

AN EVALUATION OF TIME DEPENDANT NUMERICAL
METHODS APPLIED TO A RAPIDLY CONVERGING NOZZLE

A THESIS

Presented to

The Faculty of the Graduate Division

By

Garland Eldridge Giles, Jr.

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Mechanical Engineering

Georgia Institute of Technology

March, 1973

AN EVALUATION OF TIME DEPENDANT NUMERICAL
METHODS APPLIED TO A RAPIDLY CONVERGING NOZZLE

Approved:

S. V. Shelton, Chairman

G. T. Colwell

H. M. McMahon

Date approved by Chairman: 23 May 73

ACKNOWLEDGMENTS

I would like to thank my advisor, Dr. S. V. Shelton, whose assistance and encouragement made possible this work. His ability to identify problems was very helpful during the many difficulties encountered.

Douglas E. Kooker provided enlightenment in confusing times with advice from his experience in related areas. I appreciate very much his conversations with me.

Dr. Alex Alkidas provided a change of direction at a crucial point and also helped to make this work possible.

TABLE OF CONTENTS

	Page
ACKNOWLEDGMENTS	ii
LIST OF TABLES	iv
LIST OF ILLUSTRATIONS	v
NOMENCLATURE	vi
SUMMARY	ix
Chapter	
I. INTRODUCTION	1
II. DESCRIPTION OF THE PROBLEM	2
III. HISTORICAL SURVEY	4
IV. STABILITY ANALYSIS METHOD	12
V. EXPERIMENTAL STUDIES	17
VI. ANALYTICAL DEVELOPMENT	19
Boundary Conditions	
Transformation of Coordinate System	
Non-Dimensionalization of Variables	
Differential Equation	
Finite Difference Approximations	
Stability Analysis	
VII. RESULTS AND CONCLUSIONS	47
BIBLIOGRAPHY.	66

LIST OF TABLES

Table	Page
1. Survey of Unsteady Methods	7
2. Truncation Errors From Moretti	10
3. Summary of Difference Approximations Invested	35

LIST OF ILLUSTRATIONS

Figure	Page
1. Initial Conditions	25
2. Physical Coordinate System	27
3. Transformed Coordinate System	28
4. Experimental Mach Number	53
5. Experimental Pressure Ratio	54
6. Pressure Ratio - Moretti Method	55
7. Pressure Ratio - Moretti Method	56
8. Pressure Ratio - Moretti Method	57
9. Mach Number - Moretti Method.	58
10. Percent Error of R Per Time Step - Moretti Method	59
11. Pressure Ratio - Rusanov Method	60
12. Pressure Ratio - Rusanov Method	61
13. Pressure Ratio - Rusanov Method	62
14. Mach Number - Rusanov Method	63
15. Percent Error of R Per Time Step - Rusanov Method	64
16. Time Variation of Throat Mach Number.	65

NOMENCLATURE

English Symbols

A	= UC + VD
Al	stretching parameter
a	speed of sound
B	= UK
C	= $\partial y / \partial x$
D	= $\partial y / \partial r$
E	= DG
f	dummy variable
F	= CG
G	= a^2
H	= D/y
h	specific enthalpy
I	axial grid point index
J	radial grid point index
K	= $\partial z / \partial x$
L	= KG
M	Mach number
P	pressure ratio P'/P'_0
R	$\ln p$
\bar{R}	gas constant
R_t	= r_c/r_t

NOMENCLATURE (Continued)

r	cylindrical radial coordinate
r_c	throat contour radius
r_t	radius of nozzle throat
t	time
T	total truncation error
U	axial velocity
V	radial velocity
HX	total velocity
X	physical axial coordinate
y	transformed radial coordinate
W	shock speed
Z	transformed axial coordinate

Greek Symbols

α	axial weighting function for Rusanov method
β	radial weighting function for Rusanov method
Δ	difference in variable between two grid points; forward difference operator (second definition used only in finite difference section)
$\bar{\Delta}$	backward difference operator
ξ	dummy coordinate
δ	centered difference operator
η	$= \Delta y / \Delta z$
ϵ	finite difference truncation error
γ	ratio of specific heats

NOMENCLATURE (Concluded)

ρ	density ratio ρ'/ρ_0'
σ	Courant constant
w	Rusanov variable

Superscripts

'	dimensional variable
\rightarrow	vector quantity
n	time step number

Subscripts

0	stagnation condition
n	conditions at nozzle wall
$\Delta, \bar{\Delta}$ or δ	indicates function derived with $\Delta, \bar{\Delta}$ or δ operator
x, y, z, r or t	indicates differential with respect to x, y, z, r or t
i or j	indicates grid location of variable
R	indicates function derived from finite difference equation for R

SUMMARY

To find a solution to the rapidly converging nozzle problem, a search of existing numerical methods was conducted. From the literature it was concluded that the method that was best suited for use with a computer in solving the problem was a finite difference equation approximating the time dependant flow equations. Two methods, the Moretti and Rusanov, were chosen for further study based on their performance as presented in the literature. These two methods were applied to a rapidly converging nozzle, with boundary conditions consistent with the physical problem.

A truncation error analysis was performed on both methods by using Taylor series expansions to derive the differential equation actually being represented by the finite difference equation. The difference between the desired and the actual differential equation is the truncation error. An order of magnitude study of the truncation error was performed by calculating the truncation error for the starting conditions, which were the isentropic, one-dimensional flow solutions. The results of the order of magnitude study showed that both methods were incapable of providing an adequate solution.

CHAPTER I

INTRODUCTION

In recent years there has been considerable interest in rocket nozzles with high inlet angles and small throat radius curvature. These rapidly converging nozzles can offer less weight, smaller size and less cooling requirement than conventional nozzles. It has been experimentally shown that the heat transfer in a rocket nozzle can be reduced by as much as 50 percent by using a high inlet angle. It also appears that the performance loss of the large inlet angle nozzles may be small enough to make the rapidly converging nozzles attractive.

Development of numerical techniques to aid in design of the rapidly converging nozzles has been hindered by the severe two-dimensional effects near the throat and the existence of mixed subsonic, transonic and supersonic flows.

The experimental determination of the heat transfer coefficients and propulsive performance of rapidly converging nozzles would involve large amounts of manhours, sophisticated instrumentation, and money. In addition, optimization of design can not always be performed by the cut-and-try method. The goal of this investigation is to find a method of solving the axisymmetric compressible flow equations so that designers can save money and time in designing propulsion and other rapidly converging nozzles.

CHAPTER II

DESCRIPTION OF THE PROBLEM

The greatest difficulty associated with the rapidly converging nozzle problem is the existence of coupled subsonic and transonic flows in the region of investigation. The desired technique should be valid from low subsonic into the supersonic regions. The technique must handle the elliptic subsonic, parabolic transonic, and hyperbolic supersonic flow equation. Existing methods can only solve the equations for subsonic and transonic regions separately. Since information should be allowed to be transmitted from the throat to inlet and back, by the nature of the elliptic and parabolic equations the two regions must be solved simultaneously.

In addition, the sonic line at the throat of nozzles with small contour radius can be quite concave. The Mach number at the wall can be sonic or greater while at the same axial location the mach number at the centerline can be subsonic. The higher pressure at the centerline causes the flow near the wall to turn into the wall necessitating a compressive turning of the flow to bring the flow parallel to the wall. This compressive turning can precipitate a shock formation aft of the throat.

The usual method of attempting a solution is to simplify the Navier-Stokes equations with approximations based on consideration of known physical characteristics of the flow. In the subsonic this method

allows one to reduce the Navier-Stokes equations to Laplaces' equation for the incompressible region and another equation of elliptical form for the compressible region. The solution for Laplaces' equation is well within the state of the art. The compressible region presents more of a problem and since the two regions are by nature interdependent, it would be preferable to have one method to solve the two regions simultaneously.

The reduced Navier-Stokes equations in the transonic region are of parabolic form and also by nature interdependent with both of the subsonic regions.

Due to the inability for disturbances in the supersonic region to propagate up stream, the solution to the supersonic flow can be performed when the flow variables at the entrance of the supersonic region are known. Such proven methods as the method of characteristics can be used in this region with little difficulty.

The problem and goal of this work is to find a method of solving the transonic parabolic equations and subsonic elliptical equations simultaneously.

CHAPTER III

HISTORICAL SURVEY

The first attempts to solve the nozzle problem utilized approximations in the steady state differential equation. Taylor [56],* Hooker [57], Sauer [47] and Mendelson [27] each improved on previous studies and achieved a direct solution to the transonic region. Unfortunately the use of Taylor's method and improvements required the patching of two methods to achieve the subsonic and transonic flow field solution. Also it was found that the above direct methods were applicable to only nozzles with small inlet angles and large throat radius of curvature.

Oswatitach and Rothstein [36, 37] utilized an interactive method to solve the transonic flow equations. Their method had the same short comings as the direct solutions of Taylor.

Hall [15], Moore and Hall [29], Quan and Kliegal [40], Kliegal and Levine [20], and Shelton [50] utilized an inverse series expansion method but again they found the same short comings as the direct method.

To find a numerical solution which can solve all regions simultaneously and which can handle large inlet angles the numerical solution of the time dependent differential equations was studied.

* Numbers in brackets refer to references.

The two methods of solving the time dependent developed to date are

- (1) the Method of Characteristics and
- (2) direct substitution of finite difference approximations for the partial derivatives in the continuity and momentum equations.

Due to the complexity in programming the Methods of Characteristics problem in two space and one time dimension and excessive execution time the direct substitution was examined in much more detail.

The technique of solving the time-dependent equations by substituting finite difference approximations for the partial differentials and using a scheme to integrate the resultant equations with respect to time was first suggested by von Neumann and Richtmyer [55]. These investigators used a Taylor series expansion to compute each of the flow variables at time $t + \Delta t$. The time derivatives in the Taylor series expansion were rewritten by algebraic substitution from the differential equations as space derivatives which were then approximated by finite differences. The major benefit of this method in application to mixed flow problems is that the unsteady state flow equations are of hyperbolic form for all regions of the flow. Thus one method can be used for the entire flow problem.

Lax [24], rewrote the flow equations in the conservative form, used central differences for the space derivatives, and used forward differences for the time derivative. Lax and Wendroff [24] further improved the time integration scheme in their One Step Method by averaging the first term of the Taylor series over the adjacent points

and included an additional time derivative in the expansion. Table 1 contains basic information about the schemes discussed here.

Lax and Wendroff have also developed a two step or predictor-corrector method where the parameters for time $t + \Delta t$ are calculated from the parameters at time t but the parameters at $t + 2\Delta t$ are calculated from the parameters at t and $t + \Delta t$. The Lax-Wendroff Two Step was developed to reduce computer storage and execution time requirements and to improve the stability by approximating the central differences for the time derivatives. Richtmyer [41] shows that a scheme with central differences (an implicit scheme) is inherently stable.

Most of the time-dependent numerical solutions exhibit errors described as overshoot, undershoot, or numerical instability. To smooth the effects of these errors Lax and Wendroff introduced an "artificial viscosity". The effect of the "artificial viscosity" was to smooth all large gradients such as shocks and expansion waves where the physical situation results in steep gradients. Rubin and Burnstein [43] utilize the Lax-Wendroff Two Step method with a simplified "artificial viscosity" to investigate the numerical stability in a shock tube problem. Lapidus [22] further simplified Burnstein's "artificial viscosity" and developed the method to study transonic flow around blunt bodies. Lapidus does prove that the use of "artificial viscosity" does not prohibit the solution to the time dependent equations from converging to a solution which should be close to the real solution. Lapidus demonstrated the ability of the

Table 1. Survey of Unsteady Methods

Investigator	Reference	Method	Stability Condition	Order of Accuracy
Euler	[30]	$f_0^{n+1} = f_0^n + f_t^n \Delta t$		1
von Neuman and Richtmyer	[55]	$f_0^{n+1} = f_0^n + f_t^n \Delta t + f_{tt}^n \frac{\Delta t^2}{2}$		2
Lax-Wendroff One Step	[10]	$f_0^{n+1} = f_0^n + f_t^n \Delta t + f_{tt}^n \frac{\Delta t^2}{2}$		2
Lax-Wendroff Two Step	[30]	$f_0^{n+\frac{1}{2}} = f_0^n + f_t^n \Delta t + f_{tt}^n \frac{\Delta t^2}{2}$ $f_0^{n+1} = f_0^n + f_t^{n+\frac{1}{2}} \Delta t + f_{tt}^{n+\frac{1}{2}} \frac{\Delta t^2}{2}$		2
Moretti	[30]	$f_0^{n+1} = f_0^n + f_t^n \Delta t + f_{tt}^n \frac{\Delta t^2}{2}$		2
Equations are written in non-conservational form				
MacCormack	[30]	$f_0^{n+\frac{1}{2}} = f_0^n + f_t^n \Delta t$ $f_0^{n+1} = \frac{1}{2}(f_0^n + f_0^{n+\frac{1}{2}}) + f_t^{n+\frac{1}{2}} \Delta t$		2

Table 1. (Continued)

Investigator	Reference	Method	Stability Condition	Order of Accuracy
--------------	-----------	--------	---------------------	-------------------

Rusanov	[10]	$f^{n+1} = f_0^{*n} + f_t^n \Delta t$		1
---------	------	---------------------------------------	--	---

$$f_0^{*n} = f_0^n + \frac{\Delta Z^2}{2} \frac{\partial}{\partial Z} (\alpha f_z)$$

$$+ \frac{\Delta Y^2}{2} \frac{\partial}{\partial Y} (\beta f_y)$$

$$\alpha = \frac{\sigma w (\bar{V} + a) \Delta Y^2}{(\bar{V} + a)_{\max} (\Delta Y^2 + \Delta Z^2)}$$

$$\beta = \frac{\sigma w (\bar{V} + a) \Delta Z^2}{(\bar{V} + a)_{\max} (\Delta Y^2 + \Delta Z^2)}$$

In the above methods

$$f_t = -A f_z$$

$$f = \begin{bmatrix} \ln \\ U \end{bmatrix}$$

$$A = \begin{bmatrix} U-w & \gamma \\ P/\rho & U-w \end{bmatrix}$$

\bar{f}_0^n = average values of adjacent points

Lax-Wendroff Two Step method to solve the transonic flow region. Laval [23] applied Lapidus' approach to the rapidly converging nozzle problem and presents good agreement with experimental data for the throat section. Serra [49] uses the Lax-Wendroff One Step method with Burnstein's form of "artificial viscosity" and presents good agreement for the transonic region of a rapidly converging nozzle.

Prozan and Kooker [39] developed an error minimization technique to solve the time dependent equations and also show good agreement for the transonic portion of the rapidly converging nozzle. Prozan attempted to get as good a solution for the subsonic region by running the program for long times. However, instabilities in the subsonic regions accumulated errors which propagated to the throat destroying the whole solution. This phenomenon can be exhibited by several of the numerical techniques discussed here.

Moretti [30] presents a survey of some of the above methods. In Moretti's survey he performs a truncation error analysis and numerical experiments in a shock tube problem to indicate the best method. The methods compared and their truncation errors are shown in Table 2. In the numerical experiments, Moretti determined the error at one location of a shock tube for a time span including the arrival of the shock at the point of interest. Moretti also writes all methods in the non-conservation and conservation form. Although previous investigators [Richtmyer 41, 42, Lax 24] had argued that the conservative forms were better, Moretti proceeds to prove that the conservation forms actually have greater error than the non-conservative forms. Moretti also shows that his scheme, which is a Lax-Wendroff One Step method

Table 2. Truncation Errors From Moretti

Method	Truncation Error for $f_t = A f_z$
Euler	$-\frac{1}{2} \sigma^2 f_{tt} \Delta Z^2$
Lax First Order	$-\frac{1}{2} (f_{zz} + \sigma^2 f_{tt}) \Delta Z^2$
Lax-Wendroff Two Step	$1/6 (\sigma^2 A^2 - 1) \sigma A \Delta Z^3$
Moretti and Lax-Wendroff One Step	$1/6 (A f_{zzz} + \sigma^2 f_{ttt}) \sigma \Delta Z^3$

with non-conservation form of the equations has the least error for the time span investigated in his numerical experiments.

The Moretti scheme was chosen to be studied since it was shown to have less error than the other available schemes. Difficulties in the subsonic region prompted more study and surveys by Tyler [54], Taylor, et al. [53], and Hirt [17] were found which compare the performance and stability of many different methods across shocks, contact discontinuities, and expansion waves.

Taylor, et al., compared the Rusanov and Godunov first order schemes, MacCormack and Richtmyer second order two step schemes and Rusanov's third order scheme. Of this group the Godunov scheme performed with least error across shocks, contact discontinuities, and expansion waves. The first order Rusanov was the second most desirable scheme. The second and third order scheme all exhibited overshoots and undershoots near the discontinuities.

Because the Godunov first order method is very complex to program, the Rusanov first order scheme was chosen for further examination. In addition, the Rusanov first order scheme is more flexible in use as well as simpler to program.

CHAPTER IV

STABILITY ANALYSIS METHODS

Since the most common problems in numerical solutions of time dependent differential equations are the local and global stabilites, a search was made to find a method of predicting instabilities.

The first stability conditions which must be satisfied is the well-known Courant-Fredrichs-Lewy condition which says that the distance traveled by a sound wave relative to the fluid must not exceed the distance between neighboring grid points. The exact form of the CFL condition is derived for a constant coefficient form of the time integration scheme. The applicability of the CFL derived for a constant coefficient to a problem with nonconstant coefficients has been proven only by experience for both the Moretti and Rusanov method.

Moretti [30] shows that the CFL condition for his scheme is

$$\Delta t \leq \frac{\sigma \Delta x}{(U+a)} \quad \text{and} \quad \Delta t \leq \frac{\sigma \Delta y}{(V+a)} \quad (1)$$

Kentzer [19] shows by an error propagation analysis that $\sigma = 1/\sqrt{2}$ is the optimum condition to minimize the growth of both large and small wave length error disturbances.

Rusanov [46] shows that the CFL condition, his first order

method by using the Fourier method is

$$\Delta t = \frac{\sigma \omega \Delta x \Delta y}{(\Delta x^2 + \Delta y^2)(HX + a)} \quad (2)$$

and

$$\sigma < \omega < 1/\sigma \quad (3)$$

However, the CFL stability condition alone is not sufficient to predict all the instabilities that impede the application of the time dependent numerical solution. The study of the truncation error supplies the cause of these additional instabilities.

Richtmyer [41] defined truncation error as the difference between the differential equation and the actual differential represented by the finite difference equation. To find the differential equation represented by the finite difference equation each term in the time integrator scheme is expanded by a Taylor Series as in equation (4) and (5)

$$f_i^{n+1} = f_i^n + f_t \Delta t + f_{tt} \frac{\Delta t^2}{2} + f_{ttt} \frac{\Delta t^3}{6} + f_{tttt} \frac{\Delta t^4}{24} + \dots \quad (4)$$

$$f_{i+1}^m = f_i^m + f_x \Delta x + f_{xx} \frac{\Delta x^2}{2} + f_{xxx} \frac{\Delta x^3}{6} + f_{xxxx} \frac{\Delta x^4}{24} + \dots \quad (5)$$

For example

$$f_i^n = f_i^{n-1} + f_t \Delta t \quad (6)$$

is the time integration scheme for the Euler method. The time derivative of f is replaced with

$$f_t = -A f_x \quad (7)$$

or

$$f_i^{n+1} = f_i^n - A f_x \Delta t \quad (8)$$

The space derivatives are replaced by finite difference approximations and each term in equation (9) is expanded by equation (4) or (5). This results in equation (10).

$$f_i^{n+1} = f_i^n - A \Delta t (f_{i+1} - f_{i-1}) / 2 \Delta x \quad (9)$$

$$f_i^n + f_t \Delta t + f_{tt} \frac{\Delta t^2}{2} + f_{ttt} \frac{\Delta t^3}{6} + f_{tttt} \frac{\Delta t^4}{24} = \quad (10)$$

$$(2f_x \Delta x + 2f_{xxx} \frac{\Delta x^3}{6} + 2f_{xxxxx} \frac{\Delta x^5}{120}) / 2 \Delta x$$

$$f_t \Delta t + f_{tt} \frac{\Delta t^2}{2} + f_{ttt} \frac{\Delta t^3}{6} + f_{tttt} \frac{\Delta t^4}{24} = -A1 \frac{\Delta t}{\Delta x} \quad (11)$$

$$(f_x \Delta x + f_{xxx} \frac{\Delta x^3}{6} + f_{xxxxx} \frac{\Delta x^5}{120})$$

Each time derivative is expanded by differentiation of equation (7) with respect to time. Equation (7) can be differentiated with respect to x to yield terms which when substituted into the time differentiations of equation (4) yield expressions for f_{tt} , f_{ttt} , and f_{tttt} in terms of space derivatives only. Rewriting equation (11):

$$f_t + A1 f_x = -(f_{tt} \frac{\Delta t}{2} + f_{ttt} \frac{\Delta t^2}{6} + f_{tttt} \frac{\Delta t^3}{24}) - \quad (12)$$

$$A1 (f_{xxx} \frac{\Delta x^2}{6} + f_{xxxxx} \frac{\Delta x^4}{120})$$

From the CFL condition of stability

$$\Delta t = \sigma \Delta x / a$$

$$f_t + A1 f_x = - (f_{tt} \frac{\sigma \Delta x}{2a} + f_{ttt} \frac{\sigma^2 \Delta x^2}{6a^2} + f_{tttt} \frac{\sigma^3 \Delta x^3}{24a^3}) \quad (13)$$

$$-A1 (f_{xxx} \frac{\Delta x^2}{6} + f_{xxxxx} \frac{\Delta x^4}{120})$$

The terms on the left when set to zero give exactly the differential equation for which a solution is desired. The terms on the right are the truncation error (T).

Hirt [17] suggests that the terms of T that contain even order derivatives of the function being integrated in time be considered as diffusion-like terms. The sign of these "numerical diffusion" terms can indicate instabilities. If the coefficient of the lowest even order term is negative, then "nonlinear instabilities" can occur in some area of the flow such as a contact discontinuity, shock, or expansion wave. Nonlinear instabilities are characterized by sharp spikes that grow with time and do not flip-flop with each time step. If due to some perturbation a sharp gradient is built up, the negative diffusion coefficient produces the diffusion of error in the direction of the gradient rather than away from the gradient (up-hill rather than down-hill). Thus the perturbation continues to build.

The analysis of the truncation error magnitude can show any areas where the error is a significant percent of the function. This type of analysis can show that the application of a method to a particular problem is feasible.

CHAPTER V

EXPERIMENTAL STUDIES

Back, Cuffel and Massier [3,6] summarized their own and previous investigator's data. Their summary contained nozzles with convergent half angles of 30 to 75 degrees throat contour radii of 0.25 to 2.0, and shoulder contour radii of 1.0 to 1.5. Comparing the thrust, specific impulse, and flow coefficient values with values calculated from a one dimensional nozzle of the same area ratio provides a measure of a nozzles performance. Back, et al., show that for a 75 degree nozzle the loss in thrust is between 6 and 7 percent but the loss in specific impulse is only 1 percent. The loss in thrust and specific impulse for a 45 degree nozzle is only 3 percent and 1 percent, respectively.

Back, Massier and Cuffel [2] experimentally investigated the convective heat transfer in nozzles of 10, 30, and 45 degrees throat contour radii of 2.25, 2.0, and .625, and shoulder radii of 2.0, 1.5 and 1.0, respectively. Back has shown that the reduction in convective heat transfer at the throat of a 45 degree nozzle compared to a 10 degree nozzle can be as much as 50 percent. The high acceleration of the boundary layer appears to delay the transition from laminar to turbulent at the throat which causes a lower heat transfer.

Back and Cuffel [5] investigated the shock formation aft of

the throat in the 45 degree nozzle. They pointed out that such a shock can disrupt the boundary layer and thus could increase the heat transfer at the wall. Avoidance of such a shock is possible by properly designing the transition from circular arc throat contour to conical divergent section.

CHAPTER VI

ANALYTICAL DEVELOPMENT

Boundary Conditions

The choice of the method of treating the boundary conditions can be as important as the choice of the time integration scheme as Prozan [37] points out.

Subsonic Entrance

Early investigators [22,23] utilized an inlet plane a finite distance from the throat. Moretti [33] shows that such treatment produces an ill posed or incorrect boundary condition. Numerical experiments by this and other investigators show that if the inlet plane is not at infinity and there are not enough points between the inlet and throat, disturbances will propagate upstream to the inlet and be reflected back down stream, thereby being trapped within the boundaries. Large errors can be built up by these trapped disturbances. By placing the inlet at infinity only half the problem is solved as shown below. Also, if a uniform grid spacing is used an infinite number of points are required.

If a nonsingular space transformation is used, Lapidus [22] shows that the transform does not adversely affect the solution or the problem. A transform of the form below has been used by Sheppard [52] which is a simplified version of the transform used by Laval [23].

$$Z = \frac{1 + \exp(-2/A1)}{1 + \exp(-2x/A1)} \quad (14)$$

Where X is the physical axial coordinate, Z the transformed coordinate, and A1 is a stretching parameter controlling the transform. This transform was used by this investigator and was found to produce the same errors as those produced by an inlet plane at a finite location. It was determined that at values of A1 which provided a sufficient number of grid points in the throat to provide an accurate solution in that region, the second point is too far from the inlet that it acts like an inlet plane and reflects disturbances back downstream.

Prozan [39] utilized a transform of the form

$$X = A1 \tan (\pi Z/2) \quad (15)$$

A slightly modified form was found to work satisfactorily.

$$X = A1 \tan (\pi Z1/2) \quad (16)$$

$$Z1 = [2/\pi \tan^{-1}(1/A1)+1] Z - 1 \quad (17)$$

Supersonic Exit

Since disturbances near the exit cannot propagate upstream the treatment of the supersonic exit will not effect the upstream solution.

The usual method is a simple linear extrapolation from the upstream points.

Centerline

Due to the symmetry of the problem at the centerline, the radial partial derivatives of the flow parameters vanish. Also the radial velocity vanishes at the centerline. By using a finite difference equation for the first derivative of the axial velocity with respect to the radial coordinate, and setting it equal to zero, a value of the centerline axial velocity is obtained. The forward finite difference equation which has a truncation error of one order less than the interior centered difference scheme was found to be the simplest equation which would produce a stable solution. A discussion of the finite difference techniques and their stability follows below.

The treatment of the continuity equation by the above technique produced instabilities which destroyed the solution. It was discovered that due to the transient nature of the problem the isentropic relationships for density and velocity are not valid. Also, since the flow is inviscid there are no physical reasons for the radial derivative for the density to vanish at the centerline. Therefore, a reduced form of the continuity equation, where $V = V_y = U_y = 0$, is used to integrate the density with respect to time. The term V/r in the axisymmetric equations (see equation (31)) is indeterminate but can be proven by L'Hopital's rule to vanish. When the time dependent equations reach an asymptotic solution the isentropic relationships will be valid and the solution for the density will exhibit

the vanishing radial derivatives for the density.

Wall Boundary

The physical boundary condition at the wall is satisfied if the velocity is parallel to the wall. Some investigators add additional constraints to the problem by obtaining the wall values from extrapolating the interior points with a parabolic curve fit requiring the component of velocity normal to the wall vanish. Laval [23] uses this technique which incorrectly sets a constraint on the second derivative at the wall.

Serra [49] uses a reflection technique similar to that used on centerlines. Since this technique does not model the physical situation, a more appropriate method is preferred.

Lapidus [22] used a complex system of approximating the flux of a flow variable into a cube bordering on the wall to update the flow variable. Even though the flow characteristics are considered in this approach, success has not yet been achieved.

Moretti [30, 31, 32, 34] indicated that errors generated at the wall propagate into the flow and cause instabilities in areas of the grid system where small perturbations are poorly defined, i.e., near the inlet. To eliminate the errors Moretti suggests the use of a method of characteristic approach. Since the time dependent equations are of hyperbolic nature, a characteristic approach with one time and one space dimension could be used at the wall to update the wall values. The technique involves using a Taylor series approximation of the velocity and speed of sound at the wall at time $t + \Delta t$. Using

the approximations, a characteristic line is constructed to an interior point P^* at time t . The flow values at P^* are interpolated from the surrounding grid points. The flow values at the wall at time $t + \Delta t$ are calculated by integrating the compatibility equations along the characteristic line; these new values are used to reconstruct another characteristic line. The process is repeated until the position of P^* stabilizes. Sheppard, [52] found this method used excessive execution time, did not converge in the subsonic region, and abandoned it. Since the two-dimensional method of characteristics neglects the axial variation of flow parameters this investigator felt this method would not converge in the subsonic region where axial variations become large. Utilizing the three-dimensional method of characteristics may provide an accurate solution. However, the complexity of this approach prompted a search for an alternative method.

The use of backward finite difference approximation for one sided derivatives had been used by Moretti [30] and with lack of success. However, by using a finite difference scheme with an order of accuracy greater than the interior finite difference scheme, a stable and accurate scheme can be developed. The use of a less accurate scheme can produce a negative numerical diffusion coefficient (see section on stability analysis) and therefore, an unstable method.

The backward difference scheme is used to evaluate the partial derivatives for the equations used in the time integration scheme as applied to the interior points.

Initial Conditions

Since the solution achieved from the unsteady method is the asymptotic one, the choice of initial conditions is arbitrary since the transient solutions are of little interest. Since one of the primary concerns in computer numerical analysis is the minimization of execution time, the initial conditions are chosen to be close to the expected solution. This is accomplished by solving the one-dimensional flow equations for the variables and altering the velocities to make the velocity vector parallel to the wall. The radial velocity is linearly decreased radially to zero at the center-line while maintaining the same magnitude of the total velocity vector. Figure 1 shows the initial pressure and mach number distribution used in this study.

Coordinate System

Since the problem is axisymmetric, a cylindrical coordinate system with fixed origin at the throat is used.

Because the finite difference technique is to be used a uniform grid spacing is desirable. It is also desirable to have grid points on the wall rather than having to interpolate from interior points. By means of a coordinate transformation the coordinate system in the physical plane can be mapped into a rectangular coordinate system. The radial coordinate transform is

$$y = r/r_n \quad (18)$$

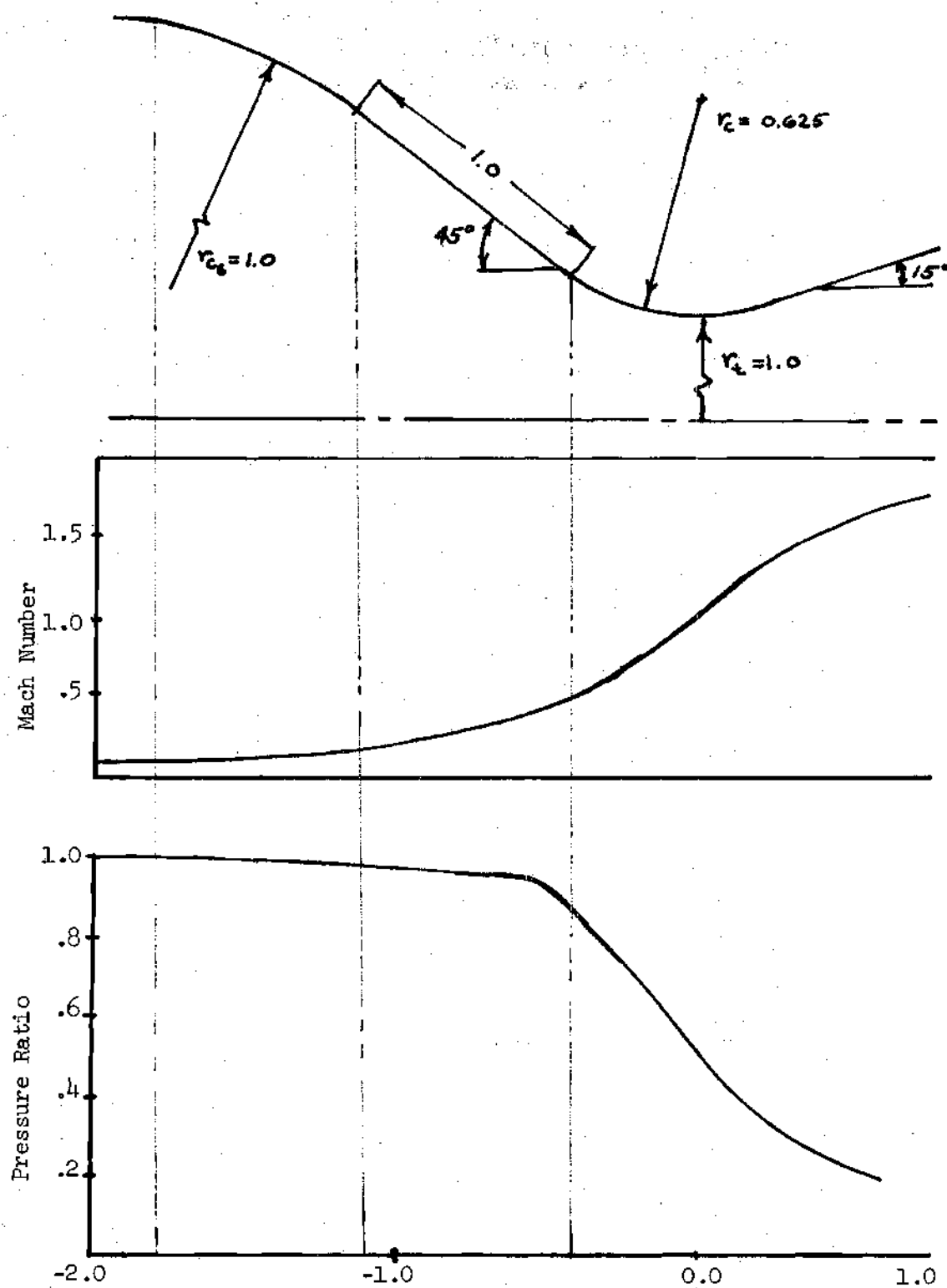


Figure 1. Initial Conditions.

To provide a subsonic inlet at infinity and to prevent extremely large storage and execution time, the axial coordinate is transformed. This transform places most of the axial grid positions in the throat region and provides sufficient axial positions between the throat and subsonic inlet to prevent too rapid a change in the physical coordinate.

$$X = A_1 \tan [\pi/2((2/\pi \tan^{-1}(1/A_1) + 1) z + 1)] \quad (19)$$

Figure 2 shows the physical coordinate system. Figure 3 shows the transformed coordinate system.

Non-Dimensionalization Procedure

To generalize the differential equations the flow parameters are non-dimensionalized. The pressure and density are non-dimensionalized by the stagnation values of p_o' . The velocities should be non-dimensionalized by the speed of sound at the inlet or $(\gamma p_o' / \rho_o')^{1/2}$, however, in the interest of simplicity $(p_o' / \rho_o')^{1/2}$ is used instead.

The non-dimensionalized parameters are as follows:

$$P = P' / p_o' \quad (20)$$

$$\rho = \rho' / \rho_o' \quad (21)$$

$$U = U' / (p_o' / \rho_o')^{1/2} \quad (22)$$

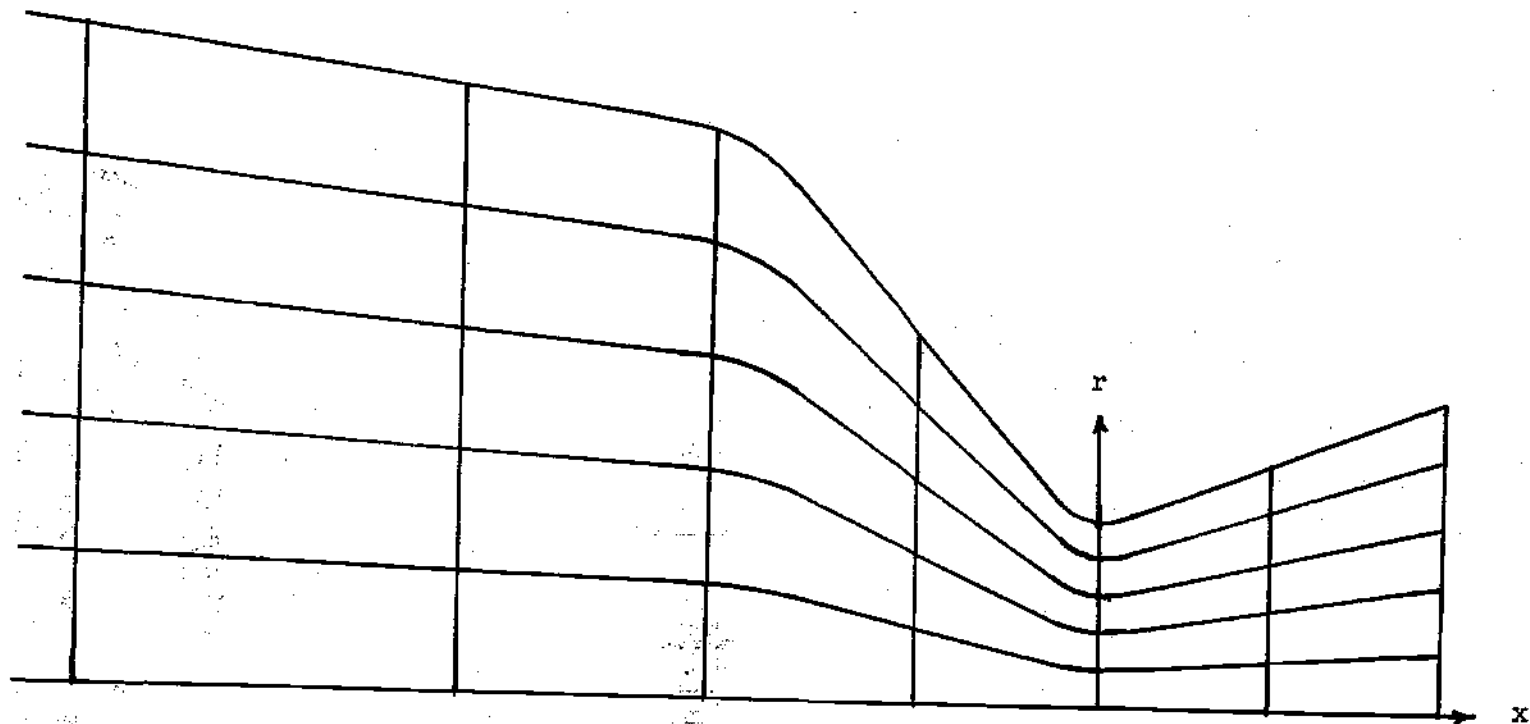


Figure 2. Physical Coordinate System.

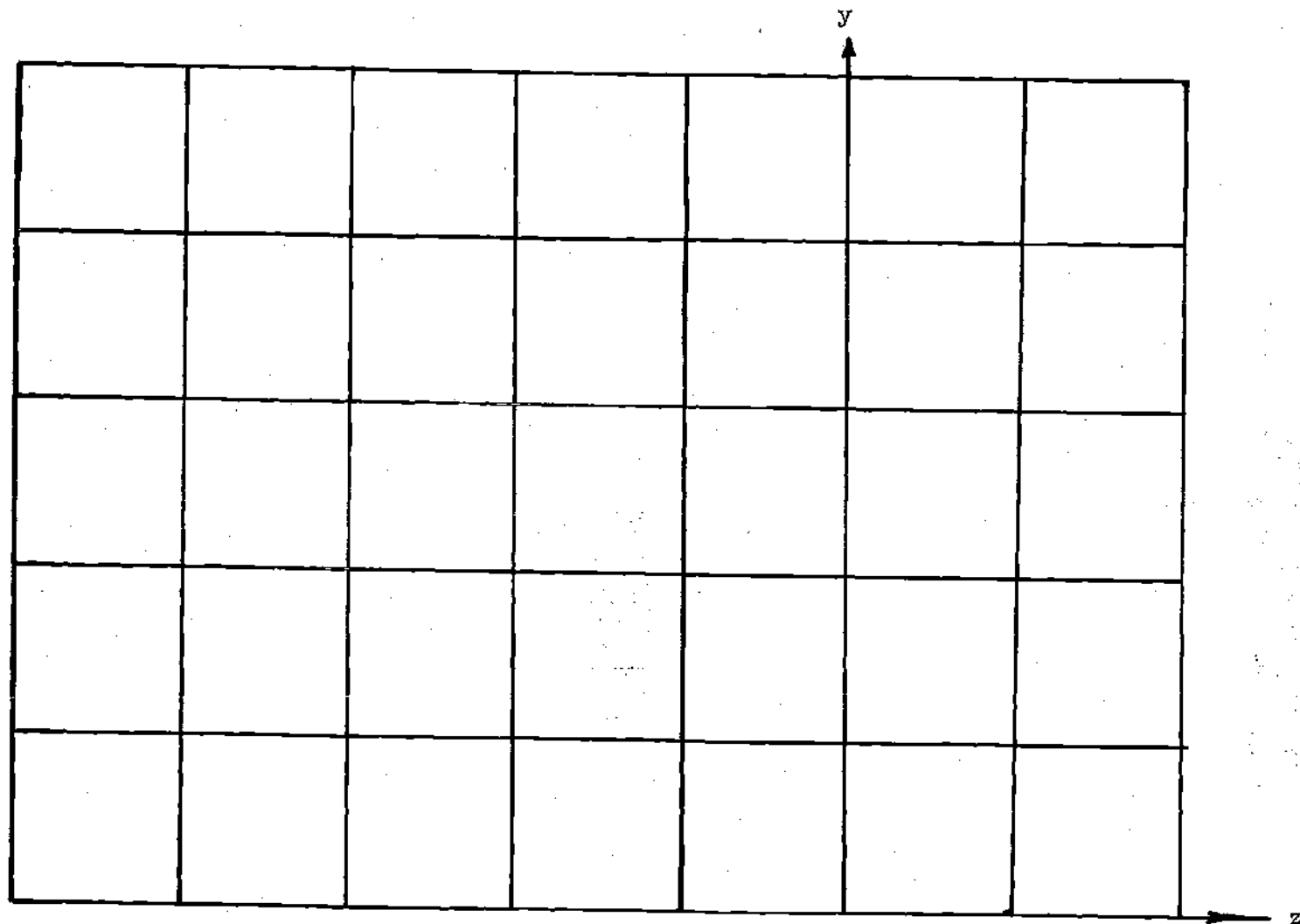


Figure 3. Transformed Coordinate System.

$$V = V' / (P_0' / \rho_0')^{\frac{1}{2}} \quad (23)$$

$$a = a' / (P_0' / \rho_0')^{\frac{1}{2}} \quad (24)$$

$$t = t' r' / (P_0' / \rho_0')^{\frac{1}{2}} \quad (25)$$

Differential Equations

The two methods of writing the time-dependent, axisymmetric differential equations for a compressible fluid dynamics are the conservational and non-conservational forms. The non-conservational form was chosen since Moretti [30] shows that this form is preferable because of ease of programming and better accuracy.

The equations necessary to define the system in the non-conservational form are below.

$$\text{Continuity Equation:} \quad D\rho/Dt + \rho \nabla \cdot \vec{V} = 0 \quad (26)$$

$$\text{Momentum Equation:} \quad \rho D\vec{V}/Dt + \nabla P = 0 \quad (27)$$

$$\text{Energy Equation:} \quad \rho Dh/Dt = DP/dt \quad (28)$$

The energy equation is used to derive an expression relating density and pressure to eliminate pressure from the continuity and momentum equations.

$$G = \left(\frac{\partial P}{\partial \rho} \right) \text{ isentropic} \quad (29)$$

$$\frac{\partial P}{\partial \xi} = G \frac{\partial \rho}{\partial \xi} \quad (30)$$

The continuity and momentum equations are expanded in cylindrical coordinates and simplified by using $R = \ln \rho$ as suggested by Moretti.

$$\text{Continuity equation: } R_t + UR_x + VR_r + U_x + V_r + \frac{V}{r} = 0 \quad (31)$$

$$\text{Axial momentum equation: } U_t + VU_r + UU_x + GR_x = 0 \quad (32)$$

$$\text{Radial momentum equation: } V_t + UV_x + VV_r + GR_r = 0 \quad (33)$$

The transformation of the coordinate system as discussed in the preceeding section is applied to the above equations. Where

$$y = \frac{r}{r_n} \quad (34)$$

$$z = (\tan^{-1}(x/A1) - \pi/2) / \tan^{-1}(1/A1) + \pi/2 \quad (35)$$

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} \frac{\partial f}{\partial z} + \frac{\partial y}{\partial x} \frac{\partial f}{\partial y} \quad (36)$$

$$\frac{\partial f}{\partial r} = \frac{\partial y}{\partial r} \frac{\partial f}{\partial y} \quad (37)$$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t} \quad (38)$$

Where

$$K = \frac{\partial z}{\partial x} \quad (39)$$

$$C = \frac{\partial y}{\partial x} = - \frac{y}{r_n} \frac{d(r_n)}{dx} \quad (40)$$

$$D = \frac{\partial y}{\partial r} = \frac{1}{r_n} \quad (41)$$

$$A = UC + VD \quad (42)$$

$$B = UK \quad (43)$$

$$E = DG \quad (44)$$

$$F = CG \quad (45)$$

$$L = GK \quad (46)$$

$$G = a^2 = (\text{Sonic Velocity})^2 \quad (47)$$

$$H = D/y = \frac{1}{r_n y} \quad (48)$$

The transformed equations after simplification with equation (34) through (48) are as follows.

Transformed continuity equation:

$$R_t + BR_z + AR_y + KU_z + CU_y + DV_y + HV = 0 \quad (49)$$

Transformed axial momentum equation

$$U_t + BU_z + AU_y + LR_z + FR_y = 0 \quad (50)$$

Transformed radial momentum equation

$$V_t + BV_z + AV_y + ER_y = 0 \quad (51)$$

Finite Difference Approximation

The choice of possible difference approximations available to the investigator is large. There are three major types of approximations, the forward (Δ), backward (∇) and centered differences (δ).

Where the form of each is:

$$\Delta_1(f) = (f_i - f_{i+1}) \quad (52)$$

$$\bar{\Delta}_1(f) = (f_{i-1} - f_i) \quad (53)$$

$$\delta_1(f) = (f_{i+1} - f_{i-1}) \quad (54)$$

Further classification of finite difference approximations involve the truncation error of the approximation, ϵ . By substituting a Taylor series expansion (see equation (1) and (2)) for each term in the finite difference approximation the order of magnitude of ϵ can be determined. For instance

$$f_x \approx \frac{\delta_1(f)}{2\Delta x} = \frac{f_{i+1} - f_{i-1}}{2\Delta x} = f_x + f_{xxx} \frac{\Delta x^2}{6} \quad (55)$$

$$+ f_{xxxxx} \frac{\Delta x^4}{120} + \dots$$

$$\epsilon_{\delta_1} = f_{xxx} \frac{\Delta x^2}{6} + f_{xxxxx} \frac{\Delta x^4}{120} + \dots \quad (56)$$

or

$$\epsilon_{\delta_1} = \text{Order of } \Delta x^2 \quad (57)$$

and

$$\epsilon_{\bar{\Delta}_1} = -\epsilon_{\Delta} = \text{Order of } \Delta x \quad (58)$$

To avoid errors being introduced at the boundaries by large truncation errors and negative "numerical diffusion," more accurate backward and forward difference must be derived.

Hildebrand [15] contains the general method of derivation of higher order approximations of Δ and $\bar{\Delta}$. Defining the shifter operator (E) and the derivative operator (D) enables the derivation of a series expansion of D in terms of Δ or $\bar{\Delta}$. This series can be utilized to derive higher order approximations and approximations for higher order derivatives as shown in Table 3.

$$E f_i = f_{i+1} \quad (59)$$

$$E^{-1} f_i = f_{i-1} \quad (60)$$

$$D f_i = \left(\frac{\partial f}{\partial x} \right)_{x=x_i} \quad (61)$$

$$\Delta = E - 1 \text{ or } E = 1 + \Delta \quad (62)$$

$$\bar{\Delta} = 1 - E^{-1} \quad (63)$$

$$\delta = E^{\frac{1}{2}} - E^{-\frac{1}{2}} \quad (64)$$

$$E^k = (1 + \Delta)^k = 1 + k\Delta + \frac{k(k-1)}{2} \Delta^2 + \dots \quad (65)$$

Table 3. Summary of Difference Approximations Investigated.

Approximation		
$\bar{\Delta}$:		ϵ :
1st	$f_z = (f_i - f_{i-1})/\Delta Z$	$-f_{zz} \frac{\Delta Z}{2} + f_{zzz} \frac{\Delta Z^2}{6}$
	$f_{zz} = (f_i - 2f_{i-1} + f_{i-2})/\Delta Z^2$	$-f_{zzz} \Delta Z + \frac{9}{12} f_{zzzz} \Delta Z^2$
2nd	$f_z = (3f_i - 4f_{i-1} + f_{i-2})/2\Delta Z$	$-\frac{2}{3} f_{zzz} \Delta Z^2 + f_{zzzz} \frac{\Delta Z^3}{2}$
	$f_{zz} = (2f_i - 5f_{i-1} + 4f_{i-2} - f_{i-3})/\Delta Z^2$	$-f_{zzzz} \frac{\Delta Z^2}{2} + f_{zzzzz} \frac{\Delta Z^3}{10}$
3rd	$f_z = (\frac{11}{6} f_i - 3f_{i-1} + \frac{3}{2} f_{i-2} - \frac{1}{3} f_{i-3})/\Delta Z$	$-f_{zzzz} \frac{\Delta Z^3}{4} + \frac{3}{10} f_{zzzzz} \Delta Z^4$
	$f_{zz} = (35f_i - 104f_{i-1} + 114f_{i-2} - 56f_{i-3} + 11f_{i-4})/24\Delta Z^2$	$-\frac{5}{12} f_{zzzzz} \Delta Z^3 + \frac{2}{3} f_{zzzzzz} \Delta Z^4$

For Δ replace all $i+n$ with $i-n$ and $\Delta Z = -\Delta Z$ in the above equations. The same truncation errors apply except $\epsilon_{\Delta} = -\epsilon_{\bar{\Delta}}$.

Table 3. (Continued)

Approximation	
$\delta:$	
1st	$f_z = (f_{i+1} - f_{i-1})/2\Delta z$ $f_{zz} = (f_{i+1} - 2f_i + f_{i-1})/4\Delta z^2$
2nd	$f_z = [8(f_{i+1} - f_{i-1}) - (f_{i+2} - f_{i-2})]/12\Delta z$ $f_{zz} = [16(f_{i+1} + f_{i-1}) - (f_{i+2} + f_{i-2}) - 30f_i]/12\Delta z^2$
	$f_{zzz} \frac{\Delta z^2}{6} + f_{zzzz} \frac{\Delta z^4}{120}$ $f_{zzzz} \frac{\Delta z^2}{12} + f_{zzzzz} \frac{\Delta z^4}{720}$ $- \frac{1}{30} \frac{\partial^5 f}{\partial z^5} \Delta z^4 - \frac{1}{252} \frac{\partial^7 f}{\partial z^7} \Delta z^6$ $- .013 \frac{\partial^6 f}{\partial z^6} \Delta z^4 - .0041 \frac{\partial^8 f}{\partial z^8} \Delta z^6$

From the Taylor series expansion

$$\begin{aligned}
 f_{i+1} &= f_i + f_x \Delta x + f_{xx} \frac{\Delta x^2}{2} + \dots \\
 &= f_i + \Delta x D f_i + \frac{\Delta x^2}{2!} D^2 f_i + \frac{\Delta x^3}{3!} D^3 f_i \\
 &= (1 + \Delta x D + \frac{(\Delta x D)^2}{2} + \dots + \frac{(\Delta x D)^k}{k!} + \dots) f_i \\
 &= e^{\Delta x D} f_i
 \end{aligned} \tag{66}$$

$$E f_i = f_{(x+\Delta x)} = f_{i+1} = e^D f_i \tag{67}$$

or

$$E = e^{\Delta x D} \tag{68}$$

$$\Delta x D = \ln E \tag{69}$$

$$\ln E = \ln(1+\Delta) = \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots + (-1)^k \frac{\Delta^k}{k} \tag{70}$$

Thus

$$\Delta x D f_i = (\Delta - \frac{1}{2} \Delta^2 + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots) f_i \tag{71}$$

or

$$Df_1 = \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \dots + (-1)^k \frac{\Delta^k}{k} \right) \frac{f_1}{\Delta x} \quad (72)$$

also

$$D^2 f_1 = D(D(f_1)) = \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} \right) \left(\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} \right) f_1 \quad (73)$$

Thus the finite difference approximation can be derived for Δ and $\bar{\Delta}$ to any desired order of accuracy by including more terms in the series in equation (72).

In order to derive higher order centered difference approximations, the Taylor series expansion for points adjacent to the point of interest must be algebraically manipulated to obtain the scheme with the required accuracy.

Two dimensional problems require the approximation of cross derivatives or derivatives with respect to two space variables. Again, reverting to operator notation these approximations to any order can be derived.

$$f_{(x,y)} = f_{1,j} \quad (74)$$

$$\delta_y(f) = (f_{0,1} - f_{0,-1}) \quad (75)$$

$$\delta_x(f) = (f_{1,0} - f_{-1,0}) \quad (76)$$

$$f_x = \frac{\partial f(x,y)}{\partial x} = \frac{\delta_x(f)}{2\Delta x} \quad (77)$$

$$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = \frac{\delta_y}{2\Delta y} \left(\frac{\partial f(x,y)}{\partial x} \right) = \frac{\delta_y(\delta_x(f))}{4\Delta x \Delta y} \quad (78)$$

or

$$f_{xy} = \frac{\partial^2 f(x,y)}{\partial x \partial y} = (f_{1,1} - f_{-1,1} - f_{+1,-1} + f_{-1,-1})/4\Delta x \Delta y \quad (79)$$

At the wall and centerline the cross derivatives can be derived by the application of δ and $\bar{\Delta}$ or Δ . The order of application makes no difference to the finite difference equations or

$$\delta(\bar{\Delta}(f)) = \bar{\Delta}(\delta(f)) \quad (80)$$

The operator Δ behaves in the same manner.

$$\delta(\bar{\Delta}(f)) = \left(\frac{11}{6}\right) (f_{1,0} - f_{-1,0}) - 3(f_{1,-1} - f_{-1,-1}) \quad (81)$$

$$+ \frac{3}{2} (f_{1,-2} - f_{-1,-2}) - \frac{1}{3} (f_{1,-3} - f_{-1,-3})/2\Delta z \Delta y$$

$$\epsilon_{\delta\bar{\Delta}} = \frac{2}{3} f_{yyyy} \Delta y^2 + \frac{2}{9} f_{zzzy} \Delta z^2 \quad (82)$$

For determination of the simplest scheme to use the stability analysis for the problem must be performed. The choice of higher order approximation can sometime improve both the stability and

truncation error of a scheme.

Numerical Techniques

The two numerical techniques compared in this study are the first order Rusanov and the second order Moretti method. These two schemes were chosen because of their proven superiority in references [30] and [53]. The Moretti method was examined since it was hoped that the higher order of Taylor series utilized would produce less error.

The Moretti method utilizes a second order Taylor series time integration scheme as shown below.

$$f_{i,j}^{n+1} = f_{i,j}^n + f_t \Delta t + f_{tt} \frac{\Delta t^2}{2} \quad (83)$$

The equation (83) is applied to R, U, and V where

$$R = \ln \rho \quad (84)$$

The first time derivatives of R, U, and V are evaluated from equations (89), (90), and (91). These equations for the first derivative become equations (92) through (105). When differentiated again with respect with time or either space coordinate. These equations can be algebraically combined and substituted into the time integration scheme equation (83) to produce equations for the advance time flow parameters in terms of spatial derivatives only.

The first order Rusanov method requires only the first time derivatives as shown below.

$$f_{i,j}^{n+1} = \bar{f}_{i,j}^n + f_t \Delta t \quad (85)$$

$$\bar{f}_{i,j}^n = f_{i,j}^n + \frac{\Delta z^2}{2} \frac{\partial}{\partial z} (\alpha_{i,j} f_z) + \frac{\Delta y^2}{2} \frac{\partial}{\partial y} (\rho_{i,j} f_y) \quad (86)$$

$$\alpha_{i,j} = \frac{\sigma \omega \Delta y^2}{(\Delta y^2 + \Delta z^2)} \frac{(HX + a)_{i,j}}{(HX + a)_{\max}} \quad (87)$$

$$\rho_{i,j} = \frac{\sigma \omega \Delta z^2}{(\Delta y^2 + \Delta z^2)} \frac{(HX + a)_{i,j}}{(HX + a)_{\max}} \quad (88)$$

$$R_t = - (BR_z + AR_y + KU_z + CU_y + DV_y + HV) \quad (89)$$

$$U_t = - (BU_z + AU_y + LR_z + FR_y) \quad (90)$$

$$V_t = - (BV_z + AV_y + ER_y) \quad (91)$$

$$R_{tt} = - [BV_{zt} + B_t R_y + AR_{zt} + A_t R_y + KU_{zt} + CU_{yt} + DV_{yt} + HV_t] \quad (92)$$

$$U_{tt} = - [BU_{zt} + B_t U_z + AU_{yt} + A_t U_y + LR_{zt} + L_t R_z + FR_{yt} + F_t R_y] \quad (93)$$

$$V_{tt} = -[BV_{zt} + B_t V_z + AV_{yt} + A_t V_y + ER_{yt} + E_t R_y] \quad (94)$$

$$R_{zt} = -[BR_{zz} + B_z R_z + AR_{yz} + A_z R_y + KU_{zz} + K_z U_z + CU_{yz} + \quad (95)$$

$$+ C_z U_y + D_z B_y + HV_z + H_z V]$$

$$R_{yt} = -[BR_{zy} + B_y R_z + AR_{yy} + A_y R_y + KV_{zy} + CU_{yy} + C_y U_y + \quad (96)$$

$$+ DV_{yy} + HV_y + H_y V]$$

$$U_{zt} = -[BU_{zz} + B_z U_z + AU_{yz} + A_z U_y + LR_{zz} + L_z R_z + FR_{yz} + F_z R_y] \quad (97)$$

$$U_{yt} = -[BU_{zy} + B_y U_z + AU_{yy} + A_y U_y + LR_{zy} + L_y R_z + FR_{yy} + F_y R_y] \quad (98)$$

$$V_{yt} = -[BV_{zy} + B_y V_z + AV_{yy} + A_y V_y + ER_{yy} + E_y R_y] \quad (99)$$

$$A_t = [U_t C + V_t D] \quad (100)$$

$$B_t = U_t K \quad (101)$$

$$E_t = DG_t \quad (102)$$

$$F_t = CG_t \quad (103)$$

$$G_t = \gamma(\gamma-1) R_t \quad (104)$$

$$L_t = KG_t \quad (105)$$

Stability Analysis

The expressions for the truncation error of the R time integration scheme of both the Moretti and Rusanov methods following the analysis of Hirt [17] and Tyler [54] are presented in equations (106) and (110).

For the Moretti method.

$$\begin{aligned} T_R = & -\frac{\Delta z^2}{6} [B R_{zzz} + KU_{zzz} + \eta^2 (AR_{yyy} + CU_{yyy} + DV_{yyy})] \quad (106) \\ & - \sigma^2 N^2 R_{ttt} - \sigma^3 N^3 R_{tttt} \\ & - \frac{\Delta z^4}{120} [B R_{zzzzz} + KU_{zzzzz} + \eta^4 (AR_{yyyyy} + CU_{yyyyy} + DV_{yyyyy})] \\ & - \sigma^4 N^4 R_{ttttt} + \text{Order of } \Delta z^5 \end{aligned}$$

Where

$$N = \frac{\Delta r \Delta x}{2(\Delta r^2 + \Delta x^2)^{\frac{1}{2}}(HX + a)_{\max}} \quad (107)$$

$$Q = \frac{\Delta Z^2 \Delta Y^2 (\Delta r^2 + \Delta x^2)^{\frac{1}{2}}}{2 \Delta r \Delta x (\Delta Z^2 + \Delta Y^2)} \quad (108)$$

$$\eta = \Delta Y / \Delta Z \quad (109)$$

Also, Δr is the difference between radial grid points at N_{\max} and Δx the difference between axial grid points at N_{\max} .

For Rusanov method

$$T_R = R_{zz} [\omega Q (HX + a) - \sigma N K^2 (U^2 + a^2)] + \quad (110)$$

$$+ R_{yy} [\omega Q (HX + a) - \sigma N (A^2 + C^2 a + D^2 a)] +$$

$$+ R_y [\omega Q (HX + a)_y - \sigma N (BA_z - A_t - B_t + HE + AA_y + KF_z +$$

$$+ CF_y + DE_y)] +$$

$$- R_z [\omega Q (HZ + a)_z - \sigma N (BB_z + AB_y + KL_z + L_y c)] -$$

$$- \sigma N [R_{yz} (2BA + KF + CL) + U_{zz} (2BK) + U_{yy} (2AC) + V_y (2AD) +$$

$$+ V_{yz} (2BD) + U_{yz} (2BC + 2AK) + U_y (BD_z + AC_y + CA_y +$$

$$+ KA_z + DA_y)] +$$

$$\begin{aligned}
& + U_z (BK_z + KB_z + CB_y) + V_y (BD_z + 2HA + DA_y) + \\
& + V_z (2HB + DB_y) + V(BH_z + AH_y)] - \\
& - \frac{\Delta z^2}{6} [BR_{zzz} + KU_{zzz} + n^2(AR_{yyy} + CU_{yyy} + DV_{yyy})] - \sigma^2 N^2 R_{ttt} + \\
& + \frac{\omega Q}{6} \Delta z^2 [(HX+a)_z R_{zzz} + (HX+a)_{zzz} R_z + (HX+a) \frac{R_{zzzz}}{4}] + \\
& + \frac{\omega Q \eta^2}{6} \Delta z^4 [(HX+a)_y R_{yyy} + (HX+a)_{yyy} R_y + (HX+a) \frac{R_{yyyy}}{2}] + \\
& - \frac{\sigma}{6} N^3 R_{tttt} + \frac{\Delta z^4}{120} [BR_{zzzzz} + KU_{zzzzz} + \\
& + \eta^4 (AR_{yyyyy} + CU_{yyyyy} + DV_{yyyyy}) + \\
& + \frac{\sigma N}{120} R_{ttttt} + \text{Order of } \Delta z^5.
\end{aligned}$$

Similar expressions for U and V can be derived and used in the complete stability analysis. These equations have the same form and have been omitted for simplicity.

The Rusanov method is a first order method, therefore, the truncation error for this method has some additional lower order terms when compared to the second order Moretti method. However, the numerical diffusion terms for the Rusanov method are contained in the

lower order terms. Thus, the Rusanov method has a numerical diffusion of larger magnitude than the Moretti method and positive where the Moretti diffusion is negative. The R_{zz} and R_{yy} terms for the Rusanov and R_{zzzz} and R_{yyyy} terms for the Moretti are the diffusion-like terms under discussion. The R_{zzzz} and R_{yyyy} terms come from the R_{tttt} term.

Since both methods in this study made use of the same centered differences portions of the truncation error terms are identical for both methods. The most important of these terms is the first set of terms in the Moretti truncation error or the triple spatial derivatives which come from the truncation error, ϵ , of the centered difference.

CHAPTER VII

RESULTS AND CONCLUSIONS

To evaluate the performance of both the Moretti and Rusanov methods this investigator performed the analysis of a rapidly converging nozzle. The nozzle consisted of a small angle conical convergent inlet section, which extended to infinity upstream, a shoulder radius, a large angle conical convergent section, a circular arc throat contour, and a moderate angle conical divergent section. Back, et al. [1,2,3,4,5,6] have performed experimental studies on this and similar nozzles. Figure 1 contains a sketch of the nozzle investigated and Figures 4 and 5 contain a summary of Back's data.

In applying the Moretti method to the nozzle in Figure 1, it was found that a finite difference approximation in the radial direction of one order of accuracy higher than the centered difference approximation (i. e., $\bar{\Delta}_3$ in Table 3) was required to maintain stable computations. If the truncation errors for the Moretti method using the centered difference δ_1 (see equation (106)) $\bar{\Delta}_2$, or $\bar{\Delta}_3$ are compared, the instabilities experienced, that rapidly destroy the solution, can be predicted.

In equation (106), the group of terms containing the triple spatial derivatives and multiplied by ΔZ^2 are the results of the finite approximation and thus are the terms that will change with the use of a different finite difference approximation. Using δ_1 in the

axial and $\bar{\Delta}_2$ or $\bar{\Delta}_3$ in the radial direction results in equation (111) for $\bar{\Delta}_2$ and (112) for $\bar{\Delta}_3$.

$$- \frac{\Delta Z^2}{6} [BR_{zzz} + KU_{zzz} - 4\eta^2 (AR_{yyy} + CU_{yyy} + DV_{yyy})] \quad (111)$$

$$- \frac{\Delta Z^2}{6} [BR_{zzz} + KU_{zzz}] + \frac{\Delta Z^3 \eta^3}{4} [AR_{yyy} + CU_{yyy} + DV_{yyy}] \quad (112)$$

The instabilities that destroyed the solution at the start were exhibited in the shoulder radius region, where the triple radial derivatives (i. e., R_{yyy} and U_{yyy}) are of the same order of magnitude as the triple axial derivatives (R_{zzz}). η was equal to five for the grid system determined most accurate by numerical experiments. Thus, the R_{yyy} and like derivatives are the most important terms for both δ_1 and $\bar{\Delta}_2$. The factor of minus four in equation (111) is sufficiently large to cause the computational instabilities found.

By using a $\bar{\Delta}_3$ approximation in the radial direction, the third order radial derivative terms are reduced to fourth order derivatives and their multiplier to ΔZ^3 (see equation (112)). The magnitude of the terms changes by using $\bar{\Delta}_3$ are less than or equal to the third order derivative terms in equation (106). Thus, the instabilities found earlier were avoided.

Using the $\bar{\Delta}_3$ approximation for the radial derivatives at the wall and centerline, the Moretti method provided very good results for the transonic and supersonic regions of the flow. Large errors

developed in the shoulder radius area and could grow in such a manner as to destroy the solution for the whole flow field after long computational times. Figures 6 and 7 show the growth of this error as time progresses. Figures 8 and 9 show the results of the Moretti method compared to the experimental results after the throat computations have stabilized (see Figure 16) and before the long term errors can destroy the solution.

These errors in the shoulder radius were found to be the results of the large axial variations in the flow parameters. In the initial conditions, these axial variations were largest in the shoulder radius area.

Because of the flexibility and the positive diffusion of the Rusanov method, this method was also applied to the nozzle in Figure 1. Figures 11 and 12 show the growth of errors in the Rusanov solution as time progresses. The Rusanov method did not perform as well as the Moretti method as shown by the lack of agreement with experimental data in Figures 13 and 14.

A truncation error order of magnitude calculation was performed for both methods to investigate the cause of errors in the solutions. The analysis was performed by calculating the terms of the truncation error expressions which were of the order of the Δy^3 term or lower. The major portion of the truncation error was retained by this approximation. Figures 10 and 15 present the percent error of R per time step caused by the truncation error at the initial conditions. The truncation error per time step of the Rusanov method (Figure 15) shows the reason for this method's lack of agreement with experimental data.

Because the Rusanov is a first order method, not only is this error much larger for the Rusanov method but it also remains large over most of the flow field while the Moretti method's error, which is a second order method, is large in only the shoulder radius area. The lack of agreement of the Moretti method in the shoulder radius section only, is predicted by the truncation error analysis very neatly.

The artificial diffusion or viscosity of the Rusanov method did not appear to help in this problem. If the Rusanov and Courant variables are adjusted to yield a lower truncation error for the Rusanov method, the coefficients of the numerical diffusion terms become very small or negative. The solution is then destroyed by the resultant constantly growing errors in the shoulder radius area. Thus, the flexibility of the Rusanov method is for naught.

The most general method of reducing the truncation error is to reduce the grid size. Since the truncation error of the Moretti method is proportional to ΔZ^2 , reducing the truncation error to 1 percent of the present value requires the reduction of ΔZ to one-tenth. If ΔZ were reduced the time step size must also be reduced in order to remain within the CFL stability condition. Thus, to calculate the time dependent solution to the same physical time as present, ten times as many time steps must be taken and ten times as many points must be considered. From numerical experiments, both the Rusanov and Moretti methods require approximately one second per each time step with 556 points. With the present methods approximately 300 time steps are required for the calculations to converge. Thus, a reduction of ΔZ

by ten would require 3×10^5 seconds for convergence. Most computing facilities will not even consider this amount of time or one case.

Another possible method of reducing the truncation error would be to use a more accurate centered finite difference. The resulting form of the Moretti method should reduce the truncation error by one order of magnitude without increasing the execution time significantly. However, numerical experiments with the higher order centered differences were destroyed by instabilities of an undetermined cause. Moretti [30] also reported similar problems and discarded this approach.

After much evaluation, neither the Rusanov or the Moretti method in the present form was capable of providing an adequate solution to the subsonic portion of the nozzle flow field. Previous investigators also show good results for the transonic region but present very little data about the subsonic region.

It was felt that, since the Moretti method was, in general, more accurate, some modified form might provide a good solution for the subsonic as well as transonic and supersonic regions. Isolating the cause of the errors of the higher order schemes could show the direction future investigators should follow.

In addition, it was noted that the point of maximum truncation error for both methods corresponded to points at which the second derivative of the nozzle wall radius with respect to the axial coordinate was discontinuous. Therefore, some form of modified nozzle wall or some method of smoothing the second derivative may help to reduce the truncation error in the shoulder radius area. A more

accurate finite difference approximation (i.e. a five point finite difference), a stretching function optimized to minimize the truncation error throughout the flow field, and utilization of wall contours with continuous derivatives might enable the Moretti method to provide an adequate solution.

Many finite difference studies fail to include a truncation error and stability analysis and failures of the techniques are contributed to non-linear instabilities which cannot be analyzed. This study concludes that many of these failures can be predicted by stability and truncation error analysis. The author strongly recommends this be carried out before attempting to obtain numerical solutions.

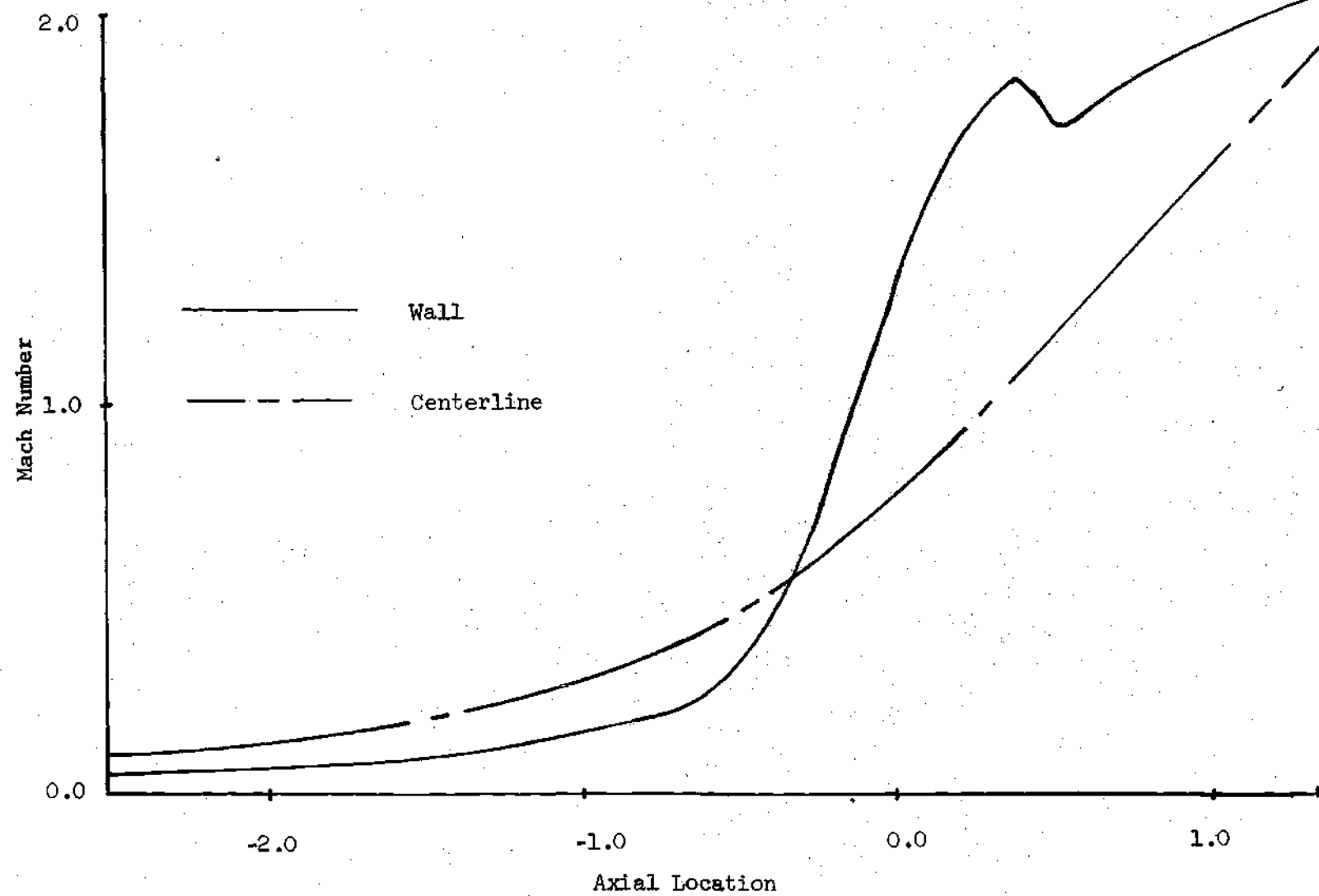


Figure 4. Experimental Mach Number.

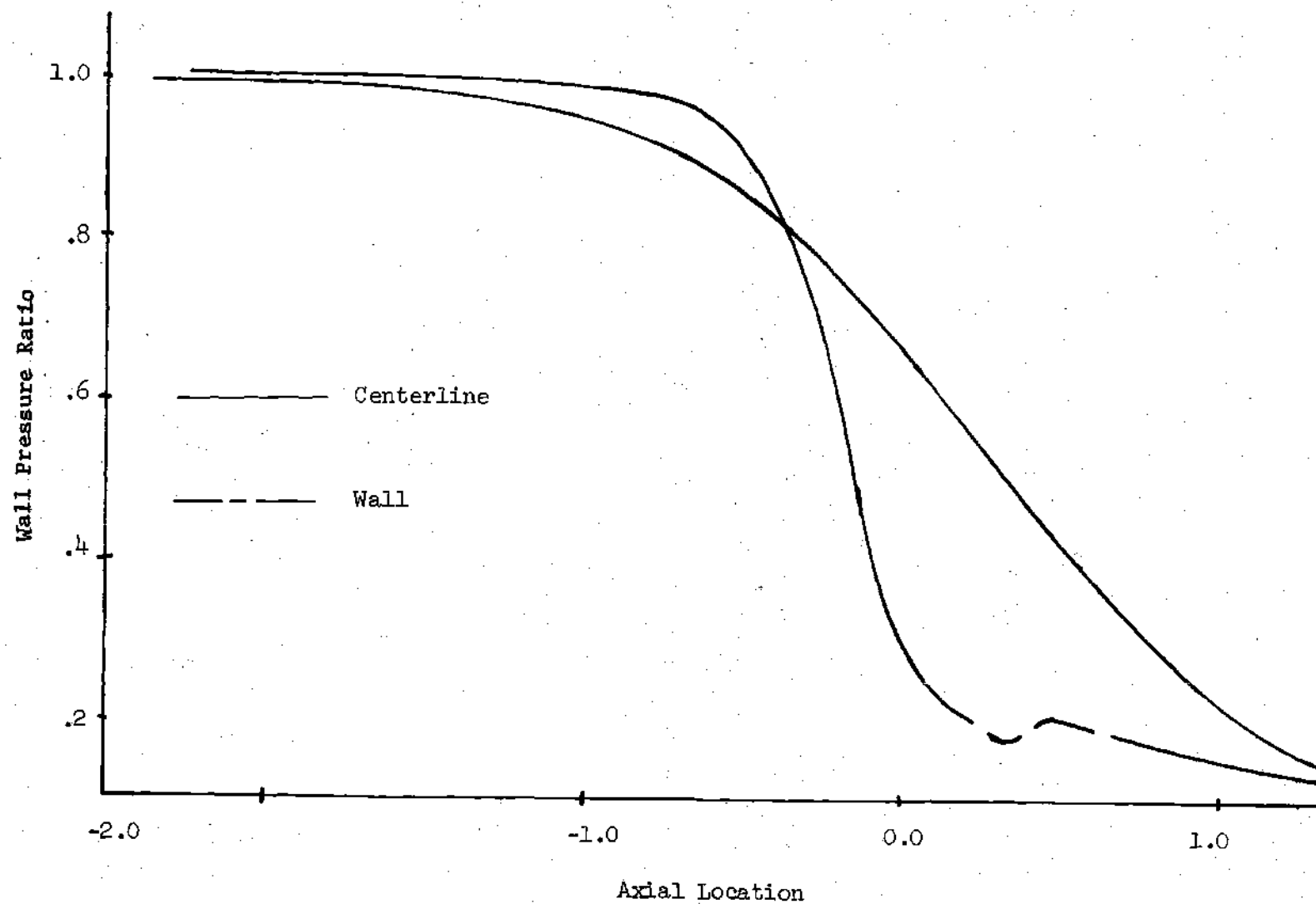


Figure 5. Experimental Pressure Ratio.

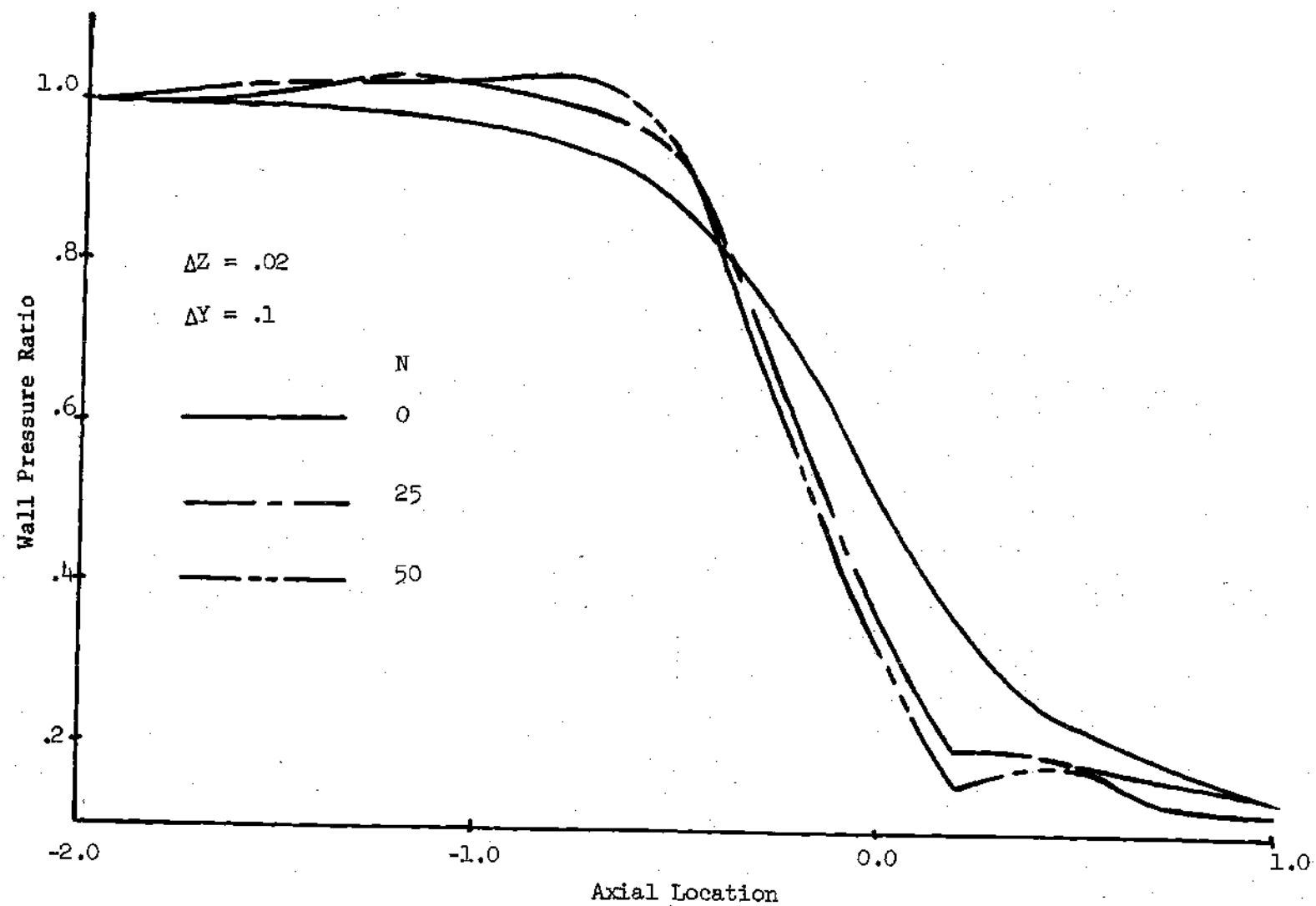


Figure 6. Pressure Ratio - Moretti Method.

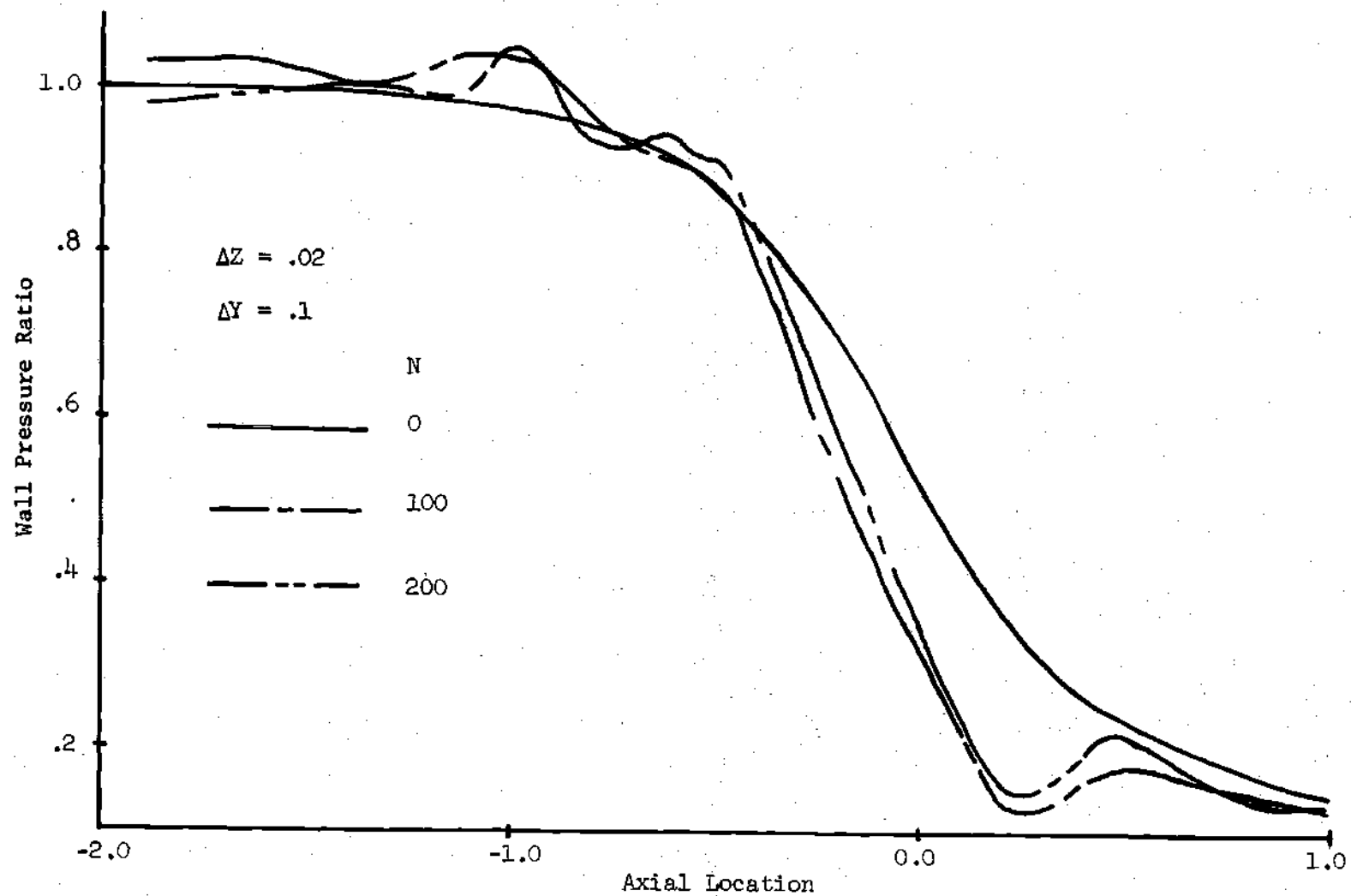


Figure 7. Pressure Ratio - Moretti Method.

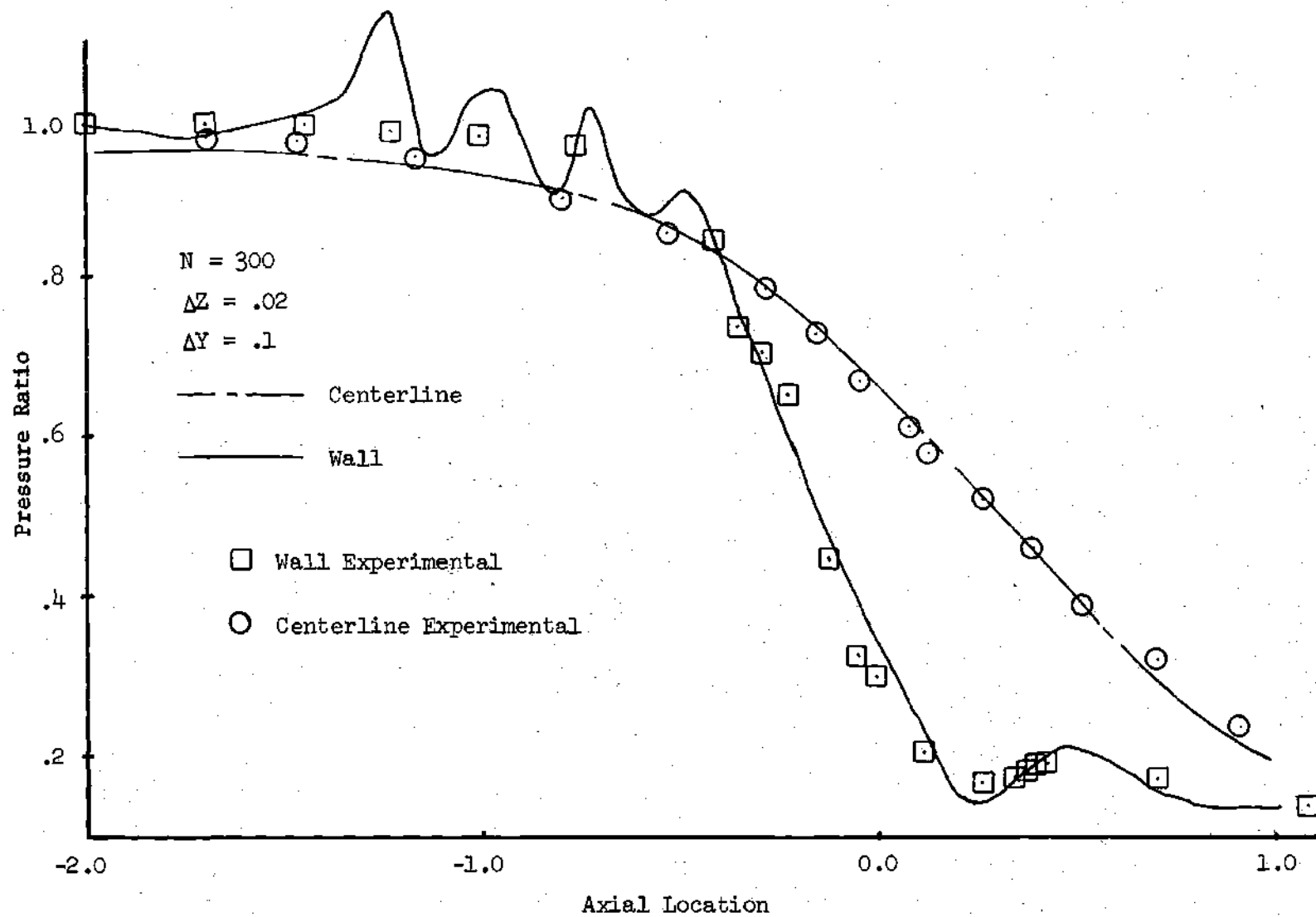


Figure 8. Pressure Ratio - Moretti Method.

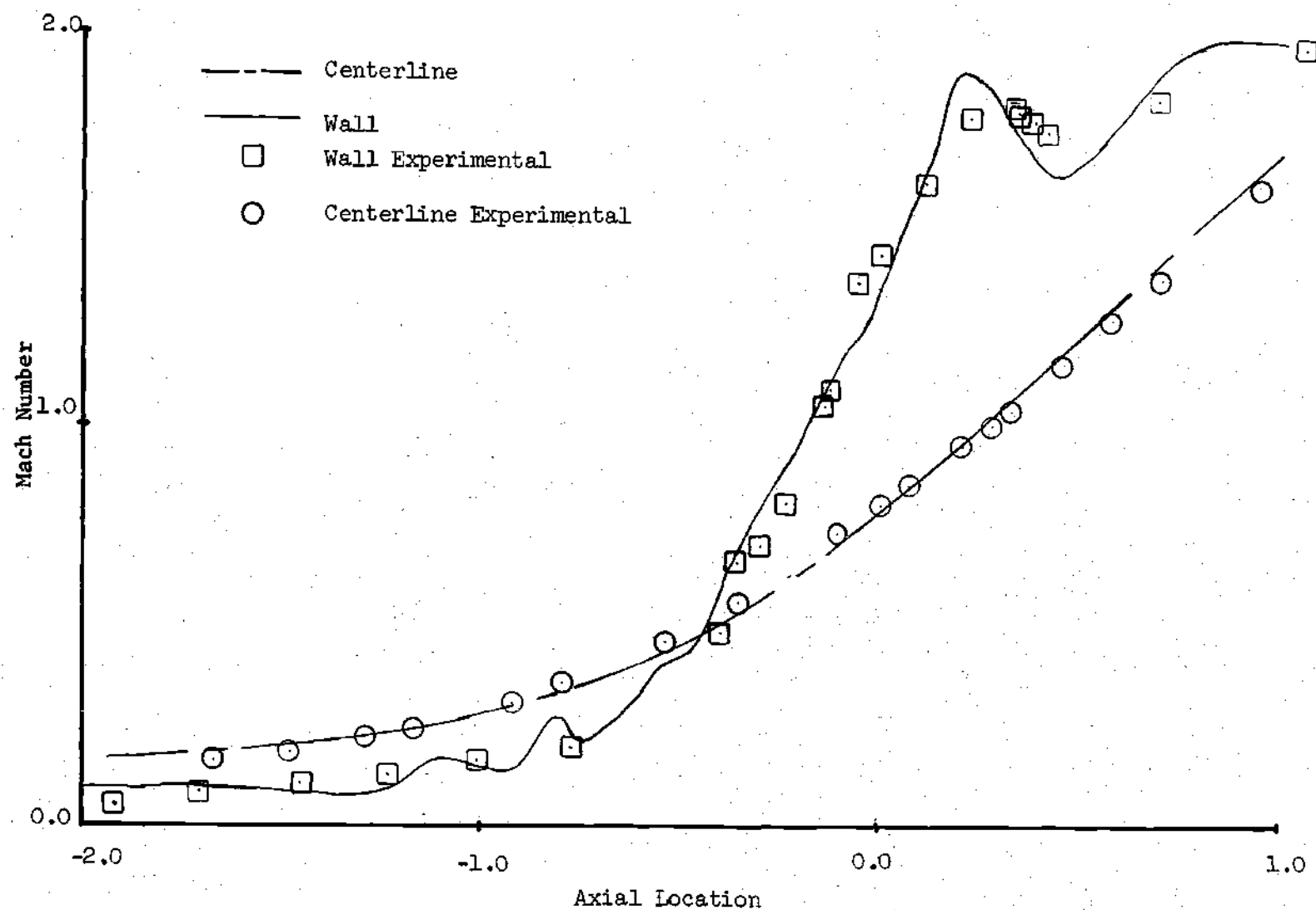


Figure 9. Mach Number - Moretti Method.

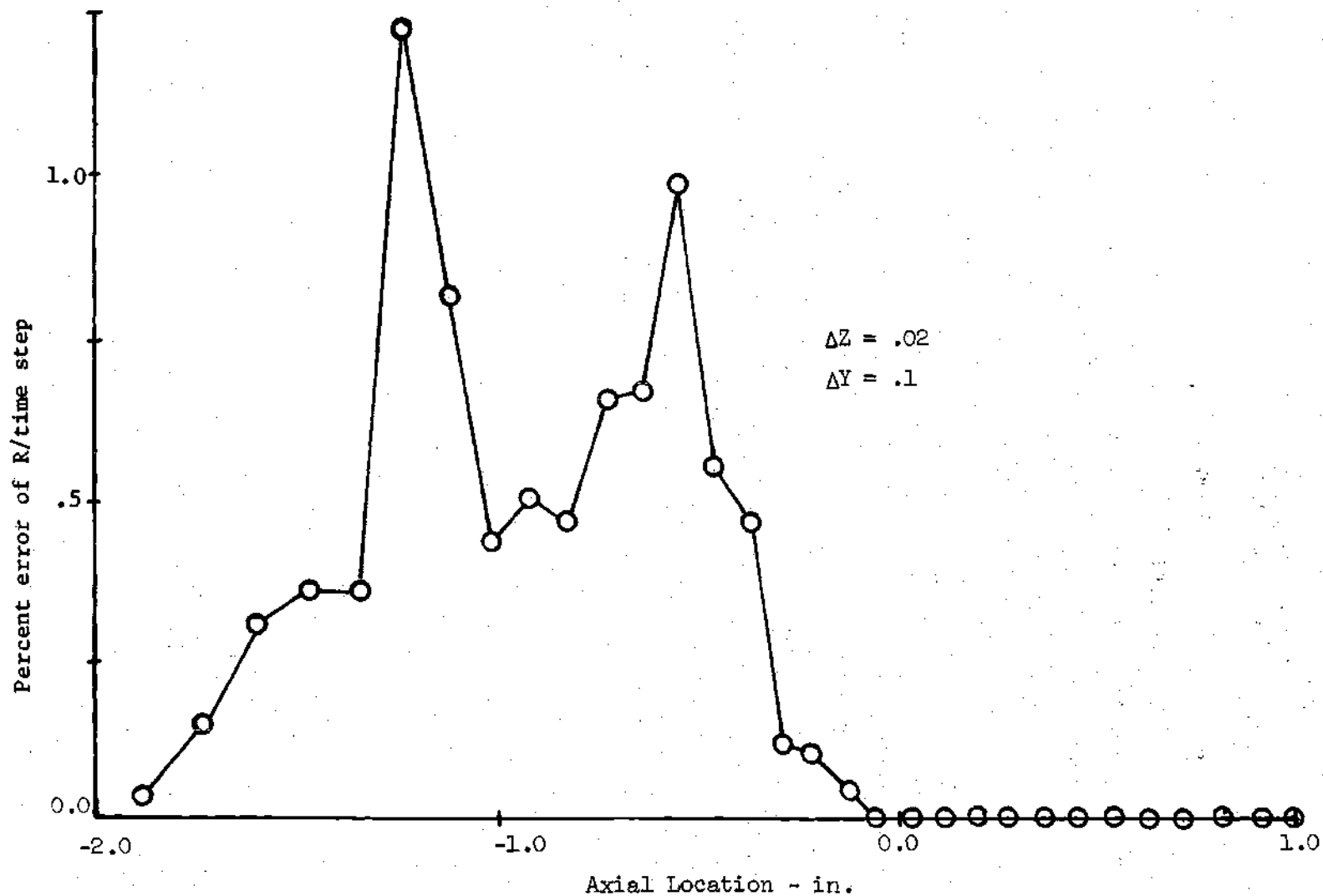


Figure 10. Percent Error of R Per Time Step - Moretti Method.

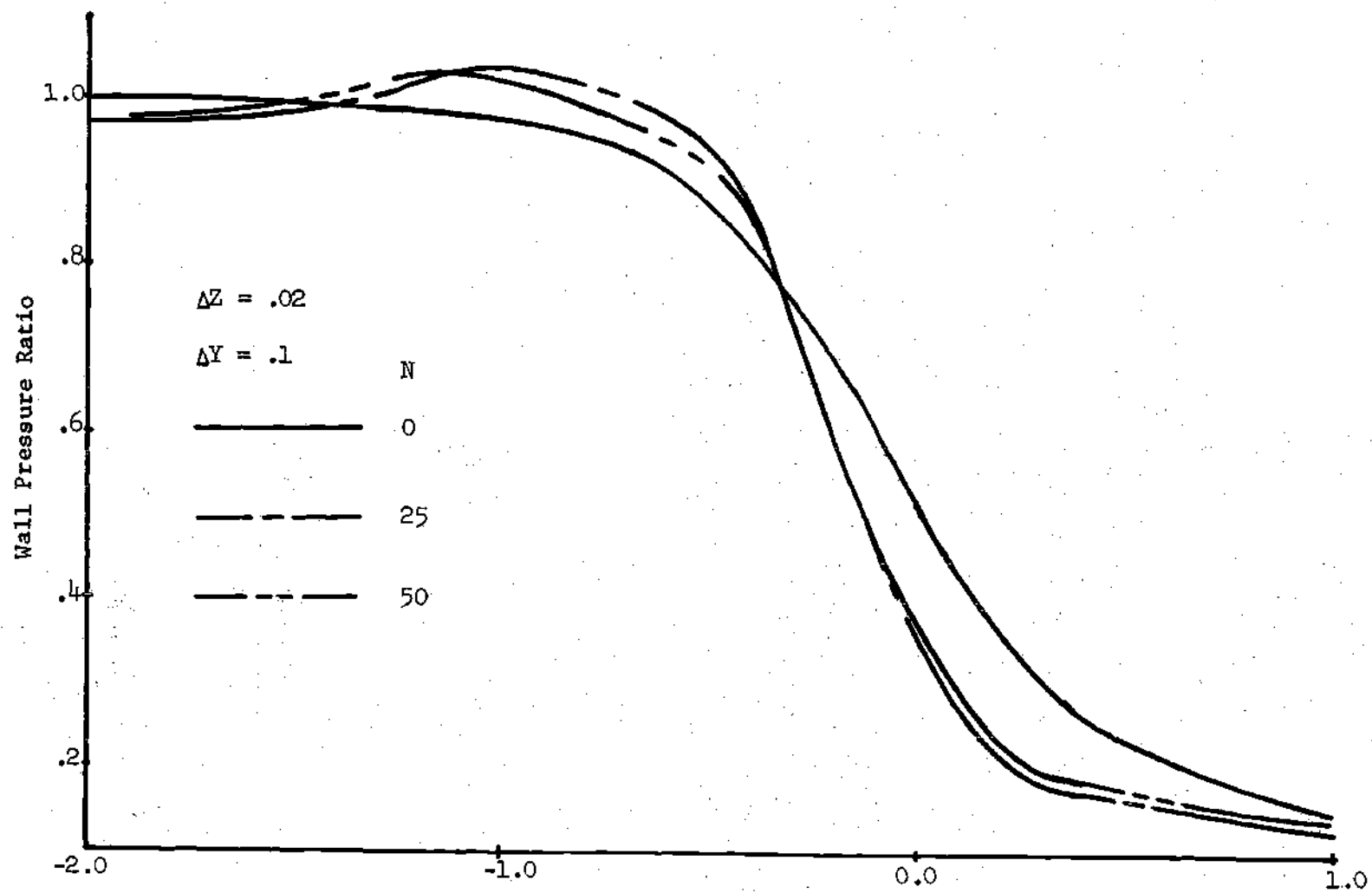


Figure 11. Pressure Ratio - Rusanov Method.

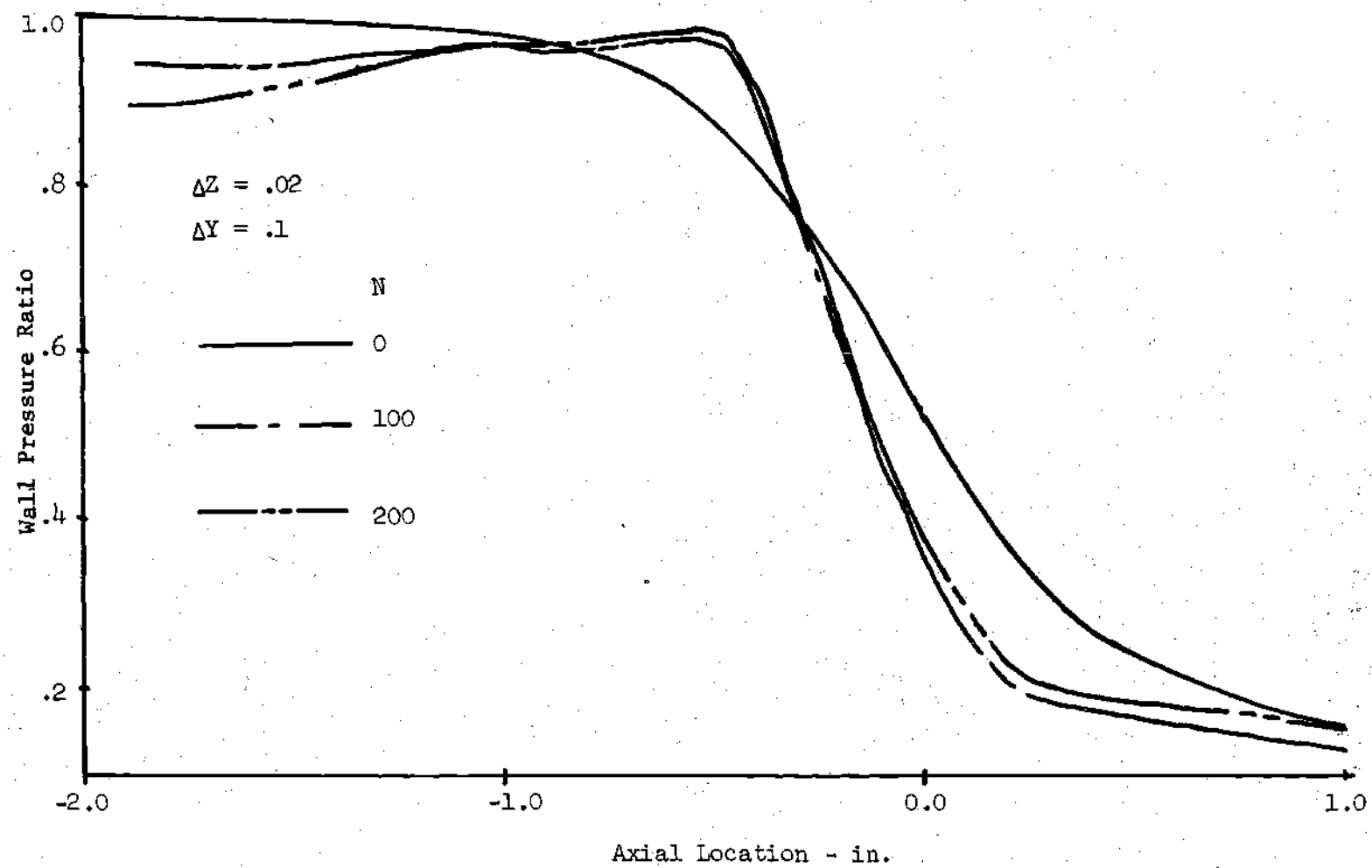


Figure 12. Pressure Ratio - Rusanov Method.

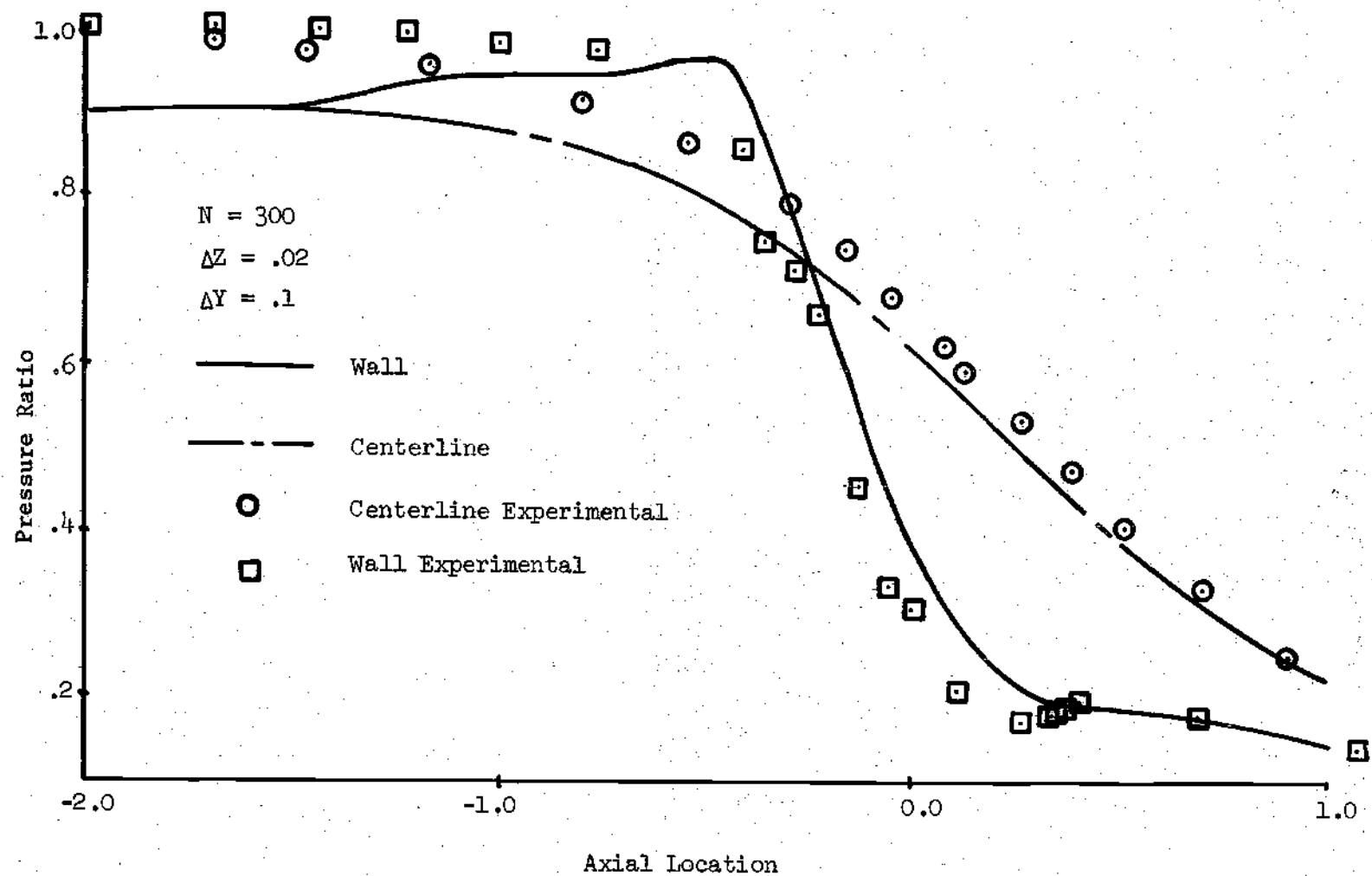


Figure 13. Pressure Ratio - Rusanov Method.

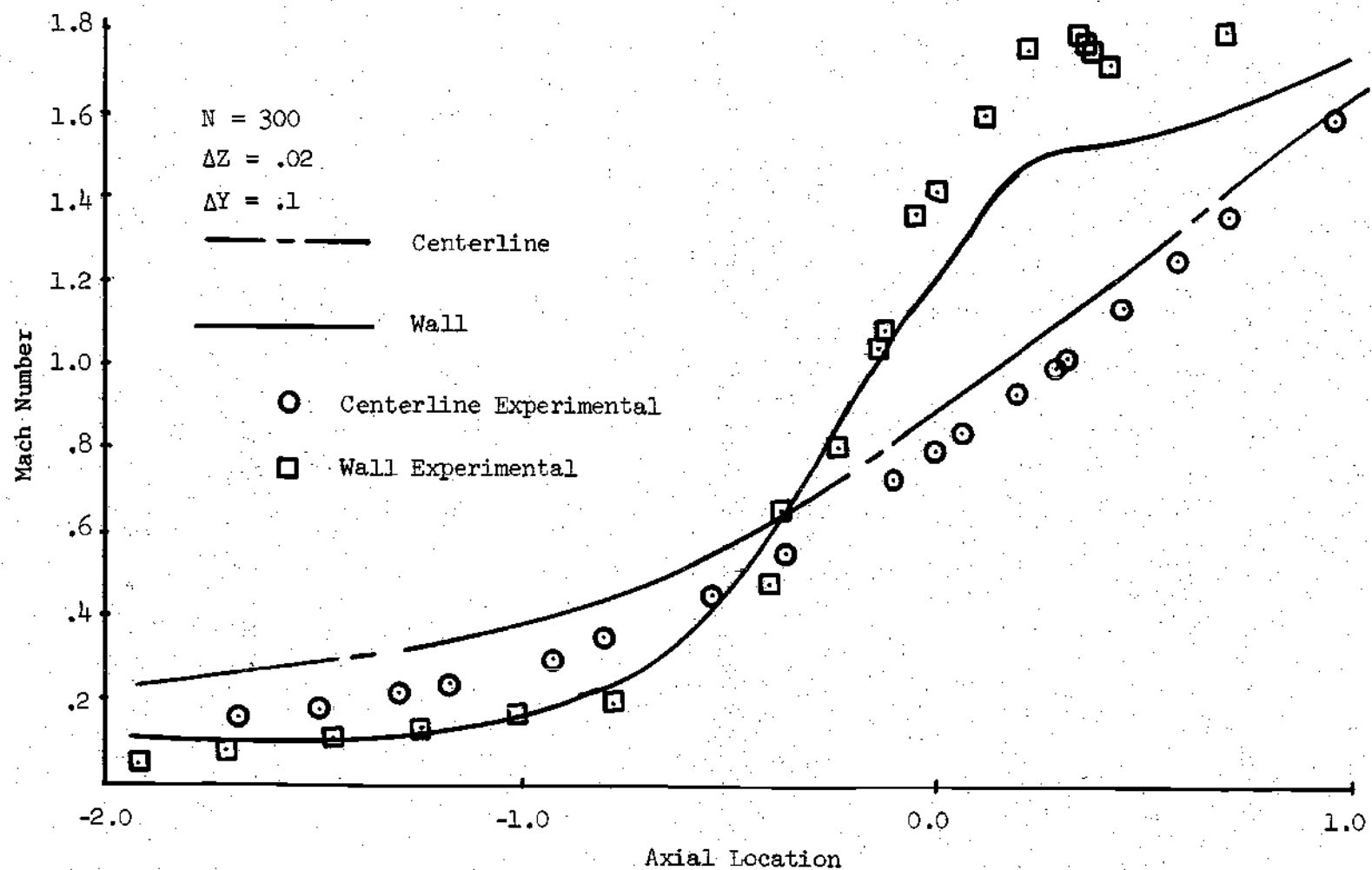


Figure 14. Mach Number - Rusanov Method.

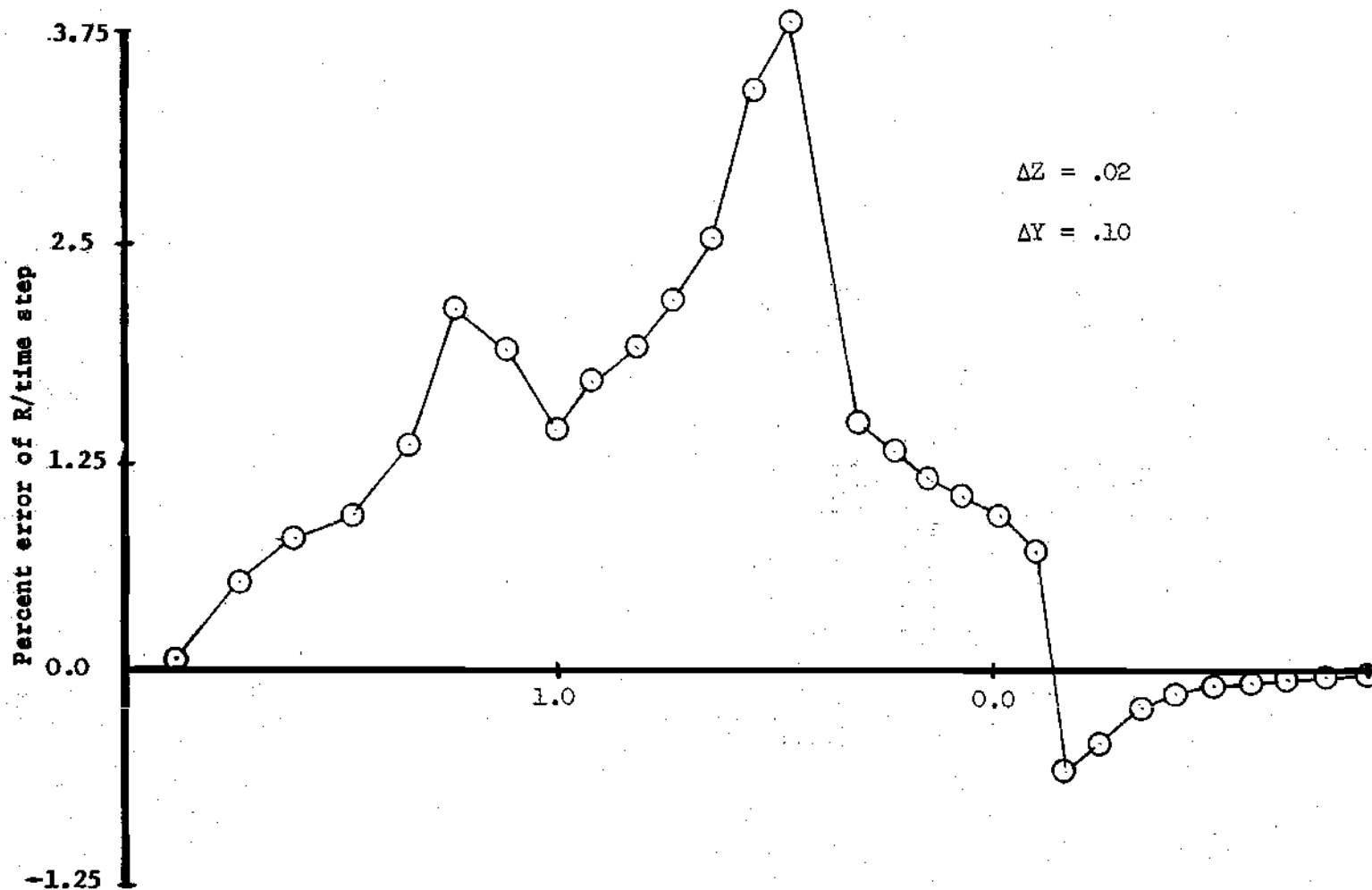


Figure 15. Percent Error of R Per Timestep - Rusanov Method.

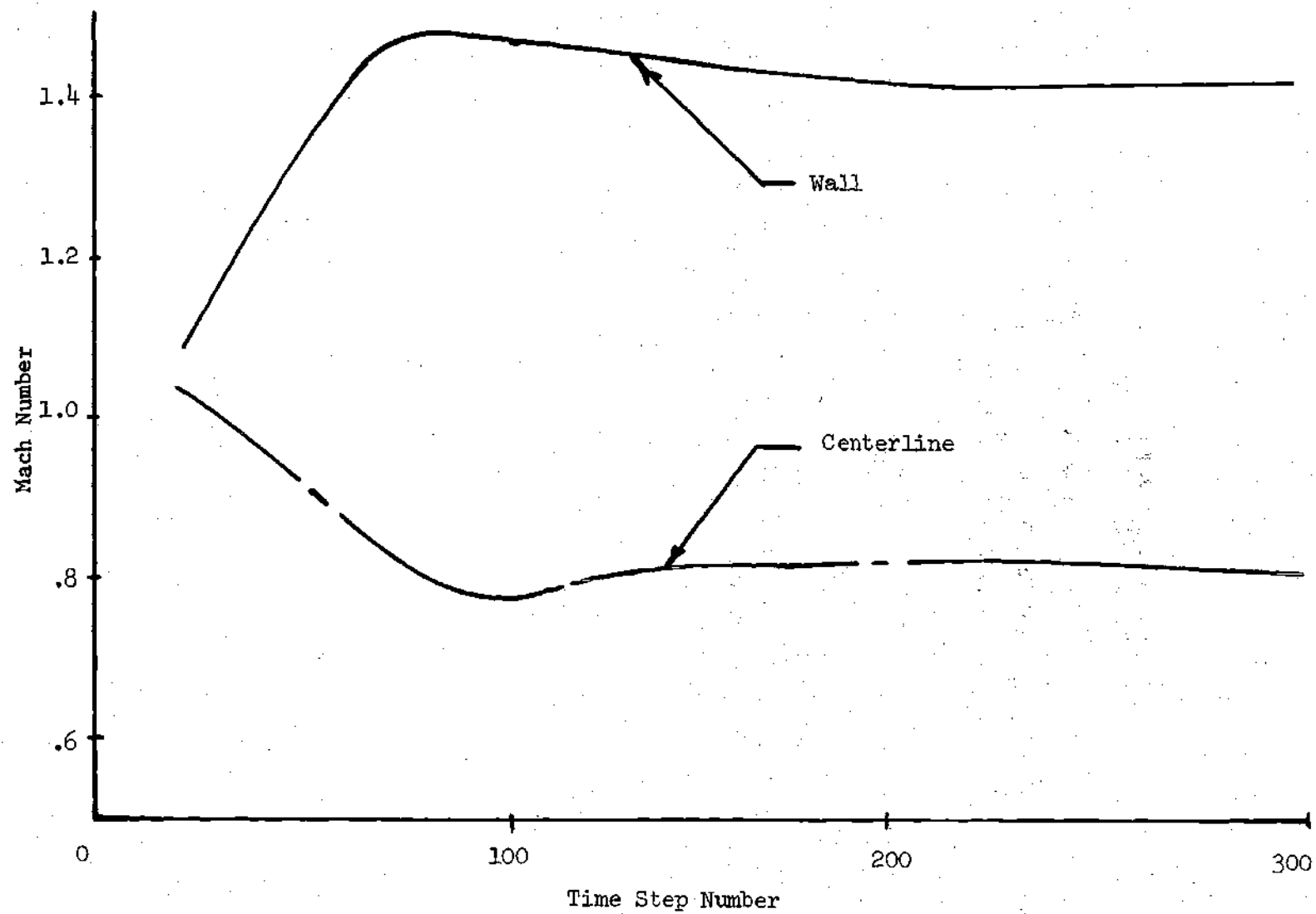


Figure 16. Time Variation of Throat Mach Number.

BIBLIOGRAPHY

1. Back, L. H., Massier, P. F., and Gier, H. L., "Comparison of Measured and Predicted Flows Through Conical Supersonic Nozzles with Emphasis on the Transonic Region," American Institute of Aeronautics and Astronautics Journal, August, 1965.
2. Back, L. H., Massier, P. F., and Cuffel, R. F., "Some Observations on Reduction of Turbulent Boundary-Layer Heat Transfer in Nozzles," Jet Propulsion Laboratory, California Institute of Technology, Pasadena, California, National Aviation and Space Administration. Contract No. NAS 7-100, 1965.
3. Back, L. H., Cuffel, R. F. and Massier, P. F., "Influence of Contraction Section Shape on Supersonic Nozzle Flow and Performance." Jet Propulsion Laboratory California Institute of Technology, Pasadena, California, NASA Contract No. NAS 7-100, 1971.
4. Back, L. H., Massier, P. F., and Gier, H. L., "Convective Heat Transfer in a Convergent-Divergent Nozzle," International Journal of Heat and Mass Transfer, Vol. 7, pp. 549-568, 1964.
5. Back, L. H., and Cuffel, R. F., "Detection of Oblique Shocks in a Conical Nozzle with a Circular - Arc Throat," AIAA Journal, Vol. 4, No. 12, pp. 2219-2221, December, 1966.
6. Back, L. H., Cuffel, R. F., and Massier, P. F., "Transonic Flow Field in a Supersonic Nozzle with Small Throat Radius of Curvature," AIAA Journal, Vol. 7, No. 7, pp. 1364-1366, July, 1969.
7. Bartz, D. R., "Turbulent Boundary-Layer Heat Transfer from Rapidly Accelerating Flow of Rocket Combustion Gases and of Heated Air," in Advances in Heat Transfer, Ed. by Irvine, T. F., Jr., and Hartnett, J. P., Vol. 2, Academic Press, 1965.
8. Burnstein, S.Z., "Numerical Calculations of Multidimensional Shocked Flow," AIAA Journal, Vol. 2, No. 12, pp. 2111-2117, December, 1964.
9. Elion, B., Gottlieb, D., and Zwas, G., "Numerical Stabilizers and Computing Time for Second-Order Accurate Schemes," Journal of Computational Physics, Vol. 9, p. 387-397, 1972.
10. Emery, A. F., "An Evaluation of Several Difference Methods for Inviscid Flow Problems," Journal of Computational Physics, Vol. 2, pp. 306-331, 1968.

11. Friedrichs, K. O., "Theoretical Studies on the Flow Through Nozzles and Related Problems," Applied Mathematics Panel Report 82-1R AMG-NYU No. 43, Applied Mathematics Group, New York University, April, 1944.
12. Glasgo, E. R. and Diveta, J. S., "Analytical and Experimental Evaluation of Performance Prediction Methods Applicable to Exhaust Nozzles," AIAA Paper No. 71-79, June, 1971.
13. Godunov, S. K., "Estimate of Errors for Approximate Solution of Simplest Equations of Gas Dynamics," AIAA Journal, Vol. 2, No. 1, p. 208-214, January, 1964.
14. Graham, R. W. and Deissler, R. G., "Prediction of Flow-Acceleration Effects on Turbulent Heat Transfer," Transactions of the American Society of Mechanical Engineers, Journal of Heat Transfer, Vol. 89, No. 4, pp. 371, November, 1967.
15. Hall, I. M., "Transonic Flow in Two-Dimensional and Axially Symmetric Nozzles," Journal of Mech. and Applied Math., Vol. XV, pp. 487-508, 1962.
16. Hildebrand, F. B., Finite-Difference Equation and Simulations, Prentice-Hall, 1967.
17. Hirt, C. W., "Heuristic Stability Theory for Finite-Difference Equations," Journal of Computational Physics, Vol. 2, pp. 339-355, 1968.
18. Kentzer, C. P., "Transonic Flow Past A Circular Cylinder," Journal of Computational Physics, Vol. 6, p. 1685, 1970.
19. Kentzer, C. P., "Computations of Time Dependent Flows on an Infinite Domain," AIAA paper No. 70-153, 1970.
20. Kliegel, J. R. and Levine, J. N., "Transonic Flow in Small Throat Radius of Curvature Nozzles," AIAA Journal, Vol. 7, No. 7, pp. 1375-1378, July, 1969.
21. Kooker, D. E., Private Communication, November, 1972.
22. Lapidus, Arnold, "A Detached Shock Calculation by Second-Order Finite Differences," Journal of Computational Physics, Vol. 2, pp. 154-177, 1967.
23. Laval, Pierre, "Time-Dependent Calculation Method for Transonic Nozzle Flows," International Conference on Numerical Methods in Fluid Dynamics, 2nd, University of California, Berkeley, Calif., Proceedings, p. 187-192, September, 15-19, 1970.

24. Lax, P. D., "Weak Solutions of Non-Linear Hyperbolic Equations and Their Numerical Computation," Communications on Pure and Applied Mathematics, Vol. VII, pp. 159-193, 1954.
25. Lax, P. D. and Wendroff, B., "Difference Schemes with High Order of Accuracy for Solving Hyperbolic Equations," Comm. on Pure and Applied Mathematics, Vol. XVII, pp. 381-398, 1964.
26. Lomax, H., "An Analysis of Finite-Difference Techniques Applied to Equations Governing Convective Transfer," Personal Correspondence, 1970.
27. Medelson, R. S., "A General Transonic Flow Analysis for Axially Symmetric Rocket Nozzles," Technical Report No. HSM-RO37, Space Division, Chrysler Corporation, Huntsville, Alabama, February 1964.
28. Migdal, D., Klein, K., and Moretti, G., "Time-Dependent Calculations for Transonic Nozzle Flow," AIAA Journal, Vol. 7, No. 2, pp. 372-374, February, 1969.
29. Moore, A. W. and Hall, I. M., "Transonic Flow in the Throat Region of an Annular Nozzle with an Arbitrary Smooth Profile," Aeronautical Research Council Reports and Memoranda No. 3480, January, 1965.
30. Moretti, Gino, "The Choice of a Time-Dependent Technique in Gas Dynamics," Polytechnic Institute of Brooklyn Report, PIBAL Report No. 69-26, 1969.
31. Moretti, Gino, "Complicated One-Dimension Flows," Polytechnic Institute of Brooklyn, PIBAL Report No. 71-25, 1971.
32. Moretti, Gino, and Abbett, Michael, "A Time-Dependent Computational Method for Blunt Body Flow," AIAA Journal, Vol. 4, No. 12, pp. 2136-2141, December, 1966.
33. Moretti, Gino, "The Importance of Boundary Conditions in the Numerical Treatment of Hyperbolic Equations," Polytechnic Institute of Brooklyn, PIBAL Report No. 68-34, November, 1968.
34. Moretti, G., and Abbett, M., "A Fast, Direct, and Accurate Technique for the Blunt Body Problems," General Applied Science Labs., Westbury, N. Y., GASL TR-583, 1966.
35. Moretti, Gino, and Bleich, Gary, "Three-Dimensional Flow Around Blunt Bodies," AIAA Journal, Vol. 5, No. 9, September, 1967.
36. Oswatitach, K. and Rothstein, W., "Flow Pattern in a Converging-Diverging Nozzle," NACA TM 1215, March, 1949.

37. Oswatitsch, K., Gas Dynamics, Translated by G. Kuerti, Academic Press, Inc., New York, 1956.
38. Prozan, R. J., "Transonic Flow in a Converging-Diverging Nozzle, Lockheed Missiles and Space Company, Contract NAS 7-743, Huntsville, Alabama.
39. Prozan, R. J. and Kooker, D. E., "The Error Minimization Technique with Application to a Transonic Nozzle Solution," J. Fluid. Mech., Vol. 43, Pt. 2, pp. 269-277, 1970.
40. Quan, V. and Kliegel, J. R., "Two-Zone Transonic Flow in Nozzles," AIAA Journal, Vol. 5, No. 12, pp. 2264-2266, December, 1967.
41. Richtmyer, R. D. and Morton, K. W., Difference Methods for Initial Value Problems, Interscience Publishers, New York, 1967.
42. Richtmyer, R. D., "A Survey of Difference Methods for Non-Steady Fluid Dynamics," NCAR Technical Notes 63-2, 1962.
43. Rubin, E. L. and Burstein, S. Z., "Difference Methods for the Inviscid and Viscous Equations of a Compressible Gas," Journal of Computational Physics, Vol. 2, pp. 178-196, 1967.
44. Rusanov, V. V., "On Difference Schemes of Third Order Accuracy for Nonlinear Hyperbolic Systems," Journal of Computational Physics, Vol. 5, p. 507-516, 1970.
45. Rusanov, V. V., "Three-Dimensional Supersonic Gas Flow Round a Blunt Body," USSR Computational Math. and Mathematical Physics, English translation, Vol. 1, pp. 156-179, 1968.
46. Rusanov, V. V., "The Calculation, the Interaction of Non-Stationary Shock Waves and Obstacles," USSR Computational Math and Mathematical Physics, English translation, Vol. 1, p. 304-320, 1961.
47. Sauer, R., "General Characteristics of the Flow Through Nozzles at Near Critical Speeds," NACA TM 1147, 1947.
48. Serra, R. A., "The Determination of Internal Gas Flows by a Transient Numerical Technique," Ph.D. Thesis, Rensselaer Polytechnic Institute, Troy, New York, June, 1970.
49. Serra, R. A., "The Determination of Internal Gas Flows by a Transient Numerical Technique," AIAA 9th Aerospace Sciences Meeting, AIAA Paper No. 71-45, January, 1971.
50. Shelton, S. V., "A Study of Two-Dimensional Nozzle Flow," Unpublished Report, 1971.

51. Shelton, S. V., Jet Propulsion Laboratory, Pasadena, California, Unpublished report, 1967.
52. Sheppard, R., Georgia Institute of Technology, Masters Thesis, 1973, in preparation.
53. Taylor, Thomas D. and Masson, Bruce S., "Applications of Unsteady Numerical Method of Godunov to Amputation of Supersonic Flows Past Bell-Shaped Bodies," Journal of Computational Physics, Vol. 5, pp. 443-454, 1970.
54. Tyler, L. D., "Heuristic Analysis of Convective Finite Difference Techniques," Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics, Vol. 8, pp. 314-319, September, 1970, Springer-Verlag, New York.
55. Von Neumann, J. and Richtmyer, R. D., "A Method for the Numerical Calculation of Hydrodynamic Shocks," Journal of Applied Physics, Vol. 21, pp. 232-237, March, 1950.
56. Taylor, G. E., "The Flow of Air At High Speeds Past Curved Surfaces," Aeronautical Research Council Reports and Memoranda No. 1381, 1930.
57. Hooker, S. G., "Transonic Flow Past Curved Surfaces," Aeronautical Research Council Reports and Memoranda No. 1382, 1930.
58. Taylor, T. D., Ndefo, E., and Masson, B. D., "A Study of Numerical Methods for Solving Viscous and Inviscid Flow Problems," Journal of Computational Physics, Vol. 9, p. 99, 1972.