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PHYSICAL CONCEPTS ASSOCIATED WITH TWO INFINITE DIFFERENTIAL SYSTEMS AND THEIR TRUNCATED FORMS

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SUMMARY

A study is made of some physical concepts associated with two infinite and two finite differential systems. One of the infinite differential systems is the mathematical model of an infinite chain (one end accessible) of identical masses coupled by identical, massless, linear springs. The masses move in one direction with no friction. At time $t = 0$, all springs are unstressed except the first, which is compressed $\underline{a}$ units; and all masses are stationary except the first, which has velocity $\underline{b}$. The second infinite differential system is the mathematical model of an infinite stack (the top accessible) of identical flat plates sliding in one dimension with viscous friction between each plate and the one beneath it. At time $t = 0$, each plate is stationary except the top one, which has velocity $\underline{b}$. The finite systems are mathematical models of the finite chain (stack) obtained by considering the motion of only the first N masses (plates) while supposing that all of the masses (plates) at and beyond the $(N + 1)^{th}$ are held stationary.

In Chapter I the differential systems and their solutions are presented. For the infinite spring-mass chain with $a = 0$ and a constant distance between reference positions of successive masses, Chapter II discusses a phenomenom similar to the propagation of a wave through a long, homogeneous, elastic bar of constant cross section. In Chapter III, expressions for the kinetic energies of individual masses and the total kinetic energy of both the finite and infinite spring-mass chains are found. Chapter IV outlines some results for almost periodic functions

which are used in Chapter V to consider the limits as $t \to \infty$, the least upper bounds, and the greatest lower bounds of the quantities found in Chapter III. In Chapter VI, the rate of transfer of energy from the $(j + 1)^{th}$ mass to the $(j + 1)^{th}$ spring in the infinite spring-mass chain with $a = 0$ is considered. In Chapter VII, expressions for the kinetic energies of the individual plates and the total kinetic energy of both the infinite and finite stacks of plates are found. In Chapter VIII, the limits as $t \to +\infty$ of the quantities found in Chapter VII are shown to be zero, and the limit as $t \to +\infty$ of the ratio of each of these quantities for the finite stack to its counterpart in the infinite stack is determined.

Some of the more interesting results are:

1) The occurrence of the first extremum of displacement for each mass in the infinite spring-mass chain with $a = 0$ and a constant distance between reference positions of adjacent masses can be viewed as a disturbance passing through the chain with a velocity that approaches a limit.

2) The limit as $t \to +\infty$ of the total kinetic energy in the infinite spring-mass chain (a not necessarily zero) is one half of the total energy in the chain.

3) In the finite spring-mass chain with $a = 0$ and $2N + 1$ a prime, the kinetic energy of the first mass has a least upper bound equal to the total energy of the chain; the kinetic energy of any other mass is less than the total energy of the chain. These facts are also true in the infinite chain with $a = 0$.

4) The ratio of the total kinetic energy in the finite stack of plates to the total kinetic energy in the infinite stack of plates has a limit of zero as $t \to +\infty$.

CHAPTER I

INTRODUCTION

In the doctoral dissertation of Mr. A. G. Law [1], which is now in preparation in the School of Mathematics, solutions are given for various infinite systems of linear ordinary differential equations with initial conditions. Included among the systems solved are the following:

$$\left.\begin{aligned} m\ddot{X}_o(t) &= k[X_1(t) - X_o(t)] , \\ m\ddot{X}_j(t) &= k[X_{j-1}(t) - 2X_j(t) + X_{j+1}(t)] , j=1,2,\ldots \\ X_o(0) &= a, \dot{X}_o(0) = b , \\ X_j(0) &= \dot{X}_j(0) = 0, \quad j = 1,2,\ldots, \end{aligned}\right\} \qquad \text{(A)}$$

where m and k are positive constants and a dot $(\cdot)$ indicates differentiation with respect to t, and

$$\left.\begin{aligned} m\dot{V}_o(t) &= \rho[V_1(t) - V_o(t)] , \\ m\dot{V}_j(t) &= \rho[V_{j-1}(t) - 2V_j(t) + V_{j+1}(t)], \ j=1,2,\ldots, \\ V_o(0) &= b, \ V_j(0) = 0, \ j=1,2,\ldots, \end{aligned}\right\} \qquad \text{(B)}$$

where m and ρ are positive constants. Systems (A) and (B) may be regarded as mathematical models, respectively, of an infinite chain of identical masses connected by identical, linear, massless springs and of an infinite stack of identical plates sliding in one dimension with viscous friction

(ρ) between them. The purpose of the present study is to investigate some of the physical implications of the solutions of (A) and (B) obtained by Mr. Law and, where appropriate, to compare the results with those which are obtained for finite differential systems similar to (A) and (B). More purely mathematical questions, such as the uniqueness of the solutions of (A) and (B), are not considered.

The study is organized in the following way. In the remainder of the Introduction, the physical systems corresponding to (A) and (B) are described more fully; the method used to obtain solutions of (A) and (B) is outlined; and a list is made of the questions to be considered in subsequent chapters. Which questions are considered in each chapter may be inferred from the Table of Contents.

Physical Prototypes of the Infinite Systems

The differential system (A) can be thought of as a mathematical model of the infinite chain of identical masses and identical linear springs pictured in Figure 1. When the system is in what will be called its reference position, the masses are so placed that each spring is unstressed; and the position which each mass occupies under these circumstances is called the reference position of that mass. At any time t (≥ 0), $X_j(t)$ ($j = 0,1,2,\ldots$) measures the rightward displacement of the $(j+1)^{th}$ mass from its reference position (notice, for example, that the mass at the left end of the chain is called the first mass, but its displacement is $X_o(t)$). The system is set in motion at t=0 with each mass except the first stationary at its reference position. The first mass is a units to the right of its reference position and has a rightward velocity of b.

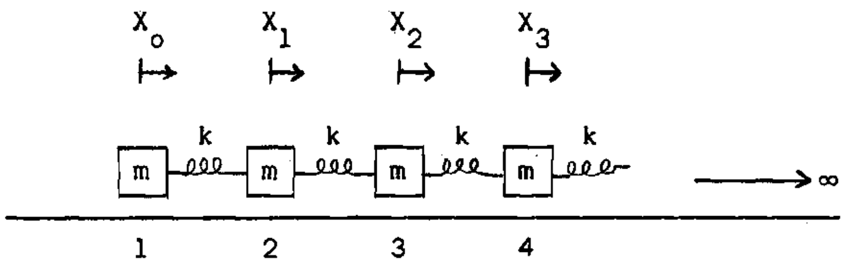

Figure 1. Physical Prototype of System (A). (The system is shown in the reference position. Friction is neglected. The masses are numbered as indicated beneath them.)

In order to compare the results for the infinite system (A) with those for a similar but finite differential system, one may suppose that in Figure 1 the $(N+1)^{th}$ mass and all masses to the right of it are held stationary. The appropriate differential system is then obtained from (A) by setting $X_j(t) \equiv 0$ $(j > N-1)$. Thus,

$$\left.\begin{aligned}
&m\ddot{X}_{N,0}(t) = k[X_{N,1}(t) - X_{N,0}(t)] \; , \\
&m\ddot{X}_{N,j}(t) = k[X_{N,j-1}(t) - 2X_{N,j}(t) + X_{N,j+1}(t)], j=1,2,\ldots,N-2, \\
&m\ddot{X}_{N,N-1}(t) = k[X_{N,N-2}(t) - 2X_{N,N-1}(t)] \; , \\
&X_{N,0}(0) = a, \;\; \dot{X}_{N,0}(0) = b \; , \\
&X_{N,0}(0) = \dot{X}_{N,0}(0) = 0, \; j = 1,2,\ldots,N-1,
\end{aligned}\right\} \quad (A')$$

where the first subscript N in $X_{N,j}(t)$ has been introduced to emphasize that a finite system of N masses is being considered.

The differential system (B) can be thought of as a mathematical model for an infinite stack of identical flat plates sliding in one dimension with viscous friction acting between any plate and the one below it (see Figure 2). $V_j(t)$ is the rightward velocity of the $(j+1)^{th}$ plate. The system is set in motion at t=0 with each plate except the first stationary. The first plate is given a rightward velocity <u>b</u>.

To compare the results for the infinite system (B) with those for a similar finite system, one may suppose that in Figure 2 all the plates at and below the $(N+1)^{th}$ are held stationary. By setting $V_j(t) \equiv 0$ $(j > N-1)$ in system (B), the appropriate differential system for the

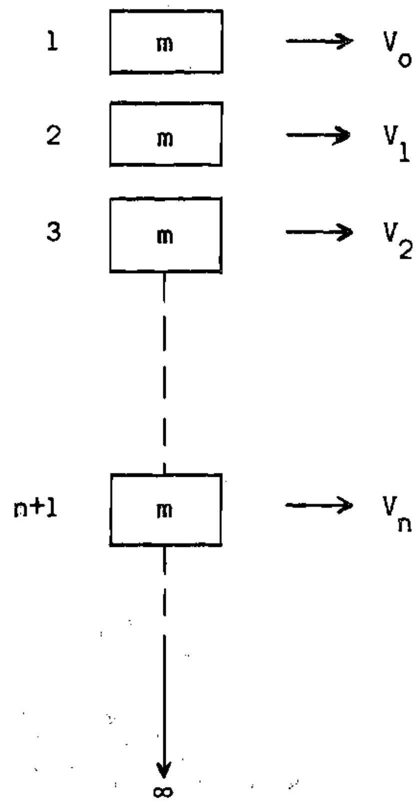

Figure 2. Physical Prototype of System (B). (The plates are numbered as indicated to the left of each plate. The coefficient of viscous friction between any plate and the one below it is ρ.)

finite case is obtained. Thus,

$$\left.\begin{aligned}
&m\dot{V}_{N,0}(t) = \rho[V_{N,1}(t) - V_{N,0}(t)] ,\\
&m\dot{V}_{N,j}(t) = \rho[V_{N,j-1}(t) - 2V_{N,j}(t) + V_{N,j+1}(t)],\ j=1,2,\ldots,N-2,\\
&m\dot{V}_{N,N-1}(t) = \rho[V_{N,N-2}(t) - 2V_{N,N-1}(t)] ,\\
&V_{N,0}(0) = b;\ V_{N,j}(0) = 0,\ j=1,2,\ldots,N-1 .
\end{aligned}\right\}\ (B')$$

It should be noted that the physical systems associated with the differential systems are not unique. The physical prototypes proposed could equally well have been electrical ladder networks, for example.

Method Used to Obtain Solutions of the Infinite Systems

Mr. Law obtains a solution of the system (A) by first considering the truncated system (A'), for which he finds that

$$X_{N,0}(t) = \frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\left\{a\cos[\omega(N,p)t] + \frac{b}{\omega(N,p)}\sin[\omega(N,p)t]\right\}, \tag{1}$$

where $\omega(N,p) = 2\sqrt{\beta}\,\sin\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)$ and $\beta = \frac{k}{m}$. By consideration of a Riemann sum for the integral

$$2\int_0^1 \cos^2\left(\frac{\pi x}{2}\right)\left\{a\cos\left[2\sqrt{\beta}\,t\,\sin\left(\frac{\pi x}{2}\right)\right] + \frac{b}{2\sqrt{\beta}\,\sin\left(\frac{\pi x}{2}\right)}\sin\left[2\sqrt{\beta}\,t\,\sin\left(\frac{\pi x}{2}\right)\right]\right\}dx, \tag{2}$$

with partition points $X_j = \frac{j}{N}$ $(j = 0,1,\ldots,N)$ and with the integrand evaluated at $X_j' = \frac{2j-1}{2N+1}$ $(j = 1,2,\ldots,N)$, it follows that $\lim_{N\to\infty} X_{N,0}(t)$ exists and is the integral (2), which is equal to

$$a[J_0(2\sqrt{\beta}\,t) + J_2(2\sqrt{\beta}\,t)] + b\int_0^t [J_0(2\sqrt{\beta}\,s) + J_2(2\sqrt{\beta}\,s)]\,ds,$$

where $J_n(z)$ is the Bessel function of the first kind of order n. By assuming that $\lim_{N\to\infty} X_{N,0}(t) = X_0(t)$ -- that is, that the first component of a solution of (A) is the limit as N approaches infinity of the first component of the solution of (A') -- and by using the equations of (A) recursively, a candidate for the j^{th} component of a solution of (A) is obtained:

$$X_j(t) = a[J_{2j}(2\sqrt{\beta}\,t) + J_{2j+2}(2\sqrt{\beta}\,t)] + b\int_0^t [J_{2j}(2\sqrt{\beta}\,s) + J_{2j+2}(2\sqrt{\beta}\,s)]\,ds, \quad j=0,1,2,\dots \tag{3}$$

Direct substitution of (3) into (A) then shows that $X_j(t)$ as given by (3) is in fact a solution of (A).

Similarly, Mr. Law finds a solution of the system (B) by first solving the truncated system (B'), then taking the limit as $N \to \infty$, and showing directly that the limit is a solution of (B). The solution of (B) so found is

$$V_j(t) = b\int_0^1 \left\{\cos[(j+1)\pi x] + \cos[j\pi x]\right\} e^{2\alpha(-1+\cos \pi x)t}\,dx, \quad j = 0,1,2,\dots, \tag{4}$$

where $\alpha = \frac{\rho}{m}$. The solution of (B') is

$$V_{N,j}(t) = \frac{4b}{2N+1} \sum_{p=1}^{N} \cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right) \cos\left[\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)(2j+1)\right] e^{\gamma_{N,p} t} ,$$

$$j=0,1,2,\ldots,N-1 , \tag{5}$$

where $$\gamma_{N,p} = 2\alpha\left[\cos\left(\frac{2p-1}{2N+1}\pi\right) - 1\right] .$$

The solutions $X_{N,j}(t)$ and $V_{N,j}(t)$ $(j=0,1,\ldots,N-1)$ of the finite systems (A') and (B') are unique. $X_j(t)$ and $V_j(t)$ $(j=0,1,2,\ldots)$, as given by (3) and (4), are known to be solutions of (A) and (B); but in the present study, the question of their uniqueness is ignored.

Questions to be Considered

1) Consider the infinite spring-mass chain with a = 0 and a constant distance between reference points of adjacent masses. Associated with this system, is there any phenomenon that is similar to the propagation of a wave through a long, homogeneous, elastic bar of constant cross-section?

2) Consider the kinetic energies of the individual masses and the total kinetic energy in both the infinite and finite spring-mass chains. For these quantities, what are the limits as $t \to \infty$ (if they exist), the least upper bounds for $t > 0$, and the greatest lower bounds for $t > 0$?

3) Consider the infinite spring-mass chain with a = 0. What can be said about the rate of transfer of energy from the $(j+1)^{th}$ mass to the $(j+1)^{th}$ spring for large values of t?

4) Consider the kinetic energies of the individual plates and the total kinetic energy in both the infinite and finite stacks of plates.

Do these quantities have limit zero as $t \to \infty$? For each of the kinetic energies in the finite stack and its counterpart in the infinite stack, what is the limit of their ratio as $t \to \infty$?

CHAPTER II

VELOCITY OF PROPAGATION OF A DISTURBANCE THROUGH THE INFINITE SPRING-MASS CHAIN

If a longitudinal displacement is initiated at one end of a long homogeneous, elastic bar of constant cross section, a wave of displacement travels down the bar with constant velocity $\sqrt{E/\rho}$, where E and ρ are, respectively, the modulus of elasticity and the density of the material of which the bar is made [2]. It seems plausible to conjecture that a similar phenomenon occurs in the infinite chain of springs and masses. The object of this chapter is to examine this conjecture for one special case.

Immediately one notices that in the physical prototype shown in Figure 1 no distance between the reference positions of successive masses is assigned. In order to discuss the velocity of propagation of a disturbance in the spring-mass chain some convention about these distances must be adopted. Since the chain is being compared with a uniform bar, let these distances have a common constant value c.

Suppose that there is some common occurrence in the displacement of each mass and that this occurrence has the following two properties: 1) it happens only once for each mass; and 2) if t_n is the time at which it happens for the $(n+1)^{th}$ mass, then $t_{n-1} < t_n$. The occurrence for each mass can be associated with the reference position of the mass, rather than with the mass itself. These occurrences can then be visualized as a disturbance which at time t_n is at the reference position of the $(n+1)^{th}$ mass.

Between times t_{n-1} and t_n there is actually no disturbance. The distance between the two reference positions where the disturbance appears at times t_{n-1} and t_n is c. The disturbance is considered to have a constant hypothetical velocity of $c/(t_n - t_{n-1})$ between these two positions during this time period. Notice that at each time t_n there may be a discontinuity of the hypothetical velocity of the disturbance.

Now considering the infinite spring-mass chain, suppose that all the masses are initially at their reference positions, that all masses except the first are stationary, and that the first mass has a rightward velocity b. For a particular occurrence -- namely, the first extremum of displacement after t = 0 -- the following questions will be answered: 1) Is $c/(t_n - t_{n-1})$ $(n = 1,2,\ldots)$ independent of n? 2) If it is not independent of n, does it have a limit as $n \to \infty$, and if it has a limit, what is the value of the limit?

The displacement of the $(n+1)^{th}$ mass in the infinite spring-mass chain with initial conditions as specified in the preceding paragraph is given by (3) with a = 0. Differentiating (3) with a = 0 gives

$$\dot{x}_n(t) = b[J_{2n}(2\sqrt{\beta}\, t) + J_{2n+2}(2\sqrt{\beta}\, t)]. \tag{6}$$

The use of the recurrence relation $J_{r-1}(z) + J_{r+1}(z) = 2rJ_r(z)/z$ [3, p. 100] in equation (6) leads to the equation

$$\dot{x}_n(t) = 2b(2n+1)\; J_{2n+1}(2\sqrt{\beta}\, t)/2\sqrt{\beta}\, t. \tag{7}$$

For ν and μ positive integers, let $j_{\nu,\mu}$ be the μ^{th} positive zero of $J_\nu(z)$. Since $J_\nu(z)$ has a sequence of isolated positive zeros [3, p. 127],

$j_{\nu,\mu}$ is well defined. Each positive zero of $J_\nu(z)$ is a simple zero [3, p. 127]; thus $\dot{X}_n(t)$ changes sign at $t = j_{2n+1,\mu}/2\sqrt{\beta}$. Hence, $X_n(t)$ has extrema at $t = j_{2n+1,\mu}/2\sqrt{\beta}$ for all positive integers μ_j and furthermore, these are the only positive values of t for which $X_n(t)$ has an extremum.

Consider the first extremum of $X_n(t)$ after $t = 0$. The existence of this extremum is an occurrence common to all the masses. Clearly there is only one first extremum for each mass. The time of occurrence t_n of the first extremum is $j_{2n+1,1}/2\sqrt{\beta}$. Since $j_{\nu,1} < j_{\nu+1,1}$ [4, p. 370], $t_{n-1} < t_n$. Hence the occurrence of the first extremum can be visualized as a disturbance with hypothetical velocity

$$c/(t_n - t_{n-1}) = 2\sqrt{\beta}\, c/(j_{2n+1,1} - j_{2n-1,1}) \qquad (8)$$

between time t_{n-1} and t_n.

1) Is $c/(t_n - t_{n-1})$ independent of n? Since to nine significant figures,

$$j_{1,1} = 3.83170\ 597,$$
$$j_{3,1} = 6.38016\ 190,$$
$$j_{5,1} = 8.77148\ 382,$$
and $j_{7,1} = 11.08637\ 002$, [5, pp. 2-8],

it follows that

$$j_{3,1} - j_{1,1} = 2.54845 \pm 10^{-5},$$
$$j_{5,1} - j_{3,1} = 2.39132 \pm 10^{-5},$$
and $j_{7,1} - j_{5,1} = 2.31488 \pm 10^{-5}$.

Hence by equation (8), $c/(t_n - t_{n-1})$ is not independent of n.

2) Does $\lim_{n\to\infty} c/(t_n - t_{n-1})$ exist, and if it does what is its value?

From equation (8), $\lim_{n\to\infty} c/(t_n - t_{n-1})$ exists if $\lim_{n\to\infty} (j_{2n+1,1} - j_{2n-1,1})$ exists and is nonzero. The asymptotic series for the first positive zero of $J_\nu(z)$ is

$$j_{\nu,1} \sim \nu + c_{1,1}\nu^{1/3} + c_{2,1}\nu^{-1/3} + c_{3,1}\nu^{-1} + \dots ,$$

where $c_{1,1}, c_{2,1}, c_{3,1}, \dots$ are constants [4, p. 371].

Let $$D(\nu) = [j_{\nu,1} - (\nu + c_{1,1}\,\nu^{1/3})] .$$

Then $$\lim_{\nu\to\infty} [\nu^{1/3}D(\nu)] = c_{2,1} .$$

Consequently, $$\lim_{n\to\infty} \left\{[2n+1]^{1/3}\, D(2n+1)\right\} = c_{2,1} \tag{9}$$

and $$\lim_{n\to\infty} \left\{[2n-1]^{1/3}\, D(2n-1)\right\} = c_{2,1} . \tag{10}$$

From equation (10) and the fact that $\lim_{n\to\infty} [(2n+1)^{1/3}/(2n-1)^{1/3}] = 1$, it follows that

$$\lim_{n\to\infty} \left\{[2n+1]^{1/3}\, D(2n-1)\right\} = c_{2,1} . \tag{11}$$

Considering the difference of equations (9) and (11) leads to

$$\lim_{n\to\infty} \left\{[2n+1]^{1/3}[D(2n+1) - D(2n-1)]\right\} = 0 . \tag{12}$$

If there is a subsequence of $\left\{D(2n+1) - D(2n-1)\right\}$ which is bounded away from zero, then the same subsequence of $\left\{[2n+1]^{1/3}[D(2n+1) - D(2n-1)]\right\}$

is unbounded. This fact would contradict equation (12). Hence

$$\lim_{n\to\infty} [D(2n+1) - D(2n-1)] = 0 . \tag{13}$$

Since $\lim_{n\to\infty} [(2n+1)^{1/3} - (2n-1)^{1/3}] = 0$, it follows from equation (13) that

$$\lim_{n\to\infty} [D(2n+1) - D(2n-1) + c_{1,1}(2n+1)^{1/3} - c_{1,1}(2n-1)^{1/3}] = 0. \tag{14}$$

But

$$\begin{aligned} D(2n+1) - D(2n-1) &= j_{2n+1,1} - (2n+1) - c_{1,1}(2n+1)^{1/3} - j_{2n-1,1} + (2n-1) \\ &\quad + c_{1,1}(2n-1)^{1/3} = j_{2n+1,1} - j_{2n-1,1} - 2 - c_{1,1}(2n+1)^{1/3} + c_{1,1}(2n-1)^{1/3}. \end{aligned} \tag{15}$$

Equations (14) and (15) imply that

$$\lim_{n\to\infty} [j_{2n+1,1} - j_{2n-1,1} - 2] = 0 ,$$

and thus

$$\lim_{n\to\infty} [j_{2n+1,1} - j_{2n-1,1}] = 2 .$$

Hence by equation (8), $\lim_{n\to\infty} c/(t_n - t_{n-1}) = c\sqrt{\beta}$, which is the limit of the hypothetical velocity of the first extremum of displacement.

The second question could be answered for the μ^{th} extremum of displacement of each mass after $t = 0$. There is clearly only one μ^{th} extremum of displacement for the $(n+1)^{th}$ mass. It occurs when $t = j_{2n+1,\mu}/2\sqrt{\beta}$. Since $j_{\nu,\mu} < j_{\nu+1,\mu}$ [4, p. 370], this occurrence can be visualized as a disturbance with a hypothetical velocity between each reference position in the chain. The asymptotic series for the μ^{th} positive zero of $J_\nu(z)$ is

$$j_{\nu,\mu} \sim \nu + c_{1,\mu}\nu^{1/3} + c_{2,\mu}\nu^{-1/3} + c_{3,\mu}\nu^{-1} + \dots,$$

where $c_{1,\mu}, c_{2,\mu}, c_{3,\mu}, \dots$ are constants [5, p. XXX]. Hence by the same reasoning as for the first extremum, the limit of the hypothetical velocity of the μ^{th} extremum of displacement after $t = 0$ is $c\sqrt{\beta}$. However it should be noted that as μ increases the asymptotic series for $j_{\nu,\mu}$ becomes progressively weaker [5, p. XXX].

The limit $c\sqrt{\beta}$ of the hypothetical velocity of the first extremum of displacement after $t = 0$ has an interesting relation to the velocity, $\sqrt{E/\rho}$, of a disturbance in a uniform bar. Let A be the cross-sectional area of the bar; let y be the distance from one end to a point on the bar when the bar is unstressed; let $u(y,t)$ be the displacement of that point as a function of time. If Y is the stress in the bar, then

$$Y = E\,\frac{\partial u}{\partial y}\,;$$

and AY is the force acting in the positive direction on the cross section of the bar:

$$F = AY = AE\,\frac{\partial u}{\partial y}\,.$$

$\frac{\partial u}{\partial y}$ is the axial strain in the bar (recall that strain is a measure of change in length per unit of original length). The counterpart of $\frac{\partial u}{\partial y}$ in the $(n+1)^{th}$ spring of the chain is $[X_{n+1}(t) - X_n(t)]/c$, and the tensile force in the $(n+1)^{th}$ spring in the chain is

$$F = k[X_{n+1}(t) - X_n(t)] = kc[X_{n+1}(t) - X_n(t)]/c .$$

kc may then be thought of as the counterpart in the chain of AE in the uniform bar. $A\rho$ for the bar is the mass per unit length of the bar. Its counterpart in the chain is m/c. The velocity of a disturbance in the bar may be written as $\sqrt{AE/A\rho}$. Replacing AE and $A\rho$ by their counterparts in the chain, $\sqrt{AE/A\rho}$ becomes

$$\sqrt{kc/(m/c)} = \sqrt{kc^2/m} = c\sqrt{k/m} = c\sqrt{\beta} .$$

Thus there is a natural parallel between the velocity $\sqrt{E/\rho}$ of a disturbance in the uniform bar and the limit $c\sqrt{\beta}$ of the hypothetical velocity in the infinite spring-mass chain.

CHAPTER III

EXPRESSIONS FOR KINETIC ENERGIES IN THE SPRING-MASS CHAINS

This chapter and Chapter IV are preliminaries to some comparisons made in Chapter V of the kinetic energies in finite and infinite spring-mass chains subjected to similar initial conditions. In the present chapter, the following quantities are calculated:

1) the displacement of the $(j+1)^{th}$ mass, $X_{N,j}(t)$ $(j=0,1,2,\ldots,N-1)$, in a finite chain of N masses for which the differential system (A') is the mathematical model,

2) the kinetic energy of the $(j+1)^{th}$ mass $(j=0,1,\ldots,N-1)$ in such a chain,

3) the total kinetic energy of the chain,

4) the kinetic energy of the $(j+1)^{th}$ mass $(j=0,1,2,\ldots)$ in an infinite spring-mass chain for which the differential system (A) is the mathematical model,

5) the total kinetic energy of this infinite chain.

The Solution of the Differential System (A')

The first component, $X_{N,0}(t)$ of the solution of (A') is given by equation (1). From this expression, the general component $X_{N,j}(t)$ can be found by using the differential equations of system (A').

Assume that

$$X_{N,j}(t) = \frac{4}{2N+1} \sum_{p=1}^{N} C(N,p,j)\, G(N,p,t), \quad j=0,1,\ldots,N-1, \tag{16}$$

where

$$G(N,p,t) = \left\{a \cos[\omega(N,p)t] + \frac{b}{\omega(N,p)} \sin[\omega(N,p)t]\right\}$$

and $C(N,p,j)$ is to be determined (recall that $\omega(N,p)$ is $2\sqrt{\beta}\,\sin\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)$).
Choose

$$C(N,p,0) = \cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right). \tag{17}$$

Then equation (16) for j=0 agrees with equation (1).

Differentiating equation (16) twice with respect to t gives

$$\ddot{X}_{N,j}(t) = \sum_{p=1}^{N} C(N,p,j)\, \frac{\partial^2}{\partial t^2}[G(N,p,t)], \; j=0,1,2,\ldots,N-1 .$$

But from the definition of $G(N,p,t)$, it follows that

$$\frac{\partial^2}{\partial t^2}[G(N,p,t)] = -[\omega(N,p)]^2\, G(N,p,t) .$$

Hence

$$\ddot{X}_{N,j}(t) = \frac{4}{2N+1} \sum_{p=1}^{N} C(N,p,j)\left\{-[\omega(N,p)]^2\right\} G(N,p,t). \tag{18}$$

The first differential equation of the system (A') (recall that $\beta = k/m$) is equivalent to

$$\ddot{X}_{N,0}(t) + \beta\, X_{N,0}(t) - \beta\, X_{N,1}(t) = 0 . \tag{19}$$

Substituting equations (16) and (18) in equation (19) and simplifying gives

$$\frac{4}{2N+1}\sum_{p=1}^{N}\left\{-\omega^2(N,p)C(N,p,0)+\beta C(N,p,0)-\beta C(N,p,1)\right\}G(N,p,t)=0.$$

Hence, choosing

$$\begin{aligned} C(N,p,1) &= \left\{1-\frac{\omega^2(N,p)}{\beta}\right\}C(N,p,0) \\ &= \left\{1-4\sin^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\right\}C(N,p,0) \end{aligned} \tag{20}$$

will guarantee that the first differential equation of system (A') is satisfied. Using the identity $2\sin^2\theta = 1-\cos 2\theta$ and equation (17) in (20) leads to

$$\begin{aligned} C(N,p,1) &= \left\{1-2+2\cos\left(\frac{2p-1}{2N+1}\,\pi\right)\right\}\cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) \\ &= \left\{2\cos\left(\frac{2p-1}{2N+1}\,\pi\right)\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)-\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\right\}\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right). \end{aligned}$$

The identity $2\cos(A)\cos(B) = \cos(A+B)+\cos(A-B)$ gives

$$\begin{aligned} C(N,p,1) &= \left\{\cos\left[\left(\frac{2p-1}{2N+1}\right)\frac{3\pi}{2}\right]+\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)-\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\right\}\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) \\ &= \cos\left[\left(\frac{2p-1}{2N+1}\right)\frac{3\pi}{2}\right]\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right). \end{aligned} \tag{21}$$

Next, to find $C(N,p,j)$ $(j=2,3,\ldots,N-1)$, consider the general equation of the system (A'), which is equivalent to

$$\ddot{X}_{N,j}(t)+2\beta X_{N,j}(t)-\beta X_{N,j-1}(t)-\beta X_{N,j+1}(t)=0\,. \tag{22}$$

Substituting equations (16) and (18) in equation (22) and simplifying gives

$$\frac{4}{2N+1}\sum_{p=1}^{N}\Big\{-[\omega(N,p)]C(N,p,j) + 2\beta C(N,p,j) - \beta C(N,p,j-1) - \beta C(N,p,j+1)\Big\}\, G(N,p,t) = 0 .$$

Choosing $C(N,p,j)$ $(j=2,3,\ldots,N)$ such that

$$C(N,p,j+1) + C(N,p,j-1) = \left\{2 - \frac{[\omega(N,p)]^2}{\beta}\right\} C(N,p,j)$$

$$= \left\{2 - 4\sin^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\right\} C(N,p,j) \tag{23}$$

guarantees that $X_{N,j}(t)$ given by equation (16) satisfies

$$m\ddot{X}_{N,j}(t) = k(X_{N,j-1}(t) - 2X_{N,j}(t) + X_{N,j+1}(t)),\quad j=1,2,\ldots,N-1 .$$

Except for the equation with $j=N-1$, these are the general equations of the differential system (A'). For the moment this discrepancy will be ignored.

The identity $1 - 2\sin^2\theta = \cos 2\theta$ and equation (23) imply

$$C(N,p,j+1) + C(N,p,j-1) = 2\cos\left(\frac{2p-1}{2N+1}\pi\right)C(N,p,j),\quad j=1,2,\ldots,N. \tag{24}$$

Standard difference-equation techniques [6, p. 241] applied to equation (24) with constraints (17) and (21) yield

$$C(N,p,j) = \cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\,\frac{\pi}{2}\,(2j+1)\right], \tag{25}$$

$$j=0,1,2,\ldots,N .$$

Now consider the discrepancy mentioned before. With $C(N,p,j)$ defined by (25), $X_{N,j}(t)$ given by (16) is known to satisfy all the differential equations of system (A') except

$$m\ddot{X}_{N,N-1}(t) = k(X_{N,N-2}(t) - 2X_{N,N-1}(t)).$$

However, (16) is known to satisfy

$$m\ddot{X}_{N,N-1}(t) = k(X_{N,N-2}(t) - 2X_{N,N-1}(t) + X_{N,N}(t)).$$

But $C(N,p,N) = \cos(\frac{2p-1}{2N+1}\frac{\pi}{2}) \cos[\frac{2p-1}{2N+1}\frac{\pi}{2}(2N+1)] = 0$, and hence $X_{N,N}(t) \equiv 0$. Therefore (16) satisfies all the differential equations of (A').

One might now ask if $X_{N,j}(t)$ with $C(N,p,j)$ given by (25) satisfies the initial conditions of (A'), $X_{N,0}(0) = a, \dot{X}_{N,0}(0) = b, X_{N,j}(0) = \dot{X}_{N,j}(0) = 0$ $(j=1,2,\ldots,N-1)$. From (16) it follows that

$$X_{N,j}(0) = \frac{4a}{2N+1} \sum_{p=1}^{N} C(N,p,j) \quad \text{and} \quad \dot{X}_{N,j}(0) = \frac{4b}{2N+1} \sum_{p=1}^{N} C(N,p,j) \ . \qquad (26a,b)$$

Whether the initial conditions are satisfied depends on the value of

$$\sum_{p=1}^{N} C(N,p,j) = \sum_{p=1}^{N} \cos(\frac{2p-1}{2N+1}\frac{\pi}{2}) \cos[(\frac{2p-1}{2N+1})\frac{\pi}{2}(2j+1)] \ . \qquad (27)$$

Because it will be useful later, consider the more general problem of finding the value of

$$S(q,r) = \sum_{p=1}^{N} \cos[(\frac{2p-1}{2N+1})\frac{\pi}{2}(2q+1)]\cos[(\frac{2p-1}{2N+1})\frac{\pi}{2}(2r+1)] \ , \qquad (28)$$

$$q,r = 0,1,\ldots,N-1 \ .$$

Using the identity $\cos(A)\cos(B) = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$, one finds that

$$S(q,r) = \frac{1}{2}\sum_{p=1}^{N} \cos\left[\left(\frac{2p-1}{2N+1}\right)\pi(q+r+1)\right] + \frac{1}{2}\sum_{p=1}^{N}\cos\left[\left(\frac{2p-1}{2N+1}\right)\frac{\pi}{2}\,(q-r)\right]. \quad (29)$$

Since $\sum_{p=1}^{N} \cos[(2p-1)\theta] = \frac{\sin 2N\theta}{2\sin\theta}$ provided that $\sin\theta \neq 0$ [7, p. 366], it follows that if $\sin\left(\frac{\nu\pi}{2N+1}\right) \neq 0$, then

$$\sum_{p=1}^{N} \cos\left[\frac{2p-1}{2N+1}\,\nu\pi\right] = \frac{\sin\left[2N\left(\frac{\nu\pi}{2N+1}\right)\right]}{2\sin\left(\frac{\nu\pi}{2N+1}\right)} = \frac{\sin\left[\nu\pi - \frac{\nu\pi}{2N+1}\right]}{2\sin\left(\frac{\nu\pi}{2N+1}\right)}$$

$$= \frac{\sin(\nu\pi)\cos\left(\frac{\nu\pi}{2N+1}\right)}{2\sin\left(\frac{\nu\pi}{2N+1}\right)} - \frac{\cos(\nu\pi)\sin\left(\frac{\nu\pi}{2N+1}\right)}{2\sin\left(\frac{\nu\pi}{2N+1}\right)}.$$

If in addition ν is an integer,

$$\sum_{p=1}^{N} \cos\left[\frac{(2p-1)\nu\pi}{2N+1}\right] = -\frac{1}{2}\cos(\nu\pi) = -\frac{1}{2}(-1)^{\nu}.$$

Now if $q \neq r$ $(q,r = 0,1,\ldots,N-1)$, $\sin\left[\frac{(q+r+1)\pi}{2N+1}\right] \neq 0$, and $\sin\left[\frac{(q-r)\pi}{2N+1}\right] \neq 0$. Therefore from equation (29) it follows that

$$S(q,r) = \frac{1}{2}\left\{\left(-\frac{1}{2}\right)(-1)^{q+r+1} + \left(-\frac{1}{2}\right)(-1)^{q-r}\right\}$$

$$= \left(-\frac{1}{4}\right)(-1)^{q}\left\{(-1)^{r+1} + \frac{1}{(-1)^{r}}\right\} = 0.$$

From (29) with $q = r$,

$$S(q,r) = \frac{1}{2}\sum_{p=1}^{N}\cos\left[\frac{2p-1}{2N+1}\,\pi\,(2q+1)\right] + \frac{1}{2}\sum_{p=1}^{N}(1);$$

and since $\sin[\frac{\pi(2q+1)}{2N+1}] \neq 0 \ (q = 0,1,\ldots,N-1)$,

$$S(q,r) = \frac{1}{2}\left\{(-\frac{1}{2})(-1)^{2q+1} + \frac{N}{2}\right\} = \frac{1}{4} + \frac{N}{2} = \frac{2N+1}{4} .$$

Hence

$$S(q,r) = \begin{cases} 0 & , \text{ if } q \neq r \\ \frac{2N+1}{4} & , \text{ if } q = r \end{cases} \qquad (30)$$

$$q,r = 0,1,2,\ldots N-1 .$$

Considering $S(0,j)$ and equations (24a), (24b), and (27) leads to the conclusion that $X_{n,j}(t)$ $(j=0,1,\ldots,N-1)$ given by (16) with $C(N,p,j)$ given by (25) satisfies the initial conditions of (A').

It is no accident that $X_{N,j}(t)$ $(j=0,1,\ldots,N-1)$ satisfies the initial conditions of (A'). It is a consequence of the fact that $X_{N,0}(t)$ given by (1) is actually the first component of the solution of (A').

<u>Kinetic Energies in the Finite Spring-Mass Chain</u>

Let $E_j^N(t)$ be the kinetic energy of the $(j+1)^{th}$ mass in a finite spring-mass chain for which the system (A') is the mathematical model. Then $E_j^N(t) = \frac{m}{2}[\mathring{X}_{N,j}(t)]^2$ where $X_{N,j}(t)$ is given by (16) with $C(N,p,j)$ given by (25). Consequently,

$$E_j^N(t) = \frac{m}{2}\left(\frac{4}{2N+1}\right)^2 \sum_{p=1}^{N}\sum_{q=1}^{N} C(N,p,j)C(N,q,j)\ \frac{\partial}{\partial t}[G(N,p,t)]\frac{\partial}{\partial t}[G(N,q,t)]. \qquad (31)$$

Let $E^N(t)$ be the total kinetic energy of the finite chain. Then

$$E^N(t) = \sum_{j=o}^{N-1} E_j^N(t)$$

$$= \frac{m}{2} \left(\frac{4}{2N+1}\right)^2 \sum_{j=o}^{N-1} \sum_{p=1}^{N} \sum_{q=1}^{N} C(N,p,j)C(N,q,j) \frac{\partial}{\partial t} [G(N,p,t)] \frac{\partial}{\partial t}[G(N,q,t)]$$

$$= \frac{m}{2} \left(\frac{4}{2N+1}\right)^2 \sum_{p=1}^{N} \sum_{q=1}^{N} \frac{\partial}{\partial t} [G(N,p,t)] \frac{\partial}{\partial t} [G(N,q,t)] \left\{ \sum_{j=o}^{N-1} C(N,p,j)C(N,q,j) \right\}.$$

But

$$\sum_{j=o}^{N-1} C(N,p,j)\ C(N,q,j) =$$

$$\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left(\frac{2q-1}{2N+1}\frac{\pi}{2}\right)\sum_{j=o}^{N-1} \cos\left[\left(\frac{2j+1}{2N+1}\right)\frac{\pi}{2}(2p-1)\right]\cos\left[\left(\frac{2j+1}{2N+1}\right)\frac{\pi}{2}(2q-1)\right]$$

$$= \cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left(\frac{2q-1}{2N+1}\frac{\pi}{2}\right)\sum_{\nu=1}^{N} \cos\left[\left(\frac{2\nu-1}{2N+1}\right)\frac{\pi}{2}(2p-1)\right]\cos\left[\left(\frac{2\nu-1}{2N+1}\right)\frac{\pi}{2}(2q-1)\right]$$

$$= \cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left(\frac{2q-1}{2N+1}\frac{\pi}{2}\right)S(p-1,\ q-1)\ ,$$

$$p,q = 1,2,\ldots,N,$$

where $S(\cdot\,,\cdot)$ is defined by equation (28). It follows from (30) that

$$E^N(t) = \frac{m}{2} \left(\frac{4}{2N+1}\right) \sum_{p=1}^{N} \cos^2 \left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right) \left\{\frac{\partial}{\partial t} G(N,p,t)\right\}^2 .$$

Now $\frac{\partial}{\partial t} [G(N,p,t)] = \left\{-a\omega(N,p) \sin[\omega(N,p)t] + b \cos[\omega(N,p)t]\right\}$;

hence,

$$E^N(t) = \frac{m}{2}\left(\frac{4}{2N+1}\right)\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\left\{-a\omega(N,p)\ \sin[\omega(N,p)t]+b\ \cos[\omega(N,p)t]\right\}^2. \tag{32}$$

Kinetic Energies in the Infinite Spring-Mass Chain

Let $E_j(t)$ be the kinetic energy of the $(j+1)^{th}$ mass in an infinite spring-mass chain for which the differential system (A) is the mathematical model. Then $E_j(t) = \frac{m}{2}[\dot{X}_j(t)]^2$, where $X_j(t)$ is given by equation (3). Differentiating (3) with respect to t gives

$$\dot{X}_j(t) = 2a\sqrt{\beta}\,[J'_{2j}(2\sqrt{\beta}\,t) + J'_{2j+2}(2\sqrt{\beta}\,t)] + b[J_{2j}(2\sqrt{\beta}\,t) + J_{2j+2}(2\sqrt{\beta}\,t)],$$

where a prime (') indicates differentiation with respect to the argument of the function. Use of the identity $J'_r(z) = \frac{1}{2}(J_{r-1}(z) - J_{r+1}(z))$ leads to

$$\dot{X}_j(t) = a\sqrt{\beta}\,[J_{2j-1}(2\sqrt{\beta}\,t) - J_{2j+3}(2\sqrt{\beta}\,t)] + b[J_{2j}(2\sqrt{\beta}\,t) + J_{2j+2}(2\sqrt{\beta}\,t)]\,.$$

Therefore,

$$E_j(t) = \frac{m}{2}\left\{a^2\beta[J_{2j-1}(2\sqrt{\beta}\,t) - J_{2j+3}(2\sqrt{\beta}\,t)]^2 + \right.$$
$$+\ 2ab\sqrt{\beta}\,[J_{2j-1}(2\sqrt{\beta}\,t) - J_{2j+3}(2\sqrt{\beta}\,t)][J_{2j}(2\sqrt{\beta}\,t) + J_{2j+2}(2\sqrt{\beta}\,t)]$$
$$\left. +\ b^2[J_{2j}(2\sqrt{\beta}\,t) + J_{2j+2}(2\sqrt{\beta}\,t)]^2\right\}. \tag{33}$$

Let E(t) be the total kinetic energy of this chain. Then

$$E(t) = \sum_{j=o}^{\infty} E_j(t) \ . \tag{34}$$

It is not immediately evident that this series converges. To find a simplified expression for the series and to prove convergence, a method which is similar to that outlined in the Introduction for finding solutions of the infinite differential systems is used. The total kinetic energy $E^N(t)$ of the truncated chain of N masses has already been found. Next $\lim_{N \to \infty} E^N(t)$ is found. Then this limit is shown to be the same as the infinite series (34), which is the total kinetic energy.

The Riemann sum for

$$m \int_o^1 \cos^2(\tfrac{\pi x}{2}) \left\{ -a \sqrt{\beta} \sin(\tfrac{\pi x}{2}) \sin[2\sqrt{\beta}\, t \sin(\tfrac{\pi x}{2})] \right. \\ \left. + b \cos[2\sqrt{\beta}\, t \sin(\tfrac{\pi x}{2})] \right\}^2 dx \tag{35}$$

with partition points $X_p = \frac{p}{N}$, $p = 0,1,\ldots,N$, and with the integrand evaluated at $X_p' = \frac{2p-1}{2N+1}$ in each interval is

$$\frac{m}{N} \sum_{p=1}^{N} \cos^2(\tfrac{2p-1}{2N+1} \tfrac{\pi}{2}) \left\{ -a\ 2\sqrt{\beta} \sin(\tfrac{2p-1}{2N+1} \tfrac{\pi}{2}) \sin[2\sqrt{\beta}\, t \sin(\tfrac{2p-1}{2N+1} \tfrac{\pi}{2})] \right. \\ \left. + b \cos[2\sqrt{\beta}\, t \sin(\tfrac{2p-1}{2N+1} \tfrac{\pi}{2})] \right\}^2 .$$

The limit of this sum as $N \to \infty$ is the integral (35). $E^N(t)$ given in equation (32) is $\frac{2N}{2N+1}$ times this sum; hence, $\lim_{N \to \infty} E^N(t)$ is the integral. By letting $y = \sin \frac{\pi x}{2}$ in (35), it follows that

$$\lim_{N\to\infty} E^N(t) = \frac{2m}{\pi}\int_0^1 \sqrt{1-y^2}\left\{- a\, 2\sqrt{\beta}\; y \sin[2\sqrt{\beta}ty] + b\cos[2\sqrt{\beta}\, t\, y]\right\}^2 dy$$

$$= \frac{8m\beta a^2}{\pi}\int_0^1 y^2\sqrt{1-y^2}\,\sin^2[2\sqrt{\beta}\,t\,y]\,dy$$

$$- \frac{8mab\sqrt{\beta}}{\pi}\int_0^1 y\sqrt{1-y^2}\;\sin[2\sqrt{\beta}\,ty]\,\cos[2\sqrt{\beta}\,ty]dy$$

$$+ \frac{2mb^2}{\pi}\int_0^1 \sqrt{1-y^2}\;\cos^2[2\sqrt{\beta}\,ty]\,dy\,. \tag{36}$$

One can show (see Appendix A) that

$$\int_0^1 y^2\sqrt{1-y^2}\;\sin^2(zy)\,dy = \frac{\pi}{32} - \frac{1}{6}\int_0^1 (1-y^2)^{3/2}\cos(2zy)\,dy$$

$$+ \frac{(2z)^2}{30}\int_0^1 (1-y^2)^{5/2}\cos(2zy)dy\,, \tag{37a}$$

$$\int_0^1 y\sqrt{1-y^2}\,\sin(zy)\,\cos(zy)dy = \frac{(2z)}{6}\int_0^1 (1-y^2)^{3/2}\cos(2zy)dy, \tag{37b}$$

and

$$\int_0^1 \sqrt{1-y^2}\,\cos^2(zy)dy = \frac{\pi}{8} + \frac{1}{2}\int_0^1 (1-y^2)^{1/2}\cos(2zy)dy\,. \tag{37c}$$

Since

$$J_\nu(2z) = \frac{(z)^\nu}{\Gamma(\frac{1}{2})\Gamma(\nu+\frac{1}{2})}\int_{-1}^1 (1-y^2)^{\nu-\frac{1}{2}}\cos(2zy)dy$$

[3, p. 114], it follows when ν is a positive integer that

$$\frac{\pi}{2}\,\frac{J_\nu(2z)}{(2z)^\nu} = \frac{1}{[1\cdot 3\cdots(2\nu-1)]}\int_0^1 (1-y^2)^{\nu-\frac{1}{2}}\cos(2zy)\,dy\ . \tag{38}$$

Using equation (38) in (37a), (37b), and (37c) leads to

$$\int_0^1 y^2\sqrt{1-y^2}\,\sin^2(zy)dy = \frac{\pi}{32} - \frac{\pi}{4}\,\frac{J_2(2z)}{(2z)^2} + \frac{\pi}{4}\,\frac{J_3(2z)}{(2z)}\ , \tag{39a}$$

$$\int_0^1 y\sqrt{1-y^2}\,\sin(zy)\,\cos(zy)\,dy = \frac{\pi}{4}\,\frac{J_2(2z)}{(2z)}\ , \tag{39b}$$

and

$$\int_0^1 \sqrt{1-y^2}\,\cos^2(zy)dy = \frac{\pi}{8} + \frac{\pi}{4}\,\frac{J_1(2z)}{(2z)}\ . \tag{39c}$$

Using the recurrence relation

$$\frac{J_{r-1}(2z) + J_{r+1}(2z)}{2r} = \frac{J_r(2z)}{2z}$$

[3, p. 100] in equation (39a) gives

$$\int_0^1 y^2\sqrt{1-y^2}\,\sin^2(zy)\,dy = \frac{\pi}{32} - \frac{\pi}{4}\,\frac{1}{(2z)}\left[\frac{J_1(2z)+J_3(2z)}{4}\right] + \frac{\pi}{4}\,\frac{J_3(2z)}{(2z)}$$

$$= \frac{\pi}{32} - \frac{\pi}{16}\,\frac{J_1(2z)}{2z} + \frac{3\pi}{16}\,\frac{J_3(2z)}{2z}\ .$$

Letting $z = 2\sqrt{\beta}\,t$ and substituting this result, together with equations (39b) and (39c), in (36) yields

$$\lim_{N\to\infty} E^N(t) = \frac{m\beta a^2}{4} - \frac{m\beta a^2}{2}\ \frac{J_1(4\sqrt{\beta}\,t)}{4\sqrt{\beta}\,t} - \frac{3J_3(4\sqrt{\beta}\,t)}{4\sqrt{\beta}\,t}$$

$$- 2mab\sqrt{\beta}\ \frac{J_2(4\sqrt{\beta}\,t)}{4\sqrt{\beta}\,t} + \frac{mb^2}{4} + \frac{mb^2}{2}\,\frac{J_1(4\sqrt{\beta}\,t)}{4\sqrt{\beta}\,t}\ .$$

By using the recurrence relation several times, one finds that

$$\lim_{N\to\infty} E^N(t) = \frac{m\beta a^2}{4}[1 - J_o(4\sqrt{\beta}\,t) + J_4(4\sqrt{\beta}\,t)]$$

$$- \frac{mab\sqrt{\beta}}{2}[J_1(4\sqrt{\beta}\,t) + J_3(4\sqrt{\beta}\,t)]$$

$$+ \frac{mb^2}{4}[1 + J_o(4\sqrt{\beta}\,t) + J_2(4\sqrt{\beta}\,t)]\ . \tag{40}$$

It remains to be shown that this limit is actually the total kinetic energy of the infinite chain. Using results (Appendix B, equations (B.1), (B.2) and (B.3)) obtained from Neumann's addition theorem for Bessel functions [4, p. 363], one can prove that

$$[1 - J_o(2z) + J_4(2z)] = 2\sum_{j=o}^{\infty}[J_{2j-1}(z) - J_{2j+3}(z)]^2\ , \tag{41a}$$

$$-[J_1(2z) + J_3(2z)] = 2\sum_{j=o}^{\infty}[J_{2j-1}(z) - J_{2j+3}(z)][J_{2j}(z) + J_{2j+2}(z)], \tag{41b}$$

and

$$[1 + J_o(2z) + J_2(2z)] = 2\sum_{j=o}^{\infty}[J_{2j}(z) + J_{2j+2}(z)]^2\ . \tag{41c}$$

These equations when considered together with equations (33), (34), and (40) imply that $\lim_{N\to\infty} E^N(t)$ is the total kinetic energy of the infinite chain. Hence

$$E(t) = \frac{m\beta a^2}{4}\left[1 - J_o(4\sqrt{\beta}\,t) + J_4(4\sqrt{\beta}\,t)\right] - \frac{mab\sqrt{\beta}}{2}\left[J_1(4\sqrt{\beta}\,t) + J_3(4\sqrt{\beta}\,t)\right] + \frac{mb^2}{4}\left[1 + J_o(4\sqrt{\beta}\,t) + J_2(4\sqrt{\beta}\,t)\right]. \quad (42)$$

CHAPTER IV

ALMOST PERIODIC FUNCTIONS

This chapter contains two sections. In the first section some definitions are made and theorems stated without proof. In the second section a proof of Kronecker's Theorem based on the theory of almost periodic functions is given. These results will be used in Chapter V to compare the kinetic energies in finite and infinite spring-mass chains subject to similar initial conditions.

Definitions and Stated Theorems

In order to give the usual definition of an almost periodic function two preliminary definitions are made.

Definition 1. A subset X of the real numbers is relatively dense if and only if there exists a real number L such that for any real number α there is an element x in the set X with the property that $\alpha \leq x \leq \alpha + L$.

Definition 2. Let f be a complex-valued function defined on the set of all real numbers. Let ε be a positive number. Then τ is a translation number of f corresponding to ε if and only if

$$|f(t) - f(t + \tau)| < \varepsilon$$

for all real t. The set of all translation numbers of a function f corresponding to a positive number ε will be denoted by $E[f; \varepsilon]$.

Definition 3. Let f be a continuous complex-valued function defined on the set of all real numbers. f is almost periodic (a.p.) if and only if for any positive number ε, $E[f; \varepsilon]$ is relatively dense.

It follows from this definition that any continuous periodic function is almost periodic.

The following theorems can be proved with this definition of an almost periodic function.

Theorem 1. If $f(t)$ and $g(t)$ are a.p. functions, then

i) $f(t) + g(t)$ is an a.p. function,

and ii) $[f(t)][g(t)]$ is an a.p. function.

Proof: See [8, pp. 36-38].

Since $A_p e^{i\lambda_p t}$ is almost periodic if λ_p is real, it follows from Theorem 1 that if $\lambda_1, \lambda_2, \ldots, \lambda_p$ are real, then $\sum_{p=1}^{N} A_p e^{i\lambda_p t}$ is almost periodic.

Theorem 2. If $f(t)$ is an a.p. function, then

$$\lim_{T \to \infty} \left[\frac{1}{T} \int_0^T f(t)dt \right] \text{ exists .}$$

Proof: See [8, pp. 39-42]. This limit is denoted by $M\{f(t)\}$ and is called the mean of f.

Theorem 3. If $f(t)$ and $g(t)$ are a.p. functions and c_1 and c_2 are constants, then

$$M\{c_1 f(t) + c_2 g(t)\} = c_1 M\{f(t)\} + c_2 M\{g(t)\} .$$

Proof: $c_1 f(t) + c_2 g(t)$ is an a.p. function by Theorem 1. The proof of Theorem 3 follows from the definition of the mean of an a.p. function.

Theorem 4. If λ_1 and λ_2 are real, then

$$M\left\{e^{i\lambda_1 t}\, e^{-i\lambda_2 t}\right\} = \begin{cases} 0 & \text{if } \lambda_1 \neq \lambda_2 \\ 1 & \text{if } \lambda_1 = \lambda_2 \end{cases} .$$

Proof: See [8, p. 48].

Theorem 5. If $\lambda_1, \lambda_2, \ldots, \lambda_N$ are distinct real numbers, and if $f(t) = \sum_{p=1}^{N} A_p e^{i\lambda_p t}$, then

$$M\left\{e^{-i\lambda t}\, f(t)\right\} = \begin{cases} 0 & \text{if } \lambda \neq \lambda_1, \lambda_2, \ldots, \text{ nor } \lambda_N \\ A_p & \text{if } \lambda = \lambda_p \text{ for some } p \ (1 \leq p \leq N). \end{cases}$$

Proof: The proof is a consequence of Theorems 3 and 4.

Theorem 6. If $f(t)$ is a real a.p. function, then the least upper bound (l.u.b.) of $f(t)$ for all real t is equal to l.u.b. $f(t)$ for positive t, and the greatest lower bound (g.l.b.) of $f(t)$ for all real t is equal to g.l.b. $f(t)$ for positive t.

Proof. The proof of this theorem follows from the fact that there are arbitrarily large translation numbers of $f(t)$ for any positive number ε.

A Proof of Kronecker's Theorem

Theorem 7. (Kronecker's Theorem). Let ε be an arbitrary positive number; let $\varphi_1, \varphi_2, \ldots, \varphi_N$ be arbitrary real numbers. If $\lambda_1, \lambda_2, \ldots, \lambda_N$ are real numbers such that for every set of rational numbers $r_1, r_2, \ldots, r_N$ which are not all zero

$$r_1\lambda_1 + r_2\lambda_2 + \ldots + r_N\lambda_N \neq 0 \text{ --}$$

that is, if the set $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ is linearly independent over the rational numbers --, then there exists a number t and integers $n_1, n_2, \ldots, n_N$ such that

$$|\lambda_p t + \varphi_p - n_p| < \varepsilon \ , \quad p = 1,2,\ldots, N \ .$$

Proof: First, two lemmas are proved.

Lemma 1. If $f(t)$ and $g(t)$ are two a.p. functions such that $|f(t)| \leq u$ for all t (where u is a constant) and $g(t)$ is real and non-negative, then

$$|M\{f(t)g(t)\}| \leq uM\{g(t)\} \ .$$

Proof: $$|M\{f(t)\ g(t)\}| = |\lim_{T\to\infty}[\frac{1}{T}\int_0^T f(t)g(t)dt]|$$

$$= \lim_{T\to\infty} |\frac{1}{T}\int_0^T f(t)g(t)dt| \ .$$

Hence

$$|M\{f(t)g(t)\}| \leq \lim_{T\to\infty} \frac{1}{T}\int_0^T |f(t)||g(t)|dt$$

$$\leq \lim_{T\to\infty} \frac{u}{T}\int_0^T g(t) \ .$$

Therefore

$$|M\{f(t)g(t)\}| \leq u\,M\{g(t)\} \ .$$

Lemma 2. If $\lambda_1, \lambda_2, \ldots \lambda_N$ are real numbers such that for every set of rational numbers $r_1, r_2, \ldots, r_N$ which are not all zero

$$r_1\lambda_1 + r_2\lambda_2 + \ldots + r_N\lambda_N \neq 0 ,$$

and if

$$f(t) = A_o + \sum_{p=1}^{N} A_p e^{i\lambda_p t} ,$$

where $A_o, A_1, A_2, \ldots, A_N$ are complex numbers, then the l.u.b. $|f(t)|$ for all real t is

$$\sum_{p=o}^{N} |A_p| .$$

<u>Proof</u>: It suffices to assume that the constant term A_o of $f(t)$ is real and non-negative, since if $A_o \neq 0$,

$$|f(t)| = \left| \frac{|A_o|}{A_o} f(t) \right|$$

for all real t, and the constant term of $\frac{|A_o|}{A_o} f(t)$ is real and non-negative (A_o is the unique constant term of $f(t)$ since if $\lambda_p = 0$ for any p, the λ_p's are not linearly independent over the rational numbers). Let U be the l.u.b. $|f(t)|$. $\sum_{p=o}^{N} |A_p|$ is clearly an upper bound of $|f(t)|$; hence

$$U \leq \sum_{p=o}^{N} |A_p| . \tag{43}$$

Now consider the q^{th}-order Fejér kernel defined by

$$K_q(t) = \sum_{\nu=-q}^{q} \left(1 - \frac{|\nu|}{q}\right) e^{-\nu t} .$$

It can be shown [8, p. 24] that

$$K_q(t) = \frac{1}{q}\left(\frac{\sin(\frac{qt}{2})}{\sin(\frac{t}{2})}\right)^2 ,$$

and thus $K_q(t)$ is real and non-negative. Let $V_1, V_2, \ldots, V_N$ be real numbers such that

$$A_p = |A_p| e^{iV_p}, \quad p = 1,2,\ldots,N .$$

Define

$$\tilde{K}_q(t) = \prod_{p=1}^{N} K_q(\lambda_p t + V_p) .$$

It follows from the properties of $K_q(t)$ that $\tilde{K}_q(t)$ is real and non-negative; hence by Lemma 1

$$|M\left\{f(t)\tilde{K}_q(t)\right\}| \leq UM\left\{\tilde{K}_q(t)\right\} . \tag{44}$$

By multiplying out the expression for $\tilde{K}_q(t)$ one obtains

$$\tilde{K}_q(t) = 1 + \frac{q-1}{q} \sum_{p=1}^{N} e^{-i\lambda_p t} e^{-iV_p} + R(t) , \tag{45}$$

where $R(t)$ is a sum of the form $\Sigma a_s e^{i\Lambda_s t}$ with none of the Λ_s equal to 0, $-\lambda_1, -\lambda_2, \ldots,$ nor $-\lambda_N$ because of the linear independence of $\lambda_1, \lambda_2, \ldots, \lambda_N$ over the rational numbers. From Theorem 5 and equation (45) it follows that

$$M\left\{\tilde{K}_q(t)\right\} = M\left\{e^{-i0t}\,\tilde{K}_q(t)\right\} = 1 .$$

This fact together with inequality (44) implies that

$$\left|M\left\{f(t)\tilde{K}_q(t)\right\}\right| \leq u . \tag{46}$$

From equation (45) and Theorem 3 it follows that

$$M\left\{f(t)\tilde{K}_q(t)\right\} = M\left\{f(t)\right\} + \frac{q-1}{q}\sum_{p=1}^{N} e^{-iV_p}\, M\left\{e^{-i\lambda_p t}\, f(t)\right\}$$

$$+ \Sigma\, a_s M\left\{e^{i\Lambda_s t}\, f(t)\right\} .$$

Use of Theorem 5, the definition of f(t), and the fact that the Λ_s's are not $0, -\lambda_1, -\lambda_2, \ldots, -\lambda_N$ reduces this equation to

$$M\left\{f(t)\tilde{K}_q(t)\right\} = A_o + \frac{q-1}{q}\sum_{p=1}^{N} A_p e^{-iV_p} .$$

Since A_o is assumed to be real and non-negative and $A_p = |A_p|\, e^{iV_p}$,

$$M\left\{f(t)\tilde{K}_q(t)\right\} = |A_o| + \frac{q-1}{q}\sum_{p=1}^{N} |A_p| .$$

Hence from inequality (46) it follows that

$$|A_o| + \frac{q-1}{q} \sum_{p=1}^{N} |A_p| \leq U .$$

Since q is an arbitrary positive integer, taking the limit as $q \to \infty$ gives

$$\sum_{p=o}^{N} |A_p| \leq U . \tag{47}$$

Together, inequalities (43) and (47) prove the lemma.

To prove Kronecker's Theorem suppose that ε is an arbitrary positive number and $\varphi_1, \varphi_2, \ldots, \varphi_N$ are arbitrary real numbers. Consider the function

$$f(t) = 1 + \sum_{p=1}^{N} e^{i2\pi\varphi_p} e^{i2\pi\lambda_p t}$$

$$= 1 + \sum_{p=1}^{N} e^{i2\pi(\lambda_p t + \varphi_p)} .$$

If $\lambda_1, \lambda_2, \ldots, \lambda_N$ are real numbers such that for every set of rational numbers $r_1, r_2, \ldots, r_N$ which are not all zero

$$r_1\lambda_1 + r_2\lambda_2 + \ldots + r_N\lambda_N \neq 0 ,$$

then $2\pi\lambda_1$, $2\pi\lambda_2, \ldots, 2\pi\lambda_N$ are real numbers which also possess this property. Hence by Lemma 2, l.u.b.$|f(t)|$ is N+1. The function f(t) equals one plus a sum of variable complex terms which is always less than or equal to N in absolute value -- that is, $f(t) = 1 + r(t)\, e^{i\varphi t}$, $r(t) \leq N$. $|f(t)|$ can be near N+1 only if f(t) is near N+1. Since l.u.b. $|f(t)|$ is N+1, there is a sequence t_j such that

$$\lim_{j \to \infty} |f(t_j) - (N+1)| = 0 . \tag{48}$$

However,

$$|f(t_j) - (N+1)| \geq |\mathrm{Re}[N+1 - f(t_j)]| ,$$

and

$$\mathrm{Re}\,[N+1 - f(t_j)] = \sum_{p=1}^{N} \left\{1 - \cos\,[2\pi(\lambda_p t_j + \varphi_p)]\right\} .$$

Since the terms of this sum are all non-negative,

$$|f(t_j) - (N+1)| \geq \left\{1 - \cos[2\pi(\lambda_p t_j + \varphi_p)]\right\} ,$$

$$p = 1,2,\ldots,N .$$

Let $n_{j,p}$ be the integers such that

$$\lambda_p t_j + \varphi_p - \frac{1}{2} \leq n_{j,p} < \lambda_p t_j + \varphi_p + \frac{1}{2} ;$$

let

$$\varepsilon_{j,p} = \lambda_p t_j + \varphi_p - n_{j,p} .$$

Then

$$|f(t_j) - (N+1)| \geq 1 - \cos[2\pi(n_{j,p} + \varepsilon_{j,p})]$$

$$= 1 - \cos 2\pi\varepsilon_{j,p} .$$

Assume with no loss in generality that $\varepsilon < \frac{1}{2}$. Suppose that for each j there exists an integer p for which $|\varepsilon_{j,p}| > \varepsilon$. Then, for this integer p

$$\varepsilon < |\varepsilon_{j,p}| < \frac{1}{2} ,$$

and hence,

$$|f(t_j) - (N+1)| \geq 1 - \cos 2\pi\varepsilon > 0$$

for each j. The fact that $|N+1 - f(t_j)|$ is bounded away from zero for all integers j contradicts (48). Hence there is a j for which

$$|\varepsilon_{j,p}| = |\lambda_p t_j + \varphi_p - n_{j,p}| < \varepsilon, \quad p = 1,2,\ldots,N .$$

Thus the theorem is proved.

A corollary will be useful in Chapter IV.

Corollary 1. Assume that $\varphi_1,\varphi_2,\ldots,\varphi_N$ and $A_o,A_1,\ldots A_N$ are arbitrary real numbers. If $\lambda_1,\lambda_2,\ldots,\lambda_N$ are real numbers such that for every set of rational numbers $r_1,r_2,\ldots,r_N$ which are not all zero

$$r_1\lambda_1 + r_2\lambda_2 + \ldots + r_N\lambda_N \neq 0 ,$$

and if

$$f(t) = A_o + \sum_{p=1}^{N} A_p \cos (\lambda_p t + \varphi_p) ,$$

then

i) l.u.b. f(t) for all positive t is

$$A_o + \sum_{p=1}^{N} |A_p|$$

and

ii) g.l.b. $f(t)$ for all positive t is

$$A_o - \sum_{p=1}^{N} |A_p| .$$

Proof: Since the proofs of part i) and part ii) are very similar, only the proof of part i) is given. By Theorem 6 it suffices to prove that l.u.b. $f(t)$ for all real t is $A_o + \sum_{p=1}^{N} |A_p|$. Obviously this quantity is an upper bound. Let ε be an arbitrary positive number. Next it is shown that there is a real t such that $|f(t) - (A_o + \sum_{p=1}^{N} |A_p|)| < \varepsilon$, and thus part i) of the corollary is proved. Let

$$\delta_p = \begin{cases} 0 & \text{if } A_p \geq 0 \\ 1 & \text{if } A_p < 0 \end{cases} \quad p = 1,2,\ldots,N .$$

Then

$$\begin{aligned} A_p \cos(X) &= A_p \cos(X + \delta_p\pi - \delta_p\pi) \\ &= A_p \cos(\delta_p\pi) \cos(X + \delta_p\pi) \\ &= |A_p| \cos(X + \delta_p\pi), \quad p = 1,2,\ldots,N . \end{aligned} \qquad (49)$$

From the definition of $f(t)$

$$\begin{aligned} |f(t) - (A_o + \sum_{p=1}^{N} |A_p|)| &= |\sum_{p=1}^{N} (A_p \cos(\lambda_p t + \varphi_p) - |A_p|)| \\ &\leq \sum_{p=1}^{N} |A_p \cos(\lambda_p t + \varphi_p) - |A_p|| , \end{aligned}$$

and from (48)

$$|f(t) - (A_o + \sum_{p=1}^{N} |A_p|)| \le \sum_{p=1}^{N} |A_p||\cos(\lambda_p t + \varphi_p + \delta_p \pi) - 1|. \qquad (50)$$

Since cos x is continuous and cos $(2n\pi) = 1$ if n is an integer, there is a positive number δ such that if x is any number with the property that there is an integer n for which $|x - 2n\pi| < \delta$, then

$$|\cos x - 1| < \frac{\varepsilon}{\sum_{p=1}^{N} |A_p|} .$$

By Kronecker's Theorem, there is a real number T and integer n_p such that

$$|\lambda_p T + \frac{\varphi_p}{2\pi} + \frac{\delta_p}{2} - n_p| < \frac{\delta}{2\pi}, \quad p = 1,2,\ldots,N .$$

Letting $t = 2\pi T$, one finds that

$$|\lambda_{pt} + \varphi_p + \delta_p \pi - 2n_p \pi| < \delta ,$$

and hence

$$|\cos(\lambda_p t + \varphi_p + \delta_p \pi) - 1| < \frac{\varepsilon}{\sum_{p=1}^{N} |A_p|} .$$

It follows from inequality (50) that

$$|f(t) - (A_o + \sum_{p=1}^{N} |A_p|)| < \varepsilon .$$

CHAPTER V

PROPERTIES OF THE KINETIC ENERGIES IN THE SPRING-MASS CHAINS

This chapter is devoted to a comparison of the kinetic energies in the infinite and finite spring-mass chains for which the systems (A) and (A') are the respective mathematical models. For the kinetic energy of the individual masses and the total kinetic energy of each system, the following questions are considered (though not completely answered in every case):

1) Does the quantity have a limit as $t \to \infty$? If it does, what is the limit?

2) What are the least upper bound and greatest lower bound of the quantity for $t > 0$?

The $(j+1)^{th}$ component $X_{N,j}(t)$ of the solution of (A') found in Chapter III is defined only for $t \geq 0$ and is therefore not an almost periodic function as defined in Chapter IV. However the expression for $X_{N,j}(t)$ given in equation (16) and the expressions for $E_j^N(t)$ in (31) and $E^N(t)$ in (32), if considered for all real t, are almost periodic functions. Corollary 1 can then be applied to these quantities if it can be shown that there is no set of rational numbers $\{r_1, r_2, \ldots, r_N\}$ not all of which are zero such that

$$r_1\omega(N,1) + r_2\omega(N,2) + \ldots + r_N\omega(N,N) = 0 .$$

Though the non-existence of such rational numbers cannot be proved in general, by using a method similar to one used by C. Hemmer in [9, pp. 21-22, 68-70] one can prove

<u>Theorem 8</u>. There exist rational numbers $r_1, r_2, \ldots, r_N$ not all of which are zero such that

$$r_1\omega(N,1) + r_2\omega(N,2) + \ldots + r_N\omega(N,N) = 0$$

if and only if 2N+1 is not a prime.

<u>Proof</u>: Several concepts introduced in the proof of this theorem are not pertinent to the discussion of the spring-mass chains, and therefore the proof is given in Appendix C.

<u>Limits of the Kinetic Energy of Individual Masses</u>

The kinetic energy $E_j^N(t)$ of the $(j+1)^{th}$ mass in the finite chain is $m/2[\dot{X}_{N,j}(t)]^2$. It follows from equations (16) and (25) that

$$\begin{aligned}\dot{X}_{N,j}(t) &= \frac{4}{2N+1}\sum_{p=1}^{N}\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\,\frac{\pi}{2}\,(2j+1)\right]\Big\{-a\omega(N,p)\sin(\omega(N,p)t) \\ &\qquad + b\cos(\omega(N,p)t)\Big\} \\ &= \frac{4}{2N+1}\sum_{p=1}^{N}\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\,\frac{\pi}{2}\,(2j+1)\right]\Big\{\sqrt{a^2\omega^2(N,p)+b^2} \\ &\qquad \times\cos(\omega(N,p)t+\varphi_{N,p})\Big\},\quad j=0,1,2,\ldots,N-1, \qquad (51)\end{aligned}$$

where $\varphi_{N,p}$ are real numbers such that

$$\cos\varphi_{N,p} = b\Big/\sqrt{a^2\omega^2(N,p)+b^2} \qquad (52a)$$

and

$$\sin \varphi_{N,p} = a\omega(N,p)\Big/\sqrt{a^2\omega^2(N,p) + b^2}\ , \qquad (52b)$$

$$p = 1,2,\ldots,N\ .$$

Suppose that $\dot{X}_{N,j}(t)$ is a constant c. From (51), by integrating from 0 to T, dividing by T, and taking the limit as $T \to \infty$, one finds that $M\{\dot{X}_{N,j}(t)\} = 0$. But $M\{c\} = c$. Hence $c = 0$. Thus if $\dot{X}_{N,j}(t)$ is a constant, then $\dot{X}_{N,j}(t) \equiv 0$; and

$$M\left\{\cos(\omega(N,1)t + \varphi_{N,1})\dot{X}_{N,j}(t)\right\} = M\left\{\cos(\omega(N,1)t + \varphi_{N,1})0\right\} = 0\ .$$

But from (51) it follows that

$$M\left\{\cos(\omega(N,1)t + \varphi_{N,1})\dot{X}_{N,j}(t)\right\} = \frac{1}{2}\cos\left(\frac{\pi}{(2N+1)^2}\right)\cos\left(\frac{2j+1}{2N+1}\,\frac{\pi}{2}\right)\sqrt{a^2\omega^2(N,1)+b^2}$$

$$\neq 0\ ,\quad j = 0,1,2,\ldots,N-1\ .$$

Hence by contradiction $\dot{X}_{N,j}(t)$ is not a constant. $E_j^N(t)$ could be a constant only if $|\dot{X}_{N,j}(t)|$ were constant. Since $\dot{X}_{N,j}(t)$ is continuous and not a constant, it follows that $E_j^N(t)$ is not a constant.

A non-constant almost periodic function f cannot have a limit as $t \to \infty$, since it will have at least two distinct values in its range and there will be arbitrarily large t's for which f(t) is arbitrarily close to each of these values. Hence the kinetic energy of each individual mass in the finite chain has no limit as $t \to \infty$.

For any integer ν, $\lim_{X \to \infty} J_\nu(X) = 0$ [10, p. 170]. It therefore follows from equation (33) that the kinetic energy of each individual mass in the infinite chain has a limit of zero as $t \to \infty$.

Least Upper Bounds of the Kinetic Energies of Individual Masses

Let

$$d_{N,j} = \underset{t>0}{\text{l.u.b.}} |\dot{X}_{N,j}(t)|, \quad j = 0,1,2,\ldots,N-1.$$

Then

$$\underset{t>0}{\text{l.u.b.}}\ E_j^N(t) = \frac{m}{2}[d_{N,j}]^2 .$$

The value of $d_{N,j}$ has not been found for the general case, but if 2N+1 is a prime, it follows from equation (51), Theorem 8, and Corollary 1 that

$$\underset{t>0}{\text{l.u.b.}}\ \dot{X}_{N,j}(t) = \frac{4}{2N+1}\sum_{p=1}^{N} \left|\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\frac{\pi}{2}(2j+1)\right]\right| \sqrt{a^2\omega^2(N,p)+b^2} \tag{53a}$$

and

$$\underset{t>0}{\text{g.l.b.}}\ \dot{X}_{N,j}(t) = -\frac{4}{2N+1}\sum_{p=1}^{N} \left|\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\frac{\pi}{2}(2j+1)\right]\right| \sqrt{a^2\omega^2(N,p)+b^2} \tag{53b}$$

$$j = 0,1,2,\ldots,N-1 .$$

A simplified expression for this sum when a and b are arbitrary has not been found; however, a more thorough analysis can be made for the special case 2N+1 a prime and a = 0. In this case the finite chain has all masses initially at their reference positions and all masses stationary except the first, which has velocity b. System (A') with a = 0 is the mathematical model of this chain. Then from (53a) and (53b)

$$d_{N,j} = \frac{4|b|}{2N+1} \sum_{p=1}^{N} \left|\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\,\cos\left[\frac{2p-1}{2N+1}\,\frac{\pi}{2}\,(2j+1)\right]\right| , \qquad (54)$$

$$j = 0,1,2,\ldots,N-1 .$$

For $N = 5,6$ $d_{N,j}/|b|$ $j=0,1,2,\ldots,N-1$ are given to four significant figures in Table 1.

Table 1. $d_{N,j}/|b|$ $(j = 0,1,2,\ldots,N-1)$

N \ j	0	1	2	3	4	5
5	1.000	0.8412	0.8576	0.8412	0.8576	--
6	1.000	0.8386	0.8301	0.8551	0.8386	0.8551

In the table $d_{N,0}/|b| = 1$ and $d_{N,j}/|b| < 1$, $j=1,2,\ldots,N-1$ -- a fact which, as will be shown immediately, can be proved in general for this case ($2N+1$ prime, $a=0$).

<u>$d_{N,0}/|b| = 1$</u>

From (54),

$$d_{N,0}/|b| = \frac{4}{2N+1} \sum_{p=1}^{N} \left|\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\,\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\right|$$

$$= \frac{4}{2N+1} \sum_{p=1}^{N} \cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)$$

$$= \frac{4}{2N+1}\, S(0,\, 0) ,$$

where $S(\cdot,\, \cdot)$ is defined by equation (28). $S(0,0)$ was found to be equal to 2N+1/4; hence

$$d_{N,0}/|b| = 1 .$$

$d_{N,j}/|b| < 1,\ j=1,2,\ldots,N-1$

For p = 1,2,...,N and j = 1,2,...,N-1, there exists an integer $q_{p,j}$ such that

$$-\frac{1}{2} \leq \frac{(2p-1)(2j+1)}{2N+1}\frac{1}{2} - q_{p,j} < \frac{1}{2} . \tag{55}$$

The possibility of equality on the left is actually excluded since both 2p-1 and 2j+1 are less than 2N+1, and 2N+1 is now assumed to be a prime. $|(2p-1)(2j+1) - 2q_{p,j}(2N+1)|$ is an odd integer, and from (55) with equality excluded it follows that

$$|(2p-1)(2j+1) - 2q_{p,j}(2N+1)| < 2N+1 .$$

Define $I(p,q) = |(2p-1)(2j+1) - 2q_{p,j}(2N+1)|$. Suppose that for some fixed j $I(p,j) = I(p',j)$ for $1 \leq p,p' \leq N$, $p \neq p'$. If

$$(2p-1)(2j+1) - 2q_{p,j}(2N+1) = (2p'-1(2j+1) - 2q_{p,j}(2N+1) ,$$

it follows that

$$(p-p')(2j+1) = (q_{p',j} - q_{p,j})(2N+1) .$$

Since the left side of this equation is not zero and 2N+1 is a prime, it follows that 2N+1 must divide either (p-p') or (2j+1). But (p-p') $(1 \leq p,p' \leq N;\ p \neq p')$ and (2j+1) $(1 \leq j \leq N-1)$ are both less than 2N+1.

Thus, this equation is a contradiction. Now if

$$I(p,j) = I(p',j)$$

and

$$(2p-1)(2j+1) - 2q_{p,j}(2N+1) \neq (2p'-1)(2j+1) - 2q_{p',j}(2N+1),$$

it follows that

$$(2p-1)(2j+1) - 2q_{p,j}(2N+1) = -(2p'-1)(2j+1) + 2q_{p',j}(2N+1) ,$$

from which one can obtain

$$(p+p')(2j+1) = (q_{p',j} + q_{p,j})(2N+1) .$$

By the same reasoning as before, this equation is a contradiction. Hence $I(p,j) \neq I(p',j)$ if $p \neq p'$.

Since $|\cos(\theta)| = |\cos(\theta + n\pi)|$ for n an integer and since $\cos\theta \geq 0$ for $-\frac{\pi}{2} \leq \theta < \frac{\pi}{2}$,

$$\left|\cos\left[\frac{(2p-1)(2j+1)}{2N+1}\,\frac{\pi}{2}\right]\right| = \left|\cos\left[\frac{(2p-1)(2j+1)}{2N+1}\,\frac{\pi}{2} - q_{p,j}\pi\right]\right|$$

$$= \cos\left[\frac{(2p-1)(2j+1) - 2q_{p,j}(2N+1)}{2N+1}\,\frac{\pi}{2}\right] .$$

Since $\cos\theta = \cos|\theta|$,

$$\left|\cos\left[\frac{(2p-1)(2j+1)}{2N+1}\,\frac{\pi}{2}\right]\right| = \cos\left[\frac{I(p,j)}{2N+1}\,\frac{\pi}{2}\right] .$$

In addition $\left|\cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\right| = \cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)$ $(p = 1,2,\ldots,N)$; and therefore from equation (54) it follows that

$$d_{N,j}/|b| = \frac{4}{2N+1} \sum_{p=1}^{N} \cos(\frac{2p-1}{2N+1} \frac{\pi}{2}) \cos(\frac{I(p,j)}{2N+1} \frac{\pi}{2}) , \tag{56}$$

$$j = 1,2,\ldots,N-1 .$$

With j fixed $I(p,j)$, $p = 1,2,\ldots,N$, are N distinct odd positive integers less than $2N+1$ -- that is, the integers $1,3,\ldots,2N-1$ (though not in that order). The sum

$$\sum_{p=1}^{N} \cos(\frac{2p-1}{2N+1} \frac{\pi}{2}) \cos (\frac{I(p,j)}{2N+1} \frac{\pi}{2})$$

can be regarded as the dot product of two vectors

$$[\cos(\frac{1}{2N+1} \frac{\pi}{2}), \cos(\frac{3}{2N+1} \frac{\pi}{2}) ,\ldots, \cos(\frac{2N-1}{2N+1} \frac{\pi}{2})]$$

and

$$[\cos(\frac{I(1,j)}{2N+1} \frac{\pi}{2}), \cos(\frac{I(2,j)}{2N+1} \frac{\pi}{2}),\ldots,\cos(\frac{I(N,j)}{2N+1} \frac{\pi}{2})] .$$

In the first vector, each component is larger than all the components following it. From the definition of $I(p,j)$, $I(1,j) = 2j+1$, and hence the first component of the second vector is $\cos(\frac{2j+1}{2N+1} \frac{\pi}{2})$. One of the $I(p,j)$, $p = 2,3,\ldots,N$, must be 1; hence there is a component of the second vector which is greater than its first component. Therefore neither of these vectors can be a constant multiple of the other. Using the Cauchy-Schwarz inequality with equality excluded yields

$$\left[\sum_{p=1}^{N} \cos(\frac{2p-1}{2N+1} \frac{\pi}{2})\cos(\frac{I(p,j)}{2N+1} \frac{\pi}{2})\right]^2 < \sum_{p=1}^{N} \cos^2(\frac{2p-1}{2N+1} \frac{\pi}{2}) \sum_{p=1}^{N} \cos^2(\frac{I(p,j)}{2N+1} \frac{\pi}{2}) .$$

Since $\quad \left\{I(p,j)\,|\,p = 1,2,\ldots,N\right\} = \left\{1,3,5,\ldots,2N-1\right\}$,

$$\sum_{p=1}^{N} \cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) = \sum_{p=1}^{N} \cos^2\left(\frac{I(p,j)}{2N+1}\,\frac{\pi}{2}\right) .$$

Hence

$$\sum_{p=1}^{N} \cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) \cos\left(\frac{I(p,j)}{2N+1}\,\frac{\pi}{2}\right) < \sum_{p=1}^{N} \cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) .$$

But

$$\sum_{p=1}^{N} \cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) = S(0,0) = \frac{2N+1}{4} ,$$

and therefore

$$\frac{4}{2N+1} \sum_{p=1}^{N} \cos\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) \cos\left[\frac{I(p,j)}{2N+1}\,\frac{\pi}{2}\right] < 1 .$$

From this equation and (56), it follows that

$$d_{N,j}/|b| < 1 \quad j=1,2,\ldots,N-1 .$$

A physical interpretation of these facts for the special case under consideration is as follows. At $t = 0$ the total kinetic energy in the chain is the kinetic energy $mb^2/2$ of the first mass. $d_{N,0}/|b| = 1$ and $d_{N,j}/|b| < 1$, $j = 1,2,\ldots,N-1$, mean, respectively, that the kinetic energy of the first mass for some $t > 0$ is arbitrarily close to the total kinetic energy of the chain, and that the kinetic energy of any other mass is bounded away from the total kinetic energy for all $t > 0$.

Now consider the kinetic energy of the $(j+1)^{th}$ mass in the infinite chain for which the system (A) is the mathematical model. An expression for $\underset{t>o}{\text{l.u.b.}}\ E_j(t)$ has not been found. However since $E_j(t) \geq 0$, it follows that $\underset{t>o}{\text{l.u.b.}}\ E_j(t) \geq 0$. Furthermore since $\lim_{t\to\infty} E_j(t) = 0$ and $E_j(t)$ is continuous, $\underset{t>o}{\text{l.u.b.}}\ E_j(t)$ is attained by $E_j(t)$ at some finite t.

As in the finite chain a more thorough analysis can be made for the infinite chain if a = 0. For this case it follows from the initial conditions that the total energy of the chain is $mb^2/2$. Since $E_o(0) = mb^2/2$ and $E_o(t)$ is continuous, $\underset{t>o}{\text{l.u.b.}}\ E_o(t) = mb^2/2$. Now suppose that for some $j > 0$ $\underset{t>o}{\text{l.u.b.}}\ E_j(t) = mb^2/2$. From the preceding paragraph it follows that there is some t_o such that $E_j(t_o) = mb^2/2$. Since $E_j(o) = 0$, $t_o > 0$. Hence at some positive t_o, all the energy in the chain is kinetic energy. Thus no spring is stretched at t_o --

$$X_n(t_o) - X_{n-1}(t_o) = 0, \quad n = 1,2,\ldots .$$

But from equation (3) with a = 0, it follows that

$$X_n(t) - X_{n-1}(t) = b\int_o^t [J_{2n}(2\sqrt{\beta}\, s) + J_{2n+2}(2\sqrt{\beta}\, s) - J_{2n-2}(2\sqrt{\beta}\, s) - J_{2n}(2\sqrt{\beta}\, s)]\, ds.$$

From the relation $2J_r'(z) = (J_{r-1}(z) - J_{r+1}(z))$ [3, p. 100] it follows that

$$X_n(t) - X_{n-1}(t) = -2b\int_o^t [J'_{2n-1}(2\sqrt{\beta}\, s) + J'_{2n+1}(2\sqrt{\beta}\, s)]ds$$
$$= -\frac{b}{\sqrt{\beta}}[J_{2n-1}(2\sqrt{\beta}\, t) + J_{2n+1}(2\sqrt{\beta}\, t)] .$$

Hence if $E_j(t_o) = \frac{mb^2}{2}$,

$$J_{2n-1}(2\sqrt{\beta}\, t_o) + J_{2n+1}(2\sqrt{\beta}\, t_o) = 0, \; n=1,2,\ldots .$$

Then

$$\sum_{p=n}^{q} (-1)^p [J_{2p-1}(2\sqrt{\beta}\, t_o) + J_{2p+1}(2\sqrt{\beta}\, t_o)] = 0$$

$$= (-1)^n J_{2n-1}(2\sqrt{\beta}\, t_o) + (-1)^q J_{2q+1}(2\sqrt{\beta}\, t_o),$$

for any positive integers n and q. Since $\lim_{v \to \infty} J_v(x) = 0$ [10, p. 176], taking the limit as $q \to \infty$ yields

$$(-1)^n J_{2n-1}(2\sqrt{\beta}\, t_o) = 0$$

for any positive integer n. But two Bessel functions of different integer order cannot have a common positive zero [11, p. 484]. Thus by contradiction,

$$E_j(t_o) \neq mb^2/2$$

for any $j > 0$ and any t_o; and hence $\text{l.u.b.}_{t>o}\, E_j(t) < mb^2/2$. As in the finite chain with $a = 0$ and $2N+1$ a prime, the least upper bound of the kinetic energy of the first mass in the infinite chain is the total energy of the chain, and the least upper bound of the kinetic energy of any other mass is less than the total energy.

<u>Greatest Lower Bounds of the Kinetic Energies of Individual Masses</u>

It was shown previously in this chapter that $M\left\{\dot{X}_{N,j}(t)\right\} = 0$, $j = 0,1,2,\ldots,N-1$. Now suppose that $|\dot{X}_{N,j}(t)|$ is bounded away from zero --

that is, suppose that there exists an $\varepsilon > 0$ such that $|\dot{X}_{N,j}(t)| > \varepsilon$ for all $t > 0$. Since $\dot{X}_{N,j}(t)$ is continuous, $\dot{X}_{N,j}(t)$ must then be positive for all $t > 0$ or negative for all $t > 0$. If $\dot{X}_{N,j}(t)$ is positive, it follows that

$$M\left\{\dot{X}_{N,j}(t)\right\} > \varepsilon .$$

If $\dot{X}_{N,j}(t)$ is negative, it follows that

$$M\left\{\dot{X}_{N,j}(t)\right\} < -\varepsilon .$$

But either of these conditions is a contradiction, since $M\left\{\dot{X}_{N,j}(t)\right\} = 0$. Hence $|\dot{X}_{N,j}(t)|$ is not bounded away from zero, and consequently

$$\underset{t>o}{\text{g.l.b.}}\; E_j^N(t) = 0 .$$

Since $E_j(t) \geq 0$ and $\lim_{t\to\infty} E_j(t) = 0$, it follows that

$$\underset{t>o}{\text{g.l.b.}}\; E_j(t) = 0 .$$

Thus in both the finite and infinite chains the greatest lower bound of the kinetic energy of each individual mass is zero.

<u>Limits of the Total Kinetic Energies</u>

From equation (32) it follows that the total kinetic energy of the finite chain of N masses is

$$E^N(t) = \frac{m}{2}\,\frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\!\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p)+b^2\right\}\cos^2(\omega(N,p)t+\varphi_{N,p}) ,$$

where $\varphi_{N,p}$, $p = 1,2,\ldots,N$, satisfy (52a) and (52b). Using the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, one finds that

$$E^N(t) = \frac{m}{4}\,\frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p) + b^2\right\} +$$

$$\frac{m}{4}\,\frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p) + b^2\right\}\cos(2\omega(N,p)t+2\varphi_{N,p}). \tag{57}$$

By reasoning similar to that used in the first of this chapter to show that $\dot{x}_{N,j}(t)$ has no limit as $t \to \infty$, it follows that the non-constant part of $E^N(t)$,

$$\frac{m}{4}\,\frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p) + b^2\right\}\cos(2\omega(N,p)t+2\varphi_{N,p}) ,$$

has no limit as $t \to \infty$. Hence $E^N(t)$ has no limit as $t \to \infty$.

Although $\lim_{t\to\infty} E^N(t)$ does not exist, it is interesting to note the value of the mean of $E^N(t)$. By integrating (57) from 0 to T, dividing by T, and taking the limit as $T \to \infty$, one finds that

$$M\left\{E^N(t)\right\} = \frac{m}{4}\,\frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p) + b^2\right\} .$$

Since $\omega(N,p) = 2\sqrt{\beta}\,\sin\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)$,

$$M\left\{E^N(t)\right\} = \frac{m}{4}\,\frac{4}{2N+1}\sum_{p=1}^{N} a^2\beta\, 4\cos^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)\sin^2\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right)$$

$$+ \frac{m}{4}\,\frac{4}{2N+1}\,b^2\,S(0,\,0) ,$$

where $S(\cdot, \cdot)$ is defined by equation (28). Using the identity $2 \cos \theta \sin \theta = \sin 2\theta$ and the value of $S(0, 0)$ given by (30), one finds that

$$M\left\{E^N(t)\right\} = \frac{m}{4}\,\frac{4a^2\beta}{2N+1}\sum_{p=1}^{N}\sin^2\left(\frac{2p-1}{2N+1}\,\pi\right) + \frac{mb^2}{4}\,. \tag{58}$$

Since

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \quad \text{and} \quad \sum_{p=1}^{N}\cos((2p-1)\theta) = \frac{\sin 2N\theta}{2\sin\theta}$$

if $\sin\theta \neq 0$ [7, p. 366] ,

$$\sum_{p=1}^{N}\sin^2\left(\frac{2p-1}{2N+1}\,\pi\right) = \sum_{p=1}^{N}\frac{1-\cos\left(\frac{2p-1}{2N+1}\,2\pi\right)}{2}$$

$$= \frac{N}{2} - \frac{\sin\left[\frac{(2N)(2\pi)}{2N+1}\right]}{4\sin\left(\frac{2\pi}{2N+1}\right)}$$

$$= \frac{N}{2} - \frac{\sin\left(2\pi - \frac{2\pi}{2N+1}\right)}{\sin\left(\frac{2\pi}{2N+1}\right)}$$

$$= \frac{2N+1}{4}\,.$$

From this identity and equation (58),

$$M\left\{E^N(t)\right\} = \frac{ma^2\beta}{4} + \frac{mb^2}{4}\,.$$

But $\beta = k/m$, and hence

$$M\left\{E^N(t)\right\} = \frac{a^2k}{4} + \frac{mb^2}{4}\,.$$

At $t = 0$, all masses in the finite chain except the first are at rest; the first has velocity b. Consequently the total kinetic energy at that time is $mb^2/2$. Also at $t = 0$ all springs except the first are unstressed; the first is stretched or compressed $|a|$ units. Therefore the total potential energy at that time is $ka^2/2$. Hence the total energy in the finite spring-mass chain is $ka^2/2 + mb^2/2$. Thus, the mean of the total kinetic energy of the finite chain, which can be regarded as the average of the total kinetic energy with respect to time, is one half the total energy of the chain.

Now consider the infinite chain. From the fact that $\lim_{X \to \infty} J_\nu(X) = 0$ for any integer ν [10, p. 170] and from equation (42), it follows that

$$\lim_{t \to \infty} E(t) = \frac{ma^2\beta}{4} + \frac{mb^2}{4} = \frac{ka^2}{4} + \frac{mb^2}{4} .$$

Examining the initial conditions of the infinite chain, one finds that it, like the finite chain, has a total energy of $ka^2/2 + mb^2/2$ at $t = 0$. Hence the limit of the total kinetic energy of the infinite chain is one half of its total energy.

Bounds of the Total Kinetic Energies

Trivial upper and lower bounds of the total kinetic energy for the finite and infinite chains are the total energy, $mb^2/2 + ka^2/2$, and zero, respectively. Expressions for the least upper bound and greatest lower bound of these kinetic energies have not been found in the general case. However for the finite chain of N masses if $2N+1$ is a prime, it follows from Theorem 8, Corollary 1, and equation (56) that

$$\underset{t>o}{\text{l.u.b.}}\ E^N(t) = \frac{m}{4}\frac{4}{2N+1}\sum_{p=1}^{N}\left|\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p)+b^2\right\}\right|$$

$$+\frac{m}{4}\frac{4}{2N+1}\sum_{p=1}^{N}\left|\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p)+b^2\right\}\right|$$

$$=\frac{m}{2}\frac{4}{2N+1}\sum_{p=1}^{N}\left|\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p)+b^2\right\}\right| \qquad (59)$$

and

$$\underset{t>o}{\text{g.l.b.}}\ E^N(t) = \frac{m}{4}\frac{4}{2N+1}\sum_{p=1}^{N}\left|\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p)+b^2\right\}\right|$$

$$-\frac{m}{4}\frac{4}{2N+1}\sum_{p=1}^{N}\left|\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p)+b^2\right\}\right|$$

$$= 0\ . \qquad (60)$$

In the previous section of this chapter, it was shown that

$$\frac{m}{4}\frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\left\{a^2\omega^2(N,p)+b^2\right\} = \frac{a^2k}{4}+\frac{mb^2}{4}\ .$$

Hence from equation (59) it follows that the least upper bound of the total kinetic energy in the finite chain of N masses with 2N+1 a prime is the total energy. Equation (60) states that the greatest lower bound of the total kinetic energy in this chain is zero.

CHAPTER VI

RATES OF TRANSFER OF ENERGY IN THE INFINITE SPRING-MASS CHAIN

In Chapter V it was shown that the kinetic energy of each mass in the infinite chain has a limit of zero as $t \to \infty$. Next consider the potential energy of the $(j+1)^{th}$ spring of this chain, $\frac{K}{2}(X_j(t) - X_{j+1}(t))^2$. From equation (3) it follows that

$$X_j(t) - X_{j+1}(t) = a[J_{2j}(2\sqrt{\beta}\,t) - J_{2j+4}(2\sqrt{\beta}\,t)] + b\int_0^t [J_{2j}(2\sqrt{\beta}\,s) - J_{2j+2}(2\sqrt{\beta}\,s) + J_{2j+2}(2\sqrt{\beta}\,s) - J_{2j+4}(2\sqrt{\beta}\,s)]\,ds\,.$$

Using the relation $J_{q-1}(z) - J_{q+1}(z) = 2J'_q(z)$ [3, p. 100] and routine integration, one finds that

$$X_j(t) - X_{j+1}(t) = a[J_{2j}(2\sqrt{\beta}\,t) - J_{2j+4}(2\sqrt{\beta}\,t)] + (b/\sqrt{\beta})[J_{2j+1}(2\sqrt{\beta}\,t) + J_{2j+3}(2\sqrt{\beta}\,t)]. \quad (61)$$

Since $\lim_{X\to\infty} J_\nu(X) = 0$ if ν is an integer [10, p. 170], it follows that the limit as $t \to \infty$ of the potential energy of any spring in the infinite chain is also zero. Thus the limit as $t \to \infty$ of the total energy in any finite portion of the chain is zero.

Consider now the infinite chain in which initially all masses are at their reference positions, all masses except the first are stationary, and the first mass has velocity b. At $t = 0$ the total energy $mb^2/2$ of this chain is kinetic energy located in the first mass. As t increases the energy dissipates into the chain. As $t \to \infty$ the amount of energy which has passed from the $(j+1)^{th}$ mass to the $(j+1)^{th}$ spring approaches $mb^2/2$.

Let $R_j(t)$ be the rate of transfer of energy from the $(j+1)^{th}$ mass to the $(j+1)^{th}$ spring in this chain. Then

$$R_j(t) = k(X_j(t) - X_{j+1}(t))\dot{X}_j(t) , \tag{62}$$

where $X_j(t)$ is given by equation (3) with $a = 0$. This expression is the product of the force exerted by the $(j+1)^{th}$ mass on the $(j+1)^{th}$ spring and the velocity of the $(j+1)^{th}$ mass. From equations (62), (61) with $a = 0$, and (6), it follows that

$$R_j(t) = \frac{kb^2}{\sqrt{\beta}}[J_{2j+1}(2\sqrt{\beta}\,t) + J_{2j+3}(2\sqrt{\beta}\,t)][J_{2j}(2\sqrt{\beta}\,t) + J_{2j+2}(2\sqrt{\beta}\,t)] .$$

Using the recurrence relation $J_{q-1}(z) + J_{q+1}(z) = \dfrac{2qJ_q(z)}{z}$ [3, p. 100], one finds that

$$R_j(t) = [\frac{kb^2}{\beta^{3/2}t^2}(2j+2)(2j+1)][J_{2j+2}(2\sqrt{\beta}\,t)][J_{2j+1}(2\sqrt{\beta}\,t)] . \tag{63}$$

Now since the positive zeros of $J_q(z)$ for any integer q form a sequence which converges to ∞ and each positive zero is simple [3, p. 127], and since $J_{2j+2}(z)$ and $J_{2j+1}(z)$ have no common positive zeros [11, p. 484], it follows from (60) that the positive zeros of $R_j(t)$ form a sequence which converges to ∞ and each positive zero of $R_j(t)$ is simple. Thus there are arbitrarily large t for which $R_j(t)$ is negative. Although the total energy in the finite portion of the chain preceding the $(j+1)^{th}$ spring has a limit of zero as $t \to \infty$, there are arbitrarily large values of t for which energy is returning into the finite portion of the chain from the infinite portion.

Now consider the amount of energy which passes from the $(j+1)^{th}$ mass to the $(j+1)^{th}$ spring between times 0 and T. This quantity can be written as

$$\int_o^T R_j(t)\, dt .$$

As $T \to \infty$ the amount of energy which has passed from the $(j+1)^{th}$ mass to the $(j+1)^{th}$ spring approaches $mb^2/2$; hence it should be the case that

$$\int_o^\infty R_j(t)dt = \frac{mb^2}{2} .$$

From equation (63),

$$\int_o^\infty R_j(t)dt = \frac{kb^2}{\beta^{3/2}} (2j+2)(2j+1) \int_o^\infty \frac{J_{2j+2}(2\sqrt{\beta}\, t) J_{2j+1}(2\sqrt{\beta}\, t)}{t^2}\, dt ,$$

if these improper integrals actually exist. Since

$$\int_o^\infty \frac{J_\mu(ct)\ J_\nu(ct)}{t^2}\, dt = \frac{(\frac{c}{2})^{\lambda-1}\ \Gamma(\lambda)\ \Gamma(\frac{\mu+\nu-\lambda+1}{2})}{2\Gamma(\frac{\lambda-\mu+\nu+1}{2})\Gamma(\frac{\lambda+\mu+\nu+1}{2})\Gamma(\frac{\lambda+\mu-\nu+1}{2})}$$

provided that the real part of $\mu+\nu+1$ is greater than the real part of λ and that the real part of λ is positive [4, p. 403], it follows that

$$\int_0^\infty R_j(t)dt = \frac{Kb^2}{\beta^{3/2}}(2j+2)(2j+1)\frac{\sqrt{\beta}}{2}\frac{(2j)!}{(2j+2)!}$$

$$= \frac{Kb^2}{2\beta}, \quad j = 0,1,2,\ldots .$$

Since $\beta = K/m$, $\int_0^\infty R_j(t)dt = \frac{mb^2}{2}$,

which is the desired result.

In Chapter II, a spring-mass chain like the one now being considered was studied under the assumption that the distance between reference positions of adjacent masses is a constant c. The occurrence of the first extremum of displacement for each mass was regarded as a disturbance passing through the chain; the disturbance was assigned a hypothetical velocity between adjacent masses; and this hypothetical velocity was shown to have a limit of $c\sqrt{\beta}$ as $t \to \infty$. Now the occurrence of the first maximum of the energy transferred from the $(j+1)^{th}$ mass to the $(j+1)^{th}$ spring can also be considered as a disturbance passing through the chain. This maximum occurs at the first positive zero of $R_j(t)$. Since the first positive zero of $J_{2j+1}(z)$ precedes the first positive zero of $J_{2j+2}(z)$ [4, p. 370], the first maximum of the energy transferred from the $(j+1)^{th}$ mass to the $(j+1)^{th}$ spring occurs when $2\sqrt{\beta}\,t$ equals the first zero of $J_{2j+1}(z)$. Thus by the same reasoning as used in Chapter II, it can be shown that the hypothetical velocity of this disturbance also approaches $c\sqrt{\beta}$ as $t \to \infty$.

CHAPTER VII

EXPRESSIONS FOR KINETIC ENERGIES IN THE STACKS OF SLIDING PLATES

In Chapter I the infinite differential system (B) and its truncated form (B') are presented. Solutions of these systems are given by equations (4) and (5), respectively. A physical prototype of the system (B) is an infinite stack of identical flat plates sliding in one dimension with viscous friction acting between any plate and the one below it (see Figure 2). $V_j(t)$, $j = 0,1,2,\ldots$ is the rightward velocity of the $(j+1)^{th}$ plate. At $t = 0$ each plate except the first is stationary, and the first plate has a rightward velocity b. A physical prototype of the finite system (B') is obtained by considering the motion of only the first N plates while supposing that all the plates at and below the $(N+1)^{th}$ are held stationary.

This chapter is preliminary to some comparisons made in Chapter VIII of the kinetic energies in the finite and infinite stacks of plates. In the present chapter the following quantities are calculated for both the finite and infinite stacks of plates:

1) the kinetic energy of each individual plate in the stack;

2) the total kinetic energy of the stack.

Kinetic Energies in the Finite Stack

$V_{N,j}(t)$ given by equation (5) is the rightward velocity of the $(j+1)^{th}$ plate in the stack of N plates. Let $E_j^N(t)$ be the kinetic energy of this plate; that is

$$E_j^N(t) = \frac{m}{2}[V_{N,j}(t)]^2$$

$$= \frac{mb^2}{2}\left(\frac{4}{2N+1}\right)^2 \sum_{p=1}^{N}\sum_{q=1}^{N}\left\{\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left(\frac{2q-1}{2N+1}\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\frac{\pi}{2}(2j+1)\right]\right.$$

$$\left.\cos\left[\frac{2q-1}{2N+1}\frac{\pi}{2}(2j+1)\right] e^{(\gamma_{N,p}+\gamma_{N,q})t}\right\}, \tag{64}$$

where

$$\gamma_{N,p} = 2\alpha\left[\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right) - 1\right] .$$

Let $E^N(t)$ be the total kinetic energy in the finite stack.

$$E^N(t) = \sum_{j=o}^{N-1} E_j^N(t)$$

$$= \frac{mb^2}{2}\left(\frac{4}{2N+1}\right)^2 \sum_{p=1}^{N}\sum_{q=1}^{N}\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left(\frac{2q-1}{2N+1}\frac{\pi}{2}\right)S(p-1,q-1)$$

$$\cdot\, e^{(\gamma_{N,p}+\gamma_{N,q})t} ,$$

where $S(p-1,q-1)$ is defined by equation (28). From the value of $S(\cdot, \cdot)$ given by equation (30), it follows that

$$E^N(t) = \frac{mb^2}{2}\frac{4}{2N+1}\sum_{p=1}^{N}\cos^2\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)e^{2\gamma_{N,p}t} . \tag{65}$$

Kinetic Energies in the Infinite Stack

$V_j(t)$ given by equation (4) is the rightward velocity of the $(j+1)^{th}$ plate in the infinite stack. From equation (4) and the identity

$$\int_0^1 \cos(\nu\pi x)e^{z \cos \pi x} dx = I_\nu(z) \ , \tag{66}$$

where $I_\nu(z)$ is the modified Bessel function of order ν and ν is an integer [4, p. 376], it follows that

$$V_j(t) = be^{-2\alpha t}[I_{j+1}(2\alpha t) + I_j(2\alpha t)] \ . \tag{67}$$

Let $E_j(t)$ be the kinetic energy of the $(j+1)^{th}$ plate in the infinite stack; that is,

$$\begin{aligned} E_j(t) &= \frac{m}{2} [V_j(t)]^2 \\ &= \frac{mb^2}{2} e^{-4\alpha t}[I_{j+1}(2\alpha t) + I_j(2\alpha t)]^2 \ . \end{aligned} \tag{68}$$

Let $E(t)$ be the total kinetic energy of the infinite stack. Then

$$\begin{aligned} E(t) &= \sum_{j=o}^{\infty} E_j(t) \\ &= \frac{mb^2}{2} e^{-4\alpha t} \sum_{j=o}^{\infty} [I_{j+1}(2\alpha t) + I_j(2\alpha t)]^2 \ , \end{aligned} \tag{69}$$

if this series converges. To find a simplified expression for $E(t)$ and to prove convergence, a procedure like that employed in Chapter III to find an expression for the total kinetic energy of the infinite chain is used. As a candidate for the simplified expression consider the integral

$$mb^2 \int_0^1 \cos^2(\frac{\pi x}{2}) \ e^{4\alpha(-1 + \cos \pi x)} dx \ .$$

$E^N(t)$, the total energy of the finite stack of plates, is the product of $\frac{2N}{2N+1}$ and the Riemann sum of this integral with partition points $X_p = p/N$ $(p = 0,1,2,\ldots,N)$ and integrand evaluated at $X_p' = \frac{2p-1}{2N+1}$ $(p=1,2,\ldots,N)$. Hence

$$\lim_{N\to\infty} E^N(t) = mb^2 \int_o^1 \cos^2\left(\frac{\pi x}{2}\right) e^{4\alpha(-1+\cos\pi x)t}\,dx$$

$$= \frac{mb^2}{2}\int_o^1 (1 + \cos\pi x)\, e^{4\alpha(-1+\cos\pi x)t}\,dx \; .$$

From equation (66) it follows that

$$\lim_{N\to\infty} E^N(t) = \frac{mb^2}{2}\, e^{-4\alpha t}[I_o(4\alpha t) + I_1(4\alpha t)] \; .$$

One can show (see Appendix B) that

$$[I_o(2z) + I_1(2z)] = \sum_{j=o}^{\infty} [I_{j+1}(z) + I_j(z)]^2 \, . \tag{70}$$

Hence it follows that

$$E(t) = \frac{mb^2}{2}\, e^{-4\alpha t}[I_o(4\alpha t) + I_1(4\alpha t)] \; . \tag{71}$$

CHAPTER VIII

PROPERTIES OF THE KINETIC ENERGIES IN THE STACKS OF PLATES

In this chapter the kinetic energies in the infinite and finite stacks of sliding plates are compared. First it is shown that for the kinetic energies of the individual plates and for the total kinetic energy, in both the finite and infinite stacks, the limit as $t \to \infty$ is zero. Then the following limits are considered:

1) the limit as $t \to \infty$ of the ratio of the kinetic energy of the $(j+1)^{th}$ plate in the finite stack to the kinetic energy of the $(j+1)^{th}$ plate in the infinite stack $(j=0,1,\ldots,N-1)$;

2) the limit as $t \to \infty$ of the ratio of the total kinetic energy in the finite stack to the total kinetic energy in the infinite stack.

Limits of Kinetic Energies

The numbers

$$\gamma_{N,p} = 2\alpha\left(\cos\left(\frac{2p-1}{2N+1}\pi\right) - 1\right) \quad p=1,2,\ldots,N ,$$

are all negative; hence from equation (5), it follows that (since $t \geq 0$)

$$\lim_{t\to\infty} V_{N,j}(t) = 0 .$$

Therefore

$$\lim_{t\to\infty} E_j^N(t) = \lim_{t\to\infty} \frac{m}{2}\,[V_{N,j}(t)]^2 = 0 .$$

That is, the limit as $t \to \infty$ of the kinetic energy of each plate in the finite stack is zero; consequently the limit as $t \to \infty$ of the total energy in the finite stack is also zero.

Now in order to show that the limits as $t \to \infty$ of the kinetic energies in the infinite stack are zero, consider the asymptotic expansion of $I_v(z)$. For v a fixed integer and for z positive

$$I_v(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left\{ 1 - \frac{4v^2-1}{8z} + \frac{(4v^2-1)(4v^2-9)}{2!\,(8z)^2} - \frac{(4v^2-1)(4v^2-9)(4v^2-25)}{3!\,(8z)^3} + \cdots \right\}$$

as $z \to \infty$ [4, p. 377]. Thus for $z \geq 0$

$$\lim_{z \to \infty} \sqrt{2\pi z}\, e^{-z} I_v(z) = 1 \; ; \tag{72}$$

and consequently for $z \geq 0$

$$\lim_{z \to \infty} e^{-z} I_v(z) = 0 \, . \tag{73}$$

Since $\alpha > 0$ it follows from equations (67) and (73) that $\lim\limits_{t \to \infty} V_j(t) = 0$ and hence that

$$\lim_{t \to \infty} E_j(t) = 0 \, .$$

From equations (71) and (73) it follows that

$$\lim_{t \to \infty} E(t) = 0 \, .$$

Thus the kinetic energy of each plate in the infinite stack and the total kinetic energy of the infinite stack have the limit zero as $t \to \infty$.

Limits of Ratios of Kinetic Energies

In order to find the limit of the ratios

$$E_j^N(t)/E_j(t)\,, \quad (j = 0,1,2,\ldots,N-1)$$

consider the ratios

$$V_{N,j}(t)/V_j(t) \quad (j=0,1,2,\ldots,N-1)\;.$$

From equations (5) and (67) it follows that

$$V_{N,j}(t)/V_j(t) = \frac{\frac{4b}{2N+1}\sum_{p=1}^{N}\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\frac{\pi}{2}(2j+1)\right]e^{\gamma_{N,p}t}}{be^{-2\alpha t}\left[I_{j+1}(2\alpha t)+I_j(2\alpha t)\right]}$$

$$= \frac{\frac{4}{2N+1}\sum_{p=1}^{N}\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\frac{\pi}{2}(2j+1)\right]\sqrt{4\alpha\pi t}\;e^{\gamma_{N,p}t}}{\sqrt{4\alpha\pi t}\;e^{-2\alpha t}\;I_{j+1}(2\alpha t)+\sqrt{4\alpha\pi t}\;e^{-2\alpha t}\;I_j(2\alpha t)}\,.$$

Considering equation (72), one finds that

$$\lim_{t\to\infty} V_{N,j}(t)/V_j(t) = \frac{1}{2}\lim_{t\to\infty}\frac{4}{2N+1}\sum_{p=1}^{N}\cos\left(\frac{2p-1}{2N+1}\frac{\pi}{2}\right)\cos\left[\frac{2p-1}{2N+1}\frac{\pi}{2}(2j+1)\right]$$
$$\sqrt{4\alpha\pi t}\;e^{\gamma_{N,p}t}\;;$$

and since

$$\lim_{t\to\infty}\sqrt{4\alpha\pi t}\;e^{\gamma_{N,p}t} = 0$$

($t \geq 0$ and $\gamma_{N,p} < 0$), it follows that

$$\lim_{t \to \infty} V_{N,j}(t)/V_j(t) = 0, \quad j = 0,1,2,\ldots,N-1 .$$

Hence

$$\lim_{t \to \infty} E_j^N(t)/E_j(t) = \lim_{t \to \infty} \frac{\frac{m}{2}[V_{N,j}(t)]^2}{\frac{m}{2}[V_j(t)]^2} = 0 .$$

Next consider the ratio

$$E^N(t)/E(t) .$$

From equations (65) and (71) it follows that

$$E^N(t)/E(t) = \frac{\frac{mb^2}{2} \frac{4}{2N+1} \sum_{p=1}^{N} \cos^2\left(\frac{2p-1}{2N+1} \frac{\pi}{2}\right) e^{2\gamma_{N,p}t}}{\frac{mb^2}{2} e^{-4\alpha t}[I_o(4\alpha t) + I_1(4\alpha t)]}$$

$$= \frac{\frac{4}{2N+1} \sum_{p=1}^{N} \cos^2\left(\frac{2p-1}{2N+1} \frac{\pi}{2}\right) \sqrt{8\alpha\pi t}\; e^{2\gamma_{N,p}t}}{\sqrt{8\alpha\pi t}\; e^{-4\alpha t} I_o(4\alpha t) + \sqrt{8\alpha\pi t}\; e^{-4\alpha t} I_1(4\alpha t)} .$$

Again by considering equation (72) and the fact that

$$\lim_{t \to \infty} \sqrt{8\alpha\pi t}\; e^{2\gamma_{N,p}t} = 0 \quad (p = 1,2,\ldots,N) ,$$

it follows that

$$\lim_{t \to \infty} E^N(t)/E(t) = 0 .$$

A physical interpretation of the value of these limits is as follows. As $t \rightarrow \infty$ the kinetic energy of any individual plate and the total kinetic energy in the finite stack of plates and the counterparts of these quantities in the infinite stack all approach zero. However they do so in a manner such that the quantity in the finite stack is less than an arbitrary positive multiple of its counterpart in the infinite stack for sufficiently large t.

APPENDIX A

DERIVATION OF CERTAIN INTEGRAL RELATIONS

In order to derive equations (37a) and (37b), the well-known formula for integration by parts

$$\int_a^b u\left(\frac{dv}{dy}\right)dy = u(b)v(b) - u(a)v(a) - \int_a^b v\left(\frac{du}{dy}\right) dy$$

is used. A sufficient condition for its validity is that u and v have continuous derivatives on [a,b] [7, pp. 195-198].

From the identity $\sin^2\theta = (1 - \cos 2\theta)/2$, one obtains

$$\int_0^1 y^2 \sqrt{1-y^2} \sin^2(2\sqrt{\beta}\, t\, y)\, dy =$$

$$\frac{1}{2}\int_0^1 y^2\sqrt{1-y^2}\, dy - \frac{1}{2}\int_0^1 y^2\sqrt{1-y^2} \cos(4\sqrt{\beta}\, t\, y)dy$$

$$= \frac{\pi}{32} - \frac{1}{2}\int_0^1 y^2 \sqrt{1-y^2} \cos(4\sqrt{\beta}\, t\, y)\, dy \; . \qquad \text{(A.1)}$$

Using the integration-by-parts formula with $V = -\frac{1}{3}(1-y^2)^{3/2}$ and $u = y \cos(4\sqrt{\beta}\, t\, y)$ yields

$$\int_0^1 y^2(1-y^2)^{1/2} \cos(4\sqrt{\beta}\, t\, y)\, dy =$$

$$\frac{1}{3}\int_0^1 (1-y^2)^{3/2} \cos(4\sqrt{\beta}\, t\, y)dy - \frac{4\sqrt{\beta}\, t}{3}\int_0^1 y(1-y^2)^{3/2} \sin(4\sqrt{\beta}\, t\, y)dy, \qquad \text{(A.2)}$$

since $v(1) = u(0) = 0$. Next choosing $u = \sin(4\sqrt{\beta}\, t\, y)$ and $v = -\frac{1}{5}(1-y^2)^{5/2}$, one finds that

$$\int_0^1 y(1-y^2)^{3/2} \sin(4\sqrt{\beta}\, t\, y)\, dy = \frac{4\sqrt{\beta}\, t}{5} \int_0^1 (1-y^2)^{5/2} \cos(4\sqrt{\beta}\, t y) dy, \tag{A.3}$$

since $v(1) = u(0) = 0$. From equations (A.1), (A.2), and (A.3) follows the equation

$$\int_0^1 y^2 \sqrt{1-y^2} \sin^2(2\sqrt{\beta}\, t\, y) dy =$$

$$\frac{\pi}{32} - \frac{1}{6} \int_0^1 (1-y^2)^{3/2} \cos(4\sqrt{\beta}\, t\, y) dy + \frac{(4\sqrt{\beta}\, t)^2}{30} \int_0^1 (1-y^2)^{5/2}$$

$$\cos(4\sqrt{\beta}\, t\, y)\, dy\ ,$$

which is equation (37a).

By the identity $\sin\theta \cos\theta = \frac{1}{2}\sin 2\theta$,

$$\int_0^1 y \sqrt{1-y^2} \sin(2\sqrt{\beta}\, t\, y) \cos(2\sqrt{\beta}\, t\, y)\, dy =$$

$$\frac{1}{2} \int_0^1 y \sqrt{1-y^2} \sin(4\sqrt{\beta}\, t\, y)\, dy\ .$$

Using the integration-by-parts formula with $u = \sin(4\sqrt{\beta}\, t\, y)$ and $V = -\frac{1}{3}(1-y^2)^{3/2}$ in the right hand side of this last equation, one obtains, since $V(1) = u(0) = 0$,

$$\int_0^1 y \sqrt{1-y^2} \sin(2\sqrt{\beta}\, t\, y) \cos(2\sqrt{\beta}\, t\, y)\, dy =$$

$$\frac{(4\sqrt{\beta}\, t)}{6} \int_0^1 (1-y^2)^{3/2} \cos(4\sqrt{\beta}\, t\, y)\, dy\ ,$$

which is equation (37b).

From the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ follows the equation

$$\int_0^1 \sqrt{1 - y^2} \cos^2(2\sqrt{\beta}\, t\, y)\, dy =$$

$$\frac{1}{2}\int_0^1 \sqrt{1 - y^2}\, dy + \frac{1}{2}\int_0^1 \sqrt{1 - y^2} \cos(4\sqrt{\beta}\, t\, y)\, dy$$

$$= \frac{\pi}{8} + \frac{1}{2}\int_0^1 (1 - y^2)^{1/2} \cos(4\sqrt{\beta}\, t\, y)\, dy ,$$

which is equation (37c).

APPENDIX B

PERTINENT RELATIONS INVOLVING BESSEL FUNCTIONS

Equations (41a), (41b), and (41c) can be derived from the relations

$$J_n(2z) = \sum_{j=o}^{n} J_j(z)J_{n-j}(z) + 2\sum_{j=1}^{\infty}(-1)^j J_j(z)J_{n+j}(z) , \qquad (B.1)$$

$$0 = \sum_{j=o}^{2n}(-1)^j J_j(z)J_{2n-j}(z) + 2\sum_{j=1}^{\infty} J_j(z)J_{2n+j}(z), \quad n \geq 1 \qquad (B.2)$$

and

$$1 = J_o^2(z) + 2\sum_{j=1}^{\infty} J_j^2(z) \qquad (B.3)$$

[4, p. 363].

Subtracting (B.1) with n = 0 from (B.3) yields

$$1 - J_o(2z) = 2\sum_{j=1}^{\infty} J_{2j-1}^2(z) + 2\sum_{j=1}^{\infty} J_{2j-1}^2(z), \qquad (B.4)$$

where the identical series are not combined for reasons that will be evident shortly. Subtracting (B.2) with n = 2 from (B.1) with n = 4 yields

$$J_4(2z) = 4J_1(z)J_3(z) - 4\sum_{j=1}^{\infty} J_{2j-1}(z)J_{2j+3}(z) . \qquad (B.5)$$

Adding (B.4) with a change of index in the second series and (B.5), one obtains

$$[1 - J_o(2z) + J_4(2z)] =$$

$$4J_1(z)J_3(z) + 2\sum_{j=1}^{\infty} J_{2j-1}^2(z) - 4\sum_{j=1}^{\infty} J_{2j-1}(z)J_{2j+3}(z) + 2\sum_{j=-1}^{\infty} J_{2j+3}^2(z)$$

$$= 2J_1^2(z) + 4J_1(z)J_3(z) + 2J_3^2(z)$$

$$+ 2\sum_{j=1}^{\infty}[J_{2j-1}^2(z) - 2J_{2j-1}(z)J_{2j+3}(z) + J_{2j+3}^2(z)] .$$

Since $J_1(z) = -J_{-1}(z)$ [4, p. 358], equation (41a) follows from this equation.

From equation (B.1) with $n = 3$,

$$J_3(2z) = 2J_o(z)J_3(z) + 2J_1(z)J_2(z) + 2\sum_{j=1}^{\infty}(-1)^j J_j(z)J_{3+j}(z)$$

$$= 2J_o(z)J_3(z) + 2J_1(z)J_2(z) - 2\sum_{j=1}^{\infty}[J_{2j-1}(z)J_{2j+2}(z) - J_{2j}(z)J_{2j+3}(3)]. \tag{B.6}$$

From equation (B.1) with $n = 1$,

$$J_1(2z) = 2J_o(z)J_1(z) + 2\sum_{j=1}^{\infty}(-1)^j J_j(z)J_{1+j}(z) ;$$

and by a moderately complicated but justifiable rearrangement

$$J_1(2z) = 2J_o(z)J_1(z) + 2J_2(z)J_3(z)$$

$$+ \sum_{j=1}^{\infty}[J_{2j-1}(z)J_{2j}(z) - J_{2j+3}(z)J_{2j+2}(z)] \tag{B.7}$$

for z real. Adding (B.6) and (B.7), one obtains

$$[J_1(2z) + J_3(2z)] = 2J_1(z)J_o(z) + 2J_1(z)J_2(z) + 2J_3(z)J_o(z) + 2J_3(z)J_2(z)$$

$$- 2 \sum_{j=1}^{\infty} [J_{2j-1}(z)J_{2j}(z) + J_{2j-1}(z)J_{2j+2}(z) - J_{2j+3}(z)J_{2j}(z)$$

$$- J_{2j+3}(z)J_{2j+2}(z)] \ .$$

Since $J_1(z) = -J_{-1}(z)$ [4, p. 358], equation (41b) follows from this equation.

Adding (B.1) with n = 0 and (B.3) yields

$$1 + J_o(2z) = 2J_o^{\,2}(z) + 2 \sum_{j=1}^{\infty} J_{2j}^2(z) + 2 \sum_{j=1}^{\infty} J_{2j}^2(z) \ . \tag{B.8}$$

Adding (B.1) with n = 2 and (B.2) with n = 1 yields

$$J_2(2z) = 4J_o(z)J_2(z) + 4 \sum_{j=1}^{\infty} J_{2j}(z)J_{2j+2}(z) \ . \tag{B.9}$$

Adding (B.8) with a change of index in the second series and (B.9), one obtains

$$1 + J_o(2z) + J_2(2z) = 2J_o^{\,2}(z) + 4\, J_o(z)J_2(z)$$

$$+ 2 \sum_{j=1}^{\infty} J_{2j}^2(z) + 4 \sum_{j=1}^{\infty} J_{2j}(z)J_{2+2j}(z) + 2 \sum_{j=o}^{\infty} J_{2j+2}^2(z)$$

$$= 2J_o^{\,2}(z) + 4J_o(z)J_2(z) + 2J_2^{\,2}(z)$$

$$+ 2 \sum_{j=1}^{\infty} [J_{2j}^2(z) + 2J_{2j}(z)J_{2j+2}(z) + J_{2j+2}^2(z)] \ ,$$

which is equation (41c).

Equation (70) can be derived from (B.1) and the definition of $I_j(z)$,

$$I_j(z) = i^{-j} J_j(z) \ .$$

From (B.1) with n = 0 and z replaced by iz ,

$$I_o(2z) = J_o(2iz) = 2J_o^2(iz) + 2 \sum_{j=1}^{\infty} (-1)^j J_j^2(iz) \, .$$

But $(-1)^j = i^{-2j}$; hence

$$I_o(2z) = I_o^2(z) + \sum_{j=1}^{\infty} I_j^2(z) + \sum_{j=1}^{\infty} I_j^2(z) \ . \tag{B.10}$$

From (B.1) with n = 1,

$$I_1(2z) = i^{-1} J_2(2iz) =$$

$$i^{-1} 2J_o(iz) J_1(iz) + 2i^{-1} \sum_{j=1}^{\infty} (-1)^j J_j(iz) J_{j+1}(iz) \ .$$

Since $i^{-1}(-1)^j = i^{-2j-1}$, it follows that

$$I_1(2z) = 2I_o(z) I_1(z) + 2 \sum_{j=1}^{\infty} I_j(z) I_{j+1}(z) \ . \tag{B.11}$$

Adding (B.10) with a change of index in the second series and (B.11), one obtains

$$I_o(2z) + I_1(2z) = I_o^2(z) + 2I_o(z)I_1(z)$$

$$+ \sum_{j=1}^{\infty} I_j^2(z) + 2 \sum_{j=1}^{\infty} I_j(z)I_{j+1}(z) + \sum_{j=o}^{\infty} I_{j+1}^2(z) \; .$$

$$= \sum_{j=o}^{\infty} [I_j^2(z) + 2I_j(z)I_{j+1}(z) + I_{j+1}^2(z)] \; ,$$

which is equation (70).

APPENDIX C

PROOF OF THEOREM EIGHT

There exist rational numbers $r_1, r_2, \ldots, r_N$ (not all zero) such that

$$r_1\omega(N,1) + r_2\omega(N,2) + \ldots + r_N\omega(N,N) = 0$$

if and only if there exist integers $n_1, n_2, \ldots, n_N$ (not all zero) such that

$$\sum_{p=1}^{N} n_p \sin\left(\frac{2p-1}{2N+1}\,\frac{\pi}{2}\right) = 0 \ . \qquad \text{(C.1)}$$

Let $\lambda = e^{\frac{i\pi}{2(2N+1)}}$. Equation (C.1) is equivalent to

$$\sum_{p=1}^{N} n_p(\lambda^{2p-1} - \lambda^{-2p+1}) = 0 \ . \qquad \text{(C.2)}$$

Assuming that there are integers $n_1, n_2, \ldots, n_N$ not all zero for which (C.2) holds, let h be the integer such that $n_h \neq 0$ and $n_p = 0$ if $h < p \leq N$. Multiplying by λ^{2h-1} ($\neq 0$) and neglecting the terms for which $p > h$ in equation (C.2), one obtains

$$\sum_{p=1}^{h} n_p(\lambda^{2(h+p-1)} - \lambda^{2(h-p)}) = 0 \ . \qquad \text{(C.3)}$$

It is easily shown that conversely if there are integers $n_1, n_2, \ldots, n_h$

such that $h \leq N$, $n_h \neq 0$, and (C.3) holds, then there are integers $n_1, n_2, \ldots, n_N$ not all of which are zero such that (C.1) holds.

A polynomial P(X) of degree n is a <u>reciprocal polynomial</u> of the first or second class when it satisfies

$$X^n P(\tfrac{1}{X}) = \begin{cases} + P(X) \\ \text{or} \\ -P(X) \end{cases}$$

respectively [9, p. 21]. It follows readily that a polynomial G(x) of degree 2h-1 is a reciprocal polynomial of the second class if and only if

$$G(x) = \sum_{p=1}^{h} C_p (X^{(n+p-1)} - X^{(h-p)}) .$$

Thus from equation (C.3) it follows that one can prove Theorem 8 by proving that 2N+1 not being a prime is a necessary and sufficient condition for the existence of a polynomial G(x) with these properties:

(i) G(x) has integer coefficients;

(ii) the degree of G(x) is $2h-1 \leq 2N-1$;

(iii) $G(\lambda^2) = 0$; and

(iv) G(x) is a reciprocal polynomial of the second class.

The remainder of this Appendix relies heavily on the theory of numbers. A basic knowledge is assumed; pertinent definitions are made; and theorems are stated without proof.

The <u>cyclotomic polynomial</u> of index n is defined by

$$F_n(X) = \prod_p (X - \mu^p) ,$$

where $\mu = e^{\frac{i2\pi}{n}}$ and p ranges over all positive integers less than n and prime relative to n. It can be shown that $F_n(X)$ has integer coefficients and that among all such polynomials, it (or an integer multiple of it) is the one of least degree for which μ is a zero [12, pp. 158-164].* Clearly the degree of $F_n(X)$ is $\varphi(n)$, where $\varphi(n)$ is the number of positive integers less than n which are prime relative to n. It can be shown that

$$\varphi(n) = n(1 - \frac{1}{p_1})(1 - \frac{1}{p_2}) \ldots (1 - \frac{1}{p_r}) , \quad (C.4)$$

where $p_1, p_2, \ldots, p_r$ are all the distinct prime factors of n [14, p. 34]. Another property of $F_n(X)$ is that it is a reciprocal polynomial of the first class [9, p. 68].

Now assume that a polynomial G(X) satisfying properties (i) through (iv) exists. Since $G(\lambda^2) = 0$ and $\lambda^2 = e^{i2\pi/2(2N+1)}$, the degree 2h-1 of G(x) must be greater than or equal to the degree of $F_{2(2N+1)}(X)$, which is $\varphi(2(2N+1))$. Now suppose in addition that 2N+1 is a prime. By equation (C.4)

$$\varphi(2(2N+1)) = 2(2N+1)(1 - \frac{1}{2})(1 - \frac{1}{2N+1})$$
$$= 2N .$$

But by property (ii) $2h-1 \leq 2N-1$; hence by contradiction 2N+1 is not prime.

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*The proof of Lemma 1 on page 161 of this reference is incorrect. However the lemma is provable and is given as an exercise in [13, p. 97].

Next assume that 2N+1 is not a prime. The polynomial

$$G(X) = (X - 1)F_{2(2N+1)}(X)$$

satisfies properties (i) through (iv).

<u>Property (i)</u>. Clearly $G(x)$ has integer coefficients since $F_{2(2N+1)}(x)$ has integer coefficients.

<u>Property (ii)</u>. The degree of $G(x)$ is $\varphi(2(2N+1)) + 1$. Let $p_1, p_\lambda, \ldots, p_r$ be all the distinct prime factors of 2N+1. Then from equation (C.4)

$$\varphi(2(2N+1)) = 2(2N+1)(1 - \tfrac{1}{2})(1 - \frac{1}{p_1})\ldots(1 - \frac{1}{p_r})$$

since 2 is not a prime factor of 2N+1. Hence,

$$\begin{aligned}\varphi(2(2N+1)) &= (2N+1)(1 - \frac{1}{p_1}) \ldots (1 - \frac{1}{p_r}) \\ &= \varphi(2N+1) .\end{aligned}$$

Since $p_1, p_2, \ldots, p_r$ are odd integers

$$\varphi(2N+1) = (2N+1)(\frac{p_1 - 1}{p_1}) \ldots (\frac{p_r - 1}{p_r})$$

is an even integer. From the definition of $\varphi(n)$, since 2N+1 is assumed not to be a prime, it follows that

$$\varphi(2N+1) \leq 2N - 1 .$$

Since φ (2N+1) is even,

$$\varphi(2N+1) \leq 2N - 2, \quad \text{and}$$

$\varphi(2N+1) + 1$ is odd. Hence the degree of $G(x)$,

$$\varphi(2N+1) + 1 = 2h-1 \leq 2N-1 ,$$

and property (ii) is satisfied.

<u>Property (iii)</u>. From the definitions of $F_n(x)$ it follows that $G(\lambda^2) = (\lambda^2 - 1)F_{2(2N+1)}(\lambda^2) = 0$. Hence property (iii) is satisfied.

<u>Property (iv)</u>. The product of two reciprocal polynomials of different classes is easily shown to be a reciprocal polynomial of the second class. Since $X - 1$ is a reciprocal polynomial of the second class and $F_{2(2N+1)}(x)$ is a reciprocal polynomial of the first class, property (iv) is satisfied. Thus if $2N+1$ is not a prime, there exists a polynomial satisfying properties (i) through (iv).

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