be the instrument in T_2' with side x_1z_1 whose other side is closest to u_2. It follows since there is no such edge z_1z_2 as above that u_2 is in F. If F is a bellows, then $z_1 = w_1$ and the claim follows. If F is a double bellows, then $z_1 \sim x_2$ and the claim follows.

So we may suppose that F is a defective double bellows. Yet x_2 must be the center of the wheel. If $x_2 \sim z_1$, the claim follows. So $x_2 \not\sim z_1$. But then there exists $z_2 \in F$ such that $z_1z_2x_2x_1$ is a 4-cycle. Yet z_2 must be in P_1 since $z_1 \not\sim u_2$ as $d(u_1, u_2) \geq 3$ by Claim 2.8.6. $\square$

An identical claim holds for the symmetric version of Claim 2.8.10.

Claim 2.8.13. *For all $i \in \{1, 2\}$, either there exists a vertex z_i in P_i adjacent to both x_1 and x_2, or, there exists adjacent vertices z_i, z_i in P_i such that $z_iz_i'x_2x_1$ is a 4-cycle.*

Proof. If Claim 2.8.10(2) holds, then the claim follows by applying Claim 2.8.12 with $i = 1$ and again with $i = 2$. So we may suppose that Claim 2.8.10(1) holds. Without loss of generality suppose Claim 2.8.10(1) holds with $i = 2$. By Claim 2.8.10, the claim holds for $i = 1$. Moreover as Claim 2.8.10(1) holds with $i = 2$, $z = v_1$ is adjacent to both x_1 and x_2, or, there exists z_1, z_2 in P_i such that $z_1z_2x_2x_1$ is a 4-cycle where $z_1 = v_1$.

Now it follows that (2) holds for the symmetric version of Claim 2.8.10 or that (1) holds with $i = 1$. By the symmetric version of Claim 2.8.10(2), the claim holds for $i = 2$. So claim holds for $i = 1$ and 2. This proves the claim. $\square$

Moreover, by these arguments, and symmetric arguments for P', we find that either $x \sim w_1$ or $x' \sim v_1$ or there exists $z_1 \neq v_1, w_1$ such that $z_1 \in V(C)$ and

$z_1 \sim x, x'$. Similarly either $x \sim w_2$ or $x' \sim v_2$ or there exists $z_2 \neq v_2, w_2$ such that $z_2 \in V(C)$ and $z_2 \sim x, x'$.

Define $L'(z) = L(z) \setminus \{L(u) : u \in \{u_1, u_2\}, u \sim z\}$. Thus $|L'(w_1)|$,$|L'(w_2)|$,$|L'(v_1)|$,$|L'(v_2)| \geq 2$ and $|L'(x_1)|, |L'(x_2)| \geq 4$. Consider the canvas $(G \setminus \{u_1, u_2\}, S, L')$ where $S = \{v_1, v_2, w_1, w_2\}$. By Lemma 2.8.1, this has an L'-coloring of $G \setminus \{u_1, u_2\}$ unless there exists $z \in \{x_1, x_2\}$ such that z is adjacent to all of S and $|L'(z)| = 4$. But an L'-coloring of $G \setminus \{u_1, u_2\}$ extends to an L-coloring of G, a contradiction. So we may assume without loss of generality that x_1 is adjacent to all of S and $|L'(x_1)| = 4$. But then x_2 is in the interior of the 4-cycle $u_2 w_1 x_1 w_2$, a contradiction. □

Theorem 2.8.14. *Let $T = (G, P \cup P', L)$ be a canvas where P, P' are paths of length at most one. Suppose that if P (resp. P') has length one, then the set of L-colorings of P (resp P') is a democracy. If P (resp P') has length zero, then suppose that the vertex of P has a list of size at most two. Further suppose that*

(1) *if $|V(P)| = 2$, $P' = u'$, $|L(u')| = 2$, then $d(P, u') > 1$;*

(2) *if $P = u$, $P' = u'$ and $|L(u)| = |L(u')| = 1$, then $d(u, u') > 2$;*

(3) *if $|V(P)| = 2$, $P' = u'$, $|L(u')| = 1$, then $d(P, P') > 3$;*

(4) *if $|V(P)| = |V(P')| = 2$ then $d(P, P') > 4$.*

If there does not exist an L-coloring of G, then there exists an essential chord of the outer walk C of G whose ends have lists of size less than five.

Proof. Note that (1) follows Theorem 2.7.8. The rest follow from Theorem 2.8.3 as an orchestra of the prescribed lengths yield chords whose ends have lists of size less than five as desired. □

2.9 Reducing a Precolored Edge to a Government

In this section, we extend Theorem 2.8.14 to the case when $|\mathcal{C}_1|, |\mathcal{C}_2| = 1$. First a definition that will be useful for the proof.

Definition (d-slicing)**.** Let $d > 0$. Let $T = (G, P \cup P', L)$ be a canvas and let P be a path such that $d(P, P') > d$.

We say that a canvas $T' = (G', P_1 \cup P', L')$ is a *d-slicing* of T with respect to P if all of the following hold:

(i) There exists a path P_0 with both ends on the outer walk of G that divides G into G_1 and G' where G_1 includes P and G' includes P'.

(ii) P_1 is a subpath of P_0 of length at most one such that for all $v \in V(P_1)$, $d(v, P) \leq d-1$ and if $|V(P_1)| = 2$, then the set of L'-colorings of P_1 contains a government.

(iii) For all $v \in V(G)$ with $d(v, P) \geq d$, $v \in V(G')$ and $L'(v) = L(v)$.

(iv) If G' has an L'-coloring, then G has an L-coloring.

Note that a d-slicing of T with respect to P is a d'-slicing of T with respect to P for all $d' \geq d$ such that $d' < d(P, P')$.

Theorem 2.9.1. *Let $d = 4$. Let $T = (G, P \cup P', L)$ be a canvas where P is path of length one and P' is a path of length at most one. Let $P = p_1p_2$ and suppose that $|L(p_1)| = |L(p_2)| = 1$. Suppose that*

(1) *if $P' = u$ and $|L(u)| = 2$, then $d(P, u) > d + 1$;*

(2) *if $P' = u$ and $|L(u)| = 1$, then $d(P, u) > d + 3$;*

(3) *if $P' = p'_1p'_2$ and $L(p'_1) = L(p'_2)$, $|L(p'_1)| = 2$, then $d(P, P') > d + 4$;*

(4) *if $P' = p'_1p'_2$ and $|L(p'_1)| = |L(p'_2)| = 1$, then $d(P, P') > 2d + 4$;*

.

If there does not exist an L-coloring of G, then there exists an essential chord of the outer walk C of G whose ends have lists of size less than five.

Proof. Suppose not. Let $T = (G, S, L)$ be counterexample with a minimum number of edges where $S = P \cup P'$. We may assume that T is L-critical; hence by Lemma 2.4.1, every cutvertex of G or chord of G is essential and there is no vertex in the interior of a 4-cycle. Let $P = p_1p_2$ and $p_1v_1, p_2v_2 \in E(C)$. As $d(P, P') > 1$, $v_1, v_2 \notin V(P')$. Let $L(p_1) = \{c_1\}$ and $L(p_2) = \{c_2\}$. Let $S(w) = L(w) \setminus \{c_i | w \sim p_i\}$.

The following claim is very useful.

Claim 2.9.2. *There does not exist a d-slicing of T with respect to P.*

Proof. Suppose not. Let $T' = (G', P_1 \cup P', L')$ be a d-slicing of T with respect to P. As T is a counterexample, there does not exist an L-coloring of G. Hence, by property (iv) of d-slicing, there does not exist an L'-coloring of G'.

First suppose there exists an essential chord of the outer walk C' of G' with both ends having lists of size less than five. Let U be such a chord of C' closest to P'. As T is a counterexample, U is not a chord of C. Hence there is an end, call it z of U such that $z \notin V(C)$. Thus $|L(z)| = 5$ and yet $|L'(z)| < 5$. By property (iii) of d-slicing, $d(z, P) \le d - 1$. Hence $d(U, P) \le d - 1$. Note then that $d(P, P') \le d(U, P) + 1 + d(U, P') \le d + d(U, P')$.

Consider $T'' = (G'', U \cup P', L')$ where U divides G' into two graphs G_1 and G'' where G'' is the one containing P'. Note that by property (ii) of d-slicing, either P_1 is one vertex or a path of length one and there exists a set of L-coloring of P_1 that is a government. Either way, by Theorem 2.6.5, it follows that there exists a government $\mathcal{C}$ for U such that every L-coloring $\phi \in \mathcal{C}$ extends to an L'-coloring of G_1. As there does not exist an L'-coloring of G', there does not exist an L'-coloring ϕ of G'' with $\phi \restriction U \in \mathcal{C}$. Further note that as U was chosen closest to P', there does not exist an essential chord of the outer walk of G'' with both ends having list of size less than five.

Now if (1),(2), or (3) holds for T, apply Theorem 2.8.14 to T'' to find that $d(U, P') \leq 1$, 3, or 4 respectively. Hence $d(P, P') \leq d+1$, $d+3$, or $d+4$, respectively, a contradiction. If (4) holds for T, apply (2) or (3) to T'' to find that $d(U, P') \leq d+4$ and hence $d(P, P') \leq 2d+4$, a contradiction.

So we may suppose there does not exist an essential chord of C' with both ends having lists of size less than five. Note again that by property (ii) of d-slicing, either P_1 is one vertex or a path of length one and there exists a set of L-coloring of P_1 that is a government. Furthermore $d(P_1, P) \leq d-1$ by property (ii) of d-slicing. Note that $d(P, P') \leq d(P_1, P') + d$.

If (1),(2), or (3) holds for T, apply Theorem 2.8.14 to T' to find that $d(P_1, P') \leq 1$, 3, or 4 respectively. Hence $d(P, P') \leq d+1$, $d+3$, or $d+4$, respectively, a contradiction. If (4) holds for T, apply (2) or (3) to T' to find that $d(U, P') \leq d+4$ and hence $d(P, P') \leq 2d+4$, a contradiction. $\square$

Claim 2.9.3. *There does not exists a chord U of C with an end v such that $d(v, P) \leq d-1$.*

Proof. Suppose not. Now U divides G into two graphs G_1, G_2 with $P \cap (G_2 \setminus U) = \emptyset$ and $P' \cap (G_1 \setminus U) = \emptyset$. Now there must be an end of U with a list of size five as otherwise T is not a counterexample. It now follows from Theorem 2.6.5 that there exists a government $\mathcal{C}'$ for U such that every L-coloring $\phi \in \mathcal{C}$ extends to an L-coloring of G_1.

Consider $T' = (G_2, U \cup P', L')$. If (1),(2), or (3) holds for T, apply Theorem 2.8.14 to T' to find that $d(U, P') \leq 1$, 3, or 4 respectively. Hence $d(P, P') \leq d+1$, $d+3$, or $d+4$, respectively, a contradiction. If (4) holds for T, apply (2) or (3) to T'' to find that $d(U, P') \leq d+4$ and hence $d(P, P') \leq 2d+4$, a contradiction. $\square$

Claim 2.9.4. *For $i \in \{1,2\}$, $|S(v_i)| = 2$.*

Proof. Suppose not. Suppose without loss of generality that $|L(v_1)| \geq 4$ or $c_1 \notin L(v_1)$. Let $G' = G \setminus p_1$ and let $L'(v) = L(v) \setminus \{c_1\}$ for all $v \in N(p_1)$ and $L' = L$ otherwise. Now $T' = (G', p_2 + P', L')$ is a canvas as there is no chord of C incident with p_1 whose other end has a list of size less than five. Yet T' is a 2-slicing of T with respect to P, contradicting Claim 2.9.2. □

Claim 2.9.5. *There does not exist a cutvertex v of G.*

Proof. Suppose there does. As T is critical, v is essential. Thus v divides G into two graphs G_1, G_2 where $V(P) \cap (V(G_2) \setminus \{v\}) = \emptyset$ and $V(P') \cap (V(G_1) \setminus \{v\}) = \emptyset$. Suppose $v \in V(P)$. But then $(G \setminus (P \setminus v), v + P', L)$ is a 1-slicing of T with respect to P, contradicting Claim 2.9.2.

So we may suppose that $v \notin V(P) \cup V(P')$. For $i \in \{1,2\}$, let $L_i(v)$ be the set of all colors c in $L(v)$ such that coloring v with c does not extend to an L-coloring of G_i. As T is critical, $|L_1(v)| + |L_2(v)| \geq |L(v)| \geq 3$. Yet by Theorem 1.4.2, $|L_i(v)| \geq 1$ for all $i \in \{1,2\}$. Let $T_1 = (G_1, P + v, L_1)$ where $L_1(v)$ is as above and $L_1 = L$ otherwise. Similarly let $T_2 = (G_2, v + P', L_2)$ where $L_2(v)$ is as above and $L_2 = L$ otherwise.

Suppose $|L_1(v)| = 2$. By (1) applied to T_1, it follows that $d(v,P) \leq 3 \leq d+1$. Yet $d(P,P') \leq d(v,P) + d(v,P')$. If (1) holds for T_2, apply Theorem 2.8.14 to T'' to find that there exists an essential chord whose ends have lists of size less than five, a contradiction. If (2) or (3) holds for T_2, apply Theorem 2.8.14 to find that $d(v,P') \leq 2$ or 3 respectively. Hence $d(P,P') \leq d+3$, or $d+4$, respectively, a contradiction. If (4) holds for T, apply (2) to T'' to find that $d(v,P') \leq d+3$ and hence $d(P,P') \leq 2d+3$, a contradiction.

So we may suppose that $|L_1(v)| = 1$. By (2) applied to T_1, it follows that $d(v,P) \leq d+3$. Hence $|L_2(v)| = 2$. If (1) or (2) holds for T, then by Theorem 2.8.14 applied to T_2, there exists an essential chord of the outer walk of G_2 whose ends have lists of size less than five, a contradiction. If (3) holds for T, then by Theorem 2.8.14 applied to T_2, $d(v, P') \leq 1$ and hence $d(P, P') \leq d + 4$, a contradiction. If (4) holds for T, then it follows from (1) applied to T_2 that $d(v, P') \leq d + 1$ and hence $d(P, P') \leq 2d + 4$, a contradiction. □

As there does not exist a chord of C with both ends having size less than five, we may consider the Thomassen reductions $T_1 = (G_1, S, L_1)$ and $T_2 = (G_2, S, L_2)$ for v_1, v_2 respectively.

Claim 2.9.6. *For $i \in \{1, 2\}$, there exists $x_i \notin V(C)$ such that $x_i \sim p_1, p_2, v_i$, $c_1, c_2 \in L(x_i)$ and $S(v_i) \subseteq S(x_i)$.*

Proof. By symmetry it suffices to prove the claim for $i = 1$. So consider T_1. As T is a minimum counterexample, there exists a chord of the outer walk C_1 of G_1 whose ends have lists in L_1 of size at most four. As T is a counterexample, such a chord of C_1 is not also a chord of C.

Let $U = u_1u_2$ be the furthest such chord of C_1 from P where $u_1 \notin V(C)$. Now U divides G into two graphs H_1, H_2 where we may assume without loss of generality that $P \cap (H_2 \setminus U) = \emptyset$. As U is not a chord of C and $|L_1(u_1)| < 5$, we find that u_1 is adjacent to v_1.

Suppose $\Phi_{T_1}(U, \mathcal{C})$ contains a government $\mathcal{C}'$. Now $d(v, P) \leq 3 \leq d$ for all $v \in U$. But then $T'' = (G'', U \cup P', L_1)$ is a d-slicing of T with respect to P, contradicting Claim 2.9.2.

So we may suppose that $\Phi_{T_1}(U, \mathcal{C})$ does not contain a government. By Theorem 2.6.5, $|\Phi_{T_1}(U, \mathcal{C}_1)| = 1$ and there exists a 1-accordion T_1' in T_1 from P to U. As p_1 is not in a chord of C by Claim 2.9.3, there exists $x_1 \sim p_2$ such that x_1p_2 is a chord of C_1 and $|L_1(x_1)| = 3$. Yet x_1p_2 is not a chord of C. Hence $x_1 \sim v_1$ and $L(x_1)$ is the disjoint of $L_1(x_1)$ and $S(v_1)$. As there is no vertex inside of the 4-cycle $p_1v_1x_1p_2$, we find given the 1-accordion T_1' that $p_1 \sim x_1$. Hence, $c_1, c_2 \in L_1(x_1)$ and thus $S(v_1) \subseteq S(x_1)$. This proves the claim. $\square$

As G is planar, it follows that $x_1 = x_2$. Call this vertex x. Hence $c_1, c_2 \in L(x)$ and $S(v_1), S(v_2) \subseteq S(x)$.

Claim 2.9.7. $|S(v_1) \cap S(v_2)| = 1$.

Proof. Suppose not. As $S(v_1), S(v_2)$ are lists of size two and both are a subset of $S(x)$, a list of size three, we find that $|S(v_1) \cap S(v_2)| \geq 1$. So we may assume that $S(v_1) = S(v_2)$. Let $S(x) \setminus S(v_1) = \{c\}$. Let $T' = (G \setminus P, x + P', L')$ where $L'(x) = \{c\}$, $L'(v_1) = S(v_1) \cup \{c\}$, $L'(v_2) = S(v_2) \cup \{c\}$ and $L' = L$ otherwise. Now $d(x, P) \leq 1 \leq d$. It follows that T' is a 2-slicing of T with respect to P, contradicting Claim 2.9.2. $\square$

So we may assume that $S(v_1) = \{c_3, c_4\}$, $S(v_2) = \{c_3, c_5\}$, and $S(x) = \{c_3, c_4, c_5\}$. For $i \in \{1, 2\}$, let $s_i \in S \setminus P$ such that s_i is closest to v_i $C \setminus P$; let P_i be the path in C from v_i to s_i avoiding P; let u_i be the neighbor of x in P_i closest to s_i, as measured in P_i. Let W_1 be the bellows with base $p_1 x u_1$ and W_2 be the bellows with base $p_2 x u_2$. Note that these are the bases of bellows as T is L-critical. Further note that as $d \geq 3$, neither u_1 nor u_2 is in a chord of C.

Claim 2.9.8. *For all $i \in \{1, 2\}$, then there are at least two colors in $S(x)$ that extend to a L-coloring of the base of W_i that does not extend to an L-coloring of W_i.*

Proof. Suppose not. We may assume without loss of generality that there exists at most one such color c for W_1. Let $L'(x) = L(x) \setminus \{c, c_1\}$ if c exists and $L'(x) = L(x) \setminus \{c_1\}$ otherwise. Let $L'(w) = L(w)$ for all $w \in G \setminus \{x\}$. Let $G' = G \setminus (W_1 \setminus \{x\})$ and $S' = S \setminus \{p_1\}$. Now $T' = (G', p_2 + P', L')$ is a 4-slicing of T with respect to P, contradicting Claim 2.9.2. $\square$

It follows that for all $i \in \{1, 2\}$, W_i is an exceptional odd fan or an exceptional even fan. Hence, $c_3, c_4 \in L(v)$ for all $v \in W_1 \setminus \{x, u_1\}$ and $c_3, c_5 \in L(v)$ for all $v \in W_2 \setminus \{x, u_2\}$.

If W_1 is an exceptional odd fan, then there exists $c_6 \in L(u_1)$ such that the only non-extendable L-colorings of p_1, x, u_1 to W_1 are c_1, c_3, c_6 and c_1, c_4, c_6 and hence $c_6 \neq c_3, c_4$. Similarly if W_2 is an exceptional odd fan, then there exists $c_7 \in L(u_2)$ such that the only non-extendable L-colorings of p_2, x, u_2 to W_2 are c_2, c_3, c_7 and c_2, c_5, c_7 and hence $c_7 \neq c_3, c_5$.

If W_1 is an exceptional even fan, then the only non-extendable L-colorings of p_1, x, u_1 to W_1 are c_1, c_3, c_4 and c_1, c_4, c_3. If W_2 is an exceptional even fan, then the only non-extendable L-colorings of p_2, x, u_2 to W_2 are c_1, c_3, c_5 and c_1, c_5, c_3.

Here are some useful claims before we break our analysis into cases.

Claim 2.9.9. *For $i \in \{1,2\}$, $|L(u_i)| = 3$.*

Proof. Suppose not. Suppose without loss of generality that $|L(u_1)| \geq 4$. Thus $u_1 \neq v_1$.

First suppose W_1 is odd. Let $L'(u_1) = L(u_1) \setminus \{c_6\}$, $L'(x) = L(x) \setminus \{c_1\}$. Let $G' = G \setminus (W_1 \setminus \{x, u_1\})$. Now $T' = (G', p_2 + P', L')$ is a 3-slicing of T with respect to P, contradicting Claim 2.9.2.

So we may suppose that W_1 is even. Let $L'(u_1) = L(u_1) \setminus \{c_3\}$, $L'(x) = \{c_4\}$, $L'(v_2) = \{c_3, c_4, c_5\}$. Let $G' = G \setminus (P \cup (W_1 \setminus \{x, u_1\}))$. Now $T' = (G', x + P', L')$ is a 3-slicing of T with respect to P, contradicting Claim 2.9.2. □

Claim 2.9.10. *If W_i is odd, then $L(u_i) \setminus \{c_{5+i}\} = S(v_i)$ or $\{c_4, c_5\}$.*

Proof. Suppose not. Suppose without loss of generality that W_1 is odd and yet $L(u_1) \setminus \{c_6\} \neq \{c_3, c_4\}$ or $\{c_4, c_5\}$. Let $G' = G \setminus (W_1 \setminus \{x\}) \setminus p_2$. First suppose $c_4 \in L(u_1) \setminus \{c_6\}$. Let $L'(v_2) = L'(x) = \{c_3, c_5\}$ and $L'(w) = L(w) \setminus (L(u_1) \setminus \{c_6\})$ for all $w \in N(u_1) \cap V(G')$ where $w \neq x$, and $L' = L$ otherwise. Now $T' = (G', x + P', L')$ is a 4-slicing of T with respect to P, contradicting Claim 2.9.2.

So we may suppose that $c_4 \notin L(u_1)$. Let $L'(v_2) = S(v_2) \cup \{c_4\}$, $L'(x) = \{c_4\}$, $L'(w) = L(w) \setminus (L(u_1) \setminus \{c_6\})$ for all $w \in N(u_1) \cap V(G')$ where $w \neq x$, and $L' = L$

otherwise. Now $T' = (G', x+P', L')$ is a 4-slicing of T with respect to P, contradicting Claim 2.9.2. $\square$

Case 1: W_1 and W_2 are odd.

Let $T' = (G', u_1 + P', L')$ be the democratic reduction of x, u_2 in $(G \setminus (P \cup W_1 \cup W_2 \setminus \{u_1, x, u_2\}), \{u_1, x, u_2\} \cup P', S)$ with respect to $L(u_2) \setminus \{c_6\}$ and centered around u_1. If every L'-coloring of G' extends to an L-coloring of G, then T' is a 4-slicing of T with respect to P, contradicting Claim 2.9.2.

But then it follows that $L(u_1) \setminus (L(u_2) \setminus \{c_7\} = c_6$. That is, $L(u_1) \setminus \{c_6\} = L(u_2) \setminus \{c_7\}$. This implies that either $v_1 \neq u_1$ or $v_2 \neq u_2$. Suppose without loss of generality that $v_1 \neq u_1$. Hence, by Claim 2.9.10, $L(u_1) \setminus \{c_6\} = \{c_3, c_4\}$ or $\{c_4, c_5\}$. But this implies then that $v_2 \neq u_2$. Hence, by Claim 2.9.10, $L(u_2) \setminus \{c_7\} = \{c_3, c_5\}$ or $\{c_4, c_5\}$. It now follows that $L(u_1) \setminus \{c_6\} = L(u_2) \setminus \{c_7\} = \{c_4, c_5\}$.

Let $yu_1 \in V(C)$ where $y \notin W_1$. Let $T' = (G', y + P', L')$ be the democratic reduction of u_1, x, u_2 in $(G \setminus (P \cup W_1 \cup W_2 \setminus \{u_1, x, u_2\}), \{v_1, x, u_2\} \cup P', S)$ with respect to $\{c_4, c_5\}$ and centered around y. Now T' is a 4-slicing of T with respect to P, contradicting Claim 2.9.2.

Case 2: One of W_1, W_2 is even and the other odd.

We may suppose without loss of generality that W_1 is even and W_2 is odd. As W_1 is an exceptional even fan, $c_3, c_4 \in L(u_1)$ and the only non-extendable L-colorings of p_1, x, u_1 to W_1 are c_1, c_3, c_4 and c_1, c_4, c_3. As W_2 is odd, $L(u_2) \setminus \{c_7\} = \{c_3, c_5\}$ or $\{c_4, c_5\}$ by Claim 2.9.10.

So we may assume that $S(v_1) \subset L(u_1)$. Suppose $c_5 \notin L(u_1)$. Let $T' = (G', u_1 + P', L')$ be the democratic reduction of x, u_2 in $(G \setminus (P \cup W_1 \cup W_2 \setminus \{u_1, x, u_2\})), \{u_1, x, u_2\} \cup P', S')$ with respect to $L(u_2) \setminus \{c_7\}$ and centered around u_1 where $S'(u_1) = L(u_1) \setminus S(x)$ and $S' = S$ otherwise. Now T' is a 4-slicing of T with respect to P, contradicting Claim 2.9.2.

So we may assume that $L(u_1) = \{c_3, c_4, c_5\}$. Let $yu_1 \in V(C)$ where $y \notin W_1$. Let

$T' = (G', y + P', L')$ be the democratic reduction of u_1, x, u_2 in $(G \setminus (P \cup W_1 \cup W_2 \setminus \{u_1, x, u_2\}), \{v_1, x, u_2\} \cup P', S)$ with respect to $L(u_2) \setminus \{c_7\}$ and centered around y. Now T' is a 4-slicing of T with respect to P, contradicting Claim 2.9.2.

Case 3: W_1 and W_2 are even.

As W_1 is an exceptional even fan, $c_3, c_4 \in L(u_1)$ and the only non-extendable L-colorings of p_1, x, u_1 to W_1 are c_1, c_3, c_4 and c_1, c_4, c_3. Similarly as W_2 is an exceptional even fan, $c_3, c_5 \in L(u_2)$ and the only non-extendable L-colorings of p_2, x, u_2 to W_2 are c_2, c_3, c_5 and c_2, c_5, c_3.

Suppose that $c_4 \in L(u_2)$. Let $T' = (G', u_1 + P', L')$ be the democratic reduction of x, u_2 in $(G \setminus (P \cup W_1 \cup W_2 \setminus \{u_1, x, u_2\})), \{u_1, x, u_2\} \cup P', S)$ with respect to $\{c_3, c_4\}$ and centered around u_1. Now T' is a 4-slicing of T with respect to P, contradicting Claim 2.9.2.

So we may suppose that $c_4 \notin L(u_2)$. Let $G' = G \setminus (P \cup W_1 \cup W_2 \setminus \{u_1, u_2\})$. Let $L'(u_1) = L(u_1) \setminus S(v_1)$, $L'(w) = L(w) \setminus \{c_4\}$ for all $w \in V(G') \cap N(x)$. Let $T' = (G', u_1 + P', L')$. Now T' is a 3-slicing of T with respect to P, contradicting Claim 2.9.2.

□

Theorem 2.9.11 (Two Precolored Edges)**.** *Let $T = (G, P \cup P', L)$ be a canvas where $P \cup P'$ are paths of length one and $d(P, P') \geq 14$. If there does not exist an L-coloring, then there exists an essential chord of the outer walk of G whose ends are not in $V(P) \cup V(P')$ but have lists of size less than five.*

Proof. Let C be the outer walk of G. If there exists an essential chord of C incident with a vertex of P, let P_1 be the essential chord incident with a vertex of P closest to P', and let $P_1 = P$ otherwise. Define P_2 similarly for P'. Apply Theorem 2.9.1 to the canvas $T' = (G', P_1 \cup P_2, L)$ between P_1 and P_2. As $d(P_1, P_2) \geq 12$ since $d(P, P') \geq 14$, there exists a chord U of the outer walk of G' whose ends have lists of size less than five. Now U is also a chord of the outer walk of G. Furthermore, U is not incident with a vertex of P or P' given how P_1 and P_2 were chosen. $\square$

2.10 Two Confederacies

In this section, we will further characterize the structure of orchestras which start with two confederacies. This will allow us to prove that orchestras contain either a harmonica or accordion whose length is proportional to that of the orchestra.

Definition. Let $\mathcal{C}$ be a collection of disjoint proper colorings of a path $P = p_1p_2$ of length one. We say $\mathcal{C}$ is an *alliance* if either

(1) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ such that $\mathcal{C}_1, \mathcal{C}_2$ are dictatorships, $|\mathcal{C}(p_1)|, |\mathcal{C}(p_2)| \leq 3$ and for all $i \in \{1, 2\}$, if z is the dictator of $\mathcal{C}_i$, then $\mathcal{C}_i(z) \cap \mathcal{C}_{3-i}(z) = \emptyset$, or,

(2) $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3$ where $\mathcal{C}_1$ is a dictatorship with dictator p_1, $\mathcal{C}_2$ is a dictatorship with dictator p_2 and $\mathcal{C}_3$ is a democracy with colors $\mathcal{C}_1(p_1) \cup \mathcal{C}_2(p_2)$ and $\mathcal{C}_2(p_1) \cap \mathcal{C}_1(p_2) = \emptyset$.

If (1) holds for $\mathcal{C}$, we say $\mathcal{C}$ is an *alliance of the first kind* and if (2) holds for $\mathcal{C}$ that $\mathcal{C}$ is an *alliance of the second kind.*

Lemma 2.10.1. *Let $T = (G, P, L)$ be a bellows with base $P = p_1p_2p_3$ and let $\mathcal{C}$ be a confederacy for p_1p_2. If $|L(p_3)| \geq 4$, then $\Phi(p_2p_3, \mathcal{C})$ contains an alliance.*

Proof. By Theorem 2.7.2, we may assume that T is a turbofan or $p_1 \sim p_3$. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ be a confederacy where $\mathcal{C}_1, \mathcal{C}_2$ are distinct governments. Suppose that T is a turbofan. By Lemma 2.3.5, there exists a unique coloring of P that does not extend to an L-coloring of G. Let $L'(p_3) = L(p_3) \setminus \{\phi(p_3)\}$. Let $c_1, c_2 \in \mathcal{C}(p_2)$. Let $\mathcal{C}_i'$ be the set of all colorings ϕ of p_2, p_3 such that $\phi(p_2) = c_i$ and $\phi(p_3) \in L'(p_3)$. Hence $\mathcal{C}_1', \mathcal{C}_2'$ are dictatorships with dictator p_2 such that $\mathcal{C}_1'(p_2) \neq \mathcal{C}_2'(p_2)$. Thus $\mathcal{C}' = \mathcal{C}_1' \cup \mathcal{C}_2'$ is an alliance as desired.

So we may suppose that $p_1 \sim p_3$. Suppose that $\mathcal{C}_1$ is a democracy and $\mathcal{C}_1(p_1) = \{c_1, c_2\}$. Let $L'(p_3)$ be a subset of $L(p_3) \setminus \{c_1, c_2\}$ of size two. For $i \in \{1, 2\}$, let $\mathcal{C}'_i$ be all colorings ϕ of p_2, p_3 such that $\phi(p_2) = c_i$ and $\phi(p_3) \in L'(p_3)$. Hence $\mathcal{C}'_1, \mathcal{C}'_2$ are dictatorships with dictator p_2 such that $\mathcal{C}'_1(p_2) \neq \mathcal{C}'_2(p_2)$. Thus $\mathcal{C}' = \mathcal{C}'_1 \cup \mathcal{C}'_2$ is an alliance as desired. So we may suppose that neither C_1 nor by symmetry $\mathcal{C}_2$ is a democracy.

Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are dictatorships. Suppose that the dictator of $\mathcal{C}_1$ is p_1. Let $L'(p_3)$ be a subset of $L(p_3) \setminus \mathcal{C}_1(p_1)$ of size three. Let $c_1, c_2 \in \mathcal{C}_1(p_2)$. For $i \in \{1, 2\}$, let $\mathcal{C}'_i$ be all colorings ϕ of p_2, p_3 such that $\phi(p_2) = c_i$ and $\phi(p_3) \in L'(p_3)$. Hence $\mathcal{C}'_1, \mathcal{C}'_2$ are dictatorships with dictator p_2 such that $\mathcal{C}'_1(p_2) \neq \mathcal{C}'_2(p_2)$. Thus $\mathcal{C}' = \mathcal{C}'_1 \cup \mathcal{C}'_2$ is an alliance as desired.

So we may suppose that p_2 is the dictator of C_1 and by symmetry also of $\mathcal{C}_2$. Let $L'(p_3)$ be a subset of $L(p_3)$ of size three. Let $c_1 = \mathcal{C}_1(p_2)$ and $c_2 = \mathcal{C}_2(p_2)$. For $i \in \{1, 2\}$, let $\mathcal{C}'_i$ be all colorings ϕ of p_2, p_3 such that $\phi(p_2) = c_i$ and $\phi(p_3) \in L'(p_3)$. Now $|\mathcal{C}'_i| \geq 2$ for $i = 1, 2$. Hence $\mathcal{C}'_1, \mathcal{C}'_2$ are dictatorships with dictator p_2 such that $\mathcal{C}'_1(p_2) \neq \mathcal{C}'_2(p_2)$. Let $\mathcal{C}' = \mathcal{C}'_1 \cup \mathcal{C}'_2$. Now by Theorem 1.4.2, all colorings in $\mathcal{C}'$ are in $\Phi(P', \mathcal{C})$. Hence $\mathcal{C}'$ is an alliance as desired. □

Lemma 2.10.2. *Let $T = (G, P, L)$ be a bellows with base $P = p_1p_2p_3$. If $\mathcal{C}$ is an alliance for p_1p_2 and $\mathcal{C}'$ is a confederacy for p_2p_3 such that $|\mathcal{C}(p_2) \cup \mathcal{C}'(p_2)| \leq 3$, then there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G.*

Proof. Note that $|\mathcal{C}(p_2)|, |\mathcal{C}(p_3)| \geq 2$. Hence as $|\mathcal{C}(p_2) \cup \mathcal{C}'(p_2)| \leq 3$, $\mathcal{C}(p_2) \cap \mathcal{C}'(p_2) \neq \emptyset$.

We claim that there exists $c \in \mathcal{C}(p_2) \cap \mathcal{C}'(p_2)$ such that there exists two colorings ϕ in $\mathcal{C}$ or two colorings ϕ in $\mathcal{C}'$ with $\phi(p_2) = c$. Suppose $|\mathcal{C}(p_2)| = 2$. Hence $\mathcal{C}$ is an alliance of the first kind. Now for all $c \in \mathcal{C}(p_2)$ there exist two colorings $\phi \in \mathcal{C}$ such that $\phi(p_2) = c$ as $\mathcal{C}$ is an alliance. Thus the claim follows with $c \in \mathcal{C}(p_2) \cap \mathcal{C}'(p_2)$.

So we may suppose that $|\mathcal{C}(p_2)| = 3$. But then there is one $c_0 \in \mathcal{C}(p_2)$ such that there exist two colorings $\phi \in \mathcal{C}$ with $\phi(p_2) = c$ as $\mathcal{C}$ is an alliance. Hence we may assume that $c_0 \notin \mathcal{C}'(p_2)$ as otherwise the claim follows with $c = c_0$. So $|\mathcal{C}'(p_2)| = 2$ as $|\mathcal{C}(p_2) \cup \mathcal{C}'(p_2)| \leq 3$. But then there exists $c \in \mathcal{C}'(p_2)$ such that there exist two colorings $\phi \in \mathcal{C}'$ with $\phi(p_2) = c$ as $\mathcal{C}$ is a confederacy. As c is also in $\mathcal{C}(p_2)$, this proves the claim.

Let $L'(p_2) = \{c\}$, $L'(p_1) = \{\phi(p_1) | \phi \in \mathcal{C}, \phi(p_2) = c\} \cup \{c\}$ and $L'(p_3) = \{\phi(p_3) | \phi \in \mathcal{C}', \phi(p_2) = c\} \cup \{c\}$. As $c \in \mathcal{C}(p_2) \cap \mathcal{C}'(p_2)$, $|L'(p_1)|, |L'(p_3)| \geq 2$. Furthermore, by the claim above either $|L'(p_1)| \geq 3$ or $|L'(p_3)| \geq 3$. Hence by Theorem 1.4.2, there exists an L'-coloring of G. But this implies there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G as desired. $\square$

Lemma 2.10.3. *Let $T = (G, P \cup P', L)$ be a double bellows, $\mathcal{C}$ be a democracy for P and $\mathcal{C}'$ be a democracy for P'. Then there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G.*

Proof. Let x be the center of T. Let $P = p_1p_2$, $P' = p_1'p_2'$ such that $T_1 = (G_1, p_1xp_1', L)$ is a bellows with base p_1xp_1', $T_2 = (G_2, p_2xp_2', L)$ is a bellows with base p_2xp_2' and $V(G_1) \cap V(G_2) = \{x\}$. Let L_1 be the colors of the democracy $\mathcal{C}$ and L_2 be the colors of the democracy $\mathcal{C}'$. Suppose that neither T_1 nor T_2 is a turbofan. Hence G is a wheel. Let $c \in L(x) \setminus L_1 \cup L_2$. Now let $L'(w) = L(w) \setminus \{c\}$ for all $w \in G \setminus (\{x\} \cup V(P) \cup V(P'))$, $L'(p_1) = L'(p_2) = L_1$ and $L'(p_1') = L'(p_2') = L_2$. Now $|L'(w)| \geq 2$ for all $w \in G \setminus \{x\}$. Hence by Theorem 1.4.3, either there exists an L'-coloring and the lemma follows or $L'(w) = L'(v)$ for all $w, v \in G \setminus \{x\}$ and G is an odd wheel. But then there exists a vertex $v \in G \setminus (\{x\} \cup V(P_1) \cup V(P_2))$. Yet $|L(v)| = 3$. So by repeating the argument above with $c \in L(x) \setminus L(v)$, it follows that there exist an L-coloring as desired. $\square$

Lemma 2.10.4. *Let $T = (G, P \cup P', L)$ be a double bellows, $\mathcal{C}$ be a dictatorship for $P = p_1p_2$ with dictator p_1 and $\mathcal{C}'$ be a democracy for $P' = p_1'p_2'$. Let $L'(p_2) = \mathcal{C}(p_2) \cup \mathcal{C}(p_1)$ and x be the center of the double bellows. If there do not exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G, then all of the following hold:*

(1) $\mathcal{C}(p_1) \cap \mathcal{C}'(p_1') = \emptyset$ *and* $\mathcal{C}(p_1), \mathcal{C}'(p_1') \subset L(x)$.

(2) p_1xp_1', p_1xp_2' *are the bases of exceptional odd fans of length at least three and there exists* $c \in \mathcal{C}'(p_1')$ *such that the non-extendable colorings are* $\phi(p_1) \in \mathcal{C}(p_1)$, $\phi(x) \in L(x) \setminus (\mathcal{C}(p_1) \cup \mathcal{C}'(p_1'))$ *and* $\phi(p_i') = c$ *for all* $i \in \{1,2\}$.

Proof. Let $T_1 = (G_1, p_1xp_1', L')$ be the bellows with base p_1xp_1', $T_2 = (G_2, p_1xp_2', L')$ the bellows with base p_1xp_2' where $L'(p_1) = \mathcal{C}(p_1)$, $L'(p_2) = \mathcal{C}(p_2) \cup \mathcal{C}(p_1)$, $L'(p_1') = \mathcal{C}'(p_1')$, $L'(p_2') = \mathcal{C}(p_2')$ and $L' = L$ otherwise. We may assume that $p_1 \not\sim p_1', p_2'$ as otherwise G is a wheel and yet $p_1 \cup P'$ either has two colorings or a coloring with only two colors; hence by Lemma 2.3.5, there exists an L'-coloring of G, a contradiction.

We claim that if T_1 is not an exceptional even fan then T_2 is an exceptional odd fan where the non-extendable colors of x are $L(x) \setminus (\mathcal{C}(p_1) \cup \mathcal{C}'(p_1')$. To see this, note that by Lemma 2.3.6, there exists $c \in L'(p_1')$ such that any L'-coloring ϕ of $G \setminus (G_1 \setminus \{p_1, x, p_1'\})$ with $\phi(p_1') = c$ can be extended to an L'-coloring of G. But there does not exist an L'-coloring of G. So color p_1' with c, then color p_2' from $L'(p_2) \setminus \{c\}$. By Lemma 2.3.8, this coloring extends to an L'-coloring of G_2 unless T_2 is an exceptional odd fan with non-extendable colors $L(x) \setminus (\mathcal{C}(p_1) \cup \mathcal{C}'(p_1')$. This proves the claim. In addition, as this is a set of size at most two, we find that $\mathcal{C}(p_1), \mathcal{C}'(p_1') \subset L(x)$ and $\mathcal{C}(p_1) \cap \mathcal{C}'(p_1') = \emptyset$ also follow from the claim.

Thus if T_1 is not an exceptional even fan, then T_2 is an exceptional odd fan by the claim. But then so is T_1 by the claim and the lemma follows. So we may suppose that T_1 is an exceptional even fan and by symmetry so is T_2. But then the non-extendable colorings of x to T_1 and to T_2 are $\mathcal{C}'(p_1')$. Hence color x from $L(x) \setminus (\mathcal{C}(p_1) \cup \mathcal{C}'(p_1'))$, then color p_1', p_2'. This extends to a L'-coloring of G as x was colored with an extendable color of T_1 and T_2, a contradiction. □

Lemma 2.10.5. *Let* $T = (G, P \cup P', L)$ *be a double bellows,* $\mathcal{C}$ *be a confederacy for* $P = p_1p_2$ *such that* $\mathcal{C}$ *is the union of two dictatorships and* $\mathcal{C}'$ *be a confederacy for*

$P' = p_1'p_2'$ such that $\mathcal{C}'$ is the union of two dictatorships. Suppose $|L(p_1')|, |L(p_2')| \geq 3$. If there do not exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G, then $\mathcal{C}, \mathcal{C}'$ are not alliances and $\Phi(P', \mathcal{C})$ contains an alliance.

Proof. Suppose not. Let x be the center of T. Let T_1 be the bellows with base p_1xp_1' and T_2 be the bellows with base p_1xp_2'. Let T_1' be the bellows with base p_2xp_1' and T_2' be the bellows with base p_2xp_2'. We may assume without loss of generality that p_1, p_1', p_2', p_2 appear in that order in the outer walk C of G. Let $\mathcal{C} \supseteq \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_1, \mathcal{C}_2$ are distinct dictatorships with distinct dictators if possible and let $\mathcal{C}' \supseteq \mathcal{C}_1' \cup \mathcal{C}_2'$ where $\mathcal{C}_1', \mathcal{C}_2'$ are distinct dictatorships with distinct dictators if possible.

Claim 2.10.6. *If $z_1 \in V(P)$ is the dictator of a dictatorship $\mathcal{C}_3 \subset \mathcal{C}$ and $z_2 \in V(P')$ is the dictator of a dictatorship $\mathcal{C}_4 \subset \mathcal{C}'$ such that $z_1 \not\sim z_2$, then one of the bellows with base z_1xz_2 is an exceptional odd fan. Furthermore, if only one of the bellows B_1 and B_2 with base z_1xz_2, say B_1, is an exceptional odd fan then $L(x)$ is the disjoint union of $\mathcal{C}_3(z_1)$, $\mathcal{C}_4(z_2)$, the two non-extendable colors of B_1 and the one non-extendable color of B_2.*

Proof. Let $B_1 \neq B_2$ be the two bellows with base z_1xz_2. We may suppose that neither B_1 nor B_2 is an exceptional odd fan. Hence by Lemma 2.3.6, for all $i \in \{1, 2\}$, there exists at most one color c_i so that coloring x with c_i does not extend to a coloring of B_i. Let $c_3 \in L(x) \setminus (\mathcal{C}_3(z_1) \cup \mathcal{C}_4(z_2) \cup \{c_1, c_2\})$. Now the coloring of z_1, x, z_2 with colors $\mathcal{C}_3(z_1), c_3, \mathcal{C}_4(z_2)$ respectively extends to colorings of B_1 and B_2, a contradiction.

Similarly suppose B_1 is an exceptional odd fan and B_2 is not an exceptional odd fan. It must be that $L(x) = \{c_1, c_2, c_3, c_4, c_5\}$ where c_1, c_2 are the non-extendable colors of x in B_1, $\mathcal{C}_3(z_1) = \{c_3\}$, $\mathcal{C}_4(z_2) = \{c_4\}$ and c_5 is the non-extendable color of B_2. □

Claim 2.10.7. *Suppose that p_1' is the dictator of $\mathcal{C}_1'$ and p_2' is the dictator of $\mathcal{C}_2'$. Further suppose that p_1 is the dictator of $\mathcal{C}_1$ and $p_1 \not\sim p_1'$.*

Let $L(x) = \{1, 2, 3, 4, 5\}$. Then all of the following hold up to permutation of the colors of $L(x)$:

(1) *T_1, T_2 are both exceptional even fans or both exceptional odd fans and $\mathcal{C}_1(p_1) = \{1\}$;*

(2) *if T_1, T_2 are both even, then $\mathcal{C}'(p_1') = \{2, 3\}$, $\mathcal{C}'(p_2') = \{4, 5\}$, the colors of the non-extendable democracy of xp_1' in T_1 are $2, 3$ and, for xp_2' in T_2 are $4, 5$;*

(3) *if T_1, T_2 are both odd, then $\mathcal{C}'(p_1') = \{2, 3\}$, $\mathcal{C}'(p_2') = \{4, 5\}$, $\mathcal{C}_1'(p_1') = \{2\}$, $\mathcal{C}_2'(p_2') = \{4\}$ the non-extendable dictatorship for T_1 has colors $3, 4$ for x and for T_2 has color $2, 5$ for x;*

(4) *p_2 is the dictator of $\mathcal{C}_2$.*

Proof. We may assume without loss of generality that $\mathcal{C}_1(p_1) = \{1\}$. By Claim 2.10.6, one of the bellows with base p_1xp_1' is odd and one of the bellows with base p_1xp_2' is odd. This implies that T_1, T_2 are either both even or both odd. This proves (1).

Suppose T_1, T_2 are both even. Thus $T_1 + p_2'$ is odd and $T_2 + p_1'$ is odd. Hence $T_1 + p_2'$ must be an odd exceptional fan by Claim 2.10.6 applied to $\mathcal{C}_1$ and $\mathcal{C}_2'$. Thus T_1, T_2 are exceptional even fans and thus their non-extendable colorings are democracies $\mathcal{C}_3, \mathcal{C}_4$ by Lemma 2.6.3. Hence $\mathcal{C}_2(p_1') = \mathcal{C}_3(p_1')$ has size two and $\mathcal{C}_1(p_2') = \mathcal{C}_4(p_1')$ has size two. Let $\mathcal{C}_2(p_1') = \{2, 3\}$. We may assume without loss of generality that $\mathcal{C}_2(p_2') = \{4\}$. But then $4, 5$ is a non-extendable coloring of T_2 and hence $\mathcal{C}_4(p_1') = \mathcal{C}_1(p_2') = \{4, 5\}$. It follows that $\mathcal{C}_1(p_1') \subset \{2, 3\}$. This proves (2).

Suppose T_1, T_2 are both odd. Hence T_1, T_2 are exceptional odd fans by Claim 2.10.6 applied to $\mathcal{C}_1$ and $\mathcal{C}_1'$. Thus their non-extendable colorings are dictatorships $\mathcal{C}_3, \mathcal{C}_4$ with dictators p_1', p_2' respectively by Lemma 2.6.3. We may assume without loss of generality that $\mathcal{C}_3(p_1') = \mathcal{C}_1(p_1') = \{2\}$ and $\mathcal{C}_3(x) = \{3, 4\}$. But then $5 \in \mathcal{C}_4(x)$ and $\mathcal{C}_1(p_2') = \{5\} \cup \mathcal{C}_2(p_2')$. Suppose without loss of generality that $\mathcal{C}_4(p_2') = \mathcal{C}_2(p_2') = \{4\}$.

But then $2 \in \mathcal{C}_4(x)$ as $2 \notin C_3(x)$. Thus $\mathcal{C}_4(p_2') = \{4,5\}$ and $\mathcal{C}_4(x) = \{2,5\}$. This proves (3).

Finally, we prove (4). Suppose that p_2 is not the dictator of $\mathcal{C}_2$. But then p_1 is the dictator of $\mathcal{C}_2$. It follows from (2) or (3) that $\mathcal{C}_2(p_1) = L(x)\setminus(\mathcal{C}'(p_1')\cup\mathcal{C}'(p_2')) = \mathcal{C}_1(p_1)$, a contradiction. □

Claim 2.10.8. *Suppose that p_1' is the dictator of $\mathcal{C}_1'$ and p_2' is the dictator of $\mathcal{C}_2'$. Then there exists $i,j \in \{1,2\}$ such that $p_i \sim p_i'$ and p_i is the dictator of $\mathcal{C}_j$.*

Proof. Suppose without loss of generality that p_1 is the dictator of $\mathcal{C}_1$. Suppose $p_1 \not\sim p_1'$. By Claim 2.10.7(1), T_1, T_2 are either both exceptional even fans or both exceptional odd fans. By Claim 2.10.7(5), p_2 is the dictator of $\mathcal{C}_2$. If $p_2 \sim p_2'$ the claim follows with $i = j = 2$. So we may suppose $p_2 \not\sim p_2'$. By Claim 2.10.7(1) T_1', T_2' are either both even or both odd. Yet T_1' and T_1 have different parity.

Without loss of generality we may suppose that T_1', T_2' are odd and T_1, T_2 even. By Claim 2.10.7(3) and (4), it follows that $|\mathcal{C}_2(p_1) \cap \mathcal{C}_2'(p_1')| = 1$ and $\mathcal{C}_1(p_2) = \mathcal{C}_1'(p_2')$ and $L(x) = \mathcal{C}_2(p_1) \cup \mathcal{C}_2'(p_1') \cup \mathcal{C}_1(p_2)$. Now color p_1, p_1' from $\mathcal{C}_2(p_1) \cap \mathcal{C}_2'(p_1')$, color $T_2' \setminus \{x\}$ from $\mathcal{C}_2(p_1)$. Color $T_1 \setminus \{x\}$ from $\mathcal{C}_2(p_1)$ and x from $\mathcal{C}_2'(p_1') \setminus \mathcal{C}_2(p_1)$. Hence there exists an L-coloring ϕ of G with $\phi \restriction P \in \mathcal{C}$ and $\phi \restriction P' \in \mathcal{C}'$, a contradiction. □

Claim 2.10.9. *$\mathcal{C}_1'$ and $\mathcal{C}_2'$ have the same dictator.*

Proof. Suppose not. We may assume without loss of generality that p_1' is the dictator for $\mathcal{C}_1'$ and p_2' is the dictator for $\mathcal{C}_2'$. By Claim 2.10.8, we may suppose without loss of generality that $p_2 \sim p_2'$ and p_2 is the dictator of $\mathcal{C}_2$. Suppose p_1 is the dictator of $\mathcal{C}_1$. Now T_1, T_2 are even. By Claim 2.10.7, we may suppose without loss of generality that $\mathcal{C}_1(p_1) = \{1\}$, $\mathcal{C}_1(p_2) = \{4,5\}$, $\mathcal{C}'(p_1') = \{2,3\}$ and $\mathcal{C}'(p_2') = \{4,5\}$. Symmetrically, we find that without loss of generality $\mathcal{C}'(p_2) = \{4,5\}$, $\mathcal{C}'(p_1) = \{1,2\}$ and $\mathcal{C}_1'(p_1') = \{3\}$.

Unfortunately, there does now not have to exist a coloring of G from extending colorings in $\mathcal{C}$ and $\mathcal{C}'$. But we have determined that $\mathcal{C}$ and $\mathcal{C}'$ are not alliances. Indeed, it is not hard to see that any other L-coloring of P' does extend to an L-coloring of G. That is $\Phi(P', \mathcal{C})$ is the set of all L-coloring of P' not in $\mathcal{C}'$. Thus we may assume that $L(p_1') = \{2, 3, x\}$ and $L(p_2') = \{4, 5, y\}$ as otherwise $\Phi(P', \mathcal{C})$ contains an alliance, a contradiction. Now $\Phi(P', \mathcal{C})$ contains a dictatorship with dictator p_1 in color x and a dictatorship with dictator p_2 in color y.

Suppose $y \neq 2$. Then $\Phi(P, \mathcal{C}')$ contains a dictatorship with dictator p_1 in color 2. Hence $\Phi(P', \mathcal{C})$ contains an alliance as there are two disjoint dictatorships with dictator p_1, one in color x and one in color 2, a contradiction. So we may suppose $y = 2$. Similarly suppose $x \neq 4$. Then $\Phi(P, \mathcal{C}')$ contains a dictatorship with dictator p_1 in color 2. Hence $\Phi(P', \mathcal{C})$ contains an alliance as there are two disjoint dictatorships with dictator p_2, one in color y and one in color 4, a contradiction. So we may suppose $x = 4$. But now $\Phi(P', \mathcal{C})$ contains the alliance of the second kind: $\{(4, 2), (4, 5), (3, 5), (2, 4)\}$, a contradiction.

So we may suppose that p_2 is the dictator of $\mathcal{C}_1$. But then there exists $i \in \{1, 2\}$ such $\mathcal{C}_i(p_2) \cap \mathcal{C}_2(p_2') = \emptyset$. But then there exists an L-coloring of G by Theorem 1.4.2, a contradiction. $\square$

So we may assume that $\mathcal{C}_1'$ and $\mathcal{C}_2'$ have the same dictator z. Suppose without loss of generality that p_1' is the dictator of $\mathcal{C}_1$ and $\mathcal{C}_2$. Note that in the proof of Claim 2.10.9, the symmetry of P and P' is only broken in the case that $\mathcal{C}_1$ and $\mathcal{C}_2$ do not have the same dictator. It follows then by symmetry that $\mathcal{C}_1$ and $\mathcal{C}_2$ have the same dictator z'.

Suppose $z \sim z'$. We may assume without loss of generality that $\mathcal{C}_1(z) \neq \mathcal{C}_2(z')$. But then there exists an L-coloring of G by Theorem 1.4.2, a contradiction.

So we may suppose that $z \not\sim z'$. Let T_3, T_4 be the distinct bellows with base zxz'. By Claim 2.10.6, at least one of T_3, T_4 is an odd exceptional fan. Suppose without loss of generality that T_3 is an odd exceptional fan. Let $x_1 = V(P) \setminus \{z\}$ and $x_2 = V(P') \setminus \{z'\}$. If T_4 is an even fan of length two, then at least one of x_1, x_2 is in T_3. Thus there exists $y \in \{x_1, x_2\}$ such that the bellows T_i containing y is not an even of length two, and hence contains a vertex $y' \neq z, z', y, x$ in its outer walk.

Suppose without loss of generality that $y = x_1$. Given y', it follows that $\mathcal{C}_1(y) \cap \mathcal{C}_2(y) \neq \emptyset$. Let $c \in \mathcal{C}_1(y) \cap \mathcal{C}_2(y)$. Now let $\mathcal{C}_3 = \{\phi_1, \phi_2\}$ where $\phi_1(y) = \phi_2(y) = c$ and $\phi_1(z) \in \mathcal{C}_1(z)$ and $\phi_2(z) \in \mathcal{C}_2(z)$. So $\mathcal{C}_3$ is a dictatorship with dictator y. But now $\mathcal{C}_1 \cup \mathcal{C}_3$ have distinct dictators and therefore contradict the choice of $\mathcal{C}_1, \mathcal{C}_2$. □

Lemma 2.10.10. *Let $T = (G, P \cup P', L)$ be a double bellows or a defective double bellows, $\mathcal{C}$ be a confederacy for P and $\mathcal{C}'$ be a confederacy for P'. If there do not exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G, then $\mathcal{C}, \mathcal{C}'$ are not alliances and $\Phi(P', C)$ contains an alliance.*

Proof. We may assume without loss of generality that T is a double bellows. Let x be the center of T. Let $P = p_1p_2$, $P' = p_1'p_2'$ such that $T_1 = (G_1, p_1xp_1', L)$ is a bellows with base p_1xp_1', $T_2 = (G_2, p_2xp_2', L)$ is a bellows with base p_2xp_2' and $V(G_1) \cap V(G_2) = \{x\}$. By Lemma 2.10.3, we may assume that $\mathcal{C}$ does not contain a democracy. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ where $\mathcal{C}_1, \mathcal{C}_2$ are dictatorships.

Suppose $\mathcal{C}'$ contains a democracy $\mathcal{C}''$. Suppose without loss of generality that p_1 is the dictator of $\mathcal{C}_1$. By Lemma 2.10.4 applied to $\mathcal{C}_1$ and $\mathcal{C}''$, we find that T_1 is an odd fan and T_2 is an even fan. Applying Lemma 2.10.4 then to $\mathcal{C}_2$ and $\mathcal{C}''$, we find that p_1 is also the dictator of $\mathcal{C}_2$. But then $\mathcal{C}_1(p_1) \cap \mathcal{C}_2(p_1) = \emptyset$. Yet the only non-extendable colorings ϕ of T_1 require $\phi(p_1) \in \mathcal{C}_1(p_1)$, a contradiction.

So we may assume that $\mathcal{C}'$ does not contain a democracy. Hence $\mathcal{C}$ is an alliance of the first kind. Let $\mathcal{C}' = \mathcal{C}_1' \cup \mathcal{C}_2'$ where $\mathcal{C}_1', \mathcal{C}_2'$ are dictatorships. But then by Lemma 2.10.5, $\Phi(P', \mathcal{C})$ contains an alliance as desired.

□

Lemma 2.10.11. *Let $T = (G, P \cup P', L)$ be a 2-connected instrumental orchestra with sides $P \cup P'$ where P, P' are paths of length one such that $d(P, P') \geq 3$. Let $\mathcal{C}$ be* *a confederacy for P and $\mathcal{C}'$ be a confederacy for P'. If there do not exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G, then T contains an accordion as a subcanvas.*

Proof. First suppose that one of the instruments of T is a double bellows or defective double bellows. Let $W = (G', P_1 \cup P_2, L)$ be such an instrument where P_1, P_2 are the sides of W. As $d(P, P') \geq 3$, we may assume without loss of generality that $P_2 \cap (P \cup P') = \emptyset$ and that P_2 separates P_1 from P'. By Theorem 2.7.2, $\Phi(P_1, \mathcal{C})$ contains a confederacy $\mathcal{C}_1$ and $\Phi(P_2, \mathcal{C}')$ contains a confederacy $\mathcal{C}_2$. If there exist $\phi \in \mathcal{C}_1$ and $\phi' \in \mathcal{C}_2$ such that $\phi \cup \phi'$ extends to an L-coloring of G', then there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}$ such that $\phi \cup \phi'$ extends to an L-coloring of G, a contradiction.

By Lemma 2.10.10 applied to W with $\mathcal{C}_1$ and $\mathcal{C}_2$, it follows that $\Phi(P_2, \mathcal{C}_1)$ contains an alliance $\mathcal{C}_1'$. Now let $W' \neq W$ be the other instrument of T with side P_2. Suppose $W' = (G'', P_2 \cup P_3, L)$ is a double bellows or defective double bellows where P_2, P_3 are the sides of W'. By Theorem 2.7.8, $\Phi(P_3, \mathcal{C}')$ contains a confederacy $\mathcal{C}_2'$. By Lemma 2.10.10 applied to W with $\mathcal{C}_1'$ and $\mathcal{C}_2'$, we find that there exist $\phi \in \mathcal{C}_1'$ and $\phi' \in \mathcal{C}_2'$ such that $\phi \cup \phi'$ extends to an L-coloring of G'', then there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}$ such that $\phi \cup \phi'$ extends to an L-coloring of G, a contradiction.

So we may suppose that W' is a bellows. By Theorem 2.7.2, $\Phi(P_3, \mathcal{C}')$ contains a confederacy $\mathcal{C}_2'$ such that $|\mathcal{C}_2'(z) \cup \mathcal{C}_1'(z)| \leq 3$ where $\{z\} = V(P_2) \cap V(P_3)$, since $z \notin V(P) \cup V(P')$. By Lemma 2.10.2 applied to W with $\mathcal{C}_1'$ and $\mathcal{C}_2'$, we find that there exist $\phi \in \mathcal{C}_1'$ and $\phi' \in \mathcal{C}_2'$ such that $\phi \cup \phi'$ extends to an L-coloring of G'', then there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}$ such that $\phi \cup \phi'$ extends to an L-coloring of G, a contradiction.

So we may assume that all the instruments in T are bellows. Let $\mathcal{C}$ be the outer walk of G. As T is not an accordion, there exists a vertex $v \in V(C) \setminus (V(P) \cup V(P'))$ such that $|L(v)| \geq 4$. Let $W = (G', P_1 \cup P_2, L)$ be a bellows of T such that P_1, P_2 are the sides of W and $v \in V(P_2) \setminus V(P_1)$. We may suppose without loss of generality that P_2 separates a vertex of P_1 from P'. By Theorem 2.7.2, $\Phi(P_1, \mathcal{C})$ contains a confederacy $\mathcal{C}_1$. By Lemma 2.10.1, $\Phi_W(P_2, \mathcal{C}_1)$ contains an alliance $\mathcal{C}_1'$.

Now let $W' \neq W$ be the other instrument of T with side P_2. Now $W' = (G'', P_2 \cup P_3, L)$ is a bellows where P_2, P_3 are the sides of W'. By Theorem 2.7.2, $\Phi(P_3, \mathcal{C}')$ contains a confederacy $\mathcal{C}_2'$ such that $|\mathcal{C}_2'(z) \cup \mathcal{C}_1'(z)| \leq 3$ where $\{z\} = V(P_2) \cap V(P_3)$, since $z \notin V(P) \cup V(P')$. By Lemma 2.10.2 applied to W with $\mathcal{C}_1'$ and $\mathcal{C}_2'$, we find that there exist $\phi \in \mathcal{C}_1'$ and $\phi' \in \mathcal{C}_2'$ such that $\phi \cup \phi'$ extends to an L-coloring of G'', then there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}$ such that $\phi \cup \phi'$ extends to an L-coloring of G, a contradiction. □

Definition (Bottleneck)**.** Let $T = (G, S, L)$ be a canvas and C be the outer walk of G. Suppose there exists chords U_1, U_2 of C with no end in S such that U_1 divides G into two graphs G_1, G_1' and U_2 divides G into G_2, G_2' where $G_1 \cap S = G_2 \cap S$. Let $G' = G \setminus (G_1 \setminus U_1) \setminus (G_2 \setminus U_2)$. If the canvas $T' = (G', U_1 \cup U_2, L)$ contains an accordion or a harmonica, call it T'', we say that T'' is a *bottleneck* of T.

Theorem 2.10.12 (Two Confederacies)**.** *Let $T = (G, P \cup P', L)$ be a canvas with P, P' distinct edges of C with $d(P, P') \geq 6$, $\mathcal{C}$ be a confederacy for P and $\mathcal{C}'$ be a confederacy for P'. If there do not exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G, then there exists a bottleneck $(G', U_1 \cup U_2, L)$ of T where $d_{G'}(U_1, U_2) \geq d_G(P, P')/2 - 3$.*

Proof. Let $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$, $\mathcal{C}' = \mathcal{C}_1' \cup \mathcal{C}_2'$ where $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_1', \mathcal{C}_2'$ are governments. We may assume that T is a counterexample with a minimum number of vertices. By Theorem 2.8.3 applied to T with $\mathcal{C}_1$ and $\mathcal{C}_1'$, there exists an orchestra $T' = (G', P_1 \cup P_2, L)$ with sides P_1', P_2' where G' is a subgraph of G, and $P_1 \subseteq P$, and $P_1 = P$ if $\mathcal{C}_1$ is a democracy, and similarly $P_2 \subseteq P'$, and $P_2 = P'$ is $\mathcal{C}_2$ is a democracy.

First suppose T' is a special orchestra. But then this implies that there exists a bottleneck $(G', U_1 \cup U_2, L)$ of T where $d_{G'}(U_1, U_2) \geq d_G(P, P')/2 - 3$ as desired.

So we may suppose that T' is an instrumental orchestra. As T is a minimum counterexample it follows that either T is a 2-connected instrumental orchestra or that there exists an essential cutvertex v of G. Suppose the former. By Lemma 2.10.11, T contains an accordion as desired. But then it follows that there exists a bottleneck $(G', U_1 \cup U_2, L)$ of T where $d_{G'}(U_1, U_2) \geq d_G(P, P') - 5 \geq d_G(P, P')/2 - 3$ as $d(P, P') \geq 4$.

So we may suppose there exists an essential cutvertex v of G. Thus v divides G into two graphs G_1, G_2 such that $P \cap (G_2 \setminus \{v\}) = \emptyset$ and $P' \cap (G_1 \setminus \{v\}) = \emptyset$. Let $L'(w) = \mathcal{C}(w)$ for $w \in V(P)$, $L'(w) = \mathcal{C}'(w)$ for $w \in V(P')$ and $L' = L$ otherwise. First suppose $v \in V(P) \cup V(P')$. Suppose then without loss of generality that $v \in P$. By Theorem 2.7.2 applied to $(G_2, P' + v, L')$, we find that there exists an L'-coloring of G_2, but this can be extended to an L'-coloring of G by Theorem 1.4.2, a contradiction.

So we may assume that $v \notin V(P) \cup V(P')$. But then by Theorem 2.7.2, there exists at most one color $c \in L(v)$ such that there does not exist a coloring ϕ of G_1 with $\phi \restriction \in \mathcal{C}$ and $\phi(v) = c$. Similarly there exists at most one color $c' \in L(v)$ such that there does not exist a coloring ϕ of G_1 with $\phi \restriction \in \mathcal{C}$ and $\phi(v) = c$. Yet $|L(v)| \geq 3$ and hence there exists $c'' \neq c, c'$ with $c'' \in L(v)$. But then there exist $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of G, a contradiction. □

Corollary 2.10.13. *Let $T = (G, P \cup P', L)$ be a critical orchestra with sides P, P' where P, P' are paths of length one. Let $\mathcal{C}_1$ be a government for P and $\mathcal{C}_2$ be government for P'. If there do not exist colorings $\phi_1 \in \mathcal{C}_1, \phi_2 \in \mathcal{C}_2$ such that $\phi_1 \cup \phi_2$ extends to an L-coloring of G, then there exist at most four vertices in $V(C) \setminus (V(P) \cup V(P'))$, where C is the outer walk of G, with lists of size at least four.*

Proof. Let C be the outer walk of G. It follows by definition that $|L(v)| < 5$ for all $v \in V(C) \setminus (V(P) \cup V(P'))$. Let $X = \{v \in V(C) \setminus (V(P) \cup V(P')) || L(v)| = 4$. Suppose to a contradiction that $|X| \geq 5$. If T is a special orchestra then $|X| = 0$ by definition. So we may suppose that T is instrumental. For all $x \in X$, it follows by definition that there exist two instruments W_1, W_2 such that $x \in W_1 \cap W_2$ and indeed that x is in a side of both W_1 and W_2.

First suppose there exists $x \in X$ such that x is a cutvertex of G. As T is critical, x is an essential curtvertex. Consider the canvases T_P from P to x and $T_{P'}$ from P' to x. By Theorem 2.7.8, there exists at least two colorings ϕ_1, ϕ_2 of T_P such that $\phi_1, \phi_2 \in \mathcal{C}$ and $\phi_1(x) \neq \phi_2(x)$. Furthermore by Theorem 2.7.8, there exists a third coloring ϕ_3 of T_P such that $\phi_3 \in \mathcal{C}$ and $\phi_3(x) \neq \phi_1(x), \phi_2(x)$ unless T_P contains a harmonica from P to u. Suppose ϕ_3 exists. Let $L'(x) = \{\phi_1(x), \phi_2(x), \phi_3(x)\}$ and $L' = L$ otherwise. By Theorem 1.4.2, there exists an L'-coloring ϕ of $T_{P'}$ with $\phi \restriction P' \in \mathcal{C}'$. But then ϕ extends to an L-coloring ϕ of T with $\phi \restriction P \in \mathcal{C}$, a contradiction.

So we may suppose that T_P contains a harmonica from P to u. As T is critical, it follows that T_P is a harmonica from P tu u. By symmetry $T_{P'}$ is a harmonica from P' to u. But then $X = \{x\}$ by the definition of harmonica, a contradiction. So we may suppose that no vertex in X is a cutvertex of G. Thus every vertex in X is in a chord of C.

Next we claim that that there exist two disjoint chords U_1, U_2 of C such that $U_1 \cap X, U_2 \cap X \neq \emptyset$. Suppose not. Thus all chords of C with an end in X must intersect. As all chords of C are essential, it follows from the planarity of G that these chords all have a common end u. But then there exist at least three distinct chords $U_1 = ux_1, U_2 = ux_2, U_3 = ux_3$ where $u \in U_1 \cap U_2 \cap U_3$ and $x_1, x_2, x_3 \in X$. We may suppose without loss of generality that U_1 separates x_2 from P and U_3 separates x_2 from P'. Hence $u_1x_2u_3$ is the base of a bellows containing x_2 as T is critical, a contradiction as $|L(x_2)| = 4$. This proves the claim.

Choose disjoint chords U_1, U_2 such that U_1 is closest to P and U_2 closest to P'. By Theorem 2.7.2, both $\Phi(U_1, \mathcal{C}_1), \Phi(U_2, \mathcal{C}_2)$ contain a confederacy. Given how U_1, U_2 were chosen and as $|X| \geq 5$, there exists a vertex $x \in X \setminus (U_1 \cup U_2)$ such that x is in the instrumental orchestra T' between U_1 and U_2. It follows from the proof of Lemma 2.10.11, that $d(U_1, U_2) \leq 2$ and that T' contains a double bellows W with U_3, U_4 such that $U_3 \cap U_1 \neq \emptyset$ and $U_4 \cap U_2 \neq \emptyset$. But then $U_1 \cup U_3$ and $U_2 \cup U_4$ are the bases of bellows. It follows then as $|L(x)| = 4$, that $x \in U_3 \setminus U_1$ or $x \in U_4 \setminus U_2$.

Suppose without loss of generality that $x \in U_3 \setminus U_1$. By Lemma 2.10.1, $\Phi(U_3, \mathcal{C}_1)$ contains an alliance. Meanwhile $\Phi(U_4, \mathcal{C}_2)$ contains a confederacy. By Lemma 2.10.11 applied to W, there exist $\phi_1 \in \Phi(U_3, \mathcal{C}_1)$, $\phi_2 \in \Phi(U_4, \mathcal{C}_2)$ such that $\phi_1 \cup \phi_2$ extends to an L-coloring of W, a contradiction. □

2.11 Bottlenecks

We conclude this chapter by proving the most substantial theorem which shows that in a canvas, if the coloring of two edges far apart does not extend to a coloring of the whole graph, then there exists a proportionally long bottleneck. We will generalize this theorem to longer paths as well as collection of more than two paths in Chapter 3. We will also use this theorem as the basis of the proofs in Chapter 4.

Theorem 2.11.1 (Bottleneck Theorem: Two Edges). *If $T = (G, P \cup P', L)$ is a canvas with P, P' distinct edges of C with $d(P, P') \geq 14$, then either there exists an L-coloring of G, or there exists a bottleneck $(G', U_1 \cup U_2, L)$ of T where $d_{G'}(U_1, U_2) \geq d_G(P, P')/6 - 22$.*

Proof. Suppose there does not exist an L-coloring of G. Let $d = (P, P')$. By Theorem 2.9.11, there exists an essential chord U_0 of C such that $d(U_0, P) \leq 13$. Now we may assume that $d \geq 132$ as otherwise U_0 is the desired bottleneck.

Similarly, there exists an essential chord U_6 of C such that $d(U_6, P) \geq d - 14$. We claim that there exists essential chords U_1, U_2, U_3, U_4, U_5 of C such that $id/6 - 7 \leq d(U_i, P) \leq id/6 + 7$. Suppose not. Let W_i be the essential chord of C with $d(W_i, P) < id/6 - 7$ and there does not exist another such chord separating a vertex of W_i from P'. Similarly let W_i' be the essential chord of C with $d(W_i', P) > id/6 + 7$ and there does not exist another such chord separating a vertex of W_i from P'. But then $d(W_i, W_i') \geq 14$. By Theorem 2.9.11, there exists an essential chord U_i of C separating a vertex of W_i from a vertex of W_i'. Given how W_i was chosen, it follows that $d(U_i, P) \geq id/6 - 7$. Similarly given how W'_i was chosen, it follows that $d(U_i, P) \leq id/6 + 7$. This proves the claim. Note that as $d/6 - 22 \geq 0$ as $d \geq 132$, it follows that all of the chords $\{U_i | 0 \leq i \leq 6\}$ are disjoint.

Let $\mathcal{C}_0 = \{\phi\}$ where ϕ is an L-coloring of P and $\mathcal{C}_6 = \{\phi'\}$ where ϕ' is an L-coloring of P'. By Theorem 2.6.5, $\Phi(U_1, \mathcal{C}_0)$ has a government or $T[P, U_1]$ is an accordion but then $T[U_0, U_1]$ is a bottleneck with $d(U_0, U_1) \geq d/6 - 21$. So we may suppose that $\Phi(U_1, \mathcal{C}_0)$ has a government $\mathcal{C}_1$. Similarly, if $\Phi(U_5, \mathcal{C}_6)$ does not have a government, then $T[U_5, P']$ is an accordion but then $T[U_5, U_6]$ is a bottleneck with $d(U_5, U_6) \geq d/6 - 22$. So we may suppose that $\Phi(U_5, \mathcal{C}_6)$ has a government $\mathcal{C}_5$.

By Theorem 2.7.2, $\Phi(U_2, \mathcal{C}_1)$ contains a confederacy unless there exists a harmonica from U_1 to U_2. But then $T[U_1, U_2]$ is a bottleneck with $d(U_1, U_2) \geq d/6 - 15$ as desired. So we may suppose that $\Phi(U_2, \mathcal{C}_1)$ contains a confederacy $\mathcal{C}_2$. Similarly we may suppose that $\Phi(U_4, \mathcal{C}_5)$ contains a confederacy $\mathcal{C}_4$.

Now $d(U_2, U_4) \geq d/3 - 15$. As $d/3 - 15 \geq 6$ since $d \geq 132$, it follows from Theorem 2.10.12 that $T[U_2, U_4]$ contains a bottleneck $T[U'_2, U'_4]$ with $d(U'_2, U'_4) \geq (d/3 - 15)/2 - 3 \geq d/6 - 22$ as desired. $\square$

CHAPTER III

LINEAR BOUND FOR ONE CYCLE

3.1 Introduction

In this chapter, we prove the following theorem which settles a conjecture of Dvorak et al [27].

Theorem 3.1.1. *Let G be a 2-connected plane graph with outer cycle C and L a 5-list-assignment for G. Then G contains a connected subgraph H with at most $29|C|$ vertices such that for every L-coloring ϕ of C either*

(i) ϕ cannot be extended to an L-coloring of H, or,

(ii) ϕ can be extended to an L-coloring of G.

Indeed, it will be necessary to prove a stronger version of Theorem 1.8.1 which bounds the the number of vertices in terms of the sum of the sizes of large faces. Another clever aspect to the proof is to incorporate the counting of neighbors of C into the stronger formula. This allows the finding of reducible configurations close to the boundary in a manner similar to the discharging method's use of Euler's formula.

In Section 3.2, we define a more general notion of criticality for graphs and canvases which will be useful for proving Theorem 1.8.1. In Section 3.3, we prove a structure theorem for said critical cycle-canvases. In Section 3.4, we prove Theorem 1.8.1.

In addition in this chapter we will prove a number of generalizations of Theorem 1.8.1. In Section 3.4, we prove that there exists a graph as in Theorem 1.8.1 with the stronger property that for every f of H every L-coloring of the boundary of f extends to the subgraph of G contained in the interior of f. In Section 3.5, we prove that if the constant in Theorem 1.8.1 is modified, then outcome (ii) can be upgraded to say that there exist $2^{c|G\setminus C|}$ extensions of ϕ for some constant c.

In Section 3.6, we show that such a linear bound implies that every vertex in H of Theorem 1.8.1 has at most logarithmic distance from C. This idea will be very crucial to the proofs in Chapter 5 and is a main reason why linear bounds are so fruitful. Logarithmic distance also implies that vertices in H exhibit exponential growth, that is, the size of the ball around a vertex grows exponentially with the radius of the ball.

In Section 3.7, 3,8 and 3.9, we extend Theorem 1.8.1, which is actually about cycle-canvases, to the more general case of path-canvases. We prove a structure theorem for critical path-canvases in Section 3.7, a linear bound in Section 3.8, and logarithmic distance and exponential growth in Section 3.9.

In Sections 3.10, 3.11 and 3.12, we extend Theorem 2.11.1 to given a canvas with any number of precolored paths of any length, then there exists a long bottleneck or the size of the canvas is linear in the number of precolored vertices.

3.2 Critical Subgraphs

Definition (T-critical)**.** Let G be a graph, $T \subseteq G$ a (not necessarily induced) subgraph of G and L a list assignment to the vertices of $V(G)$. For an L-coloring ϕ of T, we say that ϕ *extends to an L-coloring of* G if there exists an L-coloring ψ of G such that $\phi(v) = \psi(v)$ for all $v \in V(T)$. The graph G is *T-critical with respect to the list assignment* L if $G \neq T$ and for every proper subgraph $G' \subset G$ such that $T \subseteq G'$, there exists a coloring of T that extends to an L-coloring of G', but does not extend to an L-coloring of G. If the list assignment is clear from the context, we shorten this and say that G is T-critical.

We say a canvas (G, S, L) is *critical* if G is S-critical with respect to the list assignment L.

Definition. Let G be a graph and $T \subset V(G)$. For $S \subseteq G$, a graph $G' \subseteq G$ is an

S-component with respect to T of G if S is a proper subgraph of G', $T \cap G' \subseteq S$ and all edges of G incident with vertices of $V(G') \setminus V(S)$ belong to G'.

For example, if G is a plane graph with T contained in the boundary of its outer face and S is a cycle in G, then the subgraph of G drawn inside the closed disk bounded by S, which we denote by $Int_S(G)$, is an S-component of G with respect to T. Here is a useful lemma.

Lemma 3.2.1. *Let G be a T-critical graph with respect to a list assignment L. Let G' be an S-component of G with respect to T, for some $S \subseteq G$. Then G' is S-critical.*

Proof. Since G is T-critical, every isolated vertex of G belongs to T, and thus every isolated vertex of G' belongs to S. Suppose for a contradiction that G' is not S-critical. Then, there exists an edge $e \in E(G') \cap E(S)$ such that every L-coloring of S that extends to $G' \setminus e$ also extends to G'. Note that $e \notin E(T)$. Since G is T-critical, there exists a coloring Φ of T that extends to an L-coloring ϕ of $G \setminus e$, but does not extend to an L-coloring of G. However, by the choice of e, the restriction of ϕ to S extends to an L-coloring ϕ' of G'. Let ϕ'' be the coloring that matches ϕ' on $V(G')$ and ϕ on $V(G) \cap V(G')$. Observe that ϕ'' is an L-coloring of G extending Φ, which is a contradiction. $\square$

Lemma 3.2.1 has two useful corollaries. To state them, however, we need the following definitions.

Definition. Let $T = (G, S, L)$ be a canvas and $S' \subset V(G)$. If G' is a S'-component with respect to S, then we let $T[G', S']$ denote the canvas (G', S', L).

Definition. Let $T = (G, P, L)$ be a path-canvas and C be the outer walk of G. We say a path P' in G is a *span* if the ends of P' have lists of size less than five and the only internal vertices of P' with lists of size less than five are in P. Let $\delta(P')$ be the path from the ends of P' in C that does not traverse a vertex of $P \setminus P'$. We define the exterior of P', denoted by $Ext(P')$ as the set of vertices in $\delta(P') \cup Int(P' \cup \delta(P'))$.

Definition. Let $T = (G, S, L)$ be a cycle-canvas or path-canvas. If C' is a cycle in G, we let $T[C']$ denote the cycle-canvas $(Int(C'), C', L)$. If T is a path-canvas and P' is a span of G, then we let $T[P']$ denote the path-canvas $(P' \cup Ext(P'), P' \cup \delta(P'), P', L)$.

Corollary 3.2.2. *Let $T = (G, S, L)$ be a critical canvas. If C' is a cycle in G such that $Int(C') \neq C'$, then $T[C']$ is a critical cycle-canvas.*

Proof. $G' = Int(C')$ is a C'-component of G. As G is S-critical, G' is C'-critical. □

Corollary 3.2.3. *Let (G, P, L) be a critical path-canvas. If P is a span of T, then $T[P]$ is a critical path-canvas.*

Proof. $G' = P' \cup Ext(P')$ is a P'-component of G. As G is S-critical, G' is P'-critical.
□

Another useful fact is the following.

Proposition 3.2.4. *Let $T = (G, S, L)$ be a canvas such that there exists a proper L-coloring of S that does not extend to G. Then there exists a S-critical subgraph G' of G such that $S \subseteq G'$.*

Definition. Let $T = (G, S, L)$ be a canvas and $G' \subseteq G$ such that $S \subseteq G'$ and G' is connected. We define the *subcanvas* of T induced by G' to be (G', S, L).

Thus in Proposition 3.2.4, the subcanvas of T induced by G' is critical.

Corollary 3.2.5. *Let $T = (G, S, L)$ be a canvas such that there exists a proper L-coloring of S that does not extend to G. Then T contains a critical subcanvas.*

3.3 Critical Cycle-Canvases

The following theorem is an easy consequence of Theorem 1.4.2.

Theorem 3.3.1. *(Cycle Chord or Tripod Theorem)*

If $T = (G, C, L)$ is critical cycle-canvas, then either

(1) *C has a chord in G, or*

(2) *there exists a vertex of G with at least three neighbors on C, and at most one of the internal faces of $G[v \cup V(C)]$ is nonempty.*

Proof. Suppose C does not have a chord. Let X be the set of vertices with at least three neighbors on C. Let $V(G') = C \cup X$ and $E(G') = E(G[C \cup X]) - E(G[X])$.

We claim that if f is face of G' such that f is incident with at most one vertex of f, then f does not include a vertex or edge of G. Suppose not. Let C' be the boundary of f. As C has no chords and every edge with one end in X and the other in C is in $E(G')$, it follows that C' has no chords. As T is critical, there exists an L-coloring ϕ of $G \setminus Int(C')$ which does not extend to G. Hence, there exists an L-coloring ϕ of C' which does not extend to $Int(C)$. Let $G' = Int(C) \cup (C' \setminus C)$, $S' = C' \setminus C$, $L'(v) = \phi(v)$ for $v \in S$ and $L'(v) = L(v) \setminus \{\phi(x) : x \in C \cap N(v)\}$. Consider the canvas $T' = (G', S', L')$. By Theorem 1.4.2, there exists an L'-coloring of T' and hence an L-coloring of G which extends ϕ, a contradiction. This proves the claim.

As T is critical, $G \neq C$. As C has no chords, it follows from the claim above that $X \neq \emptyset$.Let $\mathcal{F}$ be the internal faces of G' incident with at least two elements of X. Consider the tree whose vertices are $X \cup \mathcal{F}$ where a vertex $x \in X$ is adjacent to $f \in \mathcal{F}$ is x is incident with f. Let v be a leaf of T. By construction, $v \in X$. Hence at most one of the internal faces of $G[v \cup V(C)]$ is incident with another vertex of X. Yet all other faces of $G[v \cup V(C)]$ are incident with only one element of X, namely v, and so by the claim above, these faces are empty as desired. □

3.3.1 Deficiency

Definition. If G is a plane, we let $\mathcal{F}(G)$ denote the set of finite faces of G. We define the deficiency of a cycle-canvas $T = (G, C, L)$ as

$$\mathrm{def}(T) = |V(C)| - 3 - \sum_{f \in \mathcal{F}(T)} (|f| - 3)$$

.

Definition. If f is a face of a graph G, let δf denote the facial walk of f and $G[f]$ denote the $Int(\delta f)$. If $T = (G, S, L)$ is a canvas and f is a face of G, let $T[f]$ denote the canvas $T[\delta f]$, that is, $(G[f], \delta f, L)$.

Lemma 3.3.2. *If T is a cycle-canvas and $T[G']$ is a subcanvas such that G' is 2-connected, then*

$$\mathrm{def}(T) = \mathrm{def}(T[G']) + \sum_{f \in \mathcal{F}(G')} \mathrm{def}(T[f])$$

.

Proof.

$$\mathrm{def}(T[G']) + \sum_{f \in \mathcal{F}(G')} \mathrm{def}(T[f]) = |V(C)| - 3 - \sum_{f \in \mathcal{F}(G')} (|f| - 3) + \sum_{f \in \mathcal{F}(G')} (|f| - 3) - \sum_{f' \in \mathcal{F}(G[f])} (|f'| - 3)$$

.

Every face of G is a face of exactly one $T[f]$, $\sum_{f \in \mathcal{F}(G')} \sum_{f' \in \mathcal{F}(G[f])} (|f'| - 3) = \sum_{f \in \mathcal{F}(G)} (|f| - 3)$. Hence,

$$|V(C)| - 3 - \sum_{f \in \mathcal{F}(G')} (|f| - 3) + \sum_{f \in \mathcal{F}(G')} (|f| - 3) - \sum_{f \in \mathcal{F}(G)} (|f| - 3).$$

As the middle terms cancel, this is just $|V(C)| - 3 - \sum_{f \in \mathcal{F}(G)} (|f| - 3) = \mathrm{def}(T)$ as desired. □

Theorem 3.3.3. *(Cycle Sum of Faces Theorem)*

If $T = (G, C, L)$ is a critical cycle-canvas, then $\mathrm{def}(T) \geq 1$.

Proof. We proceed by induction on the number of vertices of G. Note that if T is a cycle-canvas and $G = C$, then $\mathrm{def}(C) = 0$. Apply Theorem 3.3.1.

Suppose (1) holds; that is there is a chord U of C. Let C_1, C_2 be cycles such that $C_1 \cap C_2 = U$ and $C_1 \cup C_2 = C \cup U$. Hence $|V(C_1)| + |V(C_2)| = |V(C)| + 2$. Let $T_1 = T[C_1] = (G_1, C_1, L)$ and $T_2 = T[C_2] = (G_2, C_2, L)$. If $f \in \mathcal{F}(G)$, then $f \in \mathcal{F}(G_1) \cup \mathcal{F}(G_2)$. Thus by Lemma 3.3.2, $\text{def}(T) = \text{def}(T_1) + \text{def}(T_2) + 1$.

By Theorem 3.2.2, T_1 and T_2 are critical cycle-canvases (or empty). By induction, $\text{def}(T_i) \geq 1$ if $Int(C_i) \neq C_i$. As noted before, $\text{def}(T_i) = 0$ if $Int(C_i) = \emptyset$. In either case, $\text{def}(T_i) \geq 0$. Thus $\text{def}(T) \geq 0 + 0 + 1 \geq 1$ as desired.

So we may suppose that (2) holds; that is, there exists $v \notin V(C)$ such that v is adjacent to at least three vertices of C and at most one of the faces of $C \cup v$ is nonempty. First suppose that all the faces are empty, that is to say that $V(G) = V(C) \cup v$. Now v must have degree at least 5 as G is C-critical. Thus, $\text{def}(T) \geq 2$ as desired.

Let $G' = C \cup v$. So we may suppose that only one of the faces of $\mathcal{F}(G)$ is nonempty. Let C' be the boundary of the non-empty face. Now, $|V(C)| - |V(C')| \geq \sum f \in \mathcal{F}(G) \setminus \mathcal{F}(G')(|f| - 3)$. That is, $\text{def}(T) \geq \text{def}(T[C'])$. Yet by induction, $\text{def}(T[C']) \geq 1$ as $Int(C') \neq C'$. Thus $\text{def}(T) \geq 1$ as desired. □

Corollary 3.3.4. *(Cycle Bounded Face Theorem)*

Let (G, C, L) be a critical cycle-canvas. If f is an internal face of G, then $|f| < |V(C)|$.

Proof. By Theorem 3.3.3, $|V(C)| - 3 - \sum_{f \in \mathcal{F}(G)} (|f| - 3) \geq 1$. Thus $|V(C)| - 4 \geq \sum_{f \in \mathcal{F}(G)} (|f| - 3)$. As the terms on the right side are always positive, $|V(C)| - 4 \geq |f| - 3$ for any internal face f of G. Thus $|f| < |V(C)|$. □

3.4 Linear Bound for Cycles

To prove the linear bound for cycles, we shall prove a stronger statement instead. First a few definitions.

Definition. Let $T = (G, C, L)$ be a cycle-canvas. We define $v(T) = |V(G \setminus C)|$. We also define the *quasi-boundary* of T, denoted by $Q(T)$, as $\{v \notin V(C) : \exists f \in \mathcal{F}(G), v \in \delta f, \delta f \cap V(C) \neq \emptyset\}$. We let $q(T) = |Q(T)|$.

Fix $\epsilon, \alpha > 0$. Let $s(T) = \epsilon v(T) + \alpha q(T)$, $d(T) = def(T) - s(T)$.

Proposition 3.4.1. *Let T be a cycle-canvas and $T' = (G', S, L)$ be a subcanvas. The following hold:*

- $v(T) = v(T') + \sum_{f \in \mathcal{F}(G')} v(T[f])$,
- $q(T) \leq q(T') + \sum_{f \in \mathcal{F}(G')} q(T[f])$,
- $s(T) \leq s(T') + \sum_{f \in \mathcal{F}(G')} s(T[f])$,
- *If G' is 2-connected, then* $d(T) \geq d(T') + \sum_{f \in \mathcal{F}(G')} d(T[f])$.

Proof. The first follows as every vertex of $V(G \setminus C)$ is in exactly one of $V(G' \setminus C), \{V(T[f] \setminus \delta f) : f \in \mathcal{F}(G')\}$ and every vertex in one of those sets is in $V(G \setminus C)$.

The second follows from the claim that $Q(T) \subseteq Q(T[G']) \cup \bigcup_{f \in \mathcal{F}(G')} Q(T[f])$. To see this claim, suppose that $v \in Q(T)$. Now $v \in Q(T)$ if and only if there exists a path from v to a vertex u in C which is internally disjoint from G. If $v \in G'$, then P does not cross G' and yet $u \in V(C')$; hence, $v \in Q(T[G'])$. So we may assume that $v \in \delta f$ for some $f \in \mathcal{F}(G')$. Yet, it must be that $u \in \delta f$ and that P does not cross the graph $\delta f \cup Int(\delta f)$; hence, $v \in Q(T[f])$.

The third follows from the first two. The fourth follows from the third and Lemma 3.3.2. □

Corollary 3.4.2. *Let $T = (G, C, L)$ be a cycle canvas. If U is a chord of C and C_1, C_2 cycles such that $C_1 \cap C_2 = U$ and $C_1 \cup C_2 = C + U$, then*

$$d(T) \geq d(T[C_1]) + d(T[C_2]) + 1.$$

If v is a vertex with two neighbors $u_1, u_2 \in V(C)$ and C_1, C_2 cycles such that $C_1 \cap C_2 = u_1 v u_2$ and $C_1 \cup C_2 = C + u_1 v u_2$, then

$$d(T) \geq d(T[C_1]) = d(T[C_2]) - 1 - (\alpha + \epsilon).$$

Proposition 3.4.3. *Let $T = (G, C, L)$ be a 2-connected cycle-canvas.*

(i) If $G = C$, then $d(T) = 0$.

(ii) If $v(T) = 0$, then $d(T) = |E(G) \setminus E(C)|$.

(iii) If $v(T) = 1$, then $d(T) = |E(G) \setminus E(C)| - 3 - (\alpha + \epsilon)$.

Proof. (i) If $G = C$, then $v(T) = q(T) = s(T) = 0$. As $\text{def}(T) = 0$, $d(T) = 0 - 0 = 0$.

(ii) If $v(T) = 0$, then $q(T) = 0$. Thus $s(T) = 0$. As $v(T) = 0$, $\text{def}(T) = |E(G) \setminus E(C)|$ by Lemma 3.3.4. So $d(T) = |E(G) \setminus E(C)|$ as desired.

(iii) If $v(T) = 1$, then $q(T) = 1$. Thus $s(T) = \alpha + \epsilon$. Let $v \in V(G) \setminus V(C)$. As G is 2-connected, $deg(v) \geq 2$. Thus $\text{def}(T) = |E(G) \setminus E(C)| - 3$. Combining, $d(T) = |E(G) \setminus E(C)| - 3 - (\alpha + \epsilon)$ as desired. □

Corollary 3.4.4. *Let $T = (G, C, L)$ be a 2-connected cycle-canvas. If $v(T) \leq 1$, then $d(T) \geq 3 - (\alpha + \epsilon)$ unless*

(i) $v(T) = 0$ and $|E(G) \setminus E(C)| \leq 2$, or

(ii) $v(T) = 1$ and $|E(G) \setminus E(C)| \leq 5$.

We are now ready to state our generalization of the linear bound for cycles.

Theorem 3.4.5. *Let $\epsilon, \alpha, \gamma > 0$ satisfying the following:*

(1) $\epsilon \leq \alpha$,

(2) $8(\alpha + \epsilon) \leq \gamma$,

(3) $\gamma \leq 1/2 + (\alpha + \epsilon)$.

If $T = (G, C, L)$ is a critical cycle-canvas and $v(T) \geq 2$, then $d(T) \geq 3 - \gamma$.

Proof. Let $T = (G, C, L)$ be a counterexample such that $|E(G)|$ is minimized.

Let us note that as G is C-critical there does not exist a cutvertex or a separating edge, triangle, or 4-cycle in G. Furthermore, $deg(v) \geq 5$ for all $v \in V(G) \setminus V(C)$.

Claim 3.4.6. $v(T) \geq 4$

Proof. Suppose not. Suppose $v(T) = 2$. But then $|E(G) \setminus E(C)| \geq 9$. Hence, $\text{def}(T) \geq 3$ while $s(T) = 2(\alpha + \epsilon)$. Hence $d(T) \geq 3 - 2(\alpha + \epsilon)$ which is at least $3 - \gamma$ by inequality (2), contrary to the fact that T is a counterexample. So we may suppose that $v(T) = 3$. But then $|E(G) \setminus E(C)| \geq 12$. Hence, $\text{def}(T) \geq 3$ while $s(T) = 3(\alpha + \epsilon)$. Hence $d(T) \geq 3 - 3(\alpha + \epsilon)$ which is at least $3 - \gamma$ by inequality (2), contrary to the fact that T is a counterexample. □

3.4.1 Proper Critical Subgraphs

Here is a remarkably useful lemma.

Claim 3.4.7. *Suppose $T_0 = (G_0, C_0, L_0)$ is a critical cycle-canvas with $|E(G_0)| \leq |E(G)|$ and $v(T_0) \geq 2$. If G_0 contains a proper C_0-critical subgraph G', then $d(T_0) \geq 4 - \gamma$. Furthermore, if $|E(G_0) \setminus E(G')|, |E(G') \setminus E(C)| \geq 2$, then $d(T_0) \geq 4 - 2(\alpha + \epsilon)$.*

Proof. Given Proposition 3.4.3 and the fact that T is a minimum counterexample, it follows that $d(T_0[f]) \geq 0$ for all $f \in \mathcal{F}(G')$. Moreover as G' is a proper subgraph, there exists at least one f such that $Int(f) \neq \emptyset$. For such an f, $d(T[f]) \geq 1$. Furthermore, if $|E(G_0) \setminus E(G')| \geq 2$, either there exist two such f's or $d(T[f]) \geq 2 - (\alpha + \epsilon)$.

Now $d(T_0) \geq d(T[G']) + \sum_{f \in \mathcal{F}(G')} d(T_0[f])$ by Proposition 3.4.1. As noted above though, $\sum_{f \in \mathcal{F}(G')} d(T_0[f]) \geq 1$ and is at least $2 - (\alpha + \epsilon)$ if $|E(G_0) \setminus E(G')| \geq 2$.

So suppose $v(T_0[G']) > 1$. Then $d(T_0[G']) \geq 3 - \gamma$ as T is a minimum counterexample. Hence $d(T_0) \geq 4 - \gamma$ if $|E(G_0) \setminus E(G')| = 1$ and $d(T_0) \geq 5 - (\alpha + \epsilon) - \gamma$, which is at least $4 - 2(\alpha + \epsilon)$ by inequality (2), as desired.

So we may assume that $v(T_0[G']) \leq 1$. Suppose $v(T_0(G')) = 1$. Then there exists $f \in \mathcal{F}(G')$ such that $v(T_0[f]) \geq 1$. If $v(T_0[f]) \geq 2$, then $d(T_0[f]) \geq 3 - \gamma$ as T is a minimum counterexample. If $v(T_0[f]) = 1$, then $d(T_0[f]) \geq 2 - (\alpha + \epsilon)$ by Proposition 3.4.3. In either case, $d(T_0[f]) \geq 2 - (\alpha + \epsilon)$. As above, $d(T_0) \geq d(T_0[G']) + d(T[f]) \geq 2(2 - (\alpha + \epsilon)) = 4 - 2(\alpha + \epsilon)$, which is at least $4 - \gamma$ by inequality (2), and the lemma follows as desired.

So suppose $v(T_0[G']) = 0$. As $G' \neq C'$, $d(T_0[G']) \geq |E(G') \setminus E(C)|$ by Proposition 3.4.3. As $v(T_0) \geq 2$, either there exists $f \in \mathcal{F}(G')$ such that $v(T[f]) \geq 2$ or there exists $f_1, f_2 \in \mathcal{F}(G')$ such that $v(T_0[f_1]), v(T_0[f_1]) \geq 1$. Suppose the first case. Then $d(T_0[f]) \geq 3 - \gamma$ as T is a minimum counterexample. Hence $d(T_0) \geq |E(G') \setminus E(C)| + 3 - \gamma$ as desired. Thus if $|E(G') \setminus E(C)| = 1$, then $d(T_0) \geq 4 - \gamma$ as desired, and if $|E(G') \setminus E(C)| \geq 2$, then $d(T_0) \geq 5 - \gamma$ which is at least $4 - 2(\alpha + \epsilon)$ by inequality (3). So suppose the latter. Then $d(T_0[f_1]), d(T_0[f_2]) \geq 2 - \epsilon - \alpha$ and $d(T_0) \geq 1 + 2(2 - \epsilon) = 5 - 2(\epsilon + \alpha)$, as desired.

□

Claim 3.4.8. *There does not exist a proper C-critical subgraph G' of G.*

Proof. Follows from Claim 3.4.7. □

This implies that we may assume that C is precolored as follows. There exists a proper coloring ϕ of C that does not extend to G as G is C-critical. However, ϕ must extend to every proper subgraph H of G, as otherwise H contains a critical subgraph and hence G contains a proper critical subgraph contradicting Claim 3.4.8. In this sense, G is critical for every coloring of C that does not extend to G. For the rest of the proof, we fix a coloring ϕ of C which does not extend to G.

For $v \notin V(C)$, we let $S(v) = L(v) \setminus \{\phi(u) | u \in N(v) \cap V(C)\}$.

Claim 3.4.9. *There does not exist a chord of C.*

Proof. Suppose there exists a chord e of C. Let $G' = C \cup e$. As $v(T) \neq 0$, G' is a proper subgraph of G. Yet G' is critical, contradicting Claim 3.4.8. $\square$

3.4.2 Dividing Vertices

Definition. Suppose $T_0 = (G_0, C_0, L_0)$ is a cycle-canvas. Let $v \notin V(C_0)$ be a vertex and suppose there exist two distinct faces $f_1, f_2 \in \mathcal{F}(G_0)$ such that $v \in \delta f_i$ and $\delta f_i \cap V(C_0) \neq \emptyset$ for $i \in \{1, 2\}$. Let $u_i \in \delta f_i \cap V(C_0)$. Consider cycles C_1, C_2 where $C_1 \cap C_2 = u_1 v u_2$ and $C_1 \cup C_2 = C_0 \cup u_1 v u_2$; note that we might have added the edges $u_1 v, u_2 v$. If for all $i \in \{1, 2\}$, $|E(T[C_i]) \setminus E(C_i)| \geq 2$, then we say that v is a *dividing* vertex. If for all $i \in \{1, 2\}$, if $v(T[C_i]) \geq 1$, we say v is a *strong dividing* vertex. If v is a dividing vertex and the edges $u_1 v, u_2 v$ are in G, then we say that v is *true dividing* vertex.

Claim 3.4.10. *Suppose $T_0 = (G_0, C_0, L_0)$ is a critical cycle-canvas with $|E(G_0)| \leq |E(G)|$ and $v(T_0) \geq 2$. If G_0 contains a true dividing vertex v, then $d(T_0) \geq 3 - 3(\alpha + \epsilon)$.*

Proof. Let $G' = C \cup \{u_1 v, u_2 v\}$. Let C_1, C_2 be the two facial cycles of G'. Thus $|V(C)| + 4 = |V(C_1)| + |V(C_2)|$. Now $\text{def}(T[G']) = |V(C)| - 3 - (|V(C_1) - 3) - (|V(C_2) - 3) = -1$. Moreover, $s(T[G']) = \alpha + \epsilon$ and hence, $d(T[G']) = -1 - (\alpha + \epsilon)$.

Note that $T[C_1], T[C_2]$ are critical cycle-canvases. If $v(T[C_1]) = 0$, then $|E(T[C_1]) \setminus E(C_1)| \geq 2$ by the definition of dividing; hence, $d(T[C_1]) \geq 2$ by Proposition 3.4.3. If $v(T[C_1]) = 1$, then $d(T[C_1]) \geq 2 - (\alpha + \epsilon)$ by Proposition 3.4.3. If $v(T[C_1]) \geq 2$, then $d(T[C_1]) \geq 3 - \gamma$ as T is a minimum counterexample. In any case, $d(T[C_1]) \geq 2 - (\alpha + \epsilon)$ as $\gamma \leq 1 + (\alpha + \epsilon)$ by inequality (3). Similarly, $d(T[C_2]) \geq 2 - (\alpha + \epsilon)$.

By Lemma 3.4.1, $d(T) \geq d(T[G']) + d(T[C_1]) + d(T[C_2]) \geq (-1 - (\alpha + \epsilon)) + 2(2 - (\alpha + \epsilon)) = 3 - 3(\alpha + \epsilon)$. $\square$

Claim 3.4.11. *Suppose $T_0 = (G_0, C_0, L_0)$ is a critical cycle-canvas with $|E(G_0)| \leq |E(G)|$ and $v(T_0) \geq 2$. If G_0 contains a strong dividing vertex v, then $d(T_0) \geq$*

$3 - 3(\alpha + \epsilon)$.

Proof. If v is true, then the claim follows from Claim 3.4.10. So we may suppose that v is not adjacent to u_1. Note that $v(T[C_1]) \geq 1$ as v is strong. If $v(T[C_1]) = 1$, then that vertex is a true dividing vertex and the claim follows from Claim 3.4.10. So we may assume that $v(T[C_1]) \geq 2$ and similarly that $v(T[C_2]) \geq 2$.

Let $G' = C \cup \{u_1 v, u_2 v\}$. Let C_1, C_2 be the two facial cycles of G'. Thus $|V(C)| + 4 = |V(C_1)| + |V(C_2)|$. Now $\mathrm{def}(T[G']) = |V(C)| - 3 - (|V(C_1) - 3) - (|V(C_2) - 3) = -1$. Moreover, $s(T[G']) = \alpha + \epsilon$ and hence, $d(T[G']) = -1 - (\alpha + \epsilon)$.

So suppose that v is adjacent to u_2. Note that $T[C_1], T[C_2]$ are critical cycle-canvases. As $v(T[C_1]) \geq 2$, $d(T[C_1]) \geq 3 - \gamma$ as T is a minimum counterexample. Similarly, $d(T[C_2]) \geq 3 - \gamma$. By Lemma 3.4.1, $d(T + \{u_1 v\}) \geq d(T[G']) + d(T[C_1]) + d(T[C_2])$. Yet, $d(T) = d(T + \{u_1 v\}) - 1$. Hence, $d(T) \geq (-1 - \epsilon - \alpha) + 2(3 - \gamma) - 1 = 4 - (\alpha + \epsilon) - 2\gamma$. This is at least $3 - 3(\alpha + \epsilon)$ as $2\gamma \leq 1 + 2(\alpha + \epsilon)$ by inequality (3).

So we may suppose that v is not adjacent to u_2. For every $c \in L(v)$, let $\phi_c(v) = c$ and $\phi_c(x) = \phi(x)$ for all $x \in C$. For every $c \in L(v)$, ϕ_c does not extend to an L-coloring of either $Int(C_1)$ or $Int(C_2)$. Thus there exists $\mathcal{C} \subset L(v)$ with $|\mathcal{C}| = 3$ and $i \in \{1, 2\}$ such that ϕ_c does not extend to an L-coloring of $Int(C_i)$ for all $c \in \mathcal{C}$. Suppose without loss of generality that $i = 1$. Let $C_1' = C_1 \cup \{u_1 u_2\} \setminus \{u_1 v, u_2 v\}$. We claim that C_1' has a proper coloring that does not extend in. If $\phi(u_1) = \phi(u_2)$, change $\phi(u_1)$ to a new color in $L_0(u_1)$ and $L_0(x)$ for $x \in N(u_1)$. Change $L_0(v)$ to $\mathcal{C} \cup \{\phi(u_1), \phi(u_2)\}$. Now ϕ is a coloring of C_1' which does not extend to an L_0 coloring of $Int(C_1)$. Thus it contains a critical subcanvas. Using Claim 3.4.7 if necessary and as $v(T[C_1']) \geq 2$, $d(T[C_1']) \geq 3 - \gamma$ by the minimality of T. Similarly as $v(T[C_2]) \geq 2$, $d(T[C_2]) \geq 3 - \gamma$.

Let us now count deficiencies. By Lemma 3.4.1, $\mathrm{def}(T + \{u_1 v, u_2 v\}) = \mathrm{def}(T[G']) + \mathrm{def}(T[C_1]) + \mathrm{def}(T[C_2])$. Yet, $\mathrm{def}(T) = \mathrm{def}(T + \{u_1 v, u_2 v\}) - 2$. Furthermore, $\mathrm{def}(T[C_1]) = \mathrm{def}(T[C_1']) + 1$. Hence, $\mathrm{def}(T) = \mathrm{def}(T[C_1]) + \mathrm{def}(T[C_2]) - 3 =$

$\mathrm{def}(T[C_1']) + \mathrm{def}(T[C_2]) - 2$.

Next we count the function s. We claim that $s(T) \leq s(T[C_1']) + s(T[C_2])$. This follows as every vertex of $G \setminus C$ is either in $G_1' \setminus C_1'$ or $G_2 \setminus C_2$. Moreover every vertex of $Q(T)$ is either in $Q(T[C_1'])$ or $Q(T[C_2])$.

Finally putting it all together, we find that

$$d(T) \geq d(T[C_1']) + d(T[C_2]) - 2 \geq 2(3-\gamma) - 2 = 4 - 2\gamma$$

.

This is at least $3 - 3(\alpha+\epsilon)$ as $2\gamma \leq 1 + 3(\alpha+\epsilon)$ by inequality (3). □

Claim 3.4.12. *G does not have a true dividing vertex or strong dividing vertex.*

Proof. This follows from Claims 3.4.10 and 3.4.11, and as $3(\epsilon+\alpha) \leq \gamma$ by inequality (2). □

3.4.3 Tripods

Definition. Let $T_0 = (G_0, C_0, L_0)$ be a cycle-canvas. We say a vertex $v \notin V(C_0)$ is a *quadpod* if at most one face of $G_0[C_0 \cup v]$ is non-empty and $|N(v) \cap V(C_0)| \geq 4$. We say a vertex $v \notin V(C_0)$ is a *tripod* if exactly one face of $G_0[C_0 \cup v]$ is non-empty and $|N(v) \cap V(C_0)| = 3$. Letting $C_0 = c_1c_2 \dots c_k$, we then say that a vertex $v \notin V(C)$ is a *tripod for c_i* if v is a tripod, $v \sim c_i$, $c_i \in V(C)$, and the faces of $G_0[C_0 \cup v]$ incident with c_i are empty.

If v is a tripod or quadpod, we let $C_0[v]$ denote the boundary of the non-empty face of $G_0[C_0 \cup v]$. We let $W(T_0)$ denote the set of all quadpods of T_0. We let $X(T_0)$ denote the set of all tripods of T_0. If $X' \subseteq X(T_0)$, we let $C_0[X']$ denote the boundary of the non-empty face of $G_0[C_0 \cup X']$.

Let $X_1 = X(T)$, $X_2 = X(T[C[X_1]])$ and $X_3 = X(T[C[X_1][X_2]])$. Let $W_1 = W(T)$, $W_2 = W(T[C[X_1]])$ and $W_3 = W(T[C[X_1][X_2]])$.

Claim 3.4.13. *$W_1 = \emptyset$ and $X_1 \neq \emptyset$.*

Proof. By Claim 3.4.9, there does not exist a chord of C. Suppose there exists a quadpod v of C. But then v is a true dividing vertex of G, contradicting Claim 3.4.12. Hence $W_1 = \emptyset$. By Theorem 3.3.1, it follows that $X_1 \neq \emptyset$. □

Claim 3.4.14. *$C[X_1]$ does not have a chord.*

Proof. Suppose not. Let v_1v_2 be a chord of $C[X_1]$. As C has no chord by Claim 3.4.9, we may assume without loss of generality that $v_1 \notin V(C)$. Thus v_1 is a tripod of C. Hence v_2 is also a tripod, as otherwise v_1 is not a tripod. But then v_2 is a true dividing vertex for $C[v_1]$. As $v(T) \geq 3$ by Claim 3.4.6, $v(T[C[v_1]]) \geq 2$. Therefore by Claim 3.4.10, $d(T[C[v_1]]) \geq 3 - 3(\alpha + \epsilon)$. Yet $d(T) \geq d(T[C[v_1]]) - (\alpha + \epsilon)$. Thus, $d(T) \geq 3 - 4(\alpha + \epsilon)$, a contradiction as $4(\alpha + \epsilon) \leq \gamma$ by inequality (2). □

Claim 3.4.15. *$C[X_1]$ does not have a true or strong dividing vertex.*

Proof. Suppose not. Let v be a true or strong dividing vertex of $C[X_1]$. Let u_1, u_2 be as in the definition of true or strong dividing vertex. Let $U = \{u_1, u_2\} \setminus C$. Hence $|U| \leq 2$ and v is a true or strong dividing vertex of $C[U]$. Yet, $v(T[C[U]]) \geq v(T) - |U| \geq 2$ as $v(T) \geq 4$ by Claim 3.4.6. Therefore by Claims 3.4.10 and 3.4.11, $d(T[C[U]]) \geq 3-3(\alpha+\epsilon)$. Yet $d(T) \geq d(T[C[U]])-2(\alpha+\epsilon)$. Thus, $d(T) \geq 3-5(\alpha+\epsilon)$, a contradiction as $5(\alpha + \epsilon) \leq \gamma$ by inequality (2). □

Claim 3.4.16. *$W_2 = \emptyset$ and $X_2 \neq \emptyset$. Furthermore for all $x_2 \in X_2$, if x_2 is a tripod for $v \in C[X_1]$, then $v \in V(C)$.*

Proof. By Claim 3.4.14, there does not exist a chord of $C[X_1]$. Suppose there exists a quadpod v of $C[X_1]$. But then v is a true dividing vertex of $C[X_1]$, contradicting Claim 3.4.15. Hence $W_2 = \emptyset$. By Theorem 3.3.1, it follows that $X_2 \neq \emptyset$.

Let $x_2 \in X_2$. Now x_2 is tripod for some v in $C[X_1]$. If $v \in X_1$, then $d(x_1) = 4$, a contradiction. So $v \in V(C)$ as desired. □

Claim 3.4.17. *$C[X_1][X_2]$ does not have a chord and hence $X_3 \cup W_3 \neq \emptyset$.*

Proof. Suppose not. Let v_1v_2 be a chord of $C[X_1][X_2]$. As $C[X_1]$ has no chord by Claim 3.4.14, we may assume without loss of generality that $v_1 \notin V(C) \cup X_1$. Thus $v_1 \in X_2$. Hence $v_2 \in X_2$ as otherwise v_1 is not a tripod of $C[X_1]$, a contradiction. Let $U = (N(v_1) \cup N(v_2)) \cap X_1$. Let $C' = C[U][v_1, v_2]$. Let $T' = T[C']$. As $d(v_1) \geq 5$, $v(T') \geq 1$. Suppose $v(T') \geq 2$. By Claim 3.4.7, $d(T') \geq 4 - \gamma$. Yet $d(T) \geq d(T') - 6(\alpha + \epsilon)$. Thus, $d(T) \geq 3 - \gamma$, a contradiction as $6(\alpha + \epsilon) \leq 1$ by inequalities (2) and (3).

So we may suppose that $v(T') = 1$. But then $d(T') \geq 3 - (\alpha + \epsilon)$ by Proposition 3.4.3. Hence, $d(T) \geq 3 - 7(\alpha + \epsilon)$, a contradiction as $7(\alpha + \epsilon) \leq \gamma$ by inequality (2). □

Claim 3.4.18. *If $z \in X_3 \cup W_3$ and $v \in C[X_1][X_2] \cap N(z)$ such that the faces of $G[V(C) \cup X_1 \cup X_2 \cup z]$ incident with v are empty, then $v \in V(C)$.*

Proof. Suppose not. As $z \notin X_2 \cup W_2$, z is adjacent to a vertex $x_2 \in X_2$.

As $x_2 \notin X_1$, x_2 is adjacent to a vertex $x_1 \in X_1$. As noted above, x_2 is a tripod in $C[X_1]$ for vertex of C.

If $v \in X_2$, then $d(v) = 4$, a contradiction. So we may assume that $v \in X_1$. Let u_1, u_2 be the other neighbors of z in $C[X_1][X_2]$ such that the cyclic orientation of $N(z)$ is $u_1 v u_2 \ldots$.

For $i \in \{1, 2\}$, if $u_i \in X_2$ and $|N(u_i) \cap X_1| = 2$, then $x_1 \in N(u_i) \cap X_1$ as otherwise there exists a vertex of degree four, a contradiciton. Note that if $u_i \in X_1$, then u_i is not adjacent to x_1 by Claim 3.4.14.

If $u_1 \in X_2$ and $N(x_2) \cap X_1 = \{v\}$, let $\phi(u_1) \in S(u_1) \setminus S(v)$. If $u_1 \in X_2$ and $N(x_2) \cap X_1 = \{v, u_1'\}$, let $\phi(u_1') \in S(u_1')$ and $\phi(u_1) \in S(u_1) \setminus (S(v) \cup \{\phi(u_1')\})$. Choose ϕ similarly if $u_2 \in X_2$.

Let $C' = C[v, u_1, u_1', u_2, u_2']$ and $T' = (G', C', L) = T[C']$. Consider $G' \setminus vz$. We claim that $G' \setminus vz$ has a C'-critical subgraph. This follows because if ϕ extends to an L-coloring of $G' \setminus vz$ then ϕ could be extended to an L-coloring of G' and hence of G, a contradiction.

Thus G' contains a proper C'-critical subgraph G''. We claim that $v(T') \geq 2$. Suppose not. Then $v(T') = 1$ and hence $N(z) \subseteq V(C')$. But now if $u_1 \in X_2$, then $d(u_1) = 4$, a contradiction. So $u_1 \in X_1$ and similarly $u_2 \in X_1$. But then $d(v) = 4$, a contradiction. This proves the claim that $v(T') \geq 2$.

By Claim 3.4.7, we find that $d(T') \geq 4 - \gamma$. Moreover, $s(T) \leq s(T') + 5(\alpha + \epsilon)$, $\text{def}(T) = \text{def}(T')$ and hence $d(T) \geq d(T') - 5(\alpha + \epsilon)$. Thus $d(T) \geq 4 - \gamma - 5(\alpha + \epsilon)$, which is at least $3 - \gamma$ as $5(\alpha + \epsilon) \leq 1$ by inequalities (2) and (3), a contradiction. □

Claim 3.4.19. *$W_3 = \emptyset$ and $X_3 \neq \emptyset$. Furthermore if $x_3 \in X_3$, then $N(x_3) \cap (X_1 \cup X_2) = \{x_2\}$ where $x_2 \in X_2$, x_3 is not a tripod for x_2 and $N(x_2) \cap X_1 = \{x_1\}$ and x_2 is not a tripod for x_1.*

Proof. Let $z \in W_3 \cup X_3$. Let the neighbors of z in $C[X_1][X_2]$ have cyclic orientation $u_1 \dots u_k$. By Claim 3.4.18, $u_i \in V(C)$ for all i such that $2 \leq i \leq k-1$. As $z \notin X_1$, it follows that $k \leq 4$.

Suppose that $|N(z) \cap X_2| \geq 2$. Thus $u_1, u_k \in X_2$. It follows that $|N(u_1) \cap X_1| = |N(u_k) \cap X_1| = 1$. Let $N(u_1) \cap X_1 = \{u_1'\}$ and $N(u_k) \cap X_1 = \{u_k'\}$. Choose $\phi(u_1) \in S(u_1)$ and $\phi(u_k) \in S(u_k)$ such that either $\phi(u_1) = \phi(u_k)$ or at least one of $\phi(u_1), \phi(u_k)$ is not in $S(z)$. Now choose $\phi(u_1') \in S(u_1') \setminus \{\phi(u_1)\}$ and $\phi(u_k') \in S(u_k') \setminus \{\phi(u_k)\}$.

Let $C' = C[u_1, u_1', u_k, u_k']$ and $T' = (G', C', L) = T[C']$. Consider $G' \setminus vz$ where $v \in \{u_1, u_k\}$ and, if $\phi(u_1) \neq \phi(u_k)$, then $\phi(v) \notin S(z)$. We claim that $G' \setminus vz$ has a C'-critical subgraph. This follows because if ϕ extends to an L-coloring of $G' \setminus vz$ then ϕ could be extended to an L-coloring of G' and hence of G, a contradiction.

Thus G' contains a proper C'-critical subgraph G''. Now $v(T') \geq 2$ as otherwise

$d(u_1) = 4$, a contradiction. By Claim 3.4.7, we find that $d(T') \geq 4 - \gamma$. Moreover, $s(T) \leq s(T') + 4(\alpha + \epsilon)$, $\mathrm{def}(T) = \mathrm{def}(T')$ and hence $d(T) \geq d(T') - 4(\alpha + \epsilon)$. Thus $d(T) \geq 4 - \gamma - 4(\alpha + \epsilon)$, which is at least $3 - \gamma$ as $4(\alpha + \epsilon) \leq 1$ by inequalities (2) and (3), a contradiction.

So we may assume that $|N(z) \cap X_2| \leq 1$. As $z \notin X_2$, $|N(z) \cap X_2| \geq k - 2$. Thus $k = 3$ and $|N(z) \cap X_2| = 1$. This shows that $W_3 = \emptyset$; by Theorem 3.3.1, it follows that $X_3 \neq \emptyset$. We may assume without loss of generality that $u_1 \in X_2$ and hence $u_k \notin X_2$. Let $x_2 = u_1$. Hence $N(z) \cap X_2 = \{x_2\}$. It follows that $|N(x_2) \cap X_1| = 1$ as otherwise there would be a vertex of degree four, a contradiction. But then by Claim 3.4.16, x_2 is not a tripod for x_1. Thus the claim is proved if $u_k \in V(C)$.

So we may suppose that $u_k \in X_1$. But then the same argument as above produces a contradiction as we may still choose $\phi(u_1), \phi(u_k)$ such that either $\phi(u_1) = \phi(u_k)$ or at least one of $\phi(u_1), \phi(u_k)$ is not in $S(z)$, since $|S(u_1)| + |S(u_k)| = 3 + 2 > 4 = |S(z)|$.
□

By Claim 3.4.19, there exists $x_1 \in X_1, x_2 \in X_2, x_3 \in X_3$ such that $N(x_3) \cap (X_1 \cup X_2) = \{x_2\}$, $N(x_2) \cap X_1 = \{x_1\}$, x_3 is not a tripod for x_2 in $C[X_1][X_2]$ and x_2 is not a tripod for x_1 in $C[X_1]$.

Claim 3.4.20. *$deg(x_1) = deg(x_2) = 6$, $deg(x_3) \in \{5, 6\}$ and there exists adjacent vertices $z_1, z_2 \notin V(C)$ such that $z_1 \sim x_1, x_2$ and $z_2 \sim x_2, x_3$.*

Proof. We claim that $|N_{G \setminus C}(x_1) \setminus Q(T)| \leq 1$. Suppose not. As $v(T) \geq 3$ by Claim 3.4.6, $v(T[C[v]]) \geq 2$. By the minimality of T, $d(T[C[x_1]]) \geq 3 - \gamma$. Yet, $q(T) \leq q(T[C[x_1]]) - 1$ and $v(T) = v(T[C[x_1]) + 1$. Thus $s(T) \leq s(T[C[x_1]]) + \epsilon - \alpha$. As $def(T) = def(T[C[x_1]])$, we find that $d(T) \geq d(T[C[x_1]]) + \alpha - \epsilon \geq (3 - \gamma) + (\alpha - \epsilon)$. As $\alpha \geq \epsilon$ by inequality (1), $d(T) \geq 3 - \gamma$, a contradiction.

As G is C-critical, $deg(x_1) \geq 5$. So suppose $deg(x_1) \geq 7$. As shown above, $|N_{G \setminus C}(x_1) \setminus Q(T)| \leq 1$. Thus $|N_{G \setminus C}(x_1) \cap Q(T)| \geq 3$ as $deg(v) \geq 7$. Without loss of generality we may suppose that x_1 is tripod for c_2 and that the cyclic orientation of $N(v) \cap (Q(T) \cup V(C))$ is $c_1c_2c_3q_1 \ldots q \ldots q_2$ where $c_1, c_2, c_3 \in V(C)$ and $q, q_1, q_2 \in Q(T)$. Thus q is a dividing vertex of $C[x_1]$. Given the presence of q_1 and q_2, q is a strong dividing vertex of $C[x_1]$, a contradiction as in Claim 3.4.15.

Suppose $deg(x_1) = 5$. Note that $N(x_2) \cap X_1 = \{x_1\}$. Let $C' = C[x_1][x_2]$ and $T' = (G', C', L) = T[C']$. Consider $G' \setminus x_1z$ where $z \notin V(C) \cup \{x_2\}$. We claim that $G' \setminus x_1z$ has a C'-critical subgraph. To see this, choose $\phi(x_2) \in S(x_2) \setminus S(x_1)$. This set is nonempty as $|S(x_2)| = 3, |S(x_1)| = 2$. Now if ϕ extends to an L-coloring of G' then ϕ could be extended to an L-coloring of G, a contradiction, as x_1 would see at most four colors.

Thus G' contains a proper C'-critical subgraph G''. Note that $v(T') \geq 2$ as x_2 has degree at least five and thus has at least two neighbors in $Int(C') \setminus V(C')$. By Claim 3.4.7, $d(T') \geq 4 - \gamma$. Moreover, $s(T) \leq s(T') + 2(\alpha + \epsilon)$, $\mathrm{def}(T) = \mathrm{def}(T')$ and hence $d(T) \geq d(T') - 2(\alpha + \epsilon)$. Thus $d(T) \geq 4 - \gamma - 2(\alpha + \epsilon)$, which is at least $3 - \gamma$ as $2(\alpha + \epsilon) \leq 1$ by inequalities (2) and (3).

Similar arguments show that $deg(x_2) = 6$, $deg(x_3) \in \{5, 6\}$ and that $|N_{G \setminus C[X_1]}(x_2) \setminus Q(T[X_1])| \leq 1$. Moreover, $|Q(T)| = |Q(T[C[x_1]])| = |Q(T[C[x_1][x_2])|$. Let z_1, z_2 be such that $N(x_2) \setminus (C \cup \{x_1, x_3\}) = \{z_1, z_2\}$ and the cyclic orientation around x_2 is $x_1z_1z_2x_3 \ldots$. As $|Q(T[C[X_1]])| \leq |Q(T)|$, we find that $z_1 \sim x_1$. Similarly as $|Q(T[C[x_1][x_2])| \leq |Q(T)|$, we find that $z_2 \sim x_3$ as desired. $\square$

Let $\phi(x_1) \in S(x_1)$ and $C_1 = C[x_1]$. Let $S_1(z) = S(z) \setminus \{\phi(x_1)\}$ if $x_1 \sim z$ and $S_1 = S$ otherwise. We may assume by Claim3.4.7 that $T_1 = T[C_1]$ is ϕ-critical.

Claim 3.4.21. *If $z \in N_{G \setminus C_1}(x_2)$, then $S_1(x_2) \subseteq S_1(z)$.*

Proof. Suppose that $S_1(x_2) \setminus S_1(z) \neq \emptyset$. Let $C' = C[x_1][x_2]$ and $T' = (G', C', L) = T[C']$. Consider $G' \setminus x_2z$. We claim that $G' \setminus x_2z$ has a C'-critical subgraph. To see this, Choose $\phi(x_2) \in S(x_2) \setminus S(z)$. Now if ϕ extends to an L-coloring of G' then ϕ

could be extended to an L-coloring of G, a contradiction, as x_2 and z could not have the same color.

Thus G' contains a proper C'-critical subgraph G''. Note that $v(T') \geq 2$ as z has degree at least five and thus has at least two neighbors in $Int(C') \setminus V(C')$. By Claim 3.4.7, $d(T') \geq 4 - \gamma$. Moreover, $s(T) \leq s(T') + 2(\alpha + \epsilon)$, $\mathrm{def}(T) = \mathrm{def}(T')$ and hence $d(T) \geq d(T') - 2(\alpha + \epsilon)$. Thus $d(T) \geq 4 - \gamma - 2(\alpha + \epsilon)$, which is at least $3 - \gamma$ as $2(\alpha + \epsilon) \leq 1$ by inequalities (2) and (3), a contradiction. □

Claim 3.4.22. *If $z \in N_{G \setminus C_1}(x_3)$ and $z \neq x_2$, then $S_1(x_3) \subseteq S_1(z)$.*

Proof. This follows in the same manner as Claim 3.4.21. □

Claim 3.4.23. *$N(z_1) \cap V(C_1) = x_1$.*

Proof. Note that $z_1 \sim x_1$. Suppose that $|N(z_1) \cap V(C_1)| \geq 2$. But then as $deg(x_1) = 6$, z_1 is a true dividing vertex of C_1, a contradiction as in Claim 3.4.15. □

Claim 3.4.24. *$deg(x_3) = 6$.*

Proof. Suppose not. By Claim 3.4.20, $deg(x_3) = 5$. By Claim 3.4.21, $S_1(x_2) \subset S_1(x_3)$ and hence $L(z_2) \setminus (S_1(x_2) \cup S_1(x_3)) = L(z_2) \setminus S_1(x_3)$. Yet, $|L(z_2) \setminus S_1(x_3)| \geq 2$ as $|L(z_2)| = 5, |S_1(x_3)| = 3$. Let $C' = C_1[x_2][x_3] \setminus \{x_2x_3\} \cup \{x_2z_2, x_3z_2\}$ and $T' = (G', C', L) = T[C']$. Consider $G' \setminus \{x_2z_1, x_3z_3\}$ where $z_3 \notin V(C_2) \cup \{x_2\}$.

We claim that $G' \setminus \{x_2z_1, x_3z_3\}$ has a C'-critical subgraph. To see this, choose $\phi(z_2) \in L(z_2) \setminus S_1(x_3)$. If ϕ extends to an L-coloring of G', then ϕ could be extended to an L-coloring of G as x_2 would see at most one color (that of z_1) and hence ϕ could be extended to x_2 as $|S_1(x_2)| = 2$, but then ϕ could be extended to x_3 as x_3 would see at most two colors (that of x_2 and z_3) and $|S_1(x_3)| = 3$. But this contradicts that T is a counterexample.

Thus G' contains a proper C'-critical subgraph G''. Note that $v(T') \geq 2$ given z_1 and z_3. Moreover, $|E(G') \setminus E(G'')| \geq 2$. In addition, we claim that $|E(G'') \setminus E(C')| \geq 2$.

Suppose not. Then there would exist a chord of C', which would imply that z_2 is adjacent to a vertex in C. But then z_2 is a true dividing vertex of $C[x_1][x_2]$. So by Claim 3.4.10, $d(T[C[x_1][x_2]) \geq 3 - 3(\alpha + \epsilon)$. Hence $d(T) \geq 3 - 5(\alpha + \epsilon)$ which is at least $3 - \gamma$ as $5(\alpha + \epsilon) \leq \gamma$ by inequality (2), a contradiction. This proves the claim that $|E(G'') \setminus E(C')| \geq 2$.

By Claim 3.4.7, we find that $d(T') \geq 4 - 2(\alpha + \epsilon)$. Moreover, $s(T) \leq s(T') + 4(\alpha + \epsilon)$, $\mathrm{def}(T) = \mathrm{def}(T') - 1$ and hence $d(T) \geq d(T') - 1 - 4(\alpha + \epsilon)$. Thus $d(T) \geq 3 - 6(\alpha + \epsilon)$, which is at least $3 - \gamma$ as $6(\alpha + \epsilon) \leq \gamma$ by inequality (2), a contradiction. □

Let $C' = C_1[x_2][x_3] \setminus \{x_1x_2, x_2x_3\} \cup \{x_1z_1, z_1z_2, z_2x_3\}$ and $T' = (G', C', L) = T[C']$. Consider $G' \setminus \{x_3z_3, x_3z_4\}$ where $z_3 \neq z_4 \notin V(C) \cup \{x_2, z_2\}$. We claim that $G' \setminus \{x_3z_4, x_3z_4\}$ has a C'-critical subgraph. To see this, choose $\phi(z_2) \in L(z_2) \setminus S_1(x_3)$ and $\phi(z_1) \in S_1(z_1) \setminus (S_1(x_2) \cup \{\phi(z_2)\})$. If ϕ extends to an L-coloring of G', then ϕ could be extended to an L-coloring of G as x_3 would see at most two colors(that of z_3 and z_4) and hence ϕ could be extended to x_3 as $|S_1(x_3)| = 4$, but then ϕ could be extended to x_2 as x_2 would see at most one color (that of x_3) and $|S_1(x_2)| = 2$. But this contradicts that T is a counterexample.

Thus G' contains a proper C'-critical subgraph G''. Note that $v(T') \geq 2$ as given z_3, z_4. Moreover, $|E(G') \setminus E(G'')| \geq 2$. In addition, we claim that $|E(G'') \setminus E(C')| \geq 2$. Suppose not. Then there would exist a chord of C', which would imply that z_2 or z_3 is adjacent to a vertex in C. But then z_1 or z_2 is a true dividing vertex of $C[x_1][x_2]$. So by Claim 3.4.10, $d(T[C[x_1][x_2]) \geq 3 - 3(\alpha + \epsilon)$. Hence $d(T) \geq 3 - 5(\alpha + \epsilon)$ which is at least $3 - \gamma$ as $5(\alpha + \epsilon) \leq \gamma$ by inequality (2), a contradiction. This proves the claim that $|E(G'') \setminus E(C')| \geq 2$.

By Claim 3.4.7, we find that $d(T') \geq 4 - 2(\alpha + \epsilon)$. Moreover, $s(T) \leq s(T') + 5(\alpha + \epsilon)$, $\mathrm{def}(T) = \mathrm{def}(T') - 1$ and hence $d(T) \geq d(T') - 1 - 5(\alpha + \epsilon)$. Thus $d(T) \geq 3 - 5(\alpha + \epsilon)$, which is at least $3 - \gamma$ as $7(\alpha + \epsilon) \leq \gamma$ by inequality (2), a contradiction. □

Let us state Theorem 3.4.5 with explicit constants while omitting quasi-boundary from the formula.

Theorem 3.4.25. *If (G, C, L) is a critical cycle-canvas, then* $|V(G) \setminus V(C)|/28 + \sum_{f \in \mathcal{F}(G)} (|f| - 3) \leq |V(C)| - 3$.

Proof. Let $\epsilon = \alpha = 1/28$ and $\gamma = 4/7$. Then apply Theorem 3.4.5. □

We may now prove Theorem 1.8.1 which we restate in terms of critical cycle-canvases.

Theorem 3.4.26. *If (G, C, L) is a critical cycle-canvas, then* $|V(G)| \leq 29|V(C)|$.

Proof. $|V(G) \setminus V(C)| \leq 28|V(C)|$ by Corollary 3.4.25. Hence, $|V(G)| = |V(G) \setminus V(C)| + |V(C)| \leq 29|V(C)|$. □

Definition. Let $T = (G, C, L)$ be a cycle-canvas. Let $G' \subseteq G$ such that for every face $f \in \mathcal{F}(G')$, every L-coloring of the boundary walk of f extends to an L-coloring of the interior of f. We say $T' = (G', C, L)$ is an *easel* for T.

Given that the linear bound is proved in terms of deficiency, which works well for applying induction to a subgraph and its faces, we may actually prove a stronger theorem about easels which we will use in Chapter 5.

Theorem 3.4.27. *If $T = (G, C, L)$ is a cycle-canvas, then there exists an easel $T' = (G', C, L)$ for T such that* $|V(G' \setminus C)| \leq 28\text{def}(T')$.

Proof. We proceed by induction on the number of vertices of G. We may suppose that T contains a critical subcanvas $T_0 = (G_0, C, L)$, as otherwise the lemma follows with $T' = T$. By Theorem 3.4.25, $|V(G_0 \setminus C)| \leq 28\text{def}(T_0)$. For every face $f \in \mathcal{F}(G_0)$, there exists by induction an easel $T'_f = (G'_f, C_f, L)$ for $T_f = (G_f, C_f, L)$ such that $|V(G'_f \setminus C_f)| \leq 28\text{def}(T'_f)$.

Let $G' = G_0 \cup \bigcup_{f\in\mathcal{F}(G_0)} G'_f$ and let $T' = (G', C, L)$. Now for every face $f \in \mathcal{F}(G')$, every L-coloring of the boundary walk of f extends to an L-coloring of the interior of f. Moreover by Lemma 3.3.2, $\text{def}(T_0) + \sum_{f\in\mathcal{F}(G_0)} = \text{def}(T')$. Yet $|V(G_0 \setminus C)| + \sum_{f\in\mathcal{F}(G_0} |V(G_f \setminus C_f)| = |V(G' \setminus C)|$. Hence $|V(G' \setminus C)| \leq 28\text{def}(T')$ and the theorem is proved. $\square$

Let $T = (G, C, L)$ be a cycle-canvas and $T' = (G', C, L)$ be an easel for T. As $|V(G' \setminus C)| \geq 0$, Theorem 3.4.27 also implies that $\text{def}(T) \geq 0$ and hence the size of any face $f \in \mathcal{F}(G')$ is at most $|C|$. Note that G' could be equal to C though if every L-coloring of C extends to an L-coloring of G. Suppose that $T'' = (G'', C, L)$ is an easel for T'. It follows that T'' is also an easel for T'. Therefore it is of interest to consider minimal easels for T. To that end, we make the following defintion.

Definition. Let $T = (G, C, L)$ be a cycle-canvas and $T' = (G', C, L)$ an easel for T. We say that T' is a *critical easel* for T if there does not exist $T'' = (G'', C, L)$ such that $G'' \subsetneq G'$ such that T'' is an easel for T', and hence also an easel for T as noted above.

Thus Theorem 3.4.27 says that every critical easel $T' = (G', C, L)$ of $T = (G, C, L)$ satisfies $|V(G' \setminus C)| \leq 28\text{def}(T')$.

3.5 Exponentially Many Extensions of a Precoloring of a Cycle

Thomassen [51] proved that planar graphs have exponentially many 5-list-colorings from a given 5-list-assignment. Indeed, he proved a stronger statement, restated here in terms of path-canvases.

Theorem 3.5.1. *[Theorem 4 in [51]] Let $T = (G, P, L)$ be a path-canvas. Let r be the number of vertices of C, the outer walk of G, such that $|L(v)| = 3$. If T is not a bellows, then G has at least $2^{|V(G\setminus P)|/9 - r/3}$ distinct L-colorings unless $|V(P)| = 3$ and there exists $v \in V(G)$, $|L(v)| = 4$ and v is adjacent to all the vertices of P.*

Corollary 3.5.2. *Let $T = (G, C, L)$ be a cycle-canvas such that $|C| \leq 4$. Let ϕ be an L-coloring of $G[V(C)]$, then $\log E(\phi) \geq |V(G \setminus C)|/9$, where $E(\phi)$ is the number of extensions of ϕ to G, unless $|C| = 4$ and there exists a vertex not in $V(C)$ adjacent to all the vertices of C.*

Proof. Let $v \in C$. Let $G' = G \setminus \{v\}$, $P = C \setminus \{v\}$ and $L'(w) = L(w) \setminus \{\phi(v)\}$ for all $w \in N(v)$. Apply Theorem 3.5.1 to (G', P, L'). Note that there does not exist a vertex $x \in V(G')$ such that $|L'(v)| = 3$. Hence $r = 0$ and it follows that there are $2^{|V(G \setminus C)|/9}$ distinct L-colorings of G unless $|V(P)| = 3$ and there exists $x \in V(G)$ with $|L'(x)| = 4$ and x adjacent to all vertices of P. But then $|C| = 4$ and there exists a vertex adjacent to all the vertices of C. □

Lemma 3.5.3. *If $T = (G, C, L)$ is a cycle-canvas and ϕ is an L-coloring of C that extends to an L-coloring of G, then $\log E(\phi) \geq (|V(G \setminus C)| - 29(|C| - 3))/9$, where $E(\phi)$ is the number of extensions of ϕ to G.*

Proof. We proceed by induction on the number of vertices of G. As $\epsilon \leq 1/9$, we may assume that $|C| \geq 5$ by Corollary 3.5.2. It also follows from Corollary 3.5.2, that there does not exist a vertex-cut in G of size at most three as otherwise the lemma follows by induction. Thus there is no separating triangle in G. Similarly if there exists a separating 4-cycle C' in G, then there must exist a vertex in the interior of C' adjacent to all the vertices of C' as otherwise the theorem follows by induction.

First suppose there exists $v \in V(G)$ such that v has at least three neighbors on C. Suppose that v has at least four neighbors on C. Let $G' = G[V(C) \cup \{v\}]$ and $T' = (G', C, L)$. As ϕ extends to an L-coloring of G, we can extend ϕ to v. For all $f \in \mathcal{F}(G')$, it follows by induction that $\log E_{T_f}(\phi) \geq (|V(G_f \setminus C_f)| - 29(|C_f| - 3))/9$ for all $f \in \mathcal{F}(G')$. Thus $\log E(\phi) \geq \sum_{f \in \mathcal{F}(G')} \log E_{T_f}(\phi) \geq ((|V(G \setminus C)| - 1) - 29(|C| - 4))/9$ as $\text{def}(T') \geq 1$. The lemma follows.

So we may assume that v has exactly three neighbors on C. Let $S(v) = \in L(v) \setminus \{\phi(u) | u \in N(v) \cap C\}$. Hence $|S(v)| \geq 2$. Let $c_1, c_2 \in S(v)$. For $i \in \{1, 2\}$, let

$\phi_i(v) = c_i$ and $\phi_i(u) = \phi(u)$ for all $u \in V(C)$. If both ϕ_1 and ϕ_2 extend to L-colorings of ϕ, it follows by induction applied to the faces of $G[V(C) \cup \{v\}]$ that $\log E(\phi_i) \geq ((|V(G \setminus C)| - 1) - 29(|V(C)| - 3))/9$. Yet $E(\phi) \geq E(\phi_1) + E(\phi_2)$ and hence $\log E(\phi) \geq (|V(G \setminus C)| - 29(|V(C)| - 3))/9$ and the lemma follows.

So we may suppose that ϕ_1 does not extend to an L-coloring of G. Hence there exists $f \in \mathcal{F}(G')$ such that T_f contains a critical subcanvas $T'_f = (G_f, C_f, L)$. By Lemma 3.4.26, $|V(G_f \setminus C_f)| \leq 28\text{def}(T_f)$. Let $G' = G[V(G_f) \cup V(C)]$. Thus $|V(G' \setminus C)| = |V(G' \setminus C_f| + 1$, $\text{def}(T_f) \leq \text{def}(T')$. Hence $|V(G' \setminus C)| \leq 29\text{def}(T')$.

As ϕ extends to an L-coloring of G, ϕ extends to an L-coloring of G'. For all $f \in \mathcal{F}(G')$, it follows by induction that $\log E_{T_f}(\phi) \geq (|V(G_f \setminus C_f)| - 29(|C_f| - 3))/9$ for all $f \in \mathcal{F}(G')$. Thus $\log E(\phi) \geq \sum_{f \in \mathcal{F}(G')} \log E_{T_f}(\phi) \geq ((|V(G \setminus C)| - |V(G' \setminus C)|) - 29(|C| - 3 - \text{def}(T')))/9$. The lemma follows.

Suppose there exists $v \in V(G)$ such that v has two neighbors $u_1, u_2 \in V(C)$ and $d_C(u_1, u_2) \geq 3$. Consider the cycles C_1, C_2 such that $C_1 \cap C_2 = u_1 v u_2$ and $C_1 \cup C_2 = C \cup v$. Note that $|C_1| + |C_2| = |C| + 4$. Let $T_1 = (G_1, C_1, L_1)$ where $G_1 = Int(C_1)$ and $T_2 = (G_2, C_2, L)$ where $G_2 = Int(C_2)$. Now extend ϕ to v and apply induction to T_1 and T_2. By induction, $\log E_{T_i}(\phi) \geq (|V(G_i \setminus C_i)| - 29(|C_i| - 3))/9$ for all $i \in \{1, 2\}$. Hence, $\log E(\phi) \geq (|V(G \setminus C)| - 1) - 29(|C_1| + |C_2| - 6))/9$ and the lemma follows as $|C_1| + |C_2| - 6 \geq |C| - 4$.

Finally we may suppose there does not exist $v \in V(G) \setminus V(C)$ such that v has at least three neighbors in C. Let $G' = G \setminus C$ and $L'(v) = L(v) \setminus \{\phi(u) | u \in N(v) \cap C\}$ for all $v \in V(G')$. By Theorem 3.5.1, there exist $2^{|V(G')|/9 - |S|/3}$ distinct L-colorings of G extending ϕ where $S = \{v \in V(G') || L'(v)| = 3\}$.

Let $v \in S$. Then v has at least two neighbors on C as $|L'(v)| = 3$. Thus v has exactly two neighbors u_1, u_2 on C. But then $d_C(u_1, u_2) \leq 2$. Thus there exists a cycle C' of size at most four containing the vertices u_1, u_2, v and perhaps another vertex on C. But note then that there does not exists a vertex in the interior of C' as otherwise

$|C'| = 4$ and there exists a vertex adjacent to all the vertices of C', and hence adjacent to all three vertices on C, a contradiction. It now follows that $|S| \leq |C|$. Therefore $\log E(\phi) \geq |V(G')|/9 - |S|/3 \geq |V(G \setminus C)|/9 - |C|/3$. As $|C|/3 \leq 29(|C|-3)/9$ since $|C| \geq 4$, the lemma follows. $\square$

3.6 Logarithmic Distance for Cycles

Lemma 3.6.1. *If $T = (G, S, L)$ is a critical canvas such that for all $v \in V(G)$, if $|L(v)| < 5$, then $v \in S$, then $|V(G)| \leq 29|V(S)|$.*

Proof. Note that $|S| \geq 3$ by Theorem 1.4.2. Let W be the outer walk of G. Delete all instances from W of vertices not in S and remove all instances from W but for one for vertices in S. The result W' is a cycle on the vertices of S. Now add a new vertex between every two consecutive vertices in W' unless there is already an edge between those vertices that lies in the same place in the walk W. Then add edges along the new walk so as to form a cycle C. Let G' be the graph with vertex set $V(C) \cup V(G)$ and edge set $E(G) \cup E(C)$. For every $v \in V(C) \setminus V(S)$, let $L(v)$ be any set of five colors. Now (G', C', L) is a critical cycle-canvas. By Theorem 3.4.25, $|V(G') \setminus V(C')|/28 + \sum_{f \in \mathcal{F}(G')}(|f| - 3) \leq |V(C')| - 3$.

Moreover, every $v \in V(C) \setminus V(S)$ is incident with a different face in $\mathcal{F}(G')$ and these faces have size at least four because a vertex was not added if its two consecutive vertices already had an edge in the walk W. Hence, $\sum_{f \in \mathcal{F}(G')}(|f| - 3) \geq |V(C)| - |V(S)|$. Thus, $|V(G) \setminus V(S)|/28 = |V(G') \setminus V(C')|/28 \leq |V(S)| - 3$. Hence, $|V(G)| \leq 29|V(S)|$ as desired. $\square$

Theorem 3.6.2. *Let $T = (G, S, L)$ be a critical canvas and C be its outer walk. If X is a separation of G into two graphs G_1, G_2 where $S \cup V(C) \subseteq G_1$, then $|V(G_2)| \leq 29|X|$.*

Proof. Let G' be the union of X and all components of $G \setminus X$ that do not contain a vertex in S. Let X' be all vertices in G' with that are in $S \cup V(C)$ or have a neighbor in $V(G) \setminus V(G')$. Clearly, $X' \subseteq X$ as $S \cup V(C) \subseteq G_1$. Note the vertices of X' lie on the outer face of G' as they are either in S or have a path to a vertex of S through $G \setminus G'$ and yet the vertices of S lie on the outer face of G. Hence $T' = (G', X', L)$ is a canvas. Furthermore as T is critical, T' is critical by Lemma 3.2.1 as G' is an X'-component of G with respect to S. Yet every vertex in G' is either in $S \cup V(C)$ or has a list of size five. Thus every vertex in $V(G') \setminus X'$ has a list of size five. By Theorem 3.6.1, $|V(G')| \leq 29|X'|$. Hence $|V(G_2)| \leq |V(G')| \leq 29|X'| \leq 29|X|$ as desired. □

Theorem 3.6.3. *If $T = (G, C, L)$ is a critical cycle-canvas, $v_0 \in V(G)$ and $X \subseteq V(G)$ such that X separates v from C, then $d(v_0, X) \leq 58 \log |X|$ for all $v \in V(G_2) \setminus X$.*

Proof. We proceed by induction on the size of X. Let G_1, G_2 be graphs such that $X = V(G_1) \cap V(G_2)$, $C \subseteq G_1$ and $v_0 \in V(G_2)$. By Theorem 1.4.2, $|X| \geq 2$ as T is critical. Thus, we may assume that $d(v_0, X) > 58$, as otherwise the theorem follows.

Let $X_i = \{v \in V(G_2) | d(v, X) = i\}$ and let $H_i = G[\bigcup_{j \geq i} X_j]$. As $|V(G_2)| \leq 29|V(X)|$ by Theorem 3.6.2, there exists i, $1 \leq i \leq 58$ such that $|X_i| \leq |X|/2$. As $d(v_0, X) > 58$, X_i separates v_0 from C. By induction on X_i, $d(v_0, X_i) \leq 58 \log |X_i| \leq 58 \log |X| - 58$. Yet $d(v, X) \leq 58$ for all $v \in X_i$ and hence $d(v_0, X) \leq 58 \log \mathcal{C}$ as desired. □

Theorem 3.6.4. *[Logarithmic Distance for Cycle-Canvases] If $T = (G, C, L)$ is a critical cycle-canvas, then $d(v, C) \leq 58 \log |C|$ for all $v \in V(G)$.*

Proof. Follows from Theorem 3.6.3 with $X = C$. □

Theorem 3.6.5. *[Exponential Growth for Cycle-Canvases] If $T = (G, C, L)$ is a critical cycle-canvas and $v_0 \in V(G) \setminus V(C)$, then for all $k \leq d(v_0, C)$, $|N_k(v_0)| \geq 2^{k/58}$.*

Proof. Let $k \leq d(v_0, C)$. Now $N_k(v_0)$ separates v_0 from C. By Theorem 3.6.3, $k = d(v_0, N_k(v_0)) \leq 58 \log |N_k(v_0)|$. Hence $|N_k(v_0)| \geq 2^{k/58}$ as desired. □

3.6.1 Critical Easels

We use similar proofs to derive logarithmic distance and exponential growth for critical easels.

Theorem 3.6.6. *Let $T = (G, C, L)$ be a cycle-canvas and $T' = (G', C, L)$ be a critical easel for T. If X is a separation of G' into two graphs G_1, G_2 where $C \subseteq G_1$, then $|V(G_2)| \leq 29|X|$.*

Proof. Suppose not. Let G_0 be the union of X and all components of $G' \setminus X$ that do not contain a vertex in C. Let X' be all vertices in G_0 with that are in C or have a neighbor in $V(G') \setminus V(G_0)$. Clearly, $X' \subseteq X$ as $C \subseteq G_1$. Note the vertices of X' lie on the outer face of G_0 as they are either in C or have a path to a vertex of C through $G' \setminus G_0$ and yet the vertices of C lie on the outer face of G'.

Let W be the outer walk of G_0. Delete all instances from W' of vertices not in X' and remove all instances from W but for one for vertices in X'. The result W' is a cycle on the vertices of X'. Now add a new vertex between every two consecutive vertices in X' unless there is already an edge between those vertices that lies in the same place in the walk W. Then add edges along the new walk so as to form a cycle C_0. Let G_0' be the graph with vertex set $V(C_0) \cup V(G_0)$ and edge set $E(G_0) \cup E(C_0)$. For every $v \in V(C_0) \setminus V(X')$, let $L(v)$ be any set of five colors. Now $T_0 = (G_0, C_0, L)$ is a cycle-canvas. By Theorem 3.4.27, there exists an easel $T_0' = (G_0', C_0, L)$ for T_0 such that $d_0(T_0') \geq 0$, that is $|V(G_0') \setminus V(C_0)|/28 + \sum_{f \in \mathcal{F}(G_0')}(|f| - 3) \leq |V(C_0)| - 3$.

Moreover, every $v \in V(C_0) \setminus X'$ is incident with a different face in $\mathcal{F}(G_0')$ and these faces have size at least four because a vertex was not added if its two consecutive vertices already had an edge in the walk W. Hence, $\sum_{f \in \mathcal{F}(G_0')}(|f| - 3) \geq |V(C_0)| - |X'|$. Thus, $|V(G_0) \setminus X'|/28 = |V(G_0') \setminus V(C_0)|/28 \leq |X'| - 3$. Thus $G_0' \setminus (C_0 \setminus X')$ is a proper subgraph of $G_0 \setminus (C_0 \setminus X')$.

Let $G'' = G' \setminus (G_0 \setminus G_0')$. It follows that $T'' = (G'', C, L)$ is an easel for T'.

Moreover, G'' is a proper subgraph of G', contradicting that T' is a critical easel. □

Theorem 3.6.7. *Let $T = (G, C, L)$ be a cycle-canvas and $T' = (G', C, L)$ be a critical easel for T. If $v_0 \in V(G')$ and $X \subseteq V(G')$ such that X separates v from C, then $d_{G'}(v_0, X) \leq 58 \log |X|$ for all $v \in V(G_2) \setminus X$.*

Proof. See proof of Theorem 3.6.3. □

Theorem 3.6.8. *[Logarithmic Distance for Critical Easels] If $T = (G, C, L)$ is a cycle-canvas and $T' = (G', C, L)$ is a critical easel for T, then $d(v, C) \leq 58 \log |C|$ for all $v \in V(G')$.*

Proof. Follows from Theorem 3.6.7 with $X = C$. □

Theorem 3.6.9. *[Exponential Growth for Critical Easels] Let $T = (G, C, L)$ be a cycle-canvas and $T' = (G', C, L)$ be a critical easel for T. If $v_0 \in V(G') \setminus V(C)$, then for all $k \leq d(v_0, C)$, $|N_k(v_0)| \geq 2^{k/58}$.*

Proof. Let $k \leq d(v_0, C)$. Now $N_k(v_0)$ separates v_0 from C. By Theorem 3.6.7, $k = d(v_0, N_k(v_0)) \leq 58 \log |N_k(v_0)|$. Hence $|N_k(v_0)| \geq 2^{k/58}$ as desired. □

3.7 Critical Path-Canvases

We note that Theorem 2.3.4 can be restated in terms of criticality.

Theorem 3.7.1. *If $T = (G, P, L)$ is a critical path-canvas with $|V(P)| = 3$, then T is a bellows.*

We prove that something akin to Theorem 3.3.1 amazingly holds for critical path-canvases.

Theorem 3.7.2. *(Path Chord or Tripod Theorem)*

If (G, P, L) is a critical path-canvas, C is the outer walk of G, then either

(1) *there is an edge of G that is not an edge of P but has both ends in P, or*

(2) *there is a chord of C with one end in the interior of P and the other end has list of size three but is not in P, or*

(3) *there are two distinct chords of C whose common end has a list of size four but is not in P and whose other ends are in the interior of P, (and the cycle made by the chords and the subpath of P connecting their ends has empty interior)*

(4) *there exists a vertex v with list of size five with at least three neighbors on P and all of the internal faces of $G[v \cup V(P)]$ is empty.*

Proof. Suppose none of the above hold. We now claim that every L-coloring of P extends to an L-coloring of G, contrary to the fact that T is critical. To see this, fix an L-coloring ϕ of P. Let $L'(v) = L(v) \setminus \{\phi(x) : x \in V(P), x \sim v\}$ for all $v \notin V(P)$.

Let v_1, v_2 (not necessarily distinct) be the vertices of the infinite face adjacent to the ends of P. Note then that $|L'(v)| \geq 3$ for all $v \neq v_1, v_2$. This follows because if $|L(v)| = 3$, then v does not have a neighbor in P as (2) does not hold, and if $|L(v)| = 4$, then v does not have two neighbors in P as (3) does not hold. If $|L(v)| = 5$, then v has at most two neighbors in P as otherwise (4) holds by the proof of Theorem 3.3.1. Thus, $|L'(v)| \geq |L(v)| - 2 \geq 3$.

Let $G' = G \setminus P$, $S' = \{v_1, v_2\}$ and $T' = (G', S', L')$. Suppose $v_1 = v_2$, then $|L'(v_1)| \geq 1$. By Theorem 1.4.2 applied to T', there exists an L'-coloring of G' and hence ϕ extends to an L-coloring of G as desired. So we may assume that $v_1 \neq v_2$. In this case, $|L'(v_1)|, |L'(v_2)| \geq 2$. By Theorem 2.2.2 applied to T', G' has an L'-coloring and hence ϕ extends to an L-coloring of G as desired. □

3.7.1 Deficiency

Definition. (Inlets)

Let (G, P, L) be a path-canvas and C be the outer walk of G. Suppose that the path P appears only once as a subwalk of C. Decompose the subwalk $C - P$ into the sequence of subwalks between vertices with lists of size less than five that are not in the interior of P. We call any such subwalk in the sequence which has length at least two an *inlet*. If i is an inlet, we let $|i|$ denote the length of the subwalk plus one. We let $\mathcal{I}(T)$ denote the set of inlets of T.

Definition. We define the deficiency of a path-canvas T as

$$def(T) = |V(P)| - 3 - \sum_{f \in \mathcal{F}(T)} (|f| - 3) - \sum_{i \in \mathcal{I}(T)} (|i| - 3)$$

.

Theorem 3.7.3. *(Path Sum of Faces Theorem)*

If $T = (G, P, L)$ is a critical path-canvas, then $def(T) \geq 0$.

Proof. We proceed by induction on the number of vertices of G. First we claim that G is 2-connected. Suppose not. Then there exists a cutvertex v of G. As v is essential we find that $v \in V(P)$. Thus v divides G into two graphs G_1, G_2 and P into two paths P_1, P_2 with $P_1 \subset V(G_1)$ and $P_2 \subset V(G_2)$. Let $T_i = (G_i, P_i, L)$. If $G_i \neq P_i$, then T_i is P_i-critical. Thus $|V(P_1)| + |V(P_2)| = |V(P)| + 1$. Moreover, $\sum_{f \in F(T_1)} (|f| - 3) + \sum_{f \in \mathcal{F}(T_2)} (|f| - 3) = \sum_{f \in \mathcal{F}(T)} (|f| - 3)$. Therefore, we need only deduce how the inlets of T_1, T_2 and T relate. Basically, the inlets of T are just the union of the inlets of T_1 and T_2, except that there could be a new inlet at v of size three, or there is an inlet of T_1 or T_2 incident with v which then gets lengthened by one, or there is an inlet in both T_1 and T_2 incident with v and they are then combined. In all cases, we find that $\sum_{i \in \mathcal{I}(T)} (|i| - 3) \leq \sum_{i \in \mathcal{I}(T_1)} (|i| - 3) + \sum_{i \in \mathcal{I}(T_2)} (|i| - 3) + 2$. Combining all these formula shows that $def(T) \geq def(T_1) + def(T_2) \geq 0 + 0 = 0$ as desired.

Apply Theorem 3.7.2. Suppose (1) holds; that is, P has a chord in G. If the chord is actually an edge between the two ends of P, we may apply Theorem 3.3.3, to find that $def(T) \geq 0$. So we may assume the chord is not between the ends of P. Let P'

be the resulting inlet and C' be the cycle made by the chord. Consider the resulting path and cycle-canvases. Now $|V(P)| = |V(P')| + |V(C')| - 2$. As $\mathcal{F}(T) \subseteq \mathcal{F}(T[P']) \cup \mathcal{F}(T[C'])$ and $\mathcal{I}(T) \subseteq \mathcal{I}(T[P'])$, we find that $def(T) = def(T[P']) + def(T[C']) + 1$.

By Theorem 3.3.3, $def(T[C']) \geq 1$ if $Int(C') \neq \emptyset$ and $def(T[C']) = 0$ if $Int(C') = \emptyset$. Similarly if $P' = G'$ then $def(T[P']) = 0$ and if $P' \neq G'$, then $def(T[P']) \geq 0$ by induction. In either case then $def(T[C']), def(T[P']) \geq 0$ and hence $def(T) \geq 0 + 0 + 1 = 1$ as desired.

Suppose (2) or (3) holds; that is there is a chord U of C where one end is an internal vertex of P and the other end a vertex not in P with list of size less than five. Let C_1, C_2 be cycles such that $C_1 \cap C_2 = U$ and $C_1 \cup C_2 = C \cup U$; let P_1 and P_2 be paths such that $P_1 \cap P_2 = U$ and $P_1 \cup P_2 = P \cup U$ and $P_1 \subset C_1, P_2 \subset C_2$. Hence $|V(P_1)| + |V(P_2)| = |V(P)| + 3$.

By Theorem 3.2.3, $T[P_1]$ and $T[P_2]$ are critical path-canvases (or empty). As $\mathcal{F}(T) \subseteq \mathcal{F}(T[P_1]) \cup \mathcal{F}(T[P_2])$ and $\mathcal{I}(T) \subseteq \mathcal{I}(T[P_1]) \cup \mathcal{I}(T[P_2])$. Thus $def(T) = def(T[P_1]) + def(T[P_2])$. By induction, $def(T[P_k]) \geq 0$ for $k \in \{1, 2\}$ even if $G_k = P_k$. So we find that $def(T) \geq 0$ as desired.

So we may suppose that (4) holds; that is, there exists v with list of size five such that v is adjacent to at least three vertices of C and all of the internal faces of $P \cup v$ are empty. Let P' be the new path. Consider the canvas $T[P'] = (G', P', L)$. Note that $\mathcal{I}(T) \subseteq \mathcal{I}(T[P'])$. Now, $|V(P)| - |V(P')| \geq \sum_{f \in \mathcal{F}(T) \setminus \mathcal{F}(T[P'])} (|f| - 3)$. Thus $def(T) \geq def(T[P'])$. If $P' \neq G'$, then by induction $def(T[P']) \geq 0$ and so $def(T) \geq 0$ as desired. So we may suppose that $P' = G'$. In this case, $def(T[P']) = 0$. Nevertheless as $P' = G$ though, v has at least five neighbors on the boundary and so $|V(P)| - |V(P')| \geq \sum_{f \in \mathcal{F}(T) \setminus \mathcal{F}(T[P'])} (|f| - 3) + 2$ and so $def(T) \geq def(T[P']) + 2$. As $def(T[P']) = 0$, $def(T) \geq 2$ as desired. □

Corollary 3.7.4. *(Path Bounded Face Theorem)*

Let $T = (G, P, L)$ be a critical path-canvas. If f is an internal face of G, then

$|f| \leq |V(P)|$*, and if* i *is an inlet, then* $|i| \leq |V(P)|$*.*

Proof. By Theorem 3.7.3, $|V(P)| - 3 - \sum_{f \in \mathcal{F}(T)}(|f| - 3) - \sum_{i \in \mathcal{I}(T)}(|i| - 3) \geq 0$. As the terms on the right side are always positive, $|V(P)| - 3 \geq |f| - 3$ for any internal face f of G. Thus $|f| \leq |V(P)|$. Similarly $|V(P)| - 3 \geq |i| - 3$ for any inlet i. Thus $|i| \leq |V(P)|$. □

Lemma 3.7.5. *Let* $T = (G, P, L)$ *be a critical path-canvas. If* i *is an inlet with* $|i| \geq |V(P)| - 1$*, then* $|i| = |V(P)| - 1$ *and either* G *is* P *plus a bellows whose base is the first three or last three vertices of* P*, or,* G *is* P *plus an edge between two vertices of* P *which have distance two in* P*.*

Proof. We proceed by induction on the number of vertices of G. By Theorem 1.4.2, $|V(P)| \geq 3$. If $|V(P)| = 3$, then G is a bellows by Theorem 2.3.4 and the lemma follows. So we may assume that $|V(P)| \geq 4$. Apply Theorem 3.7.2 to T.

Suppose (1) holds. Then there is an edge e of G not in P but with both ends in P. Consider $P + e$ which has precisely one inlet, call it i with path P'. If the two ends do not have distance at most two in P, then $|i| = |V(P')| \leq |V(P)| - 2$. If $Ext(P') = \emptyset$, then the lemma follows immediately. Otherwise, by induction on $T[P']$, every inlet has size at most $|V(P')| - 1 \leq |V(P)| - 3$ and the lemma follows. So we may suppose that the ends have distance two in P and hence $|V(P')| = |V(P)| - 1$. If $Ext(P') = \neq \emptyset$, then by induction, every inlet has size at most $|V(P')| - 1 = |V(P)| - 2$ and the lemma follows. So we may suppose that $Ext(P') = \emptyset$ and hence $G = P + e$ and the lemma follows.

Suppose (2) or (3) holds. That is, there is a chord of C, the outer cycle of G, with one end an internal vertex of P and the other v not in P but with a list of size less than five. Let P_1, P_2 be the resulting paths. Thus $|V(P_1)| + |V(P_2)| = |V(P)| + 3$. As both P_1, P_2 have at least three vertices, $|V(P_1)|, |V(P_2)| \leq |V(P)|$. If both have size at most $|V(P)| - 1$, then the lemma follows unless one, say P_1, has size $|V(P)| - 1$

and empty exterior. In that case, the other, say P_2, has size 4 and it is not hard to see that what remains must be a bellows with base $P_2 \setminus v$.

So we may assume without loss of generality that $|V(P_1)| = |V(P)|$. It follows from Theorem 1.4.2 that v must be incident with an edge in $Ext(P_1)$ (besides the chord). Yet by induction on P_1, either G is $P_1 + e$ where e is an edge with both ends in P distance two, or G is a bellows whose base is the first or last three vertices of P_1. In the first case, v must be incident with e and hence the last three vertices must be the base of a bellows. In the latter case, again the last three vertices must be the base of a bellows.

Suppose (4) holds. That is, there is a vertex v with list of size five and at least three neighbors on P and all of the internal faces of $G[P \cup v]$ are empty. Let i be the inlet of $G[P \cup v]$ and P' its path. If v has at least five neighbors on P, then $|i| = |V(P')| \leq |V(P)| - 2$. Thus the lemma holds immediately if $Ext(P') = \emptyset$ and by induction otherwise. If v has four neighbors on P, then $|i| = |V(P')| = |V(P)| - 1$. If $Ext(P') \neq \emptyset$, the lemma follows by induction. Yet if $Ext(P') = \emptyset$, then v has degree four but a list of size five and hence G is not P-critical, a contradiction.

So we may assume that v has three neighbors on P. Moreover by a similar argument it follows that these vertices are consecutive in P and that $|i| = |V(P')| = |V(P)|$. By induction applied to P', we find that either there is an edge with both ends in P or a bellows whose base if the first or last three vertices of P. Yet as v has degree three and a list of size five, v must be incident with at least two other edges. So we may assume there is a bellows whose base is the first or last three vertices of P. Indeed, v must be in the base of that bellows. Moreover, v must be incident with two vertices, so the bellows is actually a fan. It is easy to see then that v must be a tripod for either the second or second to last vertex of P and thus T is P plus a bellows on the first or last three vertices of P. $\square$

3.8 *Linear Bound for Paths*

Definition. Let $T = (G, P, L)$ be a path-canvas. We say a vertex $v \in V(G)$ is *superfluous* if $v \notin V(P)$ and there exists a span P' in G, $|V(P')| = 3$ such that $v \in Ext(P')$. We say a vertex is *substantial* if it is not superfluous. We define the *truncation* of G, denoted by G^*, to be the subgraph of G induced by the substantial vertices of G. We define the *truncated outer walk*, denoted by C^*, to be the outer walk of G^*.

Note that if $|V(P)| = 3$, then $V(G^*) = V(P)$ trivially. We will proceed to show that the number of vertices in G^* is linear in the size of P. That is, the number of vertices not in long fans on the boundary is linear in the size of P. Such a linear bound for critical path-canvases will be instrumental for characterizing the structure of canvases when S is not just one component. We shall first prove that $|V(C^*)|$ is linear in $|V(P)|$.

Let $\overline{C} = \{v \in V(C) \| |L(v)| < 5\}$. We say that a chord of $\overline{C}$ involving an internal vertex u in P and a vertex not in P is a *short chord*, if u is adjacent to an end vertex of P. Let $R(T) = V(C^* \setminus P)$ and $r(T) = |R(T)|$.

Theorem 3.8.1. *Let $\epsilon \leq 1/19$ and $\gamma = 2\epsilon$. If $T = (G, P, L)$ is a critical path-canvas with no short chord and $|V(P)| \geq 4$, then $\epsilon r(T) \leq |V(P)| - 3 - \gamma$.*

Proof. We proceed by induction on the number of vertices. First we claim that if $|V(P)| = 4$, that $r(T) \leq 3$. We may assume that P is induced as otherwise $r(T) = 0$ and the claim follows. Now if there exists $v \notin V(P)$ such that v is adjacent to all vertices of P, then $r(T) = 1$ and the theorem follows. So now we may suppose without loss of generality that $P = p_1p_2p_3p_4$ and there does not exist a vertex adjacent to all of p_2, p_3, p_4. Let $u \neq p_3$ such that up_4 is an edge in the outer walk C of G.

Now we may assume that $|V(P)| \geq 5$ as $3\epsilon + \gamma \leq 1$. Let ϕ be a non-extendable coloring of P. Consider $T' = (G \setminus \{p_2, p_3, p_4\}, \{p_1, u\}, L')$ where $L'(v) = L(v) \setminus$

$\{\phi(p_i) | i \in \{2,3,4\}, v \sim p_i\}$. Now $|L'(u)| \geq 2$. By Theorem 2.7.8, there exists a harmonica T'' from p_1 to u where $T'' = (G', \{p_1, u\}, L')$ and $G' \subseteq G \setminus \{p_2, p_3, p_4\}$. As T'' is a harmonica, $|L'(v)| = 3$ for all $v \in G' \setminus \{p_1, u\}$. But this implies that $|V(G') \setminus V(C)| \leq 2$, because there can be one vertex in G' adjacent to p_2, p_3 and another one adjacent to p_3, p_4. However, we now find that $r(T) \leq 3$ and the claim follows.

Apply Theorem 3.7.2. Suppose (1) holds. That is, $G = C$; then $V(C^* \setminus P) = \emptyset$ and the formula follows.

Let $P = p_1 \dots p_k$. Suppose (2) or (3) holds. That is, $\overline{C}$ has a chord $U = p_i v$ where $p_i \in V(P)$. By assumption U is not a short chord; that is, $3 \leq i \leq k-2$. Let $v_1, v_2 \in \overline{C}$ be neighbors of p_i such that v_1 is closest to p_1 and v_2 is closest to p_k. Let $P_1 = p_1 \dots p_i v_1$ and $P_2 = v_2 p_i \dots p_k$. Now $|V(P)| + 3 = |V(P_1)| + |V(P_2)|$. Moreover, as v_1 chosen closest to p_1, $T[P_1]$ has no short chord. By induction, $\epsilon r(T[P_1]) \leq |V(P_1)| - 3 - \gamma$. Similarly, $\epsilon r(T[P_2]) \leq |V(P_2)| - 3 - \gamma$.

Yet $r(T) \leq r(T[P_1]) + r(T[P_2]) + 2$. Hence $\epsilon r(T) \leq |V(P_1)| - 3 + |V(P_2)| - 3 - 2\gamma + 2\epsilon = |V(P)| - 3 - \gamma + (2\epsilon - \gamma)$. As $\gamma \geq 2\epsilon$, the formula holds as desired.

So we may suppose that (4) holds. That is, T has a tripod. If T has a tripod for p_i where $3 \leq i \leq k-2$, we apply induction to $P[v]$. We find that $\epsilon r(T) \leq |V(P)| - 3 - \gamma$ as desired. So we may assume that T has no tripod for p_i, $3 \leq i \leq k-2$.

Let ϕ be a coloring of P. Let $P_{Int} = P \setminus \{p_1, p_k\}$ and $L'(v) = L(v) \setminus \{\phi(p) : p \in V(P_{Int}), p \sim v\}$ for all $v \in V(G) \setminus V(P)$. As T has no tripod for p_i, $3 \leq i \leq k-2$ and no short chord, then $T' = (G \setminus P_{Int}, \{p_1, p_k\}, L')$ is a canvas. By Theorem 2.8.3, T' contains an orchestra T'' from p_1 to p_k. Let C' be the walk in T'' from p_1 to p_k which is not a subwalk of the outer walk in T.

Suppose $T'' = (G'', \{p_1, p_k\}, L')$ is a special orchestra with cut-edge $u_1 u_2$, where u_1 separates p_1 from u_2. Thus there are harmonicas from p_1 to u_1 and from p_2 to u_2. As there are at least $d_{G''}(p_1, p_k)/2$ vertices in $V(G'') \setminus V(C)$ such that $|L'(v)| = 3$, we find that $d_{G''}(p_1, p_k) \leq 2(k-1)$. However, $r(T) \leq d_{G''}(p_1, p_k)$ unless $u_1, u_2 \notin V(C)$. In that case, we find that $r(T) \leq d_{G''}(p_1, p_k) + 3$ by applying the claim about the case when $|V(P)| = 4$ to the inlet I of $P \cup G''$ of size four with $u_1, u_2 \in V(I)$. Thus $r(T) \leq 2k+1$. Hence, $\epsilon r(T) \leq \epsilon(2k+1)$. Now this will be at most $k - 3 - \gamma$ as desired as long as $3 + 2\epsilon + \gamma \leq (1 - 2\epsilon)k$. Yet $k = |V(P)| \geq 5$ and hence it is sufficient to require that $12\epsilon + \gamma \leq 2$.

So we may suppose that T'' is an instrumental orchestra. By Lemma 2.10.13, there are at most four vertices with list of size at least four in C'. Hence, $|V(C')| \leq 3|V(P)| + 3$, because we have to account for cutvertices and cutedges in T'' which were already on the boundary of C. But then $|V(C^* \setminus P)| \leq 2|V(C')|$ as every vertex in $V(C^* \setminus P) \setminus V(C')$ would have to be the center of a double bellow or defective double bellows or the hinge of a bellows. Thus $|V(C^* \setminus P)| \leq 6|V(P)| + 6$. So $\epsilon R(T) \leq 6\epsilon|V(P)| + 6\epsilon$. Now this will be at most $|V(P)| - 3 - \gamma$ as desired as long as $3 + 6\epsilon + \gamma \leq (1 - 6\epsilon)|V(P)|$. Yet $|V(P)| \geq 5$ and hence it is sufficient to require that $36\epsilon + \gamma \leq 2$. □

Corollary 3.8.2. *If T is a critical path-canvas, then $|V(C^*)| \leq 20|V(P)|$.*

Theorem 3.8.3. *If T is a critical path-canvas, then $|V(G^*)| \leq 580|V(P)|$.*

Proof. By Corollary 3.8.2, $|V(C^*)| \leq 33|V(P)|$. But then $T[C^*]$ is a critical cycle-canvas. By the linear bound for critical cycle-canvases, $|V(G^*)| \leq 29|V(C^*)| \leq 580|V(P)|$. □

3.9 Logarithmic Distance for Paths

Definition. Let $T = (G, S, L)$ be a canvas. Let γ be a closed curve in the plane such that γ intersects G only at vertices of G. We say that γ is a *slicer* of T if there is no vertex of S in the interior of the disk whose boundary is γ. Let C be the outer walk of G. Let k be the number of times C and γ cross (as opposed to intersect). Define the *dimension* of γ to be $k/2$.

If γ is a slicer of T, we define a canvas $T_\gamma = (G', S', L)$ as follows. Let G' be the graph obtained by intersecting G with the closed disk bounded by γ. Let S' be the graph obtained by intersecting G with γ. We say that T_γ is a *slice* and define its *dimension* to be the minimum of the dimension of γ' over all slicers γ' of T such $T_\gamma = T_{\gamma'}$. We also say that S' is the *boundary* of the slice.

Note that if T' is a slice of a critical canvas T, then T' is also critical by Lemma 3.2.1.

Lemma 3.9.1. *Let $T = (G, S, L)$ be a critical canvas. Suppose there exists a path P in the outer walk of G such that $S \subseteq P$ and for all $v \in V(P) \setminus V(S)$, $|L(v)| = 5$, then $|V(G)| \leq 1160|V(S)|$.*

Proof. For every two consecutive vertices of S in P, add a new vertex adjacent to only those two vertices. Let P' be the path on the new vertices and the vertices of S. Let G' be the graph with vertex set $V(P') \cup V(G)$ and edge set $E(P') \cup E(C)$. For every $v \in V(P') \setminus V(S)$, let $L(v)$ be any set of five colors. Now (G', P', L) is a critical path-canvas. By Theorem 3.8.3, $|V(G')| \leq 580|V(P')|$. Hence, $|V(G)| \leq 1160|V(S)|$ as desired. □

Corollary 3.9.2. *Let $T = (G, S, L)$ be a critical canvas. If $T' = (G', S', L)$ is a slice of T of dimension at most one, then $|V(G')| \leq 1160|V(S')|$.*

Theorem 3.9.3. *Let $T = (G, P, L)$ is a critical path-canvas and $X \subseteq V(G)$ separate G into two graphs G_1, G_2 such that $G_2 \cap P \subseteq X$, then $|V(G_2)| \leq 1160|X|$.*

Proof. Let G' be the union of X and all components of $G \setminus X$ that do not contain a vertex in P. Let X' be all vertices in G' that are in $S \cup V(C)$ or have a neighbor in $V(G) \setminus V(G')$. Clearly, $X' \subseteq X$ as $S \cup V(C) \subseteq G_1$. Note the vertices of X' lie on the outer face of G' as they are either in P or have a path to a vertex of P through $G \setminus G'$ and yet the vertices of P lie on the outer face of G. It follows that $T' = (G', X', L)$ is a slice of T of dimension one. Furthermore T' is critical. By

Theorem 3.9.2, $|V(G')| \leq 1160|X'|$. Hence $|V(G_2)| \leq |V(G')| \leq 1160|X'| \leq 1160|X|$ as desired. $\square$

Theorem 3.9.4. *If $T = (G, P, L)$ is a critical path-canvas, $v_0 \in V(G)$ and $X \subseteq V(G)$ such that X separates v from P, then $d(v_0, X) \leq 2320 \log |X|$.*

Proof. We proceed by induction on the size of X. Let G_1, G_2 be graphs such that $X = V(G_1) \cap V(G_2)$, $C \subseteq G_1$ and $v_0 \in V(G_2)$. By Theorem 1.4.2, $|X| \geq 2$ as T is critical. Thus, we may assume that $d(v_0, X) > 2320$, as otherwise the theorem follows.

Let $X_i = \{v \in V(G_2) | d(v, X) = i\}$ and let $H_i = G[\bigcup_{j \geq i} X_j]$. As $|V(G_2)| \leq 1914|V(X)|$ by Theorem 3.9.3, there exists i, $1 \leq i \leq 320$ such that $|X_i| \leq |X|/2$. As $d(v_0, X) > 2320$, X_i separates v_0 from C. By induction on X_i, $d(v_0, X_i) \leq 1160 \log |X_i| \leq 1160 \log |X| - 1160$. Yet $d(v, X) \leq 1160$ for all $v \in X_i$ and hence $d(v_0, X) \leq 1160 \log \mathcal{C}$ as desired. $\square$

Theorem 3.9.5. *[Logarithmic Distance for Path-Canvases] If $T = (G, P, L)$ is a critical path-canvas, then $d(v, P) \leq 2320 \log |P|$ for all $v \in V(G)$.*

Proof. Follows from Theorem 3.6.3 with $X = P$. $\square$

Theorem 3.9.6. *[Exponential Growth for Path-Canvases] If $T = (G, P, L)$ is a critical path-canvas and $v_0 \in V(G) \setminus V(P)$, then for all $k \leq d(v_0, P)$, $|N_k(v_0)| \geq 2^{k/2320}$.*

Proof. Let $k \leq d(v_0, P)$. Now $N_k(v_0)$ separates v_0 from C. By Theorem 3.9.4, $k = d(v_0, N_k(v_0)) \leq 58 \log |N_k(v_0)|$. Hence $|N_k(v_0)| \geq 2^{k/2320}$ as desired. $\square$

3.10 Bottleneck Theorem for Two Paths

Definition. Let $T = (G, S, L)$ be a critical canvas. We say a vertex $v \in S$ is *relaxed* if there exist two L-coloring ϕ_1, ϕ_2 of S such that ϕ_1, ϕ_2 do not extend to G, $\phi_1(v) \neq \phi_2(v)$ and $\phi_1(w) = \phi_2(w)$ for all $w \in S \setminus \{v\}$.

Theorem 3.10.1. *If $T = (G, P_1 \cup P_2, L)$ is a connected critical canvas, where P_1, P_2 are disjoint paths of the outer walk C of G such that $d(Int(P_1), Int(P_2)) \geq \Omega(|P_1|\log|P_1| + |P_2|\log|P_2|)$, then there exists an essential chord of C whose ends have lists of size less than five and are not in $P_1 \cup P_2$. (If $|P_1|$ or $|P_2|$ at most two, then measure distance to P_1 or P_2 respectively).*

Proof. Let us proceed by induction on $|V(P_1)| + |V(P_2)|$. Suppose without loss of generality that $|V(P_1)| \geq |V(P_2)|$. We may assume that $|V(P_1)| \geq 3$ as otherwise the theorem follows from Theorem 2.11.1.

We now prove a stronger statement. For $i \in \{1, 2\}$, let R_i be the set of relaxed vertices of $Int(P_i)$ and $S_i = Int(P_i) \setminus R_i$. If $|V(P)| = 2$, let $d_r(T) = \min\{d(R_1, P_2) + 1, d(S_1, P_2)\}$. If $|V(P)| \geq 3$, let $d_r(T) = \min\{d(R_1, R_2) + 2, d(R_1, S_2) + 1, d(S_1, R_2) + 1, d(S_1, S_2)\}$.

Let $f(m_1, m_2) = 2320(m_1 \log m_1 + m_2 \log m_2)$.

We now prove that

$$d_r(T) \leq f(|V(P_1)|, |V(P_2)|) + 4.$$

Let $T = (G, P_1 \cup P_2, L)$ be a counterexample to the formula above with a minimum number of vertices where $|V(P_1)| \geq |V(P_2)|$ without loss of generality. Let $k_1 = |V(P_1)|$ and $k_2 = |V(P_2)|$. Hence $d(P_1, P_2) > f(k_1, k_2)$. Let C be the outer walk of G.

Claim 3.10.2. *For $i \in \{1, 2\}$, there does not exist $G_i \subseteq G$ such that $G_i \cap P_{3-i} = \emptyset$ and (G_i, P_i, L) is a critical canvas.*

Proof. Suppose not. Now $k_i \geq 3$ by Theorem 1.4.2. Furthermore, either P_{3-i} is contained in $Ext(I)$ where I is an inlet of G_i, or, there exists an edge e in G_i which is an essential chord U of T whose ends both have lists of size less than five. Suppose the latter. As T is a counterexample, e must be incident with a vertex of P_i. Thus

$d(U, P_i) \leq 1$. By induction, it follows that $d(P_{3-i}, U) \leq f(k_{3-i}, 2)$; hence $d(P_1, P_2) \leq f(k_{3-i}, 2) + 2 \leq f(k_{3-i}, k_i)$ as $k_i \geq 3$, a contradiction.

So suppose the former. By Lemma 3.7.5, $|I| < |V(P_i)|$. Apply induction to the canvas between I and P_{3-i}. Thus $d(I, P_{3-i}) \leq f(|I|, k_{3-i})$. By Theorem 3.9.5, $d(v, P_i) \leq 2320 \log k_i$. Hence $d(P_1, P_2) \leq f(k_1, k_2)$, a contradiction. □

Hence there does not exist an edge e not in P_i with both ends of e in P_i. Let $P_1 = p_1 p_2 \ldots p_{k_1}$. Let v_1, v_2 be the vertices of C adjacent to P_1 where $p_1 v_1, p_{k_1} v_2 \in E(C)$.

Claim 3.10.3. *$N(v_1) = \{p_1\}$, $N(v_2) = \{p_{k_1}\}$, and neither v_1 nor v_2 is in a chord of C or is a cutvertex of G (and hence $v_1 \neq v_2$).*

Proof. It suffices by symmetry to prove the claim for v_1. Suppose v_1 is a cutvertex of G. As T is critical, v is an essential cutvertex. By induction applied to the canvas between v and P_2, we find that $d(v_1, P_2) \leq f(1, k_2)$ and hence $d(P_1, P_2) \leq f(1, k_2) + 1$, a contradiction.

Similarly if v_1 is in a chord $U = v_1 v$ of C, U is an essential chord of C. Suppose that $v \neq p_2$. If $v \notin P_1$, let $P_1' = U$ and let P_1' be the union of U and the path from v to p_k otherwise. Note that $|V(P_1')| \leq k_1 - 1$ as $v \neq p_2$. Apply induction to the canvas between P_1' and P_2 to find that $d(U, P_2) \leq f(k_1 - 1, k_2)$ and hence $d(P_1, P_2) \leq f(2, k_2) + 2$, a contradiction.

So we may suppose that $v = p_2$. Let $U' = p_2 u_1$ be the chord of C with u_1 on the path from v_1 to P' and u_1 closest to P'. As T is critical, $p_1 p_2 u_1$ is the base of a bellows W. Let $P_1' = u_1 p_2 p_3 \ldots p_{k_1}$. Consider the canvas $T' = (G', P_1' \cup P_2, L)$ from P_1' to P_2. Now T' is critical. As T is a minimum counterexample, $d_r(T) \leq f(k_1, k_2) + 4$.

We claim that $R(P_1) \setminus \{p_2\} \subseteq R(P_1') \setminus \{p_2\}$. To see this, let $u \in R(P_1) \setminus \{p_2\}$. Thus there exist two L-colorings ϕ_1, ϕ_2 of $P_1 \cup P_2$ that do not extend to an L-coloring of G such that $\phi_1(u) \neq \phi_2(u)$ and $\phi_1 = \phi_2$ otherwise. By Theorem 1.4.2, ϕ_1 extends to an L-coloring ϕ of W. Let $\phi_1(u_1) = \phi_2(u_1) = \phi(u_1)$. As $\phi_1 = \phi_2(w)$ for all $w \neq u$. Now

ϕ_1, ϕ_2 are L-colorings of $P_1' \cup P_2$ that do not extend to an L-coloring of $G \setminus \{\}$ such that $\phi_1(u) \neq \phi_2(u)$ and $\phi_1 = \phi_2$ otherwise. Thus u is relaxed for T'. So $u \in R(P_1') \setminus \{p_2\}$ as claimed.

If $R(P_1) \subseteq R(P_1')$, then it follows that $d_r(T) \leq d_r(T)$ and hence $d_r(T) \leq f(k_1, k_2)+$ 4, a contradiction. By the claim of the last paragraph then, we may assume that p_2 is relaxed in T and yet p_2 is not relaxed in T'. Thus there exist two L-colorings ϕ_1, ϕ_2 of $P_1 \cup P_2$ that do not extend to an L-coloring of G such that $\phi_1(p_2) \neq \phi_2(p_2)$ and $\phi_1 = \phi_2$ otherwise. Yet ϕ_1 extends to an L-coloring ϕ_1' of W by Theorem 1.4.2. If $\phi_1'(u_1) \neq \phi_2(p_2)$, then as argued above, it follows that p_2 is relaxed, a contradiction. So $\phi_1'(u_1) = \phi_2(p_2)$. Similarly, we find that ϕ_2 extends to an L-coloring ϕ_2' of W and $\phi_2'(u_1) = \phi_1(p_2)$.

Thus $\phi_1(p_2), \phi_2(p_2) \in L(u_1)$. Note that $\phi_1(p_3) \neq \phi_1(p_2), \phi_2(p_2)$. Consider in T', the democratic reduction $T'' = (G'', P_1'' \cup P_2, L'')$ of p_2, u_1 with respect to $\{\phi_1(p_2), \phi_2(p_2)\}$ centered around p_3. Now there exists a critical subcanvas T_0 of T''.

Suppose that T_0 is connected. First suppose there exists a chord of T_0 whose ends have lists from L'' of size less than three and are not in $P_1'' \cup P_2$. Let U_0 be such a chord closest to P_2. Now U_0 is not a chord of C as T is a counterexample. Hence at least one of its ends is adjacent to either p_2 or u_1. Thus $d(P_1, U_0) \leq 2$. Yet by induction, $d(U_0, P_2) \leq f(2, k_2)$ and hence $d(P_1, P_2) \leq f(k_1, k_2)$, a contradiction. So we may suppose there is no such chord. By induction, $d(P_1'', P_2) \leq f(k_1 - 2, k_2)$, a contradiction.

So we may suppose that T_0 is not connected. Now it follows from Claim 3.10.2 that the component of T_0 containing P_2 is just P_2. But then there exists an inlet I, $|I| < |P_1'|$ of the component of T_0 containing P_1' separating P_1'' from P_2 in T''. Hence there exists a path P_0, $|V(P_0)| \leq k_1 - 1$ vertices with $d(v, P_1) \leq \log k_1$ for all vertices $v \in V(P_0)$. By induction $d(P_0, P_2) \leq f(k_1 - 1, k_2)$ and hence $d(P_1, P_2) \leq f(k_1, k_2)$, a contradiction. $\square$

Consider the Thomassen reductions $T_1 = (G_1, P_1 \cup P_2, L_1)$ and $T_2 = (G_2, P_1 \cup P_2, L_2)$, of v_1 and v_2 respectively. As T is critical, there exist critical subcanvases $T'_1 = (G'_1, P_1 \cup P_2, L_1)$ and $T'_2 = (G'_2, P_1 \cup P_2, L_2)$ of T_1 and T_2, respectively.

Claim 3.10.4. *For $i \in \{1, 2\}$, T'_i is disconnected.*

Proof. Suppose not. Suppose without loss of generality that T'_1 is connected. As T is a minimum counterexample, it follows that there exists an essential chord $U = u_1u_2$ of T'_1 whose ends have lists of size less than five and are not in $P_1 \cup P_2$. We may assume that U is such a chord closest to P_2. As T is a counterexample, U is not a chord of C. So we may suppose without loss of generality that $u_1 \notin V(C)$. Yet $|L_1(u_1)| < 5$. Thus u_1 is adjacent to v_1.

Consider the subcanvas T''_1 of T'_1 from U to P_2. Moreover, T''_1 is critical. As U was chosen closest to P_2, there does not exist an essential chord U' of T''_1 whose ends have lists of size less than five and are not in $P_1 \cup P_2$. But then as T is a minimum counterexample, we find that $d(U, P_2) \leq f(2, k_2)$. Yet $d(U, P_1) \leq 2$ as u_1 is adjacent to v_1. Hence $d(P_1, P_2) \leq f(k_1, k_2)$, a contradiction. □

It now follows that that the component G_0 of G'_i containing P_2 is just P_2. Suppose not. Then $G_0 \cap P_1 = \emptyset$ and (G_0, P_2, L) is a critical canvas. It follows from Claim 3.10.2 that there exists a vertex $u \in G_0$ such that $L(u) \neq L_1(u)$. Thus u is adjacent to v_1. So $d(u, P_1) \leq 2$ and yet $d(u, P_2) \leq 2320 \log |P_2|$ by Theorem 3.9.5. Hence, $d(P_1, P_2) \leq f(k_1, k_2)$, a contradiction.

We prove the following useful claim.

Claim 3.10.5. *For all i where $2 \leq k_1 - 1$, there does not exist a vertex v adjacent to p_{i-1}, p_i, p_{i+1}.*

Proof. Suppose not. By Claim 3.10.3 that $v \notin V(C)$ and hence $|L(v)| = 5$.

Let P'_1 be the path obtained from P_1 by replacing p_i with v. Now $|P'_1| = k_1$. Consider the canvas $T' = (G \setminus \{p_i\}, P'_1 \cup P_2, L)$ between P'_1 and P_2. Now T' is critical.

As T is a counterexample, there cannot exist a chord of the outer walk of T' whose ends have lists of size less than five and are not in $P_1' \cup P_2$. Thus as T is a minimum counterexample to the formula above, we find that $d_r(T') \leq f(k_1, k_2) + 4$.

Now we claim that v is relaxed in T'. Let ϕ be an L-coloring of $P_1 \cup P_2$ that does not extend to an L-coloring of G. Let $S(v) = L(v) \setminus \{\phi(p_{i-1}), \phi(p_i), \phi(p_{i+1})\}$. Note then that $|S(v)| \geq 2$ as $|L(v)| = 5$. Let $c_1, c_2 \in S(v)$. For $i \in \{1, 2\}$, let $\phi_i(v) = c_i$ and $\phi_i = \phi$ otherwise. Hence ϕ_1, ϕ_2 are L-colorings of $P_1' \cup P_2$ that do not extend to an L-coloring of $G \setminus \{p_i\}$ such that $\phi_1(v) \neq \phi_2(v)$ but $\phi_1 = \phi_2$ otherwise. So v is relaxed as claimed.

Next we claim that $R(P_1) \subseteq R(P_1') \setminus \{v\}$. To see this, let $u \in R(P_1)$. Thus there exist two L-colorings ϕ_1, ϕ_2 of $P_1 \cup P_2$ that do not extend to an L-coloring of G such that $\phi_1(u) \neq \phi_2(u)$ and $\phi_1 = \phi_2$ otherwise.

Suppose $u \neq p_i$. Let $S(v) = L(v) \setminus \{\phi_1(p_{i-1}), \phi_2(p_{i-1}), \phi_1(p_i), \phi_2(p_i), \phi_1(p_{i+1}), \phi_2(p_{i+1})\}$. As $\phi_1 = \phi_2(w)$ for all $w \neq u$, we find that $|S(v)| \geq 1$ as $|L(v)| = 5$. Let $c \in S(v)$ and $\phi_1(v) = \phi_2(v) = c$. Now ϕ_1, ϕ_2 are L-colorings of $P_1' \cup P_2$ that do not extend to an L-coloring of $G \setminus \{p_i\}$ such that $\phi_1(u) \neq \phi_2(u)$ and $\phi_1 = \phi_2$ otherwise. Thus u is relaxed for T'. So $u \in R(P_1') \setminus \{v\}$ as claimed.

Suppose $u = p_i$. If $\phi_1(p_{i-1}) = \phi_1(p_{i+1})$, let G' be obtained from G by deleting p_i and identifying p_{i-1} and p_{i+1} to a single vertex. If $\phi_1(p_{i-1}) \neq \phi_1(p_{i+1})$, let G' be obtained from G by deleting p_i and adding an edge between p_{i-1} and p_{i+1}. Let P_1' be the resulting path on $P_1 \setminus \{p_i\}$. Consider $T' = (G', P_1' \cup P_2, L)$. Now there does not exist an L'-coloring of G that extends ϕ_1.

Hence T' contains a critical subcanvas T''. If T'' is connected, then $d(P_1, P_2) \leq d(P_1', P_2) \leq f(k_1 - 1, k_2)$, a contradiction. If T'' is not connected, then there exists $G_1 \subseteq G$ such that $G_1 \cap P_2 = \emptyset$ and (G_1, P_1, L) is a critical canvas, contradicting Claim 3.10.2. Thus $R(P_1) \subset R(P_1') \setminus \{v\}$ as claimed.

But now it follows that $d_r(T) \leq d_r(T')$ and hence $d_r(T) \leq f(k_1, k_2) + 4$, contrary to the fact that T was a counterexample to this formula. □

Claim 3.10.6. *For $i \in \{1,2\}$, T_i' is a bellows with base $p_1p_2p_3$ or base $p_{k_1-2}p_{k_1-1}p_{k_1}$.*

Proof. Suppose not. It suffices to prove the claim for T_1'. If there exists a chord of T_1' which separates P_1 from P_2 in T_1, then we obtain a contradiction as in Claim 3.10.4. So we may assume that there is an inlet I of T_1' which separates P_1 from P_2. Suppose $|I| \le k_1 - 2$. Consequently, there is a path P_1' in G with size at most $k_1 - 1$ that separates P_1 from P_2 such that $d(v, P_1) \le 2320 \log k_1$ for all $v \in V(P_1')$. As T is a counterexample, there does not exist an essential chord whose ends have lists of size less than five and are not in $P_1 \cup P_2$. By induction, it follows that $d(P_1', P_2) \le f(|V(P_1')|, k_2)$ and hence $d(P_1, P_2) \le f(k_1, k_2)$ as $|V(P_1')| \le k_1 - 1$, a contradiction.

So we may assume that $|I| \ge k_1 - 1$. By Lemma 3.7.5, $|I| = k_1 - 1$ and T_1' is a bellows whose base is the first or last three vertices of P_1. □

Claim 3.10.7. $k_1 = 3$.

Proof. Suppose not. Hence $k_1 \ge 4$. By Claim 3.10.6, T_1' is a bellows with base $p_{i-1}p_ip_{i+1}$ for $i \in \{2, k_1 - 2\}$. By Claim 3.10.5 there does not exist a vertex v of P_1 adjacent to p_{i-1}, p_i, p_{i+1}. Thus there exists a chord $U = p_iv$ of T_1' where $v \notin V(P_1)$. By Claim 3.10.3, $v \notin V(C)$. Yet $|L_i(v)| = 3$ as T_1' is a bellows. Thus v is adjacent to v_1.

Consider the path P' from v to p_{i+1} in outer walk of T_1' avoiding p_i; let v' be the closest vertex of P' to p_{i+1}, as measured in P', such that v' is adjacent to v_1. Note that $v' \notin V(C)$ by Claim 3.10.3. Let v'' be the neighbor of v' in P' closer to p_{k_1}. Given how v' was chosen, it follows that $v'' \in V(C)$. Now $P_1'' = v_1vv''$ is a path on three vertices separating P_1 from P_2. Moreover, $d(P_1'', P_1) \le 1$ and $d(P_1'', P_2) \le f(3, k_2)$ by induction. Hence $d(P_1, P_2) \le f(k_1, k_2)$ as $k_1 \ge 4$, a contradiction. □

Thus by Claim 3.10.6, T_1' and T_2' are bellows with base $P_1 = p_1p_2p_3$. By Claim 3.10.5, there does not exist $v \notin V(C)$ such that $v \sim p_1, p_2, p_3$. Hence for $i \in \{1,2\}$, T_i' is not a turbofan and thus there exists a chord of T_i'. For $i \in \{1,2\}$, let $U_i = p_2x_i$ be a chord of T_i'. By Claim 3.10.3, $x_i \notin V(C)$. Thus x_i is adjacent to v_i. Furthermore as there are no vertices in the interior of the 4-cycles $p_2x_1v_1p_1$ and $p_2x_2v_2p_3$, we find that $x_1 \sim p_1$ and $x_2 \sim p_3$.

By Claim 3.10.5, we find that $x_1 \neq x_2$. Indeed, it follows that U_i is the only chord of T_i' for $i \in \{1,2\}$. So consider the bellows T_1'' in T_1' with base $x_1p_2p_3$. Now T_1'' must be a turbofan and hence x_2 is the center of its wheel. That is, $x_2 \sim x_1$ and $x_1x_2p_3$ is the base of an even fan. Let $x_3 \neq p_2$ such that $x_3 \sim x_1, x_2$. By symmetry there also exists a turbofan T_2'' in T_2' with base $x_2p_2p_1$ where x_1 is the center of its wheel and $x_2x_1p_1$ is the base of an even fan. Moreover, x_3 is in both T_1'' and T_2''.

If x_3 is not adjacent to v_1, then the edge in the outer walk of T_1'' incident with x_3 but not with x_1 is a chord whose ends have lists of size less than five but are not in $P_1 \cup P_2$, contrary to the fact that T is a counterexample. So $x_3 \sim v_1$ and by symmetry $x_3 \sim v_2$.

Now let ϕ be an L-coloring of P_1 that does not extend to an L-coloring of G. Let $S(x_1) = L(x_1) \setminus \{\phi(p_1), \phi(p_2)\}$ and let $S(x_2) = L(x_2) \setminus \{\phi(p_2), \phi(p_3)\}$. Given T_1'' and T_2'', we may assume that $|S(x_1)|, |S(x_2)| = 3$.

Suppose $S(x_1) \neq S(x_2)$. Hence $|S(x_1) \cap S(x_2)| \leq 2$. Let $G' = G \setminus (P \cup \{x_1, x_2\}) \cup \{z_1, z_2\}$ where $z_1 \sim z_2, x_3, v_1$ and $z_2 \sim z_1, x_3, v_2$. Let $L'(z_1) = \{c_1\}$ and $L'(z_2) = \{c_2\}$ where c_1, c_2 are brand new colors, that is not in $\bigcup_{v \in V(G)} L(v)$. Let $L'(v_1) = (L(v_1) \setminus \{\phi(p_1)\}) \cup c_1$ and $L'(v_2) = (L(v_2) \setminus \{\phi(p_3)\}) \cup c_2$. Finally let $L'(x_3) = (L(x_3) \setminus (S(x_1) \cap S(x_2))) \cup \{c_1, c_2\}$ and $L' = L$ otherwise. Now $T' = (G', P_1' \cup P_2, L')$ with $P_1' = z_1z_2$ is a canvas and there does not exist an L'-coloring of G' as there does not exist an L-coloring of G. Thus T' contains a connected critical subcanvas T''. Yet as T is counterexample, there does not exist a chord of G' whose ends have lists of size less than five and are not in $P_1' \cup P_2$. By induction $d(P_1', P_2) \leq f(2, k_2)$. But then $d(P_1, P_2) \leq f(3, k_2)$, a contradiction.

So we may assume that $S(x_1) = S(x_2)$. Let $G' = G \setminus (P \cup \{x_1, x_2\}) \cup \{v_1v_2\}$. Let $L'(v_1) = L(v_1) \setminus \{\phi(p_1)\}$, $L'(v_2) = L(v_2) \setminus \{\phi(p_3)\}$ and $L' = L$ otherwise. Now $T' = (G', P_1' \cup P_2, L')$ with $P_1' = v_1v_2$ is a canvas and there does not exist an L'-coloring of G' as there does not exist an L-coloring of G. Thus T' contains a connected critical subcanvas T''. Yet as T is counterexample, there does not exist a chord of G' whose ends have lists of size less than five and are not in $P_1' \cup P_2$. By induction $d(P_1', P_2) \leq f(2, k_2)$. But then $d(P_1, P_2) \leq f(3, k_2)$, a contradiction. □

Theorem 3.10.8. *If $T = (G, P \cup P', L)$ is a connected critical canvas, where P, P' are disjoint paths of C such that there is no chord of the outer walk of G whose ends have lists of size less than five and are not in $P \cup P'$, then $|V(G)| = O(|P| + |P'|)$.*

Proof. Let $d = d(P, P')$ and P_0 be a shortest path from P to P'. P_0 creates up to two paths P_1, P_2, whose lengths are at most $O(d + |P| + |P'|)$. Moreover $G = Ext(P_1) \cup Ext(P_2)$. Yet $(Ext(P_1), P_1, L)$ and $(Ext(P_2), P_2, L)$ are critical path-canvases. By Theorem 3.8.3, $|V(Ext(P_1))| = O(|P_1|)$ and $|V(Ext(P_2))| = O(|P_2|)$. Hence $|V(G)| = O(d + |P| + |P'|)$.

If $d \leq O(|P| + |P|')$, then $|V(G)| = O(|P| + |P'|)$ as desired. So suppose $d \geq \Omega(|P| + |P'|)$. Hence $|V(G)| \leq cd$ for some constant c. There must exist a distance i_1, i_2, $1 \leq i_1, i_2 \leq d/4$ such that $|N_{i_1}(P_1)|, |N_{i_2}(P_2)| \leq 4c$. Thus there exists a slice $T' = (G', P_3 \cup P_4, L)$ of dimension two where $P_3 \subseteq N_{i_1}(P_1)$ and $P_4 \subset N_{i_2}(P_2)$. Thus $d(P_3, P_4) \geq d/2$. By Theorem 3.10.1 applied to T', we find that $d/2 \leq d(P_3, P_4) \leq f(4c, 4c)$ and hence $d \leq 2f(4c, 4c)$. Thus $|V(G)| \leq c2f(4c, 4c))$, a constant, as desired. □

Theorem 3.10.9. *If $T = (G, P \cup P', L)$ is a connected critical canvas, where P, P' are disjoint paths of C there is no chord of the outer walk of G whose ends have lists of size less than five and are not in $P \cup P'$, then $d(P, P') \leq O(\log(|P| + |P'|))$.*

Proof. There must exist a distance i, $1 \leq i \leq 2c$ where c is the constant in Theorem 3.10.8, such that either there are at most $|P_1|/2$ vertices at distance i from P_1 or there are at most $|P_2|/2$ vertices at distance i from P_2. The corollary then follows by induction. □

Theorem 3.10.10. *[Logarithmic Distance Bottleneck Theorem: Two Paths] If $T = (G, P \cup P', L)$ is a connected critical canvas, where P, P' are disjoint paths of C such that there is no bottleneck $T' = (G', U_1 \cup U_2, L)$ of T where $d(U_1, U_2) \geq d$, then $d(P, P') \leq O(\log(|P| + |P'|)) + 6d$.*

Proof. Suppose not. By Theorem 3.10.9, there exists a chord of the outer walk of G whose ends have lists of size less than five and are not in $P \cup P'$. Let U_1 be the closest such chord to P_1 and U_2 be the closest such chord to P_2. It follows from Theorem 3.10.9 that $d(P_1, U_1) \leq O(\log |P|)$ and $d(P_2, U_2) \leq O(\log |P'|)$. Thus $d(U_1, U_2) \geq d(P, P') - O(\log(|P| + |P'|))$. Yet by Theorem 2.11.1, $d(U_1, U_2) \leq 6d + 22$ and the theorem follows. □

Using Theorem 3.8.3, we also obtain a bound on $|V(G)|$ when there is no bottleneck with sides at distance at least d as follow.

Theorem 3.10.11. *If $T = (G, P \cup P', L)$ is a connected critical canvas, where P, P' are disjoint paths of C such that there is no bottleneck $T' = (G', U_1 \cup U_2, L)$ of T where $d(U_1, U_2) \geq d$, then $|V(G)| \leq O(|P| + |P'|) + 12d$.*

Theorem 3.10.12. *[Exponential Growth Theorem: Two Paths] If $T = (G, P \cup P', L)$ is a connected critical canvas, where P, P' are disjoint paths of C such that no bottleneck $T' = (G', U_1 \cup U_2, L)$ of T where $d(U_1, U_2) \geq d$, and $v_0 \in V(G) \setminus V(P \cup P')$, then for all $k \leq d(v_0, P \cup P')$, $|N_k(v_0)| \geq 2^{\Omega(k-6d)}$.*

Proof. Let $k \leq d(v_0, P \cup P')$. Now $N_k(v_0)$ separates v_0 from C. By Theorem 3.9.4, $k = d(v_0, N_k(v_0)) \leq O(\log |N_k(v_0)|) + 6d$. Hence $|N_k(v_0)| \geq 2^{\Omega(k-6d)}$ as desired. □

3.11 Steiner Trees

Definition. Let G be a graph and $S \subset V(G)$. We say $T \subseteq G$ is a *Steiner tree* for S if T is a tree with a minimum number of edges such that $S \subset V(T)$. We let T^* denote the tree formed from T by supressing degree two vertices not in S. If $e \in E(T^*)$, we let $\psi(e)$ denote the path in T between the endpoints of e and we let $\mathrm{mid}(e)$ denote a mid-point of that path. We say that the path $\psi(e)$ is a *seam* of the tree T.

Lemma 3.11.1. *Let $T = (G, S, L)$ be a canvas. If H is a Steiner tree of G for S and we let $B(e)$ denote $N_{|e|/4-1}(\mathrm{mid}(e))$ for every seam e of H, then*

(1) *for all seams e of H, $B(e)$ is contained in a slice whose boundary is contained in $N_{|e|/4-1}(e)$, and*

(2) *for all distinct seams e, f of H, $B(e) \cap B(f) = \emptyset$.*

Proof.

Claim 3.11.2. *There cannot exist a path from an internal vertex v in a seam e of H to a vertex in $H \setminus e$ that is shorter than mimimum of the length of the paths from v to the endpoints of e.*

Proof. Otherwise, we could add such a path and delete whichever path from v to an endpoint of e that leaves H a tree. □

We now prove (1). Let e be a seam of T. It follows from the claim above that $N_{|e|/2-1}(\mathrm{mid}(e)) \cap (T \setminus \psi(e) = \emptyset$. Hence, $B(e)$ is contained in a slice whose boundary is contained in $N_{|e|/4-1}(e)$.

We now prove (2). Let e and f be distinct seams of H. Suppose $B(e) \cap B(f) \neq \emptyset$. Suppose without loss of generality that $|e| \geq |f|$. But now there exists a path of length at most $|e|/4 + |f|/4 - 2 \leq |e|/2 - 2$ between $\mathrm{mid}(e)$ and $\mathrm{mid}(f)$ which is a vertex of $H \setminus e$, contradicting the claim above. □

3.12 Bottleneck Theorem for Many Paths

Theorem 3.12.1. *[Linear Bottleneck Theorem: Many Paths] If $T = (G, S, L)$ is a connected critical canvas, where S is the union of disjoint paths of C such that there is no bottleneck $T' = (G', U_1 \cup U_2, L)$ of T where $d(U_1, U_2) \geq d$, then $|V(G)| = O(d|S|)$.*

Proof. Let T be a Steiner tree of G for S. It follows by applying Theorem 3.8.3 to all of the canvases made by T that

$$|V(G)| \leq 957(2|E(T)|)$$

. Yet, the number of seams of H is at most $2|S|$, as branch points are only necessary to span vertices in S.

As T^* was formed by supressing vertices of degree two in T, $|V(T) \setminus V(T^*)| = |E(T) \setminus E(T^*)|$. Thus,

$$|V(G)| \leq 580(4|S| + 2|V(T) \setminus V(T^*)|).$$

Let $\mathcal{E}$ be the set of all seams e of T, $\phi(e) \setminus V(T^*) \neq \emptyset$. Hence, for all $e \in \mathcal{E}$, $\text{mid}(e)$ exists. For all $e \in \mathcal{E}$, let $B(e) = N_{|e|/4-1}(\text{mid}(e))$. By Lemma 3.11.1 (i), $B(e)$ is contained in a slice whose boundary is contained in $N_{|e|/4-1}(e)$. It follows from Lemma 3.10.12 that $|B(e)| \geq 2^{c(|e|/4-1-6d)}$ for some constant c. Hence,

$$|V(G)| \geq \sum_{e \in \mathcal{E}} 2^{c(|e|/4-1-6d)} \geq |\mathcal{E}| 2^{c(\sum_{e \in \mathcal{E}}(|e|/4|\mathcal{E}|)-1-6d)}$$

where the last inequality follows from the concavity of the exponential function. Yet $|V(H) \setminus V(H^*)| \leq \sum_{e \in \mathcal{E}} |e|$. Combining, we find that

$$|\mathcal{E}| 2^{(c/4)(\sum_{e \in \mathcal{E}} |e|)/|\mathcal{E}|} / 2^{c(1+6d)} \leq |V(G)| \leq |V(G)| \leq 1160(2|S| + \sum_{e \in \mathcal{E}} |e|).$$

We may suppose that $\sum_{e \in \mathcal{E}} |e| \geq 2|S|$ as otherwise $|V(G)| \leq 4640|S|$ as desired. Hence, $|V(G)| \leq 2320 \sum_{e \in \mathcal{E}} |e|$. Letting $x = \sum_{e \in \mathcal{E}} / |\mathcal{E}|$, the average size of a seam in $|\mathcal{E}|$, we find that

$$2^{(c/4)x} \leq 2320(2^{c(1+6d)})x.$$

Let $c' = 23202^{c(1+6d)}$. Hence, $x \leq \max\{4\log(4c'/c)/c, 4/c\}$, call this constant c_0. Note that $c_0 = O(d)$. Hence,

$$|V(G)| \leq 2320c_0|\mathcal{E}| \leq 4640c_0|S|$$

as $|\mathcal{E}| \leq |E(H^*)|$. The theorem now follows with constant $\max\{4640c_0, 4640\} = O(d)$. $\square$

Corollary 3.12.2 (Logarithmic Distance Bottleneck Theorem: Many Vertices)**.** *There exists $D > 0$ such that the following holds: If $T = (G, S, L)$ is a canvas, where S is the union of disjoint vertices $v_1, v_2 \dots$ such that $d(v_i, v_j) \geq D$ and no bottleneck $T' = (G', U_1 \cup U_2, L)$ of T where $d(U_1, U_2) \geq d$, then G has an L-coloring.*

Proof. Suppose not. Then there exists $S' \subseteq S$ and $G' \subseteq G$ such that (G', S', L) is a connected critical canvas. It follows from Theorem 3.10.12, that $|B_{D/2}(v)| \geq 2^{\Omega(D)}$ for all $v \in V(S)$. Hence $|V(G')| \geq |S|2^{\Omega(D)}$, contradicting Theorem 3.12.1 for large enough D. $\square$

CHAPTER IV

TWO PRECOLORED TRIANGLES

4.1 Introduction

In this chapter, we will prove the following theorem.

Theorem 4.1.1. *[Two Precolored Triangles Theorem]*

There exists d such that following holds:

Let G be a planar graph and T_1 and T_2 triangles in G such that $d(T_1, T_2) \geq d$. Let L be a list assignment of G such that $|L(v)| \geq 5$ for all $v \in V(G)$. If ϕ is an L-coloring of $T_1 \cup T_2$, then ϕ extends to an L-coloring of G.

In Section 4.2 and 4.3, we develop a technique to color and delete a shortest path between T_1 and T_2 so that the resulting graph is a canvas (G, S, L) such that S is the union of two paths P_1, P_2 corresponding to T_1, T_2 respectively. In Section 4.4, we show that if a minimum counterexample to Theorem 4.1.1 does not have a long chain of triangles separating T_1 from T_2 where the graphs between any two consecutive triangles are one of three types then the canvas has a local L-coloring near each P_i. This then allows us to invoke Theorem 2.11.1 to produce a long bottleneck of the canvas.

In Sections 4.5 and 4.6, we show that a long bottleneck yields a similarly long chain of triangles separating T_1 from T_2 where the graphs between any two consecutive triangles just so happen to be the three types defined in Section 4.4. In Sections 4.7, 4.8 and 4.9 we develop a theory of sets of colorings, somewhat akin to that in Chapter 2, to prove that for long enough chains of triangles involving these three types of graphs any coloring of the inner and outer triangle extends to the whole graph. Finally in Section 4.10, we combine all of these results to prove Theorem 4.1.1.

4.2 Coloring a Shortest Path

Definition. Let G be a planar graph and $p_0, p_n \in V(G)$ where $d(p_0, p_n) = n$. Let $P = p_0p_1 \dots p_n$ be a shortest path between p_0 and p_n in G. We say a vertex $v \in V(G \setminus P)$ is a *mate* of $p_i \in P$ if $v \sim p_{i-1}, p_i, p_{i+1}$ and $1 \leq i \leq n-1$. We say a vertex $p \in P$ is *doubled* if p has a mate. We say a vertex $p \in P$ is *tripled* if p has two distinct mates.

We say a vertex $p \in P$ is *quadrupled* if p has three distinct mates. We will prove that in a planar graph there cannot be a quadrupled vertex.

We say a path P from p_0 to p_n is an *arrow from p_0 to p_n* if $P = p_0p_1 \dots p_n$ is a shortest path between p_0 and p_n and the following property holds: for all i, $2 \leq i \leq n-1$, if p_i is tripled, then p_{i-1} is not doubled.

Proposition 4.2.1. *Let G be a planar graph and p_0, p_n be vertices of G. Let P be a shortest path from p_0 to p_n, then no internal vertex of P is quadrupled.*

Proof. Suppose a vertex p_i of P is quadrupled. That is, p_i has three mates x_1, x_2, x_3. But that means each of x_1, x_2, x_3 is adjacent to all of p_{i-1}, p_i, p_{i+1}. Thus $G[\{x_1, x_2, x_3, p_{i-1}, p_i, p_{i+1}\}]$ contains $K_{3,3}$ as a subgraph, a contradiction since G is planar.

□

Lemma 4.2.2. *Let $n > 0$, G be a planar graph and p_0, p_{n-1}, p_n be vertices of G such that $d(p_n, p_0) = n$, $d(p_{n-1}, p_0) = n-1$, $p_{n-1} \sim p_n$. There exists an arrow $P = p_0 \dots p_{n-1}p_n$ from p_0 to p_n.*

Proof. We proceed by induction on $n = d(p_n, p_0)$. If $n = 1$, then $p_{n-1} = p_0$ as $d(p_{n-1}, p_0) = 0$. Hence $P = p_0p_1$ is an arrow from p_0 to p_1 as desired. So suppose $n \geq 2$. By induction, there exists an arrow $P' = p_0 \dots p_{n-2}p_{n-1}$ from p_0 to p_{n-1}. Now $P' + p_n$ is an arrow from p_0 to p_n as desired unless p_{n-1} is tripled and p_{n-2} is doubled. Let p'_{n-1}, p''_{n-1} be the mates of p_{n-1} and p'_{n-2} be a mate of p_{n-2} in P'.

By induction, there exists an arrow $P'' = p_0 \dots p'_{n-2}p_{n-1}$. Hence $P'' + p_n$ is an arrow as desired unless p_{n-1} is tripled and p'_{n-2} is doubled in P''. Yet the mates of p_{n-1} in P'' must be p'_{n-1}, p''_{n-1} as otherwise G contains a $K_{3,3}$ subdivision with branch points p_{n-2}, p_{n-1}, p_n and $p'_{n-1}, p''_{n-1}, p'''_{n-1}$ where p'''_{n-1} is a mate of p_{n-1} in P'' distinct from p'_{n-1}, p''_{n-1}, a contradiction to the assumption that G is planar. But then G contains a K_5 subdivision with branch points p_{n-1}, p'_{n-1}, p''_{n-1}, p_{n-2} and p'_{n-2}. □

Definition. Let G be a graph and L a list assignment for G. Let $S \subset V(G)$. We say a coloring ϕ of S is *bichromatic* if for all $v \in V(G \setminus S)$, $|\{c \in L(v) : \exists p \in V(S)$ such that $\phi(p) = c\}| \leq 2$.

Lemma 4.2.3. *Let G be a planar graph and p_0, p_n be vertices of G such that $d(p_0, p_n) = n$. Let P be an arrow from p_0 to p_n. Suppose that $|L(v)| = 5$ for all $v \in V(G) \setminus \{p_0, p_{n-1}, p_n\}$ and that $|L(p_0)| = 3$.*

(1) *If $|L(p_{n-1})| = 3$ and $|L(p_n)| = 5$, then there exists a bichromatic L-coloring of P.*

(2) *If $|L(p_{n-1})| = 5$ and $|L(p_n)| = 3$, then there exists a bichromatic L-coloring of P.*

Proof. We proceed by induction on n. If $n \leq 1$, then there is surely a bichromatic L-coloring of P. Notice that we need only consider how a coloring of P affects the mates of vertices of P.

Consider p_{n-1}. By Proposition 4.2.1, p_{n-1} is not quadrupled. We now consider three cases.

- Case 1: p_{n-1} has no mate

  Proof of (1)/(2): Apply induction using (2) to $P \setminus p_n$. There exists a bichromatic L-coloring of $P \setminus p_n$. Extend this coloring to p_n. As p_{n-1} has no mate, this coloring is bichromatic.

- Case 2: p_{n-1} has one mate v

  That is p_{n-1} is doubled.

  Proof of (1): Apply induction using (2) to $P \setminus p_n$. There exists a bichromatic L-coloring ϕ of $P \setminus p_n$. Now we need only color p_n so that v sees at most two colors from its list. We may suppose then that $\phi(p_{n-2}), \phi(p_{n-1}) \in L(v)$. But then either $\phi(p_{n-2}) \in L(p_n)$ or $L(p_n) \setminus L(v) \neq \emptyset$. Color p_n with such a color. Thus v will see at most two colors and ϕ is bichromatic.

  Proof of (2): Apply induction successively three times using (1) to $P \setminus p_n$. Thus there exists three bichromatic L-colorings ϕ_1, ϕ_2, ϕ_3 of $P \setminus p_n$ such that $\phi_i(p_{n-2}) \neq \phi_j(p_{n-2})$ for all $i \neq j \in \{1, 2, 3\}$. Let $c_i = \phi_i(p_{n-2})$. Let $\mathcal{C} = \{c_1, c_2, c_3\}$.

  If there exists i, such that $\phi_i(p_{n-1}) \notin L(v)$, then we may extend this coloring to p_n and it will be bichromatic as desired. So we may assume that for all i, $\phi_i(p_{n-1}) \in L(v)$.

  Similarly, we may assume that $c_i \in L(v)$ for all i. Now if there exists i such that $c_i \in L(p_n)$, let $\phi_i(p_n) = c_i$ and then ϕ_i is bichromatic. So we may assume that $L(p_n) \cap \mathcal{C} = \emptyset$. As $\mathcal{C} \subset L(v)$, we find that $L(p_n) \setminus L(v) \neq \emptyset$. Now let $\phi_1(p_n) \in L(p_n) \setminus L(v)$ and it follows that ϕ_1 is bichromatic.

- Case 3: p_{n-1} has two mates v_1, v_2

  As P is an arrow, p_{n-2} has no mate. Apply induction using (ii) three times to $P \setminus \{p_{n-1}, p_n\}$ to find three colorings ϕ_1, ϕ_2, ϕ_3 such that $\phi_i(p_{n-2}) \neq \phi_j(p_{n-2})$ for all $i \neq j \in \{1, 2, 3\}$. Let $c_i = \phi_i(p_{n-2})$. Let $\mathcal{C} = \{c_1, c_2, c_3\}$. It suffices to show that we may extend one of these coloring to p_{n-1}, p_n such that neither v_1 nor v_2 sees more than two colors from its list.

  Note that we may assume $L(v_1) \neq L(v_2)$ as otherwise we may proceed as if p_{n-1} had only one mate.

Proof of (1):

If there exists i such that $c_i \notin L(v_1), L(v_2)$, then we extend ϕ_i to p_{n-1} and p_n. Clearly ϕ_i is bichromatic in this case. So we may assume that $\mathcal{C} \subset L(v_1) \cup L(v_2)$.

If there exists i such that $c_i \in L(v_1) \setminus L(v_2)$, then we let $\phi_i(p_n) = c_i$ if $L(p_n) = L(v_1)$ and let $\phi_i(p_n) \in L(p_n) \setminus L(v_1)$ otherwise. We then extend ϕ_i to p_{n-1}. Now ϕ_i is bichromatic as p_{n-2} receives a color not in $L(v_2)$ and p_n receives either the same color as p_{n-2} or a color not in $L(v_1)$. So we may assume using symmetry that $\mathcal{C} \subset L(v_1) \cap L(v_2)$.

If $L(p_n) \setminus (L(v_1) \cup L(v_2))$, extend ϕ_1 to p_n using such a color and then to p_{n-1}. Now ϕ_1 is bichromatic as p_n receives a color not in $L(v_1)$ or $L(v_2)$. So we may assume that $L(p_n) \subseteq L(v_1) \cup L(v_2)$.

If there exists i such that $c_i \in L(p_n)$, let $\phi_i(p_n) = c_i$ and then extend to p_{n-1}. Now ϕ_i is bichromatic as p_n and p_{n-2} receive the same color. So we may assume that $L(p_n) \cap \mathcal{C} = \emptyset$.

But then,

$$L(p_n) \subseteq L(p_n) \setminus \mathcal{C} \subseteq (L(v_1) \cup L(v_2)) \setminus \mathcal{C} \subseteq (L(v_1) \setminus \mathcal{C}) \cup (L(v_2) \setminus \mathcal{C}).$$

However as $\mathcal{C} \subset L(v_1), L(v_2)$ and $|\mathcal{C}| = 3$, $|L(v_1) \setminus \mathcal{C}| = |L(v_2) \setminus \mathcal{C}| = 2$. Thus, $|L(p_n)| \leq 4$, a contradiction.

Proof of (2):

As $L(v_1) \neq L(v_2)$, we may assume without loss of generality that $L(p_{n-1}) \setminus L(v_2) \neq \emptyset$. Yet we may also assume that $L(p_{n-1}) \subseteq L(v_1) \cup L(v_2)$. Suppose not and let $c \in L(p_{n-1}) \setminus (L(v_1) \cup L(v_2))$. There exists i such that $c_i \neq c$. Let $\phi_i(p_{n-1}) = c$ and extend to p_n. Now ϕ_i is bichromatic as p_{n-1} receives a color not in $L(v_1)$ or $L(v_2)$.

If there exists i such that $c_i \in L(p_n)$, let $\phi_i(p_n) = c_i$ and then extend to p_{n_1}. Now ϕ_i is bichromatic as p_n and p_{n-2} receive the same color. So we may assume that $L(p_n) \cap \mathcal{C} = \emptyset$.

Let $c \in L(p_{n-1}) \setminus L(v_2)$. As $L(p_n) \cap \mathcal{C} = \emptyset$,

$$|(L(p_n) \setminus \{c\}) \cup (\mathcal{C} \setminus \{c\})| \geq 5,$$

which is larger than $|L(v_1) \setminus \{c\}| = 4$ as $c \in L(v_1)$. Thus either there exists i such that $c_i \notin L(v_1)$ or $L(p_n) \setminus L(v_1) \neq \emptyset$.

In the former case, let $\phi_i(p_{n-1}) = c$ and extend to p_n. Now extend ϕ_i to p_n. As p_{n-2} receives a color not in $L(v_1)$ and p_{n-1} receives a color not in $L(v_2)$, ϕ_i is bichromatic.

In the latter case, there exists i such that $c_i \neq c$. Let $\phi_i(p_{n-1}) = c$ and $\phi_i(p_n) \in L(p_n) \setminus L(v_1)$. As p_n receives a color not in $L(v_1)$ and p_{n-1} receives a color not in $L(v_2)$, ϕ_i is bichromatic.

□

Here is a definition which will be useful later.

Definition. Let P be an arrow from u to v of a plane graph G and $p \in P \setminus \{u, v\}$. Let p_T be the neighbor of p in P closest to v and p_B be the neighbor of p in P closest to u. We say that a neighbor z of p not in $V(P)$ is to the *right* of p if the vertices p_T, z, p_B appear in that order in the clockwise cyclic order of p and we say z is to the *left* if they appear in the order p_B, z, p_T. Similarly, we say z is a *right mate* of p if z is a mate of p that is to the right of p and we say z is *left mate* if z is a mate of p that is to the left of p. Furthermore, we say that an edge $e \notin E(P)$ incident with p is to the *left* if its other end z is to the left of p and to the *right* otherwise.

4.3 Planarizing a Prism-Canvas

We may now apply the technique of the preceding section to graphs embedded in the cylinder.

Definition. (Cylinder Cycle-Canvas, Prism-Canvas)

We say that $T = (G, C_1, C_2, L)$ is a *cylinder cycle-canvas* if G is a plane graph, C_1 is the outer facial cycle of G, C_2 is a facial cycle in G distinct from C_1 and L is a list assignment for G such that $|L(v)| \geq 5$ for all $v \in V(G \setminus (C_1 \cup C_2))$, $|L(v)| \geq 1$ for all $v \in V(C_1) \cup V(C_2)$, $C_1 \cup C_2$ has an L-coloring. We say T is a *prism-canvas* if $|C_1| = |C_2| = 3$.

Definition. (Planarization)

Suppose that $d(C_1, C_2) \geq 3$. Let $P = p_0 p_1 \ldots p_{d-1} p_d$ be a shortest path between C_1 and C_2 where $p_0 \in V(C_1)$ and $p_d \in V(C_2)$ such that $|N(p_1) \cap V(C_1)| \leq 2$ and $|N(p_{d-1}) \cap V(C_2)| \leq 2$, and $P' = P \setminus \{p_0, p_d\}$ is an arrow from p_1 to p_{d-1}.

We now define the *planarization* of $T = (G, C_1, C_2, L)$ with respect to P to be the canvas (G_0, S, L') as follows: First let $G' = G \setminus P'$. Next fix a bicoloring ϕ of P' for $G \setminus (C_1 \cup C_2)$ from the lists L where ϕ can be extended to an L-coloring of $C_1 \cup C_2$. Let $L'(v) = L(v) \setminus \{\phi(u) : u \sim v, u \in P'\}$ for all $v \in G' \setminus (C_1 \cup C_2)$. Finally if p_1 has one neighbor in C_1, cut C_1 at p_0 (i.e. split p_0 into two vertices) and let P_1 be the path between the vertices created by the split of p_0 using vertices of C_1; otherwise, let P_1 be the path between the two neighbors of p_1 in the homotopically non-trivial way and delete the homotopically trivial part. Let P_2 be defined in the same way for p_d and C_2. Let $L'(v) = L(v)$ for all $v \in P_1 \cup P_2$. Let G_0 be the resulting graph, C its outer cycle and $S = P_1 \cup P_2$. We say that Γ^* is a planarization of T is the planarization of T with respect to some such path P.

However there can be many choices of P and hence many planarizations of T. We will need to choose P such that the planarization maximizes certain structures.

Hence the following definitions.

Definition. Let (G, S, L) be a canvas. Let $U = u_1u_2$ be an essential chord of the outer walk C of G such that $|L(u_1)|, |L(u_2)| < 5$. Let $G_1 \cap G_2 = U$ and $G_1 \cup G_2 = C \cup U$. We say that U is a *stopping chord* if there exists $i \in \{1, 2\}$ such that there does not exist a vertex $z \in V(G_i)$ such $|L(z)| = 3$ and $z \sim u_1, u_2$ and neither u_1 nor u_2 is in a chord of C with a vertex in $V(G_i) \setminus U$ whose ends have lists of size three. We say that U is a *blocking chord* if there exists $i, j \in \{1, 2\}$ such that $|L(u_i)| \geq 4$, and, either $|L(u_{3-i}| \geq 4$ or u_{3-i} has at most one neighbor with a list of size three in G_j. We say that U is a *cut-edge* if u_1, u_2 are essential cutvertices of G.

Note that there does not exist a stopping chord, blocking chord, or cut-edge in the middle of a canvas containing an accordion or harmonica.

Definition. Let Γ be a cylinder canvas. Let Γ^* be a planarization of Γ and let $P = p_0p_1 \dots p_d$ where Γ^* is the planarization of Γ with respect to P. We say a cut-edge u_1u_2 of Γ^* is *dividing* if u_1 has neighbors $p_1, p_1' \in V(P)$ such that u_1 is to the left of p_1 and to the right of p_1' and similarly u_2 has neighbors $p_2, p_2' \in V(P)$ such that u_2 is to the left of p_2 and to the right of p_2'.

We say Γ^* is *good* if over all such planarizations, Γ^* maximizes $|N_{C_1}(p_1)| + |N_{C_2}(p_{d-1})|$, and subject to that Γ^* maximizes the combined total of stopping chords, blocking chords and dividing cut-edges of Γ^*.

The maximization above for an optimal planarization will prove useful precisely because accordions and harmonicas do not have stopping chords, blocking chords or cut-edges.

4.4 Bands and Band Decompositions

Recall that our goal is to prove the following:

Theorem 4.4.1 (Two Precolored Triangles Theorem)**.** *There exists d such that following holds:*

Let G be a planar graph and T_1 and T_2 triangles in G such that $d(T_1, T_2) \geq d$. If L is a list assignment of G such that $|L(v)| \geq 5$ for all $v \in V(G)$ and ϕ is a proper coloring of $T_1 \cup T_2$, then ϕ extends to an L-coloring of G.

We will make certain assumptions about a minimum counterexample to Theorem 4.1.1. The following definition will prove useful in that regard.

Definition. (Nearly Triangulated)

Let $T = (G, C_1, C_2, L)$ be a cylinder cycle-canvas. We say T is *nearly triangulated* if for every face f in G such that f is not bounded by C_1 or C_2 or a triangle, and every two nonadjacent vertices $u, v \in \delta f$, f is bounded by a cycle and $d_{G+\{uv\}}(C_1, C_2) < d_G(C_1, C_2)$.

Proposition 4.4.2. *Let $T = (G, C_1, C_2, L)$ be a nearly triangulated prism-canvas. If there is no vertex cut of size at most two separating C_1 and C_2, then G is a triangulation.*

Proof. Suppose not. Then there exists a face f not bounded by a triangle or C_1, C_2. As there is no cutvertex G, f is bounded by a cycle. Let u, v be two nonconsecutive vertices of f. As there is no vertex cut of size two, u is not adjacent to v. Yet as T is nearly triangulated, $d_{G+\{uv\}}(C_1, C_2) < d_G(C_1, C_2)$. Yet we note that there cannot exist two paths P_1, P_2 in $G + \{uv\}$ from C_1 to C_2 with length less than $d = d_G(C_1, C_2)$ where u is closer to C_1 in P_1 and v is closer to C_1 in P_2. As f is bounded by a cycle, it follows that there exists vertices u_1, u_2, v_2, v_1 that appear in f in that order such that $u_1 \sim u_2$, $v_1 \sim v_2$, and u_1 is closer to C_1 in every shortest path from C_1 to C_2 in $G + \{u_1 v_2\}$ and u_2 is closer to C_2 in every shortest path from C_1 to C_2 in $G + \{u_2 v_1\}$. Hence $d(u_1, C_1) + d(v_2, C_2) + 1 < d$ and $d(u_2, C_2) + d(v_1, C_1) + 1 < d$. But then either

$d \leq d(u_1, C_1) + d(u_2, C_2) + 1 < d$ or $d \leq d(v_1, C_1) + d(v_2, C_2) + 1 < d$, a contradiction. $\square$

Definition. (Bands and Band Decompositions)

We say that a triple (G, T_1, T_2) is a *prismatic graph* if G is a plane graph with two distinguished facial triangles $T_1 \neq T_2$ where T_1 bounds the infinite face of G. Recall that if C is a cycle in a plane graph G, then $Int(C)$ denotes the closed disk containing with boundary C. If T_3, T_4 are separating triangles in G, each separating a vertex in T_1 from a vertex in T_2 and $Int(T_3) \supseteq Int(T_4)$, then we let $G[T_3, T_4]$ denote the prismatic graph $((G \cap Int(T_3) \setminus Int(T_4)) \cup T_4, T_3, T_4)$. We say a prismatic graph (G, T_1, T_2) is a *band* if there does not exist a triangle T in G separating a vertex in T_1 from a vertex in T_2.

We say that $\Gamma = (G, C_1, C_2, L)$ is a *band* if Γ is a prism-canvas and the prismatic graph (G, C_1, C_2) is a band.

Note that every prismatic graph (G, C_1, C_2) has a unique decomposition into bands. Namely, letting $T_0 = C_1$ and $T_m = C_2$, consider the sequence of all triangles separating a vertex in C_1 from a vertex in C_2: $Int(T_0) \supset Int(T_1) \supset Int(T_2) \ldots \supset Int(T_{m-1}) \supset C_2 = T_m$. Let $B_i = (G \setminus (Ext(T_{i-1}) \cup Int(T_i)) \cup T_i, T_{i-1}, T_i)$. As the sequence contained all such triangles, B_i is a band. We define the *band decomposition* of the prismatic graph (G, C_1, C_2) to be the sequence of bands, $B_1 B_2 \ldots B_m$ produced above.

Thus if $\Gamma = (G, C_1, C_2, L)$ is a prism-canvas, we define the *band decomposition* of Γ, denoted $\mathcal{B}(\Gamma) = B_1 \ldots B_m$, where $B_i = (G_i, T_i, T_{i-1}, L)$ is the canvas - that is also a band - corresponding to the band (G_i, T_{i-1}, T_i) in the band decomposition of the prismatic graph (G, C_1, C_2) above.

Definition. (Types of Bands)

Let $B = (G, T_1, T_2, L)$ be a band. We say B is *tetrahedral* if $G = K_4$.

We say B is *octahedral* if $T_1 \cap T_2 = \emptyset$ and every vertex of T_1 has two neighbors in T_2 and vice versa.

We say B is *hexadecahedral* if $T_1 \cap T_2 = \emptyset$ and $G \setminus (T_1 \cup T_2) = C_4 = c_1c_2c_3c_4$, c_1, c_3 have two neighbors each in both of T_1 and T_2, c_2 has a neighbor in T_1 and two in T_2 while c_4 has two neighbors in T_1 and one in T_2.

Here are some useful lemmas to note.

Lemma 4.4.3. *If $\Gamma = (G, T_1 \cup T_2, L)$ is a critical prism-canvas with $T_1 \cap T_2 \neq \emptyset$, then every band in the band decomposition of Γ is tetrahedral.*

Proof. Proceed by induction on vertices of G. If $|T_1 \cap T_2| = 2$, then given that Γ is critical and that T is a triangulation, $G = K_4$ and Γ is a tetrahedral band. So we may suppose that $|T_1 \cap T_2| = 1$. By minimum counterexample (i.e. criticality), one of the outcomes of Theorem 1.5.2 holds. Of course G could be the graph induced by the walk and hence G is C plus some additional chords. But then as a Γ is a near-triangulation, we can find a tetrahedral band in the band decomposition of Γ and the lemma follows by induction. If (i) holds, that is, there is exactly one vertex v in the interior, then v is adjacent to all vertices of $T_1 \cup T_2$. So we again find a tetrahedral band in the band decomposition and the lemma follows by induction. Thus either case (ii) or (iii) holds, that is, there are either two or three pairwise adjacent vertices. But there too, we can find a tetrahedral band in the band decomposition and the lemma follows by induction. □

Lemma 4.4.4. *If Γ is a prism-canvas with $T_1 \cap T_2 = \emptyset$, $G = T_1 \cup T_2$ and G is a triangulation, then either every band in the band decomposition of Γ is tetrahedral or Γ is an octahedral band.*

Proof. If there exists $i \in \{1, 2\}$ and a vertex $v \in T_i$ such that v has three neighbors in T_{3-i}, then there is a tetrahedral band in the band decomposition. Yet then in what remains, the two triangles share the vertex v and so by Lemma 4.4.3, every band in the band decomposition of Γ is tetrahedral.

So we may suppose that every vertex in T_i has at most two neighbors in T_{3-i} for all $i \in \{1,2\}$. But then as G is a triangulation, there are six edges of G not in $E(T_1) \cup E(T_2)$. So every vertex of G must have two neighbors in the other triangle. It follows that Γ is an octahedral band. □

We are almost prepared to invoke Theorem 3.10.10 to start characterizing the bands in the band decomposition. However, we need one more lemma to handle the case when the coloring of one of the paths of the planarization does not extend locally.

Lemma 4.4.5. *Let* $\Gamma = (G, C_1, C_2, L)$ *be a critical prism-canvas such that* $d(C_1, C_2) \geq 4$. *Let* $\Gamma^* = (G^*, P_1 \cup P_2, L^*)$ *be a planarization of* Γ *with respect to an arrow* P. *For all* $i \in \{1,2\}$, *if* $G^* \setminus P_{3-i}$ *is not* L^**-colorable, then the band that contains* C_i *is either tetrahedral, octahedral, or hexadecahedral.*

Proof. It suffices by symmetry to prove the statement for $i = 1$. So we may suppose that $G^* \setminus P_2$ is not L^*-colorable and hence that there is a critical subcanvas Γ' of Γ^* containing P_1 but not P_2. Let p_1 be the end of the arrow P adjacent to C_1 and p_2 be its neighbor in P not in C_1.

Note that by the definition of optimal planarization, the vertices of P adjacent to C_1 and C_2 are chosen to maximize their number of neighbors in C_1, C_2 respectively. Let $T_1 = v_1v_2v_3$. Apply Theorem 3.7.2 to Γ'. Now (1) does not hold as by the construction of Γ^* there does not exist an edge of Γ^* with both ends in P_1 but not in P_1.

Suppose (4) holds. That is, there exists a tripod v in T'. But then there exists a separating triangle T_0 with vertices $(T_1 \setminus \{v_i\}) \cup \{v\}$ where v is a tripod for v_i and the band $G[C_1, T_0]$ is tetrahedral as desired.

Suppose (3) holds. Hence $|V(P_1)| = 4$ and there exists a vertex $v \notin P_1$ adjacent to the two vertices in the interior of P_1 such that $|L'(v)| = 4$. Suppose without loss of generality that $P_1 = v_1v_2v_3v_1$. Hence $v \sim v_2, v_3$. As $|L'(v)| = 4$, v is also adjacent to a vertex z in P. Note that $N(p_1) \cap V(C_1) = v_1$.

Suppose $z = p_1$. Hence z is adjacent to v_1. Now one of the 4-cycles v_1zvv_2 and v_1zvv_3 does not separate C_1 from C_2. Suppose without loss of generality that $C' = v_1zvv_2$ does not separate C_1 from C_2. As Γ is critical, there does not exist a vertex in the interior of C'. As Γ' is a critical path-canvas, we find that $v_1 \sim v$. But then $T_0 = v_1vv_3$ is a separating triangle and the band $G[C_1, T_0]$ is tetrahedral as desired.

So we may suppose that $z \neq p_1$. But then given v, we find that P was not chosen so that its end p_1 adjacent to C_1 had a maximum number of neighbors in C_1, a contradiction.

So we may suppose that (2) holds. We claim that $|V(P_1)| \neq 4$. Suppose not. Thus p_1 has only one neighbor in C_1. We may suppose without loss of generality that v_1 is the neighbor of p_1 in C_1. As (2) holds, there exists a vertex $v \notin P_1$ adjacent to a vertex in the interior of P_1 such that $|L'(v)| = 3$. Hence $P_1 = v_1v_2v_3v_1$. Suppose without loss of generality that $v \sim v_2$.

As $|L'(v)| = 3$, then v must be adjacent to p_1, p_2. If $v \sim v_3$, then given v, we find that P was not chosen so that its end p_1 adjacent to C_1 had a maximum number of neighbors in C_1, a contradiction. So we may assume that $v \not\sim v_3$.

Suppose $vp_1v_1v_3v_2$ is a cycle that does not separate C_1 from C_2. Suppose there exist a vertex v' in the interior of the 5-cycle. As Γ is critical, $v' \sim v_1, v_2, v_3$ and hence $v_1v'v_3$ is a separating triangle. But then the band incident with C_1 is tetrahedral as desired. So there does not exist a vertex in the interior of the 5-cycle. Now as Γ' is critical, it follows that v is adjacent to at least one of v_1, v_3. If v is adjacent to both, then there exists a separating triangle involving v and two vertices of T_0; hence C_1 is incident with a tetrahedral band as desired. So we may suppose that v

is adjacent to only one of v_1, v_2. Hence $|N(v) \cap T_1| = 2$, but then we find that P was not chosen so that its end p_1 adjacent to C_1 had a maximum number of neighbors in C_1, a contradiction.

So we may suppose that $vp_1v_1v_3v_2$ is a cycle separating C_1 from C_2. But then $vp_1v_1v_2$ is a 4-cycle that does not separate C_1 from C_2. As T is critical, there does not exist a vertex in its interior. Yet v is not adjacent to v_1, as then, since there exists a shortest path from C_1 to C_2 through v, v contradicts the choice of p_1 for the end of the arrow P adjacent to C_1.

So we may suppose that Γ' is a bellows with base $v_2v_3v_1$. If Γ' is a turbofan, then As Γ' is critical, there exists $v' \sim v_1, v_2, v_3$ and hence $v_2v'v_1$ is a separating triangle. But then the band incident with C_1 is tetrahedral as desired. $v \not\sim v_3$, there exists $v' \notin P_1$ such that $v' \sim v_3$ and $|L'(v')| = 3$. By symmetry of v_2, v_3 it follows that $v'p_1v_1v_3$ is a 4-cycle that does not separate C_1 from C_2. Furthermore, there does not exist in the interior of that cycle and yet $v' \not\sim v_1$. But then now every coloring of $P_1 \setminus \{v_1\}$ and hence of P_1 extends to an L-coloring of Γ' by Theorem 1.4.2, a contradiction to the fact that Γ' is critical.

So we can assume that $|V(P_1)| = 3$. We may assume without loss of generality that $P_1 = v_1v_2v_3$ and there exist a short chord v_2w_3 of Γ'. As $L'(w_3) = 3$, w_3 is adjacent to $p_1.p_2$. We may assume without loss of generality $v_3v_2w_3p_1$ is a 4-cycle that does not separate C_1 from C_2. Hence there is no vertex in its interior. As Γ' is a bellows, it follows that $w_3 \sim v_3$. Apply Theorem 3.7.2 to the canvas obtained to the bellows in Γ' with base $P_1' = v_1v_2w_3$. Again (1) clearly does not hold. So suppose (2) holds. That is, there is then a tripod w_1. But then $w_1 \sim v_1, v_2, w_3, p_1$. Hence $w_1w_3p_1$ is a separating triangle T_0 and the band $G[C_1, T_0]$ is octahedral as desired. So (3) holds. That is, there exists a short chord v_2w_1. But then $w_1 \sim v_1, p_1, p_2$ where we note that w_1 is not adjacent to p_i, $i \geq 3$ as then $w_1p_3p_4\dots$ is a shorter path from T_1 to T_2.

Now there cannot be another chord v_2w_2 as then w_2 is only adjacent to p_2 and thus has a list of size four, which cannot happen in a bellows. Thus $w_1v_2w_3$ is the base of a turbofan. So there is a tripod w_2, $w_2 \sim v_2, w_1, w_3$. Now $w_1w_2w_3$ is the base of an even fan as $w_1v_2w_3$ is the base of a turbofan. Hence there exist vertices x_3, x_1 where $x_3 \sim w_3, w_2, p_2, p_3$ and $x_1 \sim w_1, w_2, p_2, p_3$. But then there cannot exist another vertex x_2 in the turbo fan, otherwise x_2 has a list of size at least four as x_2 is not adjacent to p_4, because then $w_2x_2p_4$ is a shorter path from T_1 to T_2. So $x_1 \sim x_3$. But then $x_1p_2x_3$ is a separating triangle T_0 and the band $G[C_1, T_0]$ is hexadecahedral as desired. □

Lemma 4.4.6. *Let $d_0 > 0$. There exists $c_0 > 0$ such that the following holds: If $\Gamma = (G, T_1, T_2, L)$ is a counterexample to Theorem 4.1.1 with a minimum number of vertices and subject to that a maximum number of edges, then there exist triangles T_1' and T_2' of G each separating C_1 from C_2 such that either*

(1) *Γ is nearly triangulated and every planarization Γ^* of $\Gamma[T_1', T_2']$ contains a long bottleneck $\Gamma_1 = (G', P_1' \cup P_2', L)$ where $d(T_1, T_2) - 2d_0 - c_0 \leq 6d(P_1', P_2')$, or,*

(2) *$d(T_1', T_2') \geq d_0$ and every band in the band decomposition of $\Gamma[T_1', T_2']$ is a subgraph of a tetrahedral, octahedral or hexadecahedral band.*

Proof. Suppose (1) does not hold. First suppose that Γ is not nearly triangulated.

Suppose there exists a face f that is not bounded by a cycle. Hence there exists a cutvertex v of G separating T_1 from T_2. For $i \in \{1, 2\}$, let T_i' be furthest triangle from T_i such that T_i' separates T_i from v and every band in the band decomposition of $\Gamma[T_i, T_i']$ is the subgraph of a tetrahedral, octahedral or hexadecahedral band. As (2) does not hold, $d(T_i, T_i') \leq d_0$.

Let $\Gamma_i' = \Gamma[T_i', v]$. Now there does not exist a vertex in Γ_i adjacent to all vertices of T_i' as then T_i' would not be the furthest triangle. Given how T_1, T_2 were chosen, it follows from Lemma 4.4.5 that for all $i \in \{1, 2\}$ and for any planarization Γ_i^* of

Γ_i', any L-coloring of T_i' extends to an L-coloring of $\Gamma_i^* \setminus \{v\}$. Indeed more is then true, any L-coloring of T_i' extends to an L'-coloring of G where $L'(w) = L(w)$ for all $w \in \Gamma_i' \setminus \{v\}$ and $|L'(v)| \geq 3$. But now it follows that any L-coloring of $T_1' \cup T_2'$ extends to an L-coloring of $\Gamma[T_1', T_2']$, a contradiction to the fact that Γ is a counterexample.

So we may suppose there exists a face f with two nonadjacent vertices u, v where $d_{G+\{uv\}}(T_1, T_2) \geq d_G(T_1, T_2)$. But then $(G + \{uv\}, T_1, T_2, L)$ is also counterexample to Theorem 4.1.1 with same number of vertices but more edges, a contradiction.

So we may suppose that Γ is nearly triangulated. For $i \in \{1, 2\}$, let T_i' be furthest triangle from T_i such that T_i' separates T_1 from T_2 and every band in the band decomposition of $\Gamma[T_i, T_i']$ is the subgraph of a tetrahedral, octahedral or hexadecahedral band. As (2) does not hold, $d(T_i, T_i') \leq d_0$. Let $\Gamma[T_1', T_2'] = (G_0, P_1 \cup P_2, L)$. Applying Theorem 3.10.11 to $\Gamma[T_1', T_2']$, we find that $\Gamma[T_1', T_2']$ contains a long bottleneck as $d(P_1', P_2') \geq O(\log(4 + 4)) = c_0$. $\square$

We may now invoke Theorem 3.10.10 on the planarization of the prism-canvas to obtain a long harmonica or accordion as a subcanvas. Such a bottleneck will give rise to a sequence of separating triangles in the original graph. Indeed in the next section, we will show that every vertex in the middle of a long accordion is in a separating triangle and hence by the lemmas above that there is a prism-canvas in the middle where every band in its band decomposition is tetrahedral or octahedral. Similarly, in the section after that, we will show that every vertex in the middle of a long harmonica is in a separating triangle or the interior of a hexadecahedral band.

4.5 Bands for Accordions

Our goal in this section is to classify the types of bands which can occur given a long accordion in the planarization. Of course, it is not immediately clear that separating triangles are even generated or that the distance between two nearest separating triangles (and hence the size of the band) is even small. But we will show that this does indeed occur using the fact that the path P was a good planarizer! In this way, we will prove the following lemma:

Lemma 4.5.1. *Let $\Gamma = (G_0, C_1, C_2, L)$ be a cylinder-canvas, Γ^* be an optimal planarization of Γ. Suppose there exists a bottleneck $\Gamma_1 = (G, P_1 \cup P_2, L)$ of Γ^* such that Γ_1 is an accordion and $d(P_1, P_2) \geq 34$. Then there exists triangles T_1 and T_2 each separating C_1 from C_2 such that $d(T_1, T_2) \geq d(P_1, P_2) - 34$ and every band in the band decomposition of $\Gamma[T_1, T_2]$ is tetrahedral or octahedral.*

Now we will not be working with Γ_1 to prove this lemma but rather the vertices in the middle of Γ_1. We make this notion more precise with the following definition.

Definition. Suppose that $\Gamma_1 = (G, P_1 \cup P_2, L)$ is a bottleneck of a a canvas Γ with $d(P_1, P_2) \geq 32$. Let $U_1, U_2, \ldots U_m$ be a maximum collection of chords of the outer walk of Γ_1 whose ends have lists of size three and are not cutvertices of Γ_1. Let Γ_2 be the bottleneck of Γ_1 between U_5 and U_{m-4}. We say that Γ_2 is a *shortening* of Γ_1.

Lemma 4.5.2. *If $\Gamma_2 = (G', P_1' \cup P_2', L)$ is a shortening of a bottleneck $\Gamma_1 = (G, P_1 \cup P_2, L)$, then $d(P_1', P_2') \geq d(P_1, P_2) - 32$.*

Proof. Let $U_1, U_2, \ldots U_m$ be a maximum collection of chords of the outer walk of Γ_1 as in the definition of shortening where Γ_2 is the bottleneck between U_5 and U_{m-4}. As Γ_1 is an accordion or harmonica, it follows that $d(U_i, U_{i+1}) \leq 2$ for all i where $1 \leq i \leq m-1$. Similarly $d(U_1, P_1) \leq 2$ and $d(U_m, P_2) \leq 2$. Hence $d(U_5, P_1) \leq 15$ and $d(U_{m-4}, P_2) \leq 15$. Hence $d(U_1, U_2) \leq d(P_1, P_2) - 32$ as desired. $\square$

We will need the following very useful lemma.

Lemma 4.5.3. *Suppose that $\Gamma_2 = (G, P_1 \cup P_2, L)$ is the shortening of a bottleneck $\Gamma_1 = (G', P_1' \cup P_2', L)$ of an optimal planarization Γ^* of a cylinder-canvas Γ with respect to the path P. Suppose that Γ_0^* is a planarization of Γ with respect to a path P' such* *that for $V(P') \setminus V(P) \subseteq V(G)$. Then there does not exist a stopping chord, blocking chord, or dividing cut-edge U of Γ_0^* such that $U \cap V(G) \neq \emptyset$.*

Proof. Suppose not. Hence there exists a stopping chord, blocking chord, or dividing cut-edge of Γ_0^*, call it U_0, such that $U_0 \cap V(G) \neq \emptyset$.

Claim 4.5.4. *If U is a blocking chord, stopping chord, or dividing cut-edge of Γ^*, then U is not contained in $G' \setminus (P_1' \cup P_2')$.*

Proof. Suppose not. Let $U = u_1u_2$. Suppose U is a stopping chord. Suppose without loss of generality that there does not exist a vertex $z \in V(G_1)$ with a list of size three adjacent to both u_1, u_2 and neither u_1 nor u_2 are in a chord of G_1 whose ends have lists of size three. Yet as Γ is a harmonica or accordion, there exists a bellows W incident with the chord U. If U is not a fan, then Γ is an accordion. Hence Γ is 2-connected and the other side of W is a chord where both ends have lists of size three, a contradiction. So we may suppose that U is a fan. But then we may assume without loss of generality that W is a triangle and hence $z \in W \setminus U$ is in a triangle with u_1, u_2, a contradiction.

Suppose U is a blocking chord. Without loss of generality we may suppose that $|L(u_1)| \geq 4$. Hence u_1 is not contained in the harmonica or accordion. Thus Γ_2 contains a harmonica, u_2 is a cutvertex of the harmonica, and $|L(u_2)| = 3$. But then u_2 has two neighbors with lists of size three in both G_1, G_2 where $G_1 \cap G_2 = U$ and $G_1 \cup G_2 = G' \cup U$, a contradiction to the fact that U is a blocking chord.

So suppose U is a dividing cut-edge. But then u_1, u_2 are cutvertices of G'. Thus Γ_2 contains a harmonica and hence u_1, u_2 are cutvertices of the harmonica. Yet $u_1 \sim u_2$, a contradiction. □

Claim 4.5.5. *Every blocking chord, stopping chord, or dividing cut-edge of Γ^* is a blocking chord, stopping chord, or dividing cut-edge of Γ_0^*, respectively.*

Proof. Suppose not. Let $U = u_1u_2$ be a blocking chord, stopping chord, or dividing cut-edge of Γ^*. By Claim 4.5.4, there exists $i \in \{1,2\}$ such that $u_i \notin V(G') \setminus (V(P_1') \cup V(P_2'))$. We claim that $d(u_i, G) \geq 4$. Suppose not. As Γ_2 is a shortening of Γ_1, there exist chords $U_i = v_iv_i'$ for $1 \leq i \leq 4$ whose ends have lists of size three, are not cutvertices of Γ_1, and separate u_i from G in Γ^*. But then U_i is a chord of Γ^* and we may assume without loss of generality that v_i has two neighbors in P through the top and v_i' has two neighbors in P though the bottom. But then it follows, as P is a shortest path from C_1 to C_2, that the neighbors of v_4, v_4' on P closest to C_2 are not adjacent to the neighbors of v_1, v_1' on P closest to C_1. Hence there does not exist a neighbor of u_i adjacent to a neighbor of a vertex in G, a contradiction. This proves the claim.

Thus $d(u_i, P' \setminus P) \geq 4$. So $d(U, P' \setminus P) \geq 3$. As $d(U, P' \setminus P) \geq 2$ this implies that u_1, u_2 are in Γ_0^* and have the same lists in Γ_0^* as in Γ^*. Thus U is a chord of Γ_0^*. Furthermore as $d(U, P' \setminus P) \geq 3$, the neighbors of u_1, u_2 in Γ^* and the same as those in Γ_0^*. Indeed, their neighbors have the same lists in Γ_0^* as in Γ^*.

It now follows that if U is a blocking chord of Γ^*, then U is a blocking chord of Γ_0^* as desired. Similarly if U is a dividing cut-edge of Γ^*, then U is also a dividing cut-edge of Γ_0^*.

Finally note that for $i \in \{1,2\}$ if u_i in a chord U' of the outer walk of Γ_0^* with both ends having lists of size less than three, then U' is also a chord of the outer walk of Γ^* whose both ends have a list of size three. It now follows that if U is a stopping chord of Γ^*, then U is a stopping chord of Γ_0^*. $\square$

By Claim 4.5.4, U_0 is not a blocking chord, stopping chord, or cutvertex of Γ^*. But now it follows that from Claim 4.5.5 that Γ_0^* has a strictly larger sum of blocking chords, stopping chords, and dividing cut-edges than Γ^*, a contradiction to the assumption that Γ^* is an optimal planarization. $\square$

We will prove Lemma 4.5.1 by a sequence of lemmas. These lemmas require a common hypothesis which we state here.

Hypothesis 4.5.6. $\Gamma = (G_0, C_1, C_2, L)$ *is a critical nearly triangulated cylinder-canvas and* Γ^* *is an optimal planarization of* Γ *with respect to the path* $P = p_1 \dots p_d$ *where* $d = d(C_1, C_2) - 1$. *There exists a bottleneck* Γ_1 *of* Γ^* *with ends* P_1', P_2' *which is an accordion and* $d(P_1', P_2') \geq 34$. *Let* $\Gamma_2 = (G, P_1 \cup P_2, L)$ *be a shortening of* Γ_1 *and* C *be the outer walk of* G.

We will need to label various parts of the accordion Γ_2. To that end, let C be the outer walk of G. Let $U_1, U_2, \dots U_{m-1}$ be the chords of C. Let $U_0 = P_1$ and $U_m = P_2$, and $W_1, W_2, \dots W_m$ be the resulting bellows of the accordion. We will assume that no three of the U's intersect in a vertex as then we could just omit the middle chord, combining two bellows of the accordion. That is, $U_i \cap U_{i+2} = \emptyset$ for all $i, 0 \leq i \leq m-2$.

As the W_i's are bellows, $|U_i \cap U_{i+1}| = 1$ for all $i, 0 \leq i \leq m-1$. Let $x_{i+1} = U_i \cap U_{i+1}$, $x_0 = U_0 \setminus U_1$ and $x_{m+1} = U_m \setminus U_{m-1}$. Let $X = \bigcup_i x_i$.

We will say that an edge e in $E(G) \setminus E(P)$ incident with a vertex v in the interior of P is *through the bottom* if e is to the left of v. Similarly we say e is *through the top* if e is to the right of v. Similarly we say that two vertices are adjacent *through the bottom* (resp. top) if the edge incident with both of them is through the bottom (resp. top).

Let $b_L = \min\{k | p_k$ has a neighbor through the bottom in $\Gamma_2\}$, $b_R = \max\{k | p_k$ has a neighbor through the top in $\Gamma_2\}$ and let t_L, t_R be similarly defined for neighbors through the top. Let $k_L = \max\{b_L, t_L\}$ and $k_R = \min\{b_R, t_R\}$. Let $P^* = \{p_k | k_L \leq k \leq k_R\}$.

Let P_B be the minimal path in the outer walk of Γ_2 containing all the vertices which are adjacent to vertices of P^* through the bottom and similarly let P_T be the minimal path in the outer walk of Γ_2 containing all the vertices which are adjacent to vertices of P^* through the top.

Note that if $x_i \in P_B(P_T$ respectively), then $x_{i+1} \in P_T(P_B$ respectively). Moreover, as there are no cut vertices, $P_T \cap P_B = \emptyset$.

Lemma 4.5.7. *Assume Hypothesis 4.5.6. For every vertex $v \in V(C)$, $N_{G_0}(v) \cap P$ is a path of length one or two and hence $|N_{G_0}(v) \cap P| = 2$ or 3.*

Proof. As all vertices in C have a list of size three, v has at least two neighbors in P. Moreover, as there are no cutvertices of G, v is adjacent to vertices of P either through the top or through the bottom. Yet there cannot be two neighbors of v in P with distance at least three in P, as then P is not shortest. Hence the neighbors of v lie on a subpath of P of length at most two. Given that Γ is nearly triangulated, it follows that v is adjacent to all vertices on that subpath of P as desired. □

Lemma 4.5.8. *Assume Hypothesis 4.5.6. For every vertex $p \in P^*$, $N_{G_0}(p) \cap P_B$ and $N_{G_0}(p) \cap P_T$ are paths of length zero or one.*

Proof. By symmetry, it suffices to prove the lemma for $N(p) \cap P_B$. If p has at least two neighbors in P_B, then p has exactly two neighbors in P_B and they are adjacent, as otherwise, there is a vertex in P_B with at most one neighbor in P, contrary to the fact that Γ_1 is an accordion. The lemma now follows if p has a neighbor in P_B. So suppose not. Consider the face f of Γ incident with p and vertices in P_B. It is not hard to see that there must be a vertex v of P_B incident with f such that adding the edge pv does not decrease the distance from C_1 to C_2 in Γ. Yet p is not adjacent to v and so this contradicts that Γ is nearly triangulated. □

Lemma 4.5.9. *Assume Hypothesis 4.5.6. For all $i, 1 \leq i \leq m$, W_i is a fan of length at most three.*

Proof. Suppose not. First suppose that the outer cycle of W_i has length at least six. That is, there is a path $x_{i-1}v_1v_2v_3 \ldots x_{i+1}$ in C not containing x_i. Thus x_{i-1} has a neighbor u on P and x_{i+1} has a neighbor u' on P such that $d_P(u, u') \geq 5$ and yet

$ux_{i-1}x_ix_{i+1}u'$ is a path of length four in G. Thus P is not shortest, a contradiction. But then if W_i is a fan, it is fan of length at most three as desired.

So we may suppose W_i is a turbofan of length three. Suppose without loss of generality that $x_{i-1} \sim p_{j-2}, p_{j-1}$. Thus $x_{i+1} \sim p_{j+1}, p_{j+2}$. Notice that $N(x_i) \cap P \subseteq \{p_{j-1}, p_j, p_{j+1}\}$. Yet as Γ is critical, there cannot be more than three vertices in the interior of a disc bounded by 6-walk. Thus either $x_i \sim p_{j-1}, p_j$ or p_j, p_{j+1} as otherwise u_1, u_2, u_3, p_j are in the interior of the disc bounded by the 6-walk $p_{j-1}x_{i-1}x_ix_{i+1}p_{j+1}x_i$.

Let $W_i \setminus X = \{u_1, u_2, u_3\}$ where $u_1 \sim x_{i-1}, u_2 \sim x_{i+1}$ and $u_3 \sim x_{i-1}, x_i, x_{i+1}$. Thus $u_1 \sim p_{j-1}, p_j$ and $u_2 \sim p_j, p_{j+1}$.

Now consider the path P' obtained from P by replacing the vertices p_{j-1}, p_j, p_{j+1} with the vertices x_{i-1}, u_3, x_{i+1}. As Γ_1 is also an accordion, p_k is not tripled for all k, $j-3 \leq k \leq j+3$. Yet in P', x_{i-1}, x_{i+1} are not doubled and x_i is not tripled given W_i. Furthermore if p_{j-2} is tripled in P', then p_{j-3} is not doubled. Similarly if p_{j+2} is tripled in P', then p_{j+3} is not doubled. It now follows that P' is an arrow in the same direction as P.

Let $\Gamma_{P'}$ be the planarization of Γ with respect to P'. If $x_i \sim p_{j-1}, p_j$, then $U = p_{j-1}x_i$ is a stopping chord of $\Gamma_{P'}$ given that neither p_{j-1} nor x_i are in a chord in G_2 where $C_2 \subset G_2$ whose other end is a list of size three. But this contradicts Lemma 4.5.3. Similarly, if $x_i \sim p_j, p_{j+1}$, then $U = p_{j+1}x_i$ is a stopping chord of $\Gamma_{P'}$ given no chord in G_1 where $C_1 \subset G_1$. But this contradicts Lemma 4.5.3. □

Lemma 4.5.10. *Assume Hypothesis 4.5.6. For all x_i, $1 \leq i \leq m$, either the edge x_ix_{i-1} is in a separating triangle or the edge x_ix_{i+1} is in a separating triangle.*

Proof. Let p_j be the neighbor of x_{i-1} in P with j smallest and p_k be the neighbor of x_{i+1} in P with k largest. As mentioned before, $k-j \leq 4$ given the path $p_jx_{i-1}x_ix_{i+1}p_k$. Of course, $k-j \geq 2$ as x_{i-1} and x_{i+1} have at least two neighbors on P. Note that by Lemma 4.5.7, $x_{i-1} \sim p_{j+1}$ and $x_{i+1} \sim p_{k-1}$.

Note that $N(x_i) \cap P \subseteq \{p_h : k-3 \leq h \leq j+3\}$ given the paths $p_k x_{i+1} x_i$ and $p_j x_{i-1} x_i$.

If $k - j = 4$, then $N(x_i) \cap P \subseteq \{p_{j+1}, p_{j+2}, p_{j+3}\}$. Thus either p_{j+1} or p_{j+3} is a neighbor of x_i. In the former case, $x_{i-1}x_ip_{j+1}$ is a separating triangle. In the latter case, $x_ix_{i+1}p_{j+3}$ is a separating triangle.

If $k-j=3$, then $N(x_i) \subseteq \{p_j, p_{j+1}, p_{j+2}, p_{j+3}\}$. Let $p_l \sim x_i$ where $l, j \leq l \leq j+3$. If $l = j$ or $j+1$, then $x_{i-1}x_ip_l$ is a separating triangle. If $l = j+2$ or $j+3$, then $x_{i-1}x_ip_l$ is a separating triangle.

If $k - j = 2$, then $N(x_i) \subseteq \{p_{j-1}, p_j, p_{j+1}, p_{j+2}, p_{j+3}\}$. Yet $N(x_i) \cap P$ is path of length at most two by Lemma 4.5.7, so one of p_j, p_{j+1}, p_{j+2} is in $N(x_i) \cap P$. Let $p_l \sim x_i$ where $l, j \leq l \leq j+3$. If $l = j$ or $j+1$, then $x_{i-1}x_ip_l$ is a separating triangle. If $l = j+1$ or $j+2$, then $x_{i-1}x_ip_l$ is a separating triangle. □

Lemma 4.5.11. *Assume Hypothesis 4.5.6. If W_i is a fan of length one or two and $v \in W_i \setminus X$, then the edge vx_i is in a separating triangle.*

Proof. Let p_j be the neighbor of x_{i-1} in P with j smallest and p_k be the neighbor of x_{i+1} in P with k largest. As mentioned before, $k-j \leq 4$ given the path $p_jx_{i-1}x_ix_{i+1}p_k$. Of course, $k - j \geq 2$ as x_{i-1} and x_{i+1} have at least two neighbors on P.

Note that $N(x_i) \cap P \subseteq \{p_h : k-3 \leq h \leq j+3\}$ given the paths $p_k x_{i+1} x_i$ and $p_j x_{i-1} x_i$.

If $k - j = 4$, then $N(x_i) \cap P \subseteq \{p_{j+1}, p_{j+2}, p_{j+3}\}$. In this case $N(v) \cap P \supseteq \{p_{j+1}, p_{j+2}\}$ or $\{p_{j+2}, p_{j+3}\}$. Thus v, x_i is in a separating triangle as desired.

If $k - j = 3$, then $N(x_i) \subseteq \{p_j, p_{j+1}, p_{j+2}, p_{j+3}$. Yet $N(v) \cap P = \{p_{j+1}, p_{j+2}\}$. As $N(x_i) \cap P$ is a path of length one or two by Lemma 4.5.7, either p_{j+1} or p_{j+2} is a neighbor of x_i and hence vx_i is in a separating triangle as desired. □

Lemma 4.5.12. *Assume Hypothesis 4.5.6. For all vertices $p \in P^*$, there exists i, $1 \leq i \leq m$, such that $W_i \cap N(p) \cap P_B, W_i \cap N(p) \cap P_T \neq \emptyset$.*

Proof. Let $p_j \in P^*$. By Lemma 4.5.8, p has a neighbor in P_B and a neighbor in P_T. By Lemma 4.5.7, $N(v) \cap P_B$ and $N(v) \cap P_T$ are paths of length at least one. If there does not exist i as desired, then there exists i such that $U_i = x_i x_{i+1}$ that separates $N(p_j) \cap B$ from $N(p_j) \cap P_T$. As $x_i \not\sim p_j$, we find that x_i has a neighbor p_k with $k \leq j-2$. Similarly as $x_{i+1} \not\sim p_j$, x_i has a neighbor $p_{k'}$ such that $k' \geq j+2$. Yet then $p_k x_i x_{i+1} p_{k'}$ shows that P was not a shortest path, a contradiction. □

Corollary 4.5.13. *Assume Hypothesis 4.5.6. For all $p \in P^*$, p is in a separating triangle.*

Proof. If there is an edge between $N(p) \cap P_B$ and $N(p) \cap P_T$, then p is in a separating triangle. And yet these intersect the same bellows W_i by Lemma 4.5.12. However, by Lemma 4.5.9, W_i is a fan and hence all vertices of $W_i \cap P_B$ and $W_i \cap P_T$ are adjacent and the corollary follows. □

It now follows from the lemmas above that every vertex in Γ_2 is in a separating triangle in G_0. Let $V' = V(G) \cup P^*$. Let T_1 be the outermost separating triangle of G_0 with $V(T_1) \subseteq V'$ and T_2 be the innermost separating triangle of G_0 with $V(T_2) \subseteq V'$.

Lemma 4.5.14. *Assume Hypothesis 4.5.6. If $v \in \Gamma[T_1, T_2]$, then $v \in \Gamma_2 \cup P^*$.*

Proof. Let H be the subgraph of $\Gamma[T_1, T_2]$ induced by $V(\Gamma_2) \cup P^*$. It follows from Lemmas 4.5.7 and 4.5.8, that every face of H has size at most four. So by criticality there is no vertex in the interior of these faces and hence $V(\Gamma[T_1, T_2]) = V(H)$. □

Corollary 4.5.15. *Assume Hypothesis 4.5.6. If $v \in \Gamma[T_1, T_2]$, then v is in T_1, T_2 or a triangle separating a vertex of T_1 from a vertex of T_2.*

Proof. By Lemmas 4.5.10, 4.5.11 and 4.5.13, every vertex in $P^* \cup \Gamma_2$ is in a separating triangle in Γ. As triangles cannot cross, it follows that every vertex in $\Gamma[T_1, T_2]$ is in T_1, T_2 or a triangle separating a vertex of T_1 from a vertex of T_2. □

Lemma 4.5.16. *Assume Hypothesis 4.5.6. If B is a band in the band decomposition of $\Gamma[T_1, T_2]$, then B is tetrahedral or octahedral.*

Proof. Let $B = (G_B, T_3, T_4, L)$ be a band in the band decomposition of $\Gamma[T_1, T_2]$. If $G = T_3 \cup T_4$, then B is tetrahedral or octahedral by Lemma 4.4.4. So we may suppose that $G \neq T_3 \cup T_4$. But then $G_B \setminus T_3 \cup T_4$ must contain a vertex which is not in a separating triangle, contradicting Corollary 4.5.15. $\square$

Proof of Lemma 4.5.1. We may assume Hypothesis 4.5.6. By Lemma 4.5.16, every band in the band decomposition of $\Gamma[T_1, T_2]$ is tetrahedral or octahedral. Furthermore, $d(T_1, T_2) \geq d(P_1, P_2) - 2$ where $\Gamma_2 = (G, P_1 \cup P_2, L)$. As Γ_2 is a shortening of Γ_1, it follows from Lemma 4.5.2 that $d(T_1, T_2) \geq d(P_1', P_2') - 34$ where P_1', P_2' are the ends of Γ_1. $\square$

4.6 Bands for Harmonicas

Our goal in this section is to prove the following:

Lemma 4.6.1. *Let $\Gamma = (G_0, C_1, C_2, L)$ be a cylinder-canvas, Γ^* be an optimal planarization of Γ. Suppose there exists a bottleneck $\Gamma_1 = (G, P_1 \cup P_2, L)$ of Γ^* such that Γ_1 is a harmonica and $d(P_1, P_2) \geq 34$. Then there exists triangles T_1 and T_2 of G_0 each separating C_1 from C_2 such that $d(T_1, T_2) \geq d(P_1, P_2) - 32$ and every band in the band decomposition of $\Gamma[T_1, T_2]$ is tetrahedral, octahedral or hexadecahedral.*

We will prove Lemma 4.6.1 by a sequence of lemmas. These lemmas require a common hypothesis which we state here.

Hypothesis 4.6.2. $\Gamma = (G_0, C_1, C_2, L)$ *is a critical nearly triangulated cylinder-canvas and Γ^* is an optimal planarization of Γ with respect to the path $P = p_1 \ldots p_d$ where $d = d(C_1, C_2) - 1$. There exists a bottleneck Γ_1 of Γ^* with sides P_1', P_2' which is a harmonica and $d(P_1', P_2') \geq 34$. Let $\Gamma_2 = (G, P_1 \cup P_2, L)$ be a shortening of Γ_1 and C be the outer walk of G.*

Suppose Hypothesis 4.6.2 holds. We will need to label various parts of the harmonica Γ_2. To that end, let $Y = \{y_1, y_2, \ldots\}$ be the cutvertices of Γ_1 that are also in Γ_2 and F^i be the subcanvas between y_{i-1} and y_i. Let F^i_j be the jth fan of F^i. Let x^i_j be the hinge of F^i_j and let $x^i_0 = y_{i-1}$ and $x^i_{l_i} = y_i$ where l_i is the number of fans of F^i. Let $X = \bigcup_i X^i$. (Hinges are the middle vertices of bases of the fans) We say F^i is a *diamond* if it is a fan of length one (in this case, neither is more a hinge than the other; so we will say neither is a hinge). Another set of important vertices is $Z^i = N(Y) \cap F^i \setminus X^i$. Let $Z = \bigcup_i Z^i$.

We will say that an edge e in $E(G) \setminus E(P)$ incident with a vertex in the interior of P is *through the bottom* if the end not in P lies to the left of the end in P. Similarly we say e is *through the top* if the end not in P lies to the right of the end in P. Similarly we say that two vertices are adjacent *through the bottom* (resp. top) if the edge incident with both of them is through the bottom (resp. top).

Let N_B denote the set of vertices p in P with a neighbor v such that v in $P_B \setminus Y$, or v in Y and v is adjacent to p through the bottom. Similarly, let N_T denote the set of vertices p in P with a neighbor v such that $v \in P_T \setminus Y$, or v in Y and v is adjacent to p through the top. Let $b_L = \min\{k|p_k \in N_B\}$, $b_R = \max\{k|p_k \in N_B\}$, $t_L = \{\min k|p_k \in N_T\}$ and $t_R = \max\{k|p_k \in N_T\}$. Let $k_L = \max\{b_L, t_L\}$ and $k_R = \min\{b_R, t_R\}$. Let $P^* = \{p_k|k_L \le k \le k_R\}$. Let C be the outerwalk of Γ_2.

(Improve) Let P_B be the bottom path of C, that is the the path of C that contains the vertices of C who are adjacent to P through the bottom and similarly let P_T be the top path of C.

Note that if $x_i \in P_B(P_T$ respectively), then $x_{i+1} \in P_T(P_B$ respectively). Moreover, $P_T \cap P_B = Y$.

We are now ready to start proving lemmas about the vertices in Γ_2.

Lemma 4.6.3. *Suppose Hypothesis 4.6.2 holds. For every vertex $v \in V(C) \setminus Y$, $N(v) \cap P$ is a path of length one or two and hence $|N(v) \cap P| = 2$ or 3.*

Proof. As all vertices in C have a list of size three, v has at least two neighbors in P. Moreover, as v is not a cutvertex of G, v reaches these neighbors by only one homotopy type, top or bottom. Yet there cannot be two neighbors of v in P with distance at least three in P, as then P is not shortest. Hence the neighbors of v lie on a subpath of P of length at most two. Given that G is nearly-triangulated, it follows that v is adjacent to all vertices on that subpath of P as desired. □

Lemma 4.6.4. *Assume Hypothesis 4.6.2. Suppose F^i is not a diamond. If $x \in X^i \backslash Y$, then x is in a separating triangle.*

Proof. Suppose without loss of generality that $x \in P_T$. As F^i is not a diamond, x has at least two neighbors in $P_B \setminus Y$ each with two neighbors in P. Let w_1 be the neighbor of x in $P_B \setminus Y$ closest to P_1 and w_2 be the neighbor of x in $P_B \setminus Y$ closest to P_2.

Let p_j be the neighbor of w_1 in P with j smallest and p_k be the neighbor of w_2 in P with k largest. As mentioned before, $k - j \leq 4$ given the path $p_j w_1 x w_2 p_k$. Of course, $k - j \geq 2$ as w_1 and w_2 have at least two neighbors on P. Note that by Lemma 4.6.3, $w_1 \sim p_{j+1}$ and $w_2 \sim p_{k-1}$.

Note that $N(x) \cap P \subseteq \{p_h : k-3 \leq h \leq j+3\}$ given the paths $p_k w_2 x$ and $p_j w_1 x$.

If $k - j = 4$, then $N(x) \cap P \subseteq \{p_{j+1}, p_{j+2}, p_{j+3}\}$. Thus either p_{j+1} or p_{j+3} is a neighbor of x. In the former case, $w_1 x p_{j+1}$ is a separating triangle. In the latter case, $x w_2 p_{j+3}$ is a separating triangle.

If $k - j = 3$, then $N(x) \subseteq \{p_j, p_{j+1}, p_{j+2}, p_{j+3}\}$. Let $p_l \sim x$ where $j \leq l \leq j+3$. If $l = j$ or $j+1$, then $w_1 x p_l$ is a separating triangle. If $l = j+2$ or $j+3$, then $w_2 x p_l$ is a separating triangle.

If $k - j = 2$, then $N(x) \subseteq \{p_{j-1}, p_j, p_{j+1}, p_{j+2}, p_{j+3}\}$. Yet $N(x) \cap P$ is path of length at most two by Lemma 4.6.3, so one of p_j, p_{j+1}, p_{j+2} is in $N(x) \cap P$. Let $p_l \sim x$ where $j \leq l \leq j+3$. If $l = j$ or $j+1$, then $w_1 x p_l$ is a separating triangle. If $l = j+1$ or $j+2$, then $w_2 x p_l$ is a separating triangle. □

Lemma 4.6.5. *Assume Hypothesis 4.6.2. Suppose F^i is not a diamond. If $v \in F^i_j \setminus (X^i \cup Z^i)$, then the edge vx_j is in a separating triangle.*

Proof. As $v \in F^i_j \setminus X^i$, x_j has two neighbors w_1, w_2 in F^i_j such that $w_1 v w_2$ is a path in F^i_j. As $v \notin Z^i$, $w_1, w_2 \notin Y$. As F^i_j is a fan, $w_1, w_2 \sim x_i$. Let p_h be the neighbor of w_1 in P with h smallest and p_k be the neighbor of w_2 in P with k largest. As mentioned before, $k - h \leq 4$ given the path $p_h w_1 x w_2 p_k$. Given that v has two neighbors in P, $k - h \geq 3$.

If $k - h = 4$, then $N(x_j) \cap P \subseteq \{p_{h+1}, p_{h+2}, p_{h+3}\}$. Thus by Lemma 4.6.3, $p_{h+2} \sim x_j$. Yet $v \sim p_{h+2}$ and thus $vx_j p_{h+2}$ is a separating triangle.

If $k - h = 3$, then $N(x_j) \subseteq \{p_h, p_{h+1}, p_{h+2}, p_{h+3}\}$. Thus by Lemma 4.6.3, either $x_j \sim p_{h+1}$ or $x_j \sim p_{h+2}$. Yet $v \sim p_{h+1}, p_{h+2}$. So vx_j is a separating triangle. □

Let Y_2 denote the set of vertices in Y with at least two neighbors through one side (top or bottom).

Lemma 4.6.6. *Assume Hypothesis 4.6.2. If $y \in Y_2$, then y is in a separating triangle.*

Proof. Let $y_i \in Y_2$. Suppose without loss of generality that y_i has two neighbors on P where the edges go from P_B to P. Let p_j, p_k be two such neighbors of Y where we may assume $k > j$ and there does not exists h, $k > h > j$ such that p_h is a neighbor of Y through the bottom. First suppose $k = j + 1$. Let $x_1 \in N(y_i) \cap P_T \cap F^{i-1}$ and $x_2 \in N(y_i) \cap P_T \cap F^i$. Let p_{h_1} be the neighbor of x_1 in P with h_1 smallest and let p_{h_2} be the neighbor of x_2 in P with h_2 largest. By Lemma 4.6.3, $x_1 \sim p_{h_1+1}$ and $x_2 \sim p_{h_2-1}$. Hence $h_2 \geq h_1 + 2$.

If x_1 or x_2 is adjacent to p_j or p_{j+1}, then y is in a separating triangle as desired. Nevertheless, given the path $p_{h_1} x_1 y_i p_{j+1}$, we find that $h_1 \geq j - 2$ and hence $h_2 \geq j$. But then as x_2 is not adjacent to p_j or p_{j+1}, it follows that $h_2 \geq j + 3$. Similarly given the path $p_{h_2} x_2 y_i p_j$, we find that $h_2 \leq j + 3$ and hence $h_1 \leq j + 1$. As x_1 is not adjacent to p_j or p_{j+1}, it follows that $h_1 \leq j-2$. Thus $h_2 \geq h_1+5$. Yet given the path $x_1 y_i x_2$, we find that $h_2 \leq h_1+4$ as P is a shortest path, a contradiction.

So we may assume that $k \geq j+2$. As G is a nearly triangulated, y must be adjacent to p_{j+1} through the top. But then $p_j p_{j+1} y_i$ is a separating triangle as desired. □

Let Y_1 denote the vertices in Y with one neighbor in P through each side (top and bottom) such that these neighbors are adjacent.

Lemma 4.6.7. *Assume Hypothesis 4.6.2. If $y \in Y_1$, then y is in a separating triangle.*

Proof. Let $y_i \in Y_1$. Suppose without loss of generality that the neighbor of y_i on top is p_{j-1} and the neighbor of y on bottom is p_j. Then $p_{j-1} p_j y_i$ is a separating triangle. □

Let Y_1^* denote the set of vertices in Y with one neighbor in P through each such that these neighbors are not adjacent.

Lemma 4.6.8. *Assume Hypothesis 4.6.2. If $y \in Y_1^*$, then y is in a separating triangle or in the interior of a hexadecahedral band (as well as p_j,z_1,z_4).*

Proof. Let $y_i \in Y_1^*$. Without loss of generality let p_{j-1} be the neighbor of y_i on the bottom and p_{j+1} be the neighbor of y_i on top. Let $z_1 z_2 y_i$ be the triangle in F^{i-1} and $z_3 z_4 y_i$ be the triangle in F^i. We may suppose without loss of generality that $z_1, z_3 \in P_T$ and $z_2, z_4 \in P_B$. It follows that $N(z_2) \cap P = \{p_{j-2}, p_{j-1}\}$ as otherwise P is not a shortest path. Similarly $N(z_3) \cap P = \{p_{j+1}, p_{j+2}\}$. If $z_1 \sim p_{j-1}$, then $z_1 p_{j-1} y_i$ is a separating triangle as desired. Yet $z_1 \not\sim p_{j-3}$ given the path $z_1 y_i p_{j+1}$. But then by Lemma 4.6.3, it follows that $N(z_1) \cap P = \{p_j, p_{j+1}\}$. Similarly $N(z_4) \cap P = \{p_{j-1}, p_j\}$.

Meanwhile there exists $u_1 \in F^{i-1}$ such that $u_1 \neq y_i$ and $u_1 z_1 z_2$ is a triangle. Similarly there exists $u_2 \in F^i$ such that $u_2 \neq y_i$ and $u_2 z_3 z_4$ is a triangle. Now $N(u_1) \cap P \subseteq \{p_{j-2}, p_{j-1}, p_j\}$ given the path $u_1 z_1 p_{j+1}$. Similarly $N(u_2) \cap P \subseteq \{p_j, p_{j+1}, p_{j+2}\}$ given the path $u_2 z_4 p_{j-1}$.

We claim that $u_1 \sim p_{j-1}$. Suppose not. As u_1 has two neighbors on P, $u_1 \sim p_{j-2}, p_j$. Given z_2 and y_i, u_1 must be adjacent to p_j through the top. But then if u_1 is adjacent to p_{j-2} through the top, then u_1 would be adjacent to p_{j-1} as G is nearly triangulated. So we may assume that u_1 is adjacent to p_{j-2} through the bottom. Now consider the path $P' = P \setminus \{p_{j-1}, p_j\} \cup \{z_2, z_1\}$. Note then that z_1 has only one mate y_i in P', p_{j+1} has no mate in P', and as p_{j-1} is not a mate of z_2 since $p_{j-1} \not\sim z_1$, z_2 has at most one mate in P', namely u_1. Given that neither u_1 nor p_{j-1} is a mate of p_{j-2} in P', we find that p_{j-2} has no mate in P'. Combining these observations, we find that P' is an arrow.

However, the edge $p_{j-1}p_j$ is a 2-separation of the planarization $\Gamma_{P'}$ of Γ with respect to P'. Moreover, we claim that $U = p_{j-1}p_j$ is a stopping chord of $\Gamma_{P'}$ in the direction of C. Suppose not. Thus either p_j is adjacent vertex v with two neighbors in $P \setminus \{p_k : k \geq j-1\}$ such that one of those neighbors is through the top, or, p_{j-1} has a vertex with two such neighbors such that one is through the bottom. Suppose the former. Then P is not a shortest path, given v. So suppose the latter. Yet $v \neq u_1$ as p_{j-1} is not adjacent to u_1. Furthermore, u_1 is in a triangle $u_1w_1w_2$ in F^{i-2} where $w_1 \in P_T \setminus Y$ and $w_2 \in P_B \setminus Y$. It follows that $w_1 \sim p_{j-2}$ and hence $v = w_1$. But $w_1 \in P_T$ and hence does not have a neighbor on P through the bottom, a contradiction. This proves the claim.

So U is a stopping chord of $\Gamma_{P'}$ contradicting Lemma 4.5.3.

By an identical argument, we can show that $u_2 \sim p_{j+1}$. Now given z_2 and y_i, u_1 must be adjacent to p_{j-1} through the top. Similarly given z_3 and y_i, u_2 must be adjacent to p_{j+1} through the bottom. Thus $p_{j-1}p_jz_1u_1$ is a 4-cycle that does not separate C from C'. Thus there are no vertices in its interior. Yet as G is nearly triangulated, one of the edges $p_{j-1}z_1$ and p_ju_1 must be present. Yet z_1 is not adjacent to p_{j-1} and hence p_j is adjacent to u_1. A similar argument shows that p_j is adjacent to u_2. Now $T_1 = p_{j-1}u_1z_2$ and $T_2 = p_{j+1}u_2z_3$ are separating triangles. Indeed, $G[T_1, T_2]$

is a band. The internal vertices of that band are $p_j z_1 y_i z_4$ and it is not hard to check the adjacencies to see that the band is hexadecahedral as desired. □

Let P_2^* denote the set of vertices in P^* with a neighbor in Γ_2 through top and bottom. Let $P_1^* = P^* \setminus P_2^*$.

Lemma 4.6.9. *Assume Hypothesis 4.6.2. If $p \in P_2^*$, then either p is in a separating triangle or p is in the interior of a hexadecahedral band (with y as in Lemma 4.6.8).*

Proof. Suppose not. Then note that the neighbors of p in Γ_2 through the top must form a subpath as otherwise, the vertex in the middle is in Y and thus p is adjacent to it through the other side and hence p is in a separating triangle. Now if p has a neighbor on top and a neighbor on bottom that are adjacent then p is in a separating triangle as desired. So we may assume without loss of generality that the neighbors of p in Γ_2 through the top are closer to C than p's neighbors on the bottom and that the neighbors through the bottom are closer to C' than then neighbors on top.

Let u_1 be the neighbor through the top closest to C' along P_T and u_2 be the neighbor through the bottom closest to C along P_B. Let u_1' be the neighbor of u_1 in P_T closer to C'. If $u_1' \notin Y$, then consider the vertex $z_1 \in \Gamma_2$, that is in a triangle $z_1 u_1 u_1'$. Clearly, $z_1 \in P_B \setminus Y$. But then $p' z_1 u_1' p''$ yields a shorter path than P, where p' is the neighbor of z_1 on P closest to C and p'' is the neighbor of u_1' on P closest to C'. So we may suppose that u_1' in Y.

Let u_2' be the neighbor of u_2 in P_B closer to C. Similarly we find that $u_2' \in Y$. If $u_1' \neq u_2'$, then there exists $z_1 \in P_B \setminus Y$ and $z_2 \in P_T \setminus Y$ such that z_1 is adjacent to z_2 and $z_1 z_2$ is a chord of Γ_2 separating u_1 from u_2. But then $p' z_1 z_2 p''$ yields a shorter path than P, where p' is the neighbor of z_1 on P closest to C and p'' is the neighbor of z_2 on P closest to C'. So we may suppose $u_1' = u_2'$, call it y.

Let us call p, p_j, so that $p_{j-1} p_j p_{j+1}$ is a subpath of P with p_{j-1} closest to C. Now y is not adjacent to p_j as we chose u_1 closest to C' and u_2 closest to C. Let u_3 such that u_1u_3y is a triangle in Γ_2 and u_4 such that u_2u_4y is triangle in Γ_2. Now u_4 is not adjacent to p_j as u_1 was chosen closest to C. Thus there exists $k_4 \geq j+2$ such that $u_4 \sim p_{k_4}$. But this implies there does not exist $k \leq j-2$ such that $y \sim p_k$ as otherwise $p_k y u_4 p_{k_4}$ is a shortcut for P. Similarly, there exists $k_3 \leq j-2$ such that $u_3 \sim p_{k_3}$ and hence there does not exist $k \geq j+2$ such that $y \sim p_k$.

As y has at least two neighbors on P, we find that $y \sim p_{j-1}, p_{j+1}$. Indeed, y must be adjacent to p_{j+1} through the top as $u_1 \sim p_{j-1}, p_j$ through the top, and to p_{j-1} through the bottom as $u_2 \sim p_j p_{j+1}$ through the bottom. But then $u_1 p_j p_{j+1} y$ is a 4-cycle which does not separate C from C' and hence there are no vertices in its interior. As G is nearly triangulated and $y \not\sim p_j$, we find that $u_1 \sim p_{j+1}$. Similarly there are no vertices in the interior of $u_2 p_j p_{j-1} y$ and hence $u_2 \sim p_{j-1}$. Now if $u_1 \sim p_{j-1}$ and $u_2 \sim p_{j+1}$, then p_j has degree four, a contradiction.

So we may assume without loss of generality that $u_2 \not\sim p_{j+1}$. Now if $u_1 \not\sim p_{j-1}$, then it is not hard to see that y cannot be in separating triangle; thus, by Lemma 4.6.8, it follows that p_j is in the interior of the same hexadecahedral band as y and the lemma follows. So we may assume that $u_1 \sim p_{j-1}$.

Consider $P' = (P \setminus \{p_j\}) \cup \{y\}$. Now P' is also a shortest path from C to C'. Furthermore, y has exactly one mate for P', namely, u_1, p_{j-1} has exactly one mate for P', namely u_3, and p_{j+1} has exactly one mate for P', namely u_4.

Suppose P is an arrow from C to C'. Then P' is an arrow from C to C' unless p_{j+2} has two mates, call them p'_{j+2}, p''_{j+2}. Now given that u_4 is not a mate of p_{j+2} for P', either p'_{j+2} or p''_{j+2} is in a separating triangle with p_{j+1} and p_{j+2}. We may suppose without loss of generality that $p'_{j+2}p_{j+2}p_{j+1}$ is a separating triangle. Hence the edge $p'_{j+2}p_{j+2}$ is through the top and the edge $p'_{j+2}p_{j+1}$ is through the bottom. Thus $p'_{j+2} \in Y$. As Γ_1 is a harmonica, p'_{j+2} must be adjacent to u_4. Yet as Γ_1 is a harmonica, F^i must be an even fan where $y_{i-1} = y$ and $y_i = p'_{j+2}$. Thus p'_{j+2} is adjacent to u_2, a contradiction to the fact that P is a shortest path given the path

$p_{j-1}u_2p'_{j+2}p_{j+3}$. Hence if P is an arrow from C to C', P' is an arrow from C to C'.

Suppose P is an arrow from C' to C. Then P' is an arrow from C' to C unless p_{j-2} has two mates, call them p'_{j-2}, p''_{j-2}. Now given that u_3 is not a mate of p_{j-2} for P', either p'_{j-2} or p''_{j-2} is in a separating triangle with p_{j-1} and p_{j-2}. We may suppose without loss of generality that $p'_{j-2}p_{j-2}p_{j-1}$ is a separating triangle. Hence the edge $p'_{j-2}p_{j-2}$ is through the bottom and the edge $p'_{j-2}p_{j-1}$ is through the top. Thus $p'_{j-2} \in Y$. As Γ_1 is a harmonica, p'_{j-2} must be adjacent to u_1, a contradiction to the fact that P is a shortest path given the path $p_{j-3}p'_{j-2}u_1p_{j+1}$. Hence if P is an arrow from C' to C, P' is an arrow from C' to C.

So we may suppose that P' is an arrow in the same direction as P. Consider the planarization $\Gamma_{P'}$ of Γ with respect to P'. Now $U = u_1p_j$ is a dividing cut-edge in $\Gamma_{P'}$, contradicting Lemma 4.5.3. □

It now follows from the lemmas above that every vertex in Γ_2 is in a separating triangle in G_0. Let $V' = V(G) \cup P^*$. Let T_1 be the outermost separating triangle of G_0 with $V(T_1) \subseteq V'$ and T_2 be the innermost separating triangle of G_0 with $V(T_2) \subseteq V'$.

Let H be $G[V(\Gamma_2) \cup V(P^*)] \setminus \{e = uv | u, v \in V(\Gamma_2), e \notin E(\Gamma_2)\}$. Let $H' = H \setminus \{e = uv | u \in Y, v \in P^*\}$. Let $\mathcal{F}$ be the set of faces of H' incident with vertices in Y that are not triangles. Let $\mathcal{F}_i$ be faces in $\mathcal{F}$ with size i.

Proposition 4.6.10. *Assume Hypothesis 4.6.2. If $v \in \Gamma[T_1, T_2] \setminus (\Gamma_2 \cup P^*)$, then v is in the interior of a face f in $\mathcal{F}_5 \cup \mathcal{F}_6$.*

Proof. A vertex not in H must be in the interior of a face of H of size at least five. All faces of H not incident with a vertex in Y are triangles as G is a near-triangulation. Moreover, faces in H incident with a vertex in Y have size at most six. □

Proposition 4.6.11. *If $v \in P_1^*$, then v is incident with a face f in $\mathcal{F}_6$ and the edges of the boundary of f incident with v are in P.*

Proof. As $v \in P_1^*$, v does not have a neighbor in Γ_2 through either top or bottom. But then, on that side it must be incident with a face f in H of size at least four. Thus, but then f must be incident with a vertex in Y. If f has size less than six, then v has a neighbor in Γ_2. Thus f has size six and the edges of the boundary of f incident with v are in P. □

Now we may characterize the vertices in Y in a different more useful way. Namely, let $Y_{a,b}$ denote the set of vertices y in Y such that the two faces of H' that are incident with y but are not triangles have sizes a and b where $a \geq b$. Thus by the above Propositions, we may characterize the vertices of P_1^* and $\Gamma[T_1, T_2] \setminus (\Gamma_2 \cup P^*)$ by what $Y_{a,b}$ the vertex of y in those propositions belongs to.

Let $P_{a,b}$ be the vertices of P_1^* such that there exists y in $Y_{a,b}$ where p and y are both incident with a face of size six in H' as in Proposition 4.6.11. Hence $P_{a,b} = \emptyset$ if $a \leq 5$. We define $W_{a,b}$ as the vertices of $\Gamma[T_1, T_2] \setminus (V(\Gamma_2) \cup V(P^*))$ in the interior of a non-triangular face of H' incident with a vertex $y \in Y_{a,b}$.

Lemma 4.6.12. *Assume Hypothesis 4.6.2. If $a \leq 5$, then $W_{a,4} = \emptyset$.*

Proof. Suppose not. Then there exists a vertex $w \in W_{a,4}$, $a \leq 5$. Yet w must be in a face f of size at least five. Hence $a = 5$. But then the cutvertex y incident with f can have at most one neighbor in P, contradicting that Γ_2 is a harmonica. □

Lemma 4.6.13. *Assume Hypothesis 4.6.2. All vertices in $W_{5,5}$ are in a separating triangle.*

Proof. Let $w \in W_{5,5}$. Thus w is in a face f_1 of size five. Let y be the cutvertex incident with f. As y has two neighbors in P, y must be incident with two edges in the interior of f_2, the other face of size five in H' incident with y. Let p_j, p_{j+1} be these neighbors. Now if p_j or p_{j+1} is incident with f_1, then w is in a separating triangle as desired. Let p_k and p_{k+1} be the vertices of P incident with f_1. So we may suppose without loss of generality that $k \geq j+2$. Let u be such that up_{k+1} is an edge incident with f_1 and $u \neq p_k$. By Lemma 4.6.3, $u \sim p_{k+2}$. But then $p_j y u p_{k+2}$ shows that P is not a shortest path from C_1 to C_2, a contradiction. □

Lemma 4.6.14. *Assume Hypothesis 4.6.2. All vertices in* $\bigcup_{b\geq 4}(W_{6,b} \cup P_{6,b})$ *are in a separating triangle.*

Proof. Suppose not. Let z be such a vertex not in a separating triangle. If $z \in P_1^*$, let $y \in Y$ be the cutvertex opposite z as in Lemma 4.6.11; otherwise let $y \in Y$ be the cutvertex incident with the face of H' containing z in its interior as in Lemma 4.6.10.

Recall that $P = p_1 \dots p_d$ is a shortest path from C to C'. Let f_1, f_2 be the two non-triangular faces of H' incident with y. We may suppose without loss of generality that f_1 is on top and f_2 is on bottom, that $|f_1| = 6$, and f_1 is a face for z as in Lemmas 4.6.11 and 4.6.10. Let u_1u_2y and u_3u_4y be the triangles of Γ_2 incident with y. We may assume without loss of generality that $u_1, u_3 \in P_T \setminus Y$ and hence $u_2, u_4 \in P_B \setminus Y$. Let p_{j-1}, p_j, p_{j+1} be the vertices of P incident with f_1 such that the boundary of f_1 is $p_{j-1}p_jp_{j+1}u_3yu_1$. Thus $N(u_1) \cap P = \{p_{j-2}, p_{j-1}\}$ and $N(u_3) \cap P = \{p_{j+1}, p_{j+2}\}$ as otherwise P is not shortest. Therefore, we may assume that if $z \in P$, then $z = p_j$.

Claim 4.6.15. $|f_2| \neq 4$.

Proof. Suppose not. Now y is incident with at most one edge that lies in f_2. Yet y has at least two neighbors on P; thus y is incident with at least one edge that lies in f_1. Suppose that yp_j is such an edge. But then y is adjacent to p_{j-1}, p_{j+1} through edges in f_1 as G is nearly triangulated. Thus there is no vertex in the interior of f_1 or f_2. Moreover, one of u_2, u_4 is adjacent to p_j as P is a shortest path. Thus, yp_j is in a separating triangle, a contradiction as $z = p_j$.

So we may assume that yp_j is not an edge that lies in f_1. Without loss of generality, we may suppose that yp_{j+1} is an edge that lies in f_1. Suppose there is no vertex in the interior of f_1; hence $z = p_j$. Yet as p_j is not adjacent to u_1 and G is nearly traingulated, the edge $p_{j-1}y$ must lie in f_1. But then as G is nearly triangulated and yp_j is not an edge that lies in f_1, yp_j must be an edge and hence it must lie in f_2. Thus yp_jp_{j+1} is a separating triangle, contradicting that $z = p_j$.

So we may assume there exists a vertex v in the interior of f_1. Hence $N(v) = \{p_{j-1}, p_j, p_{j+1}, y, u_1\}$. Yet y must have another neighbor in P. Thus either p_j is in f_2 and the edge yp_j lies in f_2 or p_{j-1} is in f_2 and the edge yp_{j-1} lies in f_2. Suppose the former. Then yvp_j is a separating triangle, contradicting that $z \in \{v, p_j\}$. So we may suppose the latter that yp_{j-1} is an edge that lies in f_2.

Suppose P is an arrow from C to C'. Consider $P' = (P \setminus \{p_j\}) \cup \{y\}$. Now P' is also a shortest path from C to C'. Furthermore, y has exactly one mate for P', namely, v, and p_{j-1} has exactly two mates for P', namely u_1, u_2. Moreover, p_{j+1} has one mate for P', namely u_3. In addition, p_{j-2} has no mate for P' given u_1, u_2. Now P' is an arrow from C to C' unless p_{j+2} has two mates, call them p'_{j+2}, p''_{j+2}. Now either p'_{j+2} or p''_{j+2} is in a separating triangle with p_{j+1} and p_{j+2}. We may suppose without loss of generality that $p'_{j+2}p_{j+2}p_{j+1}$ is a separating triangle. Hence the edge $p'_{j+2}p_{j+2}$ is through the top and the edge $p'_{j+2}p_{j+1}$ is through the bottom because $u_3 \sim p_{j+1}, p_{j+2}$. Thus $pj + 2' \in Y$. As Γ_1 is a harmonica, p'_{j+2} must be adjacent to u_3. Yet as Γ_1 is a harmonica, F^i must be an even fan where $y_{i-1} = y$ and $y_i = p'_{j+2}$. Thus p'_{j+2} is adjacent to u_4, a contradiction to the fact that P is a shortest path given the path $p_{j-1}u_4p'_{j+2}p_{j+3}$.

So we may suppose that P' is an arrow from C to C'. Consider the planarization $\Gamma_{P'}$ of Γ with respect to P'. Now $U = vp_j$ is a dividing cut-edge of $\Gamma_{P'}$, contradicting Lemma 4.5.3.

Finally we may assume that P is an arrow from C' to C. Consider $P'' = (P \setminus \{p_j, p_{j+1}\}) \cup \{u_4, u_3\}$. Now P'' is also a shortest path from C to C'. Furthermore, u_4 has exactly one mate for P'', namely, y. Moreover, neither p_{j-1} nor p_{j-2} has a mate for P''. Note that p_{j+1} is not adjacent to u_4 as then p_j has degree four, a contradiction. Thus p_{j+1} is not a mate of u_3 for P''. So u_3 has at most one mate for P''.

Now P'' is an arrow from C' to C unless p_{j+2} has two mates, call them p'_{j+2}, p''_{j+2} and p_{j+3} has at least one mate, call it p'_{j+3}. Now either p'_{j+2} or p''_{j+2} is in a separating triangle with u_3 and p_{j+2}. We may suppose without loss of generality that $p'_{j+2}p_{j+2}u_3$ is a separating triangle. Hence the edge $p'_{j+2}p_{j+2}$ is through the bottom and the edge $p''_{j+2}p_{j+2}$ is through the top. Thus the edge $p'_{j+2}p_{j+3}$ is through the bottom as otherwise p''_{j+2} is in the interior of the 4-cycle $p'_{j+2}p_{j+3}p_{j+2}u_3$ which does not separate C from C', a contradiction.

As p_{j+3} has a mate for P'', $p''_{j+2}p_{j+3}$ cannot be through the top and so must be through the bottom. Hence $p''_{j+2} \in Y$. But then p''_{j+2} is adjacent to u_3 and $p_{j+2'}$ as Γ_1 is a harmonica. Thus F^i is an even fan where $y_{i-1} = y$ and $y_i = p''_{j+2}$. It now follows that there exists $u_5 \in \Gamma_1$ adjacent to all of u_3, u_4, p'_{j+2}. Moreover as $u_5 \in P_B \setminus Y$ and hence $N(u_5) \cap P$ is a path of length one or two. Yet $N(u_5) \cap P \subseteq \{p_j, p_{j+1}, p_{j+2}\}$. Hence $u_5 \sim p_{j+1}$. But then $u_5 \sim p_j$, as otherwise p_j has a degree at most four, a contradiction.

Note the 4-cycle $u_5p'_{j+2}p_{j+2}p_{j+1}$ that does not separate C from C'. As G is nearly triangulated, it follows that either $u_5 \sim p_{j+1}$ or $p'_{j+2} \sim p_{j+1}$. Suppose $u_5 \sim p_{j+1}$. Let $P''' = P \setminus \{p_{j+1}\} \cup \{u_5\}$. Now p_{j-1} has no mates for P'''. Furthermore, p_j has exactly one mate for P'', namely, u_4; u_5 has exactly one mate, namely p_{j+1}; p_{j+2} has exactly one mate, namely p'_{j+2}. Hence P''' is an arrow from C' to C. Yet $U = u_3p''_{j+2}$ is a dividing cut-edge of $\Gamma_{P'''}$, contradicting Lemma 4.5.3.

So we may suppose that $p'_{j+2} \sim p_{j+1}$. Let $P''' = P \setminus \{p_{j+2}\} \cup \{p'_{j+2}\}$. Now p_j has exactly one mate for P''', namely, v; p_{j+1} has no mate; p'_{j+2} has exactly one mate, namely p_{j+2}. In addition, p_{j+3} has no mate for P'''. Hence P''' is an arrow from C' to C. Yet $U = yu_3$ is a dividing cut-edge of $\Gamma_{P'''}$, contradicting Lemma 4.5.3.

So we may suppose that P'' is an arrow from C' to C. So consider the planarization $\Gamma_{P''}$ of Γ with respect to P''. Now v has a list of size at least four in $\Gamma_{P''}$ and y has a list of size at least three. Yet vy is a chord of the infinite face of $\Gamma_{P''}$ and y has only three neighbors in $\Gamma_{P''}$ with a list of size three. Hence, $U = vy$ is a blocking chord of $\Gamma_{P''}$, contradicting Lemma 4.5.3. $\square$

Claim 4.6.16. $|f_2| \neq 5$.

Proof. Suppose not. We may assume without loss of generality that the boundary of f_2 is $p_{j-1}p_j u_4 y u_2$ and that $N(u_2) \cap P = \{p_{j-2}, p_{j-1}\}$ and that $N(u_4) \cap P \supseteq \{p_j, p_{j+1}\}$. Note that p_j is not adjacent to u_1, u_2, u_3. In addition, $u_2 \not\sim u_4$ as otherwise $u_2u_4p_jp_{j-1}$ is a 4-cycle that does not separate C from C', but then as G is nearly triangulated either $u_2 \sim p_j$ or $u_4 \sim p_{j+1}$, contradicting that $|f_2| = 5$.

First suppose there exists a vertex v in the interior of f_2. As y has at least two neighbors in P, y is incident with two edges that lie in f_1. Hence there is no vertex in the interior of f_1. As G is nearly triangulated, it follows that $y \sim p_{j-1}, p_j, p_{j+1}$. Hence p_jvy is a separating triangle, a contradiction as $z \in \{v, p_j\}$.

So we may assume there does not exist a vertex in the interior of f_2. As G is nearly triangulated and $u_2 \not\sim p_j$, the edge $p_{j-1}y$ lies in f_2. Suppose $y \not\sim p_j$. Thus $y \sim p_{j+1}$ as y has two neighbors in P. Hence yp_{j+1} lies in f_1 and there is at most one vertex in the interior of f_1. But then p_j has degree at most four, a contradiction. So we may assume that $y \sim p_j$. If yp_j lies in f_1, then $p_j \sim u_1$ as G is nearly traingulated, a contradiction. So we may suppose that yp_j lies in f_2.

As P is a shortest path, p_{j-1} is not adjacent to either u_3 or p_{j+1}. Thus if there does not exist a vertex in the interior of f_1, then $u_1 \sim p_j$ as G is nearly triangulated, contradicting that $|f_1| = 6$. So we may suppose there exists a vertex in the interior of f_1. Now condition on the interior of f_1 using Theorem 1.5.2. Suppose case (i) holds. That is, there exists exactly one vertex v in the interior of f_1. As G is nearly triangulated, it follows that v is adjacent to all of $p_{j-1}, p_j, p_{j+1}, u_1, u_3, y$. Hence $p_j v y$ is a separating triangle, a contradiction as $z \in \{v, p_j\}$.

So there are at least two vertices in the interior. We claim now that u_4 is not adjacent to p_{j+2}. Suppose it is. As $u_4 \in P_B \setminus Y$, the edge $u_4 p_{j+2}$ must go through the bottom. Hence the 7-walk $C_0 = p_{j-1} y u_4 p_{j+2} u_3 y u_1$ does not separate C from C'. As case (ii) or (iii) holds for f_1, every L-coloring of the boundary of C_0 extends to its interior, a contradiction. This proves the claim that u_4 is not adjacent to p_{j+2}.

So suppose case (ii) holds. That is, there exist two adjacent vertices v_1, v_2 in the interior of f_1 such that v_1, v_2 are each adjacent to four vertices in the boundary of f_1. Suppose v_1, v_2 are both adjacent to p_j. It follows that v_1, v_2 are both adjacent to y. Hence $p_j v_1 y$ and $p_j v_2 y$ are separating triangles, a contradiction as $z \in \{v_1, v_2, p_j\}$. So we may suppose without loss of generality that v_1 is adjacent to p_{j-1}, p_j, p_{j+1} and v_2 is adjacent to u_1, y, u_3. There are two cases to consider: v_1 is adjacent to u_1 and v_2 is adjacent to p_{j+1}, or, v_1 is adjacent to u_3 and v_2 is adjacent to p_{j-1}.

Suppose that v_1 is adjacent to u_1 and v_2 is adjacent to p_{j+1}. Consider $P' = (P \setminus p_j) \cup v_1$. Now P' is also a shortest path from C to C'. Furthermore, v_1 has exactly one mate for P', namely, p_j, and p_{j-1} has exactly one mate for P', namely u_1. Moreover, p_{j+1} has no mate for P'. In addition, p_{j-2} has no mate for P' given u_1, u_2. Thus P' is an arrow in the same direction as P'. So consider the planarization $\Gamma_{P'}$ of Γ with respect to P'. Now v_2 has a list of size at least three in $\Gamma_{P'}$ and y has a list of size at least four. Yet $v_2 y$ is a chord of the infinite face of $\Gamma_{P'}$ and v_2 has only two neighbors in $\Gamma_{P'}$ with a list of size three. Hence, $U = v_2 y$ is a blocking chord of $\Gamma_{P'}$, contradicting Lemma 4.5.3.

So we may suppose that v_1 is adjacent to u_3 and v_2 is adjacent to p_{j-1}. Consider $P' = (P \setminus p_j) \cup v_1$. Now P' is also a shortest path from C to C'. Furthermore, v_1 has exactly one mate for P', namely, p_j, and p_{j+1} has exactly one mate for P', namely u_3. Moreover, p_{j-1} has no mate for P'. Suppose P' is an arrow in the same direction as P. So consider the planarization $\Gamma_{P'}$ of Γ with respect to P'. Now v_2 has a list of size at least three in $\Gamma_{P'}$ and y has a list of size at least four. Yet v_2y is a chord of the infinite face of $\Gamma_{P'}$ and v_2 has only two neighbors in $\Gamma_{P'}$ with a list of size three. Hence, $U = v_2y$ is a blocking chord of $\Gamma_{P'}$, contradicting Lemma 4.5.3.

So we may suppose that P' is not an arrow in the same direction as P. But this implies that P is an arrow from C to C' and that p_{j+2} is tripled for P'. That is, there exist two mates, call them p'_{j+2}, p''_{j+2}, of p_{j+2} for P'. We may suppose without loss of generality that $p_{j+1}p_{j+2}p'_{j+2}$ is a separating triangle. Thus $p'_{j+2} \in Y$ and as Γ_1 is a harmonica, y is adjacent to u_3 and u_4. Hence $p''_{j+2} \notin Y$ and the edge $p''_{j+2}p_{j+3}$ is through the bottom. As p''_{j+2} has degree at least five, the edge $p'_{j+2}p_{j+3}$ must be through the top. So consider $P'' = P \setminus \{p_j, p_{j+1}\} \cup \{y, u_3\}$. Now P'' is also a shortest path from C to C'. Furthermore, p_{j-2} has no mate for P''; p_{j-1} has two mates, namely u_1, u_2; y has one mate, namely v_2; u_3 has no mate as $u_4 \not\sim p_{j+2}$; p_{j+2} has one mate, namely p'_{j+2}; p_{j+3} has no mate.

Hence P'' is an arrow from C to C'. So consider the planarization $\Gamma_{P''}$ of Γ with respect to P''. Yet $U = v_1v_2$ is a dividing cut-edge of $\Gamma_{P''}$, contradicting Lemma 4.5.3.

So we may assume that case (iii) holds. That is, there exist three pairwise adjacent vertices v_1, v_2, v_3 in the interior of f_1 such that v_1, v_2, v_3 are each adjacent to three vertices in the boundary of f_1. Now we may suppose without loss of generality that either $v_1 \sim p_{j-1}, p_j, p_{j+1}$, $v_2 \sim p_{j-1}, u_1, y$ and $v_3 \sim y, u_3, p_{j+1}$, or, $v_1 \sim u_1, y, u_3$, $v_2 \sim u_1, p_{j-1}, p_j$ and $v_3 \sim p_j, p_{j+1}, u_3$. Suppose the former. Consider $P' = (P \backslash p_j) \cup v_1$. Now P' is also shortest path from C to C'. Furthermore, v_1 has exactly one mate for P', namely, p_j. Moreover, neither p_{j-1} nor p_{j+1} has a mate for P'. Hence P' is an arrow in the same direction as P. So consider the planarization $\Gamma_{P'}$ of Γ with respect to P'. Now v_2 has a list of size at least three in $\Gamma_{P'}$ and y has a list of size at least four. Yet $U = v_2y$ is a chord of the infinite face of $\Gamma_{P'}$ and v_2 has only two neighbors in $\Gamma_{P'}$ with a list of size three. Hence, v_2y is a blocking chord of $\Gamma_{P'}$, contradicting

Lemma 4.5.3.

So we may suppose the latter case that $v_1 \sim u_1, y, u_3$, $v_2 \sim u_1, p_{j-1}, p_j$ and $v_3 \sim p_j, p_{j+1}, u_3$. Consider $P'' = P \setminus \{p_j, p_{j+1}\} \cup \{y, u_3\}$. Now P'' is also a shortest path from C to C'. Furthermore, p_{j-2} has no mate for P''; p_{j-1} has two mates, namely u_1, u_2; y has no mate; u_3 has no mate as $u_4 \not\sim p_{j+2}$. However, p_{j+2} and p_{j+3} may have one or two mates for P''. Now P'' is an arrow from C to C' unless p_{j+2} has one mate and p_{j+3} has two mates. Similarly, P'' is an arrow from C' to C unless p_{j+3} has one mate and p_{j+3} has two mates.

Suppose P'' is an arrow in the same direction as P. Consider the planarization $\Gamma_{P''}$ of Γ with respect to P''. Yet v_2v_3 is a chord of $\Gamma_{P''}$ and v_2, v_3 have lists of size at least four in $\Gamma_{P''}$. Thus $U = v_2v_3$ is a blocking chord of $\Gamma_{P''}$, contradicting Lemma 4.5.3, a contradiction. So we may suppose that P'' is not an arrow in the same direction as P.

So suppose P is an arrow from C to C'. Hence P'' is not an arrow from C to C'. Therefore p_{j+2} has at least one mate p'_{j+2} for P'' and p_{j+3} has two mates, call them p'_{j+3}, p''_{j+3} for P''. Given p'_{j+2}, either $p_{j+3}p'_{j+3}p_{j+2}$ or $p_{j+3}p''_{j+3}p_{j+2}$ is a separating triangle. Suppose without loss of generality that $p_{j+3}p'_{j+3}p_{j+2}$ is a separating triangle. Now there are two cases: the edge $p_{j+2}p'_{j+3}$ is through the top or through the bottom. Suppose it is through the top. As Γ_1 is a harmonica, p'_{j+3} must be adjacent to u_3. But then $p_{j-1}yu_3p'_{j+3}p_{j+4}$ shows that P is not a shortest path, a contradiction. So we may suppose the edge $p_{j+2}p'_{j+3}$ is through the bottom. Note that as P is a shortest path, $p'_{j+3} \not\sim u_3, u_4$. Consider the vertex $u_5 \neq y$ such that $u_3u_4u_5$ is triangle in Γ_1. If $u_4 \in P_T$, then $N(u_5) \cap P = \{p_{j+2}, p_{j+3}\}$ while if $u_4 \in P_B$, then $N(u_5) \cap P = \{p_{j+1}, p_{j+2}\}$. In either case, u_5 must be adjacent to p'_{j+3}, a contradiction to the fact that Γ_1 is a harmonica as otherwise in the subcanvas F^i between $y = y_{i-1}$ and $p'_{j+3} = y_i$, the fan F^i_1 has the wrong parity - it is even when it should be odd, a contradiction.

So we may suppose that P is an arrow from C' to C. Hence P'' is not an arrow from C' to C. Therefore p_{j+3} has at least one mate p'_{j+3} for P'' and p_{j+2} has two mates, call them p'_{j+2}, p''_{j+2} for P''. Given p_{j+1}, either $p_{j+2}p'_{j+2}u_3$ or $p_{j+2}p''_{j+2}u_3$ is a separating triangle. Suppose without loss of generality that $p_{j+2}p'_{j+2}u_3$ is a separating triangle. Hence the edge $p'_{j+2}p_{j+3}$ is through the bottom. Thus the edge $p'_{j+2}p_{j+3}$ is through the bottom as otherwise p''_{j+2} has degree at most four, a contradiction. Furthermore, the edge $p''_{j+2}p_{j+2}$ must be the through the top. Yet given p'_{j+3}, the edge $p''_{j+2}p_{j+3}$ must be through the bottom. Hence $p''_{j+2} \in Y$. It follows that the edge $p''_{j+2}p'_{j+2}$ exists and that $p'_{j+2}p''_{j+2}u_3$ is a triangle in Γ_1. As Γ_1 is a harmonica, F_1^i must be an even fan where $y_{i-1} = y$ and $y_i = p''_{j+2}$. Thus there exists a vertex $u_5 \sim u_3, u_4, p'_{j+2}, p_{j+2}, p_{j+1}$. Now consider the 8-walk $C_0 = yu_3p'_{j+2}p_{j+2}u_3yu_1p_{j-1}$; C_0 does not separate C from C'. Nevertheless, every L-coloring of $G[V(C_0)]$ extends to an L-coloring of $G[V(C_0) \cup \{p_j, p_{j-1}, v_1, v_2, v_3, u_4, u_5\}]$. This follows from Theorem 2.2.2 as only v_1 and u_5 have at least three neighbors in C_0 and they have exactly three. $\square$

Claim 4.6.17. $|f_2| \neq 6$.

Proof. Suppose not. As P is a shortest path, it follows that the boundary of f_2 is $p_{j-1}p_jp_{j+1}u_4yu_2$ and that $N(u_2) \cap P = \{p_{j-2}, p_{j-1}\}$ and that $N(u_2) \cap P = \{p_{j+1}, p_{j+2}\}$. Note though that p_j is not adjacent to any of u_1, u_2, u_3, u_4 and yet p_j has degree at least five. Now p_j is adjacent to p_{j-1} and p_{j+1} and possibly y, but this implies that there must be at least two vertices contained in the interiors of f_1 and f_2.

Suppose first that $y \sim p_j$. We may assume without loss of generality that the edge yp_j is in f_2. But then there are no vertices in the interior of f_2. As p_j has at least two neighbors in the interiors of f_1 and f_2, there must be at least two vertices in the interior of f_1. So case (ii) or (iii) of Theorem 1.5.2 for f_1. But then as G is nearly triangulated, y is adjacent to p_{j-1} and p_{j+1} through f_2. Hence $p_{j-1}yp_{j+1}u_3yu_1$ is a 6-walk that does not separate C from C'. Lt G_{C_0} be the graph whose boundary is C_0 as well the edges and vertices inside the disk bounded by C_0. By Theorem 1.5.2 applied to C_0 for G_{C_0}, there are at most three vertices in its interior. As p_j is in $V(G_{C_0}) \setminus V(C_0)$ and there are are least two vertices in the interior, it follows that case (iii) holds for G_{C_0}, that there are exactly two vertices v_1, v_2 in the interior of f_1 and that p_j is adjacent to both of them. Hence case (ii) holds for f_1 and $y \sim v_1, v_2$. So $p_j v_1 y$ and $p_j v_2 y$ are separating triangles, a contradiction as $z \in \{p_j, v_1, v_2\}$.

So we may assume that y is not adjacent to p_j. Thus y is adjacent to p_{j-1} and p_{j+1}. If the edges $p_{j-1}y$ and $p_{j+1}y$ are both in f_1 or both in f_2, then y is adjacent to p_j as G is nearly triangulated, a contradiction. So we may assume without loss of generality that $p_{j-1}y$ lies in f_2 and $p_{j+1}y$ lies in f_1. By Theorem 1.5.2, there is at most one vertex in the interior of f_1 and at most one vertex in the interior of f_2. Hence p_j has degree at most four, a contradiction. $\square$

$\square$

Next we consider the vertices of Z^i.

Lemma 4.6.18. *Assume Hypothesis 4.6.2. Let $z \in Z$ and $y \in Y$ such that $y \sim z$. If $y \in Y_1 \cup Y_2$, then z is in a separating triangle. If $y \in Y_1^*$, then z is in a separating triangle or the interior of a hexadecahedral band (with y as in the proof of Lemma 4.6.8).*

Proof. Suppose not. Let x be such that xyz is a triangle in Γ_2. We may assume without loss of generality that $z \sim p_j, p_{j+1}$ and $x \sim p_{j-2}, p_{j-1}$ as otherwise zx is in a separating triangle. We may also suppose without loss of generality that $z \in P_T \setminus Y$ and $z \in P_B \setminus Y$. Furthermore $y \not\sim p_k$ where $k \geq j+2$ given the path $p_{j-2}xyp_k$. Thus $N(y) \cap P \subseteq \{p_{j-1}, p_j, p_{j+1}\}$. Yet $y \not\sim p_j$ as otherwise zyp_j is a separating triangle, a contradiction. Hence $y \sim p_{j-1}, p_{j+1}$. Thus y is adjacent to p_{j-1} through the bottom given z. But then y is adjacent to p_{j+1} through the top as G is nearly triangulated and $y \not\sim p_j$.

Let yu_1u_2 be the other triangle in Γ_2 containing y. We may assume without loss of generality that $u_1 \in P_T$ and $u_2 \in P_B$. Given the path $p_{j-2}xyu_1$, we find that $u_1 \sim p_{j+1}, p_{j+2}$. Note then that p_j is not in a separating triangle. If $p_j \sim u_2$, then $p_j \in P_2^*$ and hence p_j is in a separating triangle, a contradiction, or in the interior of a hexadecahedral band with y by Lemma4.6.9. In the latter case, z must also be in the interior of the hexadecahedral band, a contradiction. Thus $p_j \not\sim u_2$ and hence p_j does not have neighbor in Γ_2 through the bottom. Thus $p_j \in P_1^*$. But then p_j is in a separating triangle by Lemmas ..., a contradiction. □

Corollary 4.6.19. *Assume Hypothesis 4.6.2. If $v \in \Gamma[T_1, T_2]$, then v is in T_1, T_2, or a triangle separating a vertex of T_1 from a vertex of T_2, or in the interior of a hexadecahedral band.*

Proof. By Lemmas 4.6.4, 4.6.5, 4.6.7, 4.6.6, 4.6.8 and Lemmas...every vertex in $\Gamma[T_1, T_2]$ is in a separating triangle in Γ or the interior of a hexadecahedral band. As triangles cannot cross, it follows that every vertex in $\Gamma[T_1, T_2]$ is in T_1, T_2 or a triangle separating a vertex of T_1 from a vertex of T_2 or in the interior of a hexadecahedral band. □

Lemma 4.6.20. *Assume Hypothesis 4.6.2. If B is a band in the band decomposition of $\Gamma[T_1, T_2]$, then B is tetrahedral, octahedral or hexadecahedral.*

Proof. Let $B = (G_B, T_3, T_4, L)$ be a band in the band decomposition. If $G = T_3 \cup T_4$, then B is tetrahedral or octahedral by Lemma 4.4.4. So suppose that $G \neq T_3 \cup T_4$. But then $G_B \setminus T_3 \cup T_4$ must contain a vertex which is not in a separating triangle. But then by Corollary 4.6.19, that vertex is in the interior of a hexadecahedral band. So B must be hexadecahedral. □

Proof of Lemma 4.6.1. We may assume Hypothesis 4.6.2. By Lemma 4.6.20, every band in the band decomposition of $\Gamma[T_1, T_2]$ is tetrahedral, octahedral or hexadecahedral. Furthermore, $d(T_1, T_2) \geq d(P_1, P_2) - 2$ where $\Gamma_2 = (G, P_1 \cup P_2, L)$. As Γ_2 is a shortening of Γ_1, it follows from Lemma 4.5.2 that $d(T_1, T_2) \geq d(P_1, P_2) - 34$ where P_1', P_2' are the sides of Γ_1. $\square$

We may now show that a minimum counterexample to Theorem 4.1.1 has distant separating triangles such that every band in the band decomposition of the prism-canvas between them is tetrahedral, octahedral or hexadecahedral.

Lemma 4.6.21. *Let $d_0 > 0$. If $\Gamma = (G, T_1, T_2, L)$ is a counterexample to Theorem 4.1.1 with a minimum number of vertices and subject to that a maximum number of edges, then there exist triangles T_1' and T_2' of G each separating C_1 from C_2 such that $d(T_1', T_2') \geq d_0$ and every band in the band decomposition of $\Gamma[T_1', T_2']$ is a subgraph of a tetrahedral, octahedral or hexadecahedral band.*

Proof. We may assume that (1) holds for Lemma 4.4.6 as otherwise the lemma follows. That is to say Γ is nearly triangulated and there exist T_1', T_2' and every planarization Γ^* of $\Gamma[T_1', T_2']$ contains a long bottleneck $\Gamma_1 = (G', P_1 \cup P_2, L)$ where $d(T_1, T_2) - 2d_0 - c_0 \leq 6d(P_1', P_2')$. As $d(P_1, P_2) \geq 32$, it follows, from Lemma 4.5.16 if Γ_1 is an accordion and Lemma 4.6.20 if Γ_1 is a harmonica, that there exist triangles T_1'', T_2'' separating C_1 from C_2 such that $d(T_1'', T_2'') \geq d(P_1, P_2) - 34$ and every band in the band decomposition of $\Gamma[T_1'', T_2'']$ is tetrahedral, octahedral, or hexadecahedral. As Γ is a counterexample, we may assume that $d(T_1, T_2) \geq 2d_0 + c_0 + 6(34 + d_0)$ and hence $d(T_1'', T_2'') \geq d_0$ as desired. $\square$

4.7 Magic Colorings with Tetrahedral Bands

Our goal in the following sections is to prove the following:

Theorem 4.7.1. *There exists d such that the following holds:*

Let $\Gamma = (G, T_1, T_2, L)$ be a prism-canvas such that $d(T_1, T_2) \geq 14$ and every band in the band decomposition of Γ is tetrahedral, octahedral or hexadecahedral. Let L be a list assignment of G such that $|L(v)| = 5$ for all $v \in V(G)$. If ϕ is an L-coloring of $T_1 \cup T_2$, then ϕ extends to an L-coloring of G.

First in this section, we will define certain magical sets of colorings for triangles that will be useful in showing that such a coloring exist. Then we will proceed to develop the theory of magical colorings for tetrahedral bands.

Definition. Let T be a triangle and L a list assignment for T such that $|L(v)| = 5$ for all $v \in T$. We say a set of proper L-colorings $\mathcal{C}$ of T is

Magic 1: If $\mathcal{C}$ is precisely the set of proper L'-colorings of T where for some ordering v_1, v_2, v_3 of T, either

(1a) $|L'(v_1)| = |L'(v_2)| = |L'(v_3)| = 2$ and these are pariwise disjoint, or

(1b) $|L'(v_1)| = |L'(v_2)| = 2, |L'(v_3)| = 3,\ L'(v_1) \cap L'(v_3) = \emptyset$, and $L'(v_2) \subseteq L'(v_3)$.

(Alternately, $|L'(v_1)| = |L'(v_2)| = 2,\ |L'(v_3)| = 3,\ L'(v_1) \cap L'(v_2) = \emptyset$ and $L'(v_1) \cap L'(v_3) = \emptyset$.)

Magic 2: If $\mathcal{C}$ is precisely the set of proper L'-colorings of T where for some ordering v_1, v_2, v_3 of T, either

(2a) $|L'(v_1)| = |L'(v_2)| = |L'(v_3)| = 2$ and these are pariwise disjoint, or

(2b) $|L'(v_1)| = 2, |L'(v_2)| = 2, |L'(v_3)| = 5,\ L'(v_1) \cap L'(v_2) = \emptyset,\ L'(v_1), L'(v_2) \subseteq L'(v_3)$, or,

(2c) $|L'(v_1)| = 2, |L'(v_2)| = |L'(v_3)| = 3,\ L'(v_1) \cap L'(v_2) = \emptyset,\ L'(v_1) \subseteq L'(v_3)$.

Magic 3: If there exist list assignments L', L'' of T such that $\mathcal{C}$ is the set of proper colorings which are L'-colorings but not L''-colorings where the list assignments are one of the following for some ordering v_1, v_2, v_3 of T:

(3a) $|L'(v_1)| = |L'(v_2)| = |L'(v_3)| = 2$ and these are pariwise disjoint, and $L''(v) = \emptyset$ for all $v \in T$, or

(3b) $|L'(v_1)| = 2, |L'(v_2)| = 2, |L'(v_3)| = 5,\ L'(v_1) \cap L'(v_2) = \emptyset,\ L'(v_1), L'(v_2) \subseteq L'(v_3)$, and $L''(v) = \emptyset$ for all $v \in T$, or,

(3c) $|L'(v_1)| = 2,\ |L'(v_2)| = |L'(v_3)| = 5,\ L'(v_1) \subset L'(v_2), L'(v_3),\ |L'(v_2) \cap L'(v_3)| \geq 4$, $L''(v_2) = L''(v_3)$ is a subset of size two of $L'(v_2) \cap L'(v_3) \setminus L'(v_1)$ and $L''(v_1) = L'(v_1)$.

Definition. Let (G, C_1, C_2) be a prismatic graph such that every band in its band decomposition $\mathcal{B} = B_1 \dots B_m$ is tetrahedral. Let $T_0 = C_1$, $T_m = C_m$ and let $T_1 \dots T_m$ be the triangles such that $B_i = G[T_{i-1}, T_i]$. We now define a natural mapping $p : V(G) \to V(C_1)$ as follows: Let $p(v) = v$ for all $v \in C_1$. Then for successive i, define $p(v)$ where $v \in T_i \setminus T_{i-1}$ to be $p(u)$ where $u \in T_{i-1} \setminus T_i$. Note that there also exists a natural ordering of the vertices of $G \setminus C_1$, namely $x_i = T_i \setminus T_{i-1}$. We now define the *signature* of (G, C_1, C_2) to be the sequence $p(x_1)p(x_2)\dots p(x_m)$. We say a prismatic graph is *variegated* if every band in its band decomposition is tetrahedral, there does not exist $x \in C_1$ such that x appears consecutively in its signature, and there does not exist $x, y \in C_1$ such that x, y appear consecutively three times in its signature, namely, $xyxyxy$.

We say a prism-canvas is *variegated* if its underlying prismatic graph is variegated.

Proposition 4.7.2. *If $\Gamma = (G, C_1, C_2, L)$ is a critical prism canvas such that every band in the band decomposition of G is tetrahedral, then Γ is variegated.*

We say a variegated prism-canvas (G, C_1, C_2, L) is *rainbow* if all three vertices of C_1 appear in the signature of (G, C_1, C_2).

Corollary 4.7.3. *Let $\Gamma = (G, C_1, C_2, L)$ be a variegated prism-canvas. If Γ has at least six bands in its band decomposition, then there exists (essential) triangles T_1, T_2 such that $\Gamma[T_1, T_2]$ is a rainbow prism-canvas with three bands in its band decomposition. Furthermore, if Γ has at least seven bands in its band decomposition, there exists T_1, T_2 such that $\Gamma[T_1, T_2]$ is a rainbow prism-canvas with four bands in its band* *decomposition and letting $T_1 = v_1v_2v_3$, the signature of $(G[T_1, T_2], T_1, T_2)$ has one of the following forms up to permuation of $\{1, 2, 3\}$:*

(i) $v_1v_2v_3v_1$

(ii) $v_1v_2v_3v_2$

4.7.1 Magic 1

Lemma 4.7.4. *Let $B = T_0 \dots T_4$ be a rainbow sequence of tetrahedral bands of length four such that $B = T_0 \dots T_3$ is also rainbow (i.e. form of previous lemma). If ϕ is a proper coloring of T_0, then there exists a Magic 1 set $\mathcal{C}$ of colorings of T_4 such that for every $\phi' \in \mathcal{C}$, $\phi \cup \phi'$ extends to an L-coloring of B.*

Proof. We let $S(u) = L(u) \setminus \{\phi(v) : v \in N(u) \cap T_0\}$ for all $u \in G \setminus T_0$. We assume without loss of generality that $|S(u)| = |L(u)| - |N(u) \cap T_0|$.

Case (i) $v_1v_2v_3v_1$:

Let $T_1 \setminus T_0 = \{v_1'\}$, $T_2 \setminus T_1 = \{v_2'\}$, $T_3 \setminus T_2 = \{v_3'\}$ and $T_4 \setminus T_3 = \{v_1''\}$.

Suppose $S(v_1') \subseteq S(v_2')$. Let $\mathcal{C}(v_2') = S(v_1'), \mathcal{C}(v_3') = S(v_3') \setminus \mathcal{C}(v_2')$ and $\mathcal{C}(v_1'') = L(v_1'') \setminus \mathcal{C}(v_2')$. Note that $\mathcal{C}$ contains a set $\mathcal{C}'$ of Magic 1 colorings. We claim that every coloring $\phi' \in \mathcal{C}$, $\phi \cup \phi'$ extends to an L-coloring of B. To see this, simply let $\psi(v) = \phi(v)$ if $v \in T_0$, $\psi(v) = \phi'(v)$ if $v \in T_4$ and $\psi(v_1') = S(v_1') \setminus \psi(v_2')$. For the rest of the chapter, we will omit such justifications of why a specified set of colorings extends as desired. Instead, we will specify the desired set of colorings and leave it to the reader to check that they extend.

So we may assume that $|S(v_2') \setminus S(v_1')| \geq 2$. Let $\mathcal{C}(v_2')$ be a subset of $S(v_2') \setminus S(v_1')$ of size two.

If $S(v_1') \subseteq S(v_3')$, let $\mathcal{C}(v_3') = S(v_1')$ and $\mathcal{C}(v_1'') = L(v_1'') \setminus S(v_1')$. So we may assume, $S(v_1') \not\subseteq S(v_3)$. If $\mathcal{C}(v_2') \subseteq S(v_3')$, let $\mathcal{C}(v_3')$ be a subset of $S(v_3') \setminus S(v_1')$ of size 3 and hence $C(v_3') \supseteq C(v_2')$ and then let $\mathcal{C}(v_1'') = L(v_1'') \setminus \mathcal{C}(v_3')$.

So we may suppose that $|S(v_3') \setminus (S(v_1') \cup \mathcal{C}(v_2'))| \geq 2$. So let $\mathcal{C}(v_3')$ be a subset of $S(v_3') \setminus (S(v_1') \cup \mathcal{C}(v_2'))$ of size two and $\mathcal{C}(v_1'') = L(v_1'') \setminus \mathcal{C}(v_2')$.

Case (ii) $v_1v_2v_3v_2$:

Let $T_1 \setminus T_0 = \{v_1'\}$, $T_2 \setminus T_1 = \{v_2'\}$, $T_3 \setminus T_2 = \{v_3'\}$ and $T_4 \setminus T_3 = \{v_2''\}$.

Let $\mathcal{C}(v_1') = S(v_1')$. If $S(v_1') \cap S(v_2') = \emptyset$ or if $S(v_1') \subseteq S(v_2')$, let $\mathcal{C}(v_3') = S(v_3') \setminus \mathcal{C}(v_1')$ and $\mathcal{C}(v_2'') = L(v_2'') \setminus \mathcal{C}(v_1')$.

So we may assume that $|S(v_1') \cap S(v_2')| = 1$. If $S(v_2') \setminus S(v_1') \subseteq S(v_3')$, let $\mathcal{C}(v_3') = S(v_2') \setminus S(v_1')$ and $\mathcal{C}(v_2'')$ be a subset of $L(v_2'') \setminus \mathcal{C}(v_3')$ of size three. If $\mathcal{C}(v_1') \subseteq S(v_3')$, let $\mathcal{C}(v_3')$ be a subset of $S(v_3') \setminus (S(v_2') \setminus S(v_1'))$ of size three containing $\mathcal{C}(v_1')$ and let $\mathcal{C}(v_2'')$ be a subset of $L(v_2'') \setminus \mathcal{C}(v_3')$ of size two.

So we may assume that $|\mathcal{C}(v_1') \cap S(v_3')| \leq 1$. Let $\mathcal{C}(v_3')$ be a subset of $S(v_3') \setminus (S(v_1') \cup S(v_2'))$ of size two and let $\mathcal{C}(v_2'') = L(v_2'') \setminus \mathcal{C}(v_1')$.

□

Proposition 4.7.5. *If $\mathcal{C}$ is a Magic 2 set of L-colorings of T, then $\mathcal{C}$ contains a Magic 1 set of L-colorings $\mathcal{C}'$ of T.*

Lemma 4.7.6. *Let $B = T_0T_1$ be a tetrahedral band. If $\mathcal{C}$ is a Magic 1 set of colorings of T_0, then there exists a Magic 2 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Proof. Let $T_0 = v_1v_2v_3$ and $T_1 = v_1'v_2v_3$. First suppose that $\mathcal{C}$ is of the form (1a). That is, $\mathcal{C}(v_1), \mathcal{C}(v_2), \mathcal{C}(v_3)$ are pairwise disjoint lists of size two. We let $\mathcal{C}'(v_2) = \mathcal{C}(v_2)$ and $\mathcal{C}'(v_3) = \mathcal{C}(v_3)$. If $|L(v_1') \setminus (\mathcal{C}(v_2) \cup \mathcal{C}(v_3)| \geq 2$, let $\mathcal{C}'(v_1')$ be a subset of $L(v_1') \setminus (\mathcal{C}(v_2) \cup \mathcal{C}(v_3))$ of size two. Thus $\mathcal{C}'$ is of the form $(2a)$. So we may assume that $\mathcal{C}(v_2), \mathcal{C}(v_3) \subset L(v_1')$. In that case, let $\mathcal{C}'(v_1') = L(v_1')$. Thus $\mathcal{C}'$ is of the form $(2b)$ with order $v_2v_3v_1'$.

We may assume that $\mathcal{C}$ is of the form (1b). First suppose that $\mathcal{C}(v_1) \cap \mathcal{C}(v_3) = \emptyset$ and $\mathcal{C}(v_1) \cap \mathcal{C}(v_2) = \emptyset$. Without loss of generality suppose that $\mathcal{C}(v_2) \subset \mathcal{C}(v_3)$. Now let $\mathcal{C}'(v_2) = \mathcal{C}(v_2)$, $\mathcal{C}'(v_3) = \mathcal{C}(v_3)$ and $\mathcal{C}'(v_1') = L(v_1') \setminus \mathcal{C}(v_2)$. Thus $\mathcal{C}'$ is of the form $(2c)$ with order $v_2v_1'v_3$.

So we may assume without loss of generality that $\mathcal{C}(v_2) \cap \mathcal{C}(v_1) = \emptyset$ and $\mathcal{C}(v_2) \cap \mathcal{C}(v_3) = \emptyset$. Suppose that $\mathcal{C}(v_3) \subset \mathcal{C}(v_1)$. We let $\mathcal{C}'(v_2) = \mathcal{C}(v_2)$ and $\mathcal{C}'(v_3) = \mathcal{C}(v_3)$. If $|L(v_1') \setminus (\mathcal{C}(v_2) \cup \mathcal{C}(v_3))| \geq 2$, let $\mathcal{C}'(v_1')$ be a subset of $L(v_1') \setminus (\mathcal{C}(v_2) \cup \mathcal{C}(v_3))$ of size two. Thus $\mathcal{C}'$ is of the form $(2a)$. So we may assume that $\mathcal{C}(v_2), \mathcal{C}(v_3) \subset L(v_1')$. In that case, let $\mathcal{C}'(v_1') = L(v_1')$. Thus $\mathcal{C}'$ is of the form $(2b)$ with order $v_2v_3v_1'$.

Finally suppose that $\mathcal{C}(v_1) \subset \mathcal{C}(v_3)$. We let $\mathcal{C}'(v_2) = \mathcal{C}(v_2)$. If $|L(v_1') \setminus (\mathcal{C}(v_2) \cup \mathcal{C}(v_1)| \geq 2$, let $\mathcal{C}'(v_1')$ be a subset of $L(v_1') \setminus (\mathcal{C}(v_2) \cup \mathcal{C}(v_1))$ of size two and $\mathcal{C}'(v_3) = \mathcal{C}(v_1)$. Thus $\mathcal{C}'$ is of the form $(2a)$. So we may assume that $\mathcal{C}(v_2), \mathcal{C}(v_1) \subset L(v_1')$. In that case, let $\mathcal{C}'(v_1') = L(v_1') \setminus \mathcal{C}(v_1)$ and let $\mathcal{C}'(v_3) = \mathcal{C}(v_3)$. Thus $\mathcal{C}'$ is of the form $(2c)$ with order $v_2v_3v_1'$.

□

Corollary 4.7.7. *Let $B = T_0T_1$ be a tetrahedral band. If $\mathcal{C}$ is a Magic 1 set of colorings of T_0, then there exists a Magic 1 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Corollary 4.7.8. *Let $B = T_0T_1$ be a tetrahedral band. If $\mathcal{C}$ is a Magic 2 set of colorings of T_0, then there exists a Magic 2 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Corollary 4.7.9. *Let $B = T_0 \dots T_7$ be a sequence of tetrahedral bands of length seven. If ϕ is a proper coloring of T_0, then there exists a Magic 1 set $\mathcal{C}$ of colorings of T_7 such that for every $\phi' \in \mathcal{C}$, $\phi \cup \phi'$ extends to an L-coloring of B.*

4.7.2 Magic 3

Proposition 4.7.10. *If $\mathcal{C}$ is a Magic 3 set of colorings, then $\mathcal{C}$ contains a Magic 2 set of coloring $\mathcal{C}'$.*

Lemma 4.7.11. *Let $B = T_0T_1$ be a tetrahedral band. If $\mathcal{C}$ is a Magic 3 set of colorings of T_0, then there exists a Magic 3 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Proof. Let $T_0 = v_1v_2v_3$ be numbered as in the definition of Magic 3. We condition on the form of $\mathcal{C}$.

$(3a)$ As $(2a)$ implies $(3a)$ and $(2b)$ implies $(3b)$, this case was already proven in Lemma 4.7.6.

$(3b)$ First suppose that $T_1 = v_1v_2v_3'$. Let $\mathcal{C}'(v_1) = \mathcal{C}(v_1)$ and $\mathcal{C}'(v_2) = \mathcal{C}(v_2)$. If $|L(v_3') \setminus (\mathcal{C}(v_1) \cup \mathcal{C}(v_2))| \geq 2$, then let $\mathcal{C}'(v_3')$ be a subset of size two from $L(v_3') \setminus (\mathcal{C}(v_1) \cup \mathcal{C}(v_2))$ and hence $(3a)$ holds. Otherwise, we let $\mathcal{C}'(v_3') = L(v_3')$ and $(3b)$ holds for the order $v_1v_2v_3'$.

We may assume without loss of generality, given the symmetry of v_1 and v_2, that $T_1 = v_1'v_2v_3$. Let $\mathcal{C}'(v_2) = \mathcal{C}(v_2)$. If $|L(v_1') \setminus (\mathcal{C}(v_1) \cup \mathcal{C}(v_2))| \geq 2$, then let $\mathcal{C}'(v_1')$ be a subset of size two from $L(v_1') \setminus (\mathcal{C}(v_1) \cup \mathcal{C}(v_2))$. Also let $\mathcal{C}'(v_3) = \mathcal{C}(v_1)$ and hence $(3a)$ or $(3b)$ holds.

So we may suppose that $\mathcal{C}(v_1), \mathcal{C}(v_2) \subset L(v_1')$. We let $\mathcal{C}'(v_1') = L(v_1')$, $\mathcal{C}'(v_3) = \mathcal{C}(v_3)$ and $L_1(v_1') = L_1(v_3) = \mathcal{C}(v_1)$ and $L_1(v_2) = \mathcal{C}(v_2)$. Now $(3c)$ holds.

$(3c)$ First suppose that $T_1 = v_1'v_2v_3$. If $L(v_1') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})| \geq 2$, then let $\mathcal{C}'(v_1')$ be a subset of size two from $L(v_1') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})$. Then let $\mathcal{C}'(v_2) = \mathcal{C}(v_1)$ and $\mathcal{C}'(v_3) = \mathcal{F}$. Hence $(3a)$ holds. So we may suppose that $\mathcal{C}(v_1), \mathcal{F} \subset L(v_1')$. Now let $\mathcal{C}'(v_1') = \mathcal{F}$, $\mathcal{C}'(v_2) = \mathcal{C}(v_2)$ and $\mathcal{C}'(v_3) = \mathcal{C}(v_3)$. Furthermore let $\mathcal{F}' = \mathcal{C}(v_1)$. Now $(3c)$ holds with $\mathcal{F}'$ and order $v_1'v_2v_3$.

So we may suppose without loss of generality, given the symmetry of v_2 and v_3, that $T_1 = v_1v_2'v_3$. Let $\mathcal{C}'(v_1) = \mathcal{C}(v_1), \mathcal{C}'(v_3) = \mathcal{F}$. If $|L(v_2') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})| \geq 2$, let $\mathcal{C}'(v_2')$ be a subset of size two of $L(v_2') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})$. Hence $(3a)$ holds. Otherwise, let $\mathcal{C}'(v_2') = L(v_2')$ and $(3b)$ holds with order $v_1v_3v_2'$.

□

Lemma 4.7.12. *Let $B = T_0T_1T_2T_3$ be a sequence of tetrahedral bands of length three. If $\mathcal{C}$ is a Magic 2 set of colorings of T_0, then there exists a Magic 3 set $\mathcal{C}'$ of colorings of T_3 such that for every $\phi' \in \mathcal{C}$, $\phi \cup \phi'$ extends to an L-coloring of B.*

Proof. Given Lemma 4.7.11, it is sufficient to prove that there exists a Magic 3 set $\mathcal{C}'$ of colorings of T_i for some $i \in \{0, 1, 2, 3\}$.

If $\mathcal{C}$ is of the form $(2a)$ or $(2b)$, then $\mathcal{C}$ is of the form $(3a)$ or $(3b)$ respectively. Thus $\mathcal{C}$ is a Magic 3 set of colorings of T_0. So we may assume that $\mathcal{C}$ is of the form $(2c)$.

Let $T_0 = v_1v_2v_3$. We may assume without loss of generality that $\mathcal{C}$ is of the form $(2c)$ for the order $v_1v_2v_3$. That is, $|\mathcal{C}(v_1)| = 2, |\mathcal{C}(v_2)| = |\mathcal{C}(v_3)| = 3, \mathcal{C}(v_1) \cap \mathcal{C}(v_2) = \emptyset$, $\mathcal{C}(v_1) \subseteq \mathcal{C}(v_3)$. Let $\mathcal{F}$ be a subset of size two of $\mathcal{C}(v_2) \setminus \mathcal{C}(v_3)$.

First suppose that $T_1 = v_1v_2v_3'$. Let $\mathcal{C}'(v_1) = \mathcal{C}(v_1)$ and $\mathcal{C}'(v_2) = \mathcal{F}$. If $|L(v_3') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})| \geq 2$, then let $\mathcal{C}'(v_3')$ be a subset of size two of $L(v_3') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})$. Hence $(3a)$ holds for $\mathcal{C}'$. So we may suppose that $\mathcal{C}(v_1), \mathcal{F} \subset L(v_3')$. Then let $\mathcal{C}'(v_3') = L(v_3')$ and hence $(3b)$ holds for $\mathcal{C}'$ with order $v_1v_2v_3'$.

Next suppose that $T_1 = v_1'v_2v_3$. Suppose $L(v_1') \neq \mathcal{C}(v_1) \cup \mathcal{C}(v_2)$. Let $\mathcal{C}'(v_3) = \mathcal{C}(v_1)$ and let $\mathcal{C}'(v_1')$ be a subset of size two of $L(v_1') \setminus \mathcal{C}(v_1)$ that is not entirely contained in $\mathcal{C}(v_2)$. Then let $\mathcal{C}'(v_2)$ be a subset of size two of $\mathcal{C}(v_2) \setminus \mathcal{C}'(v_1')$. Now $(3a)$ holds for $\mathcal{C}'$. So we may assume that $L(v_1') = \mathcal{C}(v_1) \cup \mathcal{C}(v_2)$.

Further suppose that $T_2 = v_1'v_2'v_3$. Let $\mathcal{C}'(v_1') = \mathcal{F}$ and $\mathcal{C}'(v_3) = \mathcal{C}(v_1)$. If $|L(v_2') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})| \geq 2$, then let $\mathcal{C}'(v_2')$ be a subset of size two of $L(v_2') \setminus (\mathcal{C}(v_1) \cup \mathcal{F})$. Hence $(3a)$ holds for $\mathcal{C}'$. So we may suppose that $\mathcal{C}(v_1), \mathcal{F} \subset L(v_3')$. Then let $\mathcal{C}'(v_2') = L(v_2')$ and hence $(3b)$ holds for $\mathcal{C}'$ with order $v_1'v_3v_2'$.

So instead suppose that $T_2 = v_1'v_2v_3'$. Further suppose that $\mathcal{C}(v_2)\cap\mathcal{C}(v_3)\neq\emptyset$. Let $\mathcal{C}(v_1) = \{c_1, c_2\}$. If $|L(v_3')\setminus(\{c_1\}\cup\mathcal{C}(v_2))|\geq 2$, then let $\mathcal{C}'(v_3')$ be a subset of size two of $L(v_3')\setminus(\{c_1\}\cup\mathcal{C}(v_2))$. Also let $\mathcal{C}'(v_2) = \mathcal{F}$ and $\mathcal{C}'(v_1') = \{c_1\}\cup(\mathcal{C}(v_2)\setminus\mathcal{F})$. Hence ($3a$) holds for $\mathcal{C}'$. So we may suppose that $\{c_1\}, \mathcal{C}(v_2)\subset L(v_3')$. By symmetry of c_1, c_2, we may also suppose that $c_2\in L(v_3')$. Now let $\mathcal{C}'(v_1') = L(v_1')$, $\mathcal{C}'(v_2) = \mathcal{F}$ and $\mathcal{C}'(v_3') = \{c_1, c_2\}$. Hence ($3b$) holds for $\mathcal{C}'$ with order $v_2v_3'v_1'$.

So we may assume that $\mathcal{C}(v_2)\cap\mathcal{C}(v_3)=\emptyset$ when $T_2 = v_1'v_2v_3'$. Suppose without loss of generality that $\mathcal{C}(v_1) = \{1,2\}$, $\mathcal{C}(v_2) = \{3,4,5\}$, $\mathcal{C}(v_3) = \{1,2,6\}$, $L(v_1') = \{1,2,3,4,5\}$. If $|L(v_3')\setminus\{3,4,5,6\}|\geq 2$, let $C'(v_3')$ be a subset of size two of $L(v_3')\setminus\{3,4,5,6\}$ and let $C'(v_2) = \{3,4\}$. $|L(v_1')\setminus\mathcal{C}'(v_3')|\geq 4$, let $C'(v_1)$ be a subset of size two of $L(v_1')\setminus(\mathcal{C}'(v_2)\cup\mathcal{C}'(v_3'))$ and hence ($3a$) holds. So we may suppose that $C'(v_3')\subset L(v_1')$. So let $C'(v_1') = L(v_1')$ and hence ($3b$) holds with order $v_2v_3'v_1'$. So we may assume that $L(v_3') = \{3,4,5,6,c\}$. If $c\in\{1,2\}$, let $C'(v_3') = \{c,5\}$, $C'(v_2) = \{3,4\}$ and $C'(v_1') = L(v_1')$; hence ($3b$) holds. So $c\neq 1,2$. Let $\mathcal{C}'(v_3) = \{c,5\}$, $\mathcal{C}'(v_2) = \{3,4\}$ and $\mathcal{C}'(v_1') = \{1,2\}$. Hence ($3a$) holds.

Finally we may suppose that $T_1 = v_1v_2'v_3$. But then let $\mathcal{C}(v_2')$ be a subset of size three of $L(v_2')\setminus\mathcal{C}(v_1)$. Now $\mathcal{C}$ is of the form ($2c$) with order $v_1v_2'v_3$. Yet, $T_2\neq v_1v_2''v_3$. So by the arguments of the preceding paragraphs either T_2 or T_3 has a Magic 3 set of colorings.

□

Lemma 4.7.13. *Let $B = T_0T_1T_2T_3$ be a rainbow sequence of tetrahedral bands of length three. If $\mathcal{C}$ is a Magic 3 set of colorings of T_0 and $\mathcal{C}'$ is a Magic 3 set of colorings of T_3, then there exists $\phi\in\mathcal{C}$ and $\phi'\in\mathcal{C}'$ such that $\phi\cup\phi'$ is a proper coloring of B.*

Proof. Suppose for a contradiction that the conclusion does not hold. Let $T_0 = v_1v_2v_3$, $T_1 = v_1'v_2v_3$, $T_2 = v_1'v_2'v_3$ and $T_3 = v_1'v_2'v_3'$. Consider the degree three vertices v_1 and v_3'. We say v_1 is *good* if $\mathcal{C}$ has the form ($3a$) or $|\mathcal{C}(v_1)|\geq 5$. Similarly we say v_3' is *good* if $\mathcal{C}'$ has the form $(3a)$ or $|\mathcal{C}(v_3')| \geq 5$.

If v_1 (or similarly v_3') is good, then any proper coloring ψ of $B \setminus \{v_1\}$, such that ψ restricted to $T_0 - v_1$ extends to a coloring of T_0 in $\mathcal{C}$ and ψ restricted to T_3 in C', extends to a proper coloring ψ of B such that $\psi(T_0$ is in $\mathcal{C}$.

Claim 4.7.14. *At least one of v_1, v_3' is not good.*

Proof. Otherwise delete v_1, v_3'. Now we may assume that v_2, v_3 have disjoints list of size two and similarly for v_1', v_2'. But then we may color $v_2 v_3 v_1' v_2'$ from these lists and extend back to v_1, v_3'. □

Claim 4.7.15. *Neither v_1 nor v_3' is good.*

Proof. We may suppose without loss of generality that v_1 is not good but that v_3' is good. Delete v_3'. First suppose $\mathcal{C}$ is of the form $(3b)$. As v_1 is not good, delete the vertex x with list of size five in T_0. This is permissible as both v_2, v_3 have degree at most four in $B \setminus \{v_3'\}$. Now we may color $B \setminus \{x, v_3'\}$, which is at most a K_4 with two pairs of disjoint lists of size two.

So we may suppose that $\mathcal{C}$ is of the form $(3c)$. Now remove $\mathcal{C}(v_2) \setminus (\mathcal{F} \cup \mathcal{C}(v_1))$ from the list for v_2' and delete v_2, v_3. Then color v_2', v_1', v_1 in that order. □

First suppose that $\mathcal{C}$ and $\mathcal{C}'$ are of the form $(3b)$. If v_2 has a list of size five, we color in the following order: v_3, the lists of size two in T_3, the list of size five in T_3, v_1 and finally v_2. So v_2 has a list of size two and so does v_2' by symmetry. So color v_2 and v_2'. Now v_3, v_1' have lists of size three while v_1, v_3' have lists of size two. So we may color $v_1 v_3 v_1' v_3'$ which is a $K_4 - e$.

Next suppose that $\mathcal{C}$ is of the form $(3b)$ and $\mathcal{C}'$ is of the form $(3c)$. If v_2 has a list of size five, we color in the following order: v_3, the list of size two in T_3, the lists of size five in T_3, v_1 and finally v_2. So v_2 has a list of size two. We color in the following order: v_1' from $\mathcal{C}'(v_1') \setminus \mathcal{F}$, v_1, v_2, v_3', v_3 and finally v_2'.

Last suppose that $\mathcal{C}$ and $\mathcal{C}'$ are of the form ($3c$). Remove $\mathcal{F}'$ from $\mathcal{C}'(v_1')$ and $\mathcal{F}$ from $\mathcal{C}(v_3)$. Color v_1, v_3, v_1', v_3' which is a $K_4 - e$ with two lists of size and two lists of size three. Now color v_2, v_2'.

□

Corollary 4.7.16. *Let $B = T_0 \ldots T_6$ be a sequence of tetrahedral bands of length six. If $\mathcal{C}$ is a Magic 3 set of colorings of T_0 and $\mathcal{C}'$ is a Magic 3 set of colorings of T_6, then there exists $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ extends to an L-coloring of B.*

4.8 Magic Colorings with Octahedral Bands

The goal of this section is to incorporate octahedral bands in the theory of magic colorings.

Lemma 4.8.1. *Let $B = T_0T_1T_2$ be such that T_0T_1 and T_1T_2 are octahedral bands. If ϕ is a proper coloring of T_0, then there exists a Magic 1 set $\mathcal{C}$ of colorings of T_2 such that for every $\phi' \in \mathcal{C}$, $\phi \cup \phi'$ extends to an L-coloring of B.*

Proof. Let $T_1 = v_1v_2v_3$ and $T_2 = v_1'v_2'v_3'$ where v_i is adjacent to v_j' if and only if $i \neq j$.

Let $S(v) = L(v) \setminus \{\phi(u) | u \in T_0 \cap N(v)\}$ for a $v \notin T_0$. We may assume without loss of generality that $|S(v_i)| = 3$ and $S(v_i') = L(v_i')$ for all $i \in \{1, 2, 3, \}$. First suppose that $S(v_1) = S(v_2)$. Let $\mathcal{C}(v_3')$ be a subset of $L(v_3') \setminus S(v_1)$ of size two. Let $\mathcal{C}(v_1') = L(v_1')$ and $\mathcal{C}(v_2') = L(v_2')$. We claim that any proper coloring in $\mathcal{C}$ extends to T_0: simply color v_3 and then v_1 and v_2. This works as $v_1' \sim v_2'$ and hence cannot receive the same color. The claim follows. In addition, it is not hard to see that $\mathcal{C}$ contains a Magic 1 subset.

So we may assume that $S(v_1) \setminus S(v_2) \neq \emptyset$. By symmetry we may also assume that $S(v_2) \setminus S(v_3) \neq \emptyset$. Let $c \in S(v_1) \setminus S(v_2)$. Color v_1 with c. Let $c' \in S(v_2) \setminus S(v_3)$ and let $\mathcal{C}'(v_3') = L(v_3') \setminus \{c, c\}$, $\mathcal{C}(v_1') = L(v_1')$. If $c \in S(v_3)$, let $\mathcal{C}(v_2') = L(v_2') \setminus S(v_3)$; otherwise let $\mathcal{C}(v_2') = L(v_2') \setminus \{c\}$. We claim that any proper coloring ϕ in $\mathcal{C}$ extends to T_0. If $\phi(v_1') \neq c'$, color v_2 with c' and then color v_3. So assume $\phi(v_1') = c'$ and color v_2 and then v_3. The claim follows. Moreover, it is not hard to see that $\mathcal{C}$ contains a Magic 1 subset. □

Lemma 4.8.2. *Let $B = T_0T_1T_2T_3$ be such that T_0T_1, T_2T_3 are tetrahedral bands and T_1T_2 is an octahedral band. If ϕ is a proper coloring of T_0, then there exists a Magic 1 set $\mathcal{C}$ of colorings of T_2 such that for every $\phi' \in \mathcal{C}$, $\phi \cup \phi'$ extends to an L-coloring of B.*

Proof. Let $T_1 = v_1v_2v_3$ and suppose without loss of generality that $v_2 \notin T_0$. Let $T_2 = v_1'v_2'v_3'$ where v_i is adjacent to v_j' if and only if $i \neq j$.

First suppose that $T_3 = v_1'v_2'v_3''$. Let $S(v) = L(v) \setminus \{\phi(u) | u \in T_0 \cap N(v)\}$ for a $v \notin T_0$. We may assume without loss of generality that $|S(v_2)| = 2$, $|S(v_2')| = 3$, $|S(v_1')| = |S(v_3')| = 4$ and $S(v_3'') = L(v_3'')$. Let $\mathcal{C}(v_1')$ be a subset of size two from $S(v_1') \setminus S(v_2)$ and delete v_2. Then delete v_3'. If $S(v_2')$ has two colors disjoint from $\mathcal{C}(v_1')$, let $\mathcal{C}(v_2')$ be a subset of size two of $S(v_2') \setminus \mathcal{C}(v_1')$; then if $L(v_3'')$ has two colors disjoint from $\mathcal{C}(v_1') \cup \mathcal{C}(v_2')$, let $\mathcal{C}(v_3')$ be two such disjoint colors and hence (1a) holds. Otherwise $\mathcal{C}(v_1') \subset S(v_2')$. So let $\mathcal{C}(v_2') = S(v_2')$ and let $\mathcal{C}(v_3'')$ be a subset of size two in $L(v_3'')$ disjoint from $\mathcal{C}(v_2')$. Hence (1b) holds with order $v_3''v_1'v_2'$.

We may now assume, using the symmetry of v_1' and v_3', that $T_3 = v_1'v_2''v_3'$. Suppose that $|S(v_1') \cap S(v_2')| \leq 2$. Now let $\mathcal{C}(v_1')$ be a subset of size two of $S(v_1') \setminus S(v_2')$ and delete v_2'. If $\mathcal{C}(v_1') = S(v_2)$, let $\mathcal{C}(v_3')$ be a subset of size two of $S(v_3') \setminus S(v_2)$ and delete v_2. Then let $\mathcal{C}(v_2'') = L(v_2'') \setminus \mathcal{C}(v_1')$ and either (1a) or (1b) holds. So suppose that $S(v_2) \setminus \mathcal{C}(v_1') \neq \emptyset$. Color v_2 with a color c from $S(v_2) \setminus \mathcal{C}(v_1') \neq \emptyset$. Remove c from $S(v_3')$ and delete v_2. Now if there are two colors in $S(v_3') \setminus \{c\}$ disjoint from $\mathcal{C}(v_1')$ we finish as above. Otherwise, $\mathcal{C}(v_1) \subset S(v_3') \setminus \{c\}$. So let $\mathcal{C}(v_3') = S(v_3') \setminus \{c\}$ and $\mathcal{C}(v_2'')$ be a subset of size two in $L(v_2'') \setminus \mathcal{C}(v_3')$. Hence (1$b$) holds with order $v_2''v_1'v_3'$.

So we may assume that $|S(v_1') \cap S(v_2')| \geq 3$. That is, $S(v_2') \subset S(v_1')$. By the symmetry of v_1' and v_3', we may also assume that $S(v_2') \subset S(v_3')$. If $S(v_2) \setminus S(v_2') \neq \emptyset$, let $\mathcal{C}(v_1') = \mathcal{C}(v_3') = S(v_2')$ and delete v_2. Then let $\mathcal{C}(v_2'') = L(v_2'') \setminus S(v_2')$ and delete v_2'. Hence $(1b)$ holds with order $v_2''v_1'v_3'$. So we may suppose that $S(v_2) \subset S(v_2')$. Let $\mathcal{C}(v_1') = S(v_2)$, $\mathcal{C}(v_3') = S(v_3') \setminus S(v_2)$ and delete v_2. Let $\mathcal{C}(v_2'') = L(v_2'') \setminus S(v_2)$ and delete v_2'. Now either $(1a)$ holds or $(1b)$ holds with order $v_1'v_3'v_2''$. $\square$

Lemma 4.8.3. *Let $B = T_0T_1$ be an octahedral band. If $\mathcal{C}$ is a Magic 1 set of colorings of T_0, then there exists a Magic 3 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Proof. Let $T_0 = v_1v_2v_3$ and $T_1 = v_1'v_2'v_3'$ where v_i is adjacent to v_j' if and only if $i \neq j$. First we suppose that $\mathcal{C}$ is of the form $(1a)$. Further suppose that $|L(v_3') \setminus (\mathcal{C}(v_1) \cup \mathcal{C}(v_2))| \geq 2$. Let $\mathcal{C}'(v_3')$ be a subset of size two of $L(v_3') \setminus (\mathcal{C}(v_1) \cup \mathcal{C}(v_2))$. Now we may delete v_1 and v_2.

Further suppose there exist two colors in $L(v_1')$ disjoint from $\mathcal{C}'(v_3') \cup \mathcal{C}(v_3)$. Then let $\mathcal{C}'(v_1)$ be a subset of size two of $L(v_1') \setminus (\mathcal{C}'(v_3') \cup \mathcal{C}(v_3))$ and delete v_3. Then if $L(v_2')$ has two colors disjoint from $\mathcal{C}'(v_1') \cup \mathcal{C}'(v_3')$ let $\mathcal{C}'(v_2')$ be a subset of size two $L(v_2') \setminus (\mathcal{C}'(v_1') \cup \mathcal{C}'(v_3'))$ and hence $(3a)$ holds. So $\mathcal{C}'(v_1'), \mathcal{C}'(v_3') \subset L(v_2')$ and $(3b)$ holds with order $v_1'v_3'v_2'$.

So we may suppose instead that $\mathcal{C}'(v_3'), \mathcal{C}(v_3) \subset L(v_1')$. By symmetry, $\mathcal{C}'(v_3'), \mathcal{C}(v_3) \subset L(v_2')$. But then $(3c)$ holds with $\mathcal{F} = \mathcal{C}(v_3)$ and order $v_3'v_1'v_2'$.

So we may suppose that $\mathcal{C}(v_1), \mathcal{C}(v_2) \subset L(v_3')$. By symmetry, $\mathcal{C}(v_1), \mathcal{C}(v_3) \subset L(v_2')$ and $\mathcal{C}(v_2), \mathcal{C}(v_3) \subset L(v_1')$. Now let $\mathcal{C}'(v_3') = \mathcal{C}(v_1)$, $\mathcal{C}'(v_2') = \mathcal{C}(v_3)$ and $\mathcal{C}'(v_1') = \mathcal{C}(v_2)$. Hence $(3a)$ holds.

So we may assume that $\mathcal{C}$ is of the form $(1b)$. We may assume without loss of generality that $\mathcal{C}(v_3) \cap (\mathcal{C}(v_1) \cup \mathcal{C}(v_2)) = \emptyset$, $|\mathcal{C}(v_3)| = |\mathcal{C}(v_1)| = 2$, $|\mathcal{C}(v_2)| = 3$ and $\mathcal{C}(v_1) \subset \mathcal{C}(v_2)$.

Let $\mathcal{C}'(v_3')$ be a subset of size two of $L(v_3') \setminus \mathcal{C}(v_2)$. Now any L-coloring ϕ of $B \setminus \{v_1, v_2\}$ such that $\phi(v_3') \in \mathcal{C}'(v_3')$ and $\phi(v_3) \in \mathcal{C}(v_3)$ extends to an L-coloring ϕ of B such that ϕ restricted to T_0 is in $\mathcal{C}$; simply color v_1, then v_2. So it suffices to define

$\mathcal{C}'$ for $B\setminus\{v_1, v_2\}$ such that $\mathcal{C}'$ is a Magic 3 set of colorings and every L-coloring $\phi \in C'$, there exists $c \in \mathcal{C}(v_3)$ such setting $\phi(v_3) = c$ yields an L-coloring of $B \setminus \{v_1, v_2\}$.

Suppose there exist two colors in $L(v_1')$ disjoint from $\mathcal{C}'(v_3') \cup \mathcal{C}(v_3)$. Then let $\mathcal{C}'(v_1)$ be a subset of size two of $L(v_1') \setminus (\mathcal{C}'(v_3') \cup \mathcal{C}(v_3))$; now every L-coloring ϕ of T_1 such that $\phi(v_1') \in \mathcal{C}'(v_1)$ and $\phi(v_3') \in \mathcal{C}'(v_3')$ extends to an L-coloring of $B \setminus \{v_1, v_2\}$. Now if $L(v_2')$ has two colors disjoint from $\mathcal{C}'(v_1') \cup \mathcal{C}'(v_3')$, let $\mathcal{C}'(v_2')$ be a subset of size two $L(v_2') \setminus (\mathcal{C}'(v_1') \cup \mathcal{C}'(v_3'))$ and hence $\mathcal{C}'$ is of the form ($3a$). So $\mathcal{C}'(v_1'), \mathcal{C}'(v_3') \subset L(v_2')$ and $\mathcal{C}'$ is of the form ($3b$) with order $v_1'v_3'v_2'$.

So we may suppose instead that $\mathcal{C}'(v_3'), \mathcal{C}(v_3) \subset L(v_1')$. Noting the symmetry of v_1', v_2' in $B \setminus \{v_1, v_2\}$, we find that $\mathcal{C}'(v_3'), \mathcal{C}(v_3) \subset L(v_2')$. But then $\mathcal{C}'$ is of the form ($3c$) with $\mathcal{F} = \mathcal{C}(v_3)$ and order $v_3'v_1'v_2'$. □

Corollary 4.8.4. *Let $B = T_0T_1$ be an octahedral band. If $\mathcal{C}$ is a Magic 1 set of colorings of T_0, then there exists a Magic 1 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Corollary 4.8.5. *Let $B = T_0T_1$ be an octahedral band. If $\mathcal{C}$ is a Magic 2 set of colorings of T_0, then there exists a Magic 2 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Corollary 4.8.6. *Let $B = T_0T_1$ be an octahedral band. If $\mathcal{C}$ is a Magic 3 set of colorings of T_0, then there exists a Magic 3 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Lemma 4.8.7. *Let $B = T_0T_1$ be an octahedral band. If $\mathcal{C}$ is a Magic 3 set of colorings of T_0 and $\mathcal{C}'$ is a Magic 3 set of colorings of T_1, then there exists $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ is a proper coloring of B.*

Proof. Let $T_0 = v_1v_2v_3$, $T_1 = v_1'v_2'v_3'$ where v_i is adjacent to v_j' if $i \neq j$.

First suppose that $\mathcal{C}$ is of the form ($3b$). Delete the vertex x with list of size five as x has degree four in B. If $\mathcal{C}'$ is of the form ($3a$), then we color $B \setminus \{x\}$, which is essentially a path on five vertices each with list of size two. If $\mathcal{C}'$ is of the form $(3b)$, delete the vertex x' with the list of size five in T_3 as x has degree four in B. Now color $B \setminus \{x, x'\}$ which is a path of length three or two disjoint edges all with list of size two. So we may suppose that $\mathcal{C}'$ is of the form $(3c)$. Delete a vertex x' with list of size five adjacent to x in B. Thus x' has degree three in $B \setminus \{x\}$. Then delete the other vertex x'' with list of size five in T_3. Finally color the vertices of $B \setminus \{x, x', x''\}$. So we may assume that neither $\mathcal{C}$ nor by symmetry $\mathcal{C}'$ is of the form $(3b)$.

Next suppose $\mathcal{C}$ is of the form $(3c)$. Suppose without loss of generality that $v_1, v_2 \in T_0$ have lists of size five while $v_3 \in T_0$ has a list of size two. Delete v_1 and v_2 and color $B \setminus \{v_1, v_2\}$. If $\mathcal{C}'$ is of the form $(3a)$ such a coloring exists as we have a path of length two plus an isolated vertex all with lists of size two. If $\mathcal{C}'$ is of the form $(3c)$, we color the lists of size two and then the lists of size five. Finally we may extend such a coloring ϕ back to v_1 and v_2 unless without loss of generality $\mathcal{C}(v_1) = \{1, 2, 3, 4, 5\}$, $\mathcal{C}(v_2) = \{1, 2, 3, 4, 6\}$, $\mathcal{F} = \{3, 4\}$, $\mathcal{C}(v_3) = \{1, 2\}$ and $\phi(v_3) = 1$, $\phi(v_2') = 5$, $\phi(v_3') = 2$, $\phi(v_1') = 6$. But then change the color of $\phi(v_3)$ to 2 and extend to v_1 and v_2. So we may assume that neither $\mathcal{C}$ nor by symmetry $\mathcal{C}'$ is of the form $(3c)$.

Finally we may suppose that $\mathcal{C}$ and $\mathcal{C}'$ are of the form $(3a)$. But then B is a subgraph of a cycle of length six (after deleting edges between vertices with disjoint lists) whose vertices have lists of size two. By Theorem 1.4.3, B has a list-coloring as desired. □

4.9 Magic Colorings with Hexadecahedral Bands

The goal of this section is to incorporate hexadecahedral bands in the theory of magic colorings.

Lemma 4.9.1. *Let $B = T_0T_1$ be such that T_0T_1 is a hexadecahedral band. If ϕ is a proper coloring of T_0, then there exists a Magic 3 set $\mathcal{C}$ of colorings of T_2 such that for every $\phi' \in \mathcal{C}$, $\phi \cup \phi'$ extends to an L-coloring of B.*

Proof. Let $B \setminus (T_0 \cup T_1) = v_1v_2v_3v_4$ and $T_1 = v_1'v_2'v_3'$ where for all $i \in \{1,2,3\}$, v_i is adjacent to v_j' if and only $i \neq j$ and $v_4 \sim v_2'$. Let $S(v) = L(v) \setminus \{\phi(u) | u \in T_0 \cap N(v)\}$ for a $v \notin T_0$. We may assume without loss of generality that $|S(v_1)| = |S(v_3)| = |S(v_4)| = 3$ and $|S(v_2)| = 4$.

Suppose $S(v_4) = S(v_3)$. Let $\mathcal{C}(v_2') = L(v_2') \setminus S(v_3)$. Delete v_4, then v_3, then v_2, then v_1. Let $\mathcal{C}(v_1') = L(v_1')$ and $\mathcal{C}(v_3') = L(v_3')$. Hence $\mathcal{C}$ contains either a subset of the form $(3a)$ or $(3b)$ as desired.

So we may assume that $S(v_4) \neq S(v_3)$ and by symmetry that $S(v_4) \neq S(v_1)$. Let $\mathcal{C}(v_1') = L(v_1') \setminus S(v_3)$ and $\mathcal{C}(v_2') = L(v_2')$. Let $c \in S(v_1) \setminus S(v_4)$ and $\mathcal{C}(v_3') = L(v_3') \setminus \{c\}$. We claim that any proper coloring $\phi \in \mathcal{C}$ of T_1 extends to a S-coloring of $v_1v_2v_3v_4$. If $\phi(v_2') \neq c$, color v_1 with c, then color v_2, v_3, v_4 in that order. If $\phi(v_2') = c'$, color v_1, v_2, v_3, v_4 in that order. The claim follows.

Hence $\mathcal{C}$ contains either a subset of the form $(3a)$ or $(3b)$ as desired. To see this let $\mathcal{C}'(v_2') = \mathcal{C}(v_2')$, $\mathcal{C}'(v_3')$ be a subset of size two of $\mathcal{C}(v_3') \setminus \mathcal{C}'(v_2')$. Then if $\mathcal{C}(v_1')$ has two disjoints color from $\mathcal{C}'(v_2') \cup \mathcal{C}'(v_3')$, $(3a)$ holds; otherwise let $\mathcal{C}'(v_1') = \mathcal{C}(v_1')$ and $(3b)$ holds with order $v_2'v_3'v_1'$. $\square$

Corollary 4.9.2. *Let $B = T_0T_1$ be a hexadecahedral band. If $\mathcal{C}$ is a Magic 3 set of colorings of T_0, then there exists a Magic 3 set $\mathcal{C}'$ of colorings of T_1 such that for every $\phi' \in \mathcal{C}'$, there exists $\phi \in \mathcal{C}$ where $\phi \cup \phi'$ is an L-coloring of B.*

Lemma 4.9.3. *Let $B = T_0T_1$ be a hexadecahedral band. If $\mathcal{C}$ is a Magic 3 set of colorings of T_0 and $\mathcal{C}'$ is a Magic 3 set of colorings of T_1, then there exists $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that $\phi \cup \phi'$ is a proper coloring of B.*

Proof. Let $T_0 = v_1v_2v_3$ and $T_1 = u_1u_2u_3$. Let $B \setminus (T_0 \cup T_1) = w_1w_2w_3w_4$, where $w_1 \sim v_1, v_2, u_3$, $w_2 \sim v_2, v_3, u_3, u_1$, $w_3 \sim v_3, u_1, u_2$ and $w_4 \sim v_3, v_1, u_2, u_3$. Note that in any proper coloring of $T_0 \cup T_1$, w_2 and w_4 see at most four colors and so can be colored. The problem then is that if w_1 or w_3 then see five colors they may not be able to be colored. The solution then is as follows. It suffices to choose $\phi \in \mathcal{C}$ and $\phi' \in \mathcal{C}'$ such that w_1 and w_3 see only two colors in $\phi \cup \phi'$, because then we color w_2 and w_4 followed by w_1 and w_3.

First suppose that $\mathcal{C}$ is of the form $(3a)$. Further suppose that $\mathcal{C}'$ is of the form $(3a)$. In this case, either $(\mathcal{C}(v_1) \cup \mathcal{C}(v_2) \cup \mathcal{C}'(u_3)) \setminus L(w_1) \neq \emptyset$ or $\mathcal{C}'(u_3) \cap (\mathcal{C}(v_1) \cup \mathcal{C}(v_2)) \neq \emptyset$. Either way we may color v_1, v_2, u_3 such that w_1 sees at most two colors. Symmetrically, we may color v_3, u_1, u_2 such that w_3 sees at most two colors. These coloring together are proper as vertices in T_0 have pairwise disjoint lists and similarly for T_1.

Next suppose that $\mathcal{C}'$ is of the form $(3b)$. Suppose that $|\mathcal{C}'(u_3)| = 2$. Without loss of generality suppose that $|\mathcal{C}'(u_1)| = 5$. In this case, we may color v_1, v_2, u_3 so that w_1 sees at most two colors. Now $|\mathcal{C}'(u_1) \setminus \phi'(u_3)| \geq 4$. So either $\mathcal{C}(v_3) \cup (\mathcal{C}'(u_1) \setminus \phi'(u_3)) \setminus L(w_3) \neq \emptyset$ or $\mathcal{C}(v_3) \cap (\mathcal{C}'(u_1) \setminus \phi'(u_3)) \neq \emptyset$. Either way we may color u_1 and v_3 so that w_3 sees at most one color. Then we color u_2 and then w_3 sees at most two colors as desired. So we may suppose that $|\mathcal{C}'(u_3)| = 5$. In this case, we may color v_3, u_1, u_2 so that w_3 sees at most two colors. Then as $\mathcal{C}'(u_3) \setminus \{\phi'(u_1), \phi'(u_2)\}| \geq 3$, we may color v_1, v_2, u_3 so that w_1 sees at most two colors as desired.

We suppose that $\mathcal{C}'$ is of the form $(3c)$. Suppose that $|\mathcal{C}'(u_3)| = 2$. In this case, we may color v_1, v_2, u_3 so that w_1 sees at most two colors. Now $|\mathcal{C}'(u_1) \setminus \phi'(u_3)| \geq 4$ and we may color u_1 and v_3 so that w_3 sees at most one color. Then we color u_2 and then w_3 sees at most two colors as desired. So we may suppose that $|\mathcal{C}'(u_3)| = 5$. Now we may color v_3, u_1, u_2 so that w_3 sees at most two colors. Then u_3 has as least two colors available. So we may color v_1, v_2, u_3 so that w_1 sees at most two colors as desired. So we may assume that $\mathcal{C}$, and by symmetry $\mathcal{C}'$ is not of the form $(3a)$.

So suppose that $\mathcal{C}$ is of the form $(3b)$. Further suppose that $\mathcal{C}'$ is of the form $(3b)$. Suppose that $|\mathcal{C}(v_3)| = 5$. Then suppose that $|\mathcal{C}(u_3)| = 5$. In this case, color u_1, u_2, v_3 so that w_3 sees at most two colors. Now u_3 has three available colors and v_1, v_2 have at least three available colors combined. So we may color u_3, v_1, v_2 so that w_1 sees at most two colors as desired. So suppose that $|\mathcal{C}(u_3)| = 2$. In this case, color v_1, v_2, u_3 so that w_1 sees at most two colors. Then as v_3 has at least three available colors, and v_1, v_2 have at least four available colors combined, we may color v_3, u_1, u_2 so that w_3 sees at most two colors as desired.

So we may assume that $|\mathcal{C}(v_3)| = 2$ and by symmetry that $|\mathcal{C}'(u_3)| = 2$. Note that there are at least five colors in $\mathcal{C}'(u_1) \cup \mathcal{C}'(u_2)$ and two in $\mathcal{C}(v_3)$. So consider $(\mathcal{C}'(u_1) \cup \mathcal{C}'(u_2)) \setminus \mathcal{C}'(u_3)$ and $\mathcal{C}(v_3)$. If these sets intersect or one has a color not in $L(w_3)$, we may color v_3, u_1, u_2 such that w_3 sees at most two colors and u_3 still has two available colors. Then as v_1, v_2 have at least four available colors combined, we may color v_1, v_2, u_3 so that w_1 sees at most two colors. But then if $\mathcal{C}(v_3) \setminus \mathcal{C}'(u_3) \neq \emptyset$, we may color u_1, u_2 so that w_3 sees at most one color. Then we may color v_1, v_2, u_3 so that w_1 sees at most one color and finish by coloring v_3 and then w_3 sees at most one color. So we may assume that $\mathcal{C}(v_3) = \mathcal{C}'(u_3)$. But then we may color v_3 and the vertex with a list of size five in T_1 with one color from $\mathcal{C}(v_3)$ and u_3 and the vertex with a list of size five in T_0 with the other color from $\mathcal{C}(v_3)$. In this way, w_1 and w_3 see at most two colors as desired.

Next suppose that $\mathcal{C}'$ is of the form $(3c)$. Suppose that $|\mathcal{C}(v_3)| = 5$. Then suppose that $|\mathcal{C}'(u_3)| = 2$. In this case, color v_1, v_2, u_3 so that w_1 sees at most two colors. Now v_3 has three available colors and v_1, v_2 have four available colors combined. So we may color v_3, u_1, u_2 such that w_3 sees at most two colors as desired. So suppose without loss of generality that $|\mathcal{C}'(u_1)| = 2$. In this case, color v_1, v_2, u_3 so that w_1 sees at most two colors. Now v_3 has three available colors and v_1, v_2 have three available colors combined. So we may color v_3, u_1, u_2 such that w_3 sees at most two colors as desired.

So we may assume that $|\mathcal{C}(v_3)| = 2$. Then suppose that $|\mathcal{C}'(u_3)| = 2$. Without loss of generality, we may assume that $|\mathcal{C}(v_1)| = 5$. If $\mathcal{C}(v_1) \setminus L(w_1) \neq \emptyset$, color v_1 with such a color and then color v_3. Now u_1 can either be colored the same as v_3 or with a color not in $L(w_3)$. Color u_1 as such, then u_2, u_3, v_2. But then w_1 and w_3 see at most two colors as desired. Similarly if $\mathcal{C}'(u_3) \setminus L(w_1) \neq \emptyset$, color u_3 with such a color. Now v_3 has two available colors and u_1, u_2 have four avaialble colors combined. So color v_3, u_1, u_3 such that w_3 sees at most two colors. Color v_1, v_2 and then w_1 sees at most two colors as desired.

So we may now assume that $\mathcal{C}'(u_3) \subset \mathcal{C}(v_1)$ as $\mathcal{C}'(u_3), \mathcal{C}(v_1) \subseteq L(w_1)$. If $\mathcal{C}'(u_1) \setminus L(w_3) \neq \emptyset$, color u_1 with such a color, then color u_3, v_1, v_2 such that w_1 sees at most two colors. Then color v_3 and u_2 and thus w_3 sees at most two colors as desired. If $\mathcal{C}(v_3) \setminus L(w_3) \neq \emptyset$, color v_3 with such a color, then color u_3, v_1, v_2 such that w_1 sees at most two colors. Then color u_1, u_2 and thus w_3 sees at most two colors as desired. So we may also assume that $\mathcal{C}(v_3) \subset \mathcal{C}'(u_1)$ as $\mathcal{C}(v_3), \mathcal{C}'(u_1) \subseteq L(w_3)$. In this case, color u_3 and v_1 with the same color and then color v_3 and u_1 with the same color. Finally color v_2 and u_2 and thus w_1 and w_3 see at most two colors as desired.

So suppose that $|\mathcal{C}'(u_3)| = 5$. In this case, color v_3, u_1, u_2 such that w_3 sees at most two colors. Now v_1, v_2 have at least four available colors combined while u_3 has at least two available colors. So we may color v_1, v_2, u_3 such that w_1 sees at most two colors as desired.

So we may assume that $\mathcal{C}$, and by symmetry $\mathcal{C}'$, is of the form $(3c)$. Suppose that $|\mathcal{C}(v_3)| = |\mathcal{C}'(u_3)| = 2$. If $\mathcal{C}(v_1) \setminus L(w_1) \neq \emptyset$, color v_1 with such a color and then color v_3. Now u_1 can either be colored the same as v_3 or with a color not in $L(w_3)$. Color u_1 as such, then u_3, u_2, v_2. But then w_1 and w_3 see at most two colors as desired. Similarly if $\mathcal{C}'(u_3) \setminus L(w_1) \neq \emptyset$, color u_3 with such a color. Now v_3 has two available colors and u_1, u_2 have four available colors combined. So color v_3, u_1, u_3 such that w_3 sees at most two colors. Color v_1, v_2 and then w_1 sees at most two colors as desired. So we may now assume that $\mathcal{C}'(u_3) \subset \mathcal{C}(v_1)$ as $\mathcal{C}'(u_3), \mathcal{C}(v_1) \subseteq L(w_1)$. By symmetry, we may then assume that $\mathcal{C}(v_3) \subset \mathcal{C}'(u_1)$. In this case, color u_3 and v_1 with the same color and then color v_3 and u_1 with the same color. Finally color v_2 and u_2 and thus

w_1 and w_3 see at most two colors as desired.

Next suppose that $|\mathcal{C}(v_3)| = |\mathcal{C}'(u_3)| = 5$. We may suppose without loss of generality that $|\mathcal{C}(v_1)| = 2$. Moreover, given that the succeeding arguments are identical for u_1 and u_2, we may also assume that $|\mathcal{C}(u_1)| = 2$. If $\mathcal{C}(v_1) \setminus L(w_1) \neq \emptyset$, color v_1 with such a color. Now v_3 has four available colors and u_1 has two available colors. So either color u_1, v_3 with same color or one of them with a color not in $L(w_3)$. Then color u_3, u_2, v_2 in that order. But then w_1 and w_3 see at most two colors as desired. Similarly if $\mathcal{C}'(u_3) \setminus L(w_1) \neq \emptyset$, color u_3 with such a color. Then color u_1. Now v_3 can either be colored the same as u_1 or with a color not in $L(w_3)$. So color v_3 as such and then color v_1, v_2, u_2 in that order. But now w_1 and w_3 see at most two colors as desired. So we may now assume that $\mathcal{C}(v_1) \subset \mathcal{C}'(u_3)$ as $\mathcal{C}'(u_3), \mathcal{C}(v_1) \subseteq L(w_1)$. By symmetry, we may now assume that $\mathcal{C}'(u_1) \subset \mathcal{C}(v_3)$. In this case, color u_3 and v_1 with the same color and then color v_3 and u_1 with the same color. Finally color v_2 and u_2 and thus w_1 and w_3 see at most two colors as desired.

Finally we may suppose that one of $|\mathcal{C}(v_3)|, |\mathcal{C}'(u_3)|$ is of size two and the other of size five. By symmetry, we may assume without loss of generality that $|\mathcal{C}'(u_3)| = 2$ and $|\mathcal{C}(v_3)| = 5$. In this case, color u_3, v_1, v_2 such that w_3 sees at most two colors. Now v_3 has at least two available colors while u_1, u_2 have at least four available colors combined. So we may color u_1, u_2, v_3 such that w_1 sees at most two colors as desired.
□

We are now ready to use these lemmas about magic colorings to prove Theorem 4.7.1.

Proof of Theorem 4.7.1. Suppose not. For $i \in \{1, 2, 3\}$, let M_i be the closest triangle to T_1 such that ϕ extends to a Magic i set of colorings of M_i. Similarly for $i \in \{1, 2, 3\}$, let M_i' be the closest triangle to T_2 such that ϕ' extends to a Magic i set of colorings of M_i'. By Corollary 4.7.9 and Lemmas 4.8.2, 4.8.1 and 4.9.1, $d(T_1, M_1) \leq 2$

and $d(T_2, M_1') \leq 2$. By Lemmas 4.7.6, 4.7.12, 4.8.3 and 4.9.1, $d(M_1, M_3) \leq 2$ and $d(M_1', M_3') \leq 2$. By Corollary 4.7.16 and Lemmas 4.8.7, 4.9.3, it follows that if $d(M_3, M_3') \geq 2$, then there exist ϕ extends to an L-coloring of G. Hence $d(M_3, M_3') \leq 1$. But now $d(T_1, T_2) \leq d(T_1, M_1) + 1 + d(M_1, M_3) + 1 + d(M_3, M_3') + 1 + d(M_3', M_1') + 1 + d(M_1', T_1) \leq 13$, a contradiction. □

4.10 Proof of the Two Precolored Triangles Theorem

Proof of Theorem 4.1.1. Let $\Gamma = (G, T_1, T_2, L)$ is a counterexample to Theorem 4.1.1 with a minimum number of vertices and subject to that a maximum number of edges. By Lemma 4.6.21 with $d_0 = 14$, there exists triangles T_1' and T_2' of G each separating C_1 from C_2 such that $d(T_1', T_2') \geq 14$ and every band in the band decomposition of $\Gamma[T_1', T_2']$ is a subgraph of a tetrahedral, octahedral or hexadecahedral band.

Let ϕ be an L-coloring of $T_1 \cup T_2$. By Theorem 1.4.2, it follows that $\phi \restriction T_1$ can be extended to an L-coloring of $\Gamma[T_1, T_1']$ and similarly $\phi \restriction T_2$ can be extended to an L-coloring of $\Gamma[T_2, T_2']$. By Theorem 4.7.1, ϕ can now be extended to an L-coloring of G, a contradiction. □

CHAPTER V

A GENERAL LINEAR BOUND

5.1 Introduction

In this chapter, we prove the main results of this thesis. In Section 5.2, we generalize Theorem 4.1.1 to the case of two cycles with lists of size three a constant distance apart. Then we extend Theorems 3.4.26 and 3.6.4 to the case of two precolored cycles. In Section 5.3, we extend Theorem 3.4.27 to the case of two precolored cycles. In Section 5.4, we extend Theorem 3.5.3 to the case of two precolored cycles.

In Section 5.5, we define a useful way to planarize a graph on a surface. In Section 5.6, we proceed to develop an abstract theory for families of graphs satisfying a linear isoperimetric inequality, as in Theorem 3.4.26, for the disc and any isoperimetric inequality for the cylinder. In Section 5.7, we prove our main results holds in this abstract setting. In Section 5.8, we apply the general theory to the family of 6-list-critical graphs to derive the main results for 5-list-coloring. Finally, in Section 5.9, we apply the theory for a slightly different family to obtain the exponentially many 5-list-colorings result.

5.2 A Linear Bound for the Cylinder

Theorem 5.2.1. *[Cylinder Theorem: Cycles with Lists of Size Three] If* $\Gamma = (G, C \cup C', L)$ *is a cylinder-canvas where vertices in* $C \cup C'$ *have lists of size at least three and* $d(C, C') \geq D$*, then there exists an* L*-coloring of* G*.*

Proof. Suppose that Γ is a counterexample to the theorem with a minimum number of vertices and subject to that a maximum number of edges. Thus Γ is critical. Hence there are no degree four vertices in $G \setminus (C \cup C')$ and there do not exist vertices in the interior of triangles that do not separate a vertex of C from a vertex of C'.

Claim 5.2.2. *Either there exists a triangle C_0 separating C, C' such that $d(C_0, C \cup C') \leq 2$, or there exists a path $P = p_0p_1 \ldots p_k$ from C to C' such that all the following hold:*

(1) $p_0 \in C$, $p_k \in C'$,

(2) *P is a shortest path from C to C',*

(3) *none of $p_1, p_2, p_{k-2}, p_{k-1}$ has two mates (i.e. is tripled)*

(4) *P is an arrow from p_0 to p_k,*

(5) *if p_1 (resp. p_{k-1}) has more than one neighbor on C (resp. C'), then there either there is no neighbor of p_1 on C(resp. C') to the left or no neighbor of p_1 on C_1 to the right,*

(6) *if p_2 (resp. p_{k-2}) has a mate, then if it is a right mate, then there is no neighbor of p_1 on C (resp. C') to the left and similarly if it is a left mate, then there is no neighbor of p_1 on C to the right.*

Proof. Consider a shortest path Q from C to C'. Let $p_1 \in Q$ such that $p_1 \notin C$ and yet p_1 has a neighbor in C. Similarly let $p_{k-1} \in Q$ such that $p_1 \notin C'$ and yet p_{k-1} has a neighbor in C'. Let $P' = p_1p_2 \ldots p_{k-1}$ be an arrow from p_1 to p_{k-1}, where by Lemma 4.2.2 such a P' is guaranteed to exist. We claim that neither p_2 nor p_{k-2} has two mates in P'. For suppose not. Say p_i has two mates where $i \in \{2, k-2\}$. As p_i does not have degree four and no vertices in the interior of triangles that do not separate a vertex of C_1 from a vertex of C_2, one of the mates, call it x, of p_i is in a triangle, either $p_{i-1}xp_i$ or $p_{i+1}xp_i$, that separates a vertex of C from a vertex of C'. Call this triangle C_0 and note that $d(C_0, C \cup C') \leq 2$.

Now let p_0 be a neighbor of p_1 on C such that there is no neighbor of p_1 on C either to the left or to the right, and in addition if p_2 has a mate z in P', then p_0 has no neighbor on C to the left if z is a right mate, and, p_0 has no neighbor on C to the right if z is a left mate. Let p_k be chosen similarly. Let $P = p_0 p_1 \ldots p_{k-1} p_k$. Clearly (1) and (2) hold. Furthermore, (5) and (6) hold as p_0 and p_k were chosen to satisfy these conditions. Moreover, (3) is satisfied. For suppose not. Say p_i has two mates where $i \in \{1, 2, k-2, k-1\}$. As p_i does not have degree four and no vertices in the interior of triangles that do not separate a vertex of C_1 from a vertex of C_2, one of the mates, call it x, of p_i is in a triangle, either $p_{i-1} x p_i$ or $p_{i+1} x p_i$, that separates a vertex of C from a vertex of C'. Call this triangle C_0 and note that $d(C_0, C \cup C') \leq 2$.

Finally we show that (4) is satisfied. Suppose not. Thus there exist i, $2 \leq i \leq k-1$ such that p_i is tripled and yet p_{i-1} is doubled. Yet P' was an arrow from p_1 to p_{k-1} and hence $i \notin [3, k-2]$. Thus either $i = 2$ or $i = k-1$. If $i = 2$, then p_2 is tripled, a contradiction as (3) was shown to hold for P. If $i = k-1$, then p_{k-1} is tripled, a contradiction as (3) was shown to hold for P. Hence (4) holds and the claim is proved. □

Claim 5.2.3. *There exists D' such that if there does not exist a triangle separating C and C' and $d(C, C') \geq D'$, then G is L-colorable.*

Proof. As there is no triangle separating C and C', by Claim 5.2.2, there exist a path P satisfying (1)-(6) in Claim 5.2.2. Choose P such that p_1 and p_{k-1} have mates in P if possible. Let p_0' be the other 'most' neighbor of p_1 if p_1 has more than one neighbor on C and similarly let p_k' be the other 'most' neighbor of p_{k-1} on C'. Let $P_1 = p_0 p_1 p_0'$ if p_0' exists and $P_2 = p_k p_{k-1} p_k'$ if p_k' exists. Let $B_1 = Ext(P_1)$, the bellows with base P_1 and $B_2 = Ext(P_2)$, the bellows with base B_2.

Suppose that either p_k' does not exist or there does not exist a vertex adjacent to p_k', p_{k-1}, p_{k-2}. Similarly suppose that either p_0' does not exist or there does not exist a vertex adjacent to p_0', p_1, p_2. As P is an arrow from p_0 to p_k and $|L(p_0)|, |L(p_k)| \geq 3$, there exists a bichromatic coloring ϕ of P by Lemma 4.2.3. Let G' be the graph obtained from G by cutting along P and deleting P. Let $L'(v) = L(v) \setminus \{\phi(u)|u \in N(v) \cap P\}$. Let $S' = (N(p_0) \cup N(p_1) \cup N(p_{k-1}) \cup N(p_k)) \cap (C \cup C')$. Let $\Gamma' = (G', S', L')$ be the resulting canvas.

By Theorem 1.4.2, there exists an L'-coloring ϕ' of $G' \cap (B_1 \cup B_2)$. Let $G'' = G' \setminus (B_1 \cup B_2)$. Let $L''(v) = L'(v) \setminus \{\phi'(u)|u \in \{p_0', p_k'\}, u \sim v\}$. Let $S'' = \{v||L''(v)| < 3\}$. Consider the resulting canvas $\Gamma'' = (G'', S'', L'')$.

We claim that if $v \in S''$, then either v is a neighbor of p_0 or p_0' in C or a neighbor of p_k or p_k' in C'. Suppose not. It follows that without loss of generality that v is a neighbor of p_0' and that v has two neighbors in p_0, p_1, p_2. As there does not exist v adjacent to all of p_0', p_1, p_2, we may assume that $v \sim p_0$. If $v \sim p_1$, then either vp_1p_0 or vp_1p_0' is a triangle separating C from C', a contradiction. So we may suppose that $v \sim p_2$. But then either $vp_0p_1p_2$ or $vp_0'p_1p_2$ is a 4-cycle not separating C from C' and hence $G \cup \{vp_1\}$ is a counterexample with the same number of vertices but more edges, a contradiction.

We claim that $S'' \cap C$ consists either of at most two vertices with lists of size two or one vertex with a list of size one. This follows from the fact that we chose the p_0 such that p_1 has no neighbor either to the right or to the left by (5). So if p_0' exists there can only be one vertex adjacent to p_0 in $C \cap V(G'')$ and similarly one to p_0' in $C \cap V(G'')$ as there are no chords of C or C'. If p_0' does not exist, then there are at most two neighbors of p_0 in C. This proves the claim.

Similarly $S'' \cap C'$ consists either of at most two vertices with lists of size two or one vertex with a list of size one. As there is no triangle separating C and C' and P was a shortest path between them, there cannot exist a long bottleneck in Γ' as such a bottleneck would either have to involve many chords between C (or C') and P, or create a separating triangle between C and C'. But now the claim follows by invoking Theorem 3.12.1, as the critical subcanvas (G', S, L'), where S are the vertices of Γ' with lists of size at most two, must include a vertex from S in C and another vertex from S in C' (as there are local colorings near C and C' given either one precolored vertex or two lists of size two). But then $|V(G')| \geq d(C,C') \geq D'$ and yet $|V(G')| = O(|S|) = O(4)$, a contradiction.

So suppose that p_0' exists and there exists a vertex $v \sim p_0', p_1, p_2$. We may suppose without loss of generality that $|L(p_0)| = |L(p_0')| = 3$. As P was chosen so that p_1 had a mate if possible, we find that p_1 has a mate p_1'. As there is no triangle separating C from C', $p_1' \neq v$. First suppose that $L(p_0) \setminus L(p_1') \neq \emptyset$. In this case, let $\phi(p_0) \in L(p_0) \setminus L(p_1')$. If B_1 is a fan, let $\phi(p_1)$ in $L(p_1) \setminus (\{\phi(p_0)\} \cup L(p_0'))$; if B_1 is not a fan, let $\phi(p_1) \in L(p_1) \setminus \{\phi(p_0), \phi_1(p_1)\}$ where ϕ_1 is the unique non-extendable coloring of P_1 to B_1. Now let $L_0(p_2) = L(p_2) \setminus \{\phi(p_1)\}$ and $L_0 = L$ otherwise. Otherwise we may suppose that $L(p_0) \subset L(p_1')$. In this case, let $L_0(p_2) = L(p_2) \setminus (L(p_1') \setminus L(p_0))$ and $L_0 = L$ otherwise.

As $P \setminus \{p_0, p_1\}$ is an arrow from p_2 to p_k and $|L_0(p_2)|, |L_0(p_k)| \geq 3$, there exist a bichromatic L_0-coloring ϕ of $P \setminus \{p_0, p_1\}$ by Lemma 4.2.3. Now if $L(p_0) \subset L(p_1')$, we would like to extend ϕ to p_1 and then p_0. To that end, if B_1 is a fan, let $\phi(p_1)$ in $L(p_1) \setminus (\{\phi(p_2)\} \cup L(p_0'))$; if B_1 is not a fan, let $\phi(p_1) \in L(p_1) \setminus \{\phi(p_2), \phi_1(p_1)\}$ where ϕ_1 is the unique non-extendable coloring of P_1 to B_1. Let $\phi(p_0) = \phi(p_2)$ if $\phi(p_2) \in L(p_0)$ and otherwise let $\phi(p_0) \in L(p_0) \setminus \{\phi(p_1\}$. Hence, in either case ϕ is an L-coloring of P such that all vertices not in P see at most two colors.

Let G' be the graph obtained from G by cutting along P and deleting P. Let $L'(v) = L(v) \setminus \{\phi(u) | u \in N(v) \cap P\}$. Let $S' = (N(p_0) \cup N(p_1) \cup N(p_{k-1}) \cup N(p_k)) \cap (C \cup C')$. Let $\Gamma' = (G', S', L')$ be the resulting canvas.

By Theorem 1.4.2, there exists an L'-coloring ϕ' of $G' \cap B_2$. Furthermore note that every L'-coloring of p_0' can be extended to an L-coloring of B_1 extending ϕ. Let $G'' = G' \setminus ((B_1 \setminus p_0') \cup B_2)$. Let $L''(v) = L'(v) \setminus \{\phi'(p_k') | p_k' \sim v\}$. Let $S'' = \{v || L''(v)| < 3\}$. Consider the resulting canvas $\Gamma'' = (G'', S'', L'')$.

We claim that if $v \in S''$, then either v is a neighbor of p_0 in C or $v = p_0'$ or v is a neighbor of p_k or p'_k in C'. This claim follows in the same way as before for vertices near C' and is clear for vertices near C. We then claim that $S'' \cap C$ consists either of at most two vertices with lists of size two. This follows from the fact that we chose the p_0 such that p_1 has no neighbor either to the right or to the left by (5). So as p'_0 exists there can only be one vertex adjacent to p_0 in $C \cap V(G'')$. This proves the claim.

Now $S'' \cap C'$ consists either of at most two vertices with lists of size two or one vertex with a list of size one as before. As there is no triangle separating C and C' and P was a shortest path between them, there cannot exist a long bottleneck in Γ' as such a bottleneck would either have to involve many chords between C (or C') and P, or create a separating triangle between C and C'. But now the claim follows by invoking Theorem 3.12.1, as the critical subcanvas (G', S, L'), where S are the vertices of Γ' with lists of size at most two, must include a vertex from S in C and another vertex from S in C' (as there are local colorings near C and C' given either one precolored vertex or two lists of size two). But then $|V(G')| \geq d(C, C') \geq D'$ and yet $|V(G')| = O(|S|) = O(4)$, a contradiction.

So we may suppose that p'_k exists and there exists a vertex $v' \sim p'_k, p_{k-1}, p_{k-2}$. The same argument applies as above when either p'_0 does not exist or there does not exist $v \sim p'_0, p_1, p_2$ as we did not use the direction of the arrow in that argument. A similar argument also applies when p'_0 exists and there exists $v \sim p'_0, p_1, p_2$ by modifying L to L_0 at both ends and finding a bichromatic coloring ϕ of the arrow from p_2 to p_{k-2}. In that case, we do not either p'_0 or p'_k and then proceed as above. $\square$

Let D_0 be the distance in Theorem 4.1.1. If $|C|, |C'| \leq 3$, the theorem follows from Theorem 4.1.1 as long as $D \geq D_0$. By Claim 5.2.3, there exists D' such that if there does not exist a triangle separating a vertex of C and a vertex of C' and $d(C, C') \geq D'$, then the graph G is L-colorable. So we may suppose there exists a triangle separating a vertex of C from a vertex of C'. Let T_1 be such a triangle closest to C and T_2 be such a triangle closest to C'. By Claim 5.2.3, $d(C, T_1) \leq D'$ and $d(C', T_2) \leq D'$ and yet by Theorem 4.1.1, $d(T_1, T_2) \leq D_0$. Hence $d(C, C') \leq 2D' + D_0$, a contradiction if $D > 2D' + D_0$. $\square$

Lemma 5.2.4 (Cylinder Theorem: Linear-Log Distance). *If $\Gamma = (G, C_1 \cup C_2, L)$ is a connected critical cylinder-canvas, then $d(C_1, C_2) \leq O(|C_1| \log |C_1| + |C_2| \log |C_2|)$.*

Proof. Let us proceed by induction on $|C_1| + |C_2|$. Suppose without loss of generality that $|V(C_1)| \geq |V(C_2)|$. We may assume that $|V(C_1)| \geq 4$ as otherwise the theorem follows from Theorem 4.1.1.

We now prove a stronger statement. For $i \in \{1, 2\}$, let R_i be the set of relaxed vertices of C_i and $S_i = V(C_i) \setminus R_i$. Let $d_r(\Gamma) = \min\{d(R_1, R_2) + 2, d(R_1, S_2) + 1, d(S_1, R_2) + 1, d(S_1, S_2)\}$.

Let $f(m_1, m_2) = 58((m_1 - 3) \log m_1 + (m_2 - 3) \log m_2) + D + 2$ where D is the constant in Theorem 5.2.1.

We now prove that

$$d_r(T) \leq f(|C_1|, |C_2|) + 2.$$

Let $T = (G, C_1 \cup C_2, L)$ be a counterexample to the formula above with a minimum number of vertices, where $|C_1| \geq |C_2|$ without loss of generality. Let $k_1 = |C_1|$ and $k_2 = |C_2|$. Hence $d(C_1, C_2) > f(k_1, k_2)$. Note then that $C_1 \cap C_2 = \emptyset$ as $d(C_1, C_2) \geq 1$.

Claim 5.2.5. *For $i \in \{1, 2\}$, there does not exist $G_i \subseteq G$ such that $G_i \cap C_{3-i} = \emptyset$ and (G_i, C_i, L) is a critical canvas.*

Proof. Suppose not. Suppose without loss of generality that there exists $G_1 \subseteq G$ such that $G_1 \cap C_2 = \emptyset$ and (G_1, C_1, L) is a critical canvas. There exists a face f of G_1 such that the boundary cycle of f, call it C, separates a vertex of C_1 from C_2. By Corollary 3.3.4, $|C| < |C_1|$. By induction, it follows that $d(C, C_2) \leq f(|C|, k_2)$. Hence $d(C, C_2) \leq f(k_1 - 1, k_2)$. By Theorem 3.6.4, $d(v, C_1) \leq 58 \log k_1$ for all $v \in V(C)$. Hence $d(C_1, C_2) \leq f(k_1, k_2)$, a contradiction. $\square$

Hence there does not exist a chord U of C_1 or C_2.

Claim 5.2.6. *For $i \in \{1, 2\}$, there does not exist v with at least three neighbors in $C_1 \cup C_2$.*

Proof. As $d(C_1, C_2) \geq 3$, we may suppose without loss of generality that v has at least three neighbors on C_1. As there does not exist a chord of C_1, $v \notin V(C_1)$. Let f be the face of $G[C_1 \cup \{v\}]$ such that the boundary cycle of f, call it C, separates a vertex of C_1 from C_2. Suppose $|C| < |C_1|$. By induction, it follows that $d(C, C_2) \leq f(|C|, k_2)$. Hence $d(C, C_2) \leq f(k_1 - 1, k_2)$. By Theorem 3.6.4, $d(v, C_1) \leq 58 \log k_1$ for all $v \in V(C)$. Hence $d(C_1, C_2) \leq f(k_1, k_2)$, a contradiction.

So we may suppose that $|C| = |C_1|$. Let $C_1 = v_1 v_2 v_3 \ldots v_{k_1}$. We may assume without loss of generality that $N(v) \cap C_1 = \{v_1, v_2, v_3\}$. Hence $C = v_1 v v_3 \ldots v_{k_1}$.

Consider the canvas $\Gamma' = (G \setminus \{v_2\}, C \cup C_2, L)$. Now T' is critical. As T is a counterexample with a minimum number of vertices, we find that $d_r(T') \leq f(k_1, k_2) + 2$.

Now we claim that v is relaxed in T'. Let ϕ be an L-coloring of $C_1 \cup C_2$ that does not extend to an L-coloring of G. Let $S(v) = L(v) \setminus \{\phi(v_1), \phi(v_2), \phi(v_3)\}$. Note then that $|S(v)| \geq 2$ as $|L(v)| = 5$. Let $c_1, c_2 \in S(v)$. For $i \in \{1, 2\}$, let $\phi_i(v) = c_i$ and $\phi_i = \phi$ otherwise. Hence ϕ_1, ϕ_2 are L-colorings of $P_1' \cup P_2$ that do not extend to an L-coloring of $G \setminus \{p_i\}$ such that $\phi_1(v) \neq \phi_2(v)$ but $\phi_1 = \phi_2$ otherwise. So v is relaxed as claimed.

Next we claim that $R(C_1) \subseteq R(C) \setminus \{v\}$. To see this, let $u \in R(C_1)$. Thus there exist two L-colorings ϕ_1, ϕ_2 of $P_1 \cup P_2$ that do not extend to an L-coloring of G such that $\phi_1(u) \neq \phi_2(u)$ and $\phi_1 = \phi_2$ otherwise.

Suppose $u \neq v_2$. Let $S(v) = L(v) \setminus \{\phi_1(v_1), \phi_2(v_1), \phi_1(v_2), \phi_2(v_2), \phi_1(v_3), \phi_2(v_3)\}$. As $\phi_1 = \phi_2(w)$ for all $w \neq u$, we find that $|S(v)| \geq 1$ as $|L(v)| = 5$. Let $c \in S(v)$ and $\phi_1(v) = \phi_2(v) = c$. Now ϕ_1, ϕ_2 are L-colorings of $P_1' \cup P_2$ that do not extend to an L-coloring of $G \setminus \{p_i\}$ such that $\phi_1(u) \neq \phi_2(u)$ and $\phi_1 = \phi_2$ otherwise. Thus u is relaxed for T'. So $u \in R(P_1') \setminus \{v\}$ as claimed.

Suppose $u = v_2$. If $\phi_1(v_1) = \phi_1(v_3)$, let G' be obtained from G by deleting v_2 and identifying v_1 and v_3 to a single vertex. If $\phi_1(v_1) \neq \phi_1(v_3)$, let G' be obtained from G by deleting v_2 and adding an edge between v_1 and v_2. Let C_1' be the resulting path on $C_1 \setminus \{v_2\}$. Consider $\Gamma' = (G', C_1' \cup C_2, L)$. Now there does not exist an L'-coloring of G that extends ϕ_1.

Hence Γ' contains a critical subcanvas Γ''. If Γ'' is connected, then by induction $d(C_1, C_2) \leq d(C_1', C_2) \leq f(k_1 - 1, k_2)$, a contradiction. If Γ'' is not connected, then there exists $G_1 \subseteq G$ such that $G_1 \cap P_2 = \emptyset$ and (G_1, P_1, L) is a critical canvas, contradicting Claim 5.2.5. Thus $R(C_1) \subset R(C_1') \setminus \{v\}$ as claimed.

But now it follows that $d_r(\Gamma) \leq d_r(\Gamma')$ and hence $d_r(\Gamma) \leq f(k_1, k_2) + 2$, contrary to the fact that Γ was a counterexample to this formula. □

Let ϕ be an L-coloring of $C_1 \cup C_2$ such that ϕ does not extend to an L-coloring of G. Let $G' = G \setminus (V(C_1) \cup V(C_2))$. Let $L'(v) = L(v) \setminus \{\phi(u) | u \in V(C_1) \cup V(C_2), u \in N(v)\}$. By Claim 5.2.6, $|L'(v)| \geq 3$ for all $v \in V(G')$. Let C_1' be the boundary walk of the outer face of G' and C_2' be the face of G' containing the disk bounded by C_2. Now add edges to the outer face so that vertices with lists L' of size less than five in C_1' form a cycle C_1''. Similarly add edges inside the disk bounded by C_2' so that vertices with lists L' of size less than five in C_2' form a cycle C_2''. Now $\Gamma' = (G', C_1'' \cup C_2'', L')$ is a cylinder-canvas. Furthermore, $d(C_1'', C_2'') \geq f(k_1, k_2) - 2 \geq D$, a contradiction as then ϕ extends to an L-coloring of G by Theorem 5.2.1. □

Corollary 5.2.7 (Cylinder Theorem: Linear-Log Bound)**.** *If $\Gamma = (G, C \cup C', L)$ is a connected critical cylinder-canvas, then $|V(G)| \leq O(|C| \log |C| + |C'| \log |C'|)$.*

Proof. Let f_1 be the face of G whose boundary is C and f_2 be the face of G whose boundary is C'. Let P be a shortest path from C to C'. Let f be the face of

$G[C \cup C' \cup P]$ such that $f \neq f_1, f_2$. Let C'' be the boundary walk of f. But then G is C''-critical. As $d(C, C') \leq O(|C| \log |C| + |C'| \log |C'|)$ by Lemma 5.2.4, $C'' \leq O(|C| \log |C| + |C'| \log |C'|)$. By Corollary 3.4.26, $|V(G')| \leq O(|C''|) \leq O(|C| \log |C| + |C'| \log |C'|)$. As $|V(G)| \leq |V(G')|$, the corollary follows. □

Lemma 5.2.8 (Cylinder Theorem: Linear Distance)**.** *If $\Gamma = (G, C \cup C', L)$ is a connected critical cylinder-canvas, then $d(C, C') \leq O(|C| + |C'|)$.*

Proof. There must exist a distance i, $1 \leq i \leq 2c \log |C|$ where c is the constant in Corollary 5.2.7, such that either there are at most $|C|/2$ vertices at distance i from C or there must exist a distance j, $1 \leq j \leq 2c \log |C'|$ such that there are at most $|C'|/2$ vertices at distance i from C'. The corollary then follows by induction (actually shows $d(C, C') \leq O(\log^2 |C| + \log^2 |C'|)$). □

Theorem 5.2.9 (Cylinder Theorem: Linear Bound)**.** *If $\Gamma = (G, C \cup C', L)$ is a connected critical cylinder-canvas, then $|V(G)| \leq O(|C| + |C'|)$.*

Proof. Let f_1 be the face of G whose boundary is C and f_2 be the face of G whose boundary is C'. Let P be a shortest path from C to C'. Let f be the face of $G[C \cup C' \cup P]$ such that $f \neq f_1, f_2$. Let C'' be the boundary walk of f. But then G is C''-critical. As $d(C, C') \leq O(|C| + |C'|)$ by Lemma 5.2.8, $C'' \leq O(|C| + |C'|)$. By Corollary 3.4.26, $|V(G')| \leq O(|C''|) \leq O(|C| + |C'|)$. As $|V(G)| \leq |V(G')|$, the theorem follows. □

Theorem 5.2.10. *[Cylinder Theorem: Logarithmic Distance] If $\Gamma = (G, C \cup C', L)$ is a connected critical cylinder-canvas, then $d(v, C \cup C') \leq O(\log(|C| + |C'|))$ for all $v \in V(G)$. In particular, $d(C, C') \leq O(\log(|C| + |C'|))$.*

Proof. There must exist a distance i, $1 \leq i \leq 2c$ where c is the constant in Theorem 5.2.9, such that either there are at most $|C|/2$ vertices at distance i from C or there are at most $|C'|/2$ vertices at distance i from C'. The corollary then follows by induction. □

5.3 Easels for Cylinder-Canvases

Definition. Let $T = (G, C_1 \cup C_2, L)$ be a cylinder-canvas. Let f_1 be the face of G bounded by C_1 and f_2 be the face of G bounded by C_2. Let $G' \subseteq G$ such that for every face f of G such that $f \neq f_1, f_2$, every L-coloring of the boundary of f extends to an L-coloring of the interior of f. We say the cylinder-canvas $T' = (G', C_1 \cup C_2, L)$ is an *easel* for T.

Let $T = (G, C_1 \cup C_2, L)$ be a cylinder-canvas and $T' = (G', C_1 \cup C_2, L)$ an easel for T. We say that T' is a *critical easel* for T if there does not exist $T'' = (G'', C_1 \cup C_2, L)$ such that $G'' \subsetneq G'$ such that T'' is an easel for T', and hence also an easel for T as noted above.

We may now derive a linear bound on the size of an easel for a cylinder-canvas.

Theorem 5.3.1. *If $T = (G, C_1 \cup C_2, L)$ is a cylinder-canvas, then there exists an easel $T' = (G', C_1 \cup C_2, L)$ for T such that $|V(G')| = O(|C| + |C'|)$.*

Proof. Let f_1 be the face of G whose boundary is C_1 and f_2 be the face of G whose boundary is C_2. Let P be a shortest path from C_1 to C_2. Let f be the face of $G[C \cup C' \cup P]$ such that $f \neq f_1, f_2$. Let C_0 be the boundary walk of f.

Suppose $d(C_1, C_2) \leq O(\log(|C_1| + |C_2|))$. Consider the canvas $T_0 = (G_0, C_0, L)$. By Theorem 3.4.27, there exists an easel $T_0' = (G_0', C_0, L)$ for T_0 such that $|V(G_0')| \leq 29|V(C_0)|$. But T_0' corresponds to an easel $T' = (G', C_1 \cup C_2, L)$ for T such that $|V(G')| \leq 29(|C_1| + |C_2| + |P|) + |P| = O(|C_1| + |C_2|)$.

So we may suppose that $d(C_1, C_2) \geq \Omega(\log(|C_1| + |C_2|))$. By Theorem 3.4.27, there exists a critical easel $T_1 = (G_1, C_1, L)$ for the cycle-canvas $(G \setminus C_2, C_1, L)$ and a critical easel $T_2 = (G_2, C_2, L)$ for the cycle-canvas $(G \setminus C_1, C_2, L)$. By Theorem 3.6.8, for all $i \in \{1, 2\}$, $d(v, C_i) \leq 58 \log |C_i|$ for all $v \in V(G_i)$.

As $d(C_1, C_2) \geq \Omega(\log(|C_1| + |C_2|)$, $G_1 \cap G_2 = \emptyset$. Let C_1' be the facial cycle of G_1 separating C_1 from C_2 and similarly let C_2' be the facial cycle of G_2 separating

C_1 from C_2. It follows that $d(C_1', C_2') \geq \Omega(\log(|C_1| + |C_2|)$. By Theorem 5.2.10 applied to $T[C_1', C_2']$, we may assume that every L-coloring of $C_1' \cup C_2'$ which extends to an L-coloring of the vertices at distance $\log(|C_1'| + |C_2'|)$ extends to an L-coloring of $T[C_1', C_2']$. Yet as T_1 is an easel for $(G \setminus C_2, C_1, L)$ and T_2 , it follows that every L-coloring of $C_1' \cup C_2'$ extends to an L-coloring of $T[C_1', C_2']$. Thus $T' = (G_1 \cup G_2, C_1 \cup C_2, L)$ is an easel for T and $|V(G_1 \cup G_2)| \leq 58(|C_1| + |C_2|)$ as desired. □

5.4 Exponentially Many Extensions of Two Precolored Cycles

Lemma 5.4.1. *Let $\epsilon \leq 1/18$ such that $\epsilon = 1/(1144\alpha)$ and there exists $\alpha > \beta$ such that $\Omega(\log \beta) \leq \alpha < 2^{\beta/290}/492$ with the property that if two cycles have size at most β and are at least α distance apart, then any coloring of those cycles which extends to distance $O(\log \beta)$ extends to the graph in between.*

If (G, C, C', L) is a cylinder cycle-canvas and ϕ is an L-coloring of $C \cup C'$ that extends to an L-coloring of G, then $\log E(\phi) \geq \epsilon(|V(G \setminus (C \cup C'))| - 50(|V(C)| + |V(C')|))$, where $E(\phi)$ is the number of extensions of ϕ to G.

Proof. Suppose not. Let (G, C, C', L) be a counterexample with a minimumber of vertices. Let $d = d(C, C')$ and $P = p_0 p_1 \dots p_d$ be a shortest path from C to C'. Let f_1 be the face of G whose boundary is C and f_2 be the face of G whose boundary is C'. Let P be a shortest path from C to C'. Let f be the face of $G[C \cup C' \cup P]$ such that $f \neq f_1, f_2$. Let C'' be the boundary walk of f.

Note that $|V(G \setminus (C \cup C'))| \geq 50|V(C)| + |V(C')|$, as otherwise the formula holds, a contradiction.

Claim 5.4.2. $d \geq (|C| + |C|')/3$

Proof. Suppose not. Extend ϕ to an L-coloring ϕ' of $G[V(C) \cup V(C') \cup P']$ such that ϕ extends to an L-coloring of G. By Theorem 3.5.3, $\log E(\phi') \geq (|V(G \setminus (C \cup C' \cup$

$P)| - 29(|C''|))/9 = (|V(G \setminus (C \cup C'))| - 50(|C| + |C'))/9$ as $d \leq (|C| + |C'|)/3$ and $|C''| = |C| + |C'| + 2d$, a contradiction. □

Claim 5.4.3. *$|V(G)| \leq 143d$.*

Proof. Suppose not. Extend ϕ to an L-coloring ϕ' of $G[V(C) \cup V(C') \cup P']$ such that ϕ extends to an L-coloring of G. By Theorem 3.5.3, $\log E(\phi') \geq (|V(G \setminus (C \cup C' \cup P))| - 29(|C''|))/9 = (|V(G)| - 58d - 30(|C| + |C|')/9$ as $|C''| = |C| + |C'| + 2d$. As the formula does not hold for T, we find that $\log E(\phi) \leq (|V(G)| - 51(|C| + |C'))/18$ as $\epsilon \leq 18$. Yet $E(\phi') \leq E(\phi)$. So we find that $2|V(G)| - 116d - 60(|C| + |C'|) \leq |V(G)| - 51(|C| + |C'|)$. Hence $|V(G)| \leq 116d + 9(|C| + |C|') \leq 143d$ as $|C| + |C'| \leq 3d$ by Claim 5.4.2, a contradiction. □

Let A be the set of all i such that $p_{i\alpha}$ is in a cycle C_i of size at most β separating C from C'. Let B be the set of all i such that $B_\beta(p_{i\alpha})$ is contained in a slice H where $G \setminus H$ attaches to at most one face of H and the boundary of H is contained in $N_\beta(p_{i\alpha})$.

For all $p_{i\alpha}$, $1 \leq i \leq d/\alpha$, is either in a cycle C_i of size at most β separating C from C' or $B_\beta(p_{i\alpha})$ is contained in a slice H where $G \setminus H$ attaches to at most one face of H and the boundary of H is contained in $N_\beta(p_{i\alpha})$. Hence $|A| + |B| \geq d/\alpha$. We will consider two cases, first when $|B| \geq d/2\alpha$ and second when $|A| \geq d/2\alpha$.

Suppose that $|B| \geq d/2\alpha$. Let $B' \subseteq B$ such that for all $i, j \in B$, $|j - i| \geq 2$ and $|B'| = |B|/2 \geq d/4$. Now for all $i, j \in B$, $B_\beta(p_{i\alpha}) \cap B_\beta(p_{j\alpha}) = \emptyset$ as $\beta < \alpha$. We need the following claim.

Claim 5.4.4. *For all $i \in B$ and $0 \leq j \leq \beta$, $|B_j(p_{i\alpha})| \leq 145|N_j(p_{i\alpha})|$.*

Proof. Suppose the claim does not hold for i and j. Let $p = p_{i\alpha}$. Thus $|B_j(p)| > 145|N_j(p)|$. Let C_j be a minimal subset of $N_j(p)$ separating $B_j(p)$ from $C \cup C'$. Hence there exists a closed curve γ such that $V(G) \cap \gamma = V(C_j)$ and γ does not intersect itself. For every two vertices u, v of C_j consecutive along γ, add a vertex w on γ and edges uw and vw to obtain a cycle C'_j where $|C'_j| \leq 2|C_j|$.

Let $G' = Int(\gamma)$ and consider the cycle-canvas $T' = (G', C'_j, L)$. Let $T'' = (G'', C'_j, L)$ be a critical easel for T'. It follows from Theorem 3.5.3, that if ϕ' is an L-coloring of $G \setminus (G'' \setminus C'_j)$, then $\log E(\phi') \geq (|V(G' \setminus G'')| - 29(|C'_j| - 3))/9$, where $E(\phi')$ is the number of extensions of ϕ to G. As $|C'_j| \leq 2|C_j|$, $\log E(\phi') \geq (|V(G' \setminus G'')| - 58|C_j|)/9$.

Meanwhile, consider the cylinder cycle-canvas (G_0, C, C', L) where $G_0 = G \setminus (G' \setminus G'')$. As G is a minimum counterexample, $\log E_{G_0}(\phi) \geq 2^{\epsilon(|V(G_0 \setminus (C \cup C'))| - 50(|V(C)| + |V(C')|))}$. Hence, $\log E(\phi) \geq 2^{\epsilon(|V(G \setminus (C \cup C'))| - 50(|V(C)| + |V(C')|))} 2^{(1/9 - \epsilon)|V(G' \setminus G'')| - 58|C_j|/9}$. As G is a counterexample, we find that $|V(G' \setminus G'')| \leq 58|C_j|/(1 - 9\epsilon) \leq 116|C_j|$. As $|V(G'')| \leq 29|C_j|$, we find that $|B_j(p)| \leq 145|N_j(p)|$, a contradiction. □

Claim 5.4.5. *For all $i \in B$ and $0 \leq j \leq \beta$, $|B_j(p_{i\alpha})| \geq 2^{j/290}$.*

Proof. Let $p = p_{i\alpha}$. Proceed by induction on j. If $j \leq 290$, the claim holds as $|B_j(p)| \geq 2$ if $j \geq 1$ and is at least one if $j = 0$. So suppose $j > 290$. By induction $|B_{j-290}(p)| \geq 2^{j/290}/2$. Yet by Claim 5.4.4, $|N_k(p)| \geq |B_{j-290}(p)|/145$ for all k where $j - 290 < k \leq j$. Hence $|B_j(p)| \geq 2|B_{j-290}(p)|$ and the claim follows. □

Thus $|V(G)| \geq \sum_{i \in B} 2^{\beta/290} \geq 2^{\beta/290}|B|/2$ by Claim 5.4.5. Yet $|V(G)| \leq 143d$ by Claim 5.4.3. Thus $|B| \leq 246d/2^{\beta/290}$. Yet $|B| \geq d/2\alpha$ and hence $\alpha \geq 2^{\beta/290}/492$, a contradiction.

So we may suppose that $|A| \geq d/2\alpha$. Let $A' \subseteq A$ such that for all $i, j \in A$, $|j - i| \geq 4$ and $|A'| \geq |A|/4 \geq d/8\alpha$. Now for all $i, j \in A'$, $C_i \cap C_j = \emptyset$ and $d(C_i, C_j) \geq \alpha$ as $\beta < \alpha$. Thus any choice of L-colorings for the set of cycles $\bigcup_{i \in A'} C_i$ will extend to an L-coloring as long for each cycle the L-coloring extends to distance $O(\log \beta)$. However, by Theorem 1.4.2, there are least two L-colorings for any C_i that extend to distance $O(\log \beta)$. Hence there are at least $2^{|A'|} \geq 2^{d/8\alpha}$ L-colorings of G, which is at least $2^{|V(G)|/1144\alpha} = 2^{\epsilon|V(G)|}$ as $|V(G)| \leq 143d$ by Claim 5.4.3. $\square$

5.5 Steiner Frames

Definition. Let G be a graph and $S \subset V(G)$. We say $T \subseteq G$ is a *Steiner tree* for S if T is a tree with a minimum number of edges such that $S \subset V(T)$. We let T^* denote the tree formed from T by supressing degree two vertices not in S.

We will need a generalization of this for graphs embedded on a surface.

Definition. Let G be a graph 2-cell embedded on a surface Σ and $S \subset V(G)$. We say $H \subseteq G$ is a *frame* of G for S if H is a connected subgraph such that $S \subset V(H)$ and cutting Σ along H leaves a simply connected region. We let H^* denote the graph formed from H by supressing degree two vertices not in S (unless $S = \emptyset$ and H is a cycle in which case we let H^* denote the graph formed by suppressing all but three vertices of H). If $e \in E(H^*)$, we let $\psi(e)$ denote the path in H between the endpoints of e and we let $\text{mid}(e)$ denote a mid-point of that path. We say that the path $\psi(e)$ is a *seam* of the frame H.

We say a frame H is a *Steiner frame* of G for S if it has the minimum number of edges among all frames of G for S.

Note that a Steiner frame, and hence a frame, always exists as it is also the subgraph with the minimum number of edges such that $S \subset V(H)$ and every region formed by cutting Σ open along H is simply connected. This follows because if there existed at least two regions, then there would exist an edge of H adjacent to two distint regions. But then deleting such an edge would join the two simply connected region into one simply connected region, contradicting a minimum number of edges.

Definition. Let G be a graph embedded on a surface Σ. We say that a subgraph H of G is a *slice* if the embedding of H inherited from the embedding of G is plane and there exist a set of at most two faces of the embedding such that all vertices of H adjacent to a vertex not in H are incident with one of the faces in that set. We say that H is a *disc slice* if there exists such a set with at most one face and a *cylinder slice* otherwise. If H is a slice, then the *boundary* of H is the set of vertices of H adjacent to vertices not in H.

Lemma 5.5.1. *Let G be a graph 2-cell embedded on a surface Σ and $S \subset V(G)$. If H is a Steiner frame of G for S and we let $B(e)$ denote $N_{|e|/4-1}(\mathrm{mid}(e))$ for every seam e of H, then*

(1) *for all seams e of H, $B(e)$ is contained in a slice whose boundary is contained in $N_{|e|/4-1}(e)$, and*

(2) *for all distinct seams e, f of H, $B(e) \cap B(f) = \emptyset$.*

Proof.

Claim 5.5.2. *There cannot exist a path from an internal vertex v in a seam e of H to a vertex in $H \setminus e$ that is shorter than mimimum of the length of the paths from v to the endpoints of e.*

Proof. Otherwise, we could add such a path and delete whichever path from v to an endpoint of e that leaves the cut-open simply connected. □

We now prove (1). Let e be a seam of H. It follows from the claim above that $N_{|e|/2-1}(\mathrm{mid}(e)) \cap (H \setminus \psi(e)) = \emptyset$. Hence, the inherited embedding of $B(e)$ from G is plane if the two appearances of e in the boundary walk of the simply connected region have opposite orientations and in the projective plane if they have the same. Yet if they have the same orientation and cannot be embedded in the plane, then there is a path P, with length $|e|/2 - 1$ from the midpoint to itself passing through the simply connected region. We may then add the path P to H and delete the path from the midpoint to the endpoint of e which is longest. The resulting graph still cuts open Σ to a simply connected region as the two appearances of e had the same orientation, as well as spanning the vertices of S. But this contradicts that H had a minimum number of edges.

Thus the inherited embedding of $B(e)$ is plane. It follows that even more is true. The neighborhood $B(e)$ embeds in the plane and this embedding can be extended to an embedding of a plane graph H' such that $(H \setminus B(e)) \cap H' = \emptyset$ and there are is a set of at most two faces of H' such that every neighbor of $G \setminus H'$ in H' is in one of those faces. That is, $B(e)$ is contained in a slice whose boundary is contained in $N_{|e|/4-1}(e)$.

We now prove (2). Let e and f be distinct seams of H. Suppose $B(e) \cap B(f) \neq \emptyset$. Suppose without loss of generality that $|e| \geq |f|$. But now there exists a path of length at most $|e|/4 + |f|/4 - 2 \leq |e|/2 - 2$ between $\mathrm{mid}(e)$ and $\mathrm{mid}(f)$ which is a vertex of $H \setminus e$, contradicting the claim above. $\square$

5.6 Hyperbolic Families of Graphs

Definition. We say a pair (G, H) is a *graph with boundary* if G is a graph and H is a subgraph of G. We say two graphs with boundary (G_1, H_1) and (G_2, H_2) are *isomorphic* if there exists an isomorphism from G_1 to G_2 which is also an isomorphism from H_1 to H_2.

Let (G, H) be a graph with boundary 2-cell embedded in a surface. Let (G_1, G_2) be a separation of G such that $V(G_1) \cap V(G_2) = X$ and $V(H) \subseteq G_2$. Now let G_1' be a graph obtained from G_1 by splitting vertices of X. If the resulting graph with boundary (G_1', X') can be embedded in the plane so that all the vertices of X' lie in a common face, then we say that (G_1', X') is a *disc-excision* of (G, H).

Let (G, H) be a graph with boundary 2-cell embedded in a surface. Let G_1 be a slice of G and X its boundary. If $V(H) \subseteq X \cup (G \setminus G_1)$, then we say that (G_1, X) is a *cylinder-excision* of (G, H).

Let $\mathcal{F}$ be a family of graphs with boundary 2-cell embedded on surfaces. We say that $\mathcal{F}$ is *hyperbolic* if

(1) there exists $c_{\mathcal{F}} > 0$ such that for all disc-excisions (G, H) of a member of $\mathcal{F}$, $|V(G)| \leq c_{\mathcal{F}}|V(H)|$, and

(2) there exists $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all cylinder-excisions (G, H) of a member of $\mathcal{F}$, $|V(G)| \leq f(|V(H)|)$.

We say that $c_{\mathcal{F}}$ is the *disc Cheeger constant* for $\mathcal{F}$.

5.6.1 Logarithmic Distance, Exponential Growth for Disc-Excisions

Lemma 5.6.1 (Logarithmic Distance)**.** *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. If (G, H) is a disc-excision of a member of $\mathcal{F}$, then $d(v, H) \leq 2c_{\mathcal{F}} \log |V(H)|$ for all $v \in V(G)$.*

Proof. We proceed by induction on $|V(G)|$. There must exist a distance i, $1 \leq i \leq 2c_{\mathcal{F}}$, such that either there are at most $|H|/2$ vertices at distance i from H. The corollary then follows by induction on $(G \setminus \{v|d(v, H) < i\}, \{v|d(v, H) = i\}$. □

Corollary 5.6.2. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. If (G, H) is a disc-excision of a member of $\mathcal{F}$, then $|B_k(v)| \geq 2^{\Omega(k)}$ for all $v \in V(G)$ and $k > 0$ such that $B_{k-1}(v) \cap H = \emptyset$.*

Proof. Let $k \leq d(v, H)$. Now $N_k(v)$ separates v from C. By Theorem 3.9.4, $k = d(v, N_k(v)) \leq 2c_{\mathcal{F}} \log |N_k(v)|$. Hence $|N_k(v)| \geq 2^{k/(2c_{\mathcal{F}})}$ as desired. □

5.6.2 Linear Bound, Logarithmic Distance and Exponential Growth for Cylinder-Excisions

Lemma 5.6.3 (Linear Cylinder Bound)**.** *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. Then there exists $c_{\mathcal{F},1}$ such the following holds: If (G, H) is a cylinder-excision of a member of $\mathcal{F}$, then $|V(G)| \leq c_{\mathcal{F},1}|V(H)|$.*

Proof. Thus $H = C \cup C'$ where C, C' are facial cycles. Let f_1 be the face of G whose boundary is C and f_2 be the face of G whose boundary is C'. Take a shortest path P from C to C'. Let f be the face of $G[C \cup C' \cup P]$ such that $f \neq f_1, f_2$. Let C'' be the boundary walk of f. But then (G', C'') is a disc-excision. Let $d(C, C') = d$. If $d \leq O(|C| + |C'|)$, then by $C'' \leq |C| + |C'| + 2d$. By property (i) of hyperbolic families, $|V(G')| \leq O(|C''|) \leq O(|C| + |C'|)$. As $|V(G)| \leq |V(G')|$, the lemma follows.

So we may assume that $d \geq \Omega(|C| + |C'|)$. Yet by property (i) of a hyperbolic family, $|V(G)| \leq O(d)$. Thus there exists k, $1 \leq k \leq d/4$ such that $|N_k(C)| \leq 8c_{\mathcal{F}}$ and similarly there exists k', $3d/4 \leq k' \leq d$ such that $|N_{k'}(C')| \leq 8c_{\mathcal{F}}$. Yet $d(N_k(C), N_{k'}(C')) \geq d/2$. Consider the cylinder excision, $(G'', N_k(C) \cup N_{k'}(C'))$. By property (ii), it follows that $d/2 \leq |V(G'')| \leq f(N_k(C), N_{k'}(C'))$. Hence $|V(G)| \leq O(d)$ and yet $d \leq 2 \max_{1 \leq m,n \leq 8c_{\mathcal{F}}} f(m,n)$. □

We say that $c_{\mathcal{F},1}$ is the *cylinder Cheeger constant* of $\mathcal{F}$.

Corollary 5.6.4. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. If (G, H) is a cylinder-excision of a member of $\mathcal{F}$, then $d(v, H) \leq O(\log |V(H)|)$ for all $v \in V(G)$.*

Proof. We proceed by induction on $|V(G)|$. There must exist a distance i, $1 \leq i \leq 2c_{\mathcal{F},1}$, such that either there are at most $|H|/2$ vertices at distance i from H. The corollary then follows by induction on $(G \setminus \{v | d(v,H) < i\}, \{v | d(v,H) = i\})$. □

Corollary 5.6.5. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. If (G, H) is a cylinder-excision of a member of $\mathcal{F}$, then $|B_k(v)| \geq 2^{\Omega(k)}$ for all $v \in V(G)$ and $k > 0$ such that $B_{k-1}(v) \cap H = \emptyset$.*

Proof. Let $k \leq d(v, H)$. Now $N_k(v)$ separates v from C. By Theorem 3.9.4, $k = d(v, N_k(v)) \leq 2c_{\mathcal{F},1} \log |N_k(v)|$. Hence $|N_k(v)| \geq 2^{k/(2c_{\mathcal{F},1})}$ as desired. □

5.7 General Linear Bound for Hyperbolic Families

Theorem 5.7.1. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. If $(G,H) \in \mathcal{F}$ such that G is 2-cell embedded on a surface Σ, then $|V(G)| = O(|V(H)| + g(\Sigma))$.*

Proof. Let $(G,H) \in \mathcal{F}$. Let T be a Steiner frame of G for H. By cutting open along T, we obtain a graph G' embedded in the disk with boundary C, where $|C|$ has size at most $2|E(T)|$. As (G', C) is a disc-excision of (G,H), by Property (1) of hyperbolic families,

$$|V(G)| \leq c_{\mathcal{F}}|C| = 2c_{\mathcal{F}}|E(H)|$$

. Yet, the number of seams of H is at most $2(g(\Sigma) + |H|)$, as branch points are only necessary to cut open the surface or to span vertices in H.

As T^* was formed by supressing vertices of degree two in T, $|V(T) \setminus V(T^*)| = |E(T) \setminus E(T^*)|$. Thus,

$$|V(G)| \leq c_{\mathcal{F}}(4g(\Sigma) + 4|H| + 2|V(T) \setminus V(T^*)|).$$

Let $\mathcal{E}$ be the set of all seams e of T such that $e \setminus V(T^*) \neq \emptyset$. Hence, for all $e \in \mathcal{E}$, mid(e) exists. For all $e \in \mathcal{E}$, let $B(e) = N_{|e|/4-1}(\text{mid}(e))$. By Lemma 5.5.1 (i), $B(e)$ is contained in a slice whose boundary is contained in $N_{|e|/4-1}(e)$. As $(B(e), N_{|e|/4-1})$ is a cylinder-excision of (G,H), it follows from Lemma 5.6.5 that $|B(e)| \geq 2^{c_{\mathcal{F},1}(|e|/4-1)}$. Hence,

$$|V(G)| \geq \sum_{e \in \mathcal{E}} 2^{c_{\mathcal{F},1}(|e|/4-1)} \geq |\mathcal{E}| 2^{c_{\mathcal{F},1}(\sum_{e \in \mathcal{E}} |e|/4|\mathcal{E}|-1)}$$

where the last inequality follows from the concavity of the exponential function. Yet $|V(H) \setminus V(H^*)| \leq \sum_{e \in \mathcal{E}} |e|$. Combining, we find that

$$|\mathcal{E}| 2^{(c_{\mathcal{F},1}/4)(\sum_{e \in \mathcal{E}} |e|)/|\mathcal{E}|} / 2^{c_{\mathcal{F},1}} \leq |V(G)| \leq |V(G)| \leq 2c_{\mathcal{F}}(2g(\Sigma) + 2|H| + \sum_{e \in \mathcal{E}} |e|).$$

We may suppose that $\sum_{e \in \mathcal{E}} |e| \geq 2(g(\Sigma) + |S|)$ as otherwise $|V(G)| \leq 8c_{\mathcal{F}}(g(\Sigma) + |H|)$ as desired. Hence, $|V(G)| \leq 4c_{\mathcal{F}} \sum_{e \in \mathcal{E}} |e|$. Letting $x = \sum_{e \in \mathcal{E}} / |\mathcal{E}|$, the average size of a seam in $|\mathcal{E}|$, we find that

$$2^{(c_{\mathcal{F},1}/4)x} \le 4c_{\mathcal{F}} 2^{c_{\mathcal{F},1}} x.$$

Let $c' = 4c_{\mathcal{F}}2^{c_{\mathcal{F},1}}$. Thus $x \le \max\{4\log(4c'/c_{\mathcal{F}})/c, 4/c_{\mathcal{F}}\} = \max\{4(c_{\mathcal{F},1}+4)/c_{\mathcal{F}}, 4/c_{\mathcal{F}}\}$, call this constant c_0. Hence,

$$|V(G)| \le 4c_{\mathcal{F}}c_0|\mathcal{E}| \le 8c_{\mathcal{F}}c_0(g(\Sigma) + |H|)$$

as $|\mathcal{E}| \le |E(H^*)|$. The theorem now follows with constant $\max\{8c_{\mathcal{F}}c_0, 8c_{\mathcal{F}}\}$. □

Corollary 5.7.2. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. If $(G, H) \in \mathcal{F}$ such that G is 2-cell embedded on a surface Σ and T is a Steiner frame of G for H, then $|V(T)| = O(g(\Sigma) + |V(H)|)$.*

Proof. See proof of Theorem 5.7.1. □

5.7.1 Finitely Many Members of a Hyperbolic Family on a Fixed Surface

Corollary 5.7.3. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. Let G be a graph embedded on a surface Σ such that $(G, \emptyset) \in \mathcal{F}$, then $|V(G)| \le O(g(\Sigma))$.*

Proof. Now G has 2-cell embedding on a surface Σ' whose genus is at most the genus of Σ. But then the corollary follows from Theorem 5.7.1 with $H = \emptyset$. □

Corollary 5.7.4. *Let $\mathcal{F}$ be a hyperbolic family of graphs with boundary. Let Σ be a surface. There exist only finitely many graphs G embeddable in Σ such that $(G, \emptyset) \in \mathcal{F}$.*

Let $\mathcal{F}$ be a family of graph with boundary. We say that a graph G is $\mathcal{F}$*-free* if there does not exist $G' \subseteq G$ such that $(G', \emptyset) \in \mathcal{F}$. We say that a graph with boundary (G, H) is $\mathcal{F}$-free if there does not exist a graph with boundary $(G', H') \in \mathcal{F}$ such that G' is isomorphic to a subgraph of G and under the same isomorphism $G' \cap H$ is isomorphic to H'.

Corollary 5.7.5. *Let $\mathcal{F}$ be a hyperbolic family of graphs with boundary. Let Σ be a surface. There exists a linear-time algorithm to decide if a graph embeddable in Σ is $\mathcal{F}$-free.*

Proof. Follows from the linear time algorithm of Eppstein for testing subgraph isomorpism on a fixed surface. □

5.7.2 Logarithmic Distance and Edge-Width

Corollary 5.7.6. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. Let $(G, H) \in \mathcal{F}$ such that G is a connected graph 2-cell embedded on a surface Σ. If T is a Steiner frame of G for $V(H)$, then $d(v, T) \leq O(\log(g(\Sigma) + |V(H)|))$ for all $v \in V(G)$.*

Proof. It follows from Theorem 5.6.1 that $d(v, V(T)) \leq O(\log |T|)$ for all $v \in v(G)$. By Corollary 5.7.2, $|T| = O(g(\Sigma) + |H|)$. Yet $d(v, V(T^*) \leq O(\log(g(\Sigma) + |H|)$ for all $v \in V(T)$ as otherwise there would exist $e \in \mathcal{E}$ such that $|e| \geq \Omega(g(\Sigma) + |H|)$ and hence $|V(G)| \geq 2^{c'_{\mathcal{F}}(|e|/4-1)} \geq \Omega(g(\Sigma) + |H|)$, contradicting Theorem 5.7.1. □

Lemma 5.7.7. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. Let $(G, H) \in \mathcal{F}$ such that G is a connected graph 2-cell embedded on a surface Σ. There do not exist $s_1, \ldots, s_k$, where $\sum_{i=1}^{k} s_i \geq \Omega(|V(H)| + g(\Sigma))$, and vertices $v_1, \ldots, v_k$ such that $B_{\leq \log s_i}(v_i)$ are disjoint from each other and from S, and are contained in slices.*

Proof. As $B_{\log s_i}(v_i)$ are contained in slices disjoint from S, $(B_{\log s_i}(v_i), N_{\log s_i}(v_i))$ is a cylinder-excision of (G, H) for all i. By Lemma 5.6.2, $|B_{\log s_i}(v_i)| \geq 2^{\Omega(\log s_i)} \geq \Omega(s_i)$. Hence, $|V(G)| \geq \sum_i \Omega(s_i) \geq \Omega(|H| + g(\Sigma))$, as the neighborhoods are disjoint. But this contradicts Theorem 5.7.1. □

Corollary 5.7.8. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. Let G be a connected graph 2-cell embedded on a surface Σ and $ew(G) \geq \Omega(\log g(\Sigma))$, then G is $\mathcal{F}$-free.*

Proof. Suppose not. Then G has a subgraph G' such that $(G', \emptyset) \in \mathcal{F}$. Yet $ew(G') \geq ew(G) \geq \Omega(\log g)$. Let $v_1 \in V(G)$ and $s_1 = \Omega(\log g(\Sigma))$. As $B_{\Omega(\log g(\Sigma))}(v)$ is locally planar this contradicts Lemma 5.7.7. □

Corollary 5.7.9. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. Let (G, H) be a graph with boundary such that G is 2-cell embedded in a surface Σ, $ew(G) \geq \Omega(\log g)$ and $d(u, v) \geq \Omega(\log g(\Sigma))$ for all $u \neq v \in V(H)$, then (G, H) is $\mathcal{F}$-free.*

Proof. Suppose not. Then there exists $H' \subseteq H$ and $G' \subseteq G$ such that $(G', H') \in \mathcal{F}$. As G' is connected?, it follows that $|G'| \geq \Omega(\log g(\Sigma) + |H'|)$ if G' is non-plane and $|G'| \geq \Omega(|H'|)$ if G is plane. In either case, this contradicts Theorem 5.7.1. □

Corollary 5.7.10. *Let $\mathcal{F}$ a hyperbolic family of graphs with boundary. Let (G, H) be a graph with boundary such that G is 2-cell embedded in a surface Σ, $ew(G) \geq \Omega(\log g)$. Further suppose that H is a collection $\mathcal{C} = \{C_1, C_2, \ldots\}$ of disjoint cycles of G such that $d(C_i, C_j) \geq \Omega(\log(|C_i| + |C_j| + g(\Sigma)))$ for all $C_i \neq C_j \in \mathcal{C}$ and $G_i = B_{\Omega(\log(|C_i|+g(\Sigma)))}(C_i)$ is plane for all $C_i \in \mathcal{C}$. If (G, H) is not $\mathcal{F}$-free, then there exists i such that (G_i, C_i) is not $\mathcal{F}$-free.*

5.8 Applications to 5-List-Coloring

Theorem 5.8.1. *The family of all 6-list-critical graphs is hyperbolic.*

Proof. By Theorem 3.4.26, property (i) holds. By Theorem 5.2.9, property (ii) holds. □

Hence we may apply the theorems of the previous section when $\mathcal{F}$ is the family of all 6-list-critical graphs. Note that by Theorem 1.4.4, the family of all k-list-critical graphs is hyperbolic for $k \geq 7$ and hence the theory may also be applied to those families as well.

Here is Theorem 5.7.1 restated for 5-list-coloring.

Theorem 5.8.2. *Let G be a connected graph 2-cell embedded on a surface Σ and $S \subseteq V(G)$. If G is H-critical where H is the disjoint union of the vertices in S, then $|V(G)| = O(|S| + g(\Sigma))$.*

Here is Theorem 5.7.2 restated.

Theorem 5.8.3. *Let G be a connected graph 2-cell embedded on a surface Σ and $S \subseteq V(G)$. If G is H-critical where H is the disjoint union of the vertices in S and T is a Steiner frame of G for S, then $|V(T)| = O(g(\Sigma) + |S|)$.*

5.8.1 Finitely Many 6-List-Critical Graphs on a Fixed Surface

Here is Corollary 5.7.3 restated.

Theorem 5.8.4. *Let G be a 6-list-critical graph embedded on a surface Σ, then $|V(G)| \leq O(g(\Sigma))$.*

Moreover, Corollary 5.7.3 is best possible up to the multiplicative constant. To see this apply Hajos' construction (for reference, see pp. 117-118 in [16]) to $g(\Sigma)$ copies of K_6. By genus additivity (see [11]), the resulting graph G has genus $g(\Sigma)$ and yet $|V(G)| \geq 5g(\Sigma)$. Next we restate Corollary 5.7.4, though we also note that this is best possible.

Theorem 5.8.5. *Let Σ be a surface. There exist only finitely many 6-list-critical graphs embeddable in Σ.*

Note that this implies an algorithm as in Corollary 5.7.5.

Theorem 5.8.6. *There exists a linear-time algorithm to decide 5-list-colorability on a fixed surface.*

5.8.2 Extending Precolorings: Albertson's Conjecture on Surfaces

Here is Corollary 5.7.6 restated.

Theorem 5.8.7. *Let G be a connected graph 2-cell embedded on a surface Σ and $S \subseteq V(G)$. If G is H-critical where H is the disjoint union of the vertices in S and T is a Steiner frame of G for S, then $d(v, V(T^*)) \leq O(\log(g(\Sigma)+|S|))$ for all $v \in V(G)$.*

Next we restate Corollary 5.7.9 when $\Sigma = S_0$, which is just Conjecture 1.5.4.

Theorem 5.8.8. *There exists D such that the following holds: If G is a plane graph, $X \subset V(G)$ such that $d(u,v) > D$ for all $u \neq v \in X$ and L is a 5-list assignment for the vertices of G, then any L-coloring of X extends to an L-coloring of G.*

Proof. Suppose not. Then there exists $X' \subset X$ such that G has an a connected X'-critical subgraph G'. Now G' is a connected plane graph and yet $|V(G)| \geq |X'|(D/2)$ as the vertices in X are pariwise distance D apart. Yet by Theorem 5.7.1, $|V(G)| \leq O(|X'|)$, a contradiction if D is large enough. $\square$

Next we restate Corollary 5.7.8 which improves the bound in Theorem 1.4.6 from $2^{\Omega(g(\Sigma))}$ to $\Omega(\log g(\Sigma))$.

Theorem 5.8.9. *If G is 2-cell embedded in a surface Σ and $ew(G) \geq \Omega(\log g(\Sigma))$, then G is 5-list-colorable.*

This is best possible given the existence of Ramunjan graphs (see [40]), which have girth k, $2^{\Theta(k)}$ vertices and large fixed chromatic number and hence chromatic number at least six. But the genus of any graph is at most $|V(G)|^2$. Hence for every g, there exist graphs with girth $\Theta(\log g)$ which embed on a surface of genus g and have chromatic number - and hence list-chromatic number - at least six.

Here is Corollary 5.7.9 restated.

Theorem 5.8.10. *Let G be 2-cell embedded in a surface Σ, $ew(G) \geq \Omega(\log g)$ and L be a 5-list-assignment for G. If $X \subset V(G)$ such that $d(u,v) \geq \Omega(\log g(\Sigma))$ for all $u \neq v \in X$, then any L-coloring of X extends to an L-coloring of G.*

Now we restate Corollary 5.7.10 restated. However we strengthen to the case not only when every coloring of one of the cycles extends locally, but to when a particular coloring of the cycles extends locally.

Theorem 5.8.11. *Let G be 2-cell embedded in a surface Σ, $ew(G) \geq \Omega(\log g)$ and L be a 5-list-assignment for G. Let $\mathcal{C} = \{C_1, C_2, \ldots\}$ be a collection of disjoint cycles of G such that $d(C_i, C_j) \geq \Omega(\log(|C_i| + |C_j| + g(\Sigma)))$ for all $C_i \neq C_j \in \mathcal{C}$. Let $d_i = B_{\Omega(\log(|C_i|+g(\Sigma)))}(C_i)$ and suppose further $G_i = B_{d_i}(C_i)$ is contained in a slice whose boundary is contained in N_{d_i} for all $C_i \in \mathcal{C}$. If ϕ is an L-coloring of the cycles in $\mathcal{C}$ such that $\phi \upharpoonright C_i$ can be extended to an L-coloring of $B_{d_i}(C_i)$ for all $C_i \in \mathcal{C}$, then ϕ extends to an L-coloring of G.*

Proof. Suppose not. By Lemma 3.4.27, there exists $G_i' \subseteq G_i$ such that (G_i', C_i, L) is a critical easel for (G_i, C_i, L), that is, for every face $f \in \mathcal{F}(G_i')$, every L-coloring of the boundary walk of f extends to an L-coloring of the interior of f. By Theorem 3.6.8, $d(v, C_i) \leq 58 \log |V(C_i)|$. Now extend ϕ to a coloring of $\bigcup_i G_i$. Let C_i' be the boundary of the slice containing G_i. By Corollary 5.7.10 applied to $G' = G \setminus (\bigcup_i G_i \setminus C_i')$ with $\mathcal{C} = \{C_1', C_2', \ldots\}$ we find that ϕ can be extended to an L-coloring of G. □

As a corollary, we can derive a generalization of Theorem 1.6.1 to other surfaces while also providing an independent proof of said theorem.

Theorem 5.8.12. *Let G be drawn in a surface Σ with a set of crossings X and L be a 5-list-assignment for G. Let G_X be the graph obtained by adding a vertex v_x at every crossing $x \in X$. If $ew(G_X) \geq \Omega(\log g(\Sigma))$ and $d(v_x, v_{x'}) \geq \Omega(\log g(\Sigma))$ for all $v_x \neq v_{x'} in V(G_X) \setminus V(G)$, then G is L-colorable.*

Proof. Let G' be obtained from G_X by deleting the vertices at the crossings and adding edges if necessary such that the neighbors of v_x form a 4-cycle C_x for every $x \in X$. Now $ew(G') \geq \Omega(\log g(\Sigma))$. Note that $N_{\Omega(\log g(\Sigma))}(C_x)$ is plane and that

$d(C_x, C_{x'}) \geq \Omega(\log g(\Sigma))$ by assumption. Let $\mathcal{C} = \bigcup x \in X C_x$ and ϕ be an L-coloring of the cycles in $\mathcal{C}$ such that $\phi \restriction C_x$ is an L-coloring of $G[C_x]$ for every $x \in X$. By Theorem 1.5.2, $\phi \restriction C_x$ extends to an L-coloring of $N_{\Omega(\log g(\Sigma))}(C_x)$ for all $x \in X$. By Corollary 5.8.11, ϕ extends to an L-coloring of G. Thus G is L-colorable as desired.

□

5.9 Applications to Exponentially Many 5-List-Colorings

Definition. Let $\epsilon, \alpha > 0$. Let (G, H) be a graph with boundary embedded in a surface Σ. Suppose there exists a 5-list-assignment L of G and an L-coloring ϕ of H such that there does not exist $2^{\epsilon(|V(G)\setminus V(H)| - \alpha(g(\Sigma)+|H|))}$ distinct L-colorings of G extending ϕ but for every proper subgraph $G' \subseteq G$ such that $H \subseteq G'$ there do exist $2^{\epsilon(|V(G')\setminus V(H)| - \alpha(g(\Sigma)+|H|))}$ distinct L-colorings of G' extending ϕ. Then we say that G is (ϵ, α)-*exponentially-critical.*

Let $F_{\epsilon,\alpha}$ be the family of all (ϵ, α)-exponentially-critical graphs with boundary.

Theorem 5.9.1. *Suppose* $0 < \epsilon < 1/18$*,* $\alpha \geq 0$*. If* (G', C) *is a disc-excision of a graph with boundary* $(G, H) \in \mathcal{F}_\epsilon$*, then* $|V(G')| \leq 87|V(C)|$*.*

Proof. Suppose to a contradiction that $|V(G')| > 59|V(C')|$. Let L be a 5-list-assignment for G and ϕ an L-coloring of H as in the definition of (ϵ, α)-exponentially-critical. Let $T = (G', C, L)$. By Theorem 3.4.27, there exists a critical easel $T' = (G_0, C, L)$ for T such that $|V(G_0)| \leq 29|V(C)|$. Thus G_0 is a proper subgraph of G'.

Let $G_0' = G \setminus (G' \setminus G_0)$. Thus G_0' is a proper subgraph of G. As G is (ϵ, α)-exponentially-critical, there exist a set $\mathcal{C}$ of distinct L-colorings of G_0' extending ϕ such that $|\mathcal{C}| = 2^{\epsilon(|V(G_0'\setminus H)| - \alpha(g(\Sigma)+|H|))}$. Let $\phi' \in \mathcal{C}$. Let $f \in \mathcal{F}(G_0)$. Let $T_f = (G_f, C_f, L)$ be the canvas in the closed disk bounded by f. As T' is an easel, ϕ extends to $2^{((|V(G_f\setminus C_f)|)-29(|C_f|-3))/9}$ distinct L-colorings of G_f by Theorem 3.5.3.

Let $E(\phi')$ be the number of extensions of ϕ' to G. Thus $\log E(\phi) \geq \sum_{f\in\mathcal{F}(G_0)}(|V(G_f\setminus C_f)| - 29(|C_f|-3))/9$. As $\sum_{f\in\mathcal{F}(G_0)} |V(G_f \setminus C_f)| = |V(G \setminus G_0')|$ and $\sum_{f\in\mathcal{F}(G_0)}(|C_f| -$

$3) = |C| - 3$, we find that $\log E(\phi) \geq (|V(G \backslash G_0')| - 29(|C|-3))/9$. As $|V(G \backslash G_0')|/2 \geq 29|V(C)|$, we find that $\log E(\phi) \geq |V(G \setminus G_0')|/18$. But then as $\epsilon \leq 1/18$, there exist at least $2^{\epsilon(|V(G \backslash H)| - \alpha(g(\Sigma)+|H|)}$ distinct L-colorings of G extending ϕ, a contradiction as (G, H) is (ϵ, α)-exponentially-critical for L and ϕ. □

Theorem 5.9.2. *Let $\epsilon \leq \epsilon'/2$ where ϵ' as in Theorem 5.4.1 and $\alpha \geq 0$. If $(G', C_1 \cup C_2)$ is a cylinder-excision of a graph with boundary $(G, H) \in \mathcal{F}_{\epsilon,\alpha}$, then $|V(G')| = c(|V(C_1)| + |V(C_2)|)$ for some constant $c > 0$ not depending on α or ϵ.*

Proof. Suppose to a contradiction that $|V(G')| \geq \Omega(|V(C_1)| + |V(C_2)|)$. Let L be a 5-list-assignment for G and ϕ an L-coloring of H as in the definition of (ϵ, α)-exponentially-critical. Consider the cylinder-canvas $T = (G', C_1 \cup C_2, L)$. By Theorem 5.3.1, there exists a critical easel $T' = (G_0, C_1 \cup C_2, L)$ for T such that $|V(G_0)| \leq O(|V(C_1)| + |V(C_2)|)$. Hence we may assume that G_0 is a proper subgraph of G' and $|V(G' \setminus G_0)| \geq |V(G')|/2$.

Let $G_0' = G \setminus (G' \setminus G_0)$. Thus G_0' is a proper subgraph of G. As G is (ϵ, α)-exponentially-critical, there exist a set $\mathcal{C}$ of distinct L-colorings of G_0' extending ϕ such that $|\mathcal{C}| = 2^{\epsilon(|V(G_0')| - \alpha(g(\Sigma)+|H|)}$. Let $\phi' \in \mathcal{C}$. Let $f \in \mathcal{F}(G_0)$. Let $T_f = (G_f, C_f, L)$ be the canvas in the closed disk or cylinder bounded by f. As T' is an easel, ϕ extends to $2^{\epsilon'((|V(G_f \backslash C_f)|) - 50|C_f|)}$ distinct L-colorings of G_f by Theorem 3.5.3 if C_f is a disc and by Theorem 5.4.1 where ϵ' is as in Theorem 5.4.1.

Let $E(\phi')$ be the number of extensions of ϕ' to G. Thus $\log E(\phi) \geq \sum_{f \in \mathcal{F}(G_0)} \epsilon'(|V(G_f \setminus C_f)| - 50|C_f|)$. Note that $\sum_{f \in \mathcal{F}(G_0)} |V(G_f \setminus C_f)| = |V(G \setminus G_0')|$. Further note that as G_0 is planar and $|V(G_0)| = O(|C_1| + |C_2|)$ that $\sum_{f \in \mathcal{F}(G_0)} |C_f| = O(|C_1) + |C_2|)$.

Hence we find that $\log E(\phi) \geq \epsilon'(|V(G \setminus G_0')| - \alpha'(|C_1| + |C_2|))$ for some constant α'. As $|V(G \setminus G_0')| \geq |V(G')|/2 \geq \Omega(|C_1| + |C_2|)$, we find that $\alpha'(|C_1| + |C_2|) \leq |V(G \setminus G_0')|/2$. Hence $\log E(\phi) \geq \epsilon' |V(G \setminus G_0')|/2$. But then as $\epsilon \leq \epsilon'/2$, there exist at least $2^{\epsilon(|V(G \backslash H)| - \alpha(g(\Sigma)+|H|)}$ distinct L-colorings of G extending ϕ, a contradiction as (G, H) is (ϵ, α)-exponentially-critical for L and ϕ. □

Theorem 5.9.3. *There exists $\delta > 0$ such that following holds: For all $\epsilon > 0$ with $\epsilon \leq \delta$ and $\alpha \geq 0$, $F_{\epsilon,\alpha}$ is a hyperbolic family. Moreover, the disc Cheeger constant and cylinder Cheeger constants for $\mathcal{F}_{\alpha,\epsilon}$ do not depend on α or ϵ.*

Proof. Follows from Theorems 5.9.1 and 5.9.2. □

Corollary 5.9.4. *Let $\epsilon \leq \delta$ where δ as in Theorem 5.9.3 and $\alpha \geq 0$. If $(G, H) \in \mathcal{F}_{\epsilon,\alpha}$ is a graph embedded on a surface Σ, then $|V(G)| \leq c(g(\Sigma)+|V(H)|)$ for some constant $c > 0$ not depending on α or ϵ.*

Proof. Follows from Theorem 5.7.1 with $\mathcal{F} = \mathcal{F}_{\epsilon,\alpha}$. Moreover as c only depends on the disc and cylinder Cheegers constants for $\mathcal{F}_{\epsilon,\alpha}$ and these do not depend on α or ϵ, it follows that c does not depend on α or ϵ. □

Theorem 5.9.5. *Let δ, c be as in Corollary 5.9.4. Let G be a graph embedded in a surface Σ, $X \subseteq V(G)$ and L a 5-list-assignment for G. If ϕ is an L-coloring of $G[X]$ such that ϕ extends to an L-coloring of G, then ϕ extends to at least $2^{\delta(|V(G)|-c(g(\Sigma)+|X|))}$ distinct L-colorings of G.*

Proof. Suppose not. Thus there do not exist $2^{\delta(|V(G)|-c(g(\Sigma)+|X|)}$ distinct L-colorings of G extending ϕ. So there exists a subgraph G' of G with $H \subseteq G'$ such that (G', H) is (δ, c)-exponentially-critical. Hence $(G', H) \in \mathcal{F}_{\delta,c}$. By Corollary 5.9.4, $|V(G')| \leq c(g(\Sigma) + |X|)$. Yet as ϕ extends to an L-coloring of G, ϕ also extends to an L-coloring of G'. But then $\delta(|V(G)| - c(g(\Sigma) + |X|)) \leq 0$. So ϕ extends to at least $2^{\delta(|V(G)|-c(g(\Sigma)+|X|))}$ L-colorings, a contradiction as G' is (δ, c)-exponentially-critical. □

Note that δ and c are constants not depending on $g(\Sigma)$ or $|X|$.

5.10 Conclusion

We have developed new techniques for proving 5-list-coloring results for graphs on surfaces. Let Σ be a surface, g the Euler genus of Σ, G a graph embedded in Σ and L a 5-list-assignment for G. Our main results are:

(1) There exists only finitely many 6-list-critical graphs on a surface Σ.

(2) There exists a linear-time algorithm for deciding 5-list-colorability on Σ.

(3) If $X \subseteq V(G)$, then there exist a subgraph H of G such that $X \subseteq H$, $|V(H)| = O(|X| + g)$ and for every L-coloring ϕ of X, ϕ either extends to an L-coloring of G or does not extend to an L-coloring of H.

(4) If $ew(G) \geq \Omega(\log g)$, then G is 5-list-colorable.

(5) If $ew(G) \geq \Omega(\log g)$ and $X \subseteq V(G)$ such that $d(u, v) \geq \Omega(\log g)$ for all $u \neq v \in X$, then every L-coloring of X extends to an L-coloring of G.

(6) If G' is a graph drawn in Σ with crossings $\Omega(\log(g))$ pairwise far apart and $ew(G) \geq \Omega(\log g)$, then G' is 5-list-colorable.

(7) If G is L-colorable, then G has $2^{\Omega(|V(G)|-O(g)}$ distinct L-colorings.

(8) If $X \subseteq V(G)$ and ϕ is an L-coloring of G that extends to an L-coloring of G, then there exist $2^{\Omega(|V(G)|)-O(g+|X|)}$ distinct L-colorings of G that extend ϕ.

Moreover, in Chapter 5 we developed the general theory of hyperbolic families of graphs. That is, families whose associated graphs with boundary in the disc satisfy a linear isoperimetric inequality and whose associated graphs in the cylinder satisfy some isoperimetric inequality. We applied this theory to 5-list-coloring and for finding exponentially many 5-list-colorings. This theory however has applications to other problems.

Other examples of hyperbolic families include the family of k-list-critical graphs for $k \geq 7$ and 4-critical graphs of girth at least 5. Of special interest is 3-coloring and 3-list-coloring graphs of girth 5. A linear isoperimetric inequality has been proved for the disc and cylinder for 4-critical graphs of girth 5 by Dvorak, Kral and Thomas [21, 22, 23, 24, 25] and hence the general theory applies there as well. Meanwhile, Dvorak and Kawarabayashi [20] have proved a linear isoperimetric inequality for the disc for 4-list-critical graphs of girth at least 5.

Consequently, an important open problem is proving whether there exists any isoperimetric inequality for the cylinder for 4-list-critical graphs of girth 5. By the general theory, a number of interesting theorems would follow, such as a generalization of Dvorak's result [19] that planar graphs with $\leq$ 4-cycles pairwise far part are 3-choosable.

Notice that all the above proposed applications are examples of list homomorphisms of graphs. It now becomes an interesting research area to decide for which list homomorphism problems, the corresponding critical graphs form a hyperbolic family. Another interesting research area in this regard is the development of algorithms for hyperbolic families. For example, finding an explicit linear-time algorithm whether a graph embedded in a surface is $\mathcal{F}$-free where $\mathcal{F}$ is a hyperbolic family.

As for 5-list-coloring, open problems remain. The most interesting seems to be proving that if G is a graph with a collection of facial cycles $C_1, C_2, \ldots$ pairwise far apart, and L is a list-assignment for $V(G)$ such that $|L(v)| \geq 3$ for all $v \in V(G)$ and $|L(v)| \geq 5$ for all $v \notin \bigcup_i V(C_i)$, then G has an L-coloring. More general bottleneck theorems would also be of interest, as well as more explicit descriptions of the structure of critical cycle-canvases and path-canvases. Another open problem that remains is whether the dependence of the distance in (4) on g can be removed.

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VITA

Luke Jamison Postle was born on July 17, 1987 in Hartford, CT, the youngest of four children. He was valedictorian of the Master's School, a private Christian high school, that he graduated from at the age of 15. He attended Gordon College in Wenham, Massachusetts, where he graduated Summa Cum Laude with a triple major in Mathematics, Physics and History. While pursuing his undergraduate degree, he attended the Budapest Semester in Mathematics and participated in summer Research Experience for Undergraduates at North Carolina State University and Rutgers University.

He began his Ph.D. program at Georgia Tech in May 2007 as a research assistant to Robin Thomas, a position he held until 2009 when he was awarded a three year National Science Foundation Graduate Research Fellowship. He was awarded a Georgia Tech Institute Fellowship from 2007-2011. In 2010 he was recognized by the Georgia Tech Mathematics Department with their Top Graduate Student award. He has presented research papers at numerous conferences in the U.S., Canada and Europe.

From October to December of 2010, he was a visiting researcher at the University of Hamburg in Germany, during which time he collaborated with Reinhard Diestel on expanding the burgeoning field of infinite matroids. From April to June 2011, he was a visiting researcher at Charles University in Prague, Czech Republic, where he conducted research on graph coloring with Zdenek Dvorak.

Luke plans to complete the requirements for his Ph.D. in Algorithms, Combinatorics, and Optimization at the Georgia Institute of Technology in Atlanta, Georgia in the summer of 2012. In the fall of 2012, Luke will begin a post-doctoral position at Emory University in Atlanta, Georgia.