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AN APPROACH TO THE MINIMUM TIME CONTROL OF A SIMPLE FLEXIBLE ARM

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ABSTRACT

To enable faster robot arm motion one might reduce the arm weight and use a minimum time control. Light arms will be flexible. A simple example of a light arm is a single beam, rotating about one end, and subject to bending. This paper models such a beam in modal coordinates, linearizes the model, and formulates the control according to an algorithm developed by Plant and Athans. This algorithm moves the system state to a hypersphere near the origin in minimum time. The objective of this research is to understand the nature of the optimal control to aid in formulating a more practical suboptimal control.

INTRODUCTION

Increased demands for performance of robot arms and other mechanical systems lead one to consider new ranges of design parameters. By reducing the structural mass of an arm the movement time may be reduced, improving that aspect of performance. To achieve this improved performance vibration of the more flexible arm must be considered. In this paper one approach to this problem is considered. A simple flexible arm is modeled using the method of assumed modes. The time optimal control of this nonlinear model is sought. Due to the simple configuration of the arm, the linearized equations of motions constitute a reasonable model of the system. The linearized problem is solved numerically using the method of Plant [9]. The optimal control is then applied to the nonlinear system equations in a simulation.

By solving for the optimal control for this simple case one hopes to gain insight into the optimal control of more complicated arms as well as practical approximations to this optimal control. One can also compare the true optimum with various suboptimal controls. Alternative design and control strategies can thus be evaluated.

Dynamic Modelling of the Proposed Configuration

The flexible manipulator arm will be modeled using the method of assumed modes. Reference [3] has some details in time domain modelling of two links by applying Lagrange's equation and the assumed mode method. The same approach will be used here to form the dynamic equations of the proposed configuration, shown in Figure 1.

Let $\{\vec{U}\} = \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}$ be the unit vector of reference frame OXY ,

$\{\vec{U}_1\} = \begin{bmatrix} \vec{u}_{x1} \\ \vec{u}_{y1} \end{bmatrix}$ be the unit vector of reference from OX_1Y_1 ,

and $\{\vec{U}_1\} = [C]\ \{\vec{U}\}$

where C = the rotational transformation matrix (See Fig. 1).

$$[C] = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

With respect to the reference frame $[OX_1\ Y_1]$ the vector position of point P would be

$$\vec{R}_d = \{\vec{U}_1\}' \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} = x_1\ \vec{u}_{x1} + u_1 \vec{u}_{y1}$$

The vector position of point P can also be described with respect to reference frame [OXY] as

$$\dot{\vec{R}}_d = [-\dot{\theta} x_1 \sin(\theta) - \dot{\theta}\ u_1 \cos(\theta) - u_1 \sin(\theta)]\vec{u}_x +$$
$$[\dot{\theta}\ x_1 \cos(\theta) - \dot{\theta}\ u_1 \sin(\theta) + \dot{u}_1 \cos(\theta)]\vec{u}_y$$

Assumed mode method.

A solution of the flexible motions is assumed to be a linear combination of admissible functions multiplied by time dependent generalized coorindates.

$$u_1(x_1,t) = \sum_{i=1}^{n} \phi_i(x_1) q_i(t)$$

where the ϕ_i are admissible functions which satisfy the geometric boundary conditions and the $q_i(t)$ are generalized coordinates.

Furthermore, assumming that the amplitude of the higher modes of the flexible link are very small compared with the first one (ref 2), n = 2 will be accurate enough to represent the system.

$$u_1(x_1,t) = \phi_1(x_1)q_1(t) + \phi_2(x_2)q_2(t)$$

where ϕ_1, ϕ_2 must be orthogonal. Details on the mode shape is shown in the next section.

Kinetic energy (T), may be written as

$$T = (1/2) \int_m \dot{\vec{R}}_d \cdot \dot{\vec{R}}_d\ dm + (1/2)\ [\dot{\vec{R}}_d \cdot \dot{\vec{R}}_d]|_{x=\ell} M + (1/2)\ I_o \dot{\theta}^2 +$$
$$(1/2)\ J_p\ [\partial \dot{u}/\partial x|_{x=\ell}\ ]$$

The first term in the right hand side is the kinetic energy of the rotating beam with respect to the origin (O). The second term and the third term are kinetic energy of the payload mass and the rotating inertia respectively. The last term is the kinetic energy of the payload mass when the beam is flexible.

Potential energy (V)

$$V = mgl \sin(\theta)/2 + Mgl \sin(\theta) + (1/2) \int_o^{\ell_1} EI(\partial^2 u/\partial x_1^2)\, dx$$

The first term and the second term will be neglected if the plane of motion is horizontal. The last term is the potential stored in the beam when the beam is bent.

Lagrange's Equation

By knowing kinetic energy and potential energy, the dynamic equation can be derived by using Lagrange's equation which is

$$d(\partial T/\partial \dot{q}_r)/dt - \partial T/\partial q_r + \partial V/\partial q_r = Q_r; \quad r = 1,2,\ldots.$$

where

q_r is the generalized coordinate
Q_r is the generalized force

Dynamic equation

Horizontal motion is considered here. The kinetic energy and potential energy can be derived as follows:

$$\dot{\bar{R}}_d \cdot \dot{\bar{R}}_d = \dot{\theta}^2 x_1^2 + \dot{\theta}^2 u_1 + \dot{u}_1^2 + 2\dot{\theta}\dot{u}_1 x_1$$

$$T = 1/2\ \dot{\theta}^2 \int_m x_1^2 dm + 1/2\dot{\theta} \int_m u_1^2 dm + 1/2 \int_m \dot{u}_1^2 dm + \dot{\theta} \int_m \dot{u}_1 x_1 dm$$

$$+ 1/2\ M\dot{\theta}^2 \ell^2 + 1/2\ \dot{\theta}^2 u_{1E}^2 M + 1/2\ \dot{u}_{1E}^2 M + \dot{\theta}\dot{u}_{1E}\ell M + 1/2\ I_o \dot{\theta}^2$$

$$+ 1/2\ J_P \phi_{1E}' \dot{q}_1^2 + \phi_{1E}' \phi_{2E}' \dot{q}_1 \dot{q}_2 J_P + 1/2\ J_P \phi_2'^2\ \dot{q}_2$$

where

$$\int_m x_1^2 dm = J_o$$

$\int_m u_1^2 dm$ is very small and can be neglected

$$\int_m \dot{u}_1^2 dm = (\dot{q}_1^2 + \dot{q}_2^2) m$$

$$\int_m \dot{u}_1 x_1 dm = w_1 \dot{q}_1 + w_2 \dot{q}_2$$

$$m = \int_m \phi_1^2 dm = \int_m \phi_2^2 dm$$

$$w_1 = \int_m \phi_1 x_1 dm$$

$$w_2 = \int_m \phi_2 x_1 dm$$

$$T = \frac{1}{2}\dot{\theta}^2 J_o + \frac{1}{2} m\,[\dot{q}_1^2 + \dot{q}_2^2] + \dot{\theta}[w_1\dot{q}_1 + w_2\dot{q}_2] + \frac{1}{2} M\dot{\theta}^2\ell^2 + \frac{1}{2} M\dot{\theta}^2 u_{1E}^2$$

$$+ \frac{1}{2} M\dot{u}_{1E}^2 + M\dot{\theta}\dot{u}_{1E}\ell + \frac{1}{2} I_o\dot{\theta}^2 + \frac{1}{2} J_P\phi_{1E}'^2\dot{q}_1^2 + \phi_{1E}'\ \phi_{2E}'\dot{q}_1\dot{q}_2 J_P + \frac{1}{2} J_P\ \phi_{2E}'^2\ \dot{q}_2^2$$

$$V = \frac{1}{2}\int_o^{\ell_1} EI\ (\ \frac{\partial^2 u}{\partial x^2}\ )^2 dx$$

$$= \frac{1}{2} K_1 q_1^2 + \frac{1}{2} K_2 q_2^2$$

where

$$K_1 = EI\int_o^{\ell} (\phi_1''(x))^2 dx = \frac{EI}{\ell^3}\int_o^1 (\phi_1''(\xi))^2 d_\xi$$

$$= EI\int_o^{\ell} (\phi_2''(x))^2 dx = \frac{EI}{\ell^3}\int_o^1 (\phi_1''(\xi))^2 d_\xi$$

Substitute T and V into Lagrange's equation. The dynamic equation can be written as

$$(1)\quad [J_o + M\ell^2 + I_o + M\ (\phi_{1E}^2 q_1^2 + \ \ \phi_{2E}^2 q_2^2 + 2\phi_{1E}\ \phi_{2E} q_1 q_2)]\ddot{\theta} + 2M\phi_{1E}^2 q_1\dot{q}_1\dot{\theta}$$

$$+ 2M\phi_{2E}^2 q_2\dot{q}_2\dot{\theta} + 2\phi_{1E}\phi_{2E}\dot{q}_1 q_2\dot{\theta}M + 2M\phi_{1E}\phi_{2E} q_1\dot{q}_2\ \theta + w_1\ddot{q}_1 + w_2\ddot{q}_2\ q$$

$$+ M\ell\phi_{1E}\ddot{q}_1 + M\ell\phi_{2E}\ddot{q} = T$$

$$m\ddot{q}_1 + w_1\ddot{\theta} + M\phi_{1E}^2\ddot{q}_1 + M\phi_{1E}\phi_{2E}\ddot{q}_2 + M\ell\phi_{1E}\ddot{\theta} + J_P\phi_{1E}'^2\ddot{q}_1 + J_P\phi_{1E}'\phi_{2E}'\ddot{q}_2$$

$$- M\dot{\theta}^2\phi_{1E}^2 q_1 - M\dot{\theta}^2\phi_{1E}\phi_{2E} q_2 + K_1 q_1 = 0$$

$$m\ddot{q}_2 + w_2\ddot{\theta} + M\phi_{2E}^2\ddot{q}_2 + M\phi_{1E}\phi_{2E}\ddot{q}_1 + M\ddot{\theta}\ell\phi_{2E} + J_P\phi_{1E}'\ \phi_{2E}'\ \ddot{q}_1$$

$$+ J_P\phi_{2E}'^2\ \ddot{q}_2 - M\dot{\theta}^2\phi_{2E}^2 q_2 - M\dot{\theta}^2\phi_{1E}\phi_{2E} q_1 + K_2 q_2 = 0$$

where

m = mass of the beam
M = payload mass
J_P = inertia of payload mass about its center of gravity
I_o = rotating inertia at the joint
J_o = inertia of beam with respect to the joint

$u_{1E} = u_1(x)|_{x=\ell} = u_1(\xi)|_{\xi=1}$

$u_{2E} = u_2(x)|_{x=\ell} = u_2(\xi)|_{\xi=1}$

$$\phi_{1E} = \phi_1(x)|_{x=\ell} \qquad = \qquad \phi_1(\xi)|_{\xi=1}$$

$$\phi_{2E} = \phi_2(x)|_{x=\ell} \qquad = \qquad \phi_2(\xi)|_{\xi=1}$$

$$w_1 = \int_m \phi_1(x)x_1 dm \qquad = \qquad \rho A\ell^2 \int_o^1 \phi_1(\xi)\xi d\xi$$

$$w_2 = \int_m \phi_1(x)x_1 dm \qquad = \qquad \rho A\ell^2 \int_o^1 \phi_1(\xi)\xi d\xi$$

$$K_1 = EI \int_o^\ell (\phi_1''(x))^2 dx \qquad = \qquad \frac{EI}{\ell^3} \int_o^1 (\phi_1''(\xi))^2 d\xi$$

$$K_2 = EI \int_o^\ell (\phi_2''(x)^2 dx \qquad = \qquad \frac{EI}{\ell^3} \int_o^1 (\phi_2''(\xi))^2 d\xi$$

$$\phi'' = \frac{d^2\phi(x)}{dx^2} \quad \text{or} \quad \frac{\partial^2\phi(\xi)}{\partial\xi^2}$$

$$\xi = \frac{x}{\ell}$$

Linearization

To simulate the motion of the proposed system, the nonlinear equations (1) will be used. For the purpose of design of the minimum-time control, the linearized form of equations (1) will be obtained. All the higher degree terms in θ, q_1 and q_2 are dropped from the equations. The resulting linearized system of equations (1) can then be written as

(2) $$(J_o + M\ell^2 + I_o)\ddot{\theta} + (M\ell\phi_{1E} + w_1)\ddot{q}_1 + (M\ell\phi_{2E} + w_2)\ddot{q}_2 = T$$

$$(w_1 + M\ell\phi_{1E})\ddot{\theta} + (m + M\phi_{1E}^2 + J_p\,\phi_{1E}'^2)\ddot{q}_1 + (M\phi_{1E}\phi_{2E} + J_p\phi_{1E}'\phi_{2E}')\ddot{q}_2 = -K_1q_1$$

$$(w_2 + M\ell\phi_{2E})\ddot{\theta} + (M\phi_{1E}\,\phi_{2E} + J_p\phi_{1E}'\phi_{2E}')\ddot{q}_1 + (M\phi_{2E}^2 + J_p\phi_{2E}'^2 + m)\ddot{q}_2 = -K_2q_2$$

The above equations can be written in the form of

(3) $$\dot{\underline{X}} = \underline{A}\underline{X} + \underline{B}\,u$$

where

$$A = \left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \hline & & & 0 & 0 & 0 \\ & \underline{M}^{-1}\underline{K} & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \end{array}\right]$$

$$
B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \hline \underline{M}^{-1}\underline{b} \end{bmatrix}
$$

where

$$
M = \begin{bmatrix} J_o + M\ell^2 + I_o & M\ell\phi_{1E} + w_1 & w_2 + M\ell\phi_{2E} \\ w_1 + M\ell\phi_{1E} & m + M\phi_{1E}^2 + J_P\phi_{1E}'^2 & M\phi_{1E}\phi_{2E} + J_P\phi_{1E}'\phi_{2E}' \\ w_2 + M\ell\phi_{2E} & M\phi_{1E}\phi_{2E} + J_P\phi_{1E}'\phi_{2E}' & M\phi_{2E}^2 + J_P\phi_{2E}'^2 + m \end{bmatrix}
$$

$$
\underline{K} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -K_1 & 0 \\ 0 & 0 & -K_2 \end{bmatrix}
$$

$$
\underline{b} = \begin{bmatrix} T \\ 0 \\ 0 \end{bmatrix}
$$

$$
\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \end{bmatrix} = \begin{bmatrix} \theta_1 \\ q_1 \\ q_2 \\ \dot{\theta}_1 \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix}
$$

Mode Shape Analysis

The simple arm of these studies consists of a beam with rotary inertia at one end and payload mass at the other end.

The transverse vibration of the beam can be described as

(4) $$EI\ \partial^4 U/\partial x^4 + \rho A\ \partial^2 U/\partial t^2 = 0$$

where

$$U(X,t) = q(t)r(\xi)$$

$$\xi = x/\ell \quad .$$

Boundary Conditions on the arm (inertia, beam, and payload) are

at $\xi = 0$:

1. $r(\xi) = 0$

2. $d^2r(\xi)/d\xi^2 = -(I_o/\rho A\ell^3)\beta^4(d\ r(\xi)/d\xi|_{\xi=0})$

and, at $\xi = 1$:

1. $d^2r(\xi)/d\xi^2 = 0$

2. $d^3r(\xi)d/\xi^3 = -(M/\rho A\ell)\beta^4 r(\xi)$

The solution of the equation (4) subject to the boundary condition is

$$(5) \qquad r(\xi) = \sin(\beta\xi)-\sinh(\beta\xi) + \nu\ (\cos(\beta\xi) - \cosh(\beta\xi) + (2I^*/\beta^2)\sinh(\beta\xi))$$

where

$$\nu = (\sin(\beta) + \sinh(\beta))/((2I^*/\beta^3)\sinh(\beta) - (\cos(\beta) + \cosh(\beta))$$

$$\beta^4 = \rho A\omega^2\ell^4/EI$$

$$I^* = \rho A\ell^3/I_o$$

Modified Mode Shape

The mode shape used in the dynamic modelling, is measured from the axis without deflection. With this assumption, zero slope at the pinned end has to be achieved. Thus, the mode shape can be written as

$$\phi(\xi) = r(\xi) - \xi(d\ r(\xi)/d\xi|_{\xi=0})$$

where $r(\ \xi\ )$ is the mode shape which is derived from the previous section and satisfies the boundary conditions (i.e. Beam with concentrated mass (M) at one end ($\xi = 1$) and rotary inertia mass (I_o) at the other end ($\xi = 0$). So, the modified mode shape is

$$(6) \qquad \phi(\xi) = \sin(\beta\xi) - \sinh(\beta\xi) + \nu(\cos(\beta\xi) - \cosh(\beta\xi) + (2I^*/\beta^3)\sinh(\beta\xi) - (2I^*/\beta^2)(\xi))$$

The natural frequencies have been tested by comparing the transfer matrix (DSAP package, ref [4]) and the dynamic state variable results. The frequencies of the first mode are very close together. For the second mode, the difference is typically less than 10 percent. This difference is caused by the truncation of the higher modes in the dynamic equations.

The Time-Optimal Position Control for Linear Time-Invariant Plant

Given the dynamic system

$$(7) \qquad \dot{\underline{X}}\ (t) = \underline{AX}(t) + \underline{BU}(t)$$

where

$\underline{X}(t)$ = the state vector
$\underline{A}$ = the nxn system matrix
$\underline{B}$ = the nxr gain matrix
$\underline{U}(t)$ = the control vector

$$(8) \qquad |\ U_j(t)\ | \leq 1; \quad j = 1, 2, \ldots, r$$

Then, given that at the initial time $t = 0$, the initial state of the system (7) is $X(0) = X_0$, find the control $u^*(t)$ that transfers the system (7) from X_0 to the origin 0 in minimum time.

The cost functional is

$$(9) \quad J(u) = \int_0^T dt; \quad T \text{ free}$$

The Hamiltonian function for the problem is

$$\begin{aligned}(10) \quad H(\underline{X}(t), \underline{p}(t), \underline{U}(t)) &= 1 + \langle \underline{AX}(t), \underline{p}(t) \rangle \\ &\quad + \langle \underline{BU}(t), \underline{p}(t) \rangle \\ &= 1 + \langle \underline{AX}(t), \underline{p}(t) \rangle \\ &\quad + \langle \underline{U}(t), \underline{B'}\ \underline{p}(t) \rangle\end{aligned}$$

where

$\underline{p}(t)$ = costate vector and
$\langle \cdot \rangle$ = inner product operation

Necessary Conditions

Let $U^*(t)$ be a time-optimal control that transfers the initial state X_0 to the origin 0. Let $X^*(t)$ denote the trajectory of the system (7) corresponding to $U^*(t)$, originating at X_0 at $t_o = 0$, and hitting the origin 0 at the minimum time (T^*). Then there exists a corresponding costate vector $p^*(t)$ such that:

$$\begin{aligned}\dot{\underline{X}}^*(t) &= \partial H(\underline{X}^*(t), \underline{p}^*(t), \underline{U}^*\underline{q}(t))/\partial \underline{p}^*(t) \\ &= \underline{AX}^*(t) + \underline{BU}^*(t) \\ \dot{p}^*(t) &= - \partial H(\underline{X}^*, \underline{p}^*(t), \underline{U}^*(t))/\partial \underline{X}(t) \\ &= - \underline{A'}\underline{p}^*(t)\end{aligned}$$

with the boundary conditions
$\underline{X}^*(0) = X_0$

$\underline{X}^*(T^*) = 0$.

In addition, from the Hamiltonian function and minimum principle, the inequality

$$\begin{aligned}(11) \quad & 1 + \langle \underline{A}^*\underline{X}^*(t), \underline{p}^*(t) \rangle + \langle \underline{U}^*(t), \underline{B'}\underline{p}^*(t) \rangle \\ & < 1 + \langle \underline{A}^*\underline{X}^*(t), \underline{p}^*(t) \rangle + \langle \underline{U}(t), \underline{B'}\underline{p}^*(t) \rangle\end{aligned}$$

holds for all admissible $U(t)$ and for t $[0, T^*]$. Relation (11) yields, in turn, the relation

$$(12) \quad \underline{U}^*(t) = - \operatorname{sgn}\{\underline{B'}\underline{p}^*(t)\}$$

Because $H(\underline{X}^*(t), \underline{p}^*(t), \underline{U}^*(t))$ is not an explicit function of t

$$H(\underline{X}^*(t), \underline{p}^*(t), \underline{U}^*(t)) = 1 + \langle \underline{AX}^*(t), \underline{p}^*(t) \rangle + \langle \underline{U}^*(t), \underline{B'}\underline{p}^*(t) \rangle = 0$$

holds for all t $[0,T^*]$.
The costate function obeys

(13) $\dot{\underline{p}}^*(t) = -\underline{A}'\underline{p}^*(t)$

An important property for p*(t) is

$$p^*(t) \neq 0 \text{ for } t\ [0,T^*]$$

A necessary condition for the solution of the system (7) to exist is that the system be completely controllable. The system is completely controllable if and only if the system is normal. The necessary and sufficient conditions for the system to be normal is that the [nxn] matrices G_1, G_2, ..., G_r given by the equations

$$G_1 = [b_1\ Ab_1\ A^2b_1\ \ldots\ A^{n-1}b_1]$$
$$G_2 = [b_2\ Ab_2\ A^2b_2\ \ldots\ A^{n-1}b_2]$$

•
•
•

$$G_r = [b_r\ Ab_r\ A^2b_r\ \ldots\ A^{n-1}b_r]$$

are all nonsingular where the b_i are the component vectors of B.
If the system (7) is normal, then the time-optimal control is unique (if it exists).

Modified system

For the case of interest let $\tau = T - t$, the the system (7) can be written as:

(14) $\dot{\underline{X}}(\tau) = -\underline{AX}(\tau) - bu(\tau)$

and

$$\underline{X}(\tau = 0) = \underline{V}\ .$$

Note that b is an nxl constant matrix.

Given $p(T) = \alpha^*Q^*V$, equation (13) has the unique solution

(15) $p(t) = \alpha\, e^{-A'(t - T)}QV$

and we have

$$u^*(t) = -\text{sgn}(\langle\underline{V},\ Qe^{A\tau}\ b\rangle)ds)$$

The solution of equation (14) at $\tau = T$ is

(16) $M(V,T) = e^{-A\tau}\ \{V + \int_0^T e^{As}\ b\ \text{sgn}(\langle V,Qe^{As}\ b\rangle)ds\}$

Iterative Procedure for Solving the Minimum Time Position Control

More details for this procedure can be found in ref [9] by John B. Plant. The iterative procedure is based on equation (9) by choosing element (V,T) and checking to see if the particular choice satisfies $X_0 = M(V,T)$.

A typical step in the iterative procedure is illustrated in Figure (2) (for a second order plant) where the hypersurface PHI is an isochrone

$$(17)\quad PHI(T,X_0) = \{X : X = \exp(AT)(X_o + \int_0^T \exp(-As)bu(s)ds),\ |u| < 1\}$$

or $PHI(T,X_0)$ is the set of all states which are reachable from X_0 in time T.

One can select a suitable state-space representation and use the hypersphere as a target, or choose a suitable Q matrix and leave the state-space representation as it is initially.

Some definitions which will be used in the iterative procedure will be listed in the following. More details can be found by consulting ref. [9].

<u>Definition 1</u>

Hyperspherical coordinates, $h(\theta)$, will be defined as

$$h_1(\theta) = \cos(\theta)$$

$$h_k(\theta) = \cos(\theta_k) \prod_{j=1}^{k-1} \sin\theta_\gamma \quad (k = 2,3,\ldots,n-1)$$

$$h_n(\theta) = \prod_{j=1}^{n-1} \sin\theta_j$$

where $0 < \theta_i < \pi \quad i = 1, 2, \ldots, n-2$

$$0 < \theta_{n-1} < 2\pi$$

and the $h_\theta(\theta)$ is the first derivative of $h(\theta)$ with respect to θ, that is

$$h_\theta(\theta) = \begin{bmatrix} \frac{\partial h_1(\theta)}{\partial\theta_1} & \frac{\partial h_1(\theta)}{\partial\theta_2} & \cdots\cdots & \frac{\partial h_1(\theta)}{\partial\theta_{n-1}} \\ \vdots & & & \\ \frac{\partial h_n(\theta)}{\partial\theta_1} & \frac{\partial h_n(\theta)}{\partial\theta_2} & \cdots\cdots & \frac{\partial h_n(\theta)}{\partial\theta_{n-1}} \end{bmatrix}$$

Given V, a vector which belongs to the set of vectors from the origin to boundary of hypersphere, a unique θ can be solved from the relation $\theta = (1/r)h^{-1}(LV)$ where $\langle V,QV\rangle = r^2$ and $Q = L'L$. (Q and L can be unity matrices). The $h_\theta(\theta)$ is used merely to find the derivative of $M(V,T)$ when restricted to D.

<u>Definition 2 (D)</u>

Let D $\partial S x R$ be the set of ordered pairs in $\partial S x R$ such that if $(V,T) \in D$, then $V \in \partial S$ (the boundary of the hypersphere) is the terminal state at time $t^* = T$ of the time-optimal trajectory for some state $\underline{X}$ in the domain of controllability.

Let $G(V,T)$ be the first derivative of $M(V,T)$ restricted to D and with respect to all arguments. That is,

$$G(V,T) = [rM_V(V,T)L^{-1}h_\theta(\theta) \mid M_T(V,T)].$$

The iterative procedure can be summarized in the following. (from Ref [9]).

For the k(th) guess of $(V,T) = (V_k, T_k)$

1. $X_k = M(V_k, T_k)$

(It can be shown that $M(V,T)$ is a one-to-one mapping of each point on the boundary of the hypershpere (V) to a point along the minimum isochrone with the same minimum time T.)

2. $e_k = X_0 - X_k$

where

3. $X_0 = M(V^*, T^*)$

4. $\theta_k = (1/r)h^{-1}(v_k)$

where $h^{-1}(V_k)$ is the inverse function of $h(\theta_k)$

5. $G_k = [rM_V(V_k, T_k)\ h_\theta(\theta_k) \mid M_T(V_k, T_k)]$

6. $$\begin{bmatrix} \Delta\hat{\theta}_k \\ \Delta\hat{T} \end{bmatrix} = G_k^{-1} e_k$$

7. $\Delta\hat{\gamma}_k = h_\theta(\theta_k)\Delta\theta_k$ an n-vector

8. $$V_{k+1}(\gamma_k) = \frac{r(V_k + \gamma_k\Delta\hat{\gamma}_k)}{\|V_k + \gamma_k\Delta\hat{\gamma}_k\|}$$

9. $T_{k+1}(\gamma_k) = T_k + \gamma_k\Delta\hat{T}_k$

10. choose γ_k in (0,1) such that

$$e_{k+1} = \min_{0<\gamma_k<1} |X_0 - M\{V_k(\gamma_k), T_{k+1}(\gamma_k)\}|$$

11. $V_{k+1} = V_{k+1}(\gamma_k^*)$

12. $T_{k+1} = T_{k+1}(\gamma_k^*)$

Note: The nxn matrix $G(V,T)$ has rank n if and only if

$$-\langle V,AV\rangle + |\langle b,AV\rangle| > 0$$

In the step 10 of the iterative procedure, γ_k can be chosen by considering the fact (ref. [9]) that

$$\|e_{k+1}(\gamma_k)\| \le |1 - \gamma_k|\ \|e_k\| + \gamma_k^2 N_k$$

where a positive real number N_k can be chosen such that $e_{k+1}(\gamma_k)$ is bounded.
The value of γ_k can be chosen as follows

1. Choose $\gamma_k = 1$, evaluate $M(V_{k+1}, T_{k+1})$. When N_k is chosen to $|\hat{e}_{k+1}|$, then

$$\hat{e}_{k+1} = e_{k+1}\ (\gamma_k = 1)$$

2. If $|\hat{e}_{k+1}| < (1/2)\ |e_k|$, increment k and proceed to the next iteration. Otherwise,

$$\hat{\gamma}_k = (1/2)\ (|e_k|/|e_{k+1}|)$$

is used. $\hat{e}_{k+1}$ can be evaluated when N_k is chosen equal to $(1/2) \quad |e_k|/|\hat{\gamma}_k|$.

3. If $|\hat{e}_{k+1}| < |e_k|$ proceed to the next iteration. Otherwise

$$\gamma_k = \frac{\hat{\gamma}_k^2 |e_k|}{2(|\hat{e}_{k+1}| - (1 - \hat{\gamma}_k)|e_k|)}$$

Evaluation of M(V,T)

The value of M(V,T) can be found from the function

$$M(V,T) = \exp(-AT)\ (V + \int_0^T \exp(As)\ b\ \mathrm{sgn}(\langle V, Q \exp(A\ s)\ b\rangle) ds$$

where

$u(\tau) = -\mathrm{sgn}\ (\langle V, Q\exp(As)b\rangle)$

So $u(\tau)$ has the valute of +1 or -1

Before one can evaluate M(V,T), the switching times have to be found. The switching times $\tau_j(V)$, j = 1,2,... are a set of ordered real numbers, such that (ref. [9])

1. $\tau_j(V)\ \varepsilon(0,T)$
2. The inner product $\langle V, Q\exp(A\tau_j(V))b\rangle = 0$
3. $\mathrm{sgn}\ \{\langle V, Q\exp(\underline{A}\tau_j^+(\underline{V}))\underline{b}\rangle\} = -\mathrm{sgn}\{\langle \underline{V}, \underline{Q}\exp(A\tau_j^-(V))\underline{b}\rangle\}$
4. $\tau_j(\underline{V}) < \tau_{j+1}(\underline{V})$ for all j

Solving for the switching time consumes by far the greatest portion of the computing time.

Initial guess

This iteration method is very dependent on the initial guess. As in ref [9], Plant used the biggest time-constant of the system for the starting final time. The initial point on hypersphere (V) can be solved by using this time-constant, and starting the iteration from this point. The proposed flexible arm configuration has the eignvalues on the imaginary axis, so the time constant is equal to zero. Hence it is trivial to guess the initial point.

Normality of the system can be shown, so the uniqueness of the minimum-time poistion control solution can be proven (ref [11], Athens). In ref [9] it was shown that $-\langle V, AV\rangle > 0$ has to be achieved during the iteration for V on the hyperspere.

Accuracy

By modifying the problem to hitting a hypersphere instead of the origin, the accuracy of the final position of the end point of the flexible arm is made dependent on the radius of the target hypersphere. For the example in Fig. 1, when r = 0.1 is chosen. The final angular position is 0.18 radian instead of zero and the length of beam is 60 inches. Then the error at the end point is 60*0.18 which is equal to 10.8 inches. This is unacceptably large and work is underway to reduce the radius and maintain convergence.

Strategy

The objectives for considering the minimum-time position control are not for controlling the flexible arm in all phases of operation. The solution of the minimum-time position control can be combined with the other techniques such as "bracing" or using a viscoelastic material which has a damping property. It is believed that these additional techniques can improve the accuracy of the end point.

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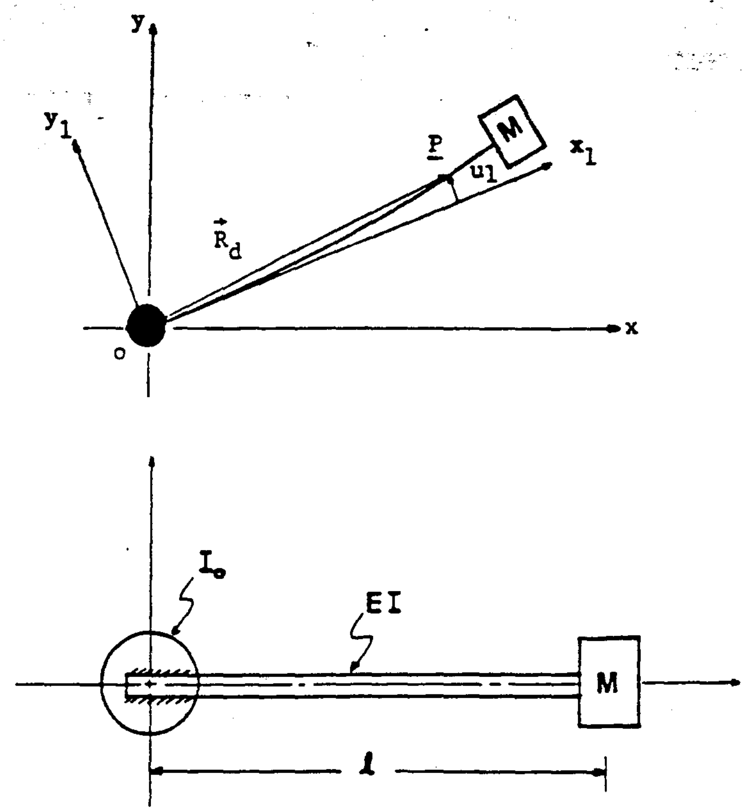

Figure 1. Coordinates and Nomenclature for the example configuration.

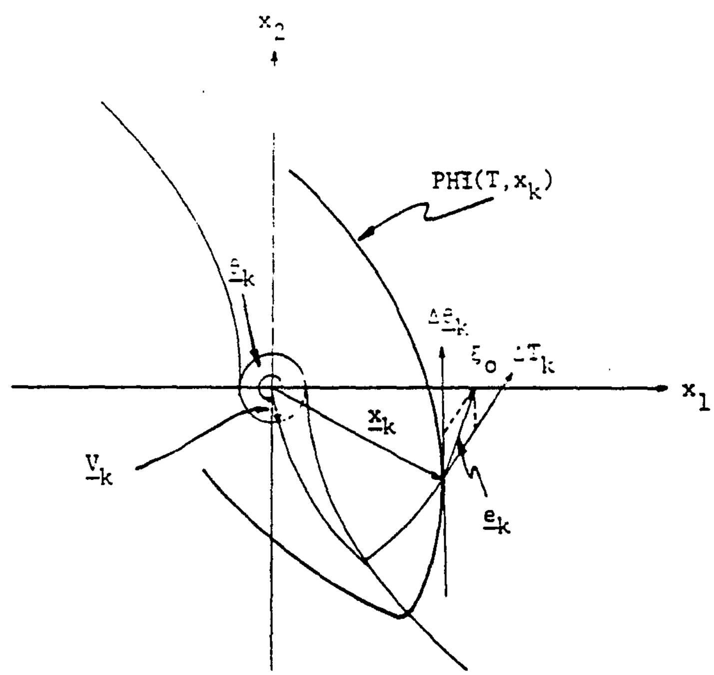

Figure 2. The state space representation of the minimum time problem for a second order plant.