COLORING GRAPHS WITH NO K_5-SUBDIVISION: DISJOINT PATHS IN GRAPHS

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Pure mathematics is, in its way, the poetry of logical ideas.

Albert Einstein

I dedicate this dissertation to my family and my friends.

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SUMMARY

A vertex coloring of a graph G, is an assignment of colors to all vertices in G, such that no two adjacent vertices are of the same color. We say G is k-colorable if G has a coloring using at most k colors. A subdivision of a graph G, also known as a topological G and denoted by TG, is a graph obtained from G by replacing certain edges of G with internally vertex-disjoint paths. This dissertation studies a problem in structural graph theory regarding the relationship between the chromatic number of a graph and subdivisions of a complete graph in the graph.

The Four Color Theorem states that every planar graph is 4-colorable. Hajós conjectured that for any positive integer k, every graph containing no TK_{k+1} is k-colorable. However, Catlin disproved Hajós conjecture for $k \geq 6$. It is not hard to prove that the conjecture is true for $k \leq 3$. Hajós' conjecture remains open for $k = 4$ and $k = 5$.

One important step to understand graphs containing TK_5 is to solve the following problem: Let H represent the tree on six vertices, two of which are adjacent and of degree 3. Let G be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of G. When does G contain a TH (i.e. an H-subdivision) in which u_1, u_2 are of degree 3, and a_1, a_2, a_3, a_4 are of degree 1? In this dissertation, we characterize graphs with no TH. (This characterization is used by He, Wang, and Yu to show that a graph containing no K_5-subdivision is planar or has a 4-cut, establishing conjecture of Kelmans and Seymour.)

Besides the topological H problem, we also consider a minimal counterexample to Hajós' conjecture for $k = 4$: a graph G, such that G contains no TK_5, G is not 4-colorable, and $|V(G)|$ is minimum. We use Hajós graph to denote such counterexample, and obtained some stuctural information of Hajós graphs.

CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Graph preliminaries

We follow the notation and terminology for graphs from [6]. We remind the reader that only simple graphs are considered in this dissertation.

A *graph* G is an ordered pair (V, E), where V is a finite set, and E is a set of 2-element subsets of V. An element of V is a *vertex* and an element of E is an *edge*. V is called the vertex set of G and often denoted by $V(G)$, and E is called the edge set of G and often denoted by $E(G)$. For convenience, we use the shorter form uv to denote the edge $\{u, v\}$ where $u, v \in V$. The edge uv is said to be $incident$ to both u and v.

Let $G = (V, E)$. We say the two vertices $u, v \in V$ are adjacent if $uv \in E$. u is a *neighbor* of v if u is adjacent to v. The *neighborhood* of a vertex u in G (denoted as $N_G(u)$) is the set of all neighbors of u in G. The *degree* of u in G (denoted as $deg_G(u)$) is the cardinality of $N_G(u)$. Let $S \subseteq V$. The *neighborhood* of S in G (denoted as $N_G(S)$) is the set of vertices in $V \backslash S$ that are adjacent to some vertex in S, and let $N_G[S] = V(S) \cup N_G(S)$. When understood, the reference to G may be dropped.

A *complete* graph on n vertices, denoted as K_n is the graph of n vertices such that every pair of vertices are adjacent. Moreover, K_n^- is the graph obtained from K_n with a single edge removed. A graph G is r-*partite* if there exists a partition of $V(G)$ into r classes $V_1, V_2, \ldots, V_r$, such that for any pair of vertices u, v in $V_i, 1 \leq i \leq r$, the two vertices u and v are not adjacent in G. A graph G is a *complete* r-*partite* graph if G is r-partite, and $uv \in E(G)\ \forall u \in V_i, v \in V_j, 1 \leq i < j \leq r$. Instead of 2-partite one usually says *bipartite*. We use $K_{m,n}$ to denote the complete bipartite graph, where m, n are the sizes of the two partite sets.

We say a graph H is a *subgraph* of G (denoted as $H \subseteq G$) if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Let $S \subseteq V(G)$. The *induced subgraph* $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in $E(G)$ that have both endpoints in S. For $H \subseteq G$, we use $G[H]$ to denote the subgraph of G induced by $V(H)$.

For $S \subseteq V(G)$, we use $G - S$ to denote the graph obtained from G by deleting all vertices in S and all edges incident to some vertices in S. For $v \in V(G)$, we use the shorter form $G - v$ to denote $G - \{v\}$. Let $S \subseteq E(G)$. We use $G - S$ to denote the graph obtained from G by deleting all edges in S. For $e \in E(G)$, we use the shorter form $G - e$ to denote $G - \{e\}$.

Let $H \subseteq G$, $S \subseteq V(G)$, and T be a collection of 2-element subsets of $V(H) \cup S$; then $H + (S \cup T)$ denotes the graph with vertex set $V(H) \cup S$ and edge set $E(H) \cup T$, and if $S = \emptyset$ and $T = \{\{x, y\}\}$ we write $H + uv$ instead of $H + \{\{u, v\}\}$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraphs of G. $G_1 \cup G_2$ is the graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$, and $G_1 \cap G_2$ is the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$.

Let $e = uv$ be an edge in $E(G)$. By G/e we denote the graph obtained from G by contracting the edge e into a new vertex v_e, where v_e is adjacent to all the vertices in $N(\{u, v\})$. For a subgraph H of G, we use G/H to denote the graph obtained from G by contracting H into a new vertex v_H, where v_H is adjacent to all the vertices in $N(H)$.

A *walk* W in G of length k is an alternating sequence of vertices and edges $v_0, e_1, v_1, e_2, v_2, \ldots, v_{k-1}, e_k, v_k$, such that $v_0, v_1, \ldots, v_k \in V(G)$, $e_1, \ldots, e_k \in E(G)$, and $e_i = v_{i-1}v_i$ for $1 \leq i \leq k$. A walk is closed if $v_0 = v_k$.

A walk is a *path* if the vertices $v_0, \ldots, v_k$ are distinct. A path is an *induced path* if it is a induced subgraph of G. Given a path P in a graph and $x, y \in V(P)$, xPy denotes the subpath of P between x and y (inclusive). We may view paths as sequences of vertices; thus if P is a path between x and y, Q is a path between y and z, and $V(P \cap Q) = \{y\}$, then PyQ denotes the path $P \cup Q$. The *ends* of the path P are the vertices of the minimum degree in P, and all other vertices of P (if any) are its *internal* vertices. A path P with ends u and v (or an u-v path) is also said to be *from u to v* or *between u and v*.

Let G be a graph. A collection of paths in G are said to be *independent* if no vertex of any path in this collection is an internal vertex of any other path in the collection. A path P in G is said to be *internally disjoint* from a subgraph H of G if no internal vertex of P belongs to H.

A graph G is *connected* if for any pair of vertices $u, v \in V(G)$, there is a path from u to v. We say G is *k-connected* if $|V(G)| \geq k+1$ and $G - S$ is connected for all $S \subseteq V(G)$ and $|S| < k$. A set $S \subseteq V(G)$ is a *k-cut* (or a *cut of size k*) in G, where k is a positive integer, if $|S| = k$ and $G - S$ is not connected. If $v \in V(G)$ and $\{v\}$ is a cut of G, then v is said to be a *cut vertex* of G.

A walk $v_0, e_1, v_1, e_2, v_2, \dots, v_{k-1}, e_k, v_k$ is a *cycle* if it's closed ($v_0 = v_k$), and the vertices $v_0, \dots, v_{k-1}$ are distinct. A cycle is an *induced cycle* if it is a induced subgraph of G. A graph G is *acyclic* if it contains no cycles. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph.

A *subdivision* of a graph G or a *G-subdivision*, also known as a *topological G* and denoted by TG, is a graph obtained from G by replacing certain edges of G with internally vertex-disjoint paths.

A *(vertex) coloring* of a graph G is an assignment of colors to all vertices in G, such that no two adjacent vertices are of the same color. A *k-coloring* of a graph is a coloring using at most k colors. We say G is *k-colorable* if G has a k-coloring. The *chromatic number* $\chi(G)$ of G is the smallest integer k such that G has a k-coloring.

A *separation* in a graph G consists of a pair of subgraphs G_1, G_2, denoted as (G_1, G_2), such that $G = G_1 \cup G_2$, $E(G_1 \cap G_2) = \emptyset$ and, for $i = 1, 2$, $V(G_i) - V(G_{3-i}) \neq \emptyset$ or $E(G_i) \neq \emptyset$. (Thus, we allow $V(G_i) - V(G_{3-i}) = \emptyset$, but if this happens we require $E(G_i) \neq \emptyset$.) Hence, if G has a separation (G_1, G_2) and $V(G_i - S) \neq \emptyset$ for $i \in \{1, 2\}$, then $V(G_1 \cap G_2)$ is a cut of G. The *order* of this separation is $|V(G_1 \cap G_2)|$, and (G_1, G_2)

is said to be a k-*separation* if its order is k.

We say that G is *planar* if G has a plane drawing, i.e. a drawing in the plane with no edges crossing. Otherwise, G is said to be *nonplanar*. Let $S \subseteq V(G)$. A *disc representation* of a graph G is a drawing of G in a closed disc in which no two edges cross. We say that (G, S) is *planar* if S are vertices in G such that G has a disc representation with S on the boundary of the disc.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. An isomorphism between graphs G_1 and G_2 is a bijection f between V_1 and V_2 such that any two vertices u and v of G_1 are adjacent if and only if $f(u)$ and $f(v)$ are adjacent in G_2. In this case, G_1, G_2 are called isomorphic and denoted as $G_1 \cong G_2$.

1.2 Main results

The Four Color Theorem [3, 4, 2, 24] states that every planar graph is 4-colorable. By Kuratowski's theorem [19], a graph is planar if and only if it contains no K_5-subdivision or $K_{3,3}$-subdivision. The structure and the chromatic number of graphs with no $K_{3,3}$-subdivision have been studied. Wagner [28] and Kelmans [16] gave a characterization of all nonplanar graphs with no $K_{3,3}$-subdivision, which is K_5, or admits a cut of size ≤ 2. Consequently, the chromatic number of a graph with no $K_{3,3}$-subdivision is at most 5. This upper bound is tight since K_5 contains no $K_{3,3}$-subdivision and is not 4-colorable.

Thus it is natural to consider the chromatic number of graphs with no K_5-subdivision. This problem could also be considered as a special case of a conjecture of Hajós. Hajós [27] conjectured that for any positive integer k, every graph containing no K_{k+1}-subdivision is k-colorable. Catlin [5] disproved Hajós' conjecture for $k \geq 6$. Subsequently, Erdős and Fajtlowicz [8] showed that Hajós' conjecture fails for almost all graphs. It's not hard to prove that the conjecture is true for $k \leq 3$. However, Hajós' conjecture remains open for $k = 4$ and $k = 5$.

We consider a minimal counterexample to Hajós' conjecture for $k = 4$: a graph G, such that G contains no K_5-subdivision, G is not 4-colorable, and $|V(G)|$ is minimum. We use *Hajós graph* to denote such counterexample. To characterize Hajós graph, we first study the connectivity of Hajós graph to derive some structural information.

A related problem is Kelmans-Seymour conjecture. Seymour [25] and, independently, Kelmans [15] conjectured that every 5-connected nonplanar graph contains a topological K_5 (i.e., subdivision of K_5). One approach to understand graphs containing K_5-subdivision is to solve the following problem: Let H represent the tree on six vertices two of which are adjacent and of degree 3. Let G be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of G. When does G contain a topological H in which u_1, u_2 are of degree 3 and a_1, a_2, a_3, a_4 are of degree 1? We say that such a topological H is *rooted* at $u_1, u_2, \{a_1, a_2, a_3, a_4\}$.

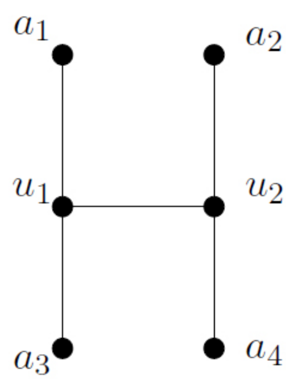


Fig. 1.1: The H graph.

For convenience, we use *quadruple* to denote (G, u_1, u_2, A) where u_1, u_2 are distinct vertices of a graph G, $A \subseteq V(G) - \{u_1, u_2\}$, and $|A| = 4$. We say that (G, u_1, u_2, A) is *feasible* if G has a topological H rooted at u_1, u_2, A. In chapter II of this dissertation we prove the following theorem:

Theorem 2.2.1. *Let (G, u_1, u_2, A) be a quadruple. Then one of the following holds:*

- *G has a topological H rooted at u_1, u_2, A.*
- *G has a separation (K, L) such that $|V(K \cap L)| \leq 2$ and for some $i \in \{1, 2\}$, $u_i \in V(K) - V(L)$ and $A \cup \{u_{3-i}\} \subseteq V(L)$.*

- *G has a separation (K, L) such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K) - V(L)$, and $A \in V(L)$.*

- *(G, u_1, u_2, A) is an obstruction of type I, II, III, or IV as described in the figure below.*

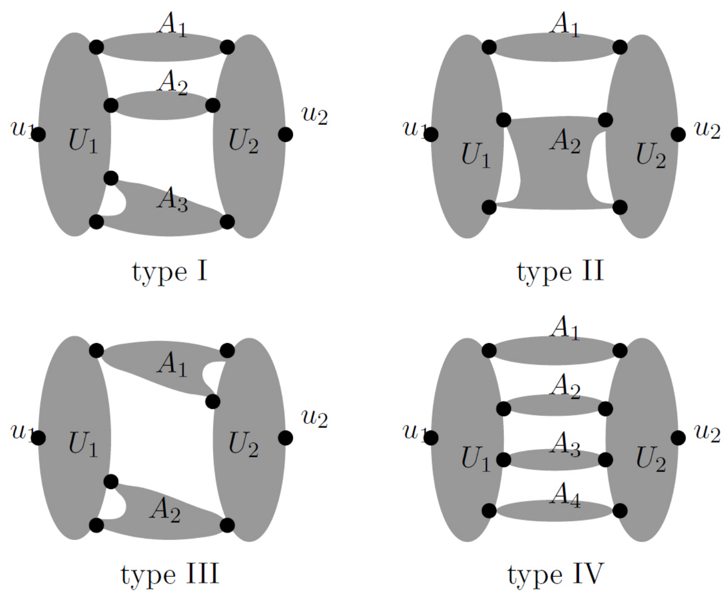


Fig. 1.2: Obstructions of 4 Types.

We refer the reader to chapter II for a precise description of these four types. The Kelmans-Seymour conjecture has been proved recently by He, Wang, and Yu [10, 11, 12, 13], and they used Theorem 2.2.1 in their proof. This conjecture implies that a Hajós graph is not 5-connected, since we know that Hajós graph is nonplanar by Four Color Theorem. Yu and Zickfeld [30] proved that Hajós graphs must be 4-connected. Thus a Hajós graph must have a 4-cut. Furthermore, Sun and Yu [26] proved that for any 4-cut T of a Hajós graph G, $G - T$ has exactly 2 components.

In chapter III of this dissertation we derive further structural information of Hajós graphs. Specifically, we recently prove of the following theorem.

Theorem 3.1.2. *No Hajós graph has a 4-separation (G_1, G_2) such that $(G_1, V(G_1 \cap G_2))$ is planar and $|V(G_1)| \geq 6$.*

Moreover we believe that Theorem 3.1.2 is very likely to lead to the proof of the following conjecture.

Conjecture 1.2.1. *No Hajós graph contains a* K_4^- *as a subgraph.*

CHAPTER 2

GRAPHS CONTAINING TOPOLOGICAL H

Let H denote the tree with six vertices two of which are adjacent and of degree three. Let G be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of G. We characterize those G that contain a topological H in which u_1, u_2 are of degree three and a_1, a_2, a_3, a_4 are of degree one. As a consequence, if G is 5-connected, then G has a topological G rooted at $u_1, u_2, \{a_1, a_2, a_3, a_4\}$.

2.1 Introduction

This work was motivated by the well known conjecture of Kelmans [15] and Seymour [25]: Every 5-connected nonplanar graph contains a topological K_5 (i.e., subdivision of K_5). Earlier, Dirac [7] conjectured an extremal function for the existence of a topological K_5: If G is a simple graph with $n \geq 3$ vertices and at least $3n - 5$ edges then G contains a topological K_5. This conjecture was established by Mader [22]. Kézdy and McGuiness [17] showed that the Kelmans-Seymour conjecture, if true, implies Mader's result.

As mentioned in Chapter 1, the Kelmans-Seymour conjecture is also related to the $k = 4$ case of the Hajós conjecture (see [5]) that every graph containing no topological K_{k+1} is k-colorable. Hajós' conjecture is false for $k \geq 6$ [5, 8] and true for $k = 1, 2, 3$, and remains open for $k = 4$ and $k = 5$.

An approach to the Kelmans-Seymour conjecture is to study the so called rooted K_4 problem. Given a graph G and four distinct vertices x_1, x_2, x_3, x_4 of G, when does G contain a topological K_4 in which x_1, x_2, x_3, x_4 are the vertices of degree three? This problem was solved for planar graphs (see [29]), and the result was used by Aigner-Horev [1] to prove the Kelmans-Seymour conjecture for apex graphs. A different and shorter proof for the apex case was found independently by Kawarabayashi [14] and Ma, Thomas and Yu [21].

One important step in [29] is to solve the following problem for planar graphs: Let H represent the tree on six vertices two of which are adjacent and of degree 3. (See Figure 2.1.) Let G be a graph and $u_1, u_2, a_1, a_2, a_3, a_4$ be distinct vertices of G. When does G contain a topological H in which u_1, u_2 are of degree 3 and a_1, a_2, a_3, a_4 are of degree 1? We say that such a topological H is *rooted* at $u_1, u_2, \{a_1, a_2, a_3, a_4\}$. For convenience, we use *quadruple* to denote (G, u_1, u_2, A) where u_1, u_2 are distinct vertices of a graph G, $A \subseteq V(G) - \{u_1, u_2\}$, and $|A| = 4$. We say that (G, u_1, u_2, A) is *feasible* if G has a topological H rooted at u_1, u_2, A.

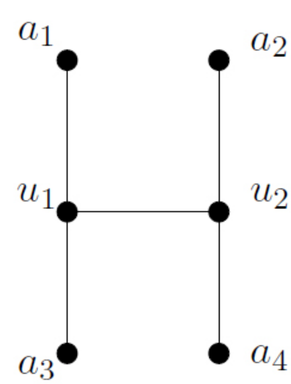


Fig. 2.1: The H graph.

The main result of this chapter is a characterization of feasible quadruples, which implies the following theorem whose proof is given after the full statement of the characterization in Section 2.2 (see Theorem 2.2.1).

Theorem 2.1.1. (G, u_1, u_2, A) *is feasible when G is 5-connected.*

The connectivity in Theorem 2.1.1 is tight. Let G be obtained from K_6 by deleting the edge between two vertices u_1, u_2, and let $A = V(G) - \{u_1, u_2\}$; then G is 4-connected and (G, u_1, u_2, A) is not feasible.

In Section 2.2, we describe the obstructions to feasibility of quadruples (there are four types) and state the main result (Theorem 2.2.1). In Section 2.3, we consider a related problem about the existence of k disjoint paths in a graph between two given sets of vertices and containing a given edge. We solve the case $k = 3$ which will be used to characterize quadruples. In Section 2.4, we deal with those quadruples (G, u_1, u_2, A) in which G admits certain cuts of size at most 3. In Section 2.5, we study quadruples containing *critical pairs*, i.e., quadruples (G, u_1, u_2, A) in which there exist distinct $x, y \in V(G) - A - \{u_1, u_2\}$ such that $(G/xy, u_1, u_2, A)$ is an obstruction (where G/xy is obtained from G by identifying x and y and removing loops or multiple edges). In Section 2.6, we deal with the case when G/xy has a certain cut of size at most 4, which reduces to the case when G has a certain cut of size 5. The proof is then completed in Section 2.7 by finding an appropriate edge $xy \in E(G - A - \{u_1, u_2\})$ such that $\{x, y\}$ is a critical pair. In Section 2.8, we describe a situation in which the theorem is used in the proof of the Kelmans-Seymour conjecture.

2.2 Obstructions

We refer the reader to Figures 2.2 and 2.3 for intuition on the following discussions about obstructions. We will show that modulo certain separations there will be just four types of obstructions.

A quadruple (G, u_1, u_2, A) is an *obstruction* if G has subgraphs U_1, U_2 (called *sides*) and $A_i, i \in [k] := \{1, 2, ..., k\}$ (called *middle parts*), such that

(1) all vertices in G are covered by the sides and the middle parts: $V(G) = V(U_1) \cup V(U_2) \cup A_{[k]}$, where $A_{[k]} = \cup_{i \in [k]} V(A_i)$;

(2) u_1 is in the side U_1, u_2 is in the side U_2, and neither is in any of the middle parts: $u_i \in V(U_i) - A_{[k]}$ for $i = 1, 2$;

(3) the middle parts are vertex-disjoint: $V(A_i) \cap V(A_j) = \emptyset, 1 \leq i < j \leq k$;

(4) all vertices in A are in the middle parts, and every middle part contains at least one vertex from A: $A \subseteq A_{[k]}$, and for any $i \in [k], V(A_i) \cap A \neq \emptyset$;

(5) if A_i contains at least 2 vertices, then $|V(A_i) \cap V(U_1 \cup U_2)| = |V(A_i) \cap A| + 1$ and $V(A_i) \cap V(U_1 \cup U_2) \cap A = \emptyset$;

(6) if A_i consists of exactly 1 vertex v, then $v \in A$ and $v \in V(U_1 \cap U_2)$, thus we may conclude $V(U_1 \cap U_2) \subseteq A$, and $V(A_i) \cap V(U_1 \cup U_2) \cap A \neq \emptyset$ iff $|V(A_i)| = 1$;

(7) for an edge $uv \in E(G)$, if $u \notin A$ and $v \notin A$, then u, v must both be in the same side or same middle part, that is, $E(G - A)$ is a disjoint union of $E(U_1 - A)$, $E(U_2 - A)$, and $E(A_i - A)$ where $i \in [k]$;

(8) if A_i contains at least 2 vertices, then $N_G(V(A_i) \cap A) \subseteq V(A_i)$. There is no restriction on $N_G(A_i \cap A)$ when A_i consists of exactly 1 vertex.

To see that obstructions are not feasible, let (G, u_1, u_2, A) be an obstruction, J a topological H in G rooted at u_1, u_2, A, and P the u_1-u_2 path in J. By definition, $V(P) \cap A = \emptyset$ and (in particular, by (4)) P has to pass through some A_i with $|V(A_i)| \geq 2$; so $|V(P) \cap V(A_i) \cap V(U_1 \cup U_2)| \geq 2$. Also $J - V(P - \{u_1, u_2\})$ contains $|V(A_i) \cap A|$ independent paths from $\{u_1, u_2\}$ to $V(A_i) \cap A$; so $|(V(J) - V(P)) \cap V(A_i) \cap V(U_1 \cup U_2)| \geq |V(A_i) \cap A|$. Thus, $|V(A_i) \cap V(U_1 \cup U_2)| \geq |V(A_i) \cap A| + 2$, contradicting (5).

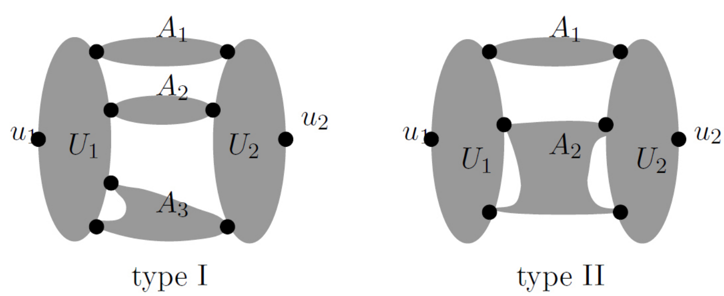


Fig. 2.2: Obstructions of type I and type II.

An obstruction (G, u_1, u_2, A) is said to be of *type I* if $k = 3$, $|V(A_i) \cap A| = 1$ for $i = 1, 2$, $|V(A_3) \cap A| = 2$, $|V(U_i \cap A_j)| = 1$ for $(i, j) \neq (1, 3)$, and $|V(U_1 \cap A_3)| = 2$.

An obstruction (G, u_1, u_2, A) is said to be of *type II* if $k = 2$, $|V(A_1) \cap A| = 1$, $|V(A_2) \cap A| = 3$, and for $i = 1, 2$, $|V(U_i \cap A_1)| = 1$ and $|V(U_i \cap A_2)| = 2$.

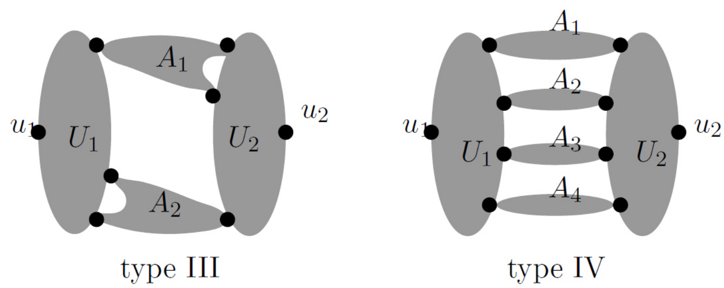


Fig. 2.3: Obstructions of type III and type IV.

An obstruction (G, u_1, u_2, A) is said to be of *type III* if $k = 2$, $|V(A_i) \cap A| = 2$ for $i = 1, 2$, $|V(U_1 \cap A_1)| = |V(U_2 \cap A_2)| = 1$, and $|V(U_1 \cap A_2)| = |V(U_2 \cap A_1)| = 2$.

An obstruction (G, u_1, u_2, A) is said to be of *type IV* if $k = 4$ and, for $1 \leq i \leq 4$ and $j \in \{1, 2\}$, $|V(A_i) \cap A| = |V(U_j \cap A_i)| = 1$.

Theorem 2.2.1. *Let (G, u_1, u_2, A) be a quadruple. Then one of the following holds:*

(i) (G, u_1, u_2, A) *is feasible.*

(ii) *G has a separation (K, L) such that $|V(K \cap L)| \leq 2$ and for some $i \in \{1, 2\}$, $u_i \in V(K) - V(L)$ and $A \cup \{u_{3-i}\} \subseteq V(L)$.*

(iii) *G has a separation (K, L) such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K) - V(L)$, and $A \subseteq V(L)$.*

(iv) (G, u_1, u_2, A) *is an obstruction of type I, or II, or III, or IV.*

Note that (ii) and (iii) both imply that (G, u_1, u_2, A) is not feasible, when $|V(K \cap L)| = 4$ the feasibility of (G, u_1, u_2, A) reduces to $(K, u_1, u_2, V(K \cap L))$.

To see that Theorem 2.2.1 implies Theorem 2.1.1, we apply Theorem 2.2.1 to the quadruple $(G - E(G[A]), u_1, u_2, A)$. Since G is 5-connected, (ii), (iii) and (iv) of Theorem 2.2.1 do not hold for $(G - E(G[A]), u_1, u_2, A)$. Hence $(G - E(G[A]), u_1, u_2, A)$ is feasible. Since any topological H in $G - E(G[A])$ rooted at u_1, u_2, A is also a topological H in G rooted at u_1, u_2, A, we see that (G, u_1, u_2, A) must be feasible.

2.3 Disjoint paths containing a given edge

In this section we prove a result about the existence of disjoint paths from three given vertices to three other given vertices such that a specific edge is used by one of these paths. This result will be used several times in the proof of Theorem 2.2.1. The problem for finding two disjoint paths between two pairs of vertices and through a given edge is equivalent to the problem for finding a cycle through three given edges. The following result is due to Lovász [20].

Lemma 2.3.1 (Lovász)**.** *Let G be a 3-connected graph and e_1, e_2, e_3 be distinct edges of G not all incident with a common vertex. Then G contains a cycle through e_1, e_2, e_3 iff $G - \{e_1, e_2, e_3\}$ is connected.*

We need an easy generalization of Lemma 2.3.1. For a subgraph K of a graph G, a *K-bridge* of G is a subgraph of G that is induced either by an edge of $G - E(K)$ with both ends in K, or by all edges in a component of $G - V(K)$ and all edges from that component to K. The K-bridges of the latter type are said to be *nontrivial*.

Lemma 2.3.2. *Let e_1, e_2, e_3 be distinct edges of a graph G not all incident with a common vertex. Then one of the following holds:*

(i) $\{e_1, e_2, e_3\}$ *is contained in a cycle in G.*

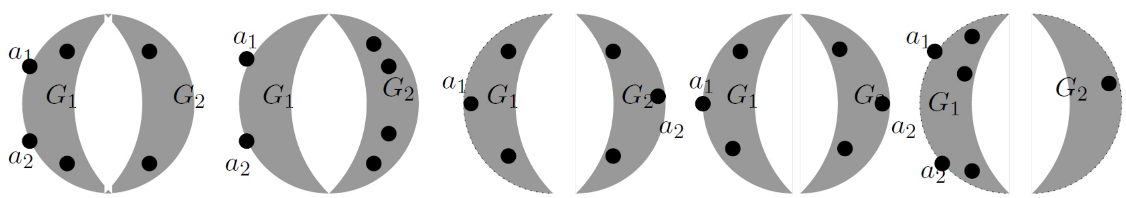
(ii) *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 2$, $V(G_i) - V(G_{3-i}) \neq \emptyset$ for $i = 1, 2$, and $|E(G_i) \cap \{e_1, e_2, e_3\}| = 1$ for some $i \in \{1, 2\}$.*

(iii) $\{e_1, e_2, e_3\}$ *is contained in a component H of G, and $H - \{e_1, e_2, e_3\}$ is not connected.*

Proof. Suppose the assertion is false, and choose a counterexample G, e_1, e_2, e_3 such that $|V(G)|$ is minimum. Then G is connected, or else (ii) holds or we get a smaller counterexample. Moreover, G is not 3-connected, as otherwise (i) or (iii) holds by Lemma 2.3.1.

So let T be a cut in G with $|T| \leq 2$. Since G has at least two nontrivial T-bridges, we may assume that B is a nontrivial T-bridge of G such that $|E(B) \cap \{e_1, e_2, e_3\}| \leq 1$. If $|E(B) \cap \{e_1, e_2, e_3\}| = 1$ then (ii) holds. So $E(B) \cap \{e_1, e_2, e_3\} = \emptyset$. If $|T| = 1$ let $G' := G - V(B - T)$, and if $|T| = 2$ let G' be obtained from $G - V(B - T)$ by adding an edge between the vertices in T. Now by the choice of G, e_1, e_2, e_3, we see that (i) or (ii) or (iii) holds for G', e_1, e_2, e_3. It is straightforward to verify that (i) or (ii) or (iii) holds for G, e_1, e_2, e_3. ∎

The following figure gives illustrations of conclusions (i) – (v) of Lemma 2.3.3. Note that there are three pairs of vertices $\{v_1, v_2\}$, $\{w_1, w_2\}$ and $\{a_1, a_2\}$ in the statement of Lemma 2.3.3. These pairs appear symmetric in the first part of the statement; however, we state the second part of the lemma according to the locations of vertices a_1, a_2, to facilitate later applications where $\{a_1, a_2\}$ will play different roles than $\{v_1, v_2\}$ and $\{w_1, w_2\}$.



(i) (ii) (iii) (iv) (v)

Fig. 2.4: The separation (G_1, G_2) in Lemma 2.3.3.

Lemma 2.3.3. *Let G be a graph and $v_1, v_2, w_1, w_2, a_1, a_2 \in V(G)$ be distinct such that $a_1a_2, v_1v_2,\ w_1w_2 \notin E(G)$. Then G has three disjoint paths with one from $\{v_1, v_2\}$ to $\{w_1, w_2\}$, one from $\{v_1, v_2\}$ to $\{a_1, a_2\}$, and another from $\{w_1, w_2\}$ to $\{a_1, a_2\}$, or G has a separation (G_1, G_2) such that one of the following holds:*

- (i) $|V(G_1 \cap G_2)| \leq 2$, $\{a_1, a_2\} \subseteq V(G_1)$, *and for some* $i \in \{1, 2\}$, $\{v_1, v_2\} \subseteq V(G_i)$ *and* $\{w_1, w_2\} \subseteq V(G_{3-i})$.
- (ii) $|V(G_1 \cap G_2)| \leq 2$, $\{a_1, a_2\} \subseteq V(G_1)$, *and* $\{v_1, v_2, w_1, w_2\} \subseteq V(G_2)$.

(*iii*) $G_1 \cap G_2 = \emptyset$, $a_1 \in V(G_1)$, $a_2 \in V(G_2)$, *and for some* $i \in \{1,2\}$, $\{v_1, v_2\} \subseteq V(G_i)$ *and* $\{w_1, w_2\} \subseteq V(G_{3-i})$.

(*iv*) $G_1 \cap G_2 = \emptyset$, $a_1 \in V(G_1)$, $a_2 \in V(G_2)$, *and for* $i \in \{1,2\}$, $|\{v_1, v_2\} \cap V(G_i)| = |\{w_1, w_2\} \cap V(G_i)| = 1$.

(*v*) $G_1 \cap G_2 = \emptyset$, $\{a_1, a_2\} \subseteq V(G_1)$, *and* $|\{v_1, v_2, w_1, w_2\} \cap V(G_1)| = 3$.

Proof. Let $G' = G + \{a_1a_2, v_1v_2, w_1w_2\}$ and apply Lemma 2.3.2 to $G', a_1a_2, v_1v_2, w_1w_2$. If Lemma 2.3.2$(i)$ holds, i.e., G' contains a cycle C containing a_1a_2, v_1v_2 and w_1w_2, then $C - \{a_1a_2, v_1, v_2, w_1w_2\}$ gives the desired paths in G. If Lemma 2.3.2(ii) holds then let (G'_1, G'_2) be a separation in G' such that $|V(G'_1 \cap G'_2)| \leq 2$, $V(G'_1) - V(G'_2) \neq \emptyset$, and $|E(G'_1) \cap \{a_1a_2, v_1v_2, w_1w_2\}| = 1$; then (i) holds if $\{v_1v_2, w_1w_2\} \cap E(G'_1) \neq \emptyset$, and (ii) holds if $a_1a_2 \in E(G'_1)$. So assume that Lemma 2.3.2(iii) holds. Then G is the disjoint union of two graphs G_1 and G_2, and one of the pairs $\{a_1, a_2\}$, $\{v_1, v_2\}$, $\{w_1, w_2\}$ has one element in G_1 and another in G_2.

Suppose $a_1 \in V(G_1)$ and $a_2 \in V(G_2)$. If there exists $i \in \{1,2\}$ such that $|\{v_1, v_2, w_1, w_2\} \cap V(G_i)| \leq 1$, then $(G_i + a_{3-i}, G_{3-i} + \{v_1, v_2, w_1, w_2\})$ shows that (ii) holds. If $|\{v_1, v_2, w_1, w_2\} \cap V(G_i)| = 2$ for $i = 1, 2$ then (iii) or (iv) holds.

So assume (by symmetry) that $a_1, a_2, v_1 \in V(G_1)$ and $v_2 \in V(G_2)$. If $|\{w_1, w_2\} \cap V(G_1)| \leq 1$ then $(G_1, G_2 + \{v_1, w_1, w_2\})$ shows that (ii) holds; if $\{w_1, w_2\} \subseteq V(G_1)$ then (v) holds. ∎

In general one could ask the following question. Given two disjoint k-sets of vertices A, B and an edge e in a graph G, when does G contain k disjoint paths from A to B and passing through e? The main result of this section is an answer to this question for $k = 3$. Note that when (i) of Lemma 2.3.4 occurs, the desired paths do not exist if $|V(G_1 \cap G_2)| \leq 2$, and the problem reduces to the smaller graphs G_1 or G_2 if $|V(G_1 \cap G_2)| = 3$.

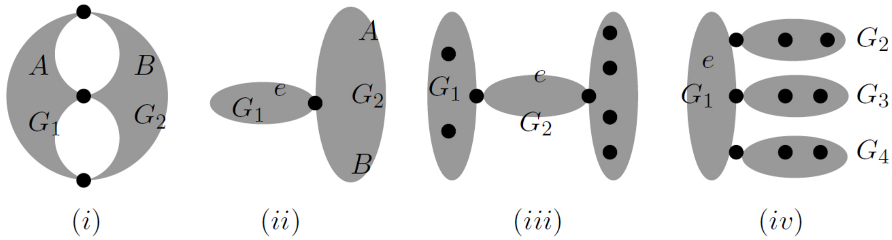


Fig. 2.5: The separations in Lemma 2.3.4.

Lemma 2.3.4. *Let G be a graph, $A, B \subseteq V(G)$ be disjoint, and $e \in E(G)$ such that $|A| = |B| = 3$ and $V(e) \cap (A \cup B) = \emptyset$. Then G has three disjoint paths from A to B and through e, or one of the following holds:*

(i) *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 3$, $A \subseteq V(G_1)$, and $B \subseteq V(G_2)$.*

(ii) *G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 1$, $e \in E(G_1)$, and $A \cup B \subseteq V(G_2)$.*

(iii) *$G = G_1 \cup G_2 \cup G_3$ such that $G_1 \cap G_3 = \emptyset$, $e \in E(G_2)$, $|V(G_i \cap G_2)| \leq 1$ for $i = 1, 3$, $|V(G_1) \cap A| = |V(G_1) \cap B| = 1$, and $|V(G_3) \cap A| = |V(G_3) \cap B| = 2$.*

(iv) *$G = G_1 \cup G_2 \cup G_3 \cup G_4$ such that $e \in E(G_1), V(G_i \cap G_j) = \emptyset$ for $2 \leq i < j \leq 4$, and $|V(G_1 \cap G_i)| = |V(G_i \cap A)| = |V(G_i \cap B)| = 1$ for $i \in \{2, 3, 4\}$.*

Proof. We may assume that A, B are independent sets in G, as otherwise (i) holds. We may also assume that G has three disjoint paths P_1, P_2, P_3 from A to B, or else (i) follows from Menger's theorem. Let $P := \bigcup_{i=1}^{3} P_i$. We may assume that $e \notin E(P)$ for any choice of P; for, otherwise, G has three disjoint paths from A to B and through e. Let L_P denote the P-bridge of G containing e. We choose P (i.e., P_1, P_2, P_3) so that

(1) L_P is maximal.

Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$ such that P_i is from a_i to b_i for $i = 1, 2, 3$. Let $x_i, y_i \in V(P_i \cap L_P)$ (if not empty) such that $x_i P_i y_i$ is maximal and a_i, x_i, y_i, b_i occur on

P_i in this order. For convenience, let $L' := L_P - V(P \cap L_P)$ and let $L_i := G[L' \cup x_i P_i y_i]$ for $i = 1, 2, 3$.

(2) If x_i, y_i are defined then no P-bridge of G intersects both $a_i P_i x_i - x_i$ and $x_i P_i b_i - x_i$, or both $a_i P_i y_i - y_i$ and $y_i P_i b_i - y_i$. For, suppose G has a P-bridge J intersecting both $a_i P_i x_i - x_i$ and $x_i P_i b_i - x_i$. Then $J \neq L_P$, and J contains a path Q_i from some $u_i \in V(a_i P_i x_i - x_i)$ to some $v_i \in V(x_i P_i b_i - x_i)$ and internally disjoint from $P \cup L_P$. Let $P' := (P - V(P_i)) \cup a_i P_i u_i Q_i v_i P_i b_i$. Then the P'-bridge of G containing e contains $L_P + x_i$, contradicting (1).

(3) If x_i, y_i are defined and L_i has a separation (L_{i1}, L_{i2}) such that $|V(L_{i1} \cap L_{i2})| = 1$, $x_i, y_i \in V(L_{i1})$, and $e \in E(L_{i2})$, we choose (L_{i1}, L_{i2}) so that L_{i2} is minimal, and let $w_i \in V(L_{i1} \cap L_{i2})$. If x_i, y_i are defined and the above separation does not exist, then we may assume $x_i = y_i$; as otherwise, L_i contains a path Q_i from x_i to y_i and through e, and hence $(P - V(P_i)) \cup a_i P_i x_i Q_i y_i P_i b_i$ gives the desired paths. In this latter case, we set $w_i = x_i = y_i$, and let L_{i1} consist of w_i only, and $L_{i2} = L_i$.

(4) We may assume that w_i, x_i, y_i, $i = 1, 2$, are defined, and $w_1 \neq w_2$. To see this, let $I = \{i : w_i, x_i, y_i \text{ are defined}\}$. If $I = \emptyset$ then the separation $(L_P, G - L_P)$ shows that (ii) holds. So assume $I \neq \emptyset$. Thus, if (4) is not true then $|I| = 1$ or $w_i = w_j$ for all $i, j \in I$; so the separation $(\cap_{i \in I} L_{i2}, G - \cap_{i \in I} V(L_{i2} - w_i))$ shows that (ii) holds.

By (4) and by the minimality of L_{i2} for $i = 1, 2$ (see (3)), $L_P - V(P - \{w_1, w_2\})$ contains a path from w_1 to w_2 through e and internally disjoint from P; hence L_{11}, L_{21} are disjoint. So for $\{i, j\} = \{1, 2\}$, L_P contains a path Q_{ij} from x_i to y_j, through e, and internally disjoint from P.

(5) We may assume that no P-bridge of G other than L_P intersects both $a_1 P_1 y_1 - y_1$ and $x_2 P_2 b_2 - x_2$, or both $a_2 P_2 y_2 - y_2$ and $x_1 P_1 b_1 - x_1$. Otherwise, by symmetry assume that some P-bridge J of G, $J \neq L_P$, intersects both $a_1 P_1 y_1 - y_1$ and $x_2 P_2 b_2 - x_2$. Then J contains a path Q from some $u \in V(a_1 P_1 y_1 - y_1)$ to some $v \in V(x_2 P_2 b_2 - x_2)$ and internally disjoint from $P \cup L_P$. Now $a_1 P_1 u Q v P_2 b_2, a_2 P_2 x_2 Q_{21} y_1 P_1 b_1, P_3$ are three disjoint

paths from A to B and through e.

Case 1. w_3, x_3, y_3 are defined.

Suppose $w_3 \notin \{w_1, w_2\}$. Then by the same argument following (4), we may assume that for any $1 \le i \ne j \le 3$, L_P has a path Q_{ij} from x_i to y_j through e and internally disjoint from P, and (5) holds for any P_i, P_j with $i \ne j$. Thus, $\{x_1, x_2, x_3\}$ or $\{y_1, y_2, y_3\}$ separates A from B (i.e. (i) holds); or $\{x_1, x_2, x_3\} = \{a_1, a_2, a_3\}$, $\{y_1, y_2, y_3\} = \{b_1, b_2, b_3\}$, and no P-bridge of G other than L_P contains two of $\{x_1, x_2, x_3\}$ or two of $\{y_1, y_2, y_3\}$. In the latter case, (iv) holds with $G_1 = L_{12} \cap L_{22} \cap L_{32}$, $L_{11} \cup P_1 \subseteq G_2$, $L_{21} \cup P_2 \subseteq G_4$, and $L_{31} \cup P_3 \subseteq G_3$. Thus, by symmetry assume $w_3 = w_2$.

Hence, again by the same argument following (4), for all $\{i, j\} \ne \{2, 3\}$, L_P has a path Q_{ij} from x_i to y_j through e and internally disjoint from P, and we may assume that

$(*)$ no P-bridge of G other than L_P intersects both $a_1P_1y_1 - y_1$ and $(x_2P_2b_2 - x_2) \cup (x_3P_3b_3 - x_3)$, or both $x_1P_1b_1 - x_1$ and $(a_2P_2y_2 - y_2) \cup (a_3P_3y_3 - y_3)$.

If no P-bridge other than L_P intersecting P_1 also intersects $P_2 \cup P_3$, then (iii) holds with $G_2 = L_{12} \cap L_{22}$, $L_{11} \cup P_1 \subseteq G_1$, and $L_{21} \cup L_{31} \cup P_2 \cup P_3 \subseteq G_3$. So assume that G has a path Q from some $u_1 \in V(P_1)$ to some $u_2 \in V(P_2 \cup P_3)$ and internally disjoint from $P \cup L_P$. Note that if for every choice of Q, we have $u_1 = x_1 = y_1$ then, since $a_1 \ne b_1$, $\{u_1, a_2, a_3\}$ or $\{u_1, b_2, b_3\}$ is a cut in G separating A from B; so (i) holds. Hence, by symmetry, assume $u_1 \in V(a_1P_1y_1 - y_1)$. Then by $(*)$, $u_2 \in V(a_2P_2x_2 \cup a_3P_3x_3)$. By symmetry, let $u_2 \in V(a_2P_2x_2)$.

First, assume that Q may be chosen so that $u_1 \in V(x_1P_1y_1 - \{x_1, y_1\})$. Then by $(*)$, $x_2 = y_2 = u_2$. Since $a_2 \ne b_2$, we may let $a_2 \ne x_2$ (by symmetry). If $\{x_1, x_2, x_3\}$ is a cut in G separating A from B then (i) holds. So by (2) and $(*)$, G has a path R internally disjoint from $L_P \cup P \cup Q$, which is from some $r \in V(a_2P_2x_2 - x_2)$ to some $s \in V(x_3P_3b_3 - x_3)$, or from some $r \in V(x_2P_2b_2 - x_2)$ to some $s \in V(a_3P_3x_3 - x_3)$. In the former case, $a_1P_1u_1Qu_2P_2b_2, a_2P_2rRsP_3b_3, a_3P_3x_3Q_{31}y_1P_1b_1$ are disjoint paths from

A to B and through e. In the latter case, $a_1P_1x_1Q_{13}y_3P_3b_3, a_2P_2u_2Qu_1P_1b_1, a_3P_3sRrP_2b_2$ are disjoint paths from A to B and through e.

Therefore, we may assume $u_1 \in V(a_1P_1x_1 - y_1)$. Thus, Q implies the existence of a path Q' in G from some $v_2 \in V(a_2P_2x_2)$ to some $v_1 \in V(a_1P_1x_1 - y_1) \cup V(a_3P_3x_3 - x_3)$ and internally disjoint from $P \cup L_P$, and we choose Q' with $v_2P_2x_2$ minimal. Let $v_3 \in P_3$ with $v_3P_3a_3$ maximal such that $v_3 = a_3$, or G contains a path R from v_3 to some $r \in V(a_1P_1x_1 - x_1) \cup V(a_2P_2v_2 - v_2)$ and internally disjoint from $P \cup L_P$.

Suppose $v_3 \in V(x_3P_3b_3 - x_3)$; so R is defined. By (2) and $(*)$, $R \cap Q' = \emptyset$; and by $(*)$, $r \in V(a_2P_2v_2 - v_2)$. If $v_1 \in V(a_1P_1x_1 - y_1)$ then $a_1P_1v_1Q'v_2P_2b_2, a_2P_2rRv_3P_3b_3, a_3P_3x_3Q_{31}y_1P_1b_1$ are disjoint paths from A to B and through e. So assume $v_1 \in V(a_3P_3x_3 - x_3)$. Then P_1, $a_2P_2rRv_3P_3b_3$, $a_3P_3v_1Q'v_2P_2b_2$ contradict the choice of P (the maximality of L_P in (1)).

Thus, $v_3 \in V(a_3P_3x_3)$. If $\{x_1, v_2, v_3\}$ is a cut in G separating A from B then (i) holds. So by (2) and $(*)$ and by the choices of v_2 and v_3, we may assume that there is a path R' from some $s' \in V(a_3P_3v_3 - v_3)$ to some $r' \in V(v_2P_2b_2 - v_2)$ and internally disjoint from P. Then R is defined, and by the minimality of $v_2P_2x_2$, $r' \in V(x_2P_2b_2 - x_2)$. So $R \cap R' = \emptyset$ by (2) and $(*)$. If $r \in V(a_2P_2v_2 - v_2)$ then $P_1, a_2P_2rRv_3P_3b_3, a_3P_3s'R'r'P_2b_2$ contradict (1); and if $r \in V(a_1P_1x_1 - x_1)$ then $a_1P_1rRv_3P_3b_3$, $a_2P_2x_2Q_{21}y_1P_1b_1$, and $a_3P_3s'R'r'P_2b_2$ are three disjoint paths from A to B and through e.

Case 2. w_3, x_3, y_3 are not defined.

Let $u \in V(P_3)$ with uP_3b_3 minimal such that $u = a_3$ or u belongs to some P-bridge of G intersecting $(a_1P_1x_1 - x_1) \cup (a_2P_2x_2 - x_2)$. We may assume $\{x_1, x_2, u\} = \{a_1, a_2, a_3\}$. For, suppose $\{x_1, x_2, u\} \neq \{a_1, a_2, a_3\}$. We further choose P_3 (while fixing P_1, P_2, L_P) so that uP_3b_3 is minimal; hence no P-bridge of G intersects both $a_3P_3u - u$ and $uP_3b_3 - u$. If G has no path from $a_3P_3u - u$ to $(x_1P_1b_1 - x_1) \cup (x_2P_2b_2 - x_2)$ and internally disjoint from $P \cup L_P$, then by (2), (5) and the choice of u, $\{x_1, x_2, u\}$ is a cut in G separating A from B, and (i) holds. So assume that G has a path Q from some $x \in V(a_3P_3u - u)$ to some $y \in V(x_1P_1b_1 - x_1) \cup V(x_2P_2b_2 - x_2)$ and internally disjoint from $P \cup L_P$. Let

R be a path in G from u to some $z \in V(a_1P_1x_1 - x_1) \cup V(a_2P_2x_2 - x_2)$ and internally disjoint from $P \cup L_P$, and by symmetry let $z \in V(a_2P_2x_2 - x_2)$. By (2) and (5), $Q \cap R = \emptyset$. Since we are in Case 2, $(L_P - P) \cap (Q \cup R) = \emptyset$. If $y \in V(x_2P_2b_2 - x_2)$ then $P_1, a_2P_2zRuP_3b_3, a_3P_3xQyP_2b_2$ contradict the choice of P (i.e., (1)). So $y \in V(x_1P_1b_1 - x_1)$. Then $a_1P_1x_1Q_{12}y_2P_2b_2, a_2P_2zRuP_3b_3, a_3P_3xQyP_1b_1$ are three disjoint paths from A to B and through e.

Similarly, let $v \in V(P_3)$ with a_3P_3v minimal such that $v = b_3$ or v belongs to some P-bridge of G intersecting $(y_1P_1b_1 - y_1) \cup (y_2P_2b_2 - y_2)$, and we may assume $\{y_1, y_2, v\} = \{b_1, b_2, b_3\}$.

If no P-bridge of G intersecting P_3 also meets P_1 (respectively, P_2) then (iii) holds with $G_2 = L_{12} \cap L_{22}$, $P_2 \cup P_3 \subseteq G_3$ and $P_1 \subseteq G_1$ (respectively, $P_1 \cup P_3 \subseteq G_3$ and $P_2 \subseteq G_1$). So assume that some P-bridge of G meets both P_2 and P_3 and some meets both P_1 and P_3.

Suppose G has a P-bridge J such that $J \cap P_i \neq \emptyset$ for $i = 1, 2, 3$. Then $J \neq L_P$ as w_3, x_3, y_3 are not defined. So by (5) and by symmetry, we may assume $V(J \cap P_i) = \{a_i\}$ for $i = 1, 2$. Let $w \in V(J \cap P_3)$ with a_3P_3w maximal. We further choose P_3 (while fixing P_1, P_2, L_P) so that wP_3b_3 is as short as possible; then no P-bridge of G intersects both $a_3P_3w - w$ and $wP_3b_3 - w$. We may assume that G has a path Q from some $x \in V(a_3P_3w - w)$ to some $y \in V(P_1 - a_1) \cup V(P_2 - a_2)$ and internally disjoint from $P \cup L_P \cup J$; for otherwise $\{a_1, a_2, w\}$ is a cut in G separating A from B, showing that (i) holds. By symmetry, assume $y \in V(P_2 - a_2)$. Let Q_1 denote a path in J from w to a_1 and internally disjoint from P. Then $a_1Q_1wP_3b_3, Q_{21}, a_3P_3xQyP_2b_2$ are three disjoint paths from A to B and through e.

So assume that no P-bridge of G intersects all P_i, $i = 1, 2, 3$. Suppose all P-bridges of G intersecting both P_3 and $P_1 \cup P_2$ meet P_3 in exactly one common vertex, say z. Assume by symmetry that $z \neq a_3$. We may further choose P_3 (while fixing P_1, P_2, L_P) so that zP_3b_3 is as short as possible. Then no P-bridge of G intersects both $a_3P_3z - z$ and $zP_3b_3 - z$. So

$\{a_1, a_2, z\}$ is a cut in G separating A from B, and (i) holds. Hence, we may assume that G has distinct P-bridges J_1 and J_2 such that $J_1 \cap P_1 \neq \emptyset$, $J_2 \cap P_2 \neq \emptyset$, and there exist $u_i \in V(J_i \cap P_3)$, $i = 1, 2$, with $u_1 \neq u_2$. By symmetry assume that a_3, u_1, u_2, b_3 occur on P_3 in order. For $i = 1, 2$, let Q_i be a path in J_i from u_i to some $v_i \in V(P_i)$ and internally disjoint from P. If $v_1 \neq a_1$ and $v_2 \neq b_2$, then $Q_{12}, a_2P_2v_2Q_2u_2P_3b_3, a_3P_3u_1Q_1v_1P_1b_1$ are three disjoint paths from A to B and through e. So by symmetry, assume $V(J_2 \cap P_2) = \{b_2\}$. By modifying P_3 (while fixing P_1, P_2, L_P) we may assume that no P-bridge of G intersects both $a_3P_3u_2 - u_2$ and $u_2P_3b_3 - u_2$. (Note that J_1 will not be used in the remaining proof.)

If no P-bridge of G intersecting $u_2P_3b_3 - u_2$ meets $(P_1 - b_1) \cup (P_2 - b_2)$, then G has separation (G_1, G_2) such that $V(G_1 \cap G_2) = \{b_1, b_2, u_2\}$, $A \subseteq V(G_1)$, and $B \subseteq V(G_2)$; so (i) holds. Hence, assume that there is a path R from some $s \in V(u_2P_3b_3 - u_2)$ to some $t \in V(P_1 - b_1) \cup V(P_2 - b_2)$. If $t \in V(P_1 - b_1)$ then $a_1P_1tRsP_3b_3, Q_{21}, a_3P_3u_2Q_2b_2$ are disjoint paths from A to B and through e. So assume $t \in V(P_2 - b_2)$. Now $P_1, a_2P_2tRsP_3b_3, a_3P_3u_2Q_2b_2$ reduce this case to Case 1. ∎

2.4 Separations of order three

We now use Lemma 2.3.4 to prove the following lemma about separations of order three.

Lemma 2.4.1. *Let (G, u_1, u_2, A) be a quadruple, and suppose G has a separation (U_1, U_2) such that $|V(U_1 \cap U_2)| \leq 3$, $V(U_1 \cap U_2) \cap A \neq \emptyset$, $u_1 \in V(U_1) - V(U_2)$, $u_2 \in V(U_2) - V(U_1)$, and $A \subseteq U_i$ for some $i \in \{1, 2\}$. Then the conclusion of Theorem 2.2.1 holds for (G, u_1, u_2, A).*

Proof. For convenience, we say a separation of G *good* if it satisfies the conditions of this lemma. We may assume that for any good separation (U_1, U_2), $|V(U_1 \cap U_2)| = 3$ (and let $V(U_1 \cap U_2) = \{v_1, v_2, v_3\}$) and U_{3-i} has three independent paths, say P_1, P_2, P_3, from u_{3-i} to v_1, v_2, v_3, respectively. For, suppose otherwise. By symmetry, let $i = 1$. If $|V(U_1 \cap U_2)| \leq 2$ let $U_{21} = U_2$ and $U_{22} = \emptyset$, and if $|V(U_1 \cap U_2)| = 3$ let (U_{21}, U_{22}) be a separation in U_2 such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in V(U_{21}) - V(U_{22})$ and $V(U_1 \cap U_2) \subseteq V(U_{22})$. Now $(U_{21}, U_{22} \cup U_1)$ is a separation in G showing that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A).

We may assume that $E(G[A]) = \emptyset$; as otherwise, it is easy to see that Theorem 2.2.1(iii) holds. Let $A = \{a_1, a_2, a_3, a_4\}$ and $a_1 = v_1$. We may assume that

$(*)$ for any good separation (U_1, U_2), $|V(U_1 \cap U_2)| = 3$, and $V(U_1 \cap U_2) \cap A = \{a_1\}$.

Again by symmetry, let $i = 1$. If $v_2, v_3 \in A$ then $U_1, U_2 + A, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV. So we may assume $v_3 \notin A$. Suppose $v_2 \in A$, say $v_2 = a_2$. Then, because of P_1, P_2, P_3, G has a topological H rooted at u_1, u_2, A if and only if $U_1 - \{a_1, a_2\}$ has three independent paths from u_1 to a_3, a_4, v_3, respectively. Thus either Theorem 2.2.1(i) holds for (G, u_1, u_2, A), or U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 4$, $a_1, a_2 \in V(U_{11} \cap U_{12})$, $u_1 \in V(U_{11}) - V(U_{12})$ and $\{a_3, a_4, v_3\} \subseteq V(U_{12})$. If $|V(U_{11} \cap U_{12})| \leq 3$ then the separation $(U_{11} \cup U_2, U_{12})$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A). So assume $|V(U_{11} \cap U_{12})| = 4$. If $a_3, a_4 \notin V(U_{11} \cap U_{12})$ then $U_{11}, U_2, \{a_1\}, \{a_2\}, U_{12} - \{a_1, a_2\}$ show that (G, u_1, u_2, A) is an obstruction of type I. So assume $a_3 \in V(U_{11} \cap U_{12})$. If $a_4 \notin V(U_{11} \cap U_{12})$ then $U_{11}, U_2 + a_3, \{a_1\}, \{a_2\}, \{a_3\}, U_{12} - \{a_1, a_2, a_3\}$ show that (G, u_1, u_2, A) is an obstruction of type IV; and if $a_4 \in V(U_{11} \cap U_{12})$ then $U_{11}, U_2 \cup U_{12}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV. This proves $(*)$.

We now look for paths in U_1 in order to form a topological H in G. Let U_1' be obtained from $(U_1 - a_1) + v_2v_3$ by duplicating u_1 twice, and denote the copies of u_1 by u_1', u_1''. We apply Lemma 2.3.4 to $U_1', \{u_1, u_1', u_1''\}, \{a_2, a_3, a_4\}, v_2v_3$. If U_1' has three disjoint paths from $\{u_1, u_1', u_1''\}$ to $\{a_2, a_3, a_4\}$ and through v_2v_3, then $(U_1 - a_1) + v_2v_3$ has three independent paths R_1, R_2, R_3 from u_1 to a_2, a_3, a_4, respectively, and through v_2v_3, and $\bigcup_{i=1}^{3}(P_i \cup R_i) - v_2v_3$ is a topological H in G rooted at u_1, u_2, A; so Theorem 2.2.1(i) holds for (G, u_1, u_2, A). Hence, assume the paths R_1, R_2, R_3 do not exist. Then one of $(i) - (iv)$ of Lemma 2.3.4 holds. Since u_1' and u_1'' are duplicates of u_1, (iii) and (iv) of

Lemma 2.3.4 do not occur here. Suppose Lemma 2.3.4(ii) holds. Then U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $a_1 \in V(U_{11} \cap U_{12})$, $A \cup \{u_1\} \subseteq V(U_{11})$, and $\{v_2, v_3\} \subseteq V(U_{12})$. Now the separation $(U_{12} \cup U_2, U_{11})$ shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A). Hence, we may assume that Lemma 2.3.4(i) holds.

Thus, U_1 has a separation (U_{11}, U_{12}) such that $a_1 \in V(U_{11} \cap U_{12})$, $|V(U_{11} \cap U_{12})| \leq 4$, $u_1 \in V(U_{11}) - V(U_{12})$, and $A \subseteq V(U_{12})$. We choose (U_{11}, U_{12}) so that U_{12} is minimal. Note that $\{v_2, v_3\} \subseteq V(U_{11})$ or $\{v_2, v_3\} \subseteq V(U_{12})$. In fact, we may assume $\{v_2, v_3\} \nsubseteq V(U_{11})$; otherwise the separation $(U_{11} \cup U_2, U_{12})$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A).

We may assume that $|V(U_{11} \cap U_{12})| = 4$ and $U_{11} - a_1$ has three independent paths Q_1, Q_2, Q_3 from u_1 to the three vertices in $V(U_{11} \cap U_{12}) - \{a_1\}$ respectively. First we may assume $|V(U_{11} \cap U_{12})| \geq 3$; otherwise the separation $(U_{11}, U_{12} \cup U_2)$ shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A). Moreover, we may assume $|V(U_{11} \cap U_{12})| = 4$; otherwise by $(*)$, $V(U_{11} \cap U_{12}) \cap (A - \{a_1\}) = \emptyset$, and $U_{11}, U_2, \{a_1\}, U_{12} - a_1$ show that (G, u_1, u_2, A) is an obstruction of type II. Now if the paths Q_1, Q_2, Q_3 do not exist, then U_{11} has a separation (U'_{11}, U''_{11}) such that $|V(U'_{11} \cap U''_{11})| \leq 3$, $a_1 \in V(U'_{11} \cap U''_{11})$, $u_1 \in V(U'_{11}) - V(U''_{11})$, and $V(U_{11} \cap U_{12}) \subseteq V(U''_{11})$. We may assume $|V(U'_{11} \cap U''_{11})| = 3$; otherwise $(U'_{11}, U''_{11} \cup U_{12} \cup U_2)$ shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A). By $(*)$, $V(U'_{11} \cap U''_{11}) \cap (A - \{a_1\}) = \emptyset$. So $U'_{11}, U_2, \{a_1\}, (U''_{11} \cup U_{12}) - a_1$ show that (G, u_1, u_2, A) is an obstruction of type II.

We may also assume that $\{v_2, v_3\} \subseteq V(U_{12}) - V(U_{11})$. Otherwise, since $\{v_2, v_3\} \nsubseteq V(U_{11})$, we may assume that $v_2 \in V(U_{11} \cap U_{12})$ and $v_3 \notin V(U_{11} \cap U_{12})$. By the minimality of U_{12}, $U_{12} - \{a_1, v_2\}$ has three disjoint paths from $\{a_2, a_3, a_4\}$ to $(V(U_{11} \cap U_{12}) - \{a_1, v_2\}) \cup \{v_3\}$. Now these paths and $\cup_{i=1}^{3}(P_i \cup Q_i)$ form a topological H in G rooted at u_1, u_2, A, and Theorem 2.2.1(i) holds for (G, u_1, u_2, A)

If $V(U_{11} \cap U_{12}) - \{a_1\} = \{a_2, a_3, a_4\}$, then $U_{11}, U_{12} \cup U_2, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV.

Suppose $|(V(U_{11} \cap U_{12}) - \{a_1\}) \cap \{a_2, a_3, a_4\}| = 2$, say $a_2 \notin V(U_{11} \cap U_{12})$. If $U_{12} - \{a_1, a_3, a_4\}$ has two disjoint paths from $\{v_2, v_3\}$ to $\{a_2\} \cup (V(U_{11} \cap U_{12}) - A)$, then these paths and $\cup_{i=1}^{3}(P_i \cup Q_i)$ form a topological H in G rooted at u_1, u_2, A; so Theorem 2.2.1(i) holds for (G, u_1, u_2, A). Hence, assume that U_{12} has a separation (S, T) such that $|S \cap T| \leq 4$, $\{a_1, a_3, a_4\} \subseteq V(S \cap T)$, $\{a_2\} \cup V(U_{11} \cap U_{12}) \subseteq V(S)$, and $\{v_2, v_3\} \subseteq V(T)$. If $a_2 \in V(S) - V(T)$ then $U_{11}, U_2 \cup T, \{a_1\}, S - \{a_1, a_3, a_4\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV; if $a_2 \in V(S \cap T)$ then $U_{11} \cup S, U_2 \cup T, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV.

Now suppose $(V(U_{11} \cap U_{12}) - \{a_1\}) \cap \{a_2, a_3, a_4\} = \emptyset$. Then we may apply Lemma 2.3.4 to $U_{12} - a_1 + v_2v_3, V(U_{11} \cap U_{12}) - \{a_1\}, \{a_2, a_3, a_4\}, v_2v_3$. If $U_{12} - a_1 + v_2v_3$ has three disjoint paths from $V(U_{11} \cap U_{12}) - \{a_1\}$ to $\{a_2, a_3, a_4\}$ and through v_2v_3, then deleting v_2v_3 from the union of these paths with $\bigcup_{i=1}^{3}(P_i \cup Q_i)$, we obtain a topological H in G rooted at u_1, u_2, A; Theorem 2.2.1(i) holds for (G, u_1, u_2, A). So assume that one of $(i) - (iv)$ of Lemma 2.3.4 holds. If Lemma 2.3.4(i) holds then U_{12} has a separation (S, T) such that $a_1 \in V(S \cap T)$, $|V(S \cap T)| \leq 4$, $V(U_{11} \cap U_{12}) \subseteq V(S)$, and $\{a_2, a_3, a_4\} \subseteq V(T)$; so $(U_{11} \cup S, T)$ contradicts the choice of (U_{11}, U_{12}). If Lemma 2.3.4(ii) holds then U_{12} has a separation (S, T) such that $a_1 \in V(S \cap T)$, $|V(S \cap T)| \leq 2$, $\{v_2, v_3\} \subseteq V(T)$, and $A \cup V(U_{11} \cap U_{12}) \subseteq V(S)$; so the separation $(U_2 \cup T, U_{11} \cup S)$ shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A). Now, suppose Lemma 2.3.4(iii) holds. Then $U_{12} - a_1 = S_1 \cup S_2 \cup S_3$ such that $S_1 \cap S_3 = \emptyset$, $\{v_2, v_3\} \subseteq V(S_2)$, $|V(S_i \cap S_2)| \leq 1$ for $i = 1, 3$, $|V(S_1) \cap \{a_2, a_3, a_4\}| = |V(S_1) \cap (V(U_{11} \cap U_{12}) - \{a_1\})| = 1$, and $|V(S_3) \cap \{a_2, a_3, a_4\}| = |V(S_3) \cap (V(U_{11} \cap U_{12}) - \{a_1\})| = 2$. Note that $|S_i \cap S_2| = 1$ for $i = 1, 3$; as otherwise, $(U_2 \cup S_2, U_{11} \cup S_1 \cup S_3)$ is a separation in G showing that Theorem 2.2.1 holds for (G, u_1, u_2, A). Therefore, since $V(S_i \cap S_2) \not\subseteq A$ for $i = 1, 3$ (by $(*)$ as $\{a_1\} \cup V((S_1 \cup S_3) \cap S_2)$ separates u_2 from $A \cup \{u_1\}$), $U_{11}, U_2 \cup S_2, \{a_1\}, S_1, S_3$ show that (G, u_1, u_2, A) is an obstruction of type I. Thus, we may assume that Lemma 2.3.4(iv) holds. Then $U_{12} - a_1 = S_1 \cup S_2 \cup S_3 \cup S_4$, $\{v_2, v_3\} \subseteq V(S_1)$, $S_i \cap S_j = \emptyset$ for $2 \leq i < j \leq 4$, and

$|V(S_i) \cap \{a_2, a_3, a_4\}| = |V(S_i) \cap (V(U_{11} \cap U_{12}) - \{a_1\})| = 1$ for $i = 2, 3, 4$. Let $A_1 = \{a_1\}$, and for $i \in \{2, 3, 4\}$, let $a_i \in V(S_i)$, $V(A_i) = \{a_i\}$ and $A'_i = S_i$ (when $a_i \in V(S_1)$), and $A_i = S_i$ and $A'_i = \emptyset$ (when $a_i \notin V(S_1)$). Now $U_{11} \cup A'_2 \cup A'_3 \cup A'_4, U_2 \cup S_1, A_1, A_2, A_3, A_4$ show that (G, u_1, u_2, A) is an obstruction of type IV.

Thus, without loss of generality, let $V(U_{11} \cap U_{12}) = \{a_1, a_4, b, c\}$, with $b, c \notin A$. Note that $U_{12} \cup U_2$ has a separation (S, T) such that $\{a_1, a_4\} \subseteq V(S \cap T)$, $|V(S \cap T)| \leq 4$, $\{a_2, a_3, b, c\} \subseteq V(S)$, and $u_2 \in V(T) - V(S)$. (For example, $S = U_{12}$ and $T = U_2 + a_4$.) Choose (S, T) to maximize T with $U_2 \subseteq T$. By $(*)$, $|V(S \cap T)| = 4$. Let $V(S \cap T) = \{a_1, a_4, v'_2, v'_3\}$. Then $T - a_4$ has three independent paths Q'_1, Q'_2, Q'_3 from u_2 to a_1, v'_2, v'_3, respectively; for otherwise, T has a separation (T_1, T_2) such that $|V(T_1 \cap T_2)| \leq 3$, $a_1, a_4 \in V(T_1 \cap T_2)$, $u_2 \in V(T_2) - V(T_1)$, and $\{v'_2, v'_3\} \subseteq V(T_1)$ (since $U_2 \subseteq T$ and because of P_1, P_2, P_3), contradicting $(*)$ (with the separation $(U_{11} \cup S \cup T_1, T_2)$).

We apply Lemma 2.3.3 to $S - \{a_1, a_4\}, b, c, v'_2, v'_3, a_2, a_3$ (with a_2, a_3 play the roles of a_1, a_2 there). If $S - \{a_1, a_4\}$ has three disjoint paths, with one from $\{b, c\}$ to $\{v'_2, v'_3\}$, one from $\{b, c\}$ to $\{a_2, a_3\}$, and another from $\{v'_2, v'_3\}$ to $\{a_2, a_3\}$, then these paths and $\bigcup_{i=1}^{3}(Q_i \cup Q'_i)$ form a topological H in G rooted at u_1, u_2, A; Theorem 2.2.1(i) holds for (G, u_1, u_2, A). So assume that $S - \{a_1, a_4\}$ has a separation (G_1, G_2) such that one of $(i) - (v)$ of Lemma 2.3.3 holds. By the minimality of U_{12} and the maximality of T, Lemma 2.3.3(i) does not occur here. If Lemma 2.3.3(ii) holds, then the separation $(U_{11} \cup T \cup G[G_2 + \{a_1, a_4\}], G_1 + \{a_1, a_4\})$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A). If Lemma 2.3.3(iii) holds, say with $\{b, c\} \subseteq V(G_1)$ and $\{v'_2, v'_3\} \subseteq V(G_2)$, then $U_{11} \cup G_1 + \{a_2, a_3\}, (T \cup G_2) + \{a_2, a_3\}, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV. If Lemma 2.3.3(iv) holds, then $U_{11}, T, \{a_1\}, \{a_4\}, G_1, G_2$ show that (G, u_1, u_2, A) is an obstruction of type IV. So assume Lemma 2.3.3(v) holds with $\{a_2, a_3\} \subseteq V(G_1)$. If $|\{v'_2, v'_3\} \cap V(G_1)| = 1$ then the separation $(T \cup G[G_2 + \{a_1, a_4\}], U_{11} \cup G[G_1 + \{a_1, a_4\}])$ contradicts $(*)$. So $|\{b, c\} \cap V(G_1)| = 1$. Then $U_{11} \cup G[G_2 + \{a_1, a_4\}], T, \{a_1\}, \{a_4\}, G_1$ show that (G, u_1, u_2, A) is an obstruction of type I. ∎

Lemma 2.4.2. *Let* (G, u_1, u_2, A) *be a quadruple, and assume that* G *has a separation* (U_1, U_2) *such that* $|V(U_1 \cap U_2)| \leq 3$*,* $|V(U_1)| \geq 5$*,* $u_1 \in V(U_1) - V(U_2)$*,* $A \cup \{u_2\} \subseteq V(U_2)$*. Suppose Theorem 2.2.1 holds for all graphs of order less than* $|V(G)|$*. Then Theorem 2.2.1 holds for* (G, u_1, u_2, A)*.*

Proof. First, we may assume that $|V(U_1 \cap U_2)| = 3$ and U_1 has three independent paths, say P_1, P_2, P_3, from u_1 to the three vertices in $V(U_1 \cap U_2)$. For, otherwise, $|V(U_1 \cap U_2)| \leq 2$ (in which case let $K = U_1$ and $L = \emptyset$), or U_1 has a separation (K, L) such that $|V(K \cap L)| \leq 2$, $u_1 \in V(K) - V(L)$ and $V(U_1 \cap U_2) \subseteq V(L)$. Then the separation $(K, L \cup U_2)$ in G shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A).

Now let G' be obtained from G by deleting $U_1 - u_1 - V(U_1 \cap U_2)$ and adding three edges from u_1 to the three vertices in $V(U_1 \cap U_2)$. By assumption, Theorem 2.2.1 holds for (G', u_1, u_2, A).

If Theorem 2.2.1(i) holds for (G', u_1, u_2, A) then let T' be a topological H in G' rooted at u_1, u_2, A. Now $(T' - u_1) \cup P_1 \cup P_2 \cup P_3$ is a topological H in G rooted at u_1, u_2, A; so Theorem 2.2.1(i) holds for (G, u_1, u_2, A).

Suppose Theorem 2.2.1(ii) holds for (G', u_1, u_2, A), and let (K, L) denote a separation in G' such that $|V(K \cap L)| \leq 2$ and, for some $i \in \{1, 2\}$, $u_i \in V(K) - V(L)$ and $A \cup \{u_{3-i}\} \subseteq V(L)$. If $i = 1$ then the separation $((K - u_1) \cup U_1, L)$ shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A). So $i = 2$. If $u_1 \notin V(K \cap L)$ then the separation $(K, (L - u_1) \cup U_1)$ shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A). So $u_1 \in V(K \cap L)$. Then the separation $(U_1 \cup K, L - u_1)$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A).

Suppose Theorem 2.2.1(iii) holds for (G', u_1, u_2, A), and let (K, L) denote a separation in G' such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K) - V(L)$ and $A \subseteq V(L)$. Now the separation $((K - u_1) \cup U_1, L)$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A).

Finally, assume Theorem 2.2.1(iv) holds for (G', u_1, u_2, A). Replacing u_1 with U_1 in that side of (G', u_1, u_2, A) containing u_1, we see that (G, u_1, u_2, A) is also an obstruction of the same type as (G', u_1, u_2, A). ∎

2.5 Quadruples with critical pairs

In this section, we consider quadruples (G, u_1, u_2, A) in which there exist $x, y \in V(G) - \{u_1, u_2\} - A$ such that $(G/xy, u_1, u_2, A)$ is an obstruction, where G/xy is obtained from G by identifying x and y. Such a pair $\{x, y\}$ is said to be *critical*. First, we need a lemma on separations of order 4 in a hypothetical minimum counterexample to Theorem 2.2.1.

Lemma 2.5.1. *Suppose (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum, and assume that G has a separation (U_1, U_2) such that $V(U_1 \cap U_2) = \{w_1, w_2, w_3, w_4\}$, $u_1 \in V(U_1) - V(U_2), u_2 \in V(U_2) - V(U_1)$, and $A \subseteq V(U_2)$. Then for any permutation $ijkl$ of $\{1, 2, 3, 4\}$,*

- (i) *$U_1 - w_l$ has three independent paths from u_1 to w_i, w_j, w_k, respectively, unless $w_l \in N(u_1)$ and $|N(u_1)| = 3$, and*
- (ii) *U_1 has three independent paths from u_1 to w_i, w_j, w_k, unless $w_l \in N(u_1)$, $|N(u_1)| = 3$, and $N(w_l) \cap V(U_1) \subseteq N[u_1]$.*

Proof. First, note that $|N(u_1)| \geq 3$, or else Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A).

Suppose $U_1 - w_l$ does not have three independent paths from u_1 to w_i, w_j, w_k, respectively. Then U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 3$, $w_l \in V(U_{11} \cap U_{12})$, $\{w_1, w_2, w_3, w_4\} \subseteq V(U_{12})$, and $u_1 \in V(U_{11}) - V(U_{12})$. Note that $|V(U_{11} \cap U_{12})| = 3$; otherwise the separation $(U_{11}, U_{12} \cup U_2)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). Now the separation $(U_{11}, U_{12} \cup U_2)$ allows us to use Lemma 2.4.2 to conclude that $V(U_{11}) = \{u_1\} \cup V(U_{11} \cap U_{12})$. Hence, $|N(u_1)| = 3$ and $N(u_1) = V(U_{11} \cap U_{12})$ (so $w_l \in N(u_1)$).

Now assume that U_1 does not have three independent paths from u_1 to w_i, w_j, w_k, respectively. Then U_1 has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in V(U_{11}) - V(U_{12})$, and $\{w_i, w_j, w_k\} \subseteq V(U_{12})$. Note that $w_l \in V(U_{11}) - V(U_{12})$ and

$|V(U_{11} \cap U_{12})| = 2$; otherwise the separation $(U_{11}, U_{12} \cup U_2 + w_l)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). Now the separation $(U_{11}, U_{12} \cup U_2 + w_l)$ allows us to use Lemma 2.4.2 to conclude that $V(U_{11}) = \{u_1, w_l\} \cup V(U_{11} \cap U_{12})$. Hence, $|N(u_1)| = 3$, $N(u_1) = \{w_l\} \cup V(U_{11} \cap U_{12})$, and $N(w_l) \cap V(U_1) \subseteq N[u_1]$. ∎

We now show, in a sequence of four lemmas, that no quadruple containing a critical pair is a minimum counterexample to Theorem 2.2.1.

Lemma 2.5.2. *Suppose (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum, and let $x, y \in V(G) - A - \{u_1, u_2\}$ be distinct. Then $(G/xy, u_1, u_2, A)$ is not an obstruction of type IV.*

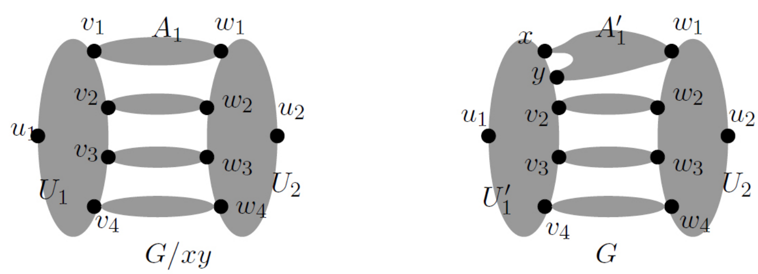


Fig. 2.6: G/xy is an obstruction of type IV.

Proof. For, suppose $(G/xy, u_1, u_2, A)$ is an obstruction of type IV, with sides U_1, U_2 and middle parts A_1, A_2, A_3, A_4. See Figure 2.6. Recall from definition of obstruction that $V(U_1 \cap U_2) \subseteq A$. Let $A := \{a_1, a_2, a_3, a_4\}$ such that $a_i \in V(A_i)$ for $1 \le i \le 4$. Let $V(U_1 \cap A_i) = \{v_i\}$ and $V(U_2 \cap A_i) = \{w_i\}$, $1 \le i \le 4$, and let $u_1 \in V(U_1) - \{v_1, v_2, v_3, v_4\}$ and $u_2 \in V(U_2) - \{w_1, w_2, w_3, w_4\}$. By definition of obstruction, if $|A_i| \ge 2$ then $a_i \in V(A_i) - \{v_i, w_i\}$ and $N(a_i) \subseteq V(A_i)$, and if $v_i = w_i$ then $\{v_i\} = \{w_i\} = \{a_i\} = V(A_i)$.

Let v be the vertex resulting from the identification of x and y. If $v \notin \{v_i, w_i : 1 \le i \le 4\}$ then (G, u_1, u_2, A) is an obstruction of type IV, a contradiction. Then by symmetry assume $v = v_1$. So $|V(A_1)| \ge 2$ and $a_1 \in V(A_1) - \{v_1, w_1\}$. Let U_1', A_1' be obtained from U_1, A_1, respectively, by unidentifying v to x and y. Note that if $xy \in E(G)$ we put xy back in exactly one of U_1' or A_1' (it does not matter which one).

Then A_1' contains disjoint paths X, Y from $\{x, y\}$ to $\{a_1, w_1\}$. For, otherwise, A_1' has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq V(A_{11})$, and $\{a_1, w_1\} \subseteq V(A_{12})$. Now $U_1' \cup A_{11}, U_2, A_{12}, A_2, A_3, A_4$ (when $a_1 \notin V(A_{11} \cap A_{12}) \neq \emptyset$), or $U_1' \cup A_{11} + a_1, U_2 \cup A_{12}, \{a_1\}, A_2, A_3, A_4$ (when $a_1 \in V(A_{11} \cap A_{12})$ or $V(A_{11} \cap A_{12}) = \emptyset$), show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

Moreover, for each $i \in \{2, 3, 4\}$, if $a_i \notin \{v_i, w_i\}$ then $A_i - v_i$ contains a path W_i' between w_i and a_i, and $A_i - w_i$ has a path V_i' between v_i and a_i. For, suppose by symmetry that $A_i - v_i$ has no path from w_i to a_i, then A_i has a separation (A_{i1}, A_{i2}) such that $V(A_{i1} \cap A_{i2}) = \{v_i\}$, $a_i \in V(A_{i1})$ and $w_i \in V(A_{i2})$. Then the separation $(G - (A_{i1} - v_i), A_{i1} + (A - \{a_i\}))$ shows that Theorem 2.2.1(iii) holds, a contradiction. Let $W_i' = V_i' = A_i$ if $a_i = v_i = w_i$.

Suppose for each $i \in \{2, 3, 4\}$, $U_1' - (A - \{v_i\})$ has three independent paths P_1^i, P_2^i, P_3^i from u_1 to x, y, v_i, respectively. If $U_2' - (A \cap \{w_2\})$ has three independent paths from u_2 to w_1, w_3, w_4, respectively, then these paths and $P_1^2, P_2^2, P_3^2, X, Y, V_2', W_3', W_4'$ form a topological H rooted at u_1, u_2, A, and Theorem 2.2.1(i) would hold. So such paths do not exist in $U_2' - (A \cap \{w_2\})$. Then by Lemma 2.5.1(i), $w_2 \in N(u_2)$ and $|N(u_2)| = 3$. Similarly, $w_3, w_4 \in N(u_2)$. Hence by Lemma 2.4.1, $w_2, w_3, w_4 \notin A$. Therefore, by Lemma 2.5.1(ii), $N(w_i) \cap V(U_2) \subseteq N[u_2]$ for $i = 2, 3, 4$. Now $G[N[u_2]] + a_1, U_1' \cup A_1' \cup (U_2 - \{u_2, w_2, w_3, w_4\}), \{a_1\}, A_2, A_3, A_4$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

Hence, we may assume by symmetry that P_1^2, P_2^2, P_3^2 do not exist. Then U_1' has a separation (U_{11}, U_{12}) such that $A \cap \{v_3, v_4\} \subseteq V(U_{11} \cap U_{12})$, $|V(U_{11} \cap U_{12})| \leq |A \cap \{v_3, v_4\}| + 2$, $u_1 \in V(U_{11}) - V(U_{12})$, and $\{x, y, v_2\} \subseteq V(U_{12})$. We choose (U_{11}, U_{12}) so that $|V(U_{11} \cap U_{12})|$ is minimum and then U_{12} is minimal.

We claim that $|V(U_{11} \cap U_{12})| = |A \cap \{v_3, v_4\}| + 2$. For, otherwise, the separation $(U_{11} + \{v_3, v_4\}, G - (U_{11} - U_{12}) + \{v_3, v_4\})$ allows us to use Lemma 2.4.1 to assume $v_3, v_4 \notin A$; so $|A \cap \{v_3, v_4\}| = 0$. Then $|V(U_{11} \cap U_{12})| = 1$ and $v_3, v_4 \in V(U_{11} - U_{12})$;

else, the separation $(U_{11}, U_{12} \cup A'_1 \cup A_2 \cup A_3 \cup A_4 \cup U_2)$ shows that Theorem 2.2.1(ii) would hold. Hence, $U_{11}, U_2, A'_1 \cup U_{12} \cup A_2, A_3, A_4$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

Let $V(U_{11} \cap U_{12}) - (A \cap \{v_3, v_4\}) = \{s_1, s_2\}$. We claim that $v_2 \notin A \cap \{s_1, s_2\}$. For, suppose $v_2 \in A \cap \{s_1, s_2\}$; so $v_2 = a_2$. Note that for each $i \in \{3, 4\}$, if $v_i \notin A$ then, since $v_2 = a_2$, we must have $v_i \notin V(U_{12})$ by Lemma 2.4.1. So U_{11}, U_2, $(U_{12} - v_2) \cup A'_1, \{v_2\}, A_3, A_4$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

Then by the minimality of U_{12}, $U_{12} - (A \cap \{v_2\})$ contains disjoint paths S_1, S_2 from $\{x, y\}$ to $\{s_1, s_2\}$. We may assume that $(U_{11} + \{v_3, v_4\}) - (A \cap \{v_4\})$ (or $(U_{11} + \{v_3, v_4\}) - (A \cap \{v_3\})$) has independent paths Q'_1, Q'_2, Q'_3 from u_1 to s_1, s_2, v_3 (or v_4), respectively. This is true if $v_3 \in \{s_1, s_2\}$ or $v_4 \in \{s_1, s_2\}$; as otherwise $U_{11} + \{v_3, v_4\}$ has a cut of size at most two separating u_1 from $\{s_1, s_2\} \cup \{v_3, v_4\}$, which gives a separation showing that Theorem 2.2.1(ii) would hold. So we may assume $v_3, v_4 \notin \{s_1, s_2\}$ and that the paths Q'_1, Q'_2, Q'_3 do not exist. Then by Lemma 2.5.1(i), $|N(u_1)| = 3$ and $v_3, v_4 \in N(u_1)$. So by Lemma 2.4.1, $v_3, v_4 \notin A$. Hence, by Lemma 2.5.1(ii), $N(v_i) \cap V(U_{11} + \{v_3, v_4\}) \subseteq N[u_1]$ for $i = 3, 4$. Now $G[N[u_1]], U_2, (U_1 - \{u_1, v_3, v_4\}) \cup A'_1 \cup A_2, A_3, A_4$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

By symmetry, assume that Q'_3 ends at v_3. Then because of S_1, S_2, we see that $U_1 - (A \cap \{v_2, v_4\})$ has independent paths Q_1, Q_2, Q_3 from u_1 to x, y, v_3, respectively. If $U_2 - (A \cap \{w_3\})$ has three independent paths from u_2 to w_1, w_2, w_4, respectively, then these paths and $Q_1, Q_2, Q_3, X, Y, V'_3, W'_2, W'_4$ form a topological H in G rooted at u_1, u_2, A, and Theorem 2.2.1(i) would hold. So such paths do not exist. Then by Lemma 2.5.1(i), $w_3 \in N(u_2)$ and $|N(u_2)| = 3$. So by Lemma 2.4.1, $w_3 \notin A$. Hence, $N(w_3) \cap V(U_2) \subseteq N[u_2]$ by Lemma 2.5.1(ii). Moreover, $v_3 \notin A$. So $v_3 \in V(U_{11} - U_{12})$ because of Q'_1, Q'_2, Q'_3. Then $v_4 \in V(U_{11} - U_{12})$; for otherwise, since $v_4 \notin \{s_1, s_2\}$ when $v_4 \in A$, $U_{11}, G[N[u_2]], A_3$, $U_{12} \cup A'_1 \cup A_2 \cup A_4 \cup (U_2 - \{u_2, w_3\})$ (and removing from the last subgraph the possible edge with both ends in $N(u_2)$) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

Suppose U_{11} does not contain independent paths from u_1 to s_1, s_2, v_4, respectively. Then by Lemma 2.5.1(ii), $v_3 \in N(u_1)$, $|N(u_1)| = 3$ and $N(v_3) \cap V(U_{11}) \subseteq N[u_1]$. Hence $G[N[u_1]], G[N[u_2]]$, $A_3, (U_1' - \{u_1, v_3\}) \cup A_1' \cup A_2 \cup A_4 \cup (U_2 - \{u_2, w_3\})$ (and removing from the last subgraph possible edges with both ends in $N(u_1)$ or $N(u_2)$) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

So let R_1, R_2, R_3 be independent paths in U_{11} from u_1 to s_1, s_2, v_4, respectively. If $U_2 - (A \cap \{w_4\})$ has independent paths from u_2 to w_1, w_2, w_3, respectively, then these paths, $R_1, R_2, R_3, S_1, S_2, X, Y, V_4', W_2', W_3'$ form a topological H in G rooted at u_1, u_2, A, and Theorem 2.2.1(i) would hold. So such paths in $U_2 - (A \cap \{w_4\})$ do not exist. Then by Lemma 2.5.1(i), $w_4 \in N(u_2)$, and by Lemma 2.4.1, $w_4 \notin A$. Hence by Lemma 2.5.1(ii), $N(w_i) \cap V(U_2) \subseteq N[u_2]$ for $i = 3, 4$. Thus, $U_{11}, G[N[u_2]], A_3, A_4, U_{12} \cup A_1' \cup A_2 \cup (U_2 - \{u_2, w_3, w_4\})$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction. ∎

Lemma 2.5.3. *Suppose (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum, and let $x, y \in V(G) - A - \{u_1, u_2\}$ be distinct. Then $(G/xy, u_1, u_2, A)$ is not an obstruction of type I.*

Proof. Suppose $(G/xy, u_1, u_2, A)$ is an obstruction of type I, with sides U_1, U_2 and middle parts A_1, A_2, A_3. Recall that $V(U_1 \cap U_2) \subseteq A$. See Figure 2.2. Let $A := \{a_1, a_2, a_3, a_4\}$ such that $a_i \in V(A_i)$ for $i = 1, 2$ and $a_3, a_4 \in V(A_3)$. Let $V(U_1 \cap A_i) = \{v_i\}$ for $i = 1, 2$, $V(U_1 \cap A_3) = \{v_3, v_4\}$, $V(U_2 \cap A_i) = \{w_i\}$ for $i = 1, 2, 3$, $u_1 \in V(U_1) - \{v_1, v_2, v_3, v_4\}$, and $u_2 \in V(U_2) - \{w_1, w_2, w_3\}$. By definition of obstruction, $a_3, a_4 \in V(A_3) - \{v_3, v_4, w_3\}$ and, for $i = 1, 2$, if $|V(A_i)| \geq 2$ then $a_i \in V(A_i) - \{v_i, w_i\}$. Note that A is independent, as otherwise the separation $(G[A], G - E(G[A]))$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A).

Let v denote the vertex resulting from the identification of x and y. Note that $v \in \{v_i : 1 \leq i \leq 4\} \cup \{w_i : 1 \leq i \leq 3\}$, for otherwise (G, u_1, u_2, A) is also an obstruction of type I, a contradiction. By symmetry, it suffices to consider four cases: $v = v_1$, $v = w_1$, $v = w_3$, and $v = v_4$. See Figure 2.7. Before distinguishing these four cases, we make observations (1), (2) and (3) below. Let $A'_i = A_i$ if $v \notin V(A_i)$, and otherwise let A'_i be obtained from A_i by unidentifying v to x and y. Similarly, let $U'_i = U_i$ if $v \notin V(U_i)$, and otherwise let U'_i be obtained from U_i by unidentifying v to x and y. When $xy \in E(G)$, we put xy back in exactly one of U'_i and A'_i.

(1) If $v \in \{v_1, v_4\}$ then $v_i, w_j \notin A$ for all i, j, and U_2 has three independent paths W_1, W_2, W_3 from u_2 to w_1, w_2, w_3, respectively.

Suppose $v \in \{v_1, v_4\}$. Note that $v_3, v_4, w_3 \notin A$ by definition of obstruction. Also, $w_1, w_2 \notin A$ by Lemma 2.4.1. Hence, $v_2 \notin A$ by definition of obstruction. Now suppose the second part of (1) fails. Then U_2 has a separation (U_{21}, U_{22}) such that $|V(U_{21} \cap U_{22})| \leq 2$, $u_2 \in V(U_{21}) - V(U_{22})$, and $\{w_1, w_2, w_3\} \subseteq V(U_{22})$. The separation $(U_{21}, U_{22} \cup U'_1 \cup A'_1 \cup A'_2 \cup A'_3)$ shows that Theorem 2.2.1(ii) holds, a contradiction.

(2) For $i \in \{1, 2\}$, if $v \notin V(A_i)$ and $|V(A_i)| \geq 2$, then $A_i - v_i$ has a path W'_i from w_i to a_i and $A_i - w_i$ has a path V'_i from v_i to a_i (and if $|V(A_i)| = 1$ then let $W'_i = V'_i = A_i$).

For, suppose W'_i does not exist. Then A_i has a separation (A_{i1}, A_{i2}) such that $V(A_{i1} \cap A_{i2}) = \{v_i\}$, $w_i \in V(A_{i1})$ and $a_i \in V(A_{i2})$. Now $(A_{i2} + A, U'_1 \cup U'_2 \cup A_{i1} \cup A'_{3-i} \cup A'_3)$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A), a contradiction. So W'_i exists. Similarly, V'_i exists.

(3) For $i \in \{3, 4\}$, if $v \notin A_3$ then $A_3 - v_{7-i}$ has disjoint paths Q_i, R_i from w_3, v_i, respectively, to $\{a_3, a_4\}$.

Otherwise, A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $v_{7-i} \in V(A_{31} \cap A_{32})$, $\{w_3, v_i\} \subseteq V(A_{31})$, and $\{a_3, a_4\} \subseteq V(A_{32})$. Now the separation $(A_{32} + \{a_1, a_2\}, U'_1 \cup U'_2 \cup A'_1 \cup A'_2 \cup A_{31})$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A), a contradiction.

Case 1. $v = v_1$.

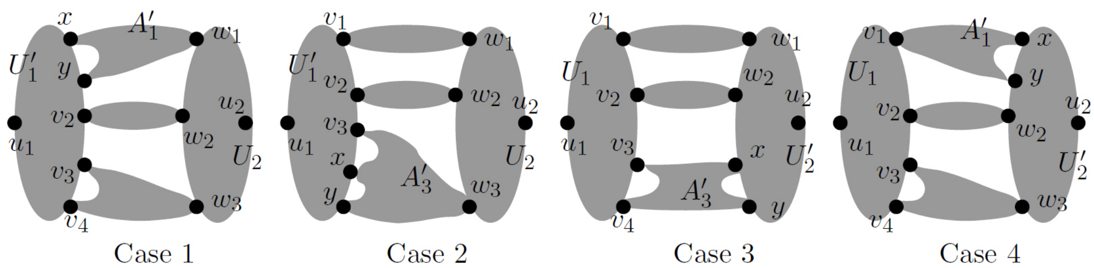


Fig. 2.7: $(G/xy, u_1, u_1, A)$ is an obstruction of type I.

Note that A_1' has disjoint paths X, Y from $\{x, y\}$ to $\{a_1, w_1\}$. Otherwise, A_1' has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq V(A_{11})$ and $\{a_1, w_1\} \subseteq V(A_{12})$. Then $U_1' \cup A_{11}, U_2, A_{12}, A_2, A_3$ (when $a_1 \notin V(A_{11} \cap A_{12}) \neq \emptyset$) or $U_1' \cup A_{11} + a_1, U_2 \cup A_{12}, \{a_1\}, A_2, A_3$ (when $V(A_{11} \cap A_{12}) \subseteq \{a_1\}$) show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

For any $i \in \{3, 4\}$, U_1' does not contain three independent paths from u_1 to x, y, v_i, respectively; as such paths and $X, Y, W_1, W_2, W_3, W_2', Q_i, R_i$ would form a topological H in G rooted at u_1, u_2, A. Thus, U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in V(U_{11}) - V(U_{12})$, and $\{x, y, v_3\} \subseteq V(U_{12})$. Choose this separation to minimize U_{12}.

Suppose $|V(U_{11} \cap U_{12})| \leq 1$. If $|V(U_{11} \cap U_{12})| = 0$ or $\{v_2, v_4\} \not\subseteq V(U_{11}) - V(U_{12})$, then the separation $(U_{11}, U_{12} \cup U_2 \cup A_1' \cup A_2 \cup A_3)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). So $|V(U_{11} \cap U_{12})| = 1$ and $v_2, v_4 \in V(U_{11}) - V(U_{12})$. Then $U_{11}, U_2, A_2, U_{12} \cup A_1' \cup A_3$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

So let $V(U_{11} \cap U_{12}) = \{s_1, s_2\}$. By the minimality of U_{12}, $U_{12} - v_3$ (when $v_3 \notin \{s_1, s_2\}$) and U_{12} (when $v_3 \in \{s_1, s_2\}$) contain disjoint paths S_1, S_2 from $\{s_1, s_2\}$ to $\{x, y\}$.

If $v_4 \notin V(U_{11}) - V(U_{12})$ then $v_2 \in V(U_{11}) - V(U_{12})$; otherwise the separation $(U_{11}, U_{12} \cup U_2 \cup A_1' \cup A_2 \cup A_3)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). But then, $U_{11}, U_2, A_2, U_{12} \cup A_1' \cup A_3$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

So $v_4 \in V(U_{11}) - V(U_{12})$. Now U_{11} does not contain three independent paths from u_1 to s_1, s_2, v_4, respectively; otherwise these paths and S_1, S_2 would form three independent paths in U_1' from u_1 to x, y, v_4, respectively (which were assumed to be nonexistent in the second paragraph of Case 1). Thus, U_{11} has a separation (K, L) such that $|V(K \cap L)| \leq 2$, $u_1 \in V(K) - V(L)$ and $\{s_1, s_2, v_4\} \subseteq V(L)$. If $v_2 \notin V(K) - V(L)$ or $|V(K \cap L)| \leq 1$ then $(K, L \cup U_{12} \cup U_2 \cup A_1' \cup A_2 \cup A_3)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). So $v_2 \in V(K) - V(L)$ and $|V(K \cap L)| = 2$. Then $K, U_2, A_2, L \cup U_{12} \cup A_1' \cup A_3$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

Case 2. $v = v_4$.

Then A_3' has three disjoint paths P_1, P_2, P_3 from $\{v_3, x, y\}$ to $\{a_3, a_4, w_3\}$. For, otherwise, A_3' has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $\{v_3, x, y\} \subseteq V(A_{31})$, and $\{a_3, a_4, w_3\} \subseteq V(A_{32})$. If $|V(A_{31} \cap A_{32})| \leq 1$, then the separation $(A_{32} + \{a_1, a_2\}, U_1' \cup U_2 \cup A_1 \cup A_2 \cup A_{31})$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A). So $|V(A_{31} \cap A_{32})| = 2$. If $V(A_{31} \cap A_{32}) \cap A = \emptyset$ then $U_1' \cup A_{31}, U_2, A_1, A_2, A_{32}$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction. If $V(A_{31} \cap A_{32}) \cap A = \{a_i\}$ for some $i \in \{3, 4\}$ then $U_1' \cup A_{31}, U_2, A_1, A_2, \{a_i\}, A_{32} - a_i$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction. Thus $a_3, a_4 \in V(A_{31} \cap A_{32})$; so $U_1' \cup A_{31}, U_2 \cup A_{32}, A_1, A_2, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

If U_1' has three independent paths from u_1 to v_3, x, y, respectively, then these paths and $P_1, P_2, P_3, W_1, W_2, W_3, W_1', W_2'$ would form a topological H in G rooted at u_1, u_2, A. Thus U_1' has a separation (U_{11}, U_{12}) such that $|V(U_{11} \cap U_{12})| \leq 2$, $u_1 \in V(U_{11}) - V(U_{12})$, and $\{v_3, x, y\} \subseteq V(U_{12})$.

Suppose $|V(U_{11} \cap U_{12})| \leq 1$. Then $|V(U_{11} \cap U_{12})| = 1$ and $\{v_1, v_2\} \subseteq V(U_{11}) - V(U_{12})$; otherwise the separation $(U_{11}, U_{12} \cup U_2 \cup A_1 \cup A_2 \cup A_3')$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). But then the separation $(U_{11} \cup U_2 \cup A_1 \cup A_2, U_{12} \cup A_3' + \{a_1, a_2\})$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A).

So $|V(U_{11} \cap U_{12})| = 2$. If $v_1, v_2 \notin V(U_{11}) - V(U_{12})$ then the separation $(U_{11}, U_{12} \cup U_2 \cup A_1 \cup A_2 \cup A_3')$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). So assume by symmetry that $v_1 \in V(U_{11}) - V(U_{12})$. If $v_2 \notin V(U_{11}) - V(U_{12})$ then $U_{11}, U_2, A_1, U_{12} \cup A_2 \cup A_3'$ show that (G, u_1, u_2, A) would be an obstruction of type II. Thus $v_2 \in V(U_{11}) - V(U_{12})$. Now $U_{11}, U_2, A_1, A_2, A_3' \cup U_{12}$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

Case 3. $v = w_3$.

In this case, there is symmetry between U_1 and U_2'. We choose U_1, U_2', A_3' (while fixing A_1 and A_2) to maximize $U_1 \cup U_2'$, subject to $\{a_3, a_4\} \subseteq V(A_3') - \{v_3, v_4, x, y\}$, $u_1 \in V(U_1) - V(U_2')$, and $u_2 \in V(U_2') - V(U_1)$. Hence, if $xy \in E(G)$ we put it in U_2', and if $v_3v_4 \in E(G)$ we put it in U_1. We apply Lemma 2.3.3 to $A_3', v_3, v_4, x, y, a_3, a_4$ (as $G, v_1, v_2, w_1, w_2, a_1, a_2$, respectively).

Suppose A_3' has a separation (G_1, G_2) such that one of $(i) - (v)$ of Lemma 2.3.3 holds. If Lemma 2.3.3(ii) holds, then the separation $(G_2 \cup U_1 \cup U_2' \cup A_1 \cup A_2, G_1 + \{a_1, a_2\})$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A). If Lemma 2.3.3(iii) holds then $(U_1 \cup G_i) + \{a_3, a_4\}, (U_2' \cup G_{3-i}) + \{a_3, a_4\}, A_1, A_2, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) would be an obstruction of type IV. If Lemma 2.3.3(iv) holds then $U_1, U_2', A_1, A_2, G_1, G_2$ show that (G, u_1, u_2, A) would be an obstruction of type IV. If Lemma 2.3.3(v) holds then $U_1 \cup G_2, U_2', A_1, A_2, G_1$ (when $\{v_3, v_4\} \cap V(G_2) \neq \emptyset$) or $U_1, U_2' \cup G_2, A_1, A_2, G_1$ (when $\{x, y\} \cap V(G_2) \neq \emptyset$) show that (G, u_1, u_2, A) would be an obstruction of type I. Thus, Lemma 2.3.3(i) holds. By symmetry between U_1 and U_2', assume $\{v_3, v_4, a_3, a_4\} \subseteq V(G_1)$ and $\{x, y\} \subseteq V(G_2)$. If $V(G_1 \cap G_2) = \{a_3, a_4\}$ then $U_1 \cup G_1, U_2' \cup G_2, A_1, A_2, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) would be an obstruction of type IV. If $V(G_1 \cap G_2) \cap \{a_3, a_4\} = \emptyset$ then we get a contradiction to the choice of U_1, U_2', A_3' (the maximality of $U_1 \cup U_2'$) by $U_1, U_2' \cup G_2, A_1, A_2$ and G_1 (when $|V(G_1 \cap G_2)| = 2$) or $G_1 + x$ (when $|V(G_1 \cap G_2)| = 1$ and $x \notin V(G_1 \cap G_2)$) or $G_1 + y$ (when $|V(G_1 \cap G_2)| = 1$ and $y \notin V(G_1 \cap G_2)$) or $G_1 + \{x, y\}$ (when $|V(G_1 \cap G_2)| = 0$). So by symmetry assume $V(G_1 \cap G_2) \cap$

$\{a_3, a_4\} = \{a_3\}$. If $V(G_1 \cap G_2) = \{a_3\}$ then $U_1 \cup G_1, (U_2' \cup G_2) + a_4, A_1, A_2, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) would be an obstruction of type IV. So $|V(G_1 \cap G_2)| = 2$. Then $(G/v_3v_4, u_1, u_2, A)$ is an obstruction of type IV with sides $(U_1 + a_3)/v_3v_4, U_2' \cup G_2$ and middle parts $A_1, A_2, \{a_3\}, (G_1 - a_3)/v_3v_4$, contradicting Lemma 2.5.2.

Hence by Lemma 2.3.3, A_3' has three disjoint paths P_1, P_2, P_3, with one from $\{v_3, v_4\}$ to $\{x, y\}$, one from $\{v_3, v_4\}$ to $\{a_3, a_4\}$, and another from $\{x, y\}$ to $\{a_3, a_4\}$.

For some $s \in \{1, 2\}$, $U_1 - (A \cap \{v_{3-s}\})$ has three independent paths S_1, S_2, S_3 from u_1 to v_s, v_3, v_4, respectively. For, otherwise, by Lemma 2.5.1(i), $v_1, v_2 \in N(u_1)$ and $|N(u_1)| = 3$. Then by Lemma 2.4.1, $N(u_1) \cap A = \emptyset$ (in particular, $v_1, v_2 \notin A$). Hence by Lemma 2.5.1(ii), $N(v_i) \cap V(U_1) \subseteq N[u_1]$ for $i = 1, 2$. Now $G[N[u_1]], U_2', A_1, A_2, A_3' \cup (U_1 - \{u_1, v_1, v_2\})$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

Similarly, for some $t \in \{1, 2\}$, $U_2' - (A \cap \{w_{3-t}\})$ has three independent paths T_1, T_2, T_3 from u_2 to w_t, x, y, respectively.

If s and t may be chosen so that $s \neq t$, then $S_1, S_2, S_3, T_1, T_2, T_3, V_s', W_t', P_1, P_2, P_3$ form a topological H in G rooted at u_1, u_2, A, a contradiction. Thus assume $s = t = 1$ is the only possibility. So by Lemma 2.5.1(i), $w_1 \in N(u_2)$ and $|N(u_2)| = 3$, and $v_1 \in N(u_1)$ and $|N(u_1)| = 3$. By Lemma 2.4.1, $(N(u_1) \cup N(u_2)) \cap A = \emptyset$. Hence by Lemma 2.5.1$(ii)$, $N(v_1) \cap V(U_1) \subseteq N[u_1]$, and $N(w_1) \cap V(U_2') \subseteq N[u_2]$. Thus, $G[N[u_1]], G[N[u_2]], A_1, A_2 \cup A_3' \cup (U_1 - \{u_1, v_1\}) \cup (U_2' - \{u_2, w_1\})$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

Case 4. $v = w_1$.

As in Case 1, we can show that A_1' has disjoint paths X, Y from $\{x, y\}$ to $\{a_1, v_1\}$. Note that $A_3 - w_3$ has disjoint paths S, T from $\{v_3, v_4\}$ to $\{a_3, a_4\}$. For otherwise A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $w_3 \in V(A_{31} \cap A_{32})$, $\{v_3, v_4\} \subseteq V(A_{31})$ and $\{a_3, a_4\} \subseteq V(A_{32})$. Hence the separation $(U_1 \cup U_2' \cup A_1' \cup A_2 \cup A_{31}, A_{32} + \{a_1, a_2\})$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A).

We claim that for some $s \in \{2, 3\}$, $U_2' - (A \cap \{w_{5-s}\})$ has three independent paths

P_1, P_2, P_3 from u_2 to x, y, w_s, respectively. First, assume $w_2 = a_2$. Then $U_2' - w_2$ has three independent paths from u_2 to x, y, w_3, respectively; else by Lemma 2.5.1(i), $w_2 \in N(u_2)$ and $|N(u_2)| = 3$, allowing us to use Lemma 2.4.1 to obtain a contradiction. So $w_2 \neq a_2$. Thus, if the claim fails then by Lemma 2.5.1(ii), $w_2, w_3 \in N(u_2)$, $|N(u_2)| = 3$, and $N(\{w_2, w_3\}) \subseteq N[u_2]$. Now $U_1, G[N(u_2)], A_1' \cup (U_2' - \{u_2, w_2, w_3\}), A_2, A_3$ show that (G, u_1, u_2, A) would be an obstruction of type I.

Suppose $s = 2$. If $U_1 - (A \cap \{v_2\})$ has three independent paths from u_1 to v_1, v_3, v_4, respectively, then these paths and $X, Y, S, T, P_1, P_2, P_3, W_2'$ would form a topological H in G rooted at u_1, u_2, A. So such paths do not exist in $U_1 - (A \cap \{v_2\})$. If $v_2 = a_2$ then by Lemma 2.5.1(i), $v_2 \in N(u_1)$ and $|N(u_1)| = 3$, which allows us to use Lemma 2.4.1 to obtain a contradiction. Thus $v_2 \neq a_2$ (and hence $w_2 \neq a_2$). Then by Lemma 2.5.1(ii), $v_2 \in N(u_1)$, $|N(u_1)| = 3$ and $N(v_2) \cap V(U_1) \subseteq N[u_1]$. Suppose U_1 has three independent paths L_1, L_2, L_3 from u_1 to v_1, v_2 and one of $\{v_3, v_4\}$, say v_3. If U_2' has three independent paths from u_2 to x, y, w_3, respectively, then these paths and $L_1, L_2, L_3, X, Y, V_2', Q_3, R_3$ (see (3)) would form a topological H in G rooted at u_1, u_2, A. So such paths do not exist in U_2'. Then by Lemma 2.5.1(ii), $w_2 \in N(u_2)$, $|N(u_2)| = 3$ and $N(w_2) \cap V(U_2) \subseteq N[u_2]$. Now $G[N[u_1]], G[N[u_2]], A_2, A_1' \cup A_3 \cup (U_1 - \{u_1, v_2\}) \cup (U_2' - \{u_2, w_2\})$ (removing from the last subgraph possible edges with both ends in $N(u_1) - \{v_2\}$ or in $N(u_2) - \{w_2\}$) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction. So these paths L_1, L_2, L_3 do not exist in U_1. Then by Lemma 2.5.1(ii), $N(u_1) = \{v_2, v_3, v_4\}$ and $N(\{v_3, v_4\}) \cap V(U_1) \subseteq N[u_1]$. Moreover, U_1 has a separation (U_{11}, U_{12}) such that $V(U_{11} \cap U_{12}) = \emptyset$, $\{u_1, v_2, v_3, v_4\} \subseteq V(U_{11})$, and $v_1 \in V(U_{12})$. Now $U_{11} + a_1, U_{12} \cup U_2' \cup A_1', \{a_1\}, A_2, A_3$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

Thus s cannot be 2 (so $s = 3$). By Lemma 2.5.1(ii), $w_3 \in N(u_2)$, $|N(u_2)| = 3$ and $N(w_3) \cap V(U_2') \subseteq N[u_2]$. If for some $i \in \{3, 4\}$, U_1 has three independent paths from u_1 to v_1, v_2, v_i, respectively, then these paths and $P_1, P_2, P_3, X, Y, V_2', Q_i, R_i$ (see (3)) would form a topological H in G rooted at u_1, u_2, A. So no such paths exist in U_1. Hence by

Lemma 2.5.1(ii), $v_3, v_4 \in N(u_1)$, $|N(u_1)| = 3$, and $N(\{v_3, v_4\}) \cap V(U_1) \subseteq N[u_1]$. Now $G[N[u_1]], G[N[u_2]], A_3, A_1' \cup A_2 \cup (U_1 - \{u_1, v_3, v_4\}) \cup (U_2' - \{u_2, w_3\})$ (removing from the last subgraph the possible edge with both ends in $N(u_2) - \{w_3\}$) show that (G, u_1, u_2, A) is an obstruction of type III, a contradiction. ∎

Lemma 2.5.4. *Suppose* (G, u_1, u_2, A) *is a counterexample to Theorem 2.2.1 with* $|V(G)|$ *minimum, and let* $x, y \in V(G) - A - \{u_1, u_2\}$ *be distinct. Then* $(G/xy, u_1, u_2, A)$ *is not an obstruction of type II.*

Proof. Suppose $(G/xy, u_1, u_2, A)$ is an obstruction of type II with sides U_1, U_2 and middle parts A_1, A_2. Let $V(U_1 \cap A_1) = \{v_1\}$, $V(U_2 \cap A_1) = \{w_1\}$, $V(U_1 \cap A_2) = \{v_2, v_3\}$, $V(U_2 \cap A_2) = \{w_2, w_3\}$, $u_1 \in V(U_1) - \{v_1, v_2, v_3\}$, and $u_2 \in V(U_2) - \{w_1, w_2, w_3\}$. Let $A := \{a_1, a_2, a_3, a_4\}$ such that $a_1 \in V(A_1)$ and $a_2, a_3, a_4 \in V(A_2) - \{v_2, v_3, w_2, w_3\}$. Then A is independent, else $(G[A], G - E(G[A]))$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A).

Let v denote the vertex resulting from the identification of x and y. If $v \notin \{v_i, w_i : 1 \le i \le 3\}$ then (G, u_1, u_2, A) would be an obstruction of type II. So by symmetry assume $v \in \{v_1, v_3\}$. Then by Lemma 2.4.1, $w_1 \notin A$ and, hence, $v_1 \notin A$. As (1) and (2) in the proof of Lemma 2.5.3, U_2 has three independent paths W_1, W_2, W_3 from u_2 to w_1, w_2, w_3, respectively, and if $v \neq v_1$ then $A_1 - v_1$ has a path W_1' from w_1 to a_1.

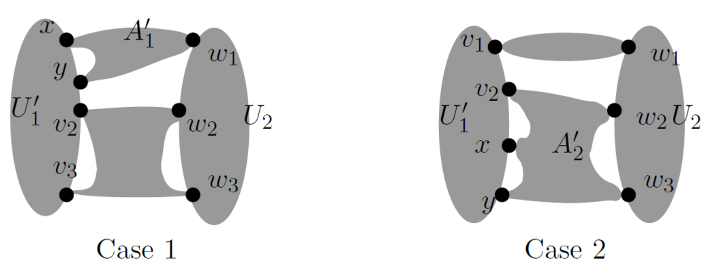


Fig. 2.8: $(G/xy, u_1, u_1, A)$ is an obstruction of type II.

Case 1. $v = v_1$.

Let U_1', A_1' be obtained from U_1, A_1, respectively, by unidentifying v to x and y. Note that A_1' has disjoint paths X, Y from $\{x, y\}$ to $\{a_1, w_1\}$; otherwise A_1' has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 1$, $\{x, y\} \subseteq V(A_{11})$ and $\{a_1, w_1\} \subseteq V(A_{12})$, and hence $U_1' \cup A_{11}, U_2, A_{12}, A_2$ (when $V(A_{11} \cap A_{12}) \not\subseteq \{a_1\}$) or $(U_1' \cup A_{11}) + a_1, U_2 \cup A_{12}, \{a_1\}, A_2$ (when $V(A_{11} \cap A_{12}) \subseteq \{a_1\}$) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

For some $s \in \{2, 3\}$, U_1' has three independent paths P_1, P_2, P_3 from u_1 to x, y, v_s, respectively. Otherwise by Lemma 2.5.1(ii), $v_2, v_3 \in N(u_1)$, $|N(u_1)| = 3$ and $N(\{v_2, v_3\}) \cap V(U_1') \subseteq N[u_1]$. Hence $G[N[u_1]], U_2, A_1' \cup (U_1' - \{u_1, v_2, v_3\}), A_2$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

Without loss of generality, let $s = 2$. If $A_2 - v_3$ has three disjoint paths from $\{a_2, a_3, a_4\}$ to $\{v_2, w_2, w_3\}$, then these paths and $P_1, P_2, P_3, W_1, W_2, W_3, X, Y$ would form a topological H in G rooted at u_1, u_2, A. So A_2 has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| \leq 3$, $v_3 \in V(A_{21} \cap A_{22})$, $\{a_2, a_3, a_4\} \subseteq V(A_{22})$, and $\{v_2, w_2, w_3\} \subseteq V(A_{21})$. Then the separation $(A_{21} \cup U_1' \cup A_1' \cup U_2, A_{22} + a_1)$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A), a contradiction.

Case 2. $v = v_3$.

Let U_1', A_2' be obtained from U_1, A_2, respectively, by unidentifying v to x and y. We choose such U_1', U_2, A_2' (while fixing A_1) to maximize $U_1' \cup U_2$ (subject to $a_2, a_3, a_4 \in V(A_2') - \{v_2, w_2, w_3, x, y\}$). Then $xy, w_2w_3 \notin E(A_2')$.

We claim that U_1' has three independent paths P_1, P_2, P_3 from u_1 to v_2, x, y, respectively. Otherwise, by Lemma 2.5.1(ii), $v_1 \in N(u_1)$, $|N(u_1)| = 3$, and $N(v_1) \cap V(U_1') \subseteq N[u_1]$. Then, $G[N[u_1]], U_2, A_1, A_2' \cup (U_1' - \{u_1, v_1\})$ (removing from the last subgraph the possible edge with both ends in $N(u_1) - \{v_1\}$) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

If $A_2'' := A_2' + w_2w_3$ has three disjoint paths from $\{a_2, a_3, a_4\}$ to $\{v_2, x, y\}$ and through w_2w_3, then these paths (deleting w_2w_3) and $P_1, P_2, P_3, W_1, W_2, W_3, W_1'$ would form a topological H in G rooted at u_1, u_2, A. So one of $(i) - (iv)$ of Lemma 2.3.4 holds, with A_2'', $\{a_2, a_3, a_4\}$, $\{v_2, x, y\}$, w_2w_3 as G, A, B, e, respectively. We use the notation in Lemma 2.3.4. See Figure 2.5.

If Lemma 2.3.4(ii) holds then the separation $(U_2 \cup (G_1 - w_2w_3), U_1' \cup G_2 \cup A_1)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A).

Suppose Lemma 2.3.4(iv) holds. For $i \in \{2, 3, 4\}$, if $V(G_i \cap G_1) \cap A \neq \emptyset$ then let $G_i' = G_i$ and $A_i' = G_i \cap G_1$, and otherwise let $G_i' = \emptyset$ and $A_i' = G_i$. Then $U_1' \cup G_2' \cup G_3' \cup G_4', U_2 \cup G_1, A_1, A_2', A_3', A_4'$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

Now suppose Lemma 2.3.4(iii) holds. If $V(G_1 \cap G_2) = \emptyset$ or $V(G_3 \cap G_2) = \emptyset$ then the separation $(U_2 \cup (G_2 - w_2w_3), U_1' \cup A_1 \cup G_1 \cup G_3)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). If $V(G_1 \cap G_2) \cap A \neq \emptyset$ or $V(G_3 \cap G_2) \cap A \neq \emptyset$ then the separation $(U_2 \cup (G_2 - w_2w_3), U_1' \cup A_1 \cup G_1 \cup G_3)$ allows us to use Lemma 2.4.1 to obtain a contradiction. So $|V(G_1 \cap G_2)| = |V(G_3 \cap G_2)| = 1$ and $V(G_2) \cap A = \emptyset$. Now $U_1', U_2 \cup (G_2 - w_2w_3), A_1, G_1, G_3$ show that (G, u_1, u_2, A) is an obstruction of type I, a contradiction.

So Lemma 2.3.4(i) holds. Then $w_2w_3 \in E(G_1)$; otherwise, $w_2w_3 \in E(G_2)$, and the separation $(A_1 \cup U_1' \cup U_2 \cup (G_2 - w_2w_3), G_1 + a_1)$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A). Also $V(G_1 \cap G_2) - A \neq \emptyset$; otherwise, $U_1' \cup G_2 + \{a_2, a_3, a_4\}, U_2 \cup (G_1 - w_2w_3), A_1, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

Suppose $|V(G_1 \cap G_2)| \leq 2$. If $V(G_1 \cap G_2) \cap A \neq \emptyset$ then the separation $(U_1' \cup G_2, U_2 \cup (G_1 - w_2w_3) \cup A_1)$ allows us to use Lemma 2.4.1 to obtain a contradiction. So $V(G_1 \cap G_2) \cap A = \emptyset$. If $|V(G_1 \cap G_2)| = 2$ then $U_1' \cup G_2, U_2, A_1, G_1 - w_2w_3$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction; and if $|V(G_1 \cap G_2)| \leq 1$ then the separation $(U_1' \cup G_2, U_2 \cup (G_1 - w_2w_3) \cup A_1)$ shows that Theorem 2.2.1(ii) holds for (G, u_1, u_2, A), a contradiction.

Thus $|V(G_1 \cap G_2)| = 3$. If $V(G_1 \cap G_2) \cap A = \emptyset$ then $U_1' \cup G_2, U_2, A_1, G_1 - w_2w_3$

contradict the choice of U_1', U_2, A_1, A_2 (the maximality of $U_1' \cup U_2$). If $V(G_1 \cap G_2) \cap A = \{a_i\}$ for some $i \in \{2, 3, 4\}$ then let $V(G_1 \cap G_2) - \{a_i\} = \{v, w\}$; now $(G/vw, u_1, u_2, A)$ is an obstruction of type I with sides $(U_1' \cup G_2)/vw, U_2 + a_i$ and middle parts A_1, $\{a_i\}$, $(G_1 - a_i - w_2w_3)/vw$, contradicting Lemma 2.5.3. Since $V(G_1 \cap G_2) - A \neq \emptyset$, $V(G_1 \cap G_2) \cap A = \{a_i, a_j\}$ for some distinct $i, j \in \{2, 3, 4\}$. Now $(G/w_2w_3, u_1, u_2, A)$ is an obstruction of type IV with sides $U_1' \cup G_2, (U_2 + \{a_i, a_j\})/w_2w_3, A_1, \{a_i\}, \{a_j\}, (G_1 - \{a_i, a_j\})/w_2w_3$, contradicting Lemma 2.5.2. ∎

Lemma 2.5.5. *Suppose* (G, u_1, u_2, A) *is a counterexample to Theorem 2.2.1 with* $|V(G)|$ *minimum, and let* $x, y \in V(G) - A - \{u_1, u_2\}$ *be distinct. Then* $(G/xy, u_1, u_2, A)$ *is not an obstruction of type III.*

Proof. Suppose $(G/xy, u_1, u_2, A)$ is an obstruction of type III with sides U_1, U_2 and middle parts A_1, A_2. Let $V(U_1 \cap A_1) = \{v_1\}$, $V(U_1 \cap A_2) = \{v_2, v_3\}$, $V(U_2 \cap A_1) = \{w_1, w_2\}$, $V(U_2 \cap A_2) = \{w_3\}$, $u_1 \in V(U_1) - \{v_1, v_2, v_3\}$, and $u_2 \in V(U_2) - \{w_1, w_2, w_3\}$. Let $A := \{a_1, a_2, a_3, a_4\}$ such that $a_1, a_2 \in V(A_1) - \{v_1, w_1, w_2\}$, and $a_3, a_4 \in V(A_2) - \{v_2, v_3, w_3\}$. As before, A is independent in G.

Let v denote the vertex resulting from the identification of x and y. Now $v \in \{v_i, w_i : 1 \leq i \leq 3\}$; otherwise (G, u_1, u_2, A) would be an obstruction of type III. Thus by symmetry, assume $v \in \{v_1, v_2\}$. Let U_1' (respectively, A_i' if $v = v_i$) be obtained from U_1 (respectively, A_i) by unidentifying v back to x and y. Let $A_i' = A_i$ when $v \notin A_i$. See Figure 2.9. We choose such U_1', U_2, A_1', A_2' to maximize $U_1' \cup U_2$. Thus if $xy \in E(G)$ then $xy \in E(U_1')$, and if $w_1w_2 \in E(G)$ then $w_1w_2 \in E(U_2)$.

As (1) in the proof of Lemma 2.5.3, U_2 contains three independent paths W_1, W_2, W_3 from u_2 to w_1, w_2, w_3, respectively.

Case 1. $v = v_2$.

We claim that A_2' has three disjoint paths Q_1, Q_2, Q_3 from $\{v_3, x, y\}$ to $\{a_3, a_4, w_3\}$. For, otherwise, A_2' has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| \leq 2$, $\{a_3, a_4, w_3\} \subseteq V(A_{22})$ and $\{v_3, x, y\} \subseteq V(A_{21})$. If $|V(A_{21} \cap A_{22})| \leq 1$ then the separation $(A_{22} \cup A_1 \cup$

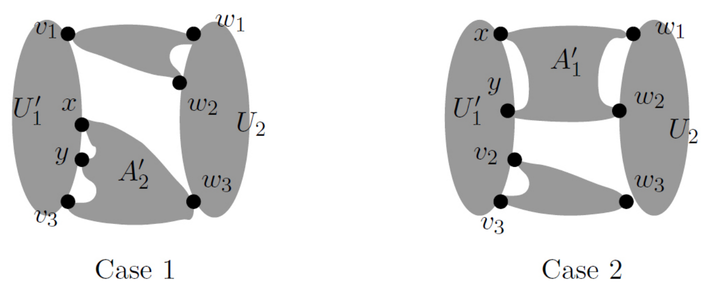


Fig. 2.9: $(G/xy, u_1, u_1, A)$ is an obstruction of type III.

$U_2 + \{a_1, a_2\}, U_1' \cup A_{21})$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). So $|V(A_{21} \cap A_{22})| = 2$. If $V(A_{21} \cap A_{22}) \cap A = \emptyset$ then $U_1' \cup A_{21}, U_2, A_1, A_{22}$ show that (G, u_1, u_2, A) is an obstruction of type III, a contradiction. So $V(A_{21} \cap A_{22}) \cap A \neq \emptyset$. Then the separation $(U_1' \cup A_{21}, U_2 \cup A_1 \cup A_{22})$ allows us to apply Lemma 2.4.1 to obtain a contradiction.

Also, $A_1 - v_1$ contains disjoint paths R_1, R_2 from $\{w_1, w_2\}$ to $\{a_1, a_2\}$. For, otherwise, A_1 has a separation (A_{11}, A_{12}) such that $|V(A_{11} \cap A_{12})| \leq 2$, $v_1 \in V(A_{11} \cap A_{12})$, $\{w_1, w_2\} \subseteq V(A_{11})$ and $\{a_1, a_2\} \subseteq V(A_{12})$. Then the separation $(U_1' \cup U_2 \cup A_{11} \cup A_2', A_{12} + \{a_3, a_4\})$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A), a contradiction.

If U_1' has three independent paths from u_1 to v_3, x, y, respectively, then these paths and $Q_1, Q_2, Q_3, R_1, R_2, W_1, W_2, W_3$ would form a topological H in G rooted at u_1, u_2, A. So such paths do not exist in U_1'. By Lemma 2.5.1(ii), $v_1 \in N(u_1)$, $|N(u_1)| = 3$ and $N(v_1) \cap V(U_1') \subseteq N[u_1]$. Hence, $G[N[u_1]], U_2, A_1, A_2' \cup (U_1' - \{u_1, v_1\})$ (removing from the last subgraph the possible edge with both ends in $N(u_1) - \{v_1\}$) show that (G, u_1, u_2, A) is an obstruction of type III, a contradiction.

Case 2. $v = v_1$.

Note that for any $i \in \{2, 3\}$, $A_2 - v_{5-i}$ contains disjoint paths Q_i, R_i from $\{w_3, v_i\}$ to $\{a_3, a_4\}$. For, otherwise, A_2 has a separation (A_{21}, A_{22}) such that $|V(A_{21} \cap A_{22})| \leq 2$, $v_{5-i} \in V(A_{21} \cap A_{22})$, $\{a_3, a_4\} \subseteq V(A_{21})$ and $\{w_3, v_i\} \subseteq V(A_{22})$. Then $(A_{21} + \{a_1, a_2\}, U_1' \cup U_2 \cup A_{22} \cup A_1')$ shows that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A), a contradiction. We apply Lemma 2.3.3 to $A_1', x, y, w_1, w_2, a_1, a_2$ (as $G, v_1, v_2, w_1, w_2, a_1, a_2$, respectively).

Suppose A_1' has three disjoint paths P_1, P_2, P_3, with one from $\{x, y\}$ to $\{w_1, w_2\}$, one from $\{x, y\}$ to $\{a_1, a_2\}$, and another from $\{w_1, w_2\}$ to $\{a_1, a_2\}$. If for some $i \in \{2, 3\}$, U_1' has three independent paths from u_1 to x, y, v_i, respectively, then these paths and $W_1, W_2, W_3, P_1, P_2, P_3, Q_i, R_i$ would form a topological H in G rooted at u_1, u_2, A. So such paths do not exist in U_1'. Then by Lemma 2.5.1(ii), $v_2, v_3 \in N(u_1)$, $|N(u_1)| = 3$, and $N(\{v_2, v_3\}) \cap V(U_1') \subseteq N[u_1]$. Now $G[N[u_1]], U_2, A_1' \cup (U_1' - \{u_1, v_2, v_3\}), A_2$ show that (G, u_1, u_2, A) is an obstruction of type III, a contradiction.

Thus, A_1' has a separation (G_1, G_2) such that one of $(i) - (v)$ of Lemma 2.3.3 holds. If Lemma 2.3.3(ii) holds then $(G_1 + \{a_3, a_4\}, U_1' \cup U_2 \cup G_2 \cup A_2)$ shows that Theorem 2.2.1(iii) would hold. If Lemma 2.3.3(iii) holds then $U_1' \cup G_i + \{a_1, a_2\}, U_2 \cup G_{3-i} + \{a_1, a_2\}, \{a_1\}, \{a_2\}, A_2$ show that (G, u_1, u_2, A) would be an obstruction of type I. If Lemma 2.3.3(iv) holds then U_1', U_2, G_1, G_2, A_2 show that (G, u_1, u_2, A) would be an obstruction of type I. Now suppose Lemma 2.3.3(v) holds. If $\{x, y\} \subseteq V(G_1)$ then $(U_2 \cup G_2, U_1' \cup G_1 \cup A_2)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). So $\{w_1, w_2\} \subseteq V(G_1)$. Then $U_1' \cup G_2, U_2, G_1, A_2$ show that (G, u_1, u_2, A) is an obstruction of type III, a contradiction.

Hence Lemma 2.3.3(i) holds. If $\{a_1, a_2\} = V(G_1 \cap G_2)$ then $U_1' \cup G_i, U_2 \cup G_{3-i}, \{a_1\}, \{a_2\}, A_2$ show that (G, u_1, u_2, A) would be an obstruction of type I. If $\{a_1, a_2\} \cap V(G_1 \cap G_2) = \emptyset$ then $U_1', U_2 \cup G_2, G_1, A_2$ (when $i = 1$) or $U_1' \cup G_2, U_2, G_1, A_2$ (when $i = 2$) contradict the choice of U_1', U_2, A_1', A_2' (the maximality of $U_1' \cup U_2$). So assume $a_1 \in V(G_1 \cap G_2)$ and $a_2 \notin V(G_1 \cap G_2)$. If $i = 1$ then $(U_2 \cup G_2, U_1' \cup G_1 \cup A_2)$ allows us to use Lemma 2.4.1 to obtain a contradiction. So $i = 2$. Then $(G/w_1w_2, u_1, u_2, A)$ is an obstruction of type I with sides $U_1' \cup G_2, U_2/w_1w_2 + a_1$ and middle parts $\{a_1\}, G_1/w_1w_2 - a_1, A_2$, contradicting Lemma 2.5.3. ∎

2.6 Separations of order five

In this section, we let (G, u_1, u_2, A) be a quadruple in which $N(u_1) \cap N(u_2) \subseteq A$, and there exist $xy \in E(G - A - \{u_1, u_2\})$ and a separation (G_1, G_2) in G such that

(1) $\{x, y\} \nsubseteq N(u_i)$ for $i \in \{1, 2\}$,

(2) $x, y \in V(G_1 \cap G_2)$, $xy \in E(G_1)$, and

(3) $|V(G_1 \cap G_2)| = 5$, $u_1, u_2 \in V(G_1) - V(G_2)$, and $A \subseteq V(G_2)$.

See Figure 2.10. Quadruples satisfying (1), (2) and (3) will occur in our proof of Theorem 2.2.1. The aim of this section is to show that such quadruples (with additional properties (4) and (5) below) cannot be a minimum counterexample to Theorem 2.2.1. First, we prove a lemma about disjoint paths in G_2, which will be used frequently in this section.

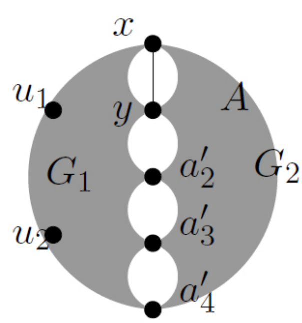


Fig. 2.10: The 5-separation (G_1, G_2).

Lemma 2.6.1. *Let (G, u_1, u_2, A) be a quadruple in which G has a separation (G_1, G_2) satisfying (1), (2) and (3) above. Suppose Theorem 2.2.1(iii) fails for (G, u_1, u_2, A), and let $v \in V(G_1 \cap G_2)$.*

(i) If $v \notin A$ then $G_2 - v$ has four disjoint paths from $V(G_1 \cap G_2) - \{v\}$ to A, and

(ii) if $v \in A$ and $N(v) \cap V(G_2) \neq \emptyset$, G_2 has four disjoint paths from $V(G_1 \cap G_2) - \{v\}$ to A.

Proof. Suppose (i) fails. Then G_2 has a separation (K, L) such that $v \in V(K \cap L)$, $|V(K \cap L)| \leq 4$, $V(G_1 \cap G_2) \subseteq V(K)$, and $A \subseteq V(L)$. Hence, the separation $(G_1 \cup K, L)$ shows that (G, u_1, u_2, A) satisfies Theorem 2.2.1(iii), a contradiction.

Now assume (ii) fails. Then G_2 has a separation (K, L) such that $|V(K \cap L)| \leq 3$, $V(G_1 \cap G_2) - \{v\} \subseteq V(K)$, and $A \subseteq V(L)$. If $V(L) \neq A$ or $E(G[A]) \neq \emptyset$ then $(G_1 \cup K, L)$ is a separation in G showing that Theorem 2.2.1(iii) holds for (G, u_1, u_2, A), a contradiction. So $V(L) = A$ and $E(G[A]) = \emptyset$. Since $N(v) \cap V(G_2) \neq \emptyset$, $v \in V(K \cap L)$ and, hence, $V(G_1 \cap G_2) \subseteq V(K)$. Therefore, $(G_1 \cup K, L)$ is a separation of order at most 3 in G, contradicting the assumption that Theorem 2.2.1(iii) fails for (G, u_1, u_2, A). ∎

We choose (G_1, G_2) such that, subject to (1), (2) and (3),

(4) G_1 is minimal.

In the rest of this section, we let $A' := V(G_1 \cap G_2) - \{x\} = \{y, a_2', a_3', a_4'\}$, and assume that

(5) $xu_1, yu_2 \in E(G)$, $N(x) \cap (V(G_1) - V(G_2)) \subseteq N[u_1]$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_2]$.

Lemma 2.6.2. *If (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum and G has a separation (G_1, G_2) satisfying (1)–(5) above, then (i), (ii) and (iii) of Theorem 2.2.1 do not hold for (G_1, u_1, u_2, A') and, moreover, (G_1, u_1, u_2, A') is an obstruction of type I, or II, or IV, with $\{y\}$ as a middle part.*

Proof. By the minimality of $|V(G)|$, Theorem 2.2.1 holds for (G_1, u_1, u_2, A'). If Theorem 2.2.1(i) holds then a topological H in G_1 rooted at u_1, u_2, A' and four disjoint paths in $G_2 - x$ from A' to A (by Lemma 2.6.1(i)) would form a topological H in G rooted at u_1, u_2, A.

Assume that Theorem 2.2.1(ii) holds and that G_1 has a separation (U_1, U_2) such that $|V(U_1 \cap U_2)| \leq 2$, $u_1 \in V(U_1) - V(U_2)$, and $A' \cup \{u_2\} \subseteq V(U_2)$. Then $|V(U_1 \cap U_2)| = 2$ and

$x \in V(U_1)-V(U_2)$; as otherwise the separation $(U_1, U_2 \cup G_2)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). Thus $y \in V(U_1 \cap U_2)$ as $xy \in E(G_1)$ and $y \in A'$. If $|V(U_1)| = 4$ then, since $\{x, y\} \not\subseteq N(u_1)$ (by (1)), Theorem 2.2.1(ii) holds. So $|V(U_1)| \geq 5$. Thus, $(U_1, U_2 \cup G_2)$ is a separation in G contradicting Lemma 2.4.2.

Now assume that Theorem 2.2.1(iii) holds; so G_1 has a separation (K, L) such that $|V(K \cap L)| \leq 4$, $u_1, u_2 \in V(K) - V(L)$, and $A' \subseteq V(L)$. Then $x \in V(K) - V(L)$ and $|V(K \cap L)| = 4$; otherwise, the separation $(K, L \cup G_2)$ shows that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A). Thus $y \in V(K \cap L)$ as $xy \in E(G_1)$ and $y \in A'$; so $(K, (L + x) \cup G_2)$ contradicts the choice of (G_1, G_2) in (4) (that G_1 is minimal).

Thus, Theorem 2.2.1(iv) holds; so (G_1, u_1, u_2, A') is an obstruction. As $y \in A'$, y belongs to some middle part. Since $y \in N(u_2)$, y belongs to the side containing u_2. Thus, by definition of obstructions, the middle part containing y is in fact $\{y\}$. As a consequence, (G_1, u_1, u_2, A') cannot be an obstruction of type III. ∎

In the next three lemmas, we consider the obstruction types of (G_1, u_1, u_2, A'), and show that (G, u_1, u_2, A) cannot be a minimum counterexample to Theorem 2.2.1.

Lemma 2.6.3. *If (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum and G has a separation (G_1, G_2) satisfying (1)–(5) above, then (G_1, u_1, u_2, A') is not an obstruction of type IV.*

Proof. Suppose (G_1, u_1, u_2, A') is an obstruction of type IV with sides U_1, U_2 and middle parts A_1, A_2, A_3, A_4. For $1 \leq i \leq 4$, let $V(U_1 \cap A_i) = \{v_i\}$ and $V(U_2 \cap A_i) = \{w_i\}$. Let $u_1 \in V(U_1) - \{v_1, v_2, v_3, v_4\}$, and $u_2 \in V(U_2) - \{w_1, w_2, w_3, w_4\}$. We choose such U_i, A_j so that $U_1 \cup U_2$ is maximal. By Lemma 2.6.2, let $V(A_1) = \{y\}$ and let $a'_i \in V(A_i)$ for $2 \leq i \leq 4$. By (5), $x \in V(U_1)$. If $x = v_i$ for some $i \in \{2, 3, 4\}$ then $(G_1 - V(A_i - \{v_i, w_i\}), G_2 \cup A_i)$ contradicts the choice of (G_1, G_2) (see (4)). So $x \notin \{v_2, v_3, v_4\}$. By (5), $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_2]$. Hence, we have symmetry between $U_1 - y, u_1, x, v_2, v_3, v_4$ and $U_2, u_2, y, w_2, w_3, w_4$. See Figure 2.11.

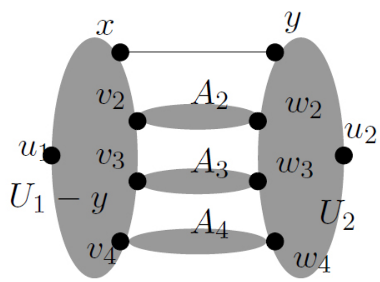


Fig. 2.11: (G_1, u_1, u_2, A') is of type IV.

(a) For each $i \in \{2, 3, 4\}$ with $|V(A_i)| \geq 2$, $A_i - v_i$ has a path W_i' from w_i to a_i', $A_i - w_i$ has a path V_i' from v_i to a_i', and $A_i - a_i'$ has a path R_i from v_i to w_i. (When $|V(A_i)| = 1$ let $W_i' = V_i' = R_i$ consist of only a_i'.) First, suppose W_i' does not exist and, without loss of generality, let $i = 2$. Then A_2 has a separation (A_{21}, A_{22}) such that $V(A_{21} \cap A_{22}) = \{v_2\}$, $w_2 \in V(A_{21})$ and $a_2' \in V(A_{22})$, and hence $(U_1 \cup U_2 \cup A_{21} \cup A_3 \cup A_4, A_{22} + A')$ shows that Theorem 2.2.1(iii) holds for (G_1, u_1, u_2, A'), contradicting Lemma 2.6.2. So W_i' does exist. Similarly, V_i' exists. Now suppose R_i does not exist. Then A_i has a separation (A_{i1}, A_{i2}) such that $V(A_{i1} \cap A_{i2}) = \{a_i'\}$, $v_i \in V(A_{i1})$ and $w_i \in V(A_{i2})$. Replacing the sides U_1, U_2 with $U_1 \cup A_{i1}, U_2 \cup A_{i2}$, and replacing the middle part A_i with $\{a_i'\}$, we get a contradiction to the maximality of $U_1 \cup U_2$.

(b) There exists a permutation ijk of $\{2, 3, 4\}$ such that $(U_1 - y) - v_k$ has three independent paths P_1, P_2, P_3 from u_1 to x, v_i, v_j, respectively, and there is a permutation rst of $\{2, 3, 4\}$ such that $U_2 - w_t$ has three independent paths Q_1, Q_2, Q_3 from u_2 to y, w_r, w_s, respectively. For, otherwise, suppose by symmetry that P_1, P_2, P_3 do not exist. By Lemma 2.5.1(i), $N(u_1) = \{v_2, v_3, v_4\}$, contradicting (5) (that $x \in N(u_1)$).

(c) $|N(u_1)| = 3 = |N(u_2)|$ and, for any choice of P_1, P_2, P_3 in (b) and any choice of Q_1, Q_2, Q_3 in (b), we have $N(u_1) \cap \{v_r, v_s\} \neq \emptyset$ and $N(u_2) \cap \{w_i, w_j\} \neq \emptyset$, where ijk and rst are permutations of $\{1, 2, 3\}$ in (b). By symmetry we only prove the claim for u_2. If $t \neq j$ for every choice of Q_1, Q_2, Q_3 above, then $U_2 - w_j$ does not have three independent paths from u_2 to y, w_i, w_k, respectively; so by Lemma 2.5.1(i), $|N(u_2)| = 3$

and $w_j \in N(u_2)$. Similarly, if $t \neq i$ for every choice of Q_1, Q_2, Q_3 above, then we have $|N(u_2)| = 3$ and $w_i \in N(u_2)$. In either case, (c) holds for u_2. Thus assume that we may choose Q_1, Q_2, Q_3 so that $t = i$ and we may choose Q_1, Q_2, Q_3 so that $t = j$. Suppose $a'_i, a'_j \in A$, $|V(A_i)| = 1$ or $N(a'_i) \cap V(G_2) = \emptyset$, and $|V(A_j)| = 1$ or $N(a'_j) \cap V(G_2) = \emptyset$. If $w_k \notin A$ then $(G/yw_k, u_1, u_2, A)$ is an obstruction of type I, with sides $U_1 - y, U_2/yw_k$ and middle parts $A_i, A_j, ((A_k \cup G_2) + xy - \{a'_i, a'_j\})/yw_k$, contradicting Lemma 2.5.3. So $w_k \in A$. Then $V(A_k) = \{w_k\}$ by the maximality of $U_1 \cup U_2$; so (G, u_1, u_2, A) is an obstruction of type IV with sides $U_1 - y, U_2$ and middle parts $A_2, A_3, A_4, G_2 - \{a'_2, a'_3, a'_4\} + xy$, a contradiction.. Hence, by symmetry we may assume that $a'_i \notin A$, or $a'_i \in A, |V(A_i)| \geq 2$ and $N(a'_i) \cap V(G_2) \neq \emptyset$. Choose Q_1, Q_2, Q_3 so that $t = j$. Then by Lemma 2.6.1, $G_2 - a'_i$ (when $a'_i \notin A$) and G_2 (when $a'_i \in A$) has four disjoint paths from $\{x, y, a'_j, a'_k\}$ to A. In either case these four paths and $P_1, P_2, P_3, Q_1, Q_2, Q_3, R_i, V'_j, W'_k$ (see (a)) form a topological H in G rooted at u_1, u_2, A, a contradiction.

(d) There exists some $\ell \in \{2, 3, 4\}$ such that $v_\ell \in N(u_1)$ and $w_\ell \in N(u_2)$. By (c), assume $w_j \in N(u_2)$. We may assume $v_j \notin N(u_1)$, as otherwise we may let $\ell = j$. Then by Lemma 2.5.1(i), $(U_1 - y) - v_j$ has three independent paths from u_1 to x, v_i, v_k, respectively; so by (c) again, $N(u_2) \cap \{w_i, w_k\} \neq \emptyset$. Hence $|N(u_2) \cap \{w_2, w_3, w_4\}| \geq 2$. By symmetry we could also prove $|N(u_1) \cap \{v_2, v_3, v_4\}| \geq 2$. Hence, ℓ exists.

Without loss of generality, let $v_3 \in N(u_1)$ and $w_3 \in N(u_2)$. By (c) and Lemma 2.4.1, $N(u_i) \cap A = \emptyset$ for $i = 1, 2$. So $|V(A_3)| \geq 2$ as $N(u_1) \cap N(u_2) \subseteq A$.

(e) There exists $b \in \{2, 4\}$ such that $v_b \in N(u_1)$ and $w_b \in N(u_2)$. Otherwise, by symmetry and by (c), since $|N(u_i)| = 3$ for $i = 1, 2$, we may assume $v_2 \notin N(u_1)$ and $w_4 \notin N(u_2)$. Then by Lemma 2.5.1(i), $(U_1 - y) - v_2$ has three independent paths P'_1, P'_2, P'_3 from u_1 to x, v_3, v_4, respectively, and $U_2 - w_4$ has three independent paths Q'_1, Q'_2, Q'_3 from u_2 to y, w_3, w_2, respectively. If $a'_3 \notin A$ then by Lemma 2.6.1(i), $G_2 - a'_3$ has four disjoint paths from $\{x, y, a'_2, a'_4\}$ to A; if $a'_3 \in A$ and $N(a'_3) \cap V(G_2) \neq \emptyset$ then by Lemma 2.6.1(ii), G_2 has four disjoint paths from $\{x, y, a'_2, a'_4\}$ to A. In either case the four paths in G_2 and

$P_1', P_2', P_3', Q_1', Q_2', Q_3', R_3, V_4', W_2'$ (see (a)) form a topological H in G rooted at u_1, u_2, A, a contradiction. So $a_3' \in A$ and $N(a_3') \cap V(G_2) = \emptyset$. Similarly, if $U_1 - y$ has three independent paths from u_1 to x, v_2, v_4 then $a_2' \in A$ and $N(a_2') \cap V(G_2) = \emptyset$. In this case, if $w_4 \notin A$ then $(G/yw_4, u_1, u_2, A)$ is an obstruction of type I with sides $U_1 - y, U_2/yw_4$ and middle parts $A_2, A_3, (G_2 \cup A_4 - \{a_2', a_3'\} + xy)/yw_4$, contradicting Lemma 2.5.3; and if $w_4 \in A$ then $V(A_2) = \{w_4\}$ by the maximality of $U_1 \cup U_2$, which implies that (G, u_1, u_2, A) is an obstruction of type IV with sides $U_1 - y, U_2$ and middle parts $A_2, A_3, A_4, G_2 - \{a_2', a_3', a_4'\} + xy$, a contradiction. Thus $U_1 - y$ has no three independent paths from u_1 to x, v_2, v_4. So by Lemma 2.5.1(ii), $N(v_3) \cap V(U_1 - y) \subseteq N([u_1])$. Similarly, we conclude that $N(w_3) \cap V(U_2) \subseteq N[u_2]$. Hence, $G[N[u_1]], G[N[u_2]], A_3, G - (A_3 + \{u_1, u_2\})$ (removing from last subgraph the possible edges with both ends in $N(u_1)$ or in $N(u_2)$) show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction.

Thus, we may assume that $N(u_1) = \{x, v_2, v_3\}$ and $N(u_2) = \{y, w_2, w_3\}$. Since $N(u_i) \cap A = \emptyset$ for $i = 1, 2$ (by Lemma 2.4.1), $|V(A_2)| \geq 2$ and $|V(A_3)| \geq 2$ (as $N(u_1) \cap N(u_2) \subseteq A$). Suppose for $i = 2, 3$, $a_i' \in A$ and $N(a_i') \cap V(G_2) = \emptyset$. If $w_4 \notin A$ then $(G/yw_4, u_1, u_2, A)$ is an obstruction of type I with sides $U_1 - y, U_2/yw_4$ and middle parts $A_2, A_3, (G_2 \cup A_4 + xy)/yw_4$, contradicting Lemma 2.5.3. So $w_4 \in A$. Then $V(A_2) = \{w_4\}$ by the maximality of $U_1 \cup U_2$; so (G, u_1, u_2, A) is an obstruction of type IV with sides $U_1 - y, U_2$ and middle parts $A_2, A_3, A_4, G_2 - \{a_2', a_3', a_4'\} + xy$, a contradiction. Hence, by symmetry we may assume $a_3' \notin A$, or $a_3' \in A$ and $N(a_3') \cap V(G_2) \neq \emptyset$.

Suppose $(U_1 - y) - \{u_1, x, v_3\}$ has a path S_2 from v_4 to v_2 or $U_2 - \{u_2, y, w_3\}$ has a path T_2 from w_4 to w_2. By symmetry, assume we have S_2. By Lemma 2.6.1, $G_2 - a_3'$ (when $a_3' \notin A$) or G_2 (when $a_3' \in A$ and $N(a_3') \cap V(G_2) \neq \emptyset$) has four disjoint paths from $V(G_1 \cap G_2) - \{a_3'\}$ to A. These paths and $u_1x, u_1v_3, S_2 + \{u_1, u_1v_2\}, u_2y, u_2w_2, u_2w_3, W_2', R_3, V_4'$ (see (a)) form a topological H in G rooted at u_1, u_2, A, a contradiction.

So neither S_2 nor T_2 exists. Then $\{x, v_3\}$ is a cut in $U_1 - y$ separating $\{u_1, v_2\}$ from $(U_1 - y) - \{u_1, v_2, x, v_3\}$, and $\{y, w_3\}$ is a cut in U_2 separating $\{u_2, w_2\}$ from

$U_2 - \{u_2, w_2, y, w_3\}$. Hence, $a_2' \notin A$, or $a_2' \in A$ and $N(a_2') \cap V(G_2) \neq \emptyset$; as otherwise, $G[N[u_1]], G[N[u_2]], A_2, (G_2 - a_2') \cup (U_1 - \{u_1, v_2\} - xv_3) \cup (U_2 - \{u_2, w_2\} - yw_3) \cup A_3 \cup A_4 + xy$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction. Moreover, $(U_1 - y) - \{u_1, x, v_2\}$ has a path S_3 from v_4 to v_3 or $U_2 - \{u_2, y, w_2\}$ has a path T_3 from w_4 to w_3; otherwise by (5), we see that $(G[N[u_1]] \cup G[N[u_2]] \cup A_2 \cup A_3, G_2 \cup G[A_4 + xy] \cup (U_1 - \{u_1, v_2, v_3\}) \cup (U_2 - \{u_2, w_2, w_3\}))$ is a a separation in G showing that Theorem 2.2.1(iii) would hold for (G, u_1, u_2, A). By symmetry, assume we have S_3. Hence $U_1 - y$ has three independent paths S_1, S_2, S_3 from u_1 to x, v_2, v_4, respectively. By Lemma 2.6.1, $G_2 - a_2'$ (when $a_2' \notin A$) or G_2 (when $a_2' \in A$ and $N(a_2') \cap V(G_2) \neq \emptyset$) has four disjoint paths from $V(G_1 \cap G_2) - \{a_2'\}$ to A. These paths and $S_1, S_2, S_3, u_2y, u_2w_2, u_2w_3, W_3', R_2, V_4'$ (see (a)) form a topological H in G rooted at u_1, u_2, A, a contradiction. ∎

Lemma 2.6.4. *If (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum and G has a separation (G_1, G_2) satisfying (1)–(5) above, then (G_1, u_1, u_2, A') is not an obstruction of type I.*

Proof. Suppose (G_1, u_1, u_2, A') is an obstruction of type I with sides U_1, U_2 and middle parts A_1, A_2, A_3. Let $V(A_1) = \{y\}$ (by Lemma 2.6.2), $V(U_1 \cap A_2) = \{v_2\}$, $V(U_1 \cap A_3) = \{v_3, v_4\}$, $V(U_2 \cap A_i) = \{w_i\}$ for $i = 2, 3$, $a_2' \in V(A_2)$, $a_3', a_4' \in V(A_3) - \{v_3, v_4, w_3\}$, $u_1 \in V(U_1) - \{y, v_2, v_3, v_4\}$, and $u_2 \in V(U_2) - \{y, w_2, w_3\}$. We choose U_i and A_j so that $U_1 \cup U_2$ is maximized.

By (5), $x \in V(U_1 - y)$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_2] \subseteq V(U_2)$. We claim that $x \notin \{v_2, v_3, v_4\}$; for, if $x = v_2$ then $(G_1 - V(A_2 - \{v_2, w_2\}), G_2 \cup A_2)$ contradicts the choice of (G_1, G_2) (see (4)), and if $x \in \{v_3, v_4\}$ then $(G_1 - V(A_3 - \{v_3, v_4, w_3\}), G_2 \cup A_3)$ contradicts the choice of (G_1, G_2) (see (4)).

By Lemma 2.4.2, $N(u_2) = \{y, w_2, w_3\}$. As in the proof of Lemma 2.6.3, if $|A_2| \geq 2$ then $A_2 - v_2$ has a path W_2' from w_2 to a_2', and $A_2 - a_2'$ has a path R_2 from w_2 to v_2 (by the maximality of $U_1 \cup U_2$). When $|A_2| = 1$, we let $W_2' = R_2 = A_2$.

For any $i \in \{3, 4\}$, $A_3 - v_{7-i}$ has two disjoint paths R_i, Q_i from $\{v_i, w_3\}$ to $\{a_3', a_4'\}$.

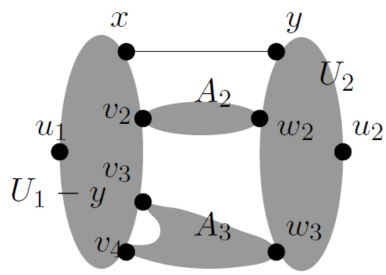


Fig. 2.12: (G_1, u_1, u_2, A') is of type I.

Otherwise, A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $v_{7-i} \in V(A_{31} \cap A_{32})$, $\{v_i, w_3\} \subseteq V(A_{31})$, and $\{a'_3, a'_4\} \subseteq V(A_{32})$. Then $(U_1 \cup U_2 \cup A_2 \cup A_{31}, A_{32} + \{y, a'_2\})$ shows that Theorem 2.2.1(iii) holds for (G_1, u_1, u_2, A'), contradicting Lemma 2.6.2.

Moreover, for any $i \in \{3, 4\}$, $A_3 - a'_{7-i}$ has two disjoint paths R'_i, Q'_i from $\{v_3, v_4\}$ to $\{w_3, a'_i\}$. For otherwise A_3 has a separation (A_{31}, A_{32}) such that $|V(A_{31} \cap A_{32})| \leq 2$, $a'_{7-i} \in V(A_{31} \cap A_{32})$, $\{v_3, v_4\} \subseteq V(A_{31})$, and $\{w_3, a'_i\} \subseteq V(A_{32})$. Then $U_1 \cup A_{31} + a'_i, U_2 \cup A_{32}, \{y\}, A_2, \{a'_3\}, \{a'_4\}$ (when $a'_i \in V(A_{31} \cap A_{32})$ or $V(A_{31} \cap A_{32}) = \{a'_{7-i}\}$) or $U_1 \cup A_{31}, U_2 + a'_{7-i}, \{y\}, A_2, \{a'_{7-i}\}, A_{32} - a'_{7-i}$ (when $a'_i \notin V(A_{31} \cap A_{32}) \neq \{a'_{7-i}\}$) show that (G_1, u_1, u_2, A') is an obstruction of type IV, contradicting Lemma 2.6.3.

Clearly, $v_3, v_4 \notin A$. We note that $v_2 \notin A$. For, otherwise, by the maximality of $U_1 \cup U_2$, $v_2 = w_2 \in N(u_2)$. So $N(u_2) \cap A \neq \emptyset$. But $|N(u_2)| = 3$, contradicting Lemma 2.4.1.

If for all $i \in \{3, 4\}$, $a'_i \in A$ and $N(a'_i) \cap V(G_2) = \emptyset$, then $(G/xv_2, u_1, u_2, A)$ is an obstruction of type III with sides $(U_1 - y)/xv_2, U_2$ and middle parts $(A_2 \cup (G_2 - \{a'_3, a'_4\}) + xy))/xv_2, A_3$, contradicting Lemma 2.5.5. Hence by symmetry, let $a'_4 \notin A$, or $a'_4 \in A$ and $N(a'_4) \cap V(G_2) \neq \emptyset$. Then by Lemma 2.6.1, $G_2 - a'_4$ (when $a'_4 \notin A$) or G_2 (when $a'_4 \in A$) has four disjoint paths S_1, S_2, S_3, S_4 from $V(G_1 \cap G_2) - \{a'_4\}$ to A.

If $a'_2 \in A$ and $N(a'_2) \cap V(G_2) = \emptyset$ then $(G/v_3v_4, u_1, u_2, A)$ is an obstruction of type II with sides $(U_1 - y)/v_3v_4, U_2$ and middle parts $A_2, (A_3/v_3v_4) \cup (G_2 - a'_2) + xy$, contradicting Lemma 2.5.4. Thus $a'_2 \notin A$, or $a'_2 \in A$ and $N(a'_2) \cap V(G_2) \neq \emptyset$. So by Lemma 2.6.1, $G_2 - a'_2$ (when $a'_2 \notin A$) or G_2 (when $a'_2 \in A$) has four disjoint paths T_1, T_2, T_3, T_4 from

$V(G_1 \cap G_2) - \{a'_2\}$ to A.

By Lemma 2.5.1(i) and the fact $u_1x \in E(G)$, there exists a permutation ijk of $\{2,3,4\}$ such that $(U_1 - y) - v_k$ has three independent paths P_1, P_2, P_3 from u_1 to x, v_i, v_j, respectively. If $\{i,j\} = \{3,4\}$ then $P_1, P_2, P_3, u_2y, u_2w_2, u_2w_3, R'_3, Q'_3, W'_2, S_1, S_2, S_3, S_4$ form a topological H in G rooted at u_1, u_2, A, a contradiction. Thus by symmetry between v_3 and v_4, assume $\{i,j\} = \{2,3\}$. Then $P_1, P_2, P_3, u_2y, u_2w_2, u_2w_3, R_3, Q_3, R_2, T_1, T_2, T_3, T_4$ form a topological H in G rooted at u_1, u_2, A, a contradiction. ∎

Lemma 2.6.5. *If (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum and G has a separation (G_1, G_2) satisfying (1)–(5) above, then (G_1, u_1, u_2, A') is not an obstruction of type II.*

Proof. Suppose (G, u_1, u_2, A) is a counterexample to Theorem 2.2.1 with $|V(G)|$ minimum, and (G_1, u_1, u_2, A') is an obstruction of type II with sides U_1, U_2 and middle parts A_1, A_2. Let $V(A_1) = \{y\}$ (by Lemma 2.6.2), $V(U_1 \cap A_2) = \{v_2, v_3\}$, $V(U_2 \cap A_2) = \{w_2, w_3\}$, $a'_2, a'_3, a'_4 \in V(A_2) - \{v_2, v_3, w_2, w_3\}$, $u_1 \in V(U_1) - \{y, v_2, v_3\}$, and $u_2 \in V(U_2) - \{y, w_2, w_3\}$. By (5), $x \in V(U_1 - y)$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_2] \subseteq V(U_2)$. Note that $x \notin \{v_2, v_3\}$; otherwise the separation $(U_1 - y, U_2 \cup A_2 \cup G_2)$ shows that Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). By Lemma 2.4.2, $V(U_1 - y) = \{u_1, v_2, v_3, x\}$ and $V(U_2) = \{u_2, w_2, w_3, y\}$. Moreover, $N(u_1) = \{v_2, v_3, x\}$ and $N(u_2) = \{w_2, w_3, y\}$, as otherwise Theorem 2.2.1(ii) would hold for (G, u_1, u_2, A). See Figure 2.13(a).

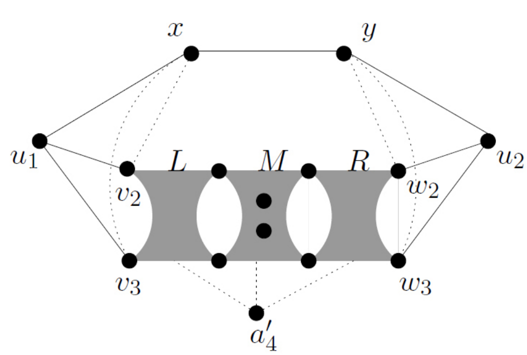


Fig. 2.13: (G_1, u_1, u_2, A') is of type II.

There exists some $i \in \{2,3,4\}$ such that $a_i' \notin A$ or $N(a_i') \cap V(G_2) \neq \emptyset$; for if this is not the case then $U_1 - y, U_2, A_2, (G_2 - \{a_2', a_3', a_4'\}) + xy$ show that (G, u_1, u_2, A) is an obstruction of type II, a contradiction. By symmetry, assume $a_4' \notin A$ or $N(a_4') \cap V(G_2) \neq \emptyset$. Then by Lemma 2.6.1, $G_2 - a_4'$ (when $a_4' \notin A$) or G_2 (when $a_4' \in A$) has four disjoint paths S_1, S_2, S_3, S_4 from $V(G_1 \cap G_2) - \{a_4'\}$ to A.

Let $A_2 - a_4' = L \cup M \cup R$ such that $|V(L \cap M)| \leq 2$, $|V(R \cap M)| \leq 2$, $V(L \cap R) \subseteq V(M)$, $\{v_2, v_3\} \subseteq V(L)$, $\{w_2, w_3\} \subseteq V(R)$, and $\{a_2', a_3'\} \subseteq V(M) - V(L \cup R)$. (Note that $L = \{v_2, v_3\}$, $M = A_2 - a_4'$ and $R = \{w_2, w_3\}$ satisfy this.) Choose L, M, R to minimize M.

Then $|V(L \cap M)| = 2$ and L has two disjoint paths from $\{v_2, v_3\}$ to $V(L \cap M)$, and $|V(R \cap M)| = 2$ and R has two disjoint paths from $\{w_2, w_3\}$ to $V(R \cap M)$. For, suppose this is not true, and assume by symmetry that $|V(L \cap M)| \leq 1$ or L has no disjoint paths from $\{v_2, v_3\}$ to $V(L \cap M)$. If $|V(L \cap M)| \leq 1$ let $L_1 = L$ and $L_2 = L \cap M$, and if $|V(L \cap M)| = 2$ then $G[L + a_4']$ has a separation (L_1, L_2) such that $|V(L_1 \cap L_2)| \leq 2$, $a_4' \in V(L_1 \cap L_2)$, $\{v_2, v_3\} \subseteq V(L_1)$, and $V(L \cap M) \subseteq V(L_2)$. Now $V(L_1 \cap L_2) \cup \{x\}$ is a cut in G separating u_1 from $A \cup \{u_2\}$, contradicting Lemma 2.4.2.

Let $V(L \cap M) = \{s_1, s_2\}$ and $V(R \cap M) = \{t_1, t_2\}$. Note that $\{s_1, s_2\} \neq \{t_1, t_2\}$; as otherwise, the separation $(G_1 - (M - \{s_1, s_2\}), G_2 \cup G[M + a_4'])$ contradicts (4). Clearly, $G[L + \{u_1, x\}]$ has three independent paths P_1, P_2, P_3 from u_1 to x, s_1, s_2, respectively, and $G[R + \{u_2, y\}]$ has three independent paths Q_1, Q_2, Q_3 from u_2 to y, t_1, t_2, respectively. If M has three disjoint paths, with one from $\{s_1, s_2\}$ to $\{t_1, t_2\}$, one from $\{s_1, s_2\}$ to $\{a_2', a_3'\}$, and another from $\{t_1, t_2\}$ to $\{a_2', a_3'\}$, then these paths and $P_1, P_2, P_3, Q_1, Q_2, Q_3, S_1, S_2, S_3, S_4$ would form a topological H in G rooted at u_1, u_2, A. So such paths in M do not exist.

We claim that $\{s_1, s_2\} \cap \{t_1, t_2\} = \emptyset$. Suppose otherwise and, without loss of generality, let $s_1 = t_1$. Then $s_2 \neq t_2$, and $M - s_1$ does not contain disjoint paths from $\{s_2, t_2\}$ to $\{a_2', a_3'\}$. Hence M has a separaion (M_1, M_2) such that $|V(M_1 \cap M_2)| \leq 2$, $s_1 \in V(M_1 \cap M_2)$, $\{s_2, t_2\} \subseteq V(M_1)$, and $\{a_2', a_3'\} \subseteq V(M_2)$. Now $(G_1 - (M_2 - V(M_1 \cap$

$M_2)), G_2 \cup G[M_2 + a_4'])$ is a separation in G contradicting (4).

By Lemma 2.3.3 (with $M, s_1, s_2, t_1, t_2, a_2', a_3'$ as $G, v_1, v_2, w_1, w_2, a_1, a_2$, respectively), M has a separation (M_1, M_2) such that one of $(i) - (v)$ of Lemma 2.3.3 holds (with M_i, $i = 1, 2$, as G_i in Lemma 2.3.3).

If Lemma 2.3.3(ii) holds, then the separation $(G_1[M_2 \cup L \cup R + \{a_4', u_1, u_2, x, y\}], M_1 + \{a_4', y\})$ shows that Theorem 2.2.1(iii) holds for (G_1, u_1, u_2, A'), contradicting Lemma 2.6.2. If Lemma 2.3.3(iii) holds, then $G_1[L \cup M_i + (A' \cup \{u_1, x\})], G_1[R \cup M_{3-i} + (A' \cup \{u_2, y\})], \{y\}, \{a_2'\}, \{a_3'\}, \{a_4'\}$ show that (G_1, u_1, u_2, A') is an obstruction of type IV, contradicting Lemma 2.6.3. If Lemma 2.3.3(iv) holds, then $G_1[L + \{a_4', u_1, x, y\}], G_1[R + \{a_4', u_2, y\}], \{y\}, \{a_4'\}, M_1, M_2$ show that (G_1, u_1, u_2, A') is an obstruction of type IV, contradicting Lemma 2.6.3. If Lemma 2.3.3(v) holds, then by symmetry assume that $\{s_1, s_2, t_1, a_2', a_3'\} \subseteq V(M_1)$; now $G_1[L + \{a_4', u_1, x, y\}]$, $G[R \cup M_2 + \{a_4', u_2, y\}]$, $\{y\}, \{a_4'\}, M_1$ show that (G_1, u_1, u_2, A') is an obstruction of type I, contradicting Lemma 2.6.4.

So Lemma 2.3.3(i) holds, and assume by symmetry that $\{s_1, s_2, a_2', a_3'\} \subseteq V(M_1)$ and $\{t_1, t_2\} \subseteq V(M_2)$. Note that $|V(M_1 \cap M_2)| = 2$ and M_2 has disjoint paths T_1, T_2 from $\{t_1, t_2\}$ to $V(M_1 \cap M_2)$; otherwise, $|V(M_1 \cap M_2)| \leq 1$ (in this case let $S := V(M_1 \cap M_2)$), or M_2 has a cut S, $|S| \leq 1$, separating $V(M_1 \cap M_2)$ from $\{t_1, t_2\}$, and hence, $S \cup \{a_4', y\}$ is a cut in G separating u_2 from $A \cup \{u_1\}$, contradicting Lemma 2.4.2.

Hence by the minimality of M, we may assume by symmetry that $V(M_1 \cap M_2) = \{a_2', z\}$. Then $z \neq a_3'$, as otherwise $G_1[L \cup M_1 + \{a_4', u_1, x, y\}], G_1[R \cup M_2 + \{a_4', u_2, y\}], \{y\}, \{a_2'\}, \{a_3'\}$, $\{a_4'\}$ show that (G_1, u_1, u_2, A') is an obstruction of type IV, contradicting Lemma 2.6.3.

If $M_1 - a_2'$ contains disjoint paths from $\{s_1, s_2\}$ to $\{a_3', z\}$ then these paths and P_1, P_2, P_3, Q_1, $Q_2, Q_3, T_1, T_2, S_1, S_2, S_3, S_4$ form a topological H in G rooted at u_1, u_2, A, a contradiction. So such paths do not exist in $M_1 - a_2'$. Then M_1 has a separation (M_{11}, M_{12}) such that $a_2' \in V(M_{11} \cap M_{12})$, $|V(M_{11} \cap M_{12})| \leq 2$, $\{s_1, s_2\} \subseteq V(M_{11})$, and $\{a_3', z\} \subseteq V(M_{12})$. If $a_3' \notin V(M_{11} \cap M_{12})$ then $G_1[L \cup M_{11} + \{a_4', u_1, x, y\}]$, $G_1[R \cup M_2 + \{a_4', u_2, y\}]$, $\{y\}, \{a_2'\}, \{a_4'\}, M_{12} - a_2'$ show that (G_1, u_1, u_2, A') is an obstruction of type IV, contradicting Lemma 2.6.3. So $a_3' \in V(M_{11} \cap M_{12})$. Then $G_1[L \cup M_{11} + \{a_4', u_1, x, y\}]$, $G_1[R \cup M_2 \cup M_{12} + \{a_4', u_2, y\}]$, $\{y\}, \{a_2'\}, \{a_3'\}, \{a_4'\}$ show that (G_1, u_1, u_2, A') is an obstruction of type IV, contradicting Lemma 2.6.3. ∎

2.7 Conclusion

We complete the proof of Theorem 2.2.1. Suppose that the assertion of Theorem 2.2.1 fails, and let (G, u_1, u_2, A) be a counterexample with $|V(G)|$ minimum.

Then $|N(u_i)| \geq 3$ (otherwise (ii) would hold for (G, u_1, u_2, A)). Also G has no separation (G_1, G_2) such that $|V(G_1 \cap G_2)| \leq 4$, $\{u_1, u_2\} \subseteq V(G_1) - V(G_2)$, and $A \subseteq V(G_2)$; for otherwise (iii) would hold for (G, u_1, u_2, A). Thus A is independent in G. Moreover, for any vertex $u \notin A \cup \{u_1, u_2\}$, the graph G', obtained from $G - u$ by duplicating u_i with u_i' ($i = 1, 2$), contains four disjoint paths from $\{u_1, u_1', u_2, u_2'\}$ to A. Now these paths give rise to four independent paths P_1, P_2, P_3, P_4 in $G - u$ from $\{u_1, u_2\}$ to A, with two from each u_i. We now prove properties (a) – (e) and use them to prove that G has a separation (G_1, G_2) satisfies (1) – (5) in Section 6.

(a) $u_1u_2 \notin E(G)$, and $N(u_1) \cap N(u_2) \subseteq A$.

For, if $u_1u_2 \in E(G)$ then P_1, P_2, P_3, P_4 and u_1u_2 would form a topological H in G rooted at u_1, u_2, A; and if there exists $u \in (N(u_1) \cap N(u_2)) - A$ then P_1, P_2, P_3, P_4 and u_1uu_2 would form a topological H in G rooted at u_1, u_2, A.

If $G - A - \{u_1, u_2\} = \emptyset$ then we see that $N(u_i) \subseteq A$. So by Lemma 2.4.1, $N(u_i) = A$ for $i = 1, 2$. Hence $G[A + u_1], G[A + u_2], \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction. Thus $G - A - \{u_1, u_2\} \neq \emptyset$. In fact,

(b) $E(G - A - \{u_1, u_2\}) \neq \emptyset$.

Otherwise, by (a), for any $x \in V(G) - A - \{u_1, u_2\}$, $N(x) \subseteq A \cup \{u_i\}$ for some $i \in \{1, 2\}$. Thus G has a separation (U_1, U_2) such that $V(U_1 \cap U_2) = A$, $u_1 \in V(U_1) - V(U_2)$, and

$u_2 \in V(U_2) - V(U_1)$. Now $U_1, U_2, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

(c) There exists $xy \in E(G - A - \{u_1, u_2\})$ such that $\{x, y\} \not\subseteq N(u_i)$ for any $i \in \{1, 2\}$.

Suppose for any $xy \in E(G - A - \{u_1, u_2\})$ we have $\{x, y\} \subseteq N(u_i)$ for some $i \in \{1, 2\}$. Then by (a), for any $v \in N(u_i) - A$, $N(v) \subseteq N[u_i] \cup A$. Thus, G has a separation (U_1, U_2) such that $V(U_1 \cap U_2) = A$, $U_1 = G[N[u_1] \cup A]$, and $U_2 = G - V(G_1 - A)$. Now $U_1, U_2, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}$ show that (G, u_1, u_2, A) is an obstruction of type IV, a contradiction.

Since $|V(G/xy)| < |V(G)|$, one of $(i)-(iv)$ of Theorem 2.2.1 holds for $(G/xy, u_1, u_2, A)$. Let v denote the vertex resulting from the contraction of xy.

(d) For any xy satisfying (c), (iii) holds for $(G/xy, u_1, u_2, A)$.

By Lemmas 2.5.2, 2.5.3, 2.5.4 and 2.5.5, (iv) does not hold for $(G/xy, u_1, u_2, A)$. If (i) holds for $(G/xy, u_1, u_2, A)$ then let K be a topological H in G/xy rooted at u_1, u_2, A; now K (when $v \notin K$) or the graph obtained from K by uncontracting v back to xy (when $v \in K$) gives a topological H in G rooted at u_1, u_2, A, a contradiction. Now suppose that (ii) holds for $(G/xy, u_1, u_2, A)$, and let (G_1', G_2') denote a separation in G/xy such that $|V(G_1' \cap G_2')| \leq 2$, $u_1 \in V(G_1') - V(G_2')$ and $A \cup \{u_2\} \subseteq V(G_2')$. Then $|V(G_1' \cap G_2')| = 2$ and $v \in V(G_1' \cap G_2')$; for otherwise (ii) would hold for (G, u_1, u_2, A). Hence G has a separation (G_1, G_2) such that $|V(G_1 \cap G_2)| = 3$, $x, y \in V(G_1 \cap G_2)$, $u_1 \in V(G_1) - V(G_2)$, and $A \cup \{u_2\} \subseteq G_2$. Since $|N(u_1)| \geq 3$ and $\{x, y\} \not\subseteq N(u_1)$, $|V(G)| \geq 5$, contradicting Lemma 2.4.2. Thus (iii) holds for $(G/xy, u_1, u_2, A)$.

By (d), for any xy satisfying (c), G/xy has a separation (G_1', G_2') such that $|V(G_1' \cap G_2')| \leq 4$, $\{u_1, u_2\} \subseteq V(G_1') - V(G_2')$, and $A \subseteq V(G_2')$. Then $v \in V(G_1' \cap G_2')$ and $|V(G_1' \cap G_2')| = 4$; or else (iii) would hold for (G, u_1, u_2, A). Hence, G has a separation (G_1, G_2) such that $x, y \in V(G_1 \cap G_2)$, $|V(G_1 \cap G_2)| = 5$, $\{u_1, u_2\} \subseteq V(G_1) - V(G_2)$, and

$A \subseteq V(G_2)$. Moreover, $N(x) \cap (V(G_1) - V(G_2)) \neq \emptyset$, and $N(y) \cap (V(G_1) - V(G_2)) \neq \emptyset$; for otherwise, (iii) would hold for (G, u_1, u_2, A). We choose xy (satisfying (c)) and (G_1, G_2) so that G_1 is minimal (subject to $xy \in E(G_1)$). Now (G, u_1, u_2, A) satisfies (1) – (4) in Section 6. We now show that (G, u_1, u_2, A), xy and (G_1, G_2) also satisfies (5) in Section 6. First, we claim that

(e) $x, y \in N(\{u_1, u_2\})$ and $(N(x) \cup N(y)) \cap (V(G_1) - V(G_2)) \subseteq N[\{u_1, u_2\}]$.

Suppose (e) fails, and assume by symmetry that it fails for x. If $x \notin N(\{u_1, u_2\})$ let $z \in N(x) \cap (V(G_1) - V(G_2))$; and if $x \in N(\{u_1, u_2\})$ then $N(x) \cap (V(G_1) - V(G_2)) \not\subseteq N[\{u_1, u_2\}]$, and let $z \in N(x) \cap (V(G_1) - V(G_2)) - N[\{u_1, u_2\}]$. Then xz satisfies (c). By the argument following (d), G has a separation (H_1, H_2) such that $\{x, z\} \subseteq V(H_1 \cap H_2)$, $|V(H_1 \cap H_2)| = 5$, $\{u_1, u_2\} \subseteq V(H_1) - V(H_2)$ and $A \subseteq V(H_2)$. Thus $u_1, u_2 \in (V(G_1) - V(G_2)) \cap (V(H_1) - V(H_2))$ and $A \subseteq V(G_2 \cap H_2)$. In particular, $|V(G_2 \cap H_2)| \geq |A \cup \{x\}| \geq 5$. Thus $|V(G_1 \cap G_2 \cap H_2) \cup V(H_1 \cap H_2 \cap G_2)| \geq 5$; as otherwise the separation $(G_1 \cup H_1, G_2 \cap H_2)$ shows that (iii) would hold for (G, u_1, u_2, A). Therefore, since $|V(G_1 \cap G_2)| + |V(H_1 \cap H_2)| = 10$, we see that $|V(G_1 \cap G_2 \cap H_1) \cup V(H_1 \cap H_2 \cap G_1)| \leq 5$. In fact, $|V(G_1 \cap G_2 \cap H_1) \cup V(H_1 \cap H_2 \cap G_1)| = 5$; otherwise, the separation $(G_1 \cap H_1, G_2 \cup H_2)$ shows that (iii) would hold for (G, u_1, u_2, A). Thus $|V(G_1 \cap G_2 \cap H_2) \cup V(H_1 \cap H_2 \cap G_2)| = 5$. By the choice of (G_1, G_2) (i.e., the minimality of G_1), the separation $(G_1 \cap H_1, G_2 \cup H_2)$ implies that $V(G_1 \cap H_2) - V(H_1) = \emptyset$ (so $V(G_1 \cap G_2 \cap H_2) = V(G_1 \cap G_2 \cap H_1 \cap H_2)$). Now since $z \notin V(G_2)$, $|V(G_1 \cap G_2 \cap H_2) \cup V(H_1 \cap H_2 \cap G_2)| = |V(H_1 \cap H_2 \cap G_2)| \leq |V(H_1 \cap H_2) - \{z\}| = 4$, a contradiction.

By (a), (c) and (e), there exists a permutation ij of $\{1, 2\}$ such that $xu_i, yu_j \in E(G)$. We now show that $N(x) \cap (V(G_1) - V(G_2)) \subseteq N[u_i]$ and $N(y) \cap (V(G_1) - V(G_2)) \subseteq N[u_j]$. Suppose this is false and, by symmetry, assume that $N(x) \cap (V(G_1) - V(G_2)) \not\subseteq N[u_i]$. Then by (a) and (e) there exists $z \in V(G_1) - V(G_2) - \{u_1, u_2\}$ such that $xz \in E(G)$ and $zu_i \notin E(G)$. By (a) again, xz satisfies (c); so by (e), $zu_j \in E(G)$. Let G_1^* be obtained from G_1 by duplicating u_k with u'_k, $k = 1, 2$. If $G_1^* - \{x, z\}$ has four disjoint paths from $\{u_1, u'_1, u_2, u'_2\}$ to $V(G_1 \cap G_2) - \{x\}$, then these paths, $u_i x z u_j$ and four disjoint paths in $G_2 - x$ from $V(G_1 \cap G_2) - \{x\}$ to A (Lemma 2.6.1(i)) give a topological H in G rooted at u_1, u_2, A, a contradiction. Thus, G_1 has a separation (G_{11}, G_{12}) such that $|V(G_{11} \cap G_{12})| \leq 5$, $\{x, z\} \subseteq V(G_{11} \cap G_{12})$, $\{u_1, u_2\} \subseteq V(G_{11})$, and $V(G_1 \cap G_2) - \{x\} \subseteq V(G_{12})$. Now the separation $(G_{11}, G_{12} \cup G_2)$ contradicts the choice of (G_1, G_2) (the minimality of G_1).

Thus, (G, u_1, u_2, A) also satisfies (5) in Section 6. Hence, we get a final contradiction by invoking Lemmas 2.6.2 – 2.6.5, completing the proof of Theorem 2.2.1.

2.8 Application

The characterization of infeasible quadruples was used by He, Wang, and Yu [10, 11, 12, 13] in their recent proof of the Kelmans-Seymour conjecture that every 5-connected non-planar graph contains a topological K_5. They applied Theorem 2.2.1 to force topological K_5 in the following case in the proof of Theorem 1.2 in [10]:

Let G be a 5-connected graph and (G_1, G_2) be a 5-separation in G. Let $V(G_1) \cap V(G_2) = \{a, a_1, a_2, a_3, a_4\}$ such that $G[\{a, a_1, a_2\}] \cong K_3$, where G is obtained from another graph by contracting a connected subgraph to the vertex a. Let u_1, u_2 be two neighbors of a, and $u_1, u_2 \in V(G_2) - V(G_1)$. The goal is to force a topological K_5 regardless of the feasibility of $(G_2, u_1, u_2, \{a_1, a_2, a_3, a_4\})$.

If $(G_2, u_1, u_2, \{a_1, a_2, a_3, a_4\})$ is feasible, then let H^* denote a topological H with root $\{u_1, u_2\}$ and ends in $\{a_1, a_2, a_3, a_4\}$. We have 2 cases:

- Case 1: H^* contains two internally vertex disjoint paths from one of u_1, u_2 to both a_1 and a_2. Since G is 5-connected, G_1 has vertex disjoint paths P_1, P_2 from $\{a_1, a_2\}$ to $\{a_3, a_4\}$, so $G[\{a, a_1, a_2, u_1, u_2\}] \cup H^* \cup P_1 \cup P_2$ is a topological K_5 with branch vertices a, a_1, a_2, u_1, u_2.

- Case 2: H^* contains two vertex disjoint paths from u_1, u_2 to a_1, a_2. By symmetry we may assume H^* contains two internally disjoint paths from u_1 to a_1, a_3, and two internally disjoint paths from u_2 to a_2, a_4. When G_1 has two vertex disjoint paths P_1 from a_1 to a_4 and P_2 from a_2 to a_3, $G[\{a, a_1, a_2, u_1, u_2\}] \cup H^* \cup P_1 \cup P_2$ is a topological K_5 with branch vertices a, a_1, a_2, u_1, u_2. If such path P_1, P_2 do not exist in G_1, then by a result of Seymour, $G_1 - a$ is planar, and hence, $G - a$ is planar. In which case, we can always find a topological K_5.

If $(G_2, u_1, u_2, \{a_1, a_2, a_3, a_4\})$ is not feasible, the characterization in Theorem 2.2.1 will provide further structural information about G, which either lead to contradictions, or the existence of topological K_5.

CHAPTER 3

PROGRESS ON HAJÓS' CONJECTURE

3.1 Introduction

Using Kuratowski's characterization of planar graphs [19], the Four Color Theorem [3, 4, 2, 24] can be stated as follows: Graphs containing no K_5-subdivision or $K_{3,3}$-subdivision are 4-colorable. Wagner [28] and Kelmans [16] showed that if a connected graph G contains no $K_{3,3}$-subdivision then G is planar, or $G \cong K_5$, or G admits a cut of size at most 2. Consequently, the chromatic number of a graph with no $K_{3,3}$-subdivision is at most 5.

What about graphs containing no K_5-subdivision? If the chromatic number of graphs containing no K_5-subdivision is at most 4, we would have a natural extension of the Four Color Theorem (as $K_{3,3}$ has chromatic number 2 and perhaps should not be excluded). This is part of a more general conjecture made by Hajós in the 1950s (see [27], although the reference [9] is often cited): For any positive integer k, every graph containing no K_{k+1}-subdivision is k-colorable. It is not hard to prove the following theorem:

Theorem 3.1.1. *Hajós' conjecture is true for $k = 1, 2, 3$.*

Proof. For $k = 1$, a graph G with no K_2-subdivision is a graph with no edges. i.e. G is a graph with only isolated vertices. G is 1-colorable. For $k = 2$, a graph G with no K_3-subdivision is a graph with no cycles. Since every acyclic graph is bipartite, G is 2-colorable.

For $k = 3$, we would like to prove a graph with no K_4-subdivision is 3-colorable. Suppose not, let G be the minimum counterexample:

(1) G has no K_4-subdivision,

(2) G is not 3-colorable, and

(3) subject to (1) and (2), $|V(G)|$ is minimum.

Then G is connected; otherwise one of the components of G must be a smaller counterexample.

Moreover, G is 2-connected. Suppose not. Let v be a cut vertex of G, and $C_1, C_2, \ldots, C_k$ be the components of $G-v$. Since G has no K_4-subdivision, $G[V(C_i) \cup \{v\}], i = 1, \ldots, k$, has no K_4-subdivision. Hence by the minimality of G, $G[V(C_i) \cup \{v\}], i = 1, \ldots, k$ are 3-colorable. Thus G must be 3-colorable, which is a contradiction.

We claim that G must be 3-connected. Suppose not. Then G must have a 2-seperation (G_1, G_2). Let $V(G_1 \cap G_2) = \{u, v\}$. We consider $G_i + uv$ for $i = 1, 2$. By the minimality of G, $G_i + uv$ either contains a K_4-subdivision, or is 3-colorable. Suppose $G_i + uv$ contain a K_4-subdivision. There must be a uv-path P in G_{3-i}, since either uv is an edge in G_{3-i}, or there must exist a vertex w in G_{3-i}, with 2 internally vertex disjoint paths from w to u, v in G_{3-i} (since G is 2-connected). Then G must have a K_4-subdivision in $G_i \cup P$. Thus, $G_i + uv, i = 1, 2$ must be 3-colorable, and hence G_i must have a 3-coloring such that u and v do not have the same color. This implies G must be 3-colorable, which is a contradiction.

Now let C be a shortest cycle in G. Then $V(C) \neq V(G)$, otherwise by the minimality of C, $E(G) \backslash E(C) = \emptyset$, which implies $G = C$, and G is 3-colorable. Let $v \in V(G) \backslash V(C)$. Since G is 3-connected, G must contain 3 internally vertex disjoint paths from v to C by Menger's Theorem [23]. Let P_1, P_2, and P_3 be 3 shortest internally vertex disjoint paths from v to C. Then $C \cup P_1 \cup P_2 \cup P_3$ is a K_4-subdivision in G.

Thus there exist no such counterexample. Hajós' conjecture is true for $k = 1, 2, 3$. ∎

However, Catlin [5] disproved Hajós' conjecture for $k \geq 6$. Subsequently, Erdős and Fajtlowicz [8] showed that Hajós' conjecture fails for almost all graphs. On the other hand, Kühn and Osthus [18] proved that Hajós conjecture holds for graphs with large girth, and Thomassen [27] pointed out interesting connections between Hajós conjecture and several important problems, including Ramsey numbers, Max-Cut, and perfect graphs. Hajós' conjecture remains open for $k = 4$ and $k = 5$.

We are concerned with Hajós' conjecture for $k = 4$. We say that a graph G is a *Hajós graph* if

(1) G contains no K_5-subdivision,

(2) G is not 4-colorable, i.e., $\chi(G) \geq 5$, and

(3) subject to (1) and (2), $|V(G)|$ is minimum.

Thus, if Hajós graph does not exist then graphs containing no K_5-subdivisions are 4-colorable.

Recently, He, Wang, and Yu [10, 11, 12, 13] proved that every 5-connected nonplanar graph contains a K_5-subdivision, establishing a conjecture of Kelmans [15] and, independently, of Seymour [25] (also see Mader [22]). Therefore, Hajós graphs cannot be 5-connected. On the other hand, Yu and Zickfeld [30] proved that Hajós graphs must be 4-connected, and Sun and Yu [26] proved that for any 4-cut T in a Hajós graph G, $G - T$ has exactly 2 components.

Our goal is to show the following theorem.

Theorem 3.1.2. *No Hajós graph has a 4-separation* (G_1, G_2) *such that* $(G_1, V(G_1 \cap G_2))$ *is planar and* $|V(G_1)| \geq 6$.

This work will be useful in modifying the recent proof of the Kelmans-Seymour conjecture in [10, 11, 12, 13] to make progress on the Hajós conjecture, in particular, for the class of graphs containing K_4^- as a subgraph, where K_4^- is the graph obtained from K_4 by removing an edge. The arguments in [11, 12] depend heavily on the assumption of 5-connectedness, and we wish to replace such arguments with coloring arguments. For this to work, we need to deal with 4-cuts with a planar bridge. (For a graph G and $T \subseteq V(G)$, a T-*bridge* of G is a subgraph of G induced by edges of a component of $G - T$ as well as edges between that component and T.)

3.2 Forcing good wheels with 4-separations

To prove Theorem 3.1.2, we consider Hajós graphs G with a 4-separation (G_1, G_2) such that $(G_1, V(G_1 \cap G_2))$ is planar and $|V(G_1)| \geq 6$. We first find a special wheel inside G_1, then extend the wheel to $V(G_1 \cap G_2)$ in G_1 by four disjoint paths, and form a K_5-subdivision with two disjoint paths in G_2. We now make this more precise. By a *wheel* we mean a graph which consists of a cycle C, a vertex v not on C (known as the center of the wheel), and some edges from v to a subset of $V(C)$. Let G be a graph, $W \subseteq G$ be a wheel with center w, $S \subseteq V(G) \setminus \{w\}$, such that $N_G(w) \subseteq V(W)$ and there exists a separation (G', G'') in G with $V(G' \cap G'') = S$, $|S| \geq 4$, and $W \subseteq G'$. We say that W is S-*good* if $S \cap V(W) \subseteq N_G(w)$. Let S' be a subset of S with size 4, we say that W is S-*extendable* (with respect to S'), if G has four paths P_1, P_2, P_3, P_4 from w to S' such that

- $V(P_i \cap P_j) = \{w\}$ for all distinct $i, j \in [4]$, and
- $|V(P_i - w) \cap V(W)| = 1$ for $i \in [4]$.

Note that the paths P_i may use more than one vertex from S.

We will first complete the proof of that G_1 has a wheel that is $V(G_1 \cap G_2)$-good. In this section, we will show the proofs of some useful lemmas to force good wheels with 4-seperations.

Lemma 3.2.1. *Let G be a Hajós graph and let (G_1, G_2) be a 4-separation in G such that $(G_1, V(G_1 \cap G_2))$ is planar. Then $|V(G_1)| \neq 6$.*

Proof. For, suppose $|V(G_1)| = 6$. Let $V(G_1) \setminus V(G_1 \cap G_2) = \{u, v\}$ and $V(G_1 \cap G_2) = \{v_i : i \in [4]\}$, and assume that v_1, v_2, v_3, v_4 occur on the boundary of a disc represention of G in clockwise order. Then, since G is 4-connected, we may further assume that $N_G(u) = \{v_1, v_2, v_3, v\}$ and $N_G(v) = \{v_1, v_3, v_4, u\}$.

Now $G' := G - \{u, v\} + v_2v_4$ contains no K_5-subdivision. For, if T' is a K_5-subdivision in G', then $(T' - v_2v_4) \cup v_2uvv_4 \subseteq G$ contains a K_5-subdivision, a contradiction.

Thus G' has a proper 4-coloring, say σ. If $\sigma(v_2) \in \{\sigma(v_1), \sigma(v_3)\}$ then we can extend σ to a proper 4-coloring of G by greedily coloring v, u in order. If $\sigma(v_2) \notin \{\sigma(v_1), \sigma(v_3)\}$ then we can extend σ to a proper 4-coloring of G by coloring v with $\sigma(v_2)$ and coloring u with a color not used by v_1, v_2, v_3, v. Either way, we obtain a contradiction to the assumption that G is a Hajós graph. ∎

Lemma 3.2.2. *Let G be a Hajós graph and let (G_1, G_2) be a 5-separation in G such that $(G_1, V(G_1 \cap G_2))$ is planar. Then $G_1 - V(G_1 \cap G_2) \not\cong K_3$.*

Proof. Suppose for a contradiction that $G_1 - V(G_1 \cap G_2) \cong K_3$. Let $u, v, w \in V(G_1) \setminus V(G_2)$ and $V(G_1 \cap G_2) = \{v_i : i \in [5]\}$, and assume that G_1 has a disc representation such that v_1, v_2, v_3, v_4, v_5 occur on the boundary of the disc in clockwise order.

Note that $N_G(v_i) \cap \{u, v, w\} \neq \emptyset$ for $i \in [5]$. For, otherwise, we may assume by symmetry that $N_G(v_5) \cap \{u, v, w\} = \emptyset$. Then, since G is 4-connected, $N_G(v_i) \cap \{u, v, w\} \neq \emptyset$ for $i \in [4]$; so by planarity, there exists some $j \in [4]$ with $|N_G(v_j) \cap \{u, v, w\}| = 1$, and without loss of generality, let $w \in N_G(v_j)$. Now G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{v_i : i \in [4] \setminus \{j\}\} \cup \{w\}$, $(H_1, V(H_1 \cap H_2))$ is planar, and $V(H_1) = \{v_i : i \in [4] \setminus \{j\}\} \cup \{u, v, w\}$, contradicting Lemma 3.2.1.

Moreover, no vertex in $\{u, v, w\}$ is adjacent to four vertices in $V(G_1 \cap G_2)$. For, suppose $vv_i \in E(G)$ for $i \in [4]$. Then, by planarity, G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{v, v_1, v_4, v_5\}$, $(H_1, V(H_1 \cap H_2))$ is planar, and $V(H_1) = \{v_1, v_4, v_5, u, v, w\}$, contradicting Lemma 3.2.1.

Also note that any two vertices of $\{u, v, w\}$ must have at least four neighbors in $V(G_1 \cap G_2)$. For, suppose u, v has at most three neighbors in $V(G_1 \cap G_2)$. Then $|(N_G(u) \cup N_G(v)) \setminus \{u, v\}| = 4$ (as G is 4-connected), and G has a separation (H_1, H_2) such that $V(H_1 \cap H_2) = (N_G(u) \cup N_G(v)) \setminus \{u, v\}$, $(H_1, V(H_1 \cap H_2))$ is planar, and $V(H_1) = N_G(u) \cup N_G(v)$ (so $|V(H_1)| = 6$), contradicting Lemma 3.2.1.

Case 1. There exists $\{a, b\} \subseteq \{u, v, w\}$ such that $V(G_1 \cap G_2) \subseteq N_G(\{a, b\})$.

Without loss of generality, we may assume that $a = v$ and $b = w$, and that $v_1, v_2 \in N_G(v)$ and $v_3, v_4, v_5 \in N_G(w)$. We may further assume that the notation is chosen so that $uv_1, uv_5 \in E(G)$ (by planarity). Moreover, $vv_3 \in E(G)$ since u and v must have at least four neighbors in $V(G_1 \cap G_2)$.

Let $G' := G - \{u, v, w\} + \{v_5v_1, v_5v_2, v_5v_3\}$. We claim that G' contains no K_5-subdivision. For, suppose T is a K_5-subdivision in G'. If $v_5v_1, v_5v_2, v_5v_3 \in E(T)$ then $T - \{v_5v_1, v_5v_2, v_5v_3\}$ and the paths wv_5, wuv_1, wvv_2, wv_3 form a K_5-subdivision in G, a contradiction. So $\{v_5v_1, v_5v_2, v_5v_3\} \not\subseteq E(T)$. Then we obtain a contradiction by forming a K_5-subdivision in G from T: replacing edges in $\{v_5v_1, v_5v_2, v_5v_3\} \cap E(T)$ with one or two paths from one of the following $\{v_5uv_1, v_5wvv_2\}$, or $\{v_5uv_1, v_5wv_3\}$, or $\{v_5uvv_2, v_5wv_3\}$.

Thus, G' has a proper 4-coloring, say σ. We extend σ to a proper 4-coloring of G by first assigning $\sigma(v_5)$ to v, and then greedily coloring w, u in order. Thus G is 4-colorable, a contradiction.

Case 2. For any $\{a, b\} \subseteq \{u, v, w\}$, $|N_G(\{a, b\}) \cap V(G_1 \cap G_2)| = 4$.

Without loss of generality, we may assume that $uv_1, uv_5, vv_2, vv_3 \in E(G)$. By symmetry and planarity, assume $wv_3, wv_4 \in E(G)$. Note that $\{wv_5, vv_1\} \not\subseteq E(G)$ as, otherwise, $V(G_1 \cap G_2) \subseteq N_G(\{v, w\})$ (but we are in Case 2). On the other hand, since any two vertices of $\{u, v, w\}$ must have at least four neighbors in $V(G_1 \cap G_2)$, $wv_5 \in E(G)$ or $vv_1 \in E(G)$. So by symmetry, we may assume $wv_5 \in E(G)$ and $vv_1 \notin E(G)$.

Let $G' := (G - \{u, v, w\}) + \{v_5v_2, v_5v_3\}$. Note that G' contain no K_5-subdivision as v_5v_2, v_5v_3 can be replaced by v_5uvv_2, v_5wv_3, respectively. Hence G' has a proper 4-coloring, say σ. By assigning $\sigma(v_5)$ to v and greedily coloring w, u in order, we obtain a proper 4-coloring of G, a contradiction. ∎

We now proceed to show the existence of a good wheel inside a nontrivial 4-separation.

Lemma 3.2.3. *Let G be a Hajós graph and let (G_1, G_2) be a 4-separation in G such that $|V(G_1)| \geq 6$ and $(G_1, V(G_1 \cap G_2))$ is planar. Then G_1 contains a $V(G_1 \cap G_2)$-good wheel.*

Proof. First, we may assume that G_1 is minimal subject the assumption in the statement of the lemma, as for any 4-separation (G_1', G_2') of G with $G_1' \subseteq G_1$, a $V(G_1' \cap G_2')$-good wheel in G_1' is also a $V(G_1 \cap G_2)$-good wheel in G_1.

By Lemma 3.2.1, $|V(G_1)| \geq 7$. Let $V(G_1 \cap G_2) = \{t_1, t_2, t_3, t_4\}$ and assume that G_1 has a disc representation with t_1, t_2, t_3, t_4 on the boundary of the disc in clockwise order. Let $D := G_1 - V(G_1 \cap G_2)$. Since G is 4-connected, $|N_G(t_i) \cap V(D)| \geq 1$ for each $i \in [4]$. In fact,

(1) $|N_G(t_i) \cap V(D)| \geq 2$ for $i \in [4]$.

For, suppose $|N_G(t_i) \cap V(D)| = 1$ for some $i \in [4]$, and let $t \in N_G(t_i) \cap V(D) = \{t\}$. Then $(G_1 - t_i, G_2 + \{t, tt_i\})$ is a separation in G that contradicts the minimality of G_1. □

(2) D is 2-connected.

Suppose to the contrary that D is not 2-connected. Then D has a 1-separation (D_1, D_2) such that $|V(D_i) \setminus V(D_{3-i})| \geq 1$ for $i = 1, 2$. Since G is 4-connected, $|N_G(D_i - D_{3-i}) \cap \{t_1, t_2, t_3, t_4\}| \geq 3$ for $i = 1, 2$.

Thus by planarity (and choosing appropriate notation for t_i), we may assume that $t_1, t_2, t_3 \in N_G(D_1 - D_2)$ and $t_3, t_4, t_1 \in N_G(D_2 - D_1)$. Since G is 4-connected, $|V(D_1 \cap D_2)| = 1$. Note that G has a separation (G_1', G_2') such that $V(G_1' \cap G_2') = \{t_1, t_2, t_3\} \cup V(D_1 \cap D_2)$ and $G_1' \subseteq G_1$, as well as a separation (G_1'', G_2'') such that $V(G_1'' \cap G_2'') = \{t_1, t_3, t_4\} \cup V(D_1 \cap D_2)$ and $G_1'' \subseteq G_2$. Thus by the choice of (G_1, G_2) (the minimality of G_1), $|V(D_i)| = 2$ for $i = 1, 2$. But then $|N_G(t_2) \cap V(D)| = 1$, contradicting (1). □

Let C be the outer cycle of D. If there exists $x \in V(D) \setminus V(C)$ then all vertices and edges of D cofacial with x (including x) form the desired wheel. Thus we may assume that

(3) $V(D) = V(C)$.

We claim that.

(4) for each $i \in [4]$, there exists $u_i \in N_G(t_i) \cap N_G(t_{i+1}) \cap V(C)$ (with $t_5 := t_1$).

For, suppose (4) fails and, without loss of generality, assume that $N_G(t_4) \cap N_G(t_1) \cap V(C) = \emptyset$. Let $v_1 \in N_G(t_1) \cap V(C)$ and $v_4 \in N_G(t_4) \cap V(C)$ such that v_4Cv_1 is minimal. Thus, $v_1t_4, v_4t_1 \notin E(G)$. By (1) and by planarity, $v_1t_2, v_1t_3, v_4t_2, v_4t_3 \notin E(G)$. Since the degree of v_1 in G is at least 4, v_1 has a neighbor v in D such that $vv_1 \in E(D) \setminus E(C)$. We choose v, such that vCv_1 is minimal. Moreover, let v' be the neighbor of v_1 on v_4Cv_1.

If $N_G(t_3) \cap V(v_1Cv - v) = \emptyset$ then G has a 4-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{t_1, t_2, v, v'\}$, $v_1 \in V(H_1) \setminus V(H_2)$, $G_2 + \{t_3, t_4\} \subseteq H_2$, $(H_1, V(H_1 \cap H_2))$ is planar, and $|V(H_1)| \geq 6$ (by (1)), contradicting the minimality of G_1.

So we may assume $N_G(t_3) \cap V(v_1Cv - v) \neq \emptyset$. Moreover, since G is 4-connected, $\{v_1, v, t_4\}$ cannot be a cut in G. Hence, $N_G(t_3) \not\subseteq V(v_1Cv)$.

Thus, G has a 4-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{t_3, t_4, v, v_1\}$, $v_4 \in V(H_1) \setminus V(H_2)$, $G_2 + \{t_1, t_2\} \subseteq H_2$, and $(H_1, V(H_1 \cap H_2))$ is planar. By (1), $|V(H_1)| \geq 6$, so (H_1, H_2) contradicts the choice of (G_1, G_2). □

We may assume that

(5) $V(C) = \{u_1, u_2, u_3, u_4\}$.

First, we may assume that $N_G(t_i) \cap (V(C) \setminus \{u_j : j \in [4]\}) = \emptyset$ for $i \in [4]$. For, suppose, without loss of generality, that there exists $u \in V(u_4Cu_1) \setminus \{u_4, u_1\}$ and $ut_1 \in E(G)$. Then the vertices and edges of G_1 cofacial with u form a $V(G_1 \cap G_2)$-good wheel.

Now suppose $V(C) \neq \{u_1, u_2, u_3, u_4\}$. Then $|V(C)| = 5$; for otherwise, G has a 4-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{u_1, u_2, u_3, u_4\}$, $H_1 = D$, $(H_1, V(H_1 \cap H_2))$ is planar, and $G_2 + \{t_i : i \in [4]\} \subseteq H_2$, contradicting the choice of (G_1, G_2).

Let $u \in V(C) \setminus \{u_1, u_2, u_3, u_4\}$ and, without loss of generality, assume that $u \in V(u_4Cu_1)$. Since $ut_1 \notin E(G)$, $uu_2, uu_3 \in E(G)$, as G is 4-connected. Hence, G has a 5-separation (H_1, H_2) such that $V(H_1 \cap H_2) = \{t_2, t_3, t_4, u_1, u_4\}$ and $H_1 - V(H_1 \cap H_2)$ is the triangle uu_2u_3u, contradicting Lemma 3.2.2. □

(6) $D \neq C$.

For, suppose $D = C$. Let σ be a proper 4-coloring of $G - \{u_1, u_2, u_3, u_4\}$ which exists as G is a Hajós graph. We can extend σ to a proper 4-coloring of G as follows: If $|\{\sigma(t_i) : i \in [4]\}| = 4$ then assign to u_1, u_2, u_3, u_4 the colors $\sigma(t_4), \sigma(t_1), \sigma(t_2), \sigma(t_3)$, respectively. If $|\{\sigma(t_i) : i \in [4]\}| \leq 3$ then assign to both u_1 and u_3 a color not in $\{\sigma(t_i) : i \in [4]\}$, and greedily color u_2, u_4 in order. This contradicts the assumption that G is a Hajós graph. $\square$

By (6) and by planarity of D, we may assume $D = C + u_2u_4$. Note that $G' := (G - \{u_1, u_2, u_3, u_4\}) + \{t_1t_2, t_2t_3, t_3t_1\}$ contains no K_5-subdivision; for, if T is a K_5-subdivision in G' then, by replacing t_1t_2, t_2t_3, t_3t_1 (whenever in T) with $t_1u_1t_2, t_2u_2t_3, t_3u_3u_4t_1$, respectively, we obtain a K_5-subdivision in G.

Thus let σ be a proper 4-coloring of G'. If $|\{\sigma(t_i) : i \in [4]\}| = 4$ then assign to u_1, u_2, u_3, u_4 the colors $\sigma(t_4), \sigma(t_1), \sigma(t_2), \sigma(t_3)$, respectively, we obtain a proper 4-coloring of G, a contradiction.

So $|\{\sigma(t_i) : i \in [4]\}| = 3$ and $\sigma(t_4) \in \{\sigma(t_i) : i \in [3]\}$. We derive a contradiction by extending σ to a proper 4-coloring of G: If $\sigma(t_4) = \sigma(t_2)$ or $\sigma(t_4) = \sigma(t_1)$ then assign $\sigma(t_1), \sigma(t_3)$ to u_2, u_4, respectively, and greedily color u_1, u_3 in order; and if $\sigma(t_4) = \sigma(t_3)$ then assign $\sigma(t_1), \sigma(t_2)$ to u_2, u_4, respectively, and greedily color u_1, u_3 in order. ∎

3.3 Sketch of the proof of Theorem 3.1.2

We will first complete the proof of that G_1 has a wheel that is $V(G_1 \cap G_2)$-good. However, we need to allow the separation (G_1, G_2) to be a 5-separation in order to deal with issues when wheels are not $V(G_1 \cap G_2)$-extendable. This is saying that when we try to extend a good wheel in the planar side of a 4-separation (G_1, G_2) by four paths to $V(G_1 \cap G_2)$ that are internally disjoint from the wheel, we encounter problems with 5-separations. For 5-separations, we characterize the situation when one cannot find a good wheel inside a 5-separation:

Lemma 3.3.1. *Let G be a Hajós graph and (G_1, G_2) a 5-separation in G such that $(G_1, V(G_1 \cap G_2))$ is planar and $V(G_1 \cap G_2)$ is independent in G_1. Then one of the following holds:*

(i) *G has a 4-separation (L_1, L_2) such that $L_1 \subseteq G_1$, $G_2 \subseteq L_2$, and $|V(L_1)| \geq 6$.*

(ii) *G_1 contains a $V(G_1 \cap G_2)$-good wheel.*

(iii) *$(G_1, V(G_1 \cap G_2))$ is one of the graphs in Figure 3.1, where G_1 is drawn in a closed disc and $V(G_1 \cap G_2)$ consists of vertices on the boundary of the disc.*

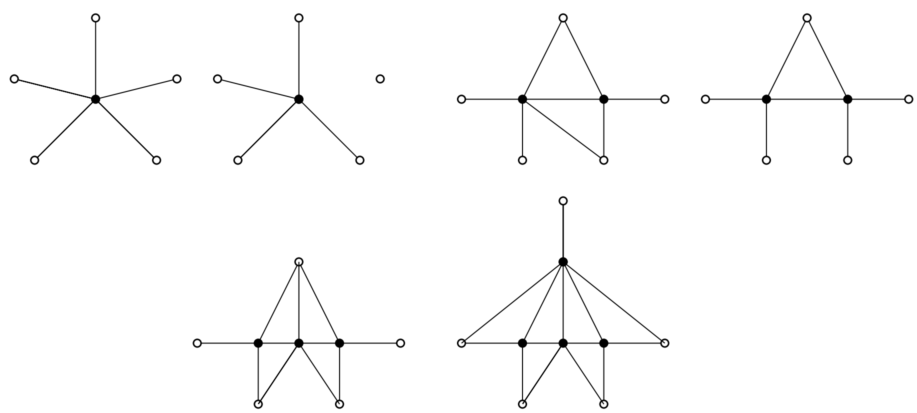


Fig. 3.1: The obstructions.

Finally we characterize the graph G when wheels are not extendable and complete the proof of Theorem 3.1.2.

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