Extremal Functions for Contractions of Graphs

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Extremal Functions for Contractions of Graphs

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For my mother,

in loving memory

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SUMMARY

In this thesis we investigate the extremal functions for complete minors (or a complete minor with one edge removed). The main results are:

(A) Every graph on $n \geq 9$ vertices and $7n - 27$ edges has a K_9 minor, or is isomorphic to $K_{2,2,2,3,3}$, or is isomorphic to a graph obtained from disjoint copies of $K_{1,2,2,2,2,2}$ by identifying cliques of size six.

(B) Every graph on $n \geq 8$ vertices and $(11n - 35)/2$ edges has a K_8^- minor, or is isomorphic to a graph obtained from disjoint copies of $K_{1,2,2,2,2}$ and/or K_7 by identifying cliques of size five.

(C) Every edge-maximal graph on $n \geq 8$ vertices without a $K_7 \cup K_1$ minor (here $K_7 \cup K_1$ stands for a disjoint union of K_7 and K_1) has at most $5n - 15$ edges, or at most 31 edges when $n = 9$, or is isomorphic to one of the following graphs: $K_4 + C_{n-4}$ when $n \neq 9$, $K_{1,2,2,2,2}$, $K_{1,3,3,3}$, $K_{2,2,2,4}$, $K_{2,3} + K_{2,3}^-$, $K_{2,2} + K_{3,3}^-$, $K_{2,3} + C_5$, $K_{3,3} + P_4$, and $K_{2,2,3,3}$.

Note that Theorem (A) extends Mader's results on the extremal function for K_p minors, where $p \leq 7$, and Jørgensen's result on K_8 minors. In Chapter 6, we also study the minimal counterexamples to the analogue of Mader's bound for K_{10} minors. However, since the proof of one of our lemmas in Theorem (A) is computer-assisted, extending our methods to K_{10} minors cannot be done without a lot of programming effort.

Theorem (B) settles a conjecture of Jakobsen from 1983 and Theorem (C) generalizes Mader's result on K_7 minors, stating that every graph on $n \geq 7$ vertices and $5n - 14$ edges has a K_7 minor; and extends Jøgensen's result on the extremal function for $K_6 \cup K_1$ minors. The proofs of Theorem (B) and Theorem (C) are computer-free.

Finally, we prove a weak bound on the extremal function for K_{10} and K_{11} minors.

CHAPTER I

INTRODUCTION

In this chapter, we introduce the basics in graph theory and some useful results that will be used in this dissertation. We will mention the background and motivation for our work.

1.1 The Basics

In this section we present the terminology and notation that will be used in this dissertation.

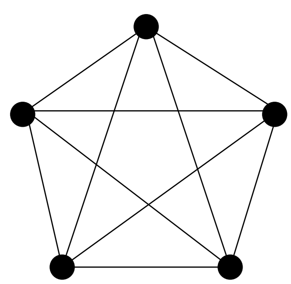
A *graph* is a pair $G = (V, E)$, where E is a set of 2-element subsets of V. The elements of V are the *vertices* of the graph G and the elements of E are called *edges*. The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by an arc if the corresponding two vertices form an edge. Just how these dots and lines are drawn is considered irrelevant: all that matters is the information which pairs of vertices form an edge and which do not.

The *cardinality* of a set S is denoted by $|S|$. For a graph G, the *order* of G is the cardinality of V and the *size* of G is the cardinality of E. We use $|G|$ and $e(G)$ to denote the order and size of G, respectively. Graphs are finite or infinite according to their order. Unless stated otherwise, all graphs in this thesis are finite.

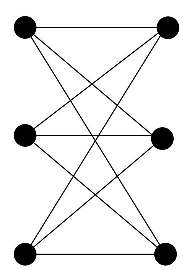
Two vertices u and v of G are said to be *adjacent* if $\{u, v\}$ forms an edge, where an edge $\{u, v\}$ is usually written as uv or vu. If $e = uv \in E(G)$, then we say that u and v are *neighbors*; we also say that u, v are the *ends* of e; and e is *incident* with u and v. Two edges e and e' of G are said to be *adjacent* if they have a common end. Pairwise non-adjacent edges are called *independent*. A set of independent edges is called a *matching* of a graph G.

If $x \in X$, $y \in Y$, then xy is an X-Y *edge*. For a graph G, $X, Y \subset V(G)$ and $x, y \in V(G)$, we will use $e_G(X, Y)$ to denote the number of X-Y edges in G and $G + xy$ to denote the graph obtained from G by adding an edge with ends x and y. If x, y are adjacent or equal, then we define $G + xy$ to be G. If $xy \in E(G)$, we denote by $G - xy$ the graph obtained from G by deleting the edge xy. The *join* $G + H$ (resp. *union* $G \cup H$) of two vertex disjoint graphs G and H is the graph having vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{xy \,|\, x \in V(G), y \in V(H)\}$ (resp. $E(G) \cup E(H)$).

For a graph $G = (V, E)$, if U is a subset of V, then $G[U]$ denotes the graph on U whose edges are precisely the edges of G with both ends in U, and $G[U]$ is called the graph *induced* by U in G. If U is any set of vertices, we write $G - U$ for $G[V - U]$; in other words, $G - U$ is obtained from G by deleting all the vertices in $U \cap V$ and their incident edges. If $U = \{u\}$ is a singleton, we write $G - u$ rather than $G - \{u\}$. Instead of $G - V(H)$, we simply write $G - H$. We call G *edge-maximal* with a given graph property if G itself has the property but no graph $G + xy$ does, for non-adjacent vertices $x, y \in V(G)$.



(a) K_5



(b) $K_{3,3}$

Figure 1: Complete and complete bipartite graphs.

For a graph G, if all the vertices of G are pairwise adjacent, then G is *complete*. A complete graph on n vertices is denoted by K_n. Let $r \geq 2$ be an integer. A graph $G = (V, E)$ is called *complete r-partite* if V can be partitioned into $V_1, V_2, \ldots, V_r$ such that for any $x \in V_i$ and $y \in V_j$, x and y are adjacent if and only if $i \neq j$. If $|V_i| = n_i$ for $i = 1, 2, \ldots, r$, then G is denoted by $K_{n_1, n_2, \ldots, n_r}$. Figure 1 shows examples of K_5 and $K_{3,3}$.

Let $G = (V, E)$ be a graph. The set of neighbors of a vertex v in G is denoted by $N_G(v)$ or briefly by $N(v)$. The *degree* of a vertex v is the number of edges incident with v and is denoted by $d(v)$ or $d_G(v)$. For a graph G we use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degrees of G, respectively. If all the vertices of G have the same degree k, then G is *k-regular*. A 3-regular graph is called *cubic*. If G is a graph and K is a subgraph of G, then by $N(K)$ we denote the set of vertices of $V(G) - V(K)$ that are adjacent to a vertex of K. If $V(K) = \{x\}$, then we use $N(x)$ to denote $N(K)$. By abusing notation we will also denote by $N(x)$ the graph induced by the set $N(x)$. We define $N[x] = N(x) \cup \{x\}$, and similarly will use the same symbol for the graph induced by that set.

A *path* is a non-empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\}$, where the x_i's are all distinct, and $x_1, x_2, \ldots, x_{k-1}$ are the inner vertices of P. The path P is also called an (x_0, x_k)-path. The integer k is called the *length* of P. We often refer to a path by the natural sequence of its vertices, say $P = x_0x_1 \ldots x_k$. If $P = x_0x_1 \ldots x_k$ is a path and $k \geq 3$, then the graph $P + x_0x_k$ is called a *cycle*. A cycle of length k is denoted by C_k.

A non-empty graph G is *connected* if there is an (x, y)-path in G for any two vertices $x, y \in V(G)$. A maximal connected subgraph of G is called a *component* of G. A graph G is called *k-connected* if $|G| > k$ and $G - X$ is connected for any $X \subset V(G)$ of size less than k.

Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We call G and G' *isomorphic* if there exists a bijection $\varphi : V \to V'$ with $xy \in E$ if and only $\varphi(x)\varphi(y) \in E'$ for all $x, y \in V$.

For a graph G, the *complement* $\overline{G}$ of a graph G has the same vertex set as G, and distinct vertices u, v are adjacent in $\overline{G}$ just when they are not adjacent in G. For an edge-transitive graph G, we denote by G^- the graph obtained from G by deleting an edge.

1.2 Background

1.2.1 Definition of Minor

In this section, we introduce the concept of a minor of a graph.

Let xy be an edge of a graph $G = (V, E)$. We denote by G/xy the graph obtained from G by *contracting* the edge xy and deleting all resulting parallel edges, where contraction of the edge xy is performed by replacing the two vertices x and y with a single new vertex, adjacent to each neighbor of x or y. We write $G > H$ if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges. In those circumstances we say that G has a minor isomorphic to H or an H *minor.*

Thus if H is a minor of a graph G, then G has a subgraph G' such that $V(G')$ can be partitioned into $|H|$ disjoint subsets indexed by the vertices of H, say $\{V_x \,|\, x \in V(H)\}$, such that $G[V_x]$ is connected for any $x \in V(H)$ and there is a V_x-V_y edge if and only if $xy \in E(H)$. The sets V_x are the *branch sets* of the minor H. Intuitively, we obtained H from G by contracting every branch set into a single vertex and deleted any parallel edges or loops that may arise.

A graph is *planar* if it can be drawn in the plane without edge crossings. Perhaps the most widely known result on graph minors is Wagner's reformulation [37] of Kuratowski's Theorem [22], the following (See Figure 1 for pictures of K_5 and $K_{3,3}$).

Theorem 1.2.1 *A graph is planar if and only if it has no K_5 or $K_{3,3}$ minor.*

This theorem will be applied in the proof of Theorem 1.3.2.

1.2.2 Mader's Result

In this dissertation, we study how global assumptions about a graph (namely, the number of edges) can force it to have a given graph H as a minor.

The graph $K_{p-2} + \overline{K_{n-p+2}}$ has n vertices and $(p-2)n - \binom{p-1}{2}$ edges and it contains no K_p minor. Mader [25] proved that for $p \leq 7$ this example is best possible, as follows.

Theorem 1.2.2 *For every integer $p = 1, 2, \ldots, 7$, a graph on $n \geq p$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

For $p \leq 5$, this was first proved by Dirac [9]. Some years later but independently of Mader, Györi [10] proved Theorem 1.2.2 for $p \leq 6$.

Mader pointed out that Theorem 1.2.2 does not hold for $p = 8$: the graph $K_{2,2,2,2,2}$ is a counterexample. However, one can construct further counterexamples by repeatedly identifying cliques of size five. That motivates the next definition.

1.2.3 Definition of (H_1, H_2, k)-cockade

For graphs H_1, H_2 and an integer k, we define an $(H_1,\ H_2,\ k)$-*cockade* recursively as follows. Any graph isomorphic to H_1 or H_2 is an (H_1, H_2, k)-cockade. Now let G_1, G_2 be $(H_1,\ H_2,\ k)$-cockades and let G be obtained from the disjoint union of G_1 and G_2 by identifying a clique of size k in G_1 with a clique of the same size in G_2. Then the graph G is also an (H_1, H_2, k)-cockade, and every $(H_1,\ H_2,\ k)$-cockade can be constructed this way. If $H_1 = H_2 = H$, then it is called $(H,\ k)$-cockade.

1.2.4 Jørgensen's Result

Jørgensen [16] generalized Theorem 1.2.2 as follows.

Theorem 1.2.3 *Every simple graph on $n \geq 8$ vertices and at least $6n - 20$ edges either has a K_8 minor, or is a $(K_{2,2,2,2,2},\ 5)$-cockade.*

To see that Theorem 1.2.3 implies Theorem 1.2.2, let G and p be as in Theorem 1.2.2, and apply Theorem 1.2.3 to the graph obtained from G by adding $8 - p$ vertices, each adjacent to every other vertex of the graph.

1.3 Main Results

1.3.1 K_9 minors

One of our main results is the following next step (see also [31]). We prove that every edge-maximal graph without a K_9 minor has at most $7n-28$ edges or is a $(K_{1,2,2,2,2,2},\,6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$. Note that every $(K_{2,2,2,3,3},\,6)$-cockade is isomorphic to $K_{2,2,2,3,3}$.

Theorem 1.3.1 *Every simple graph on $n \geq 9$ vertices and at least $7n-27$ edges either has a K_9 minor, or is a $(K_{1,2,2,2,2,2},\,6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$*

We prove Theorem 1.3.1 in Chapter 5. Note that Theorem 1.3.1 extends Mader's results on the extremal function for K_p minors, where $p \leq 7$, and Jørgensen's result on K_8 minors. In Chapter 6, we also study the minimal counterexamples to the analogue of Mader's bound for K_{10} minors. However, since the proof of one of our lemmas in Theorem 1.3.1 is computer-assisted, extending our methods to K_{10} minors cannot be done without a lot of programming effort.

1.3.2 $K_7 \cup K_1$ minors

In Chapter 4, we prove that every edge-maximal graph without a $K_7 \cup K_1$ minor has at most $5n-15$ edges or is isomorphic to one of the graphs listed in Theorem 1.3.2. Note that Theorem 1.3.2 generalizes Mader's result on the extremal function for K_7 minors (see Theorem 1.2.2) and a result of Jørgensen on the extremal function for $K_6 \cup K_1$ minors.

Theorem 1.3.2 *Let G be a graph with $n \geq 8$ vertices and*

$$e(G) \geq \begin{cases} 5n-14 & \text{if} \quad n \neq 9 \\ 32 & \text{if} \quad n = 9 \end{cases}$$

Then either $G > K_7 \cup K_1$ or G is isomorphic to $K_4 + C_{n-4}$ when $n \neq 9$, or one of the following graphs: $K_{1,2,2,2,2}$, $K_{1,3,3,3}$, $K_{2,2,2,4}$, $K_{2,3} + K^-_{2,3}$, $K_{2,2} + K^-_{3,3}$, $K_{2,3} + C_5$, $K_{3,3} + P_4$, $K_{2,2,3,3}$, where $K^-_{2,3}$ and $K^-_{3,3}$ are depicted in Figure 2 and a dotted line denotes the deleted edge.

Figure 2: $K^-_{2,3}$ and $K^-_{3,3}$

1.3.3 K_8^- minors, Jakobsen's Conjecture

The extremal functions for K_p^- minors have also been studied, where K_p^- denote the graph obtained from K_p by removing one edge. Jakobsen [13, 14] proved the following.

Theorem 1.3.3 *For $p = 5, 6, 7$, if G is a graph with $n \geq p$ vertices and at least $(p - \frac{5}{2})n - \frac{1}{2}(p-3)(p-1)$ edges, then $G > K_p^-$, or G is a $(K_{p-1}, p-3)$-cockade when $p \neq 7$, or $p = 7$ and G is a $(K_{2,2,2,2},\ K_6,\ 4)$-cockade.*

In 1983, Jakobsen [14] also conjectured that Theorem 1.3.3 extends to $p = 8$ as follows:

Conjecture 1.3.4 *If G is a graph with $n \geq 8$ vertices and at least $\frac{11n-35}{2}$ edges, then $G > K_8^-$ or G is a $(K_{1,2,2,2,2},\ K_7,\ 5)$-cockade.*

In Chapter 3 (see also [30]), we prove Conjecture 1.3.4, as follows.

Theorem 1.3.5 *If G is a graph with $n \geq 8$ vertices and at least $\frac{11n-35}{2}$ edges, then $G > K_8^-$ or G is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade.*

Jakobsen [14] pointed out that the graph $K_{2,2,2,2,3}$ contains no K_9^- minor. In fact, there are many more small counterexamples to an analogue of Conjecture 1.3.4 for $p = 9$: $K_{1,1,2,2,2,2}$, $K_{1,2,2,3,3}$, $K_{3,3,3,3}$ and $K_{2,3,3,4}$. Thus an analogue of Conjecture 1.3.4 for $p = 9$ will have to include the conclusion that G is isomorphic to one of these graphs.

1.3.4 Beyond K_9 minors

In Chapter 6, we study counterexamples to Mader's bound for $p = 10$. We believe that every edge-maximal graph without a K_{10} minor has at most $8n-36$ edges, or is a $(K_{1,1,2,2,2,2,2}, 7)$-cockade, or is isomorphic to one of the following graphs: $K_{1,2,2,2,3,3}$, or $K_{2,2,3,3,4}$, or $K_{2,3,3,3,3}$, or $K^-_{2,3,3,3,3}$, $K_{2,2,2,2,2,3}$, $K^-_{2,2,2,2,2,3}$, or the graph obtained from two disjoint copies of $K_{2,2,2,2,2,3}$ by identifying cliques of size six. Unfortunately, we are unable to prove it now. Instead, we give a weak estimate for the extremal function for K_{10} and K_{11} minors. We prove that every edge-maximal graph without K_{10} minor has at most $11n - 66$ edges and every graph without K_{11} minor has at most $13n - 89$ edges. This estimate can be improved.

1.4 Linkages

Let G be a graph and $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k \in V(G)$. The k-path problem is to determine if there exist k disjoint paths $P_1, P_2, \ldots, P_k$ such that P_i has ends s_i and t_i. A graph with at least $2k$ vertices is said to be *k-linked* if for every $2k$-tuple $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ of distinct vertices, the k-path problem is feasible.

Linkages, subdivisions and minors are related in the following sense. Larman and Mani [23] and Jung [17] noticed that if a graph G is $2k$-connected and contains a subdivision of

K_{3k}, then G is k-linked. Mader [24] showed that a graph contains a subdivision of K_{3k} if its connectivity is sufficiently large. Robertson and Seymour [27] showed that the observation of Larman and Mani and of Jung remains true under a much weaker condition that G has a K_{3k} minor. Instead of considering K_{3k}, Bollobás and Thomason [5] consider graphs containing a dense graph as a minor. Using this idea they proved that every $22k$-connected graph is k-linked, thus proving the conjecture that linear connectivity forces a graph to be k-linked. Recently, Kawarabayashi, Kostochka and Yu [18] proved that every $12k$-connected graph is k-linked, and Thomas and Wollan [32] were able to prove that every $10k$-connected graph is k-linked.

In this thesis, we shall use the following result on 2-linkages. The next theorem was first proved by Jung [17]. Seymour [29] and Thomassen [36] gave a complete characterization of all (not necessarily 4-connected) graphs that satisfy the hypothesis of the theorem.

Theorem 1.4.1 *Let G be a 4-connected graph and let x_1, x_2, y_1, y_2 be distinct vertices in G. If G does not contain an x_1-y_1 path and an x_2-y_2 path that are disjoint, then G is planar and $e(G) \le 3|G| - 7$.*

Theorem 1.4.2 is a result of Thomas and Wollan [33]. As we explain in Chapter 6, it will be useful in a possible extension of Theorem 1.2.2 to $p = 10$.

Theorem 1.4.2 *Let G be a 6-connected graph. Then G is 3-linked if $e(G) \ge 5n - 14$.*

1.5 Motivation for our work

1.5.1 Four Color Theorem

If any result in graph theory has a claim to be known to the world outside, it is the following *Four Color Theorem* (which implies that every map can be colored with at most four colors):

Theorem 1.5.1 [The Four Color Theorem] *Every planar graph is* 4*-colorable.*

The Four Color Theorem was first proved by Appel and Haken [1, 3, 2] in 1976. Their proof was computer-assisted and extremely complicated. A new proof of the Four Color Theorem was published in 1997 by Robertson, Sanders, Seymour and Thomas [26]. The new proof was still computer-assisted but the programs used in the new proof were available for independent verification.

1.5.2 Hadwiger's Conjecture

Before we state the Hadwiger's Conjecture, we need to introduce one more notion.

A graph G is said to be *k-colorable* if one can color its vertices in such a way that any two adjacent vertices must receive different colors, using at most k colors. The least integer k is called the *chromatic number* of G, and is denoted by $\chi(G)$.

Hadwiger's Conjecture [11] from 1943 suggests a far reaching generalization of the Four Color Theorem. It is considered by many as one of the deepest open problems in Graph Theory.

Conjecture 1.5.2 *For every integer $t \geq 1$, every graph with no K_{t+1} minor is t-colorable.*

Hadwiger's conjecture is trivially true for $t \leq 2$, and reasonably easy for $t = 3$, as shown by Dirac [8]. However, for $t \geq 4$, Hadwiger's conjecture implies the Four Color Theorem. (To see that, let H be a planar graph, and let G be obtained from H by adding $t-4$ vertices, each joined to every other vertex of the graph. Then G has no K_{t+1} minor, and hence is t-colorable by Hadwiger's conjecture, and hence H is 4-colorable.) Wagner [37] proved that the case $t = 4$ of Hadwiger's conjecture is, in fact, equivalent to the Four Color Theorem, and the same was shown for $t = 5$ by Robertson, Seymour, and Thomas [28]. They actually proved that a minimal counterexample to the case $t = 5$ is a graph G which has a vertex v such that $G - v$ is planar. This kind of graph is called *apex.* For $t = 6$, Kawarabayashi and

Toft [19] proved that any graph with $\chi(G) \geq 7$ has either K_7 or $K_{4,4}$ as a minor. Jakobsen [12] proved that every graph with $\chi(G) \geq 7$ contains a minor isomorphic to K_7 with two edges deleted. Hadwiger's conjecture remains open for $t \geq 6$.

1.5.3 Motivation

Our motivation was three fold. As we know, Hadwiger's Conjecture is open for $t \geq 6$. Robertson, Seymour and Thomas [28] proved that the case $t = 5$ is equivalent to the Four Color Theorem. Their proof made use of Theorem 1.2.2 for $p = 6$.

In [6] Chen, Gould, Kawarabayashi, Pfender and Wei proved that every simple graph on n vertices and at least $9n - 45$ edges has a K_9 minor, and used that to deduce that if, in addition, G is 6-connected, then it is 3-linked. It turns out [33] that the same conclusion can be obtained from a weaker bound on the number of edges by a more direct argument, but the work of Chen, Gould, Kawarabayashi, Pfender and Wei suggested that there may be interest in the extremal problem for K_9 minors.

Theorem 1.2.2 is such a nice result that it raises the question of whether it can be generalized to all values of p. But there are more depressing news than Mader's example above: for large p a graph must have at least $\Omega(p\sqrt{\log p}n)$ edges in order to guarantee a K_p minor, because, as noted by several people (Kostochka [20, 21], and Fernandez de la Vega [7] based on Bollobás, Catlin and Erdös [4]), a random graph with no K_p minor may have average degree of order $p\sqrt{\log p}$. Kostochka [20, 21] and Thomason [34] proved that this is indeed the correct order of magnitude, and in a remarkable recent result [35] Thomason was able to determine the constant of proportionality. Thus it may seem that an effort to generalize Theorem 1.2.2 will be in vain, but there is still the following possibility. The random graph examples provide only finitely many counterexamples for any given value of p. Of course, more counterexamples can be obtained by taking disjoint unions or even gluing counterexamples along small cutsets, but we know of no construction of highly connected infinite families of counterexamples. More specifically, Seymour and Thomas (see [31]) conjecture the following.

Conjecture 1.5.3 *For every $p \geq 1$ there exists a constant $N = N(p)$ such that every $(p-2)$-connected graph on $n \geq N$ vertices and at least $(p-2)n - \binom{p-1}{2} + 1$ edges has a K_p minor.*

It was this conjecture that was the third motivating factor for our research. Thus Theorem 1.3.1 implies that Conjecture 1.5.3 holds for $p \leq 9$.

CHAPTER II

$K_6^- \cup K_1$ MINORS IN GRAPHS OF SMALL ORDER

2.1 Introduction

For a real number x, we denote by $\lceil x \rceil$ the least integer $\geq x$. In the proof of Theorem 1.3.5, we shall consider graphs with n vertices and exactly $\lceil \frac{11n-35}{2} \rceil$ edges. Such graphs have vertices of degree at most 10. Since we want to consider contractions in the graph spanned by the neighbors of a vertex of degree at most 10, we need some results about contractions in graphs with at most 10 vertices. Theorem 2.1.1 and Theorem 1.3.3 will be used in proving those results, where Theorem 2.1.1 is a result of Jørgensen [16].

Theorem 2.1.1 *Let G be a graph with $n \leq 11$ vertices and $\delta(G) \geq 6$ such that for every vertex x in G, $G - x$ is not contractible to K_6. Then G is isomorphic to one of the graphs $K_{2,2,2,2}$, $K_{3,3,3}$ or the complement of the Petersen graph. In particular, $G > K_6^- \cup K_1$.*

We are now ready to prove the following two lemmas.

Lemma 2.1.2 *Let G be a graph with 8 vertices and $\delta(G) \geq 5$. Then $G > K_6^- \cup K_1$ or G is isomorphic to $\overline{C_8}$, $\overline{C_4} + \overline{C_4}$, $\overline{K_3} + C_5$, $\overline{K_2} + \overline{C_6}$, or $K_{2,3,3}$. In particular, all these graphs are edge maximal subject to not having a $K_6^- \cup K_1$ minor. Moreover, $\overline{C_8} > K_6$ and $\overline{C_4} + \overline{C_4} > K_6$.*

Proof. It is not hard to verify that the graphs listed are edge maximal subject to not having a $K_6^- \cup K_1$ minor. Thus we may assume that every edge of G is incident with a vertex of degree five. Let $x \in V(G)$ be such that $d(x) = 5$. If $e(G - x) \geq \frac{1}{2}(7|G - x| - 15) = 17$, by Theorem 1.3.3, $G - x > K_6^-$ or $G - x = K_3 + (K_2 \cup K_2)$. In the second case, x is adjacent to the four vertices of degree 4 in $K_3 + (K_2 \cup K_2)$. It is easy to check that $G > K_6^- \cup K_1$. Hence we may assume $e(G - x) \leq 16$, and so $20 \leq e(G) \leq 21$. If $e(G) = 20$, then G is 5-regular on 8 vertices. Thus $\overline{G}$ is 2-regular. It follows that $\overline{G}$ is isomorphic to C_8, $C_4 \cup C_4$, or $C_3 \cup C_5$, and so the lemma holds. If $e(G) = 21$, then G has either one vertex of degree 7 and seven vertices of degree 5 or two vertices of degree 6 and six vertices of degree 5. In the first case, let y be the vertex of degree 7. Then $G - y$ is 4-regular on 7 vertices. Thus $\overline{G - y} = C_7$ or $C_3 \cup C_4$. It is easy to check that $G - y > K_5^- \cup K_1$ and thus $G > K_6^- \cup K_1$. For the latter, let z, w be the two vertices of degree 6. Since G is edge minimal, we have $zw \notin E(G)$. It follows that $G - \{z, w\}$ is 3-regular on 6 vertices. Thus G is $\overline{K_2} + \overline{C_6}$ or $K_{2,3,3}$. The last assertion is easy to verify. ■

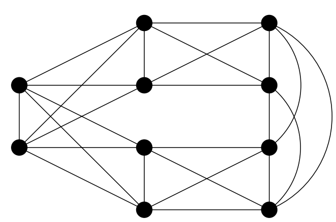


Figure 3: Graph J has no $K_6^- \cup K_1$ minor

Lemma 2.1.3 *Let G be a graph with $9 \leq n \leq 10$ vertices and $\delta(G) \geq 5$. Then $G > K_6^- \cup K_1$ or G is isomorphic to J (given in Figure 3).*

Proof. Lemma 2.1.3 can be checked by computers. However, a computer-free proof is given in the next section. ■

From Lemma 2.1.2 and Lemma 2.1.3, it follows that

Corollary 2.1.4 *Let G be a graph with $8 \leq |G| \leq 10$ and $\delta(G) \geq 5$. Then $G > K_6^- \cup K_1$ or G is isomorphic to $\overline{C_8}$, $\overline{C_4} + \overline{C_4}$, $\overline{K_3} + C_5$, $\overline{K_2} + \overline{C_6}$, $K_{2,3,3}$, or J. In particular, all these graphs are edge maximal (subject to not having a $K_6^- \cup K_1$ minor) with maximum degree $\leq |G| - 2$. Moreover, $\overline{C_8} > K_6$, $\overline{C_4} + \overline{C_4} > K_6$, and $J > K_6$.*

We need to prove two more results.

Lemma 2.1.5 *Let G be a graph on 8 vertices. Let $u, w \in V(G)$ be such that $d(u) \geq 4$, $d(w) = 7$, and $d(v) \geq 5$ for every $v \neq u, w$. Then $G > K_6^- \cup K_1$.*

Proof. Suppose $d(u) \geq 5$. Then $\delta(G) \geq 5$ and $\Delta(G) = 7$. By Lemma 2.1.2, $G > K_6^- \cup K_1$. So we may assume that $d(u) = 4$. Then $e(G) \geq \lceil \frac{4+7+5\times 6}{2} \rceil = 21$. Note that $e(G-u) = e(G) - 4 \geq 17$ and $G - u$ has at most three vertices of degree 4. By Theorem 1.3.3, $G - u > K_6^-$. ■

Lemma 2.1.6 *Let G be a graph on 9 vertices. Let $uw \in E(G)$ be such that $d(u) = 4$, $d(w) \geq 7$ and $d(v) \geq 5$ for every $v \neq u, w$. Then $G > K_6^- \cup K_1$.*

Proof. Suppose G is not contractible to $K_6^- \cup K_1$. We may assume that G is edge minimal. We claim that $d(w) = 7$. Suppose $d(w) = 8$. Since the number of odd vertices of any graph is even, there exists another vertex, say $v \in V(G)$, such that $d(v) \geq 6$. Clearly, $vw \in E(G)$ and $d_{G-vw}(w) \geq 7$, $d_{G-vw}(u) = 4$, $d_{G-vw}(v) \geq 5$ for any $v \neq u, w$, which contradicts the fact that G is edge minimal. Hence $d(w) = 7$, as claimed.

We first show that G is 4-connected. Let S be a minimal separating set of G with $|S| \leq 3$. Since $|G| = 9$ and $d(v) \geq 5$ for any $v \neq u, w$, we have $|S| = 3$. Let H_1 and H_2 be the two connected components of $G - S$. Then $|H_1| = |H_2| = 3$. We may assume that $H_1 = K_3$ and each vertex of H_1 is adjacent to all vertices of S. Note that there exists a vertex, say $a \in V(H_2)$, adjacent to all vertices in S. Let $b \in S$. Now $G/ab - V(H_2 - a) > K_6^-$. This proves that G is 4-connected.

Since $uw \in E(G)$, let $V(N(u)) = \{w, a, b, c\}$ and $A = V(G) - V(N[u]) = \{d, e, f, g\}$. We next prove the following claim.

Claim: *For any $v \in \{a, b, c\}$, if $vw \in E(G)$, then $d_{N(u)}(v) \geq 2$.*

Proof. Suppose otherwise. We may assume that $aw \in E(G)$ and $ab, ac \notin E(G)$. Let w' be the new vertex in G/ua. Then $d_{G/ua}(w) = 6$, $d_{G/ua}(w') \geq 6$ and $ww' \in E(G)$. Note that $\delta(G/ua - ww') \geq 5$. By Lemma 2.1.2, $G/ua > K_6^- \cup K_1$. ■

Suppose that w is adjacent to all vertices of A. Since $d_G(w) = 7$, we may assume that $cw \notin E(G)$. If $ca \notin E(G)$ or $d_G(a) \geq 6$, then $\Delta(G/uc) = 7$, $d_{G/uc}(b) \geq 4$ and $d_{G/uc}(v) \geq 5$ for any $v \in V(G/uc - b)$. By Lemma 6, $G/uc > K_6^- \cup K_1$. Hence $ca \in E(G)$ and $d_G(a) = 5$. Similarly, $cb \in E(G)$ and $d_G(b) = 5$. Note that $e_G(v, \{a, b, c\}) \geq 1$ for any $v \in A$. If $G[A] = K_4$ or K_4^-, then $G/ac/bc - u > K_6^-$. So we may assume that $e(G[A]) \leq 4$. Thus $e_G(A, N(u)) \geq 20 - 2e(G[A]) \geq 12$ and $e_G(\{a, b, c\}, A) = e_G(A, N(u)) - e_G(w, A) \geq 12 - 4 = 8$. Note that $d_G(a) = d_G(b) = 5$. It follows that $ab \notin E(G)$ and c is adjacent to all vertices of A. Hence $d_G(c) = 7$, $d_G(v) = 5$ for any $v \in A$, and $e(G[A]) = 4$. So $G[A] = C_4$ or $K_1 + (K_2 \cup K_1)$. In the first case, we may assume that $G[A]$ has vertices d, e, f, g in order and $ad \in E(G)$. Then by symmetry, either $af \in E(G)$ or $ae \in E(G)$. If $af \in E(G)$, then $be, bg \in E(G)$ and so $G/ad/be - u = K_6^-$. If $ae \in E(G)$, then $bf, bg \in E(G)$ and so $G/uw/de - a = K_6^-$. In the second case, we may assume that $ed, ef, eg, fg \in E(G)$. Then d is adjacent to all vertices of $N(u)$. Note that either $af, bg \in E(G)$ or $ag, bf \in E(G)$. In either case, $G/da/db - u > K_6^-$. This proves the case when w is adjacent to all vertices of A.

Suppose w is adjacent to all vertices of $N(u)$. Then $d_G(w, A) = 3$ and so $\delta(G[\{a, b, c\}]) \geq 1$ by Claim. We may assume that $ab, bc \in E(G)$. Note that $e_G(v, \{a, b, c\}) \geq 1$ for any $v \in A$. If $G[A] = K_4$, then $G/ab/bc - u > K_6^-$. So we may assume that $e(G[A]) \leq 5$. It follows that $e_G(A, N(u)) \geq 20 - 2e(G[A]) \geq 10$ and so $e_G(\{a, b, c\}, A) = e_G(A, N(u)) - e_G(w, A) \geq 10 - 3 = 7$. Thus $ca \notin E(G)$ (otherwise, since G is edge minimal, at most one of a, b, c could be of degree > 5, and so $e(\{a, b, c\}, A) \leq 4 + 1 + 1 = 6$, a contradiction). If a is adjacent to all vertices of A, then $\Delta(G/uc) = 7$, $d_{G/uc}(b) = 4$ and $d_{G/uc}(v) \geq 5$ for any $v \in V(G/uc - b)$. By Lemma 6, $G/uc > K_6^- \cup K_1$. Hence a, similarly c, is adjacent to at most three vertices of A. Thus $e_G(N(u), A) \leq 3 + 3 + 1 + 3 = 10 \leq e_G(A, N(u))$. It follows that $G[A] = K_4^-$, a (resp. c) is adjacent to exactly three vertices of A and b is adjacent to exactly one vertex of

A, all vertices of A are of degree five. Since $G[A] = K_4^-$, we may assume that $de \notin E(G)$. Note that $e_G(b, A) = 1$, we may assume that $be \notin E(G)$. Then $ew, ea, ec \in E(G)$. Observe that $e_G(d, N(u)) = 3$ and if $v \in N(u)$ is not adjacent to d, then $vf \in E(G)$ or $vg \in E(G)$, say the later. Clearly, $G/ae/dg - f > K_6^-$. ■

2.2 Proof of Lemma 2.1.3

Here we give a computer-free proof of Lemma 2.1.3.

We may assume that G is minor minimal subject to $\delta(G) \geq 5$ and $|G| \geq 9$. If $\delta(G) \geq 6$, by Theorem 2.1.1, $G > K_6^- \cup K_1$. So we may assume that $\delta(G) = 5$. We first prove two claims.

Claim 1. *Every edge of G is in at least two triangles.*

Proof. Suppose $e = uv \in E(G)$ is in at most one triangle in G. Let w be the new vertex in G/e. Then $d_G(w) \geq 7$, and $d_G(y) \geq 4$, where y is the common neighbor of u and v in G. Clearly, $wy \in E(G/e)$ and $d_{G/e}(v) \geq 5$ for any $v \neq w, y$. Since G is minor minimal, by Lemma 2.1.5 and Lemma 2.1.6, $G > G/e > K_6^- \cup K_1$. ■

Claim 2. *There is no edge of G with both ends of degree at least six in G.*

Proof. Suppose $e = uv \in E(G)$ is such that $d(u), d(v) \geq 6$. Then $\delta(G-e) \geq 5$ and $|G| \geq 9$, which contradicts the fact that G is minor minimal. ■

We next show that G is 4-connected. Let S be a minimal separating set of G with $|S| \leq 3$. Let H_1 be a component of $G - S$ with minimal order and $H_2 = G - S - H_1$. If $|S| \leq 2$, then, since $\delta(G) \geq 5$, $|H_1|, |H_2| \geq 4$, and hence $|S| = 2$, H_1 and H_2 are isomorphic to K_4, because $|G| \leq 10$. But then, clearly, $G > K_6^- \cup K_1$. Suppose $|S| = 3$. Then $H_1 = K_3$ and $3 \leq |H_2| \leq 4$. Note that every vertex of H_1 is adjacent to every vertex of S. If there is a vertex $b \in V(H_2)$ such that b is adjacent to all vertices in S, then $G/ab - V(H_2 - b) > K_6^-$, where $a \in S$. Otherwise $H_2 = K_4$. By the minimality of $|S|$, G has a matching from S

into H_2. By contracting this matching, it follows that $G > K_6^- \cup K_1$. This shows that G is 4-connected.

Since $\delta(G) = 5$, let $x \in V(G)$ be such that $d(x) = 5$. We may assume that $V(N(x)) = \{a, b, c, d, e\}$ and $A = V(G) - V(N[x]) = \{y_1, y_2, \cdots, y_{|G|-6}\}$.

Claim 3. *$N(x)$ contains no subgraph isomorphic to $K_{2,3}$.*

Proof. Suppose that $N(x)$ has a subgraph H isomorphic to $K_{2,3}$. We may assume that $d_H(a) = d_H(e) = 3$ and $d_H(b) = d_H(c) = d_H(d) = 2$, as shown in Figure 4.

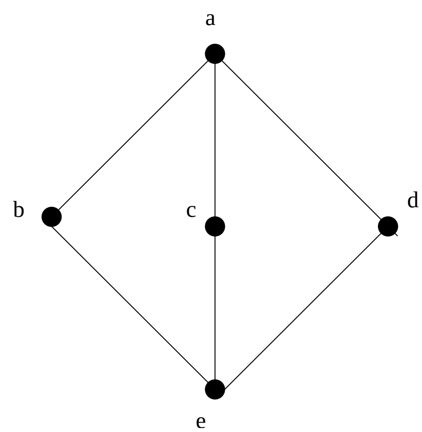


Figure 4: $K_{2,3}$

Suppose that there exists a vertex of A, say y_1, such that $y_1b, y_1c, y_1d \in E(G)$. If $G[\{b, c, d\}] \neq \overline{K_3}$, say $bc \in E(G)$, then $G/y_1d - y_2 > K_6^-$. So we may assume that $G[\{b, c, d\}] = \overline{K_3}$. If two of b, c, d, say b, c, have a common neighbor, say y_2, of $A - y_1$ in G, then $G/by_2/dy_1 - y_3 > K_6^-$. It follows that any two vertices of b, c, d have no common neighbors in A, thus there is a matching M from $\{b, c, d\}$ into $A - y_1 = \{y_2, y_3, y_4\}$, and $V(M) \cap A$ is not a stable set in G. We may assume that $y_2y_3 \in E(G)$ and $by_2, cy_3 \in M$. Now $G/by_2/y_2y_3/dy_1 - y_4 > K_6^-$. This proves that there is no vertex of A adjacent to all b, c, d in G. Next, suppose that $G[\{b, c, d\}]$ induces at least two edges, say $bc, cd \in E(G)$. We may assume that $bd, ae \notin E(G)$, otherwise $N[x] > K_6^-$. Among a, b, d, e, by Claim 2, we may assume that $d_G(e) = 5$. Let $ey_1 \in E(G)$. If $cy_1 \notin E(G)$, by Claim 1, $\delta(N(e)) \geq 2$.

Thus $by_1, dy_1 \in E(G)$ and so $G/by_1 - y_2 > K_6^-$. It follows that $cy_1 \in E(G)$. Then $d_G(c) \geq 6$. By Claim 2, $d_G(a) = d_G(b) = d_G(d) = 5$. By Claim 1 and the symmetry of b and d, we may assume that $by_1 \in E(G)$. Then $dy_1 \notin E(G)$, otherwise y_1 is adjacent to all b, c, d in G. Similarly, let $dy_2 \in E(G)$. Then $cy_2 \in E(G)$ and $by_2, ey_2 \notin E(G)$. Thus $ay_2 \in E(G)$. Now y_3 is only adjacent to c, y_1, y_2, y_4, which contradicts the fact that $d_G(y_3) \geq 5$. This proves that $G[\{b, c, d\}]$ contains at most one edge. We may assume that $bc, bd \notin E(G)$.

Suppose that $d_G(a), d_G(e) \geq 6$. Then $\delta(G/xb) \geq 5$. Since G is minor minimal, we have $|G| = 9$. Let w be the new vertex in G/xb. Then $d_{G/xb}(w) \geq 6$. If $d_{G/xb}(a) \geq 6$ or $d_{G/xb}(e) \geq 6$, say the latter, then $\delta(G/xb - ew) \geq 5$. By Lemma 2.1.2, $G/xb > K_6^- \cup K_1$. It follows that $d_G(a) = d_G(e) = 6$. Since $|G| = 9$ and the number of odd vertices of a graph is even, there exists a vertex of A, say y_1, such that $d_G(y_1) \geq 6$. Then $d_{G/xb}(y_1) \geq 6$ and $wy_1 \in E(G/xb)$. Now $\delta(G/xb - wy_1) \geq 5$. By Lemma 2.1.2, $G/xb > K_6^- \cup K_1$. Consequently, $d_G(a) = 5$ or $d_G(e) = 5$. We may assume that $d_G(a) = 5$. If $ae \in E(G)$, then, since G is 4-connected, e has at least one neighbor in A. It follows that $d_G(e) \geq 6$ and so $d_G(b) = d_G(c) = d_G(d) = 5$. Now x and b have exactly two common neighbors a and e in G. If $d_G(e) \geq 8$, then in G/xb, $\Delta(G/xb) \geq 7$, $d_{G/xb}(a) = 4$ and $d_{G/xb}(v) \geq 5$ for any $v \in V(G/xb - a)$. By Lemma 2.1.5 and Lemma 2.1.6, $G/xb > K_6^- \cup K_1$. So we may assume that e is adjacent to at most two vertices of A in G. Then $e_G(N(x), A) \leq 8$. It follows that $e_G(N(x), A) = 8$, $|A| = 4$, $G[A] = K_4$, and $G[\{b, c, d\}] = \overline{K_3}$. We may assume that $by_1, cy_4 \in E(G)$. Then $G/by_1/y_1y_2/y_2y_3 - y_4 = K_6^-$. Hence $ae \notin E(G)$. Let $ay_1 \in E(G)$. Then $cd \in E(G)$, otherwise, by Claim 1, $\delta(N(a)) \geq 2$, but then y_1 is adjacent to all b, c, d in G. Again, by Claim 1, $y_1b \in E(G)$. By symmetry of c and d, we may assume that $cy_1 \in E(G)$ and so $dy_1 \notin E(G)$ (otherwise y_1 is adjacent to all b, c, d). Let $dy_2 \in E(G)$. Then $ay_2 \notin E(G)$ and y_2 is adjacent to at most one of b and c in G. It follows that either $y_2y_1 \in E(G)$ (in this case $G/by_1/y_1y_2 - y_3 > K_6^-$) or $y_2y_3, y_2y_4 \in E(G)$ and y_1 is adjacent to at least one of y_3, y_4, say y_3 (in this case $G/by_1/y_1y_3/y_3y_2 - y_4 > K_6^-$). ■

Claim 4. *$N(x)$ contains no subgraph isomorphic to $K_1 + (K_2 \cup K_2)$.*

Proof. Suppose that $N(x)$ has a subgraph H isomorphic to $K_1 + (K_2 \cup K_2)$. We may assume that $d_H(c) = 4$, and $ab, de \in E(H)$, as depicted in Figure 5.

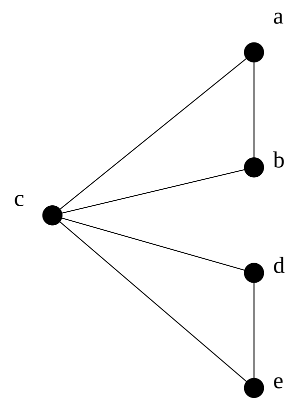


Figure 5: $K_1 + (K_2 \cup K_2)$

By Claim 3, there exists at most one edge between $\{a, b\}$ and $\{d, e\}$ in G. Suppose such an edge exists. By symmetry, we may assume that $ad \in E(G)$. By Claim 2, we may assume that $d_G(a) = 5$. Let $ay_1 \in E(G)$. By Claim 1, $\delta(N(a)) \geq 2$. By Claim 3, we may assume that $cy_1 \in E(G)$. It follows that $d_G(c) \geq 6$ and by Claim 2, $d_G(b) = d_G(d) = d_G(e) = 5$. If $e_G(c, A) \geq 3$, then $d_{G/xe}(c) \geq 7$, $d_{G/xe}(d) = 4$ and $d_{G/xe}(v) \geq 5$ for any $v \neq e$. By Lemma 2.1.5 and Lemma 2.1.6, $G > G/xe > K_6^- \cup K_1$. Hence $e_G(c, A) \leq 2$. By counting the number of edges between $N(x)$ and A in G, it follows that $e_G(A, N(x)) = 8$ and $G[A] = K_4$. Let $by_i, ey_j \in E(G)$, where y_i, y_j, y_1 could be the same. Clearly, $G/ey_j/y_jy_i/y_iy_1 - (A - \{y_1, y_i, y_j\}) = K_6^-$. This shows that there exists no edge between $\{a, b\}$ and $\{d, e\}$ in G. By Claim 2, we may assume that $d_G(b) = d_G(e) = 5$. Let $by_1, by_2 \in E(G)$.

Suppose that $d_G(c) = 5$. Then by Claim 1, $y_1y_2, ay_1, ay_2 \in E(G)$. Let y_i, y_j be the two neighbors of e in A. By Claim 1, $y_iy_j, dy_i, dy_j \in E(G)$. If $y_i = y_1$ and $y_j = y_2$, then $G/ey_1/dy_2 - y_3 > K_6^-$. If $y_i = y_1$ and $y_j \neq y_2$, we may assume that $y_j = y_3$. Then $G/ey_1/ay_3 - y_2 > K_6^-$ if $ay_3 \in E(G)$ or $G/ey_1/dy_2 - y_3 > K_6^-$ if $dy_2 \in E(G)$. It follows that $G[A] = K_4$. Now $G/ey_1/ay_2/y_2y_3 - y_4 > K_6^-$. Hence, by symmetry, we may assume that $y_i, y_j \neq y_1, y_2$ and so $ey_3, ey_4 \in E(G)$. Clearly, $G > K_6^- \cup K_1$ or G is isomorphic to J. This proves that $d_G(c) \geq 6$. By Claim 2, $d_G(a) = d_G(b) = d_G(d) = d_G(e) = 5$. If $d_G(c) \geq 8$, then $d_{G/xa}(c) \geq 7$, $d_{G/xa}(b) = 4$ and $d_{G/xa}(v) \geq 5$ for any $v \neq c, b$. By Lemma 2.1.5 and Lemma 2.1.6, $G/xa > K_6^- \cup K_1$. It follows that $6 \leq d_G(c) \leq 7$. Since $by_1, by_2 \in E(G)$, by the symmetry of a, b, d, e, we may assume that $cy_1 \notin E(G)$. By Claim 1, $y_1y_2, ay_1 \in E(G)$.

Suppose $ey_1 \in E(G)$. By Claim 1, $dy_1 \in E(G)$. If $dy_2 \in E(G)$ or $ey_2 \in E(G)$, say the latter, then $G/ay_1/by_2 - y_3 > K_6^-$. So we may assume that $dy_2, ey_2 \notin E(G)$. Let $ey_3 \in E(G)$. By Claim 1, $y_1y_3 \in E(G)$. By symmetry of a, b, d, e, $ay_3, by_3 \notin E(G)$. If $|A| = 3$, then by Claim 1, $cy_2, cy_3, ay_2, dy_3, y_2y_3 \in E(G)$ and so $G/xd/y_2y_3 - e = K_6^-$. If $|A| = 4$, since y_4 is adjacent to at least two vertices other than b, e of H, we may assume that $ay_4 \in E(G)$. Then $G/ay_4/by_1 - \{y_2, y_3\} = K_6^-$ if $dy_4 \in E(G)$, otherwise $y_3y_4 \in E(G)$ and $G/by_1/ey_3/y_3y_4 - y_2 = K_6^-$. This proves that $ey_1 \notin E(G)$ and similarly, $dy_1 \notin E(G)$. Thus $y_1y_i \in E(G)$, $i = 2, 3, 4$, and $d_G(y_1) = 5$. We claim that $G[A] = K_4$. If $ay_2 \in E(G)$, by Claim 1, $\delta(N(y_1)) \geq 2$ and so $G[A] = K_4$. If $ay_2 \notin E(G)$, we may assume that $ay_3 \in E(G)$. By Claim 1, $\delta(N(b)) \geq 2$ and so $cy_2, cy_3 \in E(G)$. Since $d_G(c) \leq 7$, we have $cy_4 \notin E(G)$ and so y_4 is adjacent to d, e, y_1, y_2, y_3. Then either $G[A] = K_4$ or $y_2y_3 \notin E(G)$ (in this case, we may assume that $ey_3 \in E(G)$. Then $G/by_1/y_1y_4/ey_3 - y_2 = K_6^-$). Hence $G[A] = K_4$, as claimed. Since $e_G(N(x), A) \geq 9$, there exists a vertex $y_i \in A$ such that $d_G(y_i) \geq 6$. Note that $d_G(y_1) = 5$, we have $y_i \neq y_1$. By Claim 2, $cy_i \notin E(G)$ and so $e_G(y_i, \{a, b, d, e\}) \geq 3$. Since $y_1e, y_1d \notin E(G)$, let $ey_j, dy_k \in E(G)$, where $y_j, y_k \neq y_i$. If $y_i \neq y_2$, then $G/ay_i/by_1/y_1y_j > K_6^- \cup K_1$. So we may assume that $y_i = y_2$. If $ay_2 \notin E(G)$, then $G/by_2/ay_1/y_1y_j > K_6^- \cup K_1$. If $ay_2 \in E(G)$, we may assume that $ey_2 \in E(G)$. Then $G/ey_2/by_1/y_1y_k > K_6^- \cup K_1$. ■

By Claim 1, $\delta(N(x)) \geq 2$. Hence, by Claim 3 and Claim 4, $N(x)$ is isomorphic to either C_5 or C_5 with exactly one chord.

Suppose that $N(x)$ is isomorphic to C_5 and $N(x)$ has vertices a, b, c, d and e in order as shown in Figure 6.

By Claim 2, $N(x)$ contains at most two vertices of degree ≥ 6. Suppose that $N(x)$ contains exactly two vertices of degree ≥ 6, say b and d. Then $\delta(G/xc) \geq 5$. Since G is

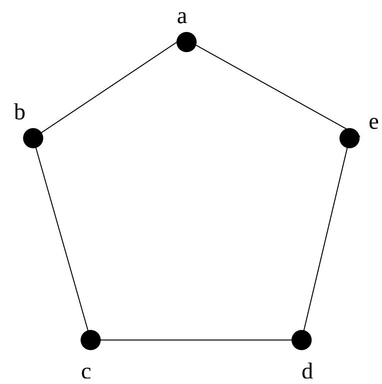


Figure 6: C_5

minor minimal, we have $|G| = 9$, $d_G(b) = d_G(d) = 6$, and by Claim 2, $d_G(v) = 5$ for any $v \in V(G - \{b, d\})$, which contradicts the fact the number of odd vertices of G is even. This implies that $N(x)$ contains at most one vertex of degree greater than five (we may assume $d_G(e) \geq 6$ if such a vertex exists). Thus $d_G(a) = d_G(b) = d_G(c) = d_G(d) = 5$. Let $cy_1, cy_2 \in E(G)$. By Claim 3 and Claim 4, $N(c)$ contains no subgraph isomorphic to $K_{2,3}$ and $K_1 + (K_2 \cup K_2)$. Thus by Claim 1, $y_1y_2 \in E(G)$. We may assume that $by_1, dy_2 \in E(G)$. Then by_2, dy_1 cannot be both in $E(G)$, otherwise $N(c) > K_{2,3}$.

Suppose $by_2, dy_1 \notin E(G)$. Since $d_G(b) = 5$, let $by_3 \in E(G)$. By Claim 1, $ay_3, y_3y_1 \in E(G)$. We claim that $dy_3 \notin E(G)$. Suppose $dy_3 \in E(G)$. By Claim 1, $y_3y_2, ey_3 \in E(G)$. Thus $d_G(y_3) \geq 6$ and so $d_G(e) = d_G(y_1) = d_G(y_2) = 5$. If $|A| = 3$, by Claim 1, $ay_1, ey_2 \in E(G)$. Clearly, $G/xb/y_1y_2 - c > K_6^-$. If $|A| = 4$, then y_4 is adjacent to a, e, y_1, y_2, y_3, and so $G/by_3/ay_4/y_4y_2 - y_1 = K_6^-$. This proves that $dy_3 \notin E(G)$. Since $d_G(d) = 5$, let $dy_4 \in E(G)$. Then by Claim 1, $ey_4, y_2y_4 \in E(G)$. If $ay_4 \notin E(G)$, then $d_G(y_4) = 5$ and $y_4y_1, y_4y_3 \in E(G)$. By Claim 1, $\delta(N(y_4)) \geq 2$ and so $ey_3 \in E(G)$. Note that a is adjacent to exactly one vertex of $\{y_1, y_2\}$. Now $G/ay_1/by_3/cd - y_2 = K_6^-$ if $ay_1 \in E(G)$ or $G/xc/xe/ay_2 - d = K_6^-$ if $ay_2 \in E(G)$. This proves that $ay_4 \in E(G)$. By Claim 1, $\delta(N(a)) \geq 2$ and so $y_3y_4 \in E(G)$. Clearly, $y_1y_4 \notin E(G)$ (otherwise $d_G(y_4) \geq 6$ and so by Claim 2, e is adjacent to exactly one of y_2 and y_3, say y_2. Then $d_G(y_3) = 4$, which is a contradiction). It follows that $ey_1 \in E(G)$ and $d_G(y_1) = 5$. By Claim 1, $\delta(N(y_1)) \geq 2$, we have $ey_2, ey_3 \in E(G)$. Now $G/xa/xc/y_3y_4 - b = K_6^-$.

Suppose $by_2 \notin E(G)$ but $dy_1 \in E(G)$. Since $d_G(b) = 5$, let $by_3 \in E(G)$. By Claim 1, $ay_3, y_3y_1 \in E(G)$. Suppose $|A| = 4$. Then y_4 is adjacent to a, e, y_1, y_2, y_3. Then $d_G(y_1) \geq 6$. By Claim 2, $d_G(y_2) = d_G(y_3) = 5$. By Claim 1, $\delta(N(y_4)) \geq 2$, we have $ey_2, ey_3 \in E(G)$. Now $G/ab/cy_2/dy_1 - y_3 = K_6^-$. So we may assume that $|A| = 3$. Since $cy_3, dy_3 \notin E(G)$, it follows that $d_G(y_3) = 5$ and $y_3e, y_3y_2 \in E(G)$. By Claim 1, $ey_2 \in E(G)$. Note that a is adjacent to exactly one vertex of y_1, y_2. Now $G/xa/y_2y_3/ - e = K_6^-$ if $ay_1 \in E(G)$ or $G/xa/y_1y_3 - b = K_6^-$ if $ay_2 \in E(G)$.

Finally, assume that $dy_1 \notin E(G)$ but $by_2 \in E(G)$. Since $d_G(d) = 5$, let $dy_3 \in E(G)$. By Claim 1, $ey_3, y_3y_2 \in E(G)$. Suppose $|A| = 4$. Then y_4 is adjacent to a, e, y_1, y_2, y_3. Thus $d_G(y_2) \geq 6$. By Claim 1, $\delta(N(y_4)) \geq 2$ and so $ay_1 \in E(G)$. Since $d_G(y_2) \geq 6$, by Claim 2, y_3 is only adjacent to d, e, y_2, y_4, which contradicts the fact that $d_G(y_3) \geq 5$. So we may assume that $|A| = 3$. Since $cy_3, by_3 \notin E(G)$, it follows that $d_G(y_3) = 5$ and $y_3a, y_3y_1 \in E(G)$. Suppose $ay_1 \in E(G)$. By Claim 2, e is adjacent to exactly one vertex of y_1, y_2. Thus $G/xe/y_2y_3/ - d = K_6^-$ if $ey_1 \in E(G)$ or $G/xe/y_1y_3 - a = K_6^-$ if $ey_2 \in E(G)$. Suppose $ay_1 \notin E(G)$. Then $ey_1, ay_2 \in E(G)$. Now $G/ay_2/ey_1 - y_3 = K_6^-$. This completes the proof that $N(x)$ is isomorphic to C_5.

It remains to consider the case when $N(x)$ is isomorphic to C_5 with exactly one chord. We may assume that $E(N(x)) = \{ab, bc, cd, de, ea, be\}$, as depicted in Figure 7.

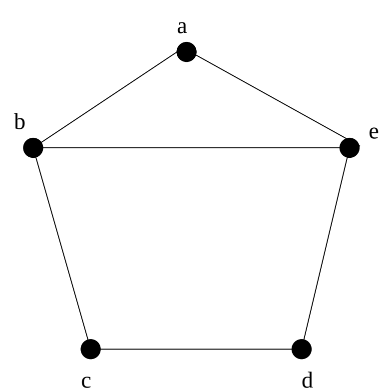


Figure 7: C_5 with one chord

By Claim 2, one of b and e, say e, is of degree five in G. Let $ey_1 \in E(G)$. By Claim

1, $\delta(N(e)) \geq 2$ and so $dy_1 \in E(G)$. Suppose $ay_1, by_1 \in E(G)$. We claim that $d_G(a) \geq 6$. Suppose $d_G(a) = 5$. Let $ay_2 \in E(G)$. By Claim 1, $\delta(N(a)) \geq 2$ and so $by_2, y_2y_1 \in E(G)$. It follows that $N(a) > K_1 + (K_2 \cup K_2)$, which contradicts Claim 4. Hence $d_G(a) \geq 6$, as claimed. By Claim 1 and Claim 2, $d_G(b) = 5$ and $cy_1 \in E(G)$. But now $N(b) > K_{2,3}$, which contradicts Claim 3. This proves that at most one of by_1, ay_1 are in $E(G)$.

Suppose $by_1 \in E(G)$ but $ay_1 \notin E(G)$. If $d_G(b) = 5$, since $\delta(N(b)) \geq 2$, $cy_1 \in E(G)$. By Claim 1, $\delta(N(a)) \geq 2$. Hence $ay_i \in E(G)$, $i = 2, 3, 4$, and $G[\{y_2, y_3, y_4\}] = K_3$. Since there is no edge between $\{b, e\}$ and $\{y_2, y_3, y_4\}$ in G, $e_G(\{y_2, y_3, y_4\}, \{c, d, y_1\}) \geq 6$. However, by Claim 2, $e_G(\{c, d, y_1\}, \{y_2, y_3, y_4\}) \leq 5$, which is a contradiction. So we may assume that $d_G(b) \geq 6$. By Claim 2, $d_G(a) = 5$. Let $ay_2, ay_3 \in E(G)$. By Claim 1, $\delta(N(a)) \geq 2$ and so $y_2y_3, by_2, by_3 \in E(G)$. Then $N(a) > K_1 + (K_2 \cup K_2)$, which contradicts Claim 4.

Finally, suppose $ay_1 \in E(G)$ but $by_1 \notin E(G)$. We claim that $d_G(d) \geq 6$. Suppose $d_G(d) = 5$. Let $dy_2 \in E(G)$. We may assume that $N(d) \neq C_5$. By Claim 1, $\delta(N(d)) \geq 2$ and so $y_1y_2, cy_1, cy_2 \in E(G)$. It follows that $G/ay_1/by_2 - y_3 > K_6^-$ if $by_2 \in E(G)$ or $G/ay_2/ey_1 - y_3 > K_6^-$ if $ay_2 \in E(G)$. So we may assume that $ay_2, by_2 \notin E(G)$. Then $y_2y_3, y_2y_4 \in E(G)$. Since $by_1, by_2 \notin E(G)$, we may assume that $by_3 \in E(G)$. Now $G/ay_1/by_3/y_3y_2 - y_4 = K_6^-$. This proves that $d_G(d) \geq 6$. By Claim 2, $d_G(c) = 5$ and so $d_G(b) \geq 6$ (otherwise, by symmetry of b and e, $d_G(c) \geq 6$). Now $\delta(G/xc) \geq 5$. Since G is minor minimal, we have $|G| = 9$. Let w be the new vertex in G/xc. Then $d_{G/xc}(w) \geq 6$. If $d_{G/xc}(b) \geq 6$ or $d_{G/xc}(d) \geq 6$, say the latter, then $\delta(G/xb - dw) \geq 5$. By Lemma 2.1.2, $G/xc > K_6^- \cup K_1$. It follows that $d_G(b) = d_G(d) = 6$. Since $|G| = 9$ and the number of odd vertices of G is even, there exists a vertex, say y_1, of A such that $d_G(y_1) \geq 6$. Note that $d_{G/xc}(y_1) \geq 6$ and $y_1c \in E(G)$. Now $y_1w \in E(G/xc)$ and $\delta(G/xc - y_1w) \geq 5$. By Lemma 2.1.2, $G/xc > K_6^- \cup K_1$. ■

CHAPTER III

THE EXTREMAL FUNCTION FOR K_8^- MINORS

In this chapter, we shall prove Conjecture 1.3.4 stated in Chapter 1.

3.1 $(K_{1,2,2,2,2},\ K_7,\ 5)$-cockades are edge-maximal

Our proof of Conjecture 1.3.4 uses induction by deleting and contracting edges of G. We need to investigate graphs G such that the new graph $G-xy$ or G/xy is a $(K_{1,2,2,2,2},\ K_7,\ 5)$-cockade, where $xy \in E(G)$. It turns out that contracting an edge of G in the proof of Conjecture 1.3.4 will not produce a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. So we only consider the case when $G - xy$ is a $(K_{1,2,2,2,2},\ K_7,\ 5)$-cockade. We do that next.

Lemma 3.1.1 *Let G be a $(K_{1,2,2,2,2},\ K_7,\ 5)$-cockade and let x and y be nonadjacent vertices in G. Then $G + xy$ is contractible to K_8^-.*

Proof. This is obviously true if G is $K_{1,2,2,2,2}$. So we may assume that G is obtained from H_1 and H_2 by identifying on K_5, where both H_1 and H_2 are $(K_{1,2,2,2,2},\ K_7,\ 5)$-cockades. If both $x, y \in V(H_i)$, then $H_i > K_8^-$ by induction. So we may assume that $x \in V(H_1) - V(H_2)$ and $y \in V(H_2) - V(H_1)$. If there exists $z \in V(H_1) \cap V(H_2)$ such that $yz \notin E(G)$, then by contracting $V(H_1) - V(H_1) \cap V(H_2)$ to z, the resulting graph will have a K_8^- minor by induction. So we may assume y is adjacent to all vertices in $V(H_1) \cap V(H_2)$. Similarly, we may assume that x is adjacent to all vertices in $V(H_1) \cap V(H_2)$. Hence there exists $w \in V(H_1)$ such that $H_1[\{w, x, V(H_1) \cap V(H_2)\}]$ is a K_7 subgraph in H_1. Clearly, $G[\{w, x, y, V(H_1) \cap V(H_2)\}] + xy > K_8^-$. ■

It is easy to observe that

Lemma 3.1.2 *Let G be a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. Then* $e(G) = \frac{11|G|-35}{2}$.

3.2 Proof of Theorem 1.3.5

In this section we prove Theorem 1.3.5 by induction on n. The only graphs G with 8 vertices and $e(G) \geq \frac{11\times 8-35}{2}$ are K_8^- and K_8. So we may assume that $n \geq 9$ and the assertion holds for smaller values of n.

Suppose G is a graph with n vertices and $e(G) \geq \frac{11n-35}{2}$ but G is not contractible to K_8^- and G is not a $(K_{1,2,2,2,2}, K_7, 5)$-cockade. By Lemma 3.1.1, we may assume that $e(G) = \lceil \frac{11n-35}{2} \rceil$, where $\lceil \frac{11n-35}{2} \rceil$ denotes the least integer $\geq \frac{11n-35}{2}$.

If G has a vertex x with $d(x) \leq 5$, then $e(G-x) \geq \frac{11n-35}{2} - 5 > \frac{11|G-x|-35}{2}$. By the induction hypothesis and Lemma 3.1.2, $G - x > K_8^-$, a contradiction. Thus

(1) $\delta(G) \geq 6$.

(2) $\delta(N(x)) \geq 5$ for any $x \in V(G)$.

Proof. Suppose that there exists $y \in N(x)$ such that $d_{N(x)}(y) \leq 4$. Then $e(G/xy) \geq \frac{11(n-1)-34}{2} > \frac{11|G/xy|-35}{2}$. By the induction hypothesis and Lemma 3.1.2, $G - x > K_8^-$, a contradiction. ■

Let S be a minimal separating set of vertices in G, and let G_1 and G_2 be proper subgraphs of G so that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$. For $i = 1, 2$, let d_i be the largest integer so that G_i contains disjoint set of vertices $V_1, V_2, ..., V_p$ so that $G_i[V_j]$ is connected and $|S \cap V_j| = 1$, $1 \leq j \leq p = |S|$, and so that the graph obtained from G_i by contracting $V_1, V_2, ..., V_p$ and deleting $V(G) - (\cup_j V_j)$ has $e(G[S]) + d_i$ edges. Let G_1' (resp. G_2') be obtained from G_1 (resp. G_2) by adding d_2 (resp. d_1) edges to $G[S]$. By (1), $|G_i| \geq 7$, $i = 1, 2$. Hence we may assume that $e(G_1) \leq \frac{11|G_1|-35}{2} - d_2$ (otherwise $e(G_1') > \frac{11|G_1'|-35}{2}$, in which case, $G_1' > K_8^-$ by induction). Similarly, we may assume that $e(G_2) \leq \frac{11|G_2|-35}{2} - d_1$. Consequently,

(3) $\frac{11n-35}{2} \le e(G) = e(G_1) + e(G_2) - e(G[S]) \le \frac{11n+11|S|-70}{2} - d_1 - d_2 - e(G[S])$, and so

(4) $11|S| \ge 35 + 2d_1 + 2d_2 + 2e(G[S])$.

(5) G is 5-connected.

Proof. It follows from (4) that $|S| \ge 4$. Note that $d_i \ge |S| - 1 - \delta(G[S])$, $i = 1, 2$, and $2e(G[S]) \ge |S|\delta(G[S])$. By (4), we have $7|S| \ge 31 + (|S| - 4)\delta(G[S])$, which implies that $|S| \ge 5$. ■

(6) There is no minimal separating set S so that $G[S]$ is complete.

Proof. Suppose that $G[S]$ is complete. By (5), $|S| \ge 5$. If $|S| \ge 6$, by contracting $V(G_1) - S$ and $V(G_2) - S$ into two new vertices, we get $G > K_8^-$. So we may assume $|S| = 5$. Note that when $G[S] = K_5$, we get equality in (3). Thus $e(G_i) = \frac{11|G_i|-35}{2}$ for $i = 1, 2$ and $e(G) = \frac{11n-35}{2}$. It follows by induction that G is a $(K_{1,2,2,2,2}, K_7, 5)$-cockade, a contradiction. ■

(7) There is no minimal separating set S with a vertex x so that $G[S - x]$ is complete.

Proof. Suppose that $G[S - x]$ is complete. By (5), $|S| \ge 5$. By (6), we may assume $\delta(G[S]) \le |S| - 2$. Then $d_1 = d_2 = |S| - 1 - \delta(G[S])$ and $2e(G[S]) = (|S| - 1)(|S| - 2) + 2\delta(G[S])$. By (4), $11|S| \ge 35 + 4(|S| - 1 - \delta(G[S])) + (|S| - 1)(|S| - 2) + 2\delta(G[S]) = |S|^2 + |S| + 33 - 2\delta(G[S]) \ge |S|^2 + |S| + 33 - 2(|S| - 2)$. It follows that $|S|^2 - 12|S| + 37 \le 0$, which is impossible. ■

(8) $7 \le \delta(G) \le 10$.

Proof. Let $x \in V(G)$ be a vertex such that $d(x) = \delta(G)$. By (1), $d(x) \ge 6$. If $d(x) = 6$, by (2), $N(x) = K_6$. Now K_6 will be a minimal separating set, which contradicts (6). Thus $\delta(G) = d(x) \ge 7$. On the other hand, since $e(G) = \lceil \frac{11n-35}{2} \rceil$, we have $d(x) \le 10$. ■

(9) $\delta(G) \geq 8$.

Proof. Suppose that $d(x) \leq 7$. By (8), $d(x) = 7$. By (2), $\delta(N(x)) \geq 5$. Thus $N(x) = K_7 - M$, where M is a matching of $N(x)$. Let K be a component of $G - N[x]$. By (7), $N(K)$ contains two nonadjacent vertices, say a and b, in $N(x)$. Let P be an $a-b$ path with interior vertices in K. If $|M| \leq 2$, then by contracting all but one of the edges of the path P, $G > K_8^-$, a contradiction. So we may assume that $|M| = 3$, that is $N(x) = K_{1,2,2,2}$.

Let $V(N(x)) = \{y, z_1, z_2, z_3, w_1, w_2, w_3\}$ so that y is adjacent to all vertices in $N(x) - y$ and $z_i w_i \notin E(G)$. Suppose that $G - N[x]$ is disconnected. Let K and K' be two components of $G - N[x]$. Since $N(x) = K_{1,2,2,2}$, by (7), $N(K)$ and $N(K')$ contain two pairs of nonadjacent vertices of $N(x)$, respectively. We may assume that $z_1, w_1 \in N(K)$ and $z_2, w_2 \in N(K')$. Let P be a z_1-w_1 path in K and P' be a z_2-w_2 path in K'. Then by contracting all but one of the edges of P and P', respectively, we get a K_8^- minor of G, a contradiction. Hence

(9a) $G - N[x]$ is connected.

(9b) There is no vertex in $G - N[x]$ that is adjacent to a pair of nonadjacent vertices in $N(x)$.

Proof. Suppose that there exists $v \in V(G) - N[x]$ adjacent to, say z_1 and w_1. Let K be a component of $G - N[x] - v$. If $N(K)$ contains a pair of nonadjacent vertices of $\{z_2, z_3, w_2, w_3\}$, say, z_2 and w_2, then there is a z_2-w_2 path P in K. Now by contracting v to z_1 and all but one of the edges of the path P, we get a K_8^- minor of G, a contradiction. Thus by (7), we may assume $z_1, w_1 \in N(K)$. Let $K' = G - N[x] - K$. Clearly, K' is connected. If $N(K')$ contains a pair of nonadjacent vertices, other that z_1 and w_1 of $N(x)$, then G would have a K_8^- minor, a contradiction. Therefore, we may assume that $w_2, w_3 \in N(K) - N(K')$ and $z_2, z_3 \in N(K') - N(K)$. Since $w_2 z_3 \in E(G)$, w_2 and z_3 have at least one common neighbor in $G - N[x]$. It follows that $vw_2, vz_3 \in E(G)$ and thus $w_2 \in N(K')$, a contradiction. ∎

Let $v \in N(x)$ and $w \in V(G - N[x])$ be such that $v \neq y$ and $vw \in E(G)$. By (2) and (9b), v and w have at most three common neighbors in $N(x)$. Hence, (9c) for any $v \in N(x) - y$, v has at least three neighbors in $G - N[x]$.

Suppose that w is a cut-vertex of $G - N[x]$. Let K be a component of $G - N[x] - w$ and let $K' = G - N[x] - K$. Then K' is connected. Since $N(x) = K_{1,2,2,2}$, by (7), $N(K)$ and $N(K')$ contain at least one pair of nonadjacent vertices of $N(x)$, respectively. If $N(K)$ and $N(K')$ contain distinct pairs of nonadjacent vertices of $N(x)$, then G would have a K_8^- minor by the existence of such two disjoint paths in K and K', respectively. So we may assume that $z_1, w_1 \in N(K) \cap N(K')$ and $N(K)$ and $N(K')$ contain no pair of nonadjacent vertices of $N(x)$ other than z_1, w_1. Thus we may assume that $z_2, z_3 \in N(K') - N(K)$ and $w_2, w_3 \in N(K) - N(K')$. Since $w_2z_3 \in E(G)$, w_2 and z_3 have at least one common neighbor in $G - N[x]$. It follows that $ww_2, wz_3 \in E(G)$, and thus $w_2 \in N(K')$, a contradiction. Therefore

(9d) $G - N[x]$ is 2-connected.

Consider the graph $H = G - \{x, y, z_3, w_3\}$. We next show that H is 4-connected.

Let S be a minimal separating set of at most three vertices in H. By (9c) and (9d), $|S| \geq 2$ and $|S \cap N(x)| \leq 1$. If $|S \cap N(x)| = 1$, we may assume that $w_1 \in S$. Since $z_1z_2, z_1w_2 \in E(G)$, z_1, z_2, w_2 are in the same component of $H - S$. Denote this component by K. If $w_1 \notin S$, then also $w_1 \in K$, and in this case we assume that S and w_1 are chosen so that $|S \cap N(w_1)|$ is maximal. We next show that there exist z_2' and w_2' in $G - N[x] - S$ adjacent to z_2 and w_2, respectively. By (9b) and (9c), we may assume that w_2 has exactly three neighbors in $G - N[x]$, say a, b, c, and $S = \{a, b, c\}$. Clearly, $w_1 \notin S$. By the assumption that $|S \cap N(w_1)|$ is maximal, it follows that w_1 is adjacent to all vertices in S. Since $w_2z_1 \in E(G)$, by (2), z_1 and w_2 have at least one common neighbor in $G - N[x]$. Since w_2 has only three neighbors a, b, c in $G - N[x]$, we may assume $z_1a \in E(G)$. Now a is adjacent to both z_1 and w_1, which contradicts (9b). This proves that there exist $z_2', w_2' \in (V(G) - N[x] - S)$ such that $z_2z_2', w_2w_2' \in E(G)$.

Clearly, $z_2', w_2' \in K$. By (9d), $G - N[x]$ contains two independent z_2'-w_2' paths. One of these paths is contained in $G[K \cup S]$.

Since G is not contractible to $N[x] + z_2w_2 + z_3w_3$, there is no z_3-w_3 path in $G[K' \cup \{z_3, w_3\}]$, where $K' \neq K$ is another component of $H - S$. But this implies that K' is separated from x by S and two adjacent vertices in $N(x)$. We may assume that such two vertices are $\{y, w_3\}$. Since G is 5-connected, $|S| = 3$. Let $S = \{s_1, s_2, s_3\}$, where $s_1 = w_1$ if $w_1 \in S$, and $S' = S \cup \{y, w_3\}$. Then S' is a minimal separating set of G. Let $H_1 = G[K' \cup S'\}$ and $H_2 = G - K'$. Let d_1 and d_2 be defined as in the paragraph following (2). Clearly, $K \cup \{x, z_3\}$ is contained in H_2. By Menger's theorem, there exist three disjoint paths between $\{x, w_1, z_2\}$ and S in $G - \{y, w_3\}$. By contracting those paths, we get $d_2 + e_G(S') = e(K_5) = 10$. By (2), $d_1 \geq 1$. By (4), $55 = 11 \times 5 \geq 35 + 2(d_2 + e(S')) + 2d_1 = 35 + 20 + 2 = 57$, a contradiction. Thus H is 4-connected.

By (9b), $d_{G-N[x]}(z_3) + d_{G-N[x]}(w_3) \leq |G| - |N[x]| = n - 8$. Since G is not contractible to K_8^-, it follows from Theorem 1.4.1 that $e(H) \leq 3|H| - 7 = 3(n-4) - 7$. Then $\frac{11n-35}{2} \leq e(G) = d(x) + (d(y) - 1) + e(H) + e_G(\{z_3, w_3\}, \{z_1, w_1, z_2, w_2\}) + d_{G-N[x]}(z_3) + d_{G-N[x]}(w_3) \leq 7 + (n-2) + 3(n-4) - 7 + 8 + n - 8 = 5n - 14$. It follows that $n \leq 7$, which contradicts the fact that $n \geq \delta(G) + 1 \geq 8$ by (8). ■

(10) Let x be a vertex such that $8 \leq d(x) \leq 10$. Then there is no component K of $G - N[x]$ such that $N(K) = N(x)$.

Proof. Suppose such a component K exists. By Corollary 2.1.4, $N(x) > K_6^- \cup K_1$ or $N(x) > K_6$ or $N(x) \in \{\overline{K_3} + C_5, K_{2,3,3}, \overline{K_2} + \overline{C_6}\}$. In the first case, there is a vertex $y \in N(x)$ such that $N(x) - y > K_6^-$. By contracting $V(K) \cup \{y\}$ to a single vertex we see that $G > K_8^-$, a contradiction. We will use this argument repeatedly later, and we shall refer to it as "contracting K onto a free vertex of $N(x)$". If $N(x) > K_6$, then we obtain the same conclusion by contracting K to a vertex. So we may assume that $N(x) \in \{\overline{K_3} + C_5, K_{2,3,3}, \overline{K_2} + \overline{C_6}\}$. We claim that $G - N[x]$ is connected. Suppose $G - N[x]$ is disconnected. Let $K' \neq K$ be another component of $G - N[x]$. By (6), $N(K')$ is not complete. Let $a, b \in N(K')$ be such that $ab \notin E(G)$. Let P be an a-b path with interior in K'. By Corollary 2.1.4, $N(x)$ is edge maximal, and so $N[x] \cup P > K_7^- \cup K_1$. By contracting K to a free vertex of $N(x) \cup P$, we get $G > K_8^-$, a contradiction. Thus $G - N[x]$ is connected, as claimed. As $N(x) \in \{\overline{K_3}+C_5, K_{2,3,3}, \overline{K_2}+\overline{C_6}\}$, by (2), $G-N[x]$ has at least two vertices. We consider the following two cases.

Case 1. $G - N[x]$ has no cut-vertex or $G - N[x]$ has only two vertices.

Suppose $N(x) \in \{\overline{K_2} + \overline{C_6}, K_{2,3,3}\}$. By (2), there exist $x_1, x_2, y_1, y_2 \in N(x)$ such that $x_1x_2, y_1y_2 \in E(G)$, x_1 and x_2 (resp. y_1 and y_2) have at least two common neighbors in $G - N[x]$, and $x_1y_1, x_2y_2 \notin E(G)$ but $N[x] + x_1y_1 + x_2y_2 > K_8^-$. Let $u_1, u_2 \in V(K)$ be two distinct common neighbors of x_1 and x_2, and $w_1, w_2 \in V(K)$ be two distinct common neighbors of y_1 and y_2, respectively. By Menger's Theorem, K contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$. Thus G has two disjoint paths with interiors in K, one with ends x_1, y_1, and the other with end x_2, y_2. Then $G > K_8^-$ by the existence of those two paths, a contradiction.

Suppose $N(x) = \overline{K_3}+C_5$. Let $V(\overline{K_3}) = \{a_1, a_2, a_3\}$ and let $\overline{C_5}$ have vertices y_1, y_2, y_3, y_4, y_5 in order. Let $w \in V(G-N[x])$. Then $G-N[x]-w$ is connected and each vertex of $N(x)$ is adjacent to at least one vertex of $G-N[x]-w$. If w is adjacent to two vertices of a_1, a_2, a_3, say a_1, a_2, then $G > N[x]+a_1a_2+y_1y_2+y_2y_3 > K_8^-$ by contracting wa_1 and $V(G-N[x]-w)$ onto y_2, respectively. Similarly, if w is adjacent to two nonadjacent vertices of $y_1, y_2, \cdots, y_5$, say y_1, y_2, then $G > N[x]+y_1y_2+y_2y_3+y_3y_4 > K_8^-$ by contracting wy_1 and $V(G-N[x]-w)$ onto y_3, respectively. So we may assume that any pair of nonadjacent vertices of $N(x)$ have no common neighbor in $G - N[x]$. By (2), there exist $w_1, w_2, w_3, w_4 \in V(G - N[x])$ such that w_i is a common neighbor of y_1 and a_i, $i = 1, 2, 3$, and w_4 a common neighbor of y_2 and y_5. Since any pair of nonadjacent vertices of $N(x)$ have no common neighbor in $G - N[x]$, we have $w_i \neq w_j$ for $i \neq j$. As $G - N[x]$ has no cut-vertex, there exist two disjoint paths, say P_1, P_2, between $\{w_1, w_4\}$ and $\{w_2, w_3\}$ in $G-N[x]$. We may assume that P_1 is a w_1-w_3 path. Now $G > N[x] + a_1a_3 + y_1y_2 + y_1y_5 > K_8^-$ by contracting a_1w_1, y_1w_2 and all but one of the edges of each of P_1, P_2, a contradiction.

Case 2. $G - N[x]$ has a cut-vertex.

In this case, $G - N[x]$ is connected. Let w be a cut-vertex of $G - N[x]$ and let H_1 be a connected component of $G - N[x] - w$ with $N(H_1)$ minimal, and let $H_2 = G - N[x] - H_1$. Clearly, H_2 is also connected. If $N(H_1) \subseteq N(H_2)$ or $N(H_2) \subseteq N(H_1)$, say the latter. Then $N(H_1) = N(K) = N(x)$. By (6), there exists $e = ab \in E(\overline{N(H_2)})$. By Corollary 2.1.4, there exists $u \in N(x)$ such that $N(x)+e-u > K_6^-$. Then $G > K_8^-$ by contracting the a-b path in H_2 and contracting $V(H_1)$ to u. So we may assume that there exist $a \in N(H_1)-N(H_2)$ and $b \in N(H_2) - N(H_1)$. By (2), any two adjacent vertices in $N(x)$ have at least one common neighbor in $G - N[x]$. Thus $ab \notin E(G)$, $N_{N(x)}(a) \subseteq N(H_1)$ and $N_{N(x)}(b) \subseteq N(H_2)$. Suppose $N(x) \in \{\overline{K_2} + \overline{C_6}, K_{2,3,3}\}$. Since $ab \notin E(G)$, there exist $x_1, y_1 \in N_{N(x)}(a)$ and $x_2, y_2 \in N_{N(x)}(b)$ such that $x_1y_1, x_2y_2 \notin E(G)$ but $N[x]+x_1y_1+x_2y_2 > K_8^-$. Then $G > K_8^-$ by the existence of x_i-y_i path in H_i, $i = 1, 2$, a contradiction. Suppose $N(x) = \overline{K_3}+C_5$. Let $V(\overline{K_3}) = \{a_1, a_2, a_3\}$ and let $\overline{C_5}$ have vertices y_1, y_2, y_3, y_4, y_5 in order. If $a, b \in \{a_1, a_2, a_3\}$, then $y_i \in (N_{N(x)}(a) \cap N_{N(x)}(b))$ for all $i = 1, 2, \cdots, 5$. Thus $G > K_8^-$ by contracting $V(H_1)$ to y_1 and $V(H_2)$ to y_2, respectively. So we may assume that $a, b \in \{y_1, \cdots, y_5\}$, say $a = y_1$ and $b = y_2$. Clearly, $a_1, a_2, a_3, y_3, y_4 \in N(H_1)$ and $a_1, a_2, a_3, y_4, y_5 \in N(H_2)$. By (2), y_3 and y_5 have at least one common neighbor, say y, in $G - N[x]$. We may assume that $y \in V(H_1)$. Then $y_5 \in N(H_1)$ and so $G > K_8^-$ by contracting $V(H_1)$ to y_4 and $V(H_2)$ to a_1, respectively, a contradiction. ∎

(11) Let x be a vertex such that $8 \leq d(x) \leq 10$. Then there is no component K of $G - N[x]$ such that $N(K') \subseteq N(K)$ for every component K' of $G - N[x]$.

Proof. Suppose such a component K exists. Among all vertices x with $8 \leq d(x) \leq 10$ for which such a component exists, choose x to be of minimal degree. By (10), $N(K) \neq N(x)$. Let $y \in N(x) - N(K)$ be of smallest degree. Then $N(y) \subseteq N[x]$. Note that $d(y) \leq d(x) \leq d(y)+2$. Suppose $d(x) = d(y)$. Then each vertex of $N(x)$ is either adjacent to all vertices in $N[x]$ or contained in $N(K)$, and $d_{N(x)}(y) = |N(x)|-1$. By Corollary 2.1.4, $N(x) > K_6^- \cup K_1$. By contracting $N(K)$ to a free vertex of $N(x)$, we obtain $G > K_8^-$, a contradiction. Next, suppose $d(x) = d(y) + 1$. Let $\{z\} = N(x) - N[y]$. Then $z \notin N(K)$, for otherwise we would have chosen y for x. By the choice of y, $d(z) = d(x)-1$. Thus $\{z\}$ is a component of $G-N[y]$ such that $N(\{z\}) = N(y)$, which contradicts (10). Finally, suppose $d(x) = d(y) + 2$. Then

$d(x) = 10$. Let $\{z, w\} = N(x) - N[y]$. Clearly, z and w are not both in $N(K)$, otherwise we would have chosen y for x. So we may assume that $z \notin N(K)$. If $zw \notin E(G)$, then $\{z\}$ is a component of $G - N[y]$ such that z is adjacent to all the vertices in $N(y)$, which contradicts (10). So we may assume $zw \in E(G)$, and thus $w \notin N(K)$ (otherwise we would have chosen y for x, because $K \cup \{z, w\}$ is a component in $G - N[y]$ satisfying (11)). By the choice of y, $d(z), d(w) \geq d(y)$. Now $e(N(x)) \geq (d(y) - 1) + (d(z) - 2) + (d(w) - 2) + 1 + \frac{4|N(x) \cap N(y)|}{2} \geq 3d(y) - 4 + 2(d(y) - 1) = 5d(y) - 6 = 5(d(x) - 2) - 6 = 5d(x) - 16 > \frac{9|N(x)|}{2} - 12$. By Theorem 1, $N(x) > K_7^-$ and so $G > N[x] > K_8^-$, a contradiction. ■

It follows from (11) that

(12) $G - N[x]$ is disconnected.

(13) Let x be a vertex such that $8 \leq d(x) \leq 10$. Then there is no component K of $G - N[x]$ with one vertex w so that $d_G(y) \geq 11$ for every vertex $y \neq w$ in K and $d_G(w) \geq d_G(x)$.

Proof. Assume that such a component K exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let d_1 be defined as in the paragraph following (2). Let G_2' be a graph with $V(G_2') = V(G_2)$ and $e(G_2') = e(G_2) + d_1$ edges obtained by contracting edges in G_1. By (9), $|G_2'| \geq 9$. If $e(G_2') > \frac{11|G_2'| - 35}{2}$, then $G > G_2' > K_8^-$ by induction, a contradiction. Thus $e(G_2) = e(G_2') - d_1 \leq \frac{11|G_2| - 35}{2} - d_1 = \frac{11|N(K)| + 11|K| - 35}{2} - d_1$. On the other hand, for any $u \in N(K)$, there exists $w \in K$ such that $uw \in E(G)$. By (2), $d_{G_2}(u) \geq 6$. Thus $e(G_2) \geq \frac{1}{2}(6 \times |N(K)| + 11(|K| - 1) + d_G(w)) \geq \frac{6|N(K)| + 11|K| - 11 + d(x)}{2}$. It follows that

(**13a**) $5|N(K)| \geq 24 + d(x) + 2d_1$ and so $|N(K)| \geq 7$ by (9).

Let $t = e_G(N(K), K)$ and $d = \delta(N(K))$. Then $e(G_2) = e(G[K]) + t + e(N(K)) \geq \frac{11(|K|-1) + d_G(w) - t}{2} + t + \frac{|N(K)| \times d}{2} \geq \frac{11|K| - 11 + d(x) + t + |N(K)| \times d}{2}$. It follows that

(**13b**) $\frac{-t + d(x)}{2} \geq d_1 + d(x) + 12 + \frac{d|N(K)| - 11|N(K)|}{2} \geq (|N(K)| - 1 - d) + d(x) + 12 + \frac{d|N(K)| - 11|N(K)|}{2} = 11 + d(x) + \frac{d(|N(K)| - 2)}{2} - \frac{9|N(K)|}{2}$.

Note that $t \geq \sum_{v \in K} d_G(v) - 2e(G[K]) \geq 11(|K| - 1) + d(w) - |K|(|K| - 1) \geq -|K|^2 + 12|K| + d(x) - 11$. If $t \leq d(x) + s$, then

(**13c**) $|K|^2 - 12|K| + 11 + s \geq 0$.

By (10), $N(K) \neq N(x)$. This, together with (13a), implies that $7 \leq |N(K)| \leq 9$. Thus $|K| \geq (\Delta(G_2)+1) - |N(K)| \geq (11+1) - 9 = 3$. We next show that $t \leq d(x) + s$, where $s = 14$.

By (2), $d \geq 5 - (|N(x)| - |N(K)|)$. If $|N(K)| = 7$, by (6) and (13a), we have $d_1 \geq 1$ and $d(x) + 2d_1 \leq 11$. Thus $d(x) \leq 9$ and $d \geq 5 - (9-7) = 3$. By (13b), $\frac{-t+d(x)}{2} \geq 1 + d(x) + 12 + \frac{3|N(K)| - 11|N(K)|}{2} \geq -7$. If $|N(K)| = 8$, then $d(x) \leq 10$ and $d \geq 5 - (10-8) = 3$. By (13b), $\frac{-t+d(x)}{2} \geq 11 + d(x) + \frac{d(|N(K)|-2)}{2} - \frac{9|N(K)|}{2} \geq 11 + d(x) + \frac{3\times(8-2)}{2} - \frac{9\times 8}{2} \geq -7$. If $|N(K)| = 9$, then $d(x) = 10$ and $d \geq 5 - (10-9) = 4$. By (13b), $\frac{-t+d(x)}{2} \geq 11 + d(x) + \frac{d(|N(K)|-2)}{2} - \frac{9|N(K)|}{2} \geq 11 + d(x) + \frac{4\times(9-2)}{2} - \frac{9\times 9}{2} > -7$. In all cases, we have $t \leq d(x) + 14$ and $s = 14$.

Since $s = 14$ and $|K| \geq 3$, by 13(c), $|K| > 8$. Note that $e(G[K]) \geq \frac{11(|K|-1)+d(w)-t}{2} \geq \frac{11(|K|-1)}{2} + \frac{-t+d(x)}{2} \geq \frac{11|K|-25}{2}$. It follows that $G[K] > K_8^-$ by induction, a contradiction. ■

By (9), G has a vertex of degree $8, 9$ or 10. Among the vertices of degree $8, 9$ or 10 for which the order of the largest component of $G - N[x]$ is maximum, choose x so that its degree is minimum. Let K be a largest component of $G - N[x]$.

By (12), there is another component K' of $G - N[x]$. By (13), there is a vertex x' in K' of degree $d_G(x') \leq 10$. By the maximality of the order of K, $N(K) \subseteq N(x') \cap N(x)$. Thus $N(K) \subseteq N(K')$ and K is also a component of $G - N[x']$. By the choice of x, $d(x') \geq d(x)$. By (13), there exists another vertex $y' \neq x'$ in K' of degree $d(x) \leq d(y') \leq 10$. Clearly, y' is adjacent to every vertex in $N(K)$. By (11), There is a third component K'' of $G - N[x]$. By symmetry, K'' has two vertices x'', y'' of degree at most 10 in G and $N(K) \subseteq N(x'') \cap N(y'')$. Let $G_1 = G - K$, $G_2 = G[N(K) \cup K]$ and let d_1 and d_2 be as in the paragraph following (2).

Since $\delta(N(x)) \geq 5$, $\delta(N(K)) \geq 5 - (10 - |N(K)|) = |N(K)| - 5$. Therefore there is a subgraph T of $N(K)$ with $|N(K)| - 5$ vertices and at least $|N(K)| - 6$ edges. Contract the vertices in $N(K)-T$ with different vertices in $\{x, x', y', x'', y''\}$, which are adjacent to every vertex in $N(K)$. It is easy to see that

$$d_1 + e(N(K)) \geq e(K_5) + 5(|N(K)| - 5) + (|N(K)| - 6) = 6|N(K)| - 21.$$

By (4), $d_1 + e(N(K)) \leq \frac{11|N(K)|-35-2d_2}{2}$. It follows that $d_2 = 1$ and $|N(K)| = 5$. However, when $|N(K)| = 5$, then $d_1 + e(N(K)) \geq e(K_5) + 5(|N(K)| - 5) = 5|N(K)| - 15 = 10$. By (4), $55 = 11|N(K)| \geq 35 + 2(d_1 + e(N(K)) + 2d_2 \geq 35 + 20 + 2 = 57$, which is impossible. This completes the proof of Theorem 1.3.5. ■

CHAPTER IV

THE EXTREMAL FUNCTION FOR $K_7 \cup K_1$ MINORS

4.1 Introduction

Mader [25] proved that every simple graph on $n \geq 7$ vertices and at least $5n - 14$ edges has a K_7 minor (see Theorem 1.2.2). In this chapter, we will prove that every edge-maximal graph on $n \geq 8$ vertices without a $K_7 \cup K_1$ minor has at most $5n - 15$ edges, or at most 31 edges when $n = 9$, or is isomorphic to one of the graphs listed in Theorem 1.3.2 in Chapter 1. Theorem 1.3.2 generalizes Theorem 1.2.2 for $p = 7$, and extends a result of Jørgensen (namely Lemma 4.1.1). Theorem 1.3.2 will be applied in the the computer-free proof of Lemma 4.3.3 for graphs on at most 11 vertices and should be helpful for a possible proof of Conjecture 6.5.1.

We need one more definition. By a $(K_{p-1}, K_{p-5} + MP, p-2)$-cockade we mean any graph that can be obtained as in the definition of a cockade, starting from K_{p-1} and graphs of the form $K_{p-5} + H$, where H is a 4-connected maximal planar graph, by identifying cliques of size $p - 2$. A $(K_{p-5} + MP, p-2)$-cockade is defined analogously.

In the proof of Theorem 1.3.2, we shall need the following two results of Jørgensen [15, 16].

Lemma 4.1.1 *Let G be a graph with $n \geq 7$ vertices and*

$$e(G) \geq \begin{cases} 4n - 9 & if \quad n \neq 8 \\ 24 & if \quad n = 8 \end{cases}$$

Then either $G > K_6 \cup K_1$, or G is isomorphic to $K_3 + C_{n-3}$ when $n \neq 8$ or $K_{2,2,2,2}$ or $K_{3,3,3}$.

Lemma 4.1.2 *For any integer p, where $5 \leq p \leq 7$, every graph G with n vertices and $(p-2)n - \binom{p-1}{2}$ edges has a K_p minor or is isomorphic to $K_{2,2,2,3}$ or is a $(K_{p-1}, K_{p-5} + MP, p-2)$-cockade.*

4.2 Proof of Theorem 1.3.2

We first prove that $\Delta(G) \leq n-2$. Suppose there exists a vertex $x \in V(G)$ such that $d(x) = n-1$. Then $e(G-x) \geq 4(n-1) - 9$ if $n \neq 9$, and $e(G-x) = 24$ if $n = 9$. By Lemma 4.1.1, $G - x > K_6 \cup K_1$ or $G - x$ is isomorphic to $K_{2,2,2,2}$, or $K_{3,3,3}$ or $K_3 + C_{n-4}$ when $n \neq 9$. Thus $G > K_7 \cup K_1$ or G is isomorphic to $K_{1,2,2,2,2}$, or $K_{1,3,3,3}$ or $K_4 + C_{n-4}$ when $n \neq 9$. Hence we may assume that

(1) $\Delta(G) \leq n-2$.

As $e(G) \geq 26$ when $n = 8$ and $e(G) \geq 32$ when $n = 9$, it follows from (1) that $n \geq 10$.

(2) For every vertex $x \in V(G)$, either $d(x) \geq 7$, or $d(x) = 6$ and $G-x$ is a (K_6, K_2+MP, 5)-cockade.

Proof. Suppose there exists $x \in V(G)$ such that $d(x) \leq 5$. Then $e(G-x) \geq 5(n-1) - 14$. By Theorem 1.2.2, $G - x > K_7$. So we may assume that $d(x) \geq 6$. Suppose $d(x) = 6$. Then $e(G-x) \geq 5(n-1) - 15$. By Lemma 4.1.2, $G - x > K_7$ or $G - x$ is a $K_{2,2,2,3}$ or $G - x$ is a (K_6, $K_2 + MP$, 5)-cockade. In the first case, we have $G > K_7 \cup K_1$. Suppose $G-x = K_{2,2,2,3}$. Then $G = K_{2,2,2,4}$ if x is not adjacent to any vertex of degree six in $G-x$. So we may assume that x is adjacent to at least one vertex of degree six in $G-x$. If x is adjacent to a K_4 subgraph in $G-x$, then it is easy to check that $G > K_7 \cup K_1$. So we may assume that there exist a pair of nonadjacent vertices, say y, z, such that $xy, xz \notin E(G)$. As we have proved that x is adjacent to at least one vertex of degree six in G, there are only two ways to join x to the vertices of $K_{2,2,2,3}$, as depicted in Figure 8, where only the complement of G is shown by dotted line. One can easily verify that $G = K_{2,2} + K_{3,3}^-$ or $G = K_{2,3} + K_{2,3}^-$, and so the theorem holds. ■

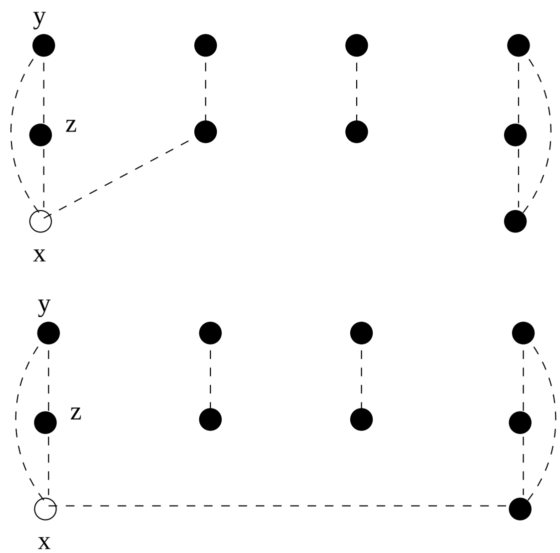


Figure 8: Two possible ways to join vertex x to $K_{2,2,2,3} - \{y, z\}$

(3) For $10 \leq n \leq 11$, if $\delta(N(x)) \geq 4$ for some $x \in V(G)$, then $d(x) \geq 7$.

Proof. Suppose there exists $x \in V(G)$ with $d(x) \leq 6$. By (2), we may assume that $d(x) = 6$. If $N(x) = K_6$, then $N[x] = K_7$ and so $G > K_7 \cup K_1$. So we may assume that $N(x) \neq K_6$. Combining this with the assumption that $\delta(N(x)) \geq 4$, we have $\delta(N(x)) = 4$. Thus $N(x) = K_6 - M$, where M is a matching of K_6. If $N(x) = K_6^-$, let $a, b \in N(x)$ be such that $ab \notin E(G)$. Suppose there is no a-b path with interior in $G - N[x]$. As $\delta(G) \geq 6$, we have $n = 11$ and $G - N[x]$ has exactly two components, say H_1 and H_2, each of size two. We may assume that $a \notin N(H_1)$. Now $G[V(H_1) \cup (N(x) - \{a\})] = K_7$ and so $G > K_7 \cup K_1$. So we may assume that there exists an a-b path with interior in $G - N[x]$. Let P be a shortest a-b path with interior in $G - N[x]$. If P has at most $|V(G) - N[x]| - 1$ interior vertices, then by contracting all but one of the edges of the path P, we see that $G > K_7 \cup K_1$. So we may assume that $G - N[x] = P - \{a, b\}$. As P is the shortest a-b path with interior in $G - N[x]$, it follows that each vertex in $V(G) - N[x]$ is adjacent to all vertices of degree five in $N(x)$. Hence $\Delta(G) = n-1$, contrary to (1). So we may assume that $N(x) = K_6 - M$ for any vertex x of degree six in G, where M is a matching of size at least 2.

Since $\delta(G) = 6$, by (2), $G-x$ is a (K_6, K_2+MP, 5)-cockade for any vertex x of degree six. If K_6 is a member of this cockade, then there exists a vertex y of degree 6 in G such that K_5 is a subgraph of $N_G(y)$, a contradiction. Thus $G-x$ is a (K_2+MP, 5)-cockade. Since every 4-connected maximal planar graph has at least six vertices, we have $G-x = K_2+H$, where H is a 4-connected maximal planar graph. Since $\Delta(G) \leq n-2$, we see that $N(x) \subset V(H)$. Let y, z be the two vertices in $(K_2 + H) - H$. Note that $N(x) = K_6 - M$, where M is a matching of size at least two in K_6. If $|M| = 2$, let z_1z_2, w_1w_2 be the two missing edges in $N(x)$. By contracting the edges zz_1, yw_1, we see that $N(x) + z_1z_2 + w_1w_2 > K_6$ and so $G - (V(H) - N(x)) > K_7$. Thus we may assume that $N(x) = K_{2,2,2}$ for any x of degree six in G. Since H is a 4-connected maximal planar graph with a $K_{2,2,2}$ subgraph, it follows that H is isomorphic to $K_{2,2,2}$. Thus $|G| \leq 9$, a contradiction. ■

(4) $n \geq 11$.

Proof. Suppose $n = 10$. Then $e(G) \geq 36$. We claim that $\delta(N(x)) \geq 4$ for any $x \in V(G)$.

Suppose there exists $y \in N(x)$ such that x and y have at most three common neighbors. Then $e(G/xy) \geq 32$ and $|G/xy| = 9$. Similarly to the case when $n = 9$, we have $G/xy > K_7 \cup K_1$ or $G/xy = K_{1,2,2,2,2}$. In the first case, $G > G/xy > K_7 \cup K_1$. Suppose $G/xy = K_{1,2,2,2,2}$. Then x and y have exactly three common neighbors, and $d(x) + d(y) \leq 13$. By (2), $d(x), d(y) \geq 6$. We may assume that $d(x) = 6$. Then $6 \leq d(y) \leq 7$. Let $V(G/xy) = \{u, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4\}$ so that $z_iw_i \notin E(G)$ for $i = 1, 2, 3, 4$, as shown in Figure 9.

Let w be the new vertex in G/xy. Suppose w is of degree seven. We may assume that $w = z_1$. Then $d(y) = 6$. Since x and y have three common neighbors, we may assume that w_2 is a common neighbor of x and y. As z_2 is adjacent to either x or y, say the latter, then $G/yz_2/z_3z_4 - x = K_7$. Suppose $w = u$. In this case, we see that $d(y) = 7$. Suppose y is adjacent to all z_i's. Since $d(y) = 7$, we may assume that $yw_1, yw_2 \in E(G)$. Then

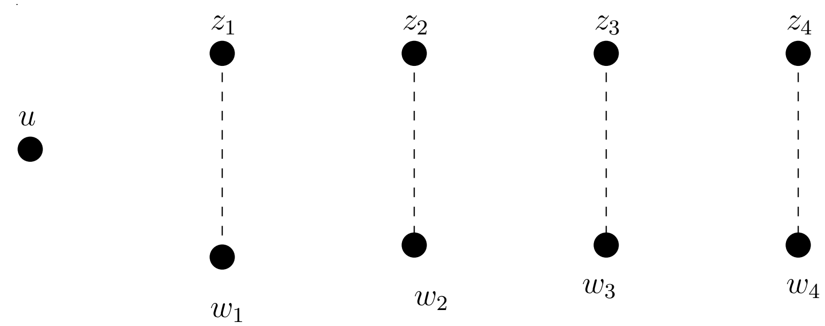


Figure 9: The complement of $K_{1,2,2,2,2}$

$G/w_1w_3/w_2w_4 - x = K_7$. So by swapping z_i with w_i, we may assume that y is adjacent to $z_1, z_2, z_3, w_1, w_2, w_3$. Thus $xz_4, xw_4 \in E(G)$. Since x and y have exactly three common neighbors, we may assume that two of those three common neighbors are z_1, z_2. If z_3 is also a common neighbor, then $G/w_1z_4/w_2w_4 - w_3 = K_7$. So we may assume that w_1 is the third common neighbor. Then $G = K_{2,3} + K_{2,3}^-$. Thus $\delta(N(x)) \geq 4$, as Claimed.

Since $\delta(N(x)) \geq 4$ for any $x \in V(G)$ as we claimed, by (3), $\delta(G) \geq 7$. Note that $e(G) \geq 36$ and $\Delta(G) \leq 8$ by (1). If $e(G) = 36$, then there exist two vertices $x, y \in V(G)$ such that $d(x) = d(y) = 8$. If $xy \notin E(G)$, then $G - x - y$ is 5-regular on 8 vertices. Thus $G - x - y$ is isomorphic to $\overline{K_3} + C_5$ or $\overline{C_4} + \overline{C_4}$ or $\overline{C_8}$. Hence $G = K_{2,3} + C_5$ or $G > K_7 \cup K_1$. If $xy \in E(G)$, then $G - xy$ is 7-regular on 10 vertices. Thus $G - xy$ is isomorphic to $\overline{K_3} + \overline{C_7}$, or $K_{3,3} + \overline{C_4}$ or $\overline{C_4} + \overline{C_6}$ or $\overline{C_5} + \overline{C_5}$. Note that all these graphs except $K_{3,3} + \overline{C_4}$ are edge maximal subject to not having a $K_7 \cup K_1$ minor. It follows that $G > K_7 \cup K_1$ or $G = K_{3,3} + P_4$. If $e(G) \geq 37$, then there exist $uw \in E(G)$ such that $d(u) = d(w) = 8$ because $\Delta(G) \leq 8$. Then $e(G - uw) \geq 36$ and $\delta(G - uw) \geq 7$. Similarly to the case when $e(G) = 36$, we have $G - uw > K_7 \cup K_1$ or $G - uw = K_{3,3} + P_4$. Thus $G > K_7 \cup K_1$ or $G = K_{2,2,3,3}$, and so the theorem holds. ■

By (4) and (2), we have $n \geq 11$ and $\delta(G) \geq 6$. We now proceed the proof by induction on n. Suppose G does not contain a $K_7 \cup K_1$ minor. We may assume that $e(G) = 5n - 14$. Suppose $n = 11$. We claim that $\delta(N(x)) \geq 4$ for any $x \in V(G)$. Suppose there exist $xy \in E(G)$ such that x and y have at most three common neighbors, then $e(G/xy) \geq 5(n-1) - 13$.

Similarly to the case when $n = 10$, we see that $G/xy > K_7 \cup K_1$ or $G/xy = K_{2,2,3,3}$. In the first case, we have $G > K_7 \cup K_1$, a contradiction. So we may assume that $G/xy = K_{2,2,3,3}$. Let w be the new vertex in G/xy. Note that $G/xy - w$ is edge maximal subject to not having a K_7 minor, and $G/xy - w$ have three independent vertices, say a, b, c, each of which is adjacent to either x or y, we may assume that $xa, xb \in E(G)$. Then $G/xa - y > K_7$, a contradiction. Thus $\delta(N(x)) \geq 4$ for any $x \in V(G)$, as claimed.

Since $\delta(N(x)) \geq 4$ for any $x \in V(G)$, by (3), $\delta(G) \geq 7$. If $\delta(G) \geq 8$, then $4n \leq e(G) = 5n - 14$. It follows that $n \geq 14$, a contradiction. Thus $\delta(G) = 7$. Let x be a vertex of degree seven in G. Suppose $\delta(N(x)) \geq 5$. We may assume that $N(x) = K_{1,2,2,2}$, otherwise $N[x] > K_7$, a contradiction. Since $n = 11$ and $\delta(G) = 7$, every vertex in $G - N[x]$ is adjacent to a pair of non-adjacent vertices in $N(x)$. It can be easily checked that $G > K_7 \cup K_1$, a contradiction. Thus we may assume that $\delta(N(x)) = 4$. Let $y \in N(x)$ be such that x and y have exactly four common neighbors. Then $e(G/xy) = 5(11 - 1) - 14$. Note that $e(G) = 41$, $\Delta(G) \leq n - 2 = 9$ by (1), and $\delta(G) = 7$. If $\Delta(G) = 9$, then G has either exactly two vertices of degree nine, one of degree eight, eight of degree seven, or exactly one vertex of degree nine, three of degree eight, and seven of degree seven. If $\Delta(G) = 8$, we see that G has exactly five vertices of degree eight and six vertices of degree seven. In either case, G/xy has at least t vertices of degree six and $4 - t$ vertices of degree seven, where $0 \leq t \leq 4$. Moreover, if $t \leq 2$, G/xy has at most $1 + t$ vertices of degree at least eight and if $t = 4$, G/xy has at least one vertex of degree at least eight. Thus G/xy is not isomorphic to any one of the graphs in $\{K_{1,3,3,3}, K_{2,2,2,4}, K_{2,3} + K_{2,3}^-, K_{2,2} + K_{3,3}^-, K_{2,3} + C_5, K_{3,3} + P_4\}$. Similarly to the case when $n = 10$, we see that $G/xy > K_7 \cup K_1$, a contradiction.

It follows that

(5) $n \geq 12$.

(6) $\delta(N(x)) \geq 5$ for any $x \in V(G)$.

Proof. Suppose there exists $y \in N(x)$ such that x and y have at most four common neighbors. Then $e(G/xy) \geq 5(n-1) - 14$ and $|G/xy| \geq 11$ by (5). Hence $G > K_7 \cup K_1$ by induction, a contradiction. ■

(7) $\delta(G) \geq 7$.

Proof. Suppose there exists a vertex $x \in V(G)$ such that $d(x) \leq 6$. Then by (6), $N[x] = K_7$ and thus $G > K_7 \cup K_1$, a contradiction. ■

Since $e(G) = 5n - 14$, we have $\delta(G) \leq 9$. Thus by (7)

(8) $7 \leq \delta(G) \leq 9$.

Let S be a minimal separating set in G. Let G_i $(i = 1, 2)$ be subgraphs in G such that $G_1 \cap G_2 = G[S]$ and $V(G_1) \cup V(G_2) = V(G)$. Then $e(G_i) \leq 5|G_i| - 15$ for $i = 1, 2$, otherwise if $e(G_i) \geq 5|G_i| - 14$, then by Theorem 1.2.2, $G_i - (G_{3-i} - S) > K_7$, a contradiction. Let $|S| = s$.

(9) $G[S]$ is not complete.

Proof. Suppose $G[S]$ is complete. Since $e(G_i) \leq 5|G_i| - 15$ for $i = 1, 2$, we have $5n - 14 = e(G) = e(G_1) + e(G_2) - e(G[S]) \leq 5(n + s) - 15 - 15 - \frac{1}{2}s(s-1)$. It follows that $s^2 - 11s + 32 \leq 0$, which is impossible. ■

(10) $s \geq 4$ and if $s = 4$, then $e(G[S]) \leq 4$.

Proof. Let G_1 and G_2 be as defined prior to (9). Since $e(G_i) \leq 5|G_i| - 15$ for $i = 1, 2$, we have $5n - 14 = e(G_1) + e(G_2) - e(G[S]) \leq 5(n + s) - 15 - 15 - e(G[S])$. It follows that $5s \geq 16 + e(G[S])$. Thus $s \geq 4$ and if $s = 4$, then $e(G[S]) \leq 4$, as desired. ■

(11) $8 \leq \delta(G) \leq 9$.

Proof. By (8), we may assume that there exists a vertex x such that $d(x) = 7$. By (6), $\delta(N(x)) \geq 5$. Since $G \not> K_7 \cup K_1$, we may assume that $N(x) = K_{1,2,2,2}$. If $G - N[x]$

contains two components, say H_1 and H_2, by (9), $N(H_1)$ is not complete. Let $a, b \in N(H_1)$ be non-adjacent. Clearly, $N(x)+ab > K_7$. By contracting $V(H_1)\cup\{a\}$ into a single vertex, we see that $G - H_2 > K_7$, a contradiction. Thus $G - N[x]$ is connected. Let P be an a-b path with interior in $G - N[x]$. We may choose a and b so that $|P|$ is minimum. Since $G \neq K_{1,2,2,2,2}$, $|G - N[x]| \geq 2$. If $G - N[x] = P - \{a, b\}$, then there would exist a vertex on P which is adjacent to a pair of non-adjacent vertices in $N(x)$, contrary to the choice of P. Thus there exists $w \in V(G - N[x]) - V(P)$. By contracting all but one of the edges of P, we see that $N[x] + ab > K_7$. Hence $G - w > K_7$, a contradiction. ∎

(12) $G - N[x]$ is disconnected for any vertex x of degree 8 or 9, and there exist $a, b \in N(x)$ such that $ab \notin E(G)$ and there is no a-b path with interior in $G - N[x]$

Proof. Suppose $G - N[x]$ is connected or every pair (a, b) has an a-b path with interior in $G - N[x]$. By (6), $\delta(N(x)) \geq 5$. By Corollary 2.1.4, $N(x) > K_6^- \cup K_1$ or $N(x) > K_6$ or $N(x) \in \{\overline{K_3} + C_5,\ \overline{K_2} + \overline{C_6},\ K_{2,3,3}\}$. In the first case, let $a, b \in N(x)$ so that $N(x) + ab > K_6 \cup K_1$. Let P be an a-b path with interior in $G - N[x]$ (we know such a path exists by assumption). Then by contracting all but one of the edges of the path P, we see that $G > N[x] + ab > K_7 \cup K_1$, a contradiction. If $N(x) > K_6$, then $N[x] > K_7$, again we obtain a contradiction. Thus $N(x) \in \{\overline{K_3}+C_5, \overline{K_2}+\overline{C_6}, K_{2,3,3}\}$. Note that $\overline{K_2}+\overline{C_6}$ and $K_{2,3,3}$ are edge maximal subject to not having a K_6 minor and $\overline{K_3}+C_5+cd > K_6$ for any $cd \notin E(C_5)$. Let u, w be two non-adjacent vertices in $N(x)$ such that $N(x) + uw > K_6$. Let P be a u-w path with interior in $G - N[x]$ (again such a path exists by assumption). We may assume that P is the shortest path among all such pairs (u, w). Then $V(G) - N[x] - V(P) \neq \emptyset$, otherwise, let u' be the unique neighbor of u on P. By (8), u' has at least six neighbors in $N(x)$. Thus u' is adjacent to a pair of non-adjacent vertices in $\overline{K_2} + \overline{C_6}$ and $K_{2,3,3}$, and a pair of non-adjacent vertices of C_5 in $\overline{K_3} + C_5$. In either case, it is contrary to the choice of P. Now by contracting all but one of the edges of the path P, we see that $G > K_7 \cup K_1$, a contradiction. ∎

Let x be a vertex of minimum degree in G. By (11), $8 \leq d(x) \leq 9$. By (12), $G - N[x]$

is disconnected. Let H_1 and H_2 be two distinct connected components in $G - N[x]$.

(13) $e(N(x)) \leq 4|N(x)| - 11$.

Proof. Suppose $e(N(x)) \geq 4|N(x)| - 10$. Let y be a vertex of minimum degree in $N(H_1)$. and let $N'(x) = N(x) + \{yw : w \in N(H_1) - N(y)\}$. By (9), $N(H_1)$ is not complete. Now by contracting $V(H_1) \cup \{y\}$ into a single vertex, we see that $e(N'(x)) \geq 4|N(x)| - 9$. By Theorem 1.2.2, $N'(x) > K_6$ and so $G[N[x] \cup N(H_1)] - H_2 > K_7$, a contradiction. ■

(14) $\delta(G) = 9$.

Proof. Suppose $d(x) = 8$. Since $\delta(N(x)) \geq 5$ by (6), we have $e(N(x)) \geq 20$. By (13), we may assume that $20 \leq e(N(x)) \leq 21$. Suppose $e(N(x)) = 20$. Then $N(x)$ is 5-regular on 8 vertices. Thus $N(x) = \overline{K_3} + C_5$ or $\overline{C_4} + \overline{C_4}$, or $\overline{C_8}$. In the latter two cases, $N(x) > K_6$ and so $N[x] > K_7$, a contradiction. So we may assume that $N(x) = \overline{K_3} + C_5$.

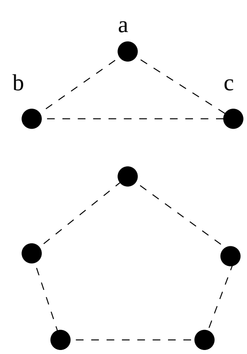


Figure 10: The complement of $\overline{K_3} + C_5$

Let $\{a, b, c\}$ be the vertex set of $\overline{K_3}$ in $\overline{K_3} + C_5$, as shown in Figure 10. If $N(H_1)$ contains a pair of non-adjacent vertices of C_5, say u and w, then $N[x] + uw > K_7$, and so $G - H_2 > K_7$, a contradiction. Thus $N(H_1)$ contains only one or two adjacent vertices of C_5. By symmetry, $N(H_2)$ also contains only one or two adjacent vertices of C_5. Then $G - N[x]$ has a third component, say H_3. By (10), $|N(H_i)| \geq 5$, for $i = 1, 2, 3$. Thus $N(H_1)$ and $N(H_2)$ must contain vertices a, b, and c. By contracting $V(H_1) \cup \{a\}$ and $V(H_2) \cup \{b\}$

into single vertices, respectively, we see that $N[x]+ab+ac+ca > K_7$, and so $G-H_3 > K_7$, a contradiction. Thus we may assume that $e(N(x)) = 21$. Let $z \in N(x)$ be of maximum degree. If $d_{N(x)}(z) = 7$, then $N(x)-z$ is 4-regular on seven vertices. Thus $N(x)-z = \overline{C_7}$ or $\overline{K_3} + \overline{C_4}$. In either case, $N[x] > K_7$, a contradiction. Thus $d_{N(x)}(z) = 6$. As $e(N(x) = 21$, there is another vertex $z' \neq z$ such that z' is of degree six in $N(x)$. If $zz' \in E(G)$, then $N(x) - zz'$ is 5-regular on eight vertices. Similarly to the case when $e(N(x)) = 20$, we see that $N(x)$ is edge maximal subject to not having a K_6 minor. By contracting H_1 onto a vertex of minimum degree in $N(H_1)$. We get $G - H_2 > K_7$, a contradiction. Thus $zz' \notin E(G)$. Then $N(x) - z - z'$ is 3-regular on six vertices. Thus $N(x) = K_{2,3,3}$ or $\overline{K_2} + \overline{C_6}$. Note that those two graphs are edge maximal subject to not having a K_6 minor. Now by contracting $V(H_1)$ onto a vertex of minimum degree in $N(H_1)$ (because $N(H_1)$ is not complete by (9)), we see that $G - H_2 > K_7$, a contradiction. ■

Let x be a vertex of minimum degree in G. From (8) and (13), we see that $d(x) = 9$ and $\delta(N(x)) \geq 5$. Thus $e(N(x)) \geq 23$. By (14), we have $23 \leq e(N(x)) \leq 25$.

(15) There is no component K of $G - N[x]$ such that K has at most two vertices of degree 9.

Proof. Suppose such a component K exists. We may assume that $K \neq H_1$. Let $G_1 = G[K \cup N(K)]$. Then by (6), $d_{G_1}(v) \geq 6$ for any $v \in N(K)$. Thus $e(G_1) \geq 5(|K|-2)+9+3|N(K)|$. On the other hand, let G_1' be obtained from G_1 by contracting x onto a vertex of minimum degree in $N(K)$. Let $d = \delta(N(K))$. Then $e(G_1') = e(G_1) + (|N(K) - 1 - d)$. We may assume that $e(G_1') \leq 5(|K|+|N(K)|)-15$, otherwise by Theorem 1.2.2, $G_1' > K_7$, and so $G-H_1 > K_7 \cup K_1$, a contradiction. Thus $e(G_1) \leq 5(|K|+|N(K)|)-15-(|N(K)|-1-d)$. The two inequalities of $e(G_1)$ imply that $d + |N(K)| \geq 13$. By (9), $d \leq |N(K)| - 2$. It follows that $|N(K)| \geq 8$. If $|N(K)| = 8$, then $d \geq 13 - |N(K)| = 5$. Let $N'(x)$ be obtained from $N(x)$ by contracting K onto a vertex of minimum degree in $N(K)$. Then the edge-set of $N'(x)$ consists of edges incident with the vertex in $N(x) - N(K)$ and the edges in $N(K)$ in $N'(x)$. Thus

$$e(N'(x)) \geq 5 + e(N(K)) + (7-d) \geq 5 + 4d + 7 - d = 12 + 3d \geq 27.$$

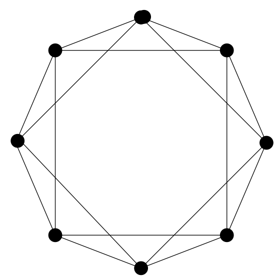
By Theorem 1.2.2, $N'(x) > K_6$ and thus $G - H_1 > K_7$, a contradiction. Thus $N(K) = N(x)$, and so for any $uw \notin N(x)$, there is a u-w path with interior in K, which contradicts (12). ∎

Among the vertices of degree 9, choose x so that the order of the largest component of $G - N[x]$ is maximum. Let K be a largest component of $G - N[x]$. By (12), $G - N[x]$ is disconnected. Let $K' \neq K$ be a connected component of $G - N[x]$. By (15), K' has at least three vertices, say x_1, y_1, z_1, of degree nine. By the choice of x, each of x_1, y_1, z_1 is adjacent to every vertex of $N(K)$. Thus $N(K) \subset N(H_1)$. By (12), there is a third component, say K'' of $G - N[x]$. By symmetry, K'' has at least three vertices, say x_2, y_2, z_2, of degree nine, each of which is adjacent to every vertex of $N(K)$, and $N(K) \subset N(K'')$. By (12), We may assume that there exist $a \in N(K') - N(K'')$ and $b \in N(K'') - N(K)$. By (10), $|N(K)| \geq 4$. If $|N(K)| \geq 5$, then by contracting the edge ax_1, and five independent edges, each with one end in $N(K)$ and the other end in $\{y_1, z_1, x_2, y_2, z_2\}$, we see that $G - K > K_7$, a contradiction. Thus $|N(K)| = 4$. If there exists an a-b path with interior in $N(x) - N(K)$, then by contracting all but one of the edges of the path P, and edges ax_1, bx_2 and four independent edges, each with one end in $N(K)$ and the other end in $\{y_1, z_1, y_2, z_2\}$, we see that $G - K > K_7$, a contradiction. Thus $N(x) - N(K)$ is disconnected. Clearly, $N(x) - N(K)$ has exactly two components and one of them has exactly two vertices. We may assume that $\{a, c\}$ is the component of size two in $N(x) - N(K)$, where $c \neq b$. Since $\delta(N(x)) \geq 5$, a (resp. c) is adjacent to every vertex of $N(K)$. By contracting three independent edges, each with one end in $N(K)$ and the other end in $\{x_1, y_1, z_1\}$, we see that $G - K - H_2 > K_7$, a contradiction. This completes the proof of Theorem 1.3.2. ∎

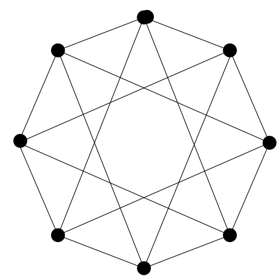
4.3 Graphs with at most 13 vertices and minimum degree 7

In the proof of Theorem 1.3.1, we need to examine graphs with at most 13 vertices and minimum degree seven. It is annoying that we are unable to find all such graphs without a

$K_7 \cup K_1$ minor without the use of computers. To see the difficulty, we give a computer-free proof for graphs with at most 11 vertices and minimum degree seven. We shall need to find all 4-regular graphs on eight vertices and cubic graphs on ten vertices.



(a) graph $\overline{V_8}$



(b) graph $K_{4,4}$

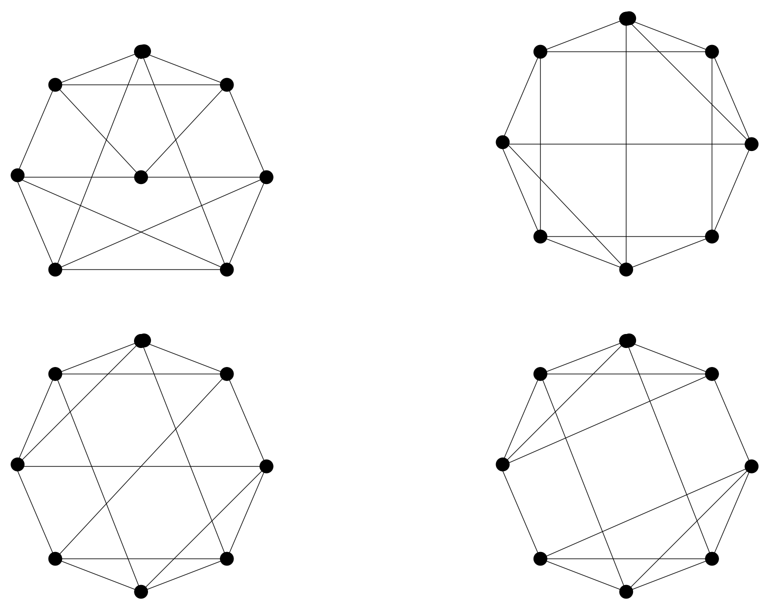


Figure 11: 4-regular graphs on eight vertices

Lemma 4.3.1 *Suppose H is 4-regular on eight vertices. Then H is isomorphic to one of the 6 graphs depicted in Figure 11.*

Note that the first graph in Figure 11 is the complement of V_8, where V_8 denotes the graph obtained from C_8 by joining all four pairs of diagonally opposite vertices. The following lemma can be verified by a routine case-checking.

Lemma 4.3.2 *Suppose H is a cubic graph on ten vertices. Then H is isomorphic to one of the 21 graphs shown in Appendix A.*

Let P be the Petersen graph, and let P' denote the graph obtained from P by subdividing one edge. We are now ready to state the following lemma.

Lemma 4.3.3 *Let G be a graph on $n \leq 13$ vertices and $\delta(G) \geq 7$. Then either $G > K_7 \cup K_1$ or G is isomorphic to one of the following graphs: $K_{1,2,2,2,2}$, $K_{1,3,3,3}$, $K_{3,3}+P_4$, $K_{3,3}+\overline{C}_4$, $K_{2,2,3,3}$, $K_{2,3}+C_5$, C_5+C_5, $\overline{K}_3+C_7$, $K_{3,4,4}$, $\overline{K}_3+\overline{V}_8$, $K_1+\overline{P}$, $\overline{P'}$, $\overline{J_1}$ and K_1+J_2, where the graph J_2 and the complement of J_1 are depicted in Figure 17.*

Proof. By a computer search we have found all graphs G on 9, 10, 11, 12 and 13 vertices such that every vertex has degree at least seven, every edge is incident with a vertex of degree seven, and for every vertex v of G the graph $G - v$ has no K_7 minor. To do this we have used the package nauty, version 2.2, written by Brendan McKay. Our small program serves to weed out graphs that do not satisfy the above conditions; it is available on the author's web-site for independent verification. Note that there are exactly one graph on 12 vertices and one graph on 13 vertices which satisfy the three conditions mentioned above. Here we give a computer-free proof of the lemma for graphs with $n \leq 11$ vertices.

By Theorem 1.3.2, we may assume that $e(G) \leq 5n - 15$, otherwise $G > K_7 \cup K_1$ or G is isomorphic to $K_{1,2,2,2,2}$, $K_{2,2,3,3}$, $G = K_{3,3} + P_4$, $K_{1,3,3,3}$, or $K_{2,3} + \overline{C_5}$. Since $\delta(G) \geq 7$, we have $7n \leq 2e(G) \leq 2(5n - 15)$. It follows that $n \geq 10$. Suppose $n = 10$. Then $35 \leq e(G) \leq 5 \times 10 - 15 = 35$. Thus $e(G) = 35$ and so G is 7-regular on ten vertices. Thus G is isomorphic to $\overline{K_3} + \overline{C_7}$, or $K_{3,3} + \overline{C_4}$, or $\overline{C_5} + \overline{C_5}$ or $\overline{C_4} + \overline{C_6}$. In the latter case, $G > K_7 \cup K_1$. Note that all those graphs are edge maximal, except $K_{3,3} + \overline{C_4}$, subject to not having a $K_7 \cup K_1$ minor.

Suppose $n = 11$. We may assume that G is edge maximal subject to not having a $K_7 \cup K_1$ minor and $\delta(G) = 7$. If $\Delta(G) = 10$, let y be a vertex of maximum degree. Then

$\delta(G-y) \geq 6$. By Theorem 2.1.1, $G-y > K_6 \cup K_1$ or $G-y = \overline{P}$, where P is the Petersen graph. Thus $G > K_7 \cup K_1$ or $G = K_1 + \overline{P}$. So we may assume that $\Delta(G) \leq 9$. Note that $39 \leq e(G) \leq 5 \times 11 - 15 = 40$ by Theorem 1.3.2. We consider the following two cases.

Case 1. $e(G) = 39$.

In this case, there exists $x \in V(G)$ such that $d(x) = 8$ and all the other vertices are of degree seven. Let y, z be the two non-neighbors of x in G. Suppose $yz \in E(G)$. Then $G-x-yz$ is 6-regular on ten vertices and so the complement of $G-x-yz$ is cubic on ten vertices. By Lemma 4.3.2, there are exactly twenty one cubic graphs on ten vertices. One can check that $G > K_7 \cup K_1$ or $G-x-yz = \overline{P}$ (and thus $G = \overline{P'}$). So we may assume that $yz \notin E(G)$.

If $N(y) = N(z)$, then there exists a vertex, say $u \in N(x)$, such that $uy, uz \notin E(G)$. Since $d(u) = 7$, u has a non-neighbor, say w, in $N(x)$. Now $N(x) - \{u, w\}$ has two vertices of degree three and four of degree two, and is isomorphic to one of the graphs depicted in Figure 12. It can be easily checked that $N(x) - \{u, w\}$ can be partitioned into two vertex disjoint subgraphs, say H_1 and H_2, such that $H_1 > K_3$ and $H_2 > K_2$. Let a, b be two adjacent vertices in H_2. Now by contracting the edges ay, bz, we see that $G - w > K_7$. Thus we may assume that $N(y)$ and $N(z)$ have exactly six common neighbors.

Let $u, w \in N(x)$ be such that $uy, wz \notin E(G)$. Clearly, $uz, wy \in E(G)$. If $uw \in E(G)$, then $N(x) - uw$ is isomorphic to a 4-regular graph on eight vertices. By checking those six graphs in Figure 11, we see that $G > K_7 \cup K_1$. So we may assume that $uw \notin E(G)$. If u and w have a common non-neighbor, say a, in $N(x)$, then u and w are adjacent to each vertex in $N(x) - \{a, u, w\}$. Thus $N(x) - \{u, w\}$ has exactly one vertex of degree four and five of degree two, and so $N(x) - \{u, w\}$ is isomorphic to the unique graph depicted in Figure 13.

If u and w have no common non-neighbor in $N(x)$, then $N(x) - \{u, w\}$ has exactly two vertices of degree three and four of degree two, and thus is isomorphic to one of the graphs

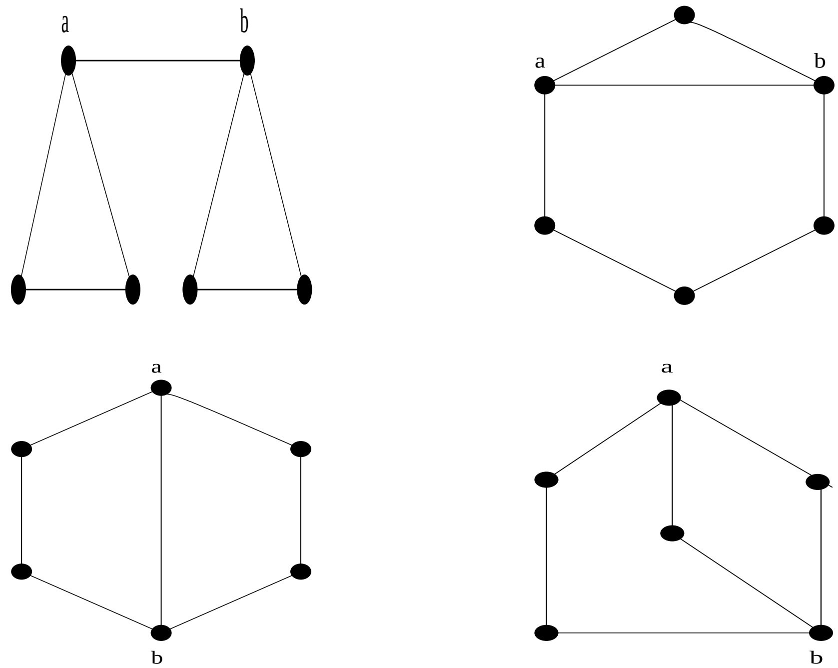


Figure 12: graph with two vertices of degree 3 and four of degree 2

depicted in Figure 12, where the vertices labeled as a and b are non-neighbors of u and w, respectively. In either case, it can be easily verified that $G > K_7 \cup K_1$.

Case 2. $e(G) = 40$.

Suppose $e(G) = 40$. Let x be a vertex of maximum degree in G. Clearly, $d(x) \geq 8$. Thus $8 \leq d(x) \leq 9$. If $d(x) = 9$, then there exists $y \neq x$ in $V(G)$ such that $d(y) = 8$ and $xy \in E(G)$, which is contrary to the assumption that G is edge-maximal. Thus we may

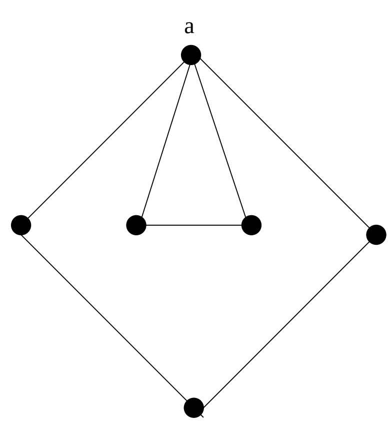


Figure 13: graph with one vertex of degree 4 and five of degree 2

assume that $d(x) = 8$. Since $e(G) = 40$ and $\delta(G) = 7$, G has another two vertices, say y, z, of degree eight in G. As G is edge maximal, x, y, z are pairwise not adjacent. It follows that each of x, y, z is adjacent to all vertices in $G - \{x, y, z\}$. Thus $G - \{x, y, z\}$ is 4-regular on eight vertices. By Lemma 4.3.1, there are exactly six 4-regular graphs on eight vertices, as shown in Figure 11. It can be easily checked that $G > K_7 \cup K_1$ or $G - \{x, y, z\}$ is isomorphic to the two graphs in the first row in Figure 11. Thus $G > K_7 \cup K_1$ or G is isomorphic to $\overline{K_3} + \overline{V_8}$ or $K_{3,4,4}$. $\blacksquare$

CHAPTER V

THE EXTREMAL FUNCTION FOR K_9 MINORS

In this chapter, we prove that every edge-maximal graph on $n \geq 9$ vertices without a K_9 minor has at most $7n - 28$ edges, or is a $(K_{1,2,2,2,2,2,2}, 6)$-cockade, or is isomorphic to $K_{2,2,2,3,3}$.

5.1 Outline of the proof of Theorem 1.3.1

Suppose for a contradiction that G is a counterexample to Theorem 1.3.1 with minimum number of vertices, say n. Since deletion or contraction of edges does not produce smaller counterexamples, it follows easily that G has minimum degree at least eight, and with some effort it can be shown that every edge of G is in at least seven triangles. It also follows by a straightforward counting argument that G is 6-connected. Also $e(G) = 7n - 27$, and hence G has a vertex x of degree at least eight and at most thirteen. Fix such a vertex, and let K be a component of $G - N[x]$. Assume for a moment that every vertex of $N(x)$ has a neighbor in K. If there exists a vertex $y \in N(x)$ such that $N(x) - y > K_7$, then by contracting the connected set $V(K) \cup \{y\}$ to a single vertex, we see that $G > K_9$. Thus $G - y \not> K_7$ for every vertex $y \in N(x)$. On the other hand, $N(x)$ has minimum degree at least seven and at most thirteen vertices. Those properties are fairly restrictive: there are only fourteen such graphs, and so they can be found explicitly. It turns out that they all have two properties in common (conditions (A) and (B) stated prior to Lemma 5.4.1) that allow us to find a K_9 minor in G in a different way. This is how we deal with the case when there is a component K of $G - N[x]$ satisfying $N(x) = N(K)$. In fact, the argument extends to the situation when there exists a component K of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$, where M is the set of all vertices of $N(x)$ that are not adjacent to every other vertex of $N(x)$.

Thus we may assume that for no vertex x of degree at most thirteen such a component exists. In the next step we prove a lemma analogous to Claim (15) of [16], namely that there is no component K of $G - N[x]$ such that $d_G(v) \geq 14$ for all vertices $v \in V(K)$, except possibly two, and the exceptional vertices have degree at least $d(x)$. This follows by counting edges, for if such a component exists, then we exhibit a proper minor of G with $n' < n$ vertices and more than $7n' - 27$ edges. That minor of G has a K_9 minor by the minimality of G, and hence G has a K_9 minor, a contradiction. Finally, in the last step, following Jørgensen [16], we select a vertex $x \in V(G)$ of degree at most thirteen to maximize the size of a component of $G - N[x]$, and, subject to that, to minimize the degree of x. Let K be the largest component of $G - N[x]$. From the previous results we know there is another component K' of $G - N[x]$, and that component has at least three vertices of degree at most thirteen (but at least $d(x)$, by the choice of x). The choice of x implies that all three of these vertices are adjacent to every vertex of $N(K)$. Thus $N(K) \subseteq N(K')$, and so there is a third component of $G - N[x]$. The same argument applies to it, and hence there are six distinct vertices of $G - N[x]$ that are adjacent to every vertex of $N(K)$. On the other hand, $|N(K)| \geq 6$ because G is 6-connected, and so it is now easy to construct a K_9 minor in G.

5.2 $(K_{1,2,2,2,2,2}, 6)$-cockades are edge maximal

As noted in Section 5.1, our proof uses induction by deleting and contracting edges of G. Thus we need to investigate graphs G such that the new graph $G - xy$ or G/xy is a $(K_{1,2,2,2,2,2}, 6)$-cockade or is isomorphic to $K_{2,2,2,3,3}$. We do that next.

Lemma 5.2.1 *Let G be $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade and let x and y be nonadjacent vertices in G. Then $G + xy$ is contractible to K_9.*

Proof. This is easily checked if $G = K_{2,2,2,3,3}$ or $G = K_{1,2,2,2,2,2}$. So we may assume that G is obtained from H_1 and H_2 by identifying cliques of size 6, where H_1 and H_2 are

$(K_{1,2,2,2,2,2,2}, 6)$-cockades. If x and y are both in H_1 or H_2, then $H_1 + xy > K_9$ or $H_2 + xy > K_9$ by induction. So we may assume that $x \in V(H_1) - V(H_2)$ and $y \in V(H_2) - V(H_1)$. Note that no $(K_{1,2,2,2,2,2,2}, 6)$-cockade contains K_7 as a subgraph. Therefore there exists $z \in V(H_1) \cap V(H_2)$ such that $zy \notin V(G)$. Now by contracting $V(H_1) - V(H_2)$ to the vertex z in $G + xy$, the resulting graph is $H_2 + zy$. By induction, $H_2 + zy > K_9$. ■

5.3 Preliminaries

In this section, we consider contractions in a graph G, which has two adjacent vertices x and y such that x and y have exactly six common neighbors and G/xy is a $(K_{1,2,2,2,2,2,2}, 6)$-cockade or is isomorphic to $K_{2,2,2,3,3}$.

Lemma 5.3.1 *Let G be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly six common neighbors. If G/xy is isomorphic to $K_{2,2,2,3,3}$, then $G > K_9$.*

Proof. Let w be the new vertex in G/xy. Since x and y have exactly six common neighbors, there exist distinct vertices $w_1, w_2, w_3, w_4 \in V(G/xy) - w$ such that $w_1w_2, w_3w_4 \notin E(G/xy)$, and w_1, w_2, w_3 are common neighbors of x and y in G. Moreover, w_4 is adjacent to x or y, say to y, in G. By contracting the edges xw_2 and yw_4 we see that G has a K_9 minor, as desired. ■

Lemma 5.3.2 *Let G be a graph and let x, y be adjacent vertices of G with exactly six common neighbors. If G/xy is isomorphic to $K_{1,2,2,2,2,2,2}$, then G has a K_9 minor, unless G is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in G.*

Proof. Let w be the new vertex of G/xy, and let $z, x_1, y_1, \ldots, x_5, y_5$ be the vertices of G/xy numbered so that x_i is not adjacent to y_i. Assume first that $w \neq z$, say $w = x_1$. Since x and y have six common neighbors, we may assume that x_2, y_2, x_3 are common neighbors of

x and y. Moreover, y_3 is adjacent to x or y, say to y. By contracting the edges xy_2, yy_3 and y_4y_5 we see that G has a K_9 minor, as desired.

Thus we may assume that $w = z$. Since x, y have six common neighbors, their degree is at least seven. Assume for a moment that $d_G(x) = 7$. Since x, y have six common neighbors in G, we deduce that y is adjacent to all other vertices of G and there exists an index i such that x_i, y_i are common neighbors of x, y. We may assume that $i = 1$. By contracting the edges xx_1, x_2x_3 and x_4x_5, we obtain a K_9 minor of G. Hence we may assume that $d(x), d(y) \geq 8$. We may also assume that G is not isomorphic to $K_{2,2,2,3,3}$ with x, y of degree nine, and so it follows that one of x, y is adjacent to x_i or y_i for every $i = 1, 2, 3, 4, 5$. Thus we may assume (by swapping x_i and y_i) that x is adjacent to all of X, where $X = \{x_1, \dots, x_5\}$. Moreover, we may assume that if y is also adjacent to every vertex of X, then $d(x) \leq d(y)$. Let $Y = \{y_1, \dots, y_5\}$. Since y has degree at least eight, there is some i such that y is adjacent to x_i and y_i. We claim that y is adjacent to at least three vertices of Y. For if not, then x is adjacent to at least three vertices of Y (the non-neighbors of y) and, since $d(y) \geq 8$, y is adjacent to all vertices of X. But then $d(x) > d(y)$, a contradiction. Thus y is adjacent to at least three vertices of Y.

Thus there exist distinct indices i, j, k such that y is adjacent to x_i, y_i, y_j, y_k. Choose such indices so that, if possible, x is not adjacent to y_i. We may assume that $i = 1$, $j = 2$ and $k = 3$. We claim that x is adjacent to at least two vertices of $Y - \{y_1\}$. For if not, then y has at least four neighbors in Y, and hence x, y have at least four common neighbors in X, and so the indices i, j, k above can be chosen so that x is not adjacent to y_i. Thus x is not adjacent to y_1, and hence x has at most one neighbor in Y, implying that $d(x) = 7$, a contradiction. Thus x has at least two neighbors in $Y - \{y_1\}$, and so we may assume that x has a neighbor in $\{y_2, y_4\}$ and a neighbor in $\{y_3, y_5\}$. By contracting the edges yy_1, y_2y_4 and y_3y_5 we see that G has a K_9 minor, as required. ■

Lemma 5.3.3 *Let G be a graph with $\delta(G) \geq 7$. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly six common neighbors. If G/xy is a $(K_{1,2,2,2,2,2}, 6)$-cockade, then either $G > K_9$, or G is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in G.*

Proof. We proceed by induction on $|G|$. By Lemma 5.3.2 we may assume that $G/xy = H_1 \cup H_2$, where $H_1 \cap H_2$ is a complete graph on six vertices and both H_1 and H_2 are $(K_{1,2,2,2,2,2}, 6)$-cockades. Let w be the new vertex of G/xy. For $i = 1, 2$ let $H_i^* = G[(V(H_i) - \{w\}) \cup \{x, y\}]$. If $w \in V(H_1) - V(H_2)$, then $H_1^* \neq K_{2,2,2,3,3}$ (because the latter graph has no K_6 subgraph) and the result follows by induction applied to H_1^*. From the symmetry we may assume that $w \in V(H_1) \cap V(H_2)$. Let $S = V(H_1) \cap V(H_2) - \{w\}$; thus $V(H_1^*) \cap V(H_2^*) = S \cup \{x, y\}$. Let Z denote the set of six common neighbors of x and y in G. If $Z \subseteq V(H_1^*)$, then by induction applied to H_1^* we may assume that H_1^* is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in H_1^*. Since H_1^* has no K_6 subgraph one of x, y, say x, is not adjacent to some $s \in S$ and x has at least one neighbor in $V(H_2) - V(H_1)$. By using a path with ends x and s and interior in $H_2^* - V(H_1^*)$ we deduce that $G > H_1^* + sx > K_9$ by Lemma 5.2.1, as desired.

Thus we may assume that $Z - V(H_1^*) \neq \emptyset \neq Z - V(H_2^*)$. Since H_2 is a $(K_{1,2,2,2,2,2}, 6)$-cockade, it is 6-connected. Let $k = |Z - V(H_1)|$. Since $|Z \cap V(H_2)| \leq 5$ we have $|S - Z| = 5 - |Z \cap S| \geq k$. Thus there exist k disjoint paths $P_1, P_2, \ldots, P_k$ in $H_2 - (Z \cap S) - w$ between $Z \cap V(H_2 - S)$ and $S - Z$. Consequently H_1^* has a supergraph H_1' on the same vertex set such that $H_1' < G$ and x, y have exactly six common neighbors in H_1'. By induction H_1' is isomorphic to $K_{2,2,2,3,3}$ and x, y have degree nine in H_1'. By symmetry the same holds for the analogous graph H_2'. It follows that in H_1' the vertex x has a unique non-neighbor in S, say x'. Then $x' \notin V(P_1 \cup \cdots \cup P_k)$. From the symmetry between H_1 and H_2 we may assume that $k \leq 3$. (In fact, $|Z - V(H_1)| = |Z - V(H_2)| = 3$.) It follows that the k disjoint paths $P_1, \ldots, P_k$ can each be chosen of length one, and that there exists a common neighbor of x and x' in $V(H_2^*)$, say u, that does not belong to any of the paths. Thus by contracting the edge ux' and all the edges of the paths $P_1, \ldots, P_k$ we deduce that $G > H_1' + xx' > K_9$ by Lemma 5.2.1, as desired. ■

5.4 Graphs without $K_7 \cup K_1$ minor

As pointed out in Section 5.3, we need to examine graphs G such that $|V(G)| \leq 13$, $\delta(G) \geq 7$ and $G \not> K_7 \cup K_1$. The next lemma shows that those graphs G satisfy the following properties:

(A) either G is isomorphic to $K_{1,2,2,2,2}$, or G has four distinct vertices a, b, c, d such that the pairs of vertices (a, b) and (c, d) are adjacent and have at most four common neighbors in G and $G + ac + bd > K_8$,

(B) for any two sets $A, B \subseteq V(G)$ of size at least five such that neither is complete and $A \cup B$ includes all vertices of G of degree at most $|G| - 2$, either

(B1) there exist $a \in A$ and $b \in B$ such that $G' > K_8$, where G' is obtained from G by adding all edges aa' and bb' for $a' \in A - \{a\}$ and $b' \in B - \{b\}$, or

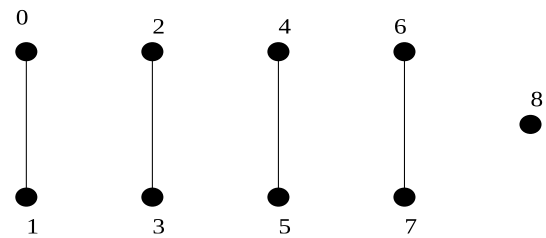
(B2) there exist $a \in A - B$ and $b \in B - A$ such that $ab \in E(G)$ and the vertices a and b have at most five common neighbors in G, or

(B3) one of A and B contains the other and $G + ab > K_7 \cup K_1$ for all nonadjacent vertices $a, b \in A \cap B$.

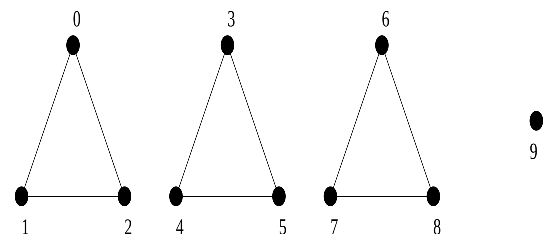
Lemma 5.4.1 *Let n be an integer satisfying $9 \leq n \leq 13$ and let G be a graph on n vertices with $\delta(G) \geq 7$. Then either $G > K_7 \cup K_1$ or G satisfies* (A) *and* (B).

Proof. By Lemma 4.3.3 the graphs G with $9 \leq n \leq 13$ vertices, $\delta(G) \geq 7$ and $G \not> K_7 \cup K_1$ are the following ones: $K_{1,2,2,2,2}$, $K_{1,3,3,3}$, $K_{3,3}+P_4$, $K_{3,3}+\overline{C}_4$, $K_{2,2,3,3}$, $K_{2,3}+C_5$, C_5+C_5, $\overline{K}_3 + C_7$, $K_{3,4,4}$, $\overline{K}_3 + \overline{V}_8$, $K_1 + \overline{P}$, $\overline{P'}$, $\overline{J_1}$ and $K_1 + J_2$, depicted in Figures 14-17. In the following we assume that the vertices of those graphs are labeled as in the above-mentioned figures.

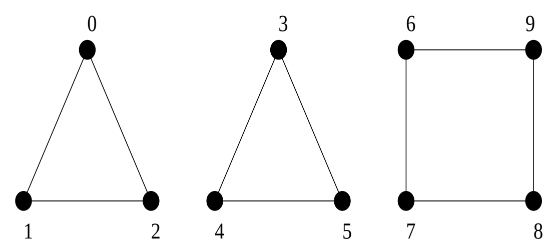
To see that condition (A) holds for those graphs G we may assume that G is not isomorphic to $K_{1,2,2,2,2}$. Let $(a, b, c, d) = (0, 3, 1, 4)$ for $K_{1,2,2,2,2}$, $K_{1,3,3,3}$, $K_{3,3} + P_4$,



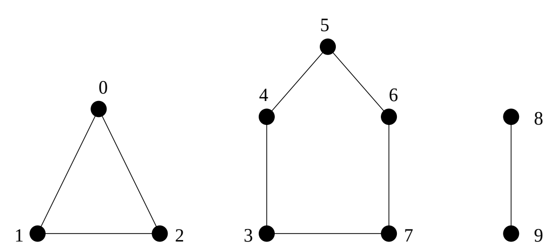
(a) Complement of $K_{1,\,2,\,2,\,2,\,2}$



(b) Complement of $K_{1,\,3,\,3,\,3}$

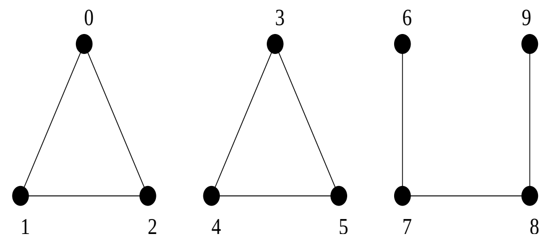


(c) Complement of $K_{3,\,3} + \overline{C}_4$

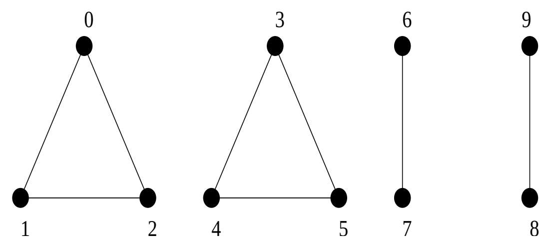


(d) The complement of $K_{2,\,3} + C_5$

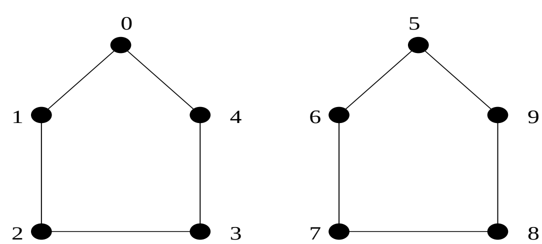
Figure 14: Graphs with no $K_7 \cup K_1$ minors.



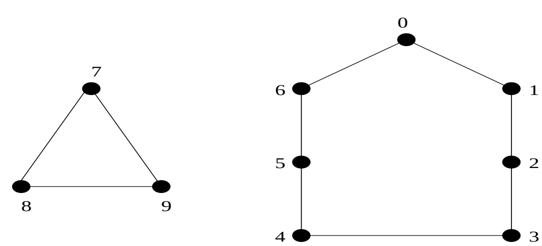
(a) Complement of $K_{3,3} + P_4$



(b) The complement of $K_{2,2,3,3}$

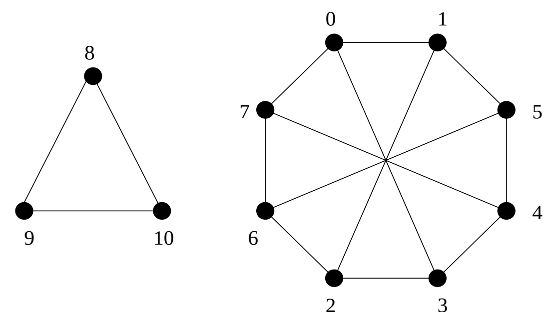


(c) Complement of $C_5 + C_5$

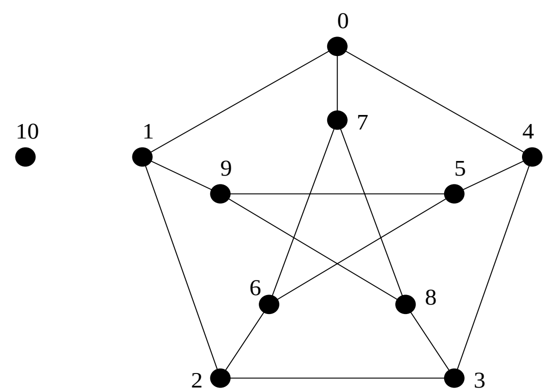


(d) Complement of $\overline{K_3} + \overline{C}_7$

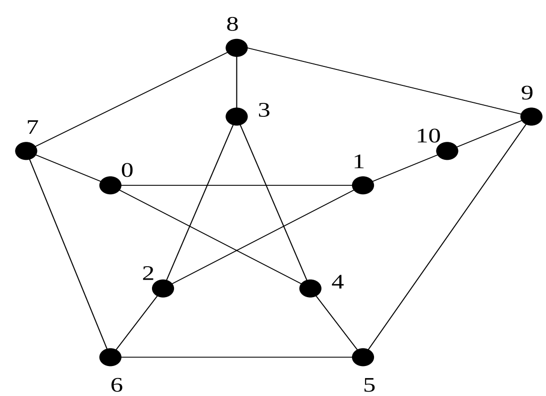
Figure 15: Graphs with no $K_7 \cup K_1$ minors.



(a) Complement of $\overline{K_3} + \overline{V}_8$

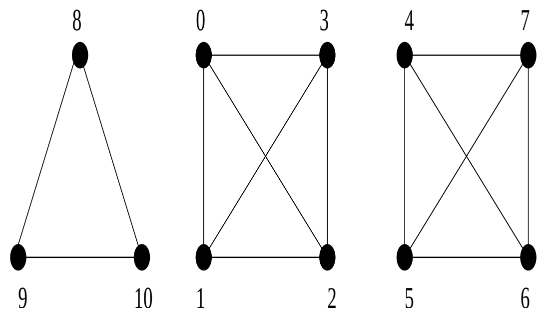


(b) The complement of $K_1 + \overline{P}$

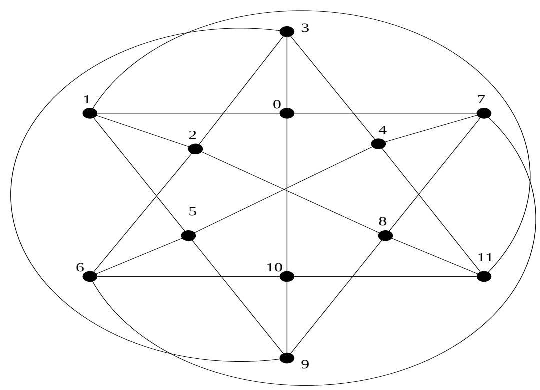


(c) Complement of $\overline{P'}$

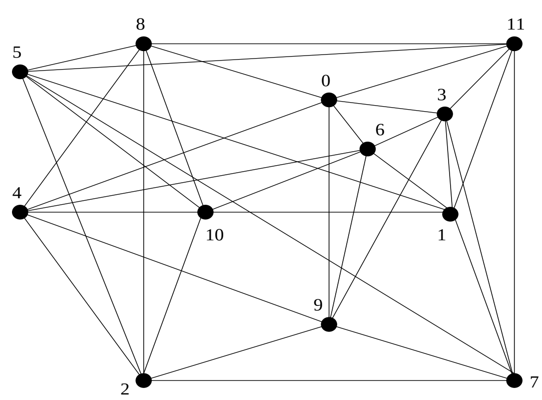
Figure 16: Graphs with no $K_7 \cup K_1$ minors



(a) Complement of $K_{3,3,4}$



(b) Complement of J_1



(c) Graph J_2

Figure 17: Graphs with no $K_7 \cup K_1$ minors.

$K_{3,3} + \overline{C}_4$, and $K_{2,2,3,3}$, let $(a, b, c, d) = (0, 11, 2, 10)$ for $K_1 + J_2$, and let $(a, b, c, d) = (0, 9, 1, 8)$ for the rest of the graphs.

To prove that condition (B) holds for all those graphs G let the sets A and B be as stated in condition (B), and assume that (B2) does not hold. A vertex in a graph is universal if it is adjacent to every other vertex of the graph. Let M be the set of all vertices of G that are not universal. For notational reasons we define $G + (u, v)$ to mean $G + uv$.

Assume first that A is a subset of B. Then M is a subset of B because M is included in the union of A and B. We may assume that G is $K_{3,3} + P_4$, $K_{3,3} + \overline{C}_4$, or $K_{2,2,3,3}$, for otherwise (B3) holds for any pair of nonadjacent vertices a, b of G. In particular, G has no vertex adjacent to every other vertex, and so $B = \{0, 1, ..., 9\}$. If A intersects both $\{0, 1, 2\}$ and $\{3, 4, 5\}$ in at most one element, then (since A has at least five elements) we may assume that $6, 7, 8 \in A$, and on letting $(a, b) = (7, 0)$ we deduce that (B1) holds (because $G + (0, 1) + (6, 7) + (7, 8) > K_8$). Thus we may assume that 0 and 1 belong to A, and by letting $(a, b) = (0, 3)$ we deduce that (B1) holds (because $G + (0, 1) + (3, 4) > K_8$).

So we may assume that there exist $x \in A - B$ and $y \in B - A$. We first dispose of the case when x and y cannot be chosen so that neither is universal. In that case we may assume that x is universal; that limits G to $K_{1,2,2,2,2}$, $K_{1,3,3,3}$, $K_1 + \overline{P}$ or $K_1 + J_2$. Note that each of them has a unique universal vertex. It follows that M is a subset of B, for otherwise a vertex in $M - B$ can replace x, yielding a choice of (x, y), where neither is universal. Let u, v be nonadjacent vertices in A. Then $G + (u, v) > K_7 \cup K_1$. (We said above that there are only two graphs, namely $K_{3,3} + \overline{C_4}$ and $K_{3,3} + P_4$, that are not edge-maximal, and those two graphs have no universal vertex.) Thus for some w the graph $G + (u, v) - w$ has a K_7 minor. If w is universal, then $G + (u, v)$ has a K_8 minor, and we are done. Otherwise $w \in M$, and hence $w \in B$, and so graph obtained from $G + (u, v)$ by adding all missing edges wb for $b \in B$ has a K_8 minor, as desired.

Thus we may assume that x and y can be chosen so that neither is universal. As A and B do not satisfy property (B2), it follows that for any $a \in N(x) - A$, the vertices x

and a have at least six common neighbors in G. This is rarely true for the graphs we are dealing with, and that is how, in the analysis below, it follows that $a \in A$. The argument that $b \in B$ is analogous. Furthermore, if x and y are adjacent, then they have at least six common neighbors, and that eliminates many cases. Here are the choices for a and b to show that (B1) holds. We list, up to symmetry, all pairs (x, y) of nonadjacent vertices and all pairs (x, y) of adjacent vertices with at least six common neighbors such that neither is universal.

$K_{1,2,2,2,2}$: we may assume that $(x, y) = (0, 1)$. Let $(a, b) = (2, 4)$.

$K_{1,3,3,3}$: by symmetry we may assume that $(x, y) = (0, 1)$. Let $(a, b) = (3, 7)$ and note that $G + (3, 4) + (7, 8) > K_8$.

$K_{3,3} + P_4$, $K_{3,3} + \overline{C}_4$ and $K_{2,2,3,3}$: if $(x, y) = (0, 1)$ let $(a, b) = (3, 6)$; if $(x, y) = (6, 7)$ or $(x, y) = (6, 8)$ let $(a, b) = (0, 3)$.

$K_{2,3} + C_5$: if $(x, y) = (0, 1)$ let $(a, b) = (3, 8)$; if $(x, y) = (3, 4)$ let $(a, b) = (0, 8)$; if $(x, y) = (8, 9)$ let $(a, b) = (0, 3)$.

$C_5 + C_5$: we may assume that $(x, y) = (0, 1)$. Let $(a, b) = (2, 5)$.

$\overline{K}_3 + C_7$: if $(x, y) = (0, 1)$ let $(a, b) = (3, 7)$; if $(x, y) = (7, 8)$ let $(a, b) = (0, 2)$.

$K_{3,4,4}$: if $(x, y) = (0, 1)$ let $(a, b) = (4, 8)$; if $(x, y) = (8, 9)$ let $(a, b) = (0, 4)$.

$\overline{K_3} + \overline{V_8}$: if $(x, y) = (0, 1)$ or $(x, y) = (0, 3)$ let $(a, b) = (4, 8)$; if $(x, y) = (8, 9)$ let $(a, b) = (0, 2)$.

$K_1 + \overline{P}$: we may assume that $(x, y) = (0, 1)$. Let $(a, b) = (5, 4)$.

$\overline{P'}$: we may assume that (x, y) is one of $(0, 1), (2, 3), (3, 4), (1, 10), (8, 3)$. Let$(a, b) = (5, 7)$. Notice that $G + (5, 4) + (7, 6)$ and $G + (5, 6) + (7, 8)$ both have K_8 minors.

J_12: we may assume that (x, y) is one of $(1, 0)$, $(1, 2)$, $(1, 11)$. Let $(a, b) = (7, 9)$ and notice that $G + (7, 8) + (7, 4) + (9, 5)$ has a K_8 minor (contract the edges $(0, 4)$, $(1, 3)$, $(2, 10)$, and $(6, 8)$).

K_1+J_1: we may assume that (x, y) is one of $(4,3)$, $(8,6)$, $(8,3)$. Let $(a, b) = (0,1)$ and notice that $G+(0,10)+(1,9)$ has a K_8 minor (contract the edges$(0,11)$, $(2,10)$, $(3,6)$, $(4,8)$, $(4,5)$).

■

5.5 Proof of Theorem 1.3.1

In this section, we are going to prove Theorem 1.3.1 by induction on n. The only graph G with 9 vertices and $e(G) \geq 7 \times 9 - 27 = 36$ is K_9. Thus we may assume that $n \geq 10$ and that the assertion holds for smaller values of n. Throughout this section we assume that G is a graph with n vertices and $e(G) \geq 7n - 27$ but G is not contractible to K_9 and G is not $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}, 6)$-cockade. By Lemma 5.2.1, we may assume that $e(G) = 7n - 27$.

Suppose that G has a vertex x of degree at most 6. Then $e(G - x) \geq 7(n-1) - 26$, and hence $G > G - x > K_9$ by induction, a contradiction. Suppose now that G has two adjacent vertices x, y with at most five common neighbors. Then $e(G/xy) \geq 7(n-1) - 26$. By induction, $G > K_9$, a contradiction. Thus $\delta(G) \geq 7$ and $\delta(N(x)) \geq 6$. If G has a vertex x of degree 7, then $N(x) = K_7$ and $e(G - x) \geq 7(n-1) - 27$. Note that neither a $(K_{1,2,2,2,2,2}, 6)$-cockade nor $K_{2,2,2,3,3}$ contain K_7 as a subgraph. Thus, by induction, $G - x > K_9$, a contradiction. Hence

(1) $\delta(G) \geq 8$ and $\delta(N(x)) \geq 6$ for any $x \in V(G)$.

Let S be a separating set of vertices in G, and let G_1 and G_2 be proper subgraphs of G so that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$. Let $m_i = 7|G_i| - 27 - e(G_i)$, $i = 1, 2$. Then $7n - 27 = e(G) = e(G_1) + e(G_2) - e(G[S]) = 7n + 7|S| - 54 - m_1 - m_2 - e(G[S])$, and so

(2) $7|S| = 27 + m_1 + m_2 + e(G[S])$.

For $i = 1, 2$, let d_i be the maximum number of edges that can be added to G_{3-i} by contracting edges of G with at least one end in G_i. More precisely, let d_i be the largest integer so that G_i contains disjoint set of vertices $V_1, V_2, \ldots, V_p$ so that $G_i[V_j]$ is connected,

$|S \cap V_j| = 1$ for $1 \leq j \leq p = |S|$, and so that the graph obtained from G_i by contracting $V_1, V_2, \ldots, V_p$ and deleting $V(G) - (\bigcup_j V_j)$ has $e(G[S]) + d_i$ edges. By (1), $\delta(G) \geq 8$. Thus $|G_i| \geq 9$, $i = 1, 2$. By induction, $d_1 \leq m_2$ and $d_2 \leq m_1$. By (2),

(3) $7|S| \geq 27 + d_1 + d_2 + e(G[S])$.

In particular, $|S| \geq 4$. If S is a minimal separating set, then let $v \in S$ be a vertex of minimum degree in $G[S]$. By choosing $V_1 = V(G_i) - (S - \{v\})$ and the rest of the sets V_j to be singletons, we see that $d_i \geq |S| - 1 - \delta(G[S])$ for $i = 1, 2$. Thus

(4) if S is a minimal separating set, then

$$5|S| \geq 25 + e(G[S]) - 2\delta(G[S])) \geq 25 + \frac{1}{2}(|S| - 4)\delta(G[S]).$$

Lemma 5.5.1 *G is 6-connected.*

Proof. Suppose G is not 6-connected. Let S be a minimal separating set of G, and let G_1, G_2, d_1, d_2 be as above. By (4) G is 5-connected and $G[S] = \overline{K_5}$. We next show that $d_1 \geq 5$. Let x and y be distinct vertices in $G_1 \backslash S$. By Menger's theorem, there exist five x-S paths $P_1, P_2, \ldots, P_5$ in G_1 which have only the vertex x in common. If all these paths have length 1, then, since there are at least four internally disjoint y-S paths in $G_1 \backslash \{x\}$, by contracting these paths we deduce that $d_1 \geq 7$. We may now assume that P_1 has length at least 2. Let $V(P_1) \cap S = \{z\}$. As $\{x, z\}$ is not a separating set in G, there is a path P from a vertex on $P_1 \backslash \{x, z\}$ to a vertex on some $P_i \backslash \{x\}$, $i \neq 1$, so that only the end vertices of P belong to $\bigcup_{j=1}^{5} P_j$. By contracting a suitable subset of the edges of $P \cup P_1 \cup \cdots \cup P_5$ we deduce that $d_1 \geq 5$, as claimed.

By symmetry, $d_2 \geq 5$ and so $d_1 + d_2 \geq 10$. However, by (3), $d_1 + d_2 \leq 8$, which is a contradiction. ∎

Lemma 5.5.2 *There is no separating set S with a vertex x so that $G[S - x]$ is complete.*

Proof. Suppose that $G[S - x]$ is complete and let G_1, G_2 be as above. We may assume that S is a minimal separating set. By Lemma 5.5.1, $|S| \geq 6$. If $|S| \geq 8$, by contracting

$V(G_1) - S$ to x and $V(G_2) - S$ to a new vertex, we get a K_9 minor, a contradiction. So we may assume that $|S| = 6$ or $|S| = 7$.

If $|S| = 6$, by (4), $5|S| \geq 25 + e(G[S]) - 2\delta(G[S])) \geq 25 + 10 + \delta(G[S]) - 2\delta(G[S])$, which implies that $G[S] = K_6$. By induction, we may assume $e(G_i) \leq 7|G_i| - 27$, $i = 1, 2$. Since $7n - 12 = 7n - 27 + 15 = e(G) + 15 = e(G_1) + e(G_2) \leq 7|G_1| - 27 + 7|G_2| - 27 = 7n - 12$, it follows that $e(G_i) = 7|G_i| - 27$, $i = 1, 2$. Since $K_{2,2,2,3,3}$ does not contain K_6 as a subgraph, by induction, $G_i > K_9$ or G_i is a $(K_{1,2,2,2,2,2}, 6)$-cockade. Thus $G > K_9$ or G is a $(K_{1,2,2,2,2,2}, 6)$-cockade, a contradiction.

If $|S| = 7$, by (4), $5|S| \geq 25 + e(G[S]) - 2\delta(G[S])) \geq 25 + 15 + \delta(G[S]) - 2\delta(G[S])$, which implies that $G[S]$ is isomorphic to K_7 or K_7 with an edge deleted. Let $e(G[S]) = 21 - t$, where $t = 0$ or 1. Suppose $e(G_1) \geq 7|G_1| - 27 - t$. Let G_1' be obtained from G by contracting $V(G_2) - S$ to x. Then $e(G_1') = e(G_1) + t \geq 7|G_1'| - 27$. Since G_1' contains a K_7 subgraph, it is not $K_{2,2,2,3,3}$ or a $(K_{1,2,2,2,2,2}$, 6)-cockade, and hence by induction, $G > G_1' > K_9$. Thus $e(G_1) \leq 7|G_1| - 28 - t$. Similarly, we have $e(G_2) \leq 7|G_2| - 28 - t$. But now $e(G) = e(G_1) + e(G_2) - e(G[S]) \leq 7(n + 7) - 28 - t - 28 - t - 21 + t = 7n - 28 - t$, which is a contradiction. ∎

Lemma 5.5.3 $\delta(N(x)) \geq 7$ *for any* $x \in V(G)$.

Proof. Suppose $\delta(N(x)) \leq 6$. By (1) there exists a vertex $y \in N(x)$ such that x and y have exactly six common neighbors. Then $e(G/xy) = 7(n - 1) - 27$. Since $G \not> K_9$, the minimality of $|G|$ implies that G/xy is isomorphic to $K_{2,2,2,3,3}$ or is a $(K_{1,2,2,2,2,2,2}$, 6)-cockade. In either case, by Lemma 5.3.1 or Lemma 5.3.3, $G > K_9$ or $G = K_{2,2,2,3,3}$, a contradiction. ∎

Lemma 5.5.4 $\delta(G) \geq 9$.

Proof. Let $x \in V(G)$ be such that $d(x) = \delta(G) \leq 8$. By Lemma 5.5.3, $N(x) = K_8$ and so $G > N[x] = K_9$, a contradiction. ∎

Lemma 5.5.5 *If $G - N[x]$ is 2-connected or has at most two vertices, then $N(x) \neq K_{1,2,2,2,2}$.*

Proof. Suppose for a contradiction that $N(x) = K_{1,2,2,2,2}$. Let $V(N(x)) = \{y, z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4\}$ so that y is adjacent to all vertices in $N(x) - y$ and $z_i w_i \notin E(G)$.

We next show that z_i and w_i have no common neighbor in $G - N[x]$ for $i = 1, 2, 3, 4$. To this end suppose that there exists a vertex $v \in V(G - N[x])$ adjacent to, say z_1 and w_1. Let $K = G - N[x] - v$. Then K is not null by Lemma 5.5.4, because G is not isomorphic to $K_{1,2,2,2,2,2}$. Since $G - N[x]$ has no cut vertex, K is connected. If $z_i, w_i \in N(K)$ for some $i \in \{2, 3, 4\}$, then let P be a path with ends z_i and w_i and interior in K. By contracting the edge $z_1 v$ and all but one of the edges of P we see that $G > N[x] + z_1 v_1 + z_i w_i > K_9$, a contradiction. Thus we may assume that $w_2, w_3, w_4 \notin N(K)$. Let $i \in \{2, 3, 4\}$. It follows from Lemma 5.5.3 applied to w_i that v is adjacent to w_i. By Lemma 5.5.3 the edge vw_i is in at least seven triangles, and hence z_2, z_3, z_4 are all adjacent to v. By Lemma 5.5.2 the set $N(K) - \{v\}$ is not complete, and hence $z_1, w_1 \in N(K)$. By contracting the edge vw_2 and all but one edge of a z_1-w_1 path with interior in K we deduce that $G > N[x] + z_1 w_1 + z_2 w_2 > K_9$, a contradiction. This proves that the vertices z_i and w_i have no common neighbor in $G - N[x]$.

Let $u \in V(G) - N[x]$ be a neighbor of z_1. By Lemma 5.5.3 the vertices u and z_1 have at least seven common neighbors, and so by the result of the previous paragraph z_1 has at least four neighbors in $G - N[x]$. By symmetry the same holds for all z_i and w_i.

Let $H = G - \{x, y, z_3, w_3, z_4, w_4\}$. We next show that H is 4-connected. Suppose for a contradiction that S is a minimal separating set of at most three vertices in H. Since $G - N[x]$ has no cut vertex, $|S| \geq 2$ and $|S \cap N(x)| \leq 1$. If $|S \cap N(x)| = 1$, we may assume that $w_1 \in S$. Since $z_1 z_2, z_1 w_2 \in E(G)$, z_1, z_2, w_2 are in the same component of $H - S$. Denote this component by K. If $w_1 \notin S$, then also $w_1 \in K$. Since z_2, w_2 have at least four neighbors in $G - N[x]$, there exist z_2' and w_2' in $G - N[x] - S$ adjacent to z_2 and w_2, respectively. Clearly, z_2' and w_2' belong to K. As $G - N[x]$ has no cut vertex, $G - N[x]$ contains two independent z_2'-w_2' paths. One of these paths is contained in $G[K \cup S]$.

Since G is not contractible to $N[x] + z_2w_2 + z_iw_i > K_9$ for $i = 3, 4$, there is no z_i-w_i path in $G[K' \cup \{z_i, w_i\}]$, where $K' \neq K$ is another component of $H - S$. But this implies that K' is separated from x by S and three pairwise adjacent vertices. We may assume that such three vertices are y, w_3, w_4. Since G is 6-connected, $|S| = 3$. Let $S = \{s_1, s_2, s_3\}$, where $s_1 = w_1$ if $w_1 \in S$, and let $S' = S \cup \{y, w_3, w_4\}$. Then S' is a minimal separating set of G. Let $H_1 = G[K' \cup S']$ and $H_2 = G - K'$. Let d_1 and d_2 be defined as in the paragraph prior to (3). Clearly, $K \cup \{x, z_3, z_4\}$ is contained in H_2. By Menger's theorem, there exist three disjoint paths between $\{x, w_1, w_2\}$ and S in $G - \{y, w_3, w_4\}$. Now by contracting those paths, we get $d_2 + e(G[S']) = e(K_6) = 15$. By Lemma 5.5.2, $d_1 \geq 1$. By (3), $42 = 7|S'| \geq 27 + 1 + 15 = 43$, a contradiction. Thus H is 4-connected.

Since G is not contractible to K_9, it follows from Theorem 1.4.1 applied to the vertices z_1, z_2, w_1, w_2 that $e(H) \leq 3|H| - 7 = 3(n-6) - 7$. Since for $i \in \{3, 4\}$ the vertices z_i and w_i have no common neighbor in $G - N[x]$, they together have at most $|G| - |N[x]| = n - 10$ neighbors in $G - N[x]$. The vertices $\{z_3, w_3, z_4, w_4\}$ are incident with 20 edges of $N[x]$. Thus

$$\begin{aligned} 7n - 27 = e(G) &\leq d(x) + d(y) - 1 + e(H) + 2(n-10) + 20 \\ &\leq 9 + n - 2 + 3(n-6) - 7 + 2(n-10) + 20 = 6n - 18. \end{aligned}$$

It follows that $n \leq 9$, a contradiction. ∎

Lemma 5.5.6 *Let $x \in V(G)$ be such that $9 \leq d(x) \leq 13$. Then there is no component K of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$, where M is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$.*

Proof. Assume such a component K exists. Among all vertices x with $9 \leq d(x) \leq 13$ for which such a component exists, choose x to be of minimal degree. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M - N(K) \neq \emptyset$, and let $y \in M - N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y) < d(x)$. Let J be the component of $G - N[y]$ containing K. Since $d(y) < d(x)$ the choice of x implies that $N(x) - N[y] \not\subseteq V(J)$, and hence some component H of $N(x) - N[y]$ is disjoint from $N(K)$. We have $d_G(z) \geq d_G(y)$ for all $z \in V(H)$ by the choice of y. Let $t = |V(H)|$. Then $t \geq 2$, for otherwise the vertex y and component H contradict the choice of x. On the other hand $t \leq d(x) - d(y) \leq 13 - 9 = 4$. From Lemma 5.5.3 applied to y we deduce that $N(y) \cap N(x)$ has minimum degree at least six. Let L be the subgraph of G induced by $(N[y] \cap N(x)) \cup V(H)$. Then the edge-set of L consists of edges of $N(x) \cap N(y)$, edges incident with y, and edges incident with $V(H)$. Thus

$$\begin{aligned} e(L) &\geq 3(d(y)-1) + d(y) - 1 + t(d(y)-1) - \frac{1}{2}t(t-1) \\ &\geq 6(d(y)+t) + (t-2)d(y) - 4 - 7t - \frac{1}{2}t(t-1) \geq 6|V(L)| - 20, \end{aligned}$$

because $d(y) \geq 9$ and $2 \leq t \leq 4$. Since $11 \leq |V(L)| \leq 13$ the graph L is not a $(K_{2,2,2,2,2}, 5)$-cockade, and hence $N(x) > L > K_8$ by Theorem 1.2.3. Thus $G > K_9$, a contradiction. This proves that $M \subseteq N(K)$.

If $N(x) > K_7 \cup K_1$, then $N(x)$ has a vertex y such that $N(x) - y > K_7$. If $y \notin M$, then $N(x) > K_8$. Otherwise, by contracting the connected set $V(K) \cup \{y\}$ we can contract K_8 onto $N(x)$. Thus in either case $G > K_9$, a contradiction. Thus by Lemma 5.4.1, we may assume that $N(x)$ satisfies properties (A) and (B).

If $G - N[x]$ is 2-connected or has at most two vertices, then by Lemma 5.5.5, we may assume that $N(x) \neq K_{1,2,2,2,2}$. Then by property (A) and Lemma 5.5.3 the set $N(x)$ has four distinct vertices x_1, y_1, x_2, y_2 such that $N(x) + x_1y_1 + x_2y_2 > K_8$ and the pairs (x_1, x_2) and (y_1, y_2) have at least two common neighbors in $G - N[x]$. Let u_1, u_2 (resp. w_1, w_2) be two distinct common neighbors of x_1 and x_2 (resp. y_1 and y_2) in $G - N[x]$. By Menger's Theorem, $G - N[x]$ contains two disjoint paths from $\{u_1, u_2\}$ to $\{w_1, w_2\}$ and so $G > N[x] + x_1y_1 + x_2y_2 > K_9$, a contradiction.

Thus we may assume that $G - N[x]$ has at least three vertices and is not 2-connected. If $G - N[x]$ is disconnected, let $H_1 = K$ and H_2 be another connected component of $G - N[x]$. If $G - N[x]$ has a cut-vertex, say w, let H_1 be a connected component of $G - N[x] - w$ and let $H_2 = G - N[x] - V(H_1)$. In either case, H_1 and H_2 are disjoint connected subgraphs of $G - N[x]$ such that $M \subseteq N(H_1) \cup N(H_2)$ (because we have shown that $M \subseteq N(K)$). For $i = 1, 2$ let $A_i = N(H_i) \cap N(x)$. By Lemma 5.5.2 and Lemma 5.5.1, A_i is not complete and

$|A_i| \geq 5$ for $i = 1, 2$. By property (B), A_1 and A_2 satisfy properties (B1), (B2) or (B3).

Suppose first that A_1 and A_2 satisfy property (B1). Then there exist $a_i \in A_i$ such that $N(x) + \{a_1a : a \in A_1 - \{a_1\}\} + \{a_2a : a \in A_2 - \{a_2\}\} > K_8$. By contracting the connected sets $V(H_1) \cup \{a_1\}$ and $V(H_2) \cup \{a_2\}$ to single vertices, we see that $G > K_9$, a contradiction. Suppose next that A_1 and A_2 satisfy property (B2). Then there exist $a_1 \in A_1 - A_2$ and $a_2 \in A_2 - A_1$ such that $a_1a_2 \in E(G)$ and the vertices a_1 and a_2 have at most five common neighbors in $N(x)$. Thus $a_1, a_2 \in M$ by Lemma 5.5.3, and by another application of the same lemma there exists a common neighbor $u \in V(G) - N[x]$ of a_1 and a_2. But $a_1 \notin A_2$ and $a_2 \notin A_1$, and hence $u \notin V(H_1) \cup V(H_2)$. Thus $G - N[x]$ is disconnected and $H_1 = K$. But then $a_2 \in M \subseteq N(K) = N(H_1)$, a contradiction. Thus we may assume that A_1 and A_2 satisfy (B3), and hence $A_i \subseteq A_{3-i}$ for some $i \in \{1, 2\}$. As $M \subseteq A_1 \cup A_2$, we have $M \subseteq N(H_{3-i})$. Since A_i is not complete, let $a, b \in A_i$ be not adjacent. By property (B3), $N(x) + ab > K_7 \cup K_1$. Let P be an a-b path with interior in H_i. By contracting all but one of the edges of the path P and by contracting H_{3-i} similarly as above, we see that $G > K_9$, a contradiction. ■

Lemma 5.5.7 *$G - N[x]$ is disconnected for every vertex $x \in V(G)$ of degree at most 13.*

Proof. If $G - N[x]$ is not null, then it is disconnected by Lemma 5.5.6. Thus we may assume that x is adjacent to every other vertex of G. Let $H = G - x$. Then $e(H) = e(G) - n + 1 = 7n - 27 - n + 1 = 6|H| - 20$. By Theorem 1.2.3 applied to H the graph G has a K_9 minor or is a $(K_{1,2,2,2,2,2}, 6)$-cockade, a contradiction. ■

Lemma 5.5.8 *$\delta(G) \geq 10$.*

Proof. Let $x \in V(G)$ be such that $d(x) = \delta(G) = 9$. By Lemma 5.5.3, $\delta(N(x)) \geq 7$. Thus $\Delta(N(x)) = 8$. Let K, K' be two components of $G - N[x]$. By Lemma 5.5.2, $N(K)$ and $N(K')$ contain distinct pairs of nonadjacent vertices of $N(x)$, say a, b and c, d, respectively. Now $e(N(x) + ab + cd) \geq \frac{1}{2}(8 + 8 \times 7) + 2 = 34 = 6|N(x)| - 20$. Since $|N(x)| = 9$, Theorem 1.2.3 implies that $N(x) + ab + cd > K_8$, and so $G > N[x] + ab + cd > K_9$ by the existence of internally disjoint a-b and c-d paths with interiors in K,K' respectively, a contradiction. ■

Lemma 5.5.9 *Let $x \in V(G)$ be such that $10 \leq d(x) \leq 13$. Then there is no component K of $G - N[x]$ with two vertices w and w' so that $d_G(y) \geq 14$ for every vertex $y \in K - \{w, w'\}$ and $d_G(w), d_G(w') \geq d_G(x)$.*

Proof. Assume that such a component K exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let d_1 be defined as in the paragraph prior to (3). Let G_2' be a graph with $V(G_2') = V(G_2)$ and $e(G_2') = e(G_2) + d_1$ edges obtained from G by contracting edges in G_1. By Lemma 5.5.8, $|G_2'| \geq 11$. If $e(G_2') \geq 7|G_2'| - 26$, then by induction $G > G_2' > K_9$, a contradiction. Thus $e(G_2) = e(G_2') - d_1 \leq 7|G_2| - 27 - d_1 = 7|N(K)| + 7|K| - 27 - d_1$. On the other hand, every $u \in N(K)$ has a neighbor in K. From Lemma 5.5.3 applied to u we deduce that $d_{G_2}(u) \geq 8$.

Let $t = e_G(N(K), K)$ and $d = \delta(N(K))$. We have $e(G_2) = e(K) + t + e(N(K))$ and

$$2e(K) \geq 14(|K| - 2) + 2d(x) - t, \qquad (*)$$

and hence

$$e(G_2) \geq 7|K| - 14 + d(x) + \frac{t}{2} + \frac{1}{2}d|N(K)|.$$

By contracting the edge xz, where $z \in N(K)$ has minimum degree in $N(K)$, we see that $d_1 \geq |N(K)| - d - 1$. By combining this with the two inequalities for $e(G_2)$ we get

$$d(x) - \frac{t}{2} \geq 12 - 6|N(K)| + 2d(x) + \frac{1}{2}d(|N(K)| - 2).$$

Let $q = d(x) - |N(K)|$. Since $N(x)$ has minimum degree at least seven, it follows that $d \geq 7 - q$. Thus $d(x) - \frac{t}{2} \geq 5 - \frac{1}{2}(q+1)|N(K)| + 3q \geq -6$ because $|N(K)| = d(x) - q \leq 13 - q$. By $(*)$ $e(K) \geq 7|K| - 14 + d(x) - \frac{t}{2} \geq 7|K| - 20$. Since G is not contractible to K_9, we deduce by induction that $|K| < 9$. As $e(K) \geq 7|K| - 20$ we have, in fact, $|K| \leq 3$.

If K has a vertex z such that $d_G(z) \leq 13$ and z is adjacent to every other vertex of K, then z and the component of $G - N[z]$ containing x contradict Lemma 5.5.6. Thus K has no such vertex. Since $N(K) \neq N(x)$ by Lemma 5.5.6 applied to x and K, it follows that

$|K| = 3$. Thus K has a vertex of degree 14, and either K has only two edges, or every vertex of K has degree 14. Therefore $e(K) + e(K, N(x)) \geq 14 + 13 + 13 - 2 = 38$. It also follows that $d(x) = 13$, $|N(K)| = 12$, and hence $d \geq \delta(N(x)) - 1 \geq 6$. Let us recall that $d_1 \geq |N(K)| - d - 1 \geq 11 - d$. Since $d_G(z) \geq d(x) = 13$ for every $z \in V(K)$ we have

$$e(G_2') \geq e(G_2) + d_1 \geq \frac{1}{2}\, d|N(K)| + 38 + 11 - d \geq 7|G_2'| - 26,$$

and hence $G > G_2' > K_9$ by induction, a contradiction. ■

By Lemma 5.5.8 and the fact that $e(G) = 7n - 27$ there is a vertex of degree $10, 11, 12$ or 13 in G. Among the vertices of degree $10, 11, 12$ or 13 for which the order of the largest component of $G - N[x]$ is maximum, choose x so that its degree is minimum. Let K be a largest component of $G - N[x]$.

By Lemma 5.5.7, there is another component K' of $G - N[x]$. By Lemma 5.5.9, there is a vertex x' in K' of degree $d_G(x') \leq 13$. By the maximality of the order of K, $N(K) \subseteq N(x') \cap N(x)$. Thus $N(K) \subseteq N(K')$ and K is also a component of $G - N[x']$. By the choice of x, $d(x') \geq d(x)$. Thus every vertex of K' has degree in G at least $d(x)$. By Lemma 5.5.9, there are two distinct vertices $y' \neq x'$ and $z' \neq x'$ in K' of degree $d_G(y'), d_G(z') \leq 13$. Similarly, y' and z' are adjacent to every vertex in $N(K)$. By Lemma 5.5.6, there is a third component K'' of $G - N[x]$. By symmetry between K' and K'', K'' has three vertices x'', y'', z'' of degree at most 13 in G and each of them is adjacent to every vertex of $N(K)$. Let $G_1 = G - K$, $G_2 = G[N(K) \cup K]$ and let d_1 and d_2 be as in the paragraph after (2).

Since $\delta(N(x)) \geq 7$, $\delta(N(K)) \geq 7 - (13 - |N(K)|) = |N(K)| - 6$. Thus there is a subgraph T of $N(K)$ with $|N(K)| - 6$ vertices and at least $|N(K)| - 7$ edges. By contracting six independent edges, each with one end in $N(K) - T$ and the other end in $\{x', y', z', x'', y'', z''\}$, we see that $d_1 + e(N(K)) \geq e(K_6) + 6(|N(K)| - 6) + |N(K)| - 7 = 7|N(K)| - 28$. This, together with (3), implies that $d_2 \leq 1$. Thus the complement of $N(K)$ is a matching, and hence $\delta(N(K)) \geq |N(K)| - 2$. Assume that $|N(K)| \geq 7$. Then there exists a set A of seven vertices of $N(K)$ such that $G[A]$ contains a subgraph H isomorphic to $K_{1,2,2,2}$. Since K' is connected, there is a x'-y' path P with interior in K'. Now by contracting three independent edges, each with one end in A and the other in $\{x'', y'', z''\}$ and all but one of the edges of the path P, we get $G > G[V(H) \cup \{x', y'\}] + E(\overline{H}) + x'y' > K_9$, a contradiction. So we may assume that $|N(K)| = 6$ and thus $N(K) = K_6 - M$, where M is a matching. Then $e(N(K)) = 15 - |M|$ and $d_1 = |M|$ (by using x', y', z'). By Lemma 5.5.2, $d_2 \geq 1$. By (3), $7 \times 6 \geq 27 + |M| + 1 + 15 - |M| = 43$, a contradiction. This completes the proof of Theorem 1.3.1. ■

CHAPTER VI

BEYOND K_9 MINORS

6.1 Minimal counterexamples to Mader's bound when $p = 10$

We believe that our methods can be used to extend Theorem 1.2.2 for the case when $p = 10$. To apply our methods, one needs to find all graphs G without a $K_8 \cup K_1$ minor, where G has at most fifteen vertices and minimum degree eight. However, we do not see a way to do this without a lot of programming effort. Still, it may be possible to prove the following Conjecture 6.1.1, which we believe is the right extension for Theorem 1.2.2 when $p = 10$.

Conjecture 6.1.1 *Every graph on $n \geq 10$ vertices and at least $8n - 35$ edges has a K_{10} minor, or is a $(K_{1,1,2,2,2,2,2,2}, 7)$-cockade, or is isomorphic to one of the following graphs: $K_{1,2,2,2,3,3}$, $K_{2,2,3,3,4}$, $K_{2,3,3,3,3}$, $K^-_{2,3,3,3,3}$, $K_{2,2,2,2,2,3}$, $K^-_{2,2,2,2,2,3}$, or J_0, where J_0 is obtained from two disjoint copies of $K_{2,2,2,2,2,3}$ by identifying cliques of size six.*

6.2 Some evidence for Conjecture 6.1.1

As noted in Section 5.3, our proof of Theorem 1.3.1 uses induction by deleting and contracting edges of G. To apply our methods to prove Conjecture 6.1.1, one needs to investigate graphs G such that the new graph $G - xy$ or G/xy is a $(K_{1,1,2,2,2,2,2,2}, 7)$-cockade or is isomorphic to one of the graphs listed in Conjecture 6.1.1. We do that next, which will establish some evidence for Conjecture 6.1.1.

Lemma 6.2.1 *Let G be a $(K_{1,1,2,2,2,2,2,2}, 7)$-cockade, or let G be isomorphic to the graph J_0 or one of the following graphs: $K_{1,2,2,2,3,3}$, $K_{2,2,3,3,4}$, $K_{2,3,3,3,3}$, $K^-_{2,3,3,3,3}$, $K_{2,2,2,2,2,3}$,* *or $K^-_{2,2,2,2,2,3}$, and let x and y be nonadjacent vertices in G. Then $G+xy$ is contractible to K_{10} or $G+xy$ is isomorphic to $K_{2,3,3,3,3}$ or $K_{2,2,2,2,2,3}$.*

Proof. This is easily checked if G is isomorphic to one of the following graphs: J_0, $K_{1,2,2,2,3,3}$, $K_{2,2,3,3,4}$, $K_{2,3,3,3,3}$, $K^-_{2,3,3,3,3}$, $K_{2,2,2,2,2,3}$ $K^-_{2,2,2,2,2,3}$, and $K_{1,1,2,2,2,2,2}$. So we may assume that G is obtained from H_1 and H_2 by identifying cliques of size seven, where H_1 and H_2 are $(K_{1,1,2,2,2,2,2}, 7)$-cockades. If x and y are both in H_1 or H_2, then $H_1+xy > K_{10}$ or $H_2+xy > K_{10}$ by induction. So we may assume that $x \in V(H_1)-V(H_2)$ and $y \in V(H_2)-V(H_1)$. Note that no $(K_{1,1,2,2,2,2,2}, 7)$-cockade contains K_8 as a subgraph. Therefore there exists $z \in V(H_1) \cap V(H_2)$ such that $zy \notin V(G)$. Now by contracting $V(H_1)-V(H_2)$ to the vertex z in $G+xy$, the resulting graph is H_2+zy. By induction, $H_2+zy > K_{10}$. ■

Now we consider contractions in a graph G, which has two adjacent vertices x and y such that x and y have six or seven common neighbors and G/xy is a $(K_{1,1,2,2,2,2,2}, 7)$-cockade or is isomorphic to one of the graphs listed in Conjecture 6.1.1.

Lemma 6.2.2 *Let G be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with six or seven common neighbors. If G/xy is isomorphic to $K_{2,3,3,3,3}$ or $K_{2,2,2,2,2,3}$, then $G > K_{10}$.*

Proof. Let w be the new vertex in G/xy. Since x and y have at least six common neighbors, there exist distinct vertices $w_1, w_2, w_3, w_4 \in V(G/xy)-w$ such that $w_1w_2, w_3w_4 \notin E(G/xy)$, $G+w_1w_2+w_3w_4 > K_{10}$, and w_1, w_2, w_3 are common neighbors of x and y in G. Moreover, w_4 is adjacent to x or y, say to y, in G. By contracting the edges xw_2 and yw_4 we see that G has a K_{10} minor, as desired. ■

Lemma 6.2.3 *Let G be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly seven common neighbors. If G/xy is isomorphic to J_0, then $G > K_{10}$.*

Proof. Let w be the new vertex in G/xy and let J_0 be obtained from H_1 and H_2, where $H_i = K_{2,2,2,2,2,3}$, $i = 1, 2$. Let S be the set of seven common neighbors of x and y. If $w \in V(H_i) - V(H_{3-i})$ for some i, then $S \subset V(H_i)$. By Lemma 6.2.2, $G > H_i > K_{10}$. So we may assume that $w \in V(H_1) \cap V(H_2)$ and $|S \cap V(H_1)| \leq |S \cap V(H_2)|$. Then $|S \cap V(H_1)| \leq 3$. Let H_2' be obtained from G by deleting the vertices in $V(H_1) - (V(H_2) \cup S)$ and contracting $|S \cap V(H_1)|$ independent edges, each with one end in $S \cap V(H_1)$ and the other in $V(H_1) \cap V(H_2) - \{w\}$. Then $H_2' = K_{2,2,2,2,2,3}$ and x and y have exactly seven common neighbors. By Lemma 6.2.2, $G > H_2' > K_{10}$, as desired. ■

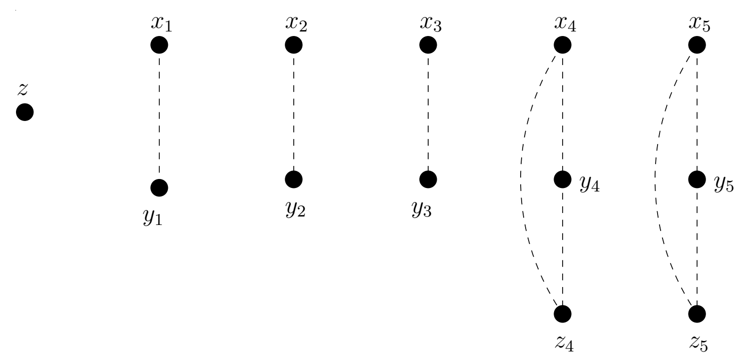


Figure 18: The complement of $K_{1,2,2,2,3,3}$.

Lemma 6.2.4 *Let G be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly seven common neighbors. If G/xy is isomorphic to $K_{1,2,2,2,3,3}$, then $G > K_{10}$ or G is isomorphic to $K_{2,2,3,3,4}$ or $K^-_{2,3,3,3,3}$.*

Proof. Let w be the new vertex in G/xy and let $z \in V(G/xy)$ be such that $d_{G/xy}(z) = 12$. If $w \neq z$, then since x and y have exactly seven common neighbors, there exist distinct vertices $w_1, w_2, w_3, w_4 \in V(G/xy) - w$ such that $w_1w_2, w_3w_4 \notin E(G/xy)$, and w_1, w_2, w_3 are common neighbors of x and y in G. Moreover, w_4 is adjacent to x or y, say to y, in G. By contracting the edges xw_2 and yw_4 we see that G has a K_{10} minor, as desired. So we may assume that $w = z$. Let $V(G/xy)$ be labeled as depicted in Figure 18.

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$. Suppose x is complete to X. If y is also complete to X, we may assume that $d_G(x) \geq d_G(y)$. Then x has at least three neighbors, say a, b, c, in $\{y_1, y_2, y_3, y_4, y_5, z_4, z_5\}$. Now by contracting four independent edges, each with one end in $\{y, a, b, c\}$ and the other in $\{y_1, y_2, y_3, y_4, y_5, z_4, z_5\} - \{a, b, c\}$, we see that $G > K_{10}$. So we may assume that there exists a color class of G/xy that are non-neighbors of x, similarly, there exists a color class of G/xy that are non-neighbors of y. Since x and y have exactly seven common neighbors, we may assume that $xx_1, xy_1 \notin E(G)$. If $yx_4, yy_4, yz_4 \notin E(G)$, then G is isomorphic to $K_{2,2,3,3,4}$. So we may assume that $yx_2, yy_2 \notin E(G)$. Since x and y have exactly seven common neighbors, we may assume that either y_3 or z_4 are not common neighbors. In either case, we may assume that $yy_3, yz_4 \notin E(G)$. It's easy to see that G is isomorphic to $K^-_{2,3,3,3,3}$, see Figure 19. ■

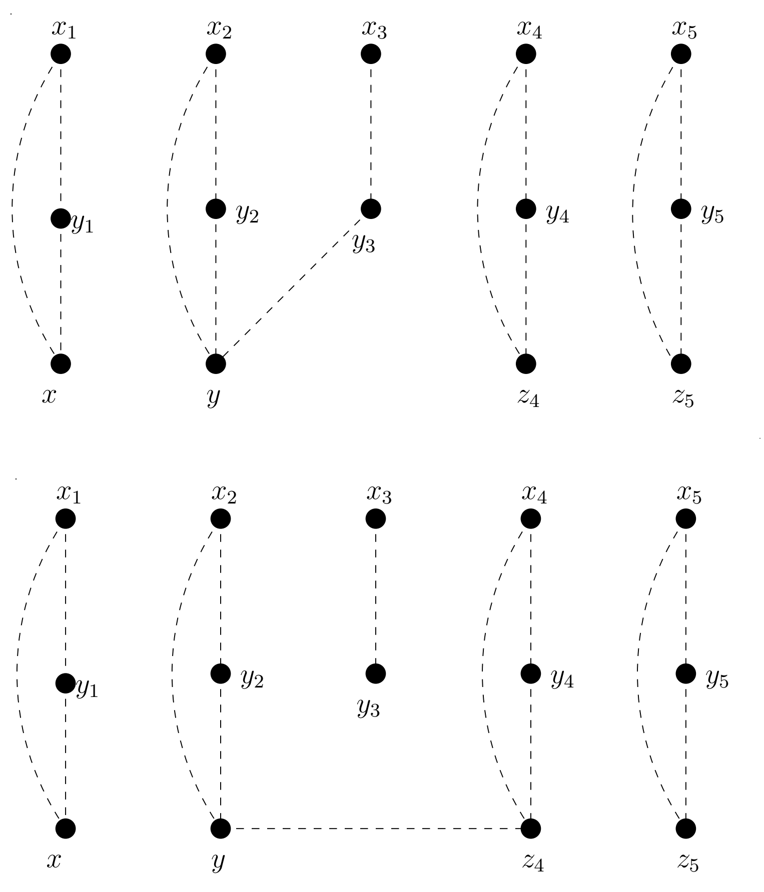


Figure 19: The complements of $K^-_{2,3,3,3,3}$.

Lemma 6.2.5 *Let G be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly seven common neighbors. If G/xy is isomorphic to $K_{2,2,3,3,4}$, $K^-_{2,3,3,3,3}$, or $K^-_{2,2,2,2,2,3}$, then $G > K_{10}$.*

Proof. Let w be the new vertex in G/xy. Since x and y have exactly seven common neighbors, there exist distinct vertices $w_1, w_2, w_3, w_4 \in V(G/xy) - w$ such that $w_1w_2, w_3w_4 \notin E(G/xy)$, $G + w_1w_2 + w_3w_4 > K_{10}$, and w_1, w_2, w_3 are common neighbors of x and y in G. Moreover, w_4 is adjacent to x or y, say to y, in G. By contracting the edges xw_2 and yw_4 we see that G has a K_{10} minor, as desired. ■

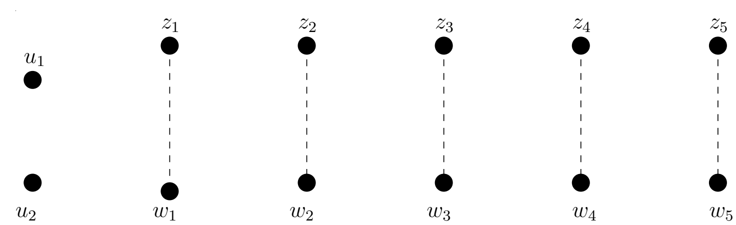


Figure 20: The complement of $K_{1,1,2,2,2,2,2}$.

Lemma 6.2.6 *Let G be a graph and let x, y be adjacent vertices of G with exactly seven common neighbors. If G/xy is isomorphic to $K_{1,1,2,2,2,2,2}$, then G has a K_{10} minor, unless G is isomorphic to $K_{1,2,2,2,3,3}$ or $K^-_{2,2,2,2,2,3}$ and in either case, $d_G(x) + d_G(y) = 20$.*

Proof. Let w be the new vertex in G/xy. Let the vertices of G/xy be as depicted in Figure 20.

Assume first that $w \neq u_1, u_2$, say $w = z_1$. Since x and y have seven common neighbors, we may assume that z_2, w_2, z_3 are common neighbors of x and y. Moreover, w_3 is adjacent to x or y, say to y. By contracting the edges xz_2, yz_3 and z_4z_5 we see that G has a K_{10} minor, as desired.

Thus we may assume that $w = u_1$. If u_2 is a common neighbor of x and y, then by Lemma 5.3.2, we have $G > K_{10}$ or G is isomorphic to $K_{1,2,2,2,3,3}$ and $d_G(x) + d_G(y) = 20$. So we may assume that $u_2x \in E(G)$ but $u_2y \notin E(G)$. If x is adjacent to z_i for $1 \le i \le 5$, then since x and y have seven common neighbors, we may assume that $xw_4, xw_5 \in E(G)$, and y has at last one neighbor in $\{w_1, w_2, w_3\}$. By contracting three independent edges, each with one end in $\{w_1, w_2, w_3\}$ and the other in $\{y, w_4, w_5\}$, we see that $G > K_{10}$. So we may assume that $xz_1, xw_1 \notin E(G)$ (by swapping z_i with w_i). Then $yz_1, yw_1 \in E(G)$. Since x and y have exactly seven common neighbors, we may assume that $z_2x \in E(G)$ or $yz_2 \in E(G)$ but not both. In either case, it can be easily checked that G is isomorphic to $K^-_{2,2,2,2,2,3}$ and $d_G(x) + d_G(y) = 20$, see Figure 21. ■

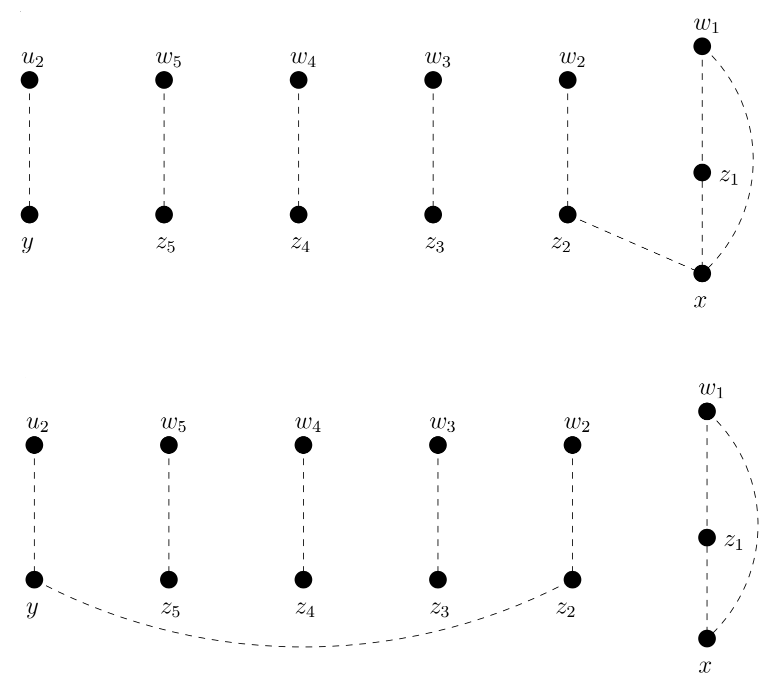


Figure 21: The complements of $K^-_{2,2,2,2,2,3}$.

Lemma 6.2.7 *Let G be a graph. Let $x, y \in V(G)$ be such that $xy \in E(G)$ with exactly seven common neighbors. If G/xy is a $(K_{1,1,2,2,2,2,2}, 7)$-cockade, then either $G > K_{10}$, or G is isomorphic to $K_{1,2,2,2,3,3}$ or $K^-_{2,2,2,2,2,3}$ and in either case, $d_G(x) + d_G(y) = 20$.*

Proof. We proceed by induction on $|G|$. By Lemma 6.2.6 we may assume that $G/xy = H_1 \cup H_2$, where $H_1 \cap H_2$ is a complete graph on seven vertices and both H_1 and H_2 are $(K_{1,1,2,2,2,2,2}, 7)$-cockades. Let w be the new vertex of G/xy. For $i = 1, 2$ let $H_i^* = G[(V(H_i) - \{w\}) \cup \{x, y\}]$. If $w \in V(H_1) - V(H_2)$, then $H_1^* \neq K_{1,2,2,2,3,3}$ or $K^-_{2,2,2,2,2,3}$ (because the latter graph has no K_7 subgraph) and the result follows by induction applied to H_1^*. From the symmetry we may assume that $w \in V(H_1) \cap V(H_2)$. Let $S = V(H_1) \cap V(H_2) - \{w\}$; thus $V(H_1^*) \cap V(H_2^*) = S \cup \{x, y\}$. Let Z denote the set of seven common neighbors of x and y in G. If $Z \subseteq V(H_1^*)$, then by induction applied to H_1^* we may assume that H_1^* is isomorphic to $K_{1,2,2,2,3,3}$ or $K^-_{2,2,2,2,2,3}$, and in either case, $d_{H_1^*}(x) + d_{H_1^*}(y) = 20$. Since H_1^* has no K_7 subgraph one of x, y, say y, is not adjacent to some $s \in S$ such that $H_1^* + sy > K_{10}$ and y has at least one neighbor in $V(H_2) - V(H_1)$ (To see such an edge ys exists, note that $K_{1,2,2,2,3,3}$ is edge maximal. If H_1^* is isomorphic to $K^-_{2,2,2,2,2,3}$, from the proof of Lemma 6.2.6, we see that $K^-_{2,2,2,2,2,3} + u_2 y > K_{10}$, where u_2 and y are labeled in the proof of Lemma 6.2.6). By using a path with ends y and s and interior in $H_2^* - V(H_1^*)$ we deduce that $G > H_1^* + sy > K_{10}$ by Lemma 6.2.1, as desired.

Thus we may assume that $Z - V(H_1^*) \neq \emptyset \neq Z - V(H_2^*)$. Since H_2 is a $(K_{1,1,2,2,2,2,2}, 7)$-cockade, it is 7-connected. Let $k = |Z - V(H_1)|$. Since $|Z \cap V(H_2)| \leq 6$ we have $|S - Z| = 6 - |Z \cap S| \geq k$. Thus there exist k disjoint paths $P_1, P_2, \ldots, P_k$ in $H_2 - (Z \cap S) - w$ between $Z \cap V(H_2 - S)$ and $S - Z$. Consequently H_1^* has a supergraph H_1' on the same vertex set such that $H_1' < G$ and x, y have exactly seven common neighbors in H_1'. By induction H_1' is isomorphic to $K_{1,2,2,2,3,3}$ or $K^-_{2,2,2,2,2,3}$ and in either case, $d_{H_1'}(x) + d_{H_1'}(y) = 20$. By symmetry the same holds for the analogous graph H_2'. It follows that in H_1' the vertex x has a unique non-neighbor in S, say x' such that $H_1' + xx' > K_{10}$. Then $x' \notin V(P_1 \cup \cdots \cup P_k)$. From the symmetry between H_1 and H_2 we may assume that $k \leq 3$. It follows that the k disjoint paths $P_1, \ldots, P_k$ can each be chosen of length one, and that there exists a common neighbor of x and x' in $V(H_2^*)$, say u, that does not belong to any of the paths. Thus by contracting the edge ux' and all the edges of the paths $P_1, \ldots, P_k$ we deduce that $G > H_1' + xx' > K_{10}$ by Lemma 6.2.1, as desired. ∎

6.3 An extremal function for K_{10} and K_{11} minors

In this section, we shall establish a weak bound on the extremal functions for K_{10} and K_{11} minors. We prove the following.

Theorem 6.3.1 *For $10 \le p \le 11$, every graph on $n \ge p$ vertices and at least $11n - 65 + (p-10)(2n-23) = (2p-9)n - 23p + 165$ edges has a K_p minor (i.e. every graph on $n \ge 10$ vertices and at least $11n - 65$ edges has a K_{10} minor, and every graph on $n \ge 11$ vertices and $13n - 88$ edges has a K_{11} minor).*

As noted in previous chapters, we need to examine graphs on $2p - 7 \le n \le 4p - 19$ vertices and $\delta(G) \ge 2p - 9$. Lemma 6.4.4 shows that all those graphs have a $K_{p-2} \cup K_1$ minor. This allows us to give a computer-free proof for Theorem 6.3.1 in the next section.

Lemma 6.3.2 *For any integer $10 \le p \le 11$, every graph on $2p - 7 \le n \le 4p - 19$ vertices and $\delta(G) \ge 2p - 9$ has a $K_{p-2} \cup K_1$ minor.*

Proof. Let x be a vertex of minimum degree in G. If $p = 10$, then $d(x) \ge 11$ and $n \le 21$. Thus $e(G - x)) \ge \frac{1}{2} d(x)n - d(x) = \frac{1}{2} d(x)(n-2) \ge \frac{11(n-2)}{2} \ge 6(n-1) - 19$ because $n \le 21$. By Theorem 1.2.3, $G - x > K_8$. If $p = 11$, then $d(x) \ge 13$ and $n \le 25$. Hence $e(G - x) \ge \frac{13(n-2)}{2} > 7(n-1) - 27$. By Theorem 1.3.1, $G - x > K_9$. In either case, we obtain a $K_{p-2} \cup K_1$ minor, as desired. ■

6.4 Proof of Theorem 6.3.1

In this section, we are going to prove Theorem 6.3.1 by induction on n. Since $e(G) \ge (2p-9)n - 23p + 165$, namely,

$$e(G) \geq \begin{cases} 11n - 65 & \text{if} \quad p = 10 \\ 13n - 88 & \text{if} \quad p = 11, \end{cases}$$

one can easily check that the only graph G with p vertices and $e(G) \geq (2p-9)n - 23p + 165$ is K_p. So we may assume that $n \geq p+1$ and that the assertion holds for smaller values of n. Throughout this section we assume that G is a graph with n vertices and $e(G) \geq (2p-9)n - 23p + 165$ but G is not contractible to K_p. We may assume that $e(G) = (2p-9)n - 23p + 165$. Suppose that G has a vertex x of degree at most $2p-9$. Then $e(G-x) \geq (2p-9)(n-1) - 23p + 165$, and hence $G > G - x > K_p$ by induction, a contradiction. Suppose now that G has two adjacent vertices x, y with at most $2p-10$ common neighbors. Then $e(G/xy) \geq (2p-9)(n-1) - 23p + 165$. By induction, $G > K_p$, a contradiction. Thus $\delta(G) \geq 2p-8$ and $\delta(N(x)) \geq 2p-9$ for any vertex x in G. If x is of degree $2p-8$ in G, then we see that $N[x] = K_{2p-7} > K_p$, a contradiction. Hence

(1) $\delta(G) \geq 2p-7$ and $\delta(N(x)) \geq 2p-9$ for any $x \in V(G)$.

Let S be a separating set of vertices in G, and let G_1 and G_2 be proper subgraphs of G so that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[S]$. Let $m_i = (2p-9)|G_i| - 23p + 165 - e(G_i)$, $i = 1, 2$. Then $(2p-9)n - 23p + 165 = e(G) = e(G_1) + e(G_2) - e(G[S]) = (2p-9)n + (2p-9)|S| - 46p + 165 + 165 - m_1 - m_2 - e(G[S])$, and so

(2) $(2p-9)|S| = -165 + 23p + m_1 + m_2 + e(G[S])$. That is,

$$\begin{cases} 11|S| = 65 + m_1 + m_2 + e(G[S]) & \text{if} \quad p = 10 \\ 13|S| = 88 + m_1 + m_2 + e(G[S]) & \text{if} \quad p = 11. \end{cases}$$

For $i = 1, 2$, let d_i be the maximum number of edges that can be added to G_{3-i} by contracting edges of G with at least one end in G_i. More precisely, let d_i be the largest integer so that G_i contains disjoint set of vertices $V_1, V_2, \ldots, V_p$ so that $G_i[V_j]$ is connected, $|S \cap V_j| = 1$ for $1 \leq j \leq p = |S|$, and so that the graph obtained from G_i by contracting

$V_1, V_2, \ldots, V_p$ and deleting $V(G) - (\bigcup_j V_j)$ has $e(G[S]) + d_i$ edges. By (1), $\delta(G) \geq 2p - 7$. Thus $|G_i| \geq 2p - 6$ for $i = 1, 2$. By induction, $d_1 + 1 \leq m_2$ and $d_2 + 1 \leq m_1$. Thus by (2)

(3) $(2p - 9)|S| \geq -163 + 23p + d_1 + d_2 + e(G[S])$. That is,

$$\begin{cases} 11|S| \geq 67 + d_1 + d_2 + e(G[S]) & \text{if} \quad p = 10 \\ 13|S| \geq 90 + d_1 + d_2 + e(G[S]) & \text{if} \quad p = 11. \end{cases}$$

In particular, $|S| \geq 7$. If S is a minimal separating set, then let $v \in S$ be a vertex of minimum degree in $G[S]$. By choosing $V_1 = V(G_i) - (S - \{v\})$ and the rest of the sets V_j to be singletons, we see that $d_i \geq |S| - 1 - \delta(G[S])$ for $i = 1, 2$. Thus

(4) if S is a minimal separating set, then

$$(2p - 11)|S| \geq -165 + 23p + e(G[S]) - 2\delta(G[S])) \geq -165 + 23p + \frac{1}{2}(|S| - 4)\delta(G[S]).$$

That is,

$$\begin{cases} 9|S| \geq 65 + d_1 + d_2 + e(G[S]) - 2\delta(G[S])) \geq 65 + \frac{1}{2}(|S| - 4)\delta(G[S]) & \text{if} \quad p = 10 \\ 11|S| \geq 88 + d_1 + d_2 + e(G[S]) - 2\delta(G[S])) \geq 88 + \frac{1}{2}(|S| - 4)\delta(G[S]) & \text{if} \quad p = 11. \end{cases}$$

From (4) it follows that

Lemma 6.4.1 *G is 8-connected.*

Lemma 6.4.2 *There is no separating set S with a vertex x so that $G[S - x]$ is complete.*

Proof. Suppose that $G[S - x]$ is complete and let G_1, G_2 be as above. We may assume that S is a minimal separating set. By Lemma 6.4.1, $|S| \geq 8$. If $|S| \geq p - 1$, by contracting $V(G_1) - S$ to x and $V(G_2) - S$ to a new vertex, we get a K_p minor, a contradiction. So we may assume that $8 \leq |S| \leq p-2 \leq 9$. By (4),

$$\begin{cases} 9 \times 8 = 9|S| \geq 65 + (e(K_7) + \delta(G[S])) - 2\delta(G[S])) \geq 81 + \delta(G[S]) & \text{if} \quad p = 10 \\ 11 \times 9 \geq 11|S| \geq 88 + (e(K_7) + \delta(G[S])) - 2\delta(G[S])) \geq 109 + \delta(G[S]) & \text{if} \quad p = 11. \end{cases}$$

From this it follows that $\delta(G[S]) \geq 9$, which is impossible because $\delta(G[S]) \leq |S| - 1 \leq 8$. ■

As $e(G) = (2p-9)n - 23p + 165$, we have $\delta(G) \leq 4p - 19$. By (1),

Lemma 6.4.3 *$2p - 7 \leq \delta(G) \leq 4p - 19$.*

Lemma 6.4.4 *Let $x \in V(G)$ be such that $2p - 7 \leq d(x) \leq 4p - 19$. Then there is no component K of $G - N[x]$ such that $N(K') \cap M \subseteq N(K)$ for every component K' of $G - N[x]$, where M is the set of vertices of $N(x)$ not adjacent to all other vertices of $N(x)$.*

Proof. Assume such a component K exists. Among all vertices x with $2p - 7 \leq d(x) \leq 4p - 19$ for which such a component exists, choose x to be of minimal degree. We first prove that $M \subseteq N(K)$. Suppose for a contradiction that $M - N(K) \neq \emptyset$, and let $y \in M - N(K)$ be such that $d(y)$ is minimum. Clearly, $d(y) < d(x)$. Let J be the component of $G - N[y]$ containing K. Since $d(y) < d(x)$ the choice of x implies that $N(x) - N[y] \not\subseteq V(J)$, and hence some component H of $N(x) - N[y]$ is disjoint from $N(K)$. We have $d_G(z) \geq d_G(y)$ for all $z \in V(H)$ by the choice of y. Let $t = |V(H)|$. Then $t \geq 2$, for otherwise the vertex y and component H contradict the choice of x. On the other hand $t \leq d(x) - d(y) \leq 2p - 12$. From (1) applied to y we deduce that $N(y) \cap N(x)$ has minimum degree at least $2p - 9$. Thus

$$
\begin{aligned}
e(N(x) \cap N(y)) &\geq \frac{1}{2}\,(2p-9)|N(x) \cap N(y)| \\
&\geq \begin{cases} (6|N(x) \cap N(y)| - 19) + (19 - \frac{|N(x) \cap N(y)|}{2}) & \text{if} \quad p = 10 \\ (7|N(x) \cap N(y)| - 26) + (26 - \frac{|N(x) \cap N(y)|}{2}) & \text{if} \quad p = 11 \end{cases} \\
&\geq \begin{cases} 6|N(x) \cap N(y)| - 19 & \text{if} \quad p = 10 \\ 7|N(x) \cap N(y)| - 26 & \text{if} \quad p = 11, \end{cases}
\end{aligned}
$$

because $|N(x) \cap N(y)| \leq 25$. By Theorem 1.2.3 and Theorem 1.3.1, $N(x) \cap N(y) > K_{p-2}$ and so $G[N[x] \cap N[y]] > K_p$, a contradiction. ■

It follows from Lemma 6.4.4 that

Lemma 6.4.5 *For any vertex x with $2p - 7 \leq d_G(x) \leq 4p - 19$, $G - N(x)$ is disconnected and each component of $G - N[x]$ has at least three vertices.*

Proof. Suppose $G - N[x]$ has a component H with $|H| \leq 2$. By Lemma 6.4.4, $N(H) \neq N(x)$. Let $u \in V(H)$. Then $d_G(u) \leq d(x)$. Now u and the component of $G - N[u]$ containing x contradict Lemma 6.4.4. ■

Lemma 6.4.6 *Let $x \in V(G)$ be such that $2p - 7 \leq d(x) \leq 4p - 19$. Then there is no component K of $G - N[x]$ with three vertices w_1, w_2, and w_3 so that $d_G(y) \geq 4p - 18$ for every vertex $y \in V(K) - \{w_1, w_2, w_3\}$ and $d_G(w_i) \geq d_G(x)$ for $i = 1, 2, 3$.*

Proof. Assume that such a component K exists. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let d_1 be defined as in the paragraph prior to (3). Let G_2' be a graph with $V(G_2') = V(G_2)$ and $e(G_2') = e(G_2) + d_1$ edges obtained from G by contracting edges in G_1. By Lemma 6.4.3, $|G_2'| \geq 2p - 6$. If $e(G_2') \geq (2p-9)|G_2'| - 23p + 165$, then by induction $G > G_2' > K_p$, a contradiction. Thus $e(G_2) = e(G_2') - d_1 \leq (2p-9)|G_2| - 23p + 164 - d_1 = (2p-9)|N(K)| +$

$(2p-9)|K| - 23p + 164 - d_1$. On the other hand, let $t = e_G(N(K), K)$ and $d = \delta(N(K))$. We have $e(G_2) = e(K) + t + e(N(K))$ and

$$2e(K) \geq (4p-18)(|K|-3) + 3d(x) - t, \tag{a}$$

and hence

$$e(G_2) \geq (2p-9)|K| - 3(2p-9) + \frac{t+3d(x)}{2} + \frac{1}{2}\, d|N(K)|.$$

By contracting the edge xz, where $z \in N(K)$ has minimum degree in $N(K)$, we see that $d_1 \geq |N(K)| - d - 1$. By combining this with the two inequalities for $e(G_2)$ we get

$$\frac{3d(x)-t}{2} \geq 17p - 138 - (2p-10)|N(K)| + 3d(x) + \frac{1}{2}\, d(|N(K)|-2). \tag{b}$$

Let $q = d(x) - |N(K)|$. Since $N(x)$ has minimum degree at least $2p-9$, it follows that $d \geq 2p - 9 - q$. We claim that $3 \leq |K| \leq 4$. We consider the following two cases.

Case 1. $p = 10$.

In this case, $d \leq 11 - q$ and $13 \leq d(x) \leq 21$. Thus by (b), $\frac{3d(x)-t}{2} \geq 11 - \frac{1}{2}(q+1)|N(K)| + 4q \geq -23$ because $|N(K)| = d(x) - q \leq 21 - q$. By (a) $e(K) \geq 11|K| - 33 + \frac{3d(x)-t}{2} \geq 11|K| - 56$. Since G is not contractible to K_{10}, we deduce by induction that $|K| < 10$. As $e(K) > 11|K| - 56$ we have, in fact, $|K| \leq 7$. Since $|K| \geq 3$, we have $3 \leq |K| \leq 7$. Suppose $5 \leq |K| \leq 7$. Let L be the subgraph of G induced by $N[x] \cup V(K)$. Then the edge-set of L consists of edges of $N(x)$, edges incident with x, and edges incident with $V(K)$. Note that $\delta(N(x)) \geq 11$. Thus

$$\begin{aligned} e(L) \geq\ & \frac{11d(x)}{2} + d(x) + 3(d(x) - (|K|-1)) + (|K|-3)(22 - (|K|-1)) + \frac{1}{2}|K|(|K|-1) \\ & \geq 11(d(x) + 1 + |K|) + \frac{1}{2}(-3d(x) + 23|K| - |K|^2 - 77) > 11|V(L)| - 65, \end{aligned}$$

because $d(x) \leq 21$ and $5 \leq |K| \leq 7$. Thus $L > K_{10}$ by induction, a contradiction. This proves that $3 \leq |K| \leq 4$, as claimed.

Case 2. $p = 11$.

In this case $d \leq 13 - q$ and $15 \leq d(x) \leq 25$. Thus by (b), $\frac{3d(x)-t}{2} \geq 36 - \frac{1}{2}(q+1)|N(K)| + 4q > -45$ because $|N(K)| = d(x) - q \leq 25 - q$. By (a) $e(K) \geq 13|K| - 39 + \frac{3d(x)-t}{2} \geq$

$13|K| - 84$. Since G is not contractible to K_{11}, we deduce by induction that $|K| \leq 10$. Since $|K| \geq 3$, we have $3 \leq |K| \leq 10$. Suppose $5 \leq |K| \leq 10$. Let L be the subgraph of G induced by $N[x] \cup V(K)$. Then the edge-set of L consists of edges of $N(x)$, edges incident with x, and edges incident with $V(K)$. Note that $\delta(N(x)) \geq 11$. Thus

$$\begin{aligned}
e(L) &\geq \frac{13d(x)}{2} + d(x) + 3(d(x) - (|K|-1)) + (|K|-3)(26 - (|K|-1)) + \frac{1}{2}|K|(|K|-1) \\
&\geq 13(d(x) + 1 + |K|) + \frac{1}{2}(-5d(x) + 27|K| - |K|^2 - 156) > 13|V(L)| - 88,
\end{aligned}$$

because $d(x) \leq 25$ and $5 \leq |K| \leq 10$. Thus $L > K_{11}$ by induction, a contradiction. This proves that $3 \leq |K| \leq 4$, as claimed.

From the above, we have $3 \leq |K| \leq 4$. We now show that K has a vertex of degree at most $4p-19$. Suppose each vertex of K is of degree at least $4p-18$. Then $|N(K)| \geq |N(x)| - 2$. Thus $\delta(N(K)) \geq (2p-9) - 2 = 2p - 11$ and so $e(N(K)) \geq \frac{(2p-11)|N(K)|}{2} > (p-5)|N(K)| - \binom{p-2}{2} + 2$ because $|N(K)| \leq 4p - 20$. By Theorem 1.2.2 and Theorem 1.2.3, $N(K) > K_{p-3}$. If each vertex of K is adjacent to all other vertices in $K \cup N(K)$, then $K > K_3$ (because $\delta(K) \geq 2$) and thus $G[K \cup N(K)] > K_p$, a contradiction. Thus $|K| = 4$ and $d(x) = 4p - 19$ and $|N(K)| = 4p - 20$. Clearly, $d \geq (2p-9) - 1 = 2p - 10$. It can be easily checked that K can be partitioned into two connected components, say K_1 and K_2, such that each vertex of $N(K)$ has at least one neighbor in K_1 and K_2, respectively, and there is an edge between K_1 and K_2 in K. Let $N'(x)$ be obtained from $N(x)$ by contracting x onto a vertex of minimum degree in $N(x)$. Then $e(N'(x)) \geq \frac{d|N(K)|}{2} + (|N(K)| - 1 - d) = (p-4)|N(K)| - 2p + 9 > (p-4)|N(K)| - \binom{p-1}{2} + 2$ because $d \geq 2p - 10$ and $10 \leq p \leq 11$. By Theorem 1.2.3 and Theorem 1.3.1, $N'(x) > K_{p-2}$. Now by contracting K_1 and K_2 into different vertices, we see that $G[K \cup N'(x)] > K_p$, a contradiction. This proves that K has a vertex, say z, of degree at most $4p - 19$ in G.

We may choose z to be of minimal degree among all vertices of K of degree at most $4p - 19$ in G. If z is adjacent to every other vertex of K, then z and the component of $G - N[z]$ containing x contradict Lemma 6.4.4. So we may assume that there is a vertex $z' \neq z$ in K such that $zz' \notin E(G)$. If z is adjacent to every other vertex of $K - z'$, then either z and the component $\{z'\}$ in $G - N[z]$ (in this case z' is adjacent to every vertex in $N(z)$) or z and the component of $G - N[z]$ containing x contradict Lemma 6.4.4. Thus z has another non-neighbor, say $z'' \neq z'$ in K. Then $|K| = 4$, and $d_G(z) = 4p - 19$ and $|N(z) \cap N(x)| = |N(K)| = |N(x)| - 1$. If z' is adjacent to every vertex in $N(z)$, then z and the component $\{z'\}$ contradict Lemma 6.4.4. Thus $d_G(z') = 4p - 19$ and $z'z'' \in E(G)$, but now z' and the component $K - \{z', z''\}$ contradict Lemma 6.4.4. ■

By Lemma 6.4.3 and the fact that $e(G) = (2p - 9)n - 23p + 165$ there is a vertex of degree $2p - 7, \ldots,$ or $4p - 19$ in G. Among the vertices of degree $2p - 7, \ldots,$ or $4p - 19$ for which the order of the largest component of $G - N[x]$ is maximum, choose x so that its degree is minimum. Let K be a largest component of $G - N[x]$. By Lemma 6.4.5, there is another component K' of $G - N[x]$. By Lemma 6.4.6, there is a vertex x' in K' of degree $d_G(x') \leq 4p - 19$. By the maximality of the order of K, $N(K) \subseteq N(x') \cap N(x)$. Thus $N(K) \subseteq N(K')$ and K is also a component of $G - N[x']$. By the choice of x, $d(x') \geq d(x)$. Thus every vertex of K' has degree in G at least $d(x)$. By Lemma 6.4.6, there are three distinct vertices $y' \neq x'$, $z' \neq x'$ and $w' \neq x'$ in K' of degree $d_G(y'), d_G(z'), d_G(w') \leq 4p-19$. Similarly, y', z' and w' are adjacent to every vertex in $N(K)$. By Lemma 6.4.4, there is a third component K'' of $G - N[x]$ such that $N(K'') - N(K) \neq \emptyset$. By symmetry between K' and K'', K'' has four vertices x'', y'', z'', w'' of degree at most $4p - 19$ in G and each of them is adjacent to every vertex of $N(K)$. By Lemma 6.4.1, $|N(K)| \geq 8$. Since $K'' \neq K'$. Let $a \in N(K'') - N(K')$ and $b \in N(K)$. If $|N(K)| \geq p - 2$, then by contracting K onto b, the edge ax'' and seven independent edges, each with one end in $N(K) - \{b\}$ and the other end in $\{x', y', z', w', y'', z'', w''\}$, we see that $G > K_p$, a contradiction. So we may assume that $8 \leq |N(K)| \leq p - 3 \leq 8$. Thus $p = 11$ and $|N(K)| = 8$. Let $G_1 = G - K$ and $G_2 = G[K \cup N(K)]$. Let d_1 and d_2 be defined as prior to (4). Then by using $\{x', y', z', w', y'', z'', w''\}$, we have $d_1 + e(G[N(K)]) = e(K_8) = 28$. By Lemma 6.4.2, $d_2 \geq 1$. Now by (3), $13|N(K)| \geq 90 + 1 + 28$, which is impossible because $|N(K)| = 8$. This completes the proof of Theorem 6.3.1. ■

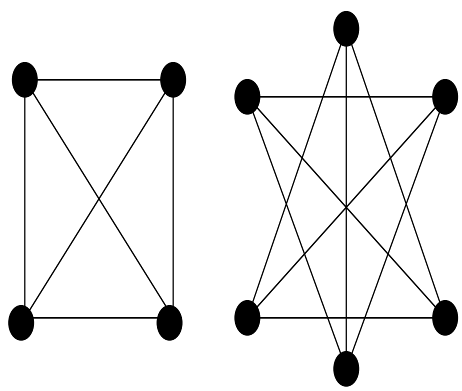
6.5 Further work

From Conjecture 6.1.1, we see the difficulty in extending Mader's result (namely Theorem 1.2.2) for the case when $p = 10$. As noted in Chapter 5, we need to deal with the case when the neighborhood of a vertex of degree ten is $K_{2,2,2,2,2}$, in which case three independent edges are needed in order to have a K_{10} minor. Thus the edge bound on 3-linkages (see Theorem 1.4.2) is needed, which will make the proof very technical. However, a computer-free proof might be possible if we increase the edge bound to $10n - 55$. Finally, we conjecture the following generalization of Theorem 1.3.2.

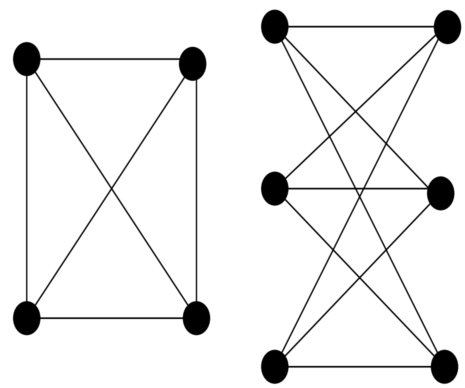
Conjecture 6.5.1 *Let G be a graph on $n \geq 9$ vertices and at least $6n - 19$ edges. Then either $G > K_8 \cup K_1$ or G is isomorphic to one of the following graphs: $K_{1,1,2,2,2,2}$, $K_{1,2,2,3,3}$, $K_{2,2,2,2,3}$, $K^-_{2,2,2,2,3}$, $K_{3,3,3,3}$, $K^-_{3,3,3,3}$, or $K_{2,3,3,4}$.*

APPENDIX A

CUBIC GRAPHS ON 10 VERTICES



(a) $K_4 \cup \overline{C_6}$



(b) $K_4 \cup K_{3,3}$

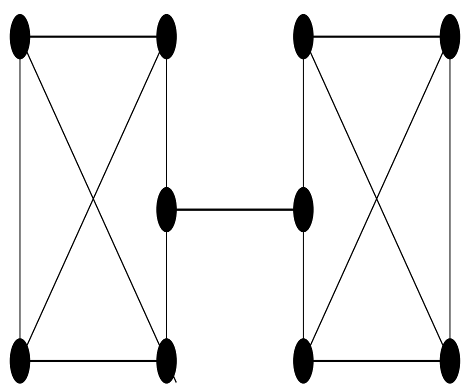


Figure 22: Cubic graphs on 10 vertices.

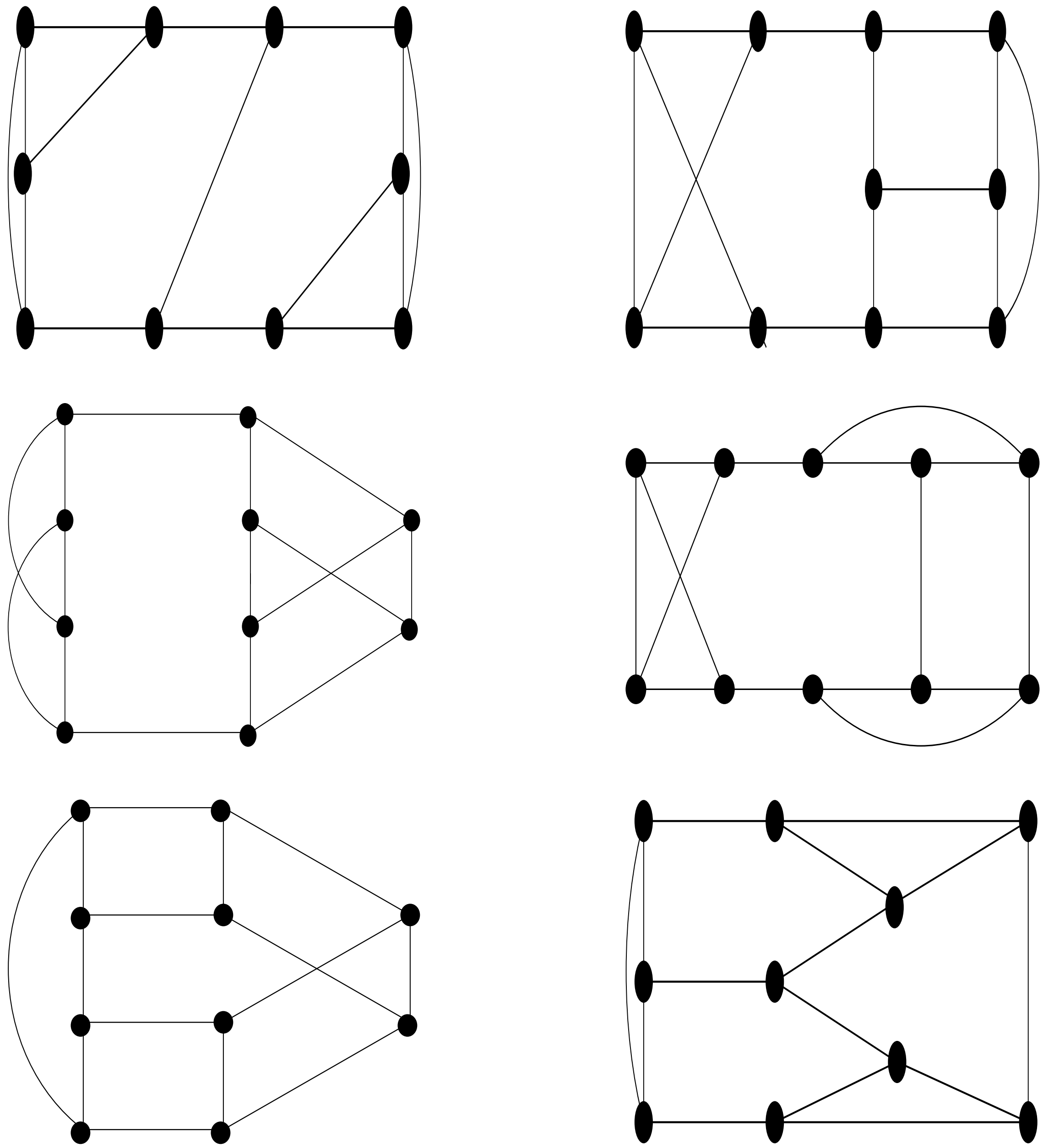


Figure 23: Cubic graphs on 10 vertices.

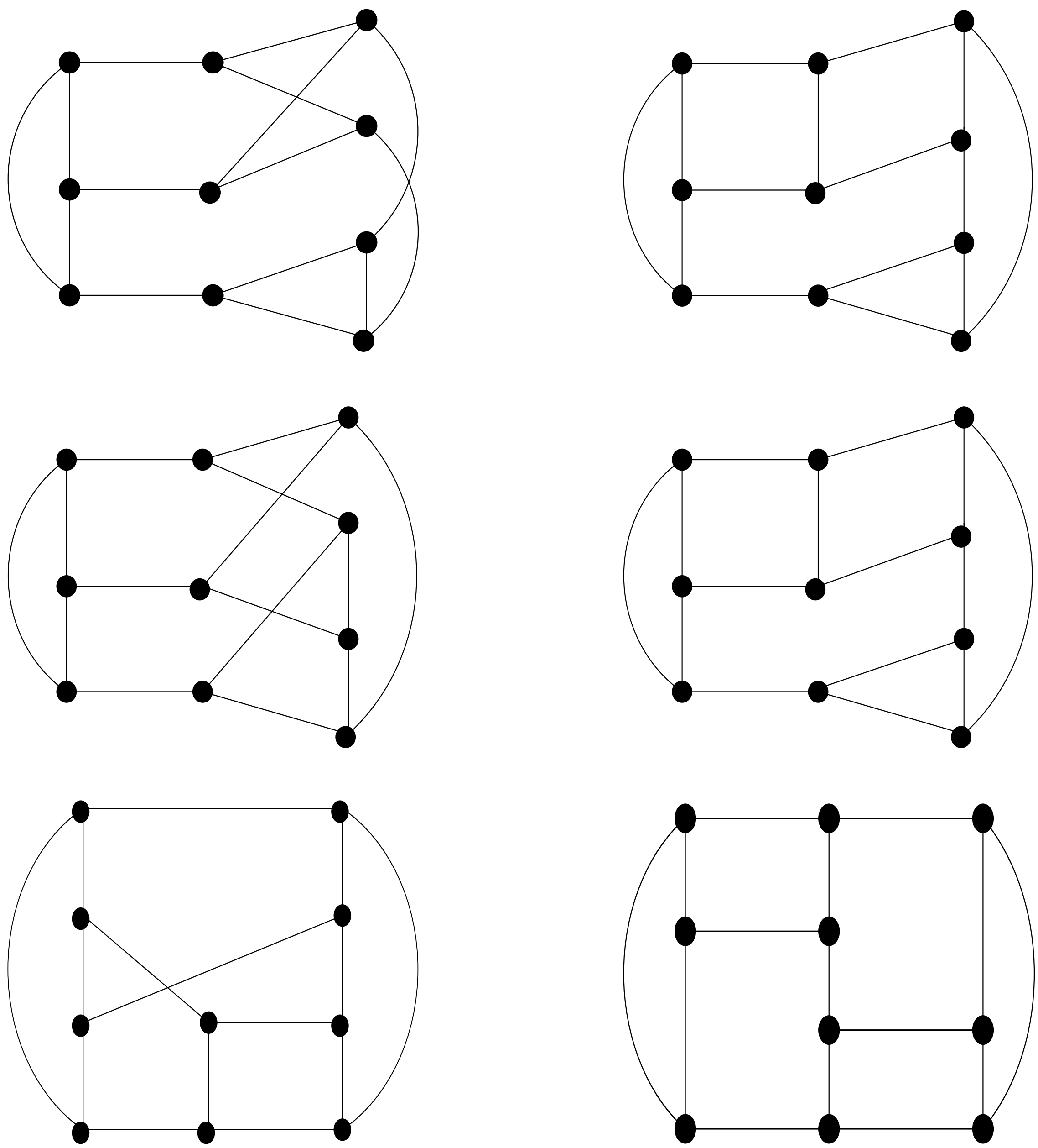
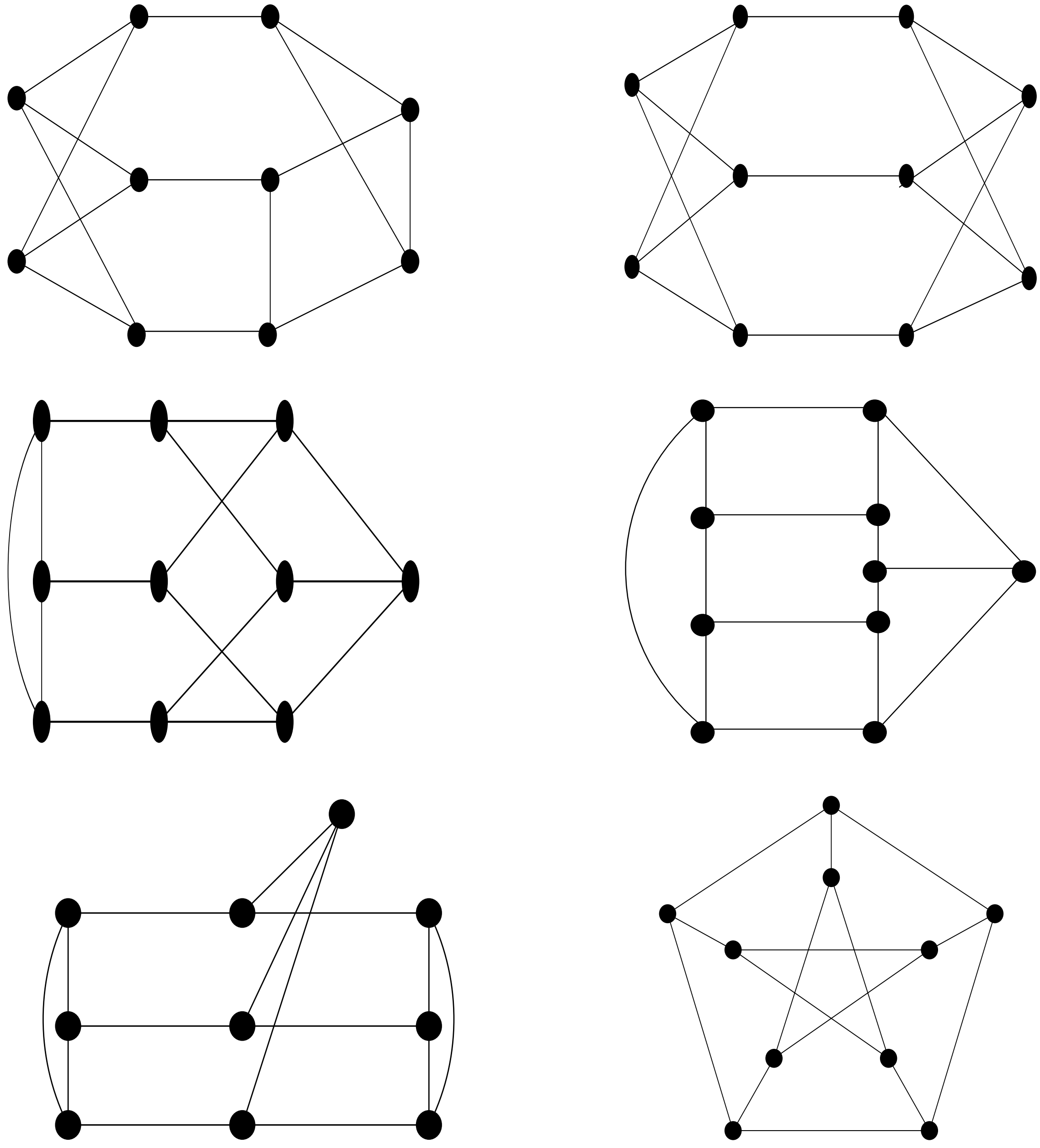


Figure 24: Cubic graphs on 10 vertices



(f) Petersen graph

Figure 25: Cubic graphs on 10 vertices.

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