Final Report for Period: 09/2008 - 08/2009 **Submitted on:** 10/02/2009

Principal Investigator: Zeng, Chongchun . **Award ID:** 0627842

Organization: GA Tech Res Corp - GIT

Submitted By:

Zeng, Chongchun - Principal Investigator

Title:

CAREER: Perturbation Problems in PDE Dynamics

Project Participants

Senior Personnel

Name: Zeng, Chongchun

Worked for more than 160 Hours: Yes

Contribution to Project:

Post-doc

Name: Deng, Shengfu

Worked for more than 160 Hours: Yes

Contribution to Project:

Dr. Deng obtained his Ph. D. degree with Prof. Shuming Sun at Virginia Polytechnic Institute in 2008. He worked with me as a postdoc with half of the support from this grant in the academic year 2008--2009. He works on the dynamics of surface water waves.

Graduate Student

Name: Lu, Nan

Worked for more than 160 Hours: Yes

Contribution to Project:

Mr. Lu was a graduate student at University of Virginia in the fall 2005 and then transferred to Georgia Institute of Technology as a graduate student beginning in the spring 2006. He was supported by this grant for the fall semester 2005, the summer of 2006, 2007, 2009, and the spring semester 2008. He works on differential equations and dynamical systems.

Undergraduate Student

Name: Martin, Charles

Worked for more than 160 Hours: No

Contribution to Project:

Mr. Martin is a senior undergraduate student at Georgia Institute of Technology. In the summer 2006, he will work with me on immersed invariant manifolds for dynamical systems.

Technician, Programmer

Other Participant

Research Experience for Undergraduates

Organizational Partners

Other Collaborators or Contacts

Prof. Jalal Shatah, Courant Institute, New York University. We collaborate on the research on Hamiltonian PDEs and their dynamics, including blow-up solutions, singular perturbations, gauge transformations, free boundary problems related to water waves and etc.

Prof. Peter W. Bates, Dept. of Math., Michigan State University. We collaborate on the research on invariant manifolds theory in general Banach spaces and singular perturbation problems on nonlinear parabolic equations.

Prof. Kening Lu, Dept. of Math., Brigham Young University. We collaborate on the research on invariant manifolds theory in general Banach spaces and singular perturbation problems on nonlinear parabolic equations.

Prof. Andrea Nahmod, Dept. of Math., University of Massachusetts at Amherst. We collaborate on Ishimori systems and gauge transforms for Schrodinger maps.

Prof. Luis Vega, Universidad del Pais Vasco, Spain. We collaborate on gauge transforms for Schrodinger maps.

Prof. Weiyue Ding, School of Math., Peking University, China. We collaborate on blow-up related problems of Schroedinger maps.

Prof. Hongyan Tang, Dept. of Math., Tsinghua University, China. We collaborate on blow-up related problems of Schroedinger maps.

Dr. Pierre Germain, Courant Institute, New York University. We collaborate on blow-up related problems of Schroedinger maps.

Prof. Junping Shi, Dept. of Math., College of William and Mary. We are seeking potential collaborations on reaction-diffusion equations.

Prof. Shui-Nee Chow, School of Math., Georgia Institute of Technology. We collaborate on dynamical system problems of PDEs.

Prof. Yi Wang, Dept. of Math., University of Sciences and Technologies of China. We collaborate on dynamical system problems of PDEs.

Activities and Findings

Research and Education Activities: (See PDF version submitted by PI at the end of the report)
See attached file.

Findings: (See PDF version submitted by PI at the end of the report)
See attached file.

Training and Development:
This grant supported my Ph.D student Mr. Nan Lu in the summer of 2007 and spring 2008. Mr. Lu is a PhD student who started in GaTech in spring 2006. With the support from this grant, he was released of teaching during those periods. Under my supervision, he studies problems in dynamical systems and differential equations. I had assigned him reading and exploring some classical papers on normally hyperbolic type geometric singular perturbation problems, which laid a solid foundation for his future study on other types of singular perturbation problems and applications, such as from fluid mechanics. Currently he is working on dynamics in normally elliptic type singular perturbation problems.

The grant supported the research of the senior undergraduate student Mr. Charles Martin in the summer of 2006. Mr Martin as an excerllent undergraduate is interested in rigorous anaylysis. He will study low dimensional dynamical systems in details in the summer. The main topic will be perturbations of immersed invariant manifolds for ODEs and the related bifurcations.

I serve as a member of the PhD dissertation committee of Mr. Viktor Grigoryan, a PhD student at University of Massachusetts since 2005. He graduated in summer 2008 and will becomes a postdoc at UC Santa Barbara.

I served as a member of the PhD dissertation committee of Mr. Jean-Philippe Lessard, a Ph.D student at School of Math., Georgia Institute of Technology, who defended his PhD thesis in June 2007.

I served as a member of the PhD dissertation committee of Mr. Jiil Choi, a PhD student at School of Physics, Georgia Institute of Technology, who defended his PhD thesis in June 2007.

I served as a member of the oral exam committee of Mr. Robert Pruvenok, a master student at Georgia Institute of Technology, in 2007.

I hosted and served as a mentor of Ms. Mei Ming, a visiting Ph. D. graduate student during the fall 2008.

Outreach Activities:

I am serving as a member of the Scientific Committee member of 'The Third International Conference on Recent Advances in Applied Dynamical Systems', to be held at Guangzhou University, China, Dec. 2009.

I served on the organizing committee of 'International Conference on the Mathematical Theory and Applications of Liquid Crystal, Ferromagnetism and Related Topics', Southern China Normal University, China, July, 2009.

I served as one of the organizers of the workshop `Asymptotics and Singularities in Nonlinear and Geometric Dispersive Equations' at Banff International Research Center, to be held 8/24 -- 8/29/2008.

I served as a member of the Scientific Committee member of `International Conference on PDEs and Stochastic Dynamical System', held in Kunming, China, 1/21--1/31/2008.

I am serving as a guest editor of the special of the journal `Discrete and Continuous Dynamical Systems' for Peter Bates' 60th birthday.

I serve as the co-chair of the organizing committee of the Second International Conference on Recent Advances in Applied Dynamical Systems, held in in Zhejiang Normal University, Jinhua, Zhejiang Province, People's Republic of China during June 4-8, 2007.

I serve as an affliated faculty member at the Center of Nonlinear Sciences for the year 2007 -- 2008, which is a research center primarily based in the School of Physics at Georgia Institute of Technology with affliated members from many departments, such as Physics, Chemistry, Biomedical Engineering, Mechanical Engineering, and etc.

In the years 2005 -- 2007, I serve as the organizer of both the Colloquia of the School of Mathematics and the Colloquia of the Center of Dynamical Systems and Nonlinear Studies, at Georgia Institute of Technology. I have been involved in inviting speaker in all fields such as physics, engineering, and etc.

Journal Publications

Jalal Shatah and Chongchun Zeng, "Schr?dinger maps and anti-ferromagnetic chains", Communications in Mathematical Physics, p. 299, vol. 262, (2006). Published,

Jalal Shatah and Chongchun Zeng, "Geometry and a priori estimates for free boundary problems of the Euler's equation", Communication on Pure and Applied Mathematics, p. , vol. 61, (2008). Published,

Jalal Shatah and Chongchun Zeng, "A priori estimates for fluid Interface problems", Communication on Pure and Applied Mathematics, p. 84, vol. 61, (2008). Published,

Andrea Nahmod, Jalal Shatah, Luis Vega, and Chongchun Zeng, "Schr\"odinger Maps and their associated Frame Systems", International Mathematics Research Notices, p. 29 pages, vol. 2007, (2007). Published,

Peter Bates, Kening Lu, and Chongchun Zeng, "Approximately Invariant Manifolds and Global Dynamics of Spike States", Inventiones Mathematicae, p. , vol. 174, (2008). Published,

Weiyue Ding, Hongyan Tang, and Chongchun Zeng, "Self-similar Solutions of Schroedinger Flows", Calculus of Variations and Partial Differential Equations, p. 267, vol. 34, (2008). Published,

Pierre Germain, Jalal Shatah, and Chongchun Zeng, "Self-similar solutions for the Schroedinger map equation", Math. Z., p. , vol. , (2008). Accepted,

Jalal Shatah and Chongchun Zeng, "Local well-posedness for the fluid interface problems", Achives of Rational Mechanics and Analysis, p. , vol. , (2008). Submitted,

Books or Other One-time Publications

Web/Internet Site

Other Specific Products

Contributions

Contributions within Discipline:

Along with my collaborators, we are carrying out analysis on dynamics of partial differential equations which are often used to describe the evolution of physical and other natural systems.

For hyperbolic type dynamical systems, we have systematically developed the graph transform method for normally hyperbolic invariant manifolds and invariant foliations in the most general setting (up to date) in infinite dimensional spaces. In this way, we had studied the persistence, smoothness, and other issues related to invariant manifolds and invariant foliations. As these are fundamental tools in studying the qualitative structures, particularly singular perturbation problems, our results and methods enable one to apply these tools to study the structures of evolutionary PDEs which can be viewed as infinite dimensional dynamical systems. Partially supported by this grant, we recently have been working on approximate invariant manifolds. Our results could 1.) conceptually justify the numerical observations of invariant manifolds and 2.) make it possible to prove the existence of certain special but representative dynamical structures in singular perturbation problems.

As a truly infinite dimensional singular perturbation problem, singular parabolic PDEs have attracted much attention recently due to its origin in material sciences, math biology, and other subjects. There have been substantial amount of work on the existence of stationary spike solutions, which are special steady states spatially very much localized. Applying our results on approximate normally hyperbolic invariant manifolds, spike solutions with moving spikes and their global dynamics are obtained. This problem also demonstrates an effective approach in handling infinite dimensional singular perturbation problems.

Geometric PDEs such as wave maps and Schr\'oding maps have both geometric and PDE characters and our work is aimed at their rich dynamical structure. As the first step, we have been analyzing the basic well-posedness and regularity results for Schrodinger maps. As a basic and natural tool, the gauge transformation under our consideration reveals rich information contained in the gauge invariance of these systems. It also provides very useful tools in the study of global well-posedness and blow-up solutions. Usually, the gauge transformation is defined for smooth solutions and weak or rough solutions often appear in the PDE analysis. Our work on the gauge transformations for rough solutions substantially extends the power of this method.

In order to study the possible blow-up or global-in-time existence problems of Schroedinger maps, the PI also study self-similar equivariant solutions with collaborators. We have obtained those solutions 1.) in higher spatial dimensions targeted on complex projective spaces with finite local energy and 2.) in 2 space dimensions (critical) in weak energy space. Though they have showed to always have infinite energy, some of their detailed qualitative properties are derived as well.

Our another interest on Schr\'odinger maps is when some strong potential is present and restricts the motions to a neighborhood of some Lagrangian submanifolds. This problem with concrete background intrinsically connects symplectic geometry, applied mathematics, and multiple-scale singular perturbations (of non-normally-hyperbolic type). Our rigorous proof of the convergence to wave maps justifies the well-known (but non-trivial) results derived from formal analysis. We expect further investigation in these topics will show an exciting picture of geometry, dynamical systems, PDEs, and applied math together.

The free moving surface of the Euler's equation is a fundamental problem well known to the scientific community for a long time. Various progresses have been achieved from the applied math or PDE point of view. Our approach by considering this problem as a Lagrangian system on an abstract infinite dimensional Riemannian manifold, with the estimates involving substantial amount of Riemannian geometry, is quite different from the others'. Our success in systematically deriving new estimates has demonstrated the power of this approach. The local-in-time well-posedness of these problems have also obtained based on these considerations. Moreover, this approach covers a very broad area. Many physical quantities such as vorticity, gravity, and surface tension, can be included and handled in this framework. In particular, the local well-posedness and energy estimates of the free boundary problem of incompressible magneto-hydrodynamics systems are obtained by this method. All these results provide a very encouraging scenario of connecting the classical PDEs and analysis to Riemannian geometry at a completely different level for fluid problems and it can be a very exciting field for future investigation.

Contributions to Other Disciplines:

We are looking into the applications of invariant manifold theory to PDEs and applied math. In particular, these tools can be applied to material interface and math biology problems as described by the Allen-Cahn and Cahn-Hilliard equations and to the study on the transitional structures in chemistry. Our application illustrates how these problems can be treated systematically by using invariant manifolds.

Schrodinger maps, including its most famous example, the Landau-Lifschtz equation, comes originally from the ferro-magnetic theory. The Schrodinger map with strong potentials we considered are directly from anti-ferro-magnetic chains. We rigorously proved the well-accepted conclusion by physists that the continuum limit of the Heisenberg model of anti-ferro-magnetic chains is the sigma model. Also, our results on these singular perturbation problems justifies some formal multi-scale analysis widely used in physics and engineering.

Our work on the incompressible Euler's equation and possible future extension to compressible or viscose fluids provide a new angle in studying the fluid mechanics. In this framework, a.) some well-known phenomena such as Raighley-Taylor and Kelvin-Helmholtz instabilities can be viewed as completely geometric phenomena; b.) we expect to see new understanding of the transition between the slightly compressible to incompressible fluids; c.) it provides a new angle to to study the interaction between magnetic field, electrical field, and fluid velocity field.

Contributions to Human Resource Development:

This grant partially supports some graduate students (Nan Lu, Mei Ming) so that they could invest more of their time and energy into the development of their career at the appropriate stages.

The grant supports many external speakers and visitors to our department. The graduates and postdoc benefited greatly from the lectures of this visitor as well as from the personal interaction with these visitors.

The grant partially supports some undergraduate research activities, an example of which is Mr. Charles Martin, a senior undergraduate student at Georgia Institute of Technology. He indicated the intention to continue his graduate study in mathematics.

Based on the work supported by this grant, I am invited to be a member of the PhD dissertation committee of Mr. Viktor Grigoryan, a PhD student at University of Massachusetts and of Mr. Jean-Philippe Lessard, a Ph.D student at Georgia Institute of Technology, Mr. Jiil Choi, a PhD student at School of Physics, Georgia Institute of Technology, and the oral committee of Robert Pruvenok, a master student at Georgia Institute of Technology.

Contributions to Resources for Research and Education:

Contributions Beyond Science and Engineering:

Conference Proceedings

Categories for which nothing is reported:

Organizational Partners

Any Book

Any Web/Internet Site

Any Product

Contributions: To Any Resources for Research and Education

Contributions: To Any Beyond Science and Engineering

Any Conference

PROJECT ACTIVITIES

This project is concerned with various problems in evolutionary partial differential equations. Having started with regular and singular perturbation problems in PDE dynamics at the beginning, we have broadened our study to the various types of structural problems of evolutionary PDE systems. Our interests include local in time well-posedness, asymptotic behavior of PDE dynamics, rigorous and formal asymptotic expansions, variational structures, geometric structures and application in physics and engineering. Since a progress report was submitted in Jan. 2006 when this grant was transferred from University of Virginia to Georgia Institute of Technology with a different grant No., here I focus on the activities since Jan. 2006.

1. Research activities

With my collaborators, we have conducted research on the following topics:

1.1. **Free surface problems of incompressible invicid fluid systems.** With Prof. Jalal Shatah at Courant Institute, New York University, we considered various types of free boundary problems of the incompressible Euler's equation such as the motions of water waves in the vacuum, the interfaces between incompressible fluids, and interfaces problems in magneto-hydrodynamics. In addition to vorticity, surface tension and gravity may also be included in our consideration.

The water waves problem has been studied extensively using various very involved techniques ranging from complex analysis, Clifford algebra, to classical PDEs methods. Breakthroughs in local-in-time well-posedness have been accomplished in recent years. Some of these work were based on Eulerian coordinates and some others are based on the Lagrangian coordinates. Our approach is from a quite different point of view compared with the others'. In particular, at an abstract level we view this PDE problem as Lagrangian flows on an infinite dimensional Riemannian manifold of volume preserving diffeomorphisms with various properties, while the finite dimensional Riemannian geometry is also heavily involved at a concrete level as the free boundary is in motion. Our goal is to establish the relationship between (1) (both the abstract and concrete) geometric aspects, such as the Riemannian connection, Riemannian curvature tensor, geodesic flows, Jacobi fields, and etc., and (2) the PDE aspects, such as well-posedness, a priori estimates, regularities, and etc. In details, we study a priori energy estimates and the regularity of solutions under the guidance of geometry as the first step. We worked in a very effective general geometric framework which can be adapted for various types of problems including both the one-fluid-surface and two-fluid-interface problems in rather general settings involving vorticity, gravity, magnetic fields, and/or surface tension. Particularly, we focus on the mathematical significance of the surface tension and the vorticity. The former has always been considered as physically helpful but rigorously it induces nonlinear higher order derivative terms which can be troublesome if not handled not carefully. The latter substantially raises the complexity of the problem as it prevents the problem from being able to be reduce to the moving boundary only. Moreover, it may cause the Taylor instability, which is examined carefully from our geometric point of view.

With a priori estimates obtained, we also established the well-posedness of these free boundary problems from this geometric approach and systematically extending to other problems such as the interface problems in magneto-hydrodynamics and possibly compressible fluid interfaces.

1.2. **Rotating waves in the free boundary problems of Euler's equation.** With my postdoc fellow Dr. Shengfu Deng, we also studied qualitative dynamics of free surfaces of incompressible and invicid fluid. In particular, we considered a droplet or a star of incompressible and invicid fluid moving in the space under the impact of surface tension and self-gravitation. We are interested in finding special type of waves. We started with the problem of surface waves and velocity fields both rotation around certain center with constant speed. The first case work on does not include vorticity for simplicity. This involves methods from functional analysis, PDEs, and dynamical systems methods and careful study of the Dirichlet-Neuman operator and Poisson equation.

1.3. **Free boundary problems of incompressible, elastic, and viscous fluids.** With Prof. Ping Zhang and my visiting Ph. D. student, Mei Ming, both at Chinese Academy of Sciences, the latter being supported by this grant in part, we studied the free boundary problems of incompressible, elastic, and viscous fluids. Our study is again based on the geometric consideration in the infinite dimensional function spaces. We are interested in a priori estimates and the local well-posedness.

1.4. **Schrödinger maps.** Motivated by the classical Landau-Lifschitz equation, we are interested in general Schrödinger maps targeted on Kähler manifolds, including the local and global well-posedness, blow-up solutions, singular perturbation problems, and etc.

First, J. Shatah and I consider the Heisenberg model of anti-ferromagnetic chains. While it is rather obvious to derive the Landau-Lifschitz equation as the continuum limit of the the Heisenberg model of ferromagnetic lattices, the one in the anti-ferromagnetic material is much less obvious though it is widely accepted that the continuum limit should be the σ-model from formal analysis. We formulated this problem into a general setting of Schrödinger maps targeted on Kähler manifolds with strong potentials which restricts the motion to neighborhoods of Lagrangian submanifolds of the target manifolds. This problem involves multiple scale dynamics, geometric PDEs, and applied math. We worked on the limit of this normally elliptic type singular perturbation problem.

With J. Shatah, Prof. Luis Vega of Universidad del Pais Vasco, Spain, and Prof. Andrea Nahmod of University of Massachusetts at Amherst, we continued to work on the gauge transformation of the Schrödinger maps. Since the Schrödinger maps are targeted on Kähler manifolds and highly nonlinearity, the gauge transformation based on the gauge invariance of the geometric PDE, transforms smooth Schrödinger maps into solution of some coupled nonlinear PDE systems to which some powerful analytic tools can be applied. We worked on gauge transformations for Schrödinger maps which are not as smooth.

An important aspect of Schrödinger maps is whether there exists blow-up phenomenon, which still remains open. From the comparison between Schrödinger map equations and nonlinear Schrödinger equations based on the gauge transformation, blow-up solutions are widely expected for Schrödinger maps. To understand the mechanism, the PI started with equivariant and self-similar solutions. With Weiyue Ding of Peking University and Hongyan Tang of Tsinghua University, China, we studied the cases of Schrd̈inger maps from $C^n \to CP^n$, $n \geq 2$. With Pierre Germain and Jalal Shatah of New York University, we studied the case from $R^2 \to S^2$ which is the energy critical case.

With A. Nahmod, we also study the Ishimori systems from the ferromagnetism, which is a 2-dim model for the topological vortices. At this stage, we are looking for special solutions such as the equivalent of the radial solutions in the Schrödinger equations. Since the Ishimori systems have the box operator, which is the Lapalace in the 2-dim (space) under the hyperbolic metric, instead of the Euclidean metric, the radial sysmmtry becomes the hyperbolic 'radial' symmetry. We start this problem with the appropriate formulation of function spaces.

1.5. **Invariant manifold.** With Prof. Peter Bates, at Michigan State University, and Prof. Kening Lu, at Brigham Young University, we have been studying approximately invariant manifolds for general infinite dimensional dynamical systems. We are interested in the existence of true invariant manifolds near approximate invariant manifolds, which may come from either nearby systems or numerical observations. The former corresponds to perturbation problems and the latter is conceptually important for numerical simulations. We also worked on the qualitative structure of the dynamical system in neighborhoods of these manifolds.

1.6. **Parabolic PDEs.** In classical finite dimensional dynamical systems, invariant manifold has been a fundamental and effective tools in singular perturbation problems. With our abstract theory of invariant manifolds in infinite dimensional spaces, it is very natural to extend their applications to PDE singular perturbation problems. We consider a singular parabolic PDE and seek dynamic spike solution with moving peaks. Previously, many results have been obtained on stationary peak solutions as they could be constructed using reduction method and Implicit Function Theory very much locally in the phase space. Since the dynamic peak solutions do not stay in a small neighborhood in the phase space and are not dynamically stable, the construction of dynamic peak solutions are completely different from the steady ones. Since the spike states are expected to form an invariant manifold in the infinite dimensional phase space, we applied our invariant manifold theory and successfully constructed dynamic spike solutions which keep the spike profile for all time. Moreover, the global dynamics of their spike locations has also been obtained up to the leading order.

With Prof. Junping Shi, at College of William and Mary, we are seeking potential collaborations on reaction-diffusion equations. We are exploring transition layer solution of some systems from material sciences.

1.7. **Talks, trips, and visitors.** The talks I have given since 01/2006:

- Energy estimates of the free boundary problem of the Euler's equation (45 min invited session talk), AMS sectional meeting, Florida International University, Miami, FL (2006 Spring Southeastern Meeting), Apr. 2006.
- Energy estimates of the free boundary problem of the Euler's equation, invited, University of Miami, April 2006.
- Energy estimates of the free boundary problem of the Euler's equation, Colloquium talk at Florida State University, Tallahassee, FL, Apr. 2006.
- Hamiltonian PDEs with strong constraining potentials, Mini-course (3X 90 min), Chengdu, China, June 2006.
- Energy estimates of the free boundary problem of the Euler's equation, (60 min) International conference on nonlinear and stochastic dynamics, Sichuan Univ., Chengdu, China, June 2006.
- Energy estimates of the free boundary problem of the Euler's equation, (60 min) ECNU Workshop on Nonlinear PDEs, Eastern China Normal University, June 2006.
- Perturbations of Homoclinic orbits for PDEs, Shanghai Jiaotong University, June 2006.
- Homoclinic Orbits and Chaos for Dissipative Near Integrable PDEs, University of Sciences and Technology of China, June 2006.
- Periodic traveling waves for nonlinear wave equations, Kunming University of Sciences and Technology, Kunming, China, June 2006.
- Hydrodynamical instabilities and energy estimates of free boundary problems of the Euler equation, (60 min) International Conference on Nonlinear and Harmonic Analysis - The Second Nankai-Edinburgh Joint Symposia (2006), Nankai University, Tianjin, China, Sept. 2006.
- Hydrodynamical instabilities and energy estimates of free boundary problems of the Euler equation (45 min), Special Session on "Theory and Applications of Infinite Dimensional Dynamical Systems", the 2006 Spring Western AMS Sectional Meeting, Salt Lake City, UT, Oct. 2006.
- Hydrodynamical instabilities and energy estimates of free boundary problems of the Euler equation, University of Toronto, Oct. 2006.
- Hydrodynamical instabilities and energy estimates of free boundary problems of the Euler equation, Brown University, Nov. 2006.
- Hydrodynamical instabilities and energy estimates of free boundary problems of the Euler equation (45 min), Workshop on 'Mathematical Theory of Water Waves', Mathematisches Forschungsinstitut Oberwolfach, Germany, Nov. 2006.
- Hydrodynamical instabilities and energy estimates of free boundary problems of the Euler equation (40 min), International Conference on Bifurcation Theory

of Dynamical Systems and Related Topics, Peking University, Beijing, China, Dec. 2006.
- Perturbations to homoclinic orbits of partial differential equations, Beijing University of Technology, Beijing, China, Dec. 2006.
- Free boundary problems of the Euler equation: hydrodynamical instabilities and energy estimates, seminar talk at Courant Institute of Mathematical Sciences, March, 2007.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation (60 min), Section of 'Dynamics, Patterns and Structures', MAMOS workshop, University of Texas, Austin, TX, Oct. 2007.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, George Washington University, Oct. 2007.
- Free boundary problems of the Euler equation and hydrodynamical instabilities (30 min), special session on 'Mathematical topics in oceanography', SIAM conference on analysis of PDEs, Mesa, AZ, Dec. 2007.
- Normally hyperbolic invariant manifolds and invariant foliations, seminar talk at Kunming University of Sciences and Technologies, Kunming, China, Dec. 2007.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, seminar talk at Yunnan Normal University, Kunming, China, Dec. 2007.
- On the free boundary problem of incompressible Euler equations, (12 hours lectures) Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing, China, Dec. 2007 – Jan. 2008.
- Schrodinger maps and anti-ferromegnatic chains, seminar talk at the Institute of Applied Physics and Computational Mathematics, Beijing, China, Jan. 2008.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, Geometry and Analysis of Dynamical Systems Conference to Celebrate the Mathematical Contributions of Xiao-Biao Lin and Stephen Schecter, North Carolina State University, Feb. 2008.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, Special Session on 'Dynamical Systems and Applications', The 7th AIMS International Conference on Dynamical Systems, Differential Equations and Applications, Arlington, TX, May. 2008.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, Special Session on 'Long Time Behavior of Hamiltonian and Dissipative Systems', The 7th AIMS International Conference on Dynamical Systems, Differential Equations and Applications, Arlington, TX, May. 2008.
- Normally hyperbolic invariant manifolds and invariant foliations, Workshop on dynamical systems and bifurcations, Shanghai Normal University, Shanghai, China, May 2008.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, Workshop on dynamical systems and bifurcations, Shanghai Normal University, Shanghai, China, May 2008.

- Hamiltonian ODEs and wave equations with strong potentials, Shanghai Jiaotong University, Shanghai, China, May 2008.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, Eastern China Normal University, Shanghai, China, May 2008.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, Fudan University, Shanghai, China, May 2008.
- Invariant manifolds and their applications, (3 hours lectures), Zhejiang Normal University Jinhua, Zhejiang, China, May 2008.
- Free boundary problem of incompressible Euler equations, Morningside Center of Mathematics, Chinese Academy of Sciences, Beijing, China, Jun. 2008.
- Free boundary problems of the Euler equation and hydrodynamical instabilities: a geometric point of view (60 min.), International Conference on Nonlinear Partial Differential Equations and Applications, Harbin, China, Jul. 2008.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, (2 hours lectures), Sun Yat-Sen University, Guangzhou, China, Jul. 2008.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, colloquium, Sun Yat-Sen University, Guangzhou China, Jul. 2008.
- Invariant manifolds and their applications, (4 hours lectures), Kunming University of Sciences and Technologies, Kunming, China, Jul. 2008.
- Unstable manifolds and nonlinear instability of the Euler equation (40 min), invited talk at the International Conference on Infinite Dimensional Dynamical Systems, York University, Toronto, Canada, Sept. 2008.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation (30 min), invited talk at the special session on "Dynamics and Applications of Differential Equations", AMS Southeastern Sectional Meeting, Huntsville, AL, Oct. 2008.
- Euler equation with fixed or free boundaries: from a Lagrangian point of view, colloquium, Georgia Institute of Technology, Nov. 2008.
- Euler equation with fixed or free boundaries: from a Lagrangian point of view, CNA seminar, Carnegie Mellon University, Pittsburgh, PA, Dec. 2008.
- Unstable manifolds and nonlinear instability of the Euler equation (30 min), invited talk at the special session on "Recent Developments in Nonlinear Dispersive Wave Theory", the Joint Meeting of AMS and Shanghai Mathematical Society, Shanghai, China, Dec. 2008.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, colloquium, Shanghai Normal University, Shanghai, China, Dec. 2008.
- Unstable manifolds and nonlinear instability of the Euler equation, seminar talk, Kunming University of Sciences and Technologies, Kunming, China, Dec. 2008.
- Invariant manifolds and their applications, seminar talk, Yunnan Normal University, Kunming, China, Dec. 2008.

- Unstable manifolds and nonlinear instability of the Euler equation, seminar talk, Yunnan University, Kunming, China, Dec., 2008.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, AMS sectional meeting, San Francisco State University, April, 2009.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities (50 min), International Conference on Variational Methods(ICVAM-2), Nankai University, Tianjin, China, May, 2009.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, seminar talk, University of sciences and technologies of China, China, May 2009.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, lecture at summer school, Sichuan University, May 2009.
- Unstable manifolds and nonlinear instability of the Euler equation, International Conference on Nonlinear and Stochastic Dynamics, Chengdu, China, June 2009.
- Perturbation to homoclinic orbits, seminar, Talk, Yunnan Normal University, June, 2009.
- Invariant manifolds and applications, seminar talk, Yunnan University of Finance and Economics, Kunming, China, June 2009.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, seminar talk, Institute of Mathematical Sciences, Chinese University of Hong Kong, Hong Kong, China, June, 2009.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation (40 min), 'International Conference on the Mathematical Theory and Applications of Liquid Crystal, Ferromagnetism and Related Topics', Southern China Normal University, China, July, 2009.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation (30 min), PDE session, First PRIMA Congress, Sydney, Australia, July, 2009.
- Free boundary problems in fluid equations, mini-course of 20 hours, Chinese National Summer School of Applied Mathematics and Statistics of 2009, Tsinghua University, China, July, 2009.
- Invariant manifold of dynamic spike solutions to a singular parabolic equation, seminar talk, Capital Normal University, Beijing, China, July, 2009.
- Free boundary problems of the Euler equation: local well-posedness and hydrodynamical instabilities, seminar talk, Institute of Applied Physics and Computational Mathematics, Beijing, China, July, 2009.
- Wave equations under strong constraining potentials, seminar talk, Morning Center of Mathematics, Chinese Academy of Sciences, July, 2009.

Other trips I made since 01/2006, which are not previously reported:

- Visit Courant Institute, New York University, to work with Prof. Shatah, May. 2006.

- Visit Courant Institute, New York University, to work with Prof. Shatah, Aug. 2006.
- Visit University of Massachusetts to work with Prof. Andrea Nahmod, Aug. 2006.
- Visit Courant Institute, New York University, to work with Prof. Shatah, March. 2007.
- Visit Courant Institute, New York University, to work with Prof. Shatah, March. 2008.
- Attended NONLINEAR WAVES CONFERENCE at Brown University, May 2007.
- Visit University of Massachusetts to work with Prof. Andrea Nahmod and to attend the PhD thesis defense of Mr. Viktor Grigoryan.
- Visit Courant Institute, New York University, to work with Prof. Shatah, Oct. 2008.
- Visit Courant Institute, New York University, to work with Prof. Shatah, May 2009.
- Visit Courant Institute, New York University, to work with Prof. Shatah, Aug. 2009.

Guests partially supported by this grant since 01/2006:

- Prof. Andrea Nahmod, University of Massachusetts,Feb. 2006 and April 2009.
- Prof. Rafael de la Llave, University of Texas at Austin, Feb. 2006.
- Prof. Alax Haro, March, 2006.
- Prof. Junping Shi, College of William and Mary, March, 2006.
- Prof. Jalal Shatah, New York University, March, 2006.
- Prof. Jianshe Yu, Guangzhou University, P. R. China, Apr. 2006 and April 2009.
- Prof. Xinchu Fu, Shanghai University, May, 2006.
- Prof. Dongmei Xiao, Shanghai Jiaotong University, March, 2007.
- Prof. Tom Bridges, University of Surrey, April, 2007.
- Prof. Prof. Rafael de la Llave, University of Texas at Austin, April, 2007.
- Prof. Kening Lu, Brigham Young University, April, 2007.
- Prof. Peter Bates, Michigan State University, April, 2007.
- Prof. Xijun Hu, Chinese Academy of Sciences, March – May, 2007.
- Prof. Li, Jibin, Kunming University of Sciences and Technologies, July – Aug. 2007 and Nov. 2008.
- Prof. Xiaoping Yuan, Fudan University, Oct. 2007.
- Dr. Shengfu Deng, Virginia Polytechnic Institute, May 2008.
- Ms. Mei Ming, Chinese Academy of Sciences, Aug. – Dec. 2008.
- Prof. Maoan Han, Shangha Normal University, Nov. 2009.
- Prof. Tiancheng Ouyang, Brigham Young University, April 2009.

2. Educational activities

- This grant provided partial support for my Ph.D student Mr. Nan Lu for the summer 2006, the summer 2007, and spring 2008. He work on analysis, differential equations, and dynamical systems.
- This grant supported the summer research of a senior undergraduate student Charles Martin on the perturbation problems related to immersed invariant manifolds for ODEs in the summer 2006.
- I supervised a master student Mr. Robert Pruvenok for summer reading in 2006. He worked on dynamical systems of maps. I also served on his oral committee in April 2007.
- I am a member of the PhD dissertation committee of Mr. Viktor Grigoryan, a PhD student at University of Massachusetts. He works on wave maps and their stabilities.
- I served as a member of the PhD dissertation committee of Mr. Jean-Philippe Lessard, a Ph.D student at Georgia Institute of Technology whose defense took place in June 2007.
- I served as a member of the PhD dissertation committee of Mr. Jiil Choi, a Ph.D student at School of Physics, Georgia Institute of Technology whose defense took place in June 2007.

3. Other Activities

- I served as a member of the Scientific Committee member of 'The Third International Conference on Recent Advances in Applied Dynamical Systems', to be held at Guangzhou University, China, Dec. 2009.
- I served on the organizing committee of 'International Conference on the Mathematical Theory and Applications of Liquid Crystal, Ferromagnetism and Related Topics', Southern China Normal University, China, July, 2009.
- I served as a member of the organizing committee of the workshop "Asymptotics and Singularities in Nonlinear and Geometric Dispersive Equations", Banff International Research Station (BIRS), Canada, Aug. 24 – 29, 2008.
- I served as a member of the scientific committee of "International Conference on PDEs and Stochastic Dynamical Systems", Kunming, China, Jan. 21-31, 2008.
- I served as the co-chair of the organizing committee of the Second International Conference on Recent Advances in Applied Dynamical Systems, held in in Zhejiang Normal University, Jinhua, Zhejiang Province, People's Republic of China during June 4-8, 2007.
- I serve as an affliated faculty member at the Center of Nonlinear Sciences for the year 2007 – 2008, which is a research center primarily based in the School of Physics at Georgia Institute of Technology.
- In the years 2005 – 2007, I serve as the organizer of both the Colloquia of the School of Mathematics and the Colloquia of the Center of Dynamical Systems and Nonlinear Studies, at Georgia Institute of Technology. I have been involved

in inviting speaker in all fields such as physics, engineering, and etc, and many of them were partially supported by this grant.

FINDINGS

0.1. **Free Boundary problems of the Euler's equation.** With Prof. Jalal Shatah at Courant Institute, New York University, we consider the free boundary problem of the incompressible Euler's equation. Our main results have been obtained for the problems on single fluid surface and fluid interface problem (two fluids). Vorticity, surface tension, and/or gravity are included in our investigation. From a geometric point of view, this problem can be considered as a Lagrangian system on the infinite dimensional manifold of the volume preserving diffeomorphisms equipped with the L^2 metric on its tangent bundle, which is due to the structure of the kinetic part of the Lagrangian. We first considered the one fluid problem with surface tension. Though surface tension is regularizing and helpful from the physical point of view. However, there has been less result in this case and until very recently, they were all on the irrotational case where the problem can be completely reduced to the boundary. The difficulty is that it appears as a nonlinear and nonlocal term involving the highest order derivative in the equation. When we look at this problem from the geometric and Lagrangian point of view, the surface tensions turns out to be the gradient of a potential, the surface area. It generates a semi-positive (not positive) definite quasi-linear operator in the leading order. It has an infinite dimensional kernel which corresponds to the vorticity. With this process, 1.) we could obtain a family of higher order energies of this problem when the fluid is with or without vorticity. These energies characterize the regularity of the velocity fields and the regularity of the geometry of the fluid domain in a completely geometric and coordinate independent form. We derived a priori estimates of this family of energies if the initial data is in reasonable Sobolev spaces. 2.) We also obtained the local-in-time well-posedness of the these free boundary PDEs based on this approach. In the construction of the local in time solutions, the estimates on the vorticity is not sufficient if the surface topology is not trivial. This difficulty is overcome by Neorther's Theorem and the observation that the vorticity is closely related to the symmetry of particle relabeling.

When there is no surface tension, the previous results by others either assumed (1) there is no vorticity and reduce the problem to the boundary or (2) applied very technical approach, such as the Nash-Moser method, in the case when the Raighley-Taylor instability is absent. The Raighley-Taylor sign condition just appears in the process of some very heavy estimates. In our approach, the free boundary water wave problem can be viewed as the geodesic flow on an infinite dimensional manifold. We went through its linearization, the Jacobi field equation on the infinite dimensional manifold, the Taylor sign condition becomes transparent as the positivity of the curvature in the flow direction. In both case, the relationship between the regularities of the free surface and the velocity field appears naturally. This systematic approach also allowed us to obtain uniform estimates as the surface converges to zero, assuming the Raighley-Taylor sign condition. In this way, the local-in-time well-posed is obtained by taking limit of vanishing surface tension problems.

The fluid interface problem also turns out to fit in this setting. It can also be formulated as a Lagrangian system on an appropriate infinite dimensional Riemannian manifold, for which our framework can be nicely modify to work. In particular, the well-known Kelvin-Helmholtz instability turns out to be the natural consequence of the negativity of the curvature tension of this manifold and it is strong then the Reighlay-Taylor instability, but suppressed by the surface tension. This same framework for one-fluid problem works here as well.

Our framework also incorporate the gravity and magnetic fields (magneto-hydro-dynamical system) in a natural way.

0.2. **Rotating waves in the free boundary problems of Euler's equation.** With my postdoc fellow Dr. Shengfu Deng, we also studied qualitative dynamics of free surfaces of incompressible and invicid fluid. In particular, we considered a droplet or a star of incompressible and invicid fluid moving in the space under the impact of surface tension and self-gravitation. We are interested in finding special type of waves. We started with the problem of surface waves and velocity fields both rotation around certain center with constant speed. The first result we obtained is the existence of such rotating waves which do not have any vorticity, by converting this problem, through Lyapunov-Schmidt reduction, to a pitch-fork bifurcation.

0.3. **Free boundary problems of incompressible, elastic, and viscous fluids.** With Prof. Ping Zhang and my visiting Ph. D. student, Mei Ming, both at Chinese Academy of Sciences, the latter being supported by this grant in part, we studied the free boundary problems of incompressible, elastic, and viscous fluids. Our study is again based on the geometric consideration in the infinite dimensional function spaces. We first obtained a priori estimates of the problem based our geometric approach. It involves some thorough elliptic estimates of an operator generated by the viscosity with natural boundary conditions which requires the continuity of the stress tensors across the free surface. In studying such operator, we also obtained a Korn's type inequality of higher order.

0.4. **Schrödinger maps.** With Jalal Shatah, we have been working on Schrödinger map targeted on Kähler manifolds. Motivated by the Heisenberg model of anti-ferromagnetic chains, we first considered Schrödinger maps with strong constraining potential which tend to restrict the motions of the system near some Lagrangian submanifolds of the target Kähler manifolds. We proved the convergence of this type of Schrödinger maps to some (possible modified) wave maps. In particular, the well-accepted formal continuum limit of the Heisenberg model of anti-ferromagnetic chains, the σ-model is justified as a special case.

With J. Shatah, Prof. Luis Vega of Universidad del Pais Vasco, Spain, and Prof. Andrea Nahmod of University of Massachusetts at Amherst, we worked on the gauge transformation of the Schrödinger maps. We have basically finished the construction of the gauge transformation in the weak formulation for schrödinger maps. We gave the gauge transformation in the general setting and then focused on the special cases when the target is S^2 or the hyperbolic plane H^2 which is non-compact. We prove the validity of the gauge transformation for Schrödinger maps with rather low regularity. In particular, we also proved the global existence of weak schrödinger maps from R^n to compact Hermitian symmetric spaces or from R^2 to hyperbolic plane.

With various collaborators, the Pi also studied the blow-up phenomenon for Schrödinger maps which is widely expected due to the comparison between Schrödinger map equations and nonlinear Schrödinger equations based on the gauge transformation. To understand the mechanism, the PI started with equivariant and self-similar solutions. 1.) With Weiyue Ding of Peking University and Hongyan Tang of Tsinghua University, China, we obtained all smooth self-similar equivariant Schrdinger maps from $C^n \to CP^n$, $n \geq 2$, which are proved to have infinite total energy and finite local energy. 2.) With Pierre Germain and Jalal Shatah of New York University, in the critical dimension R^2 to any rotational symmetric surface, we obtained a.) all smooth self-similar equivariant Schrdinger maps and proved that their gradient fall into the weak energy space; b.) existence of Schrödinger maps in a function space containing the above maps.

With A. Nahmod, we also started to study the Ishimori systems and our initiaal goal is to look for the equivalent of the radial solutions in the Schrödinger equations. Due to the ox operator in the Ishimori systems, which is the Lapalace in the 2-dim (space) under the hyperbolic metric, the radial symmetry has to be in the hyperbolic metric. We considered the appropriate definition of function spaces for hyperbolic metric. Since the hyperbolic metric is degenerate in certain directions, we have to consider functions which always have some singularities. They can be formulated, at least in the 'radial' case, in a way so that the singularity only occurs near the degenerate directions.

0.5. **Invariant manifold.** With Prof. Peter Bates, at Michigan State University, and Prof. Kening Lu, at Brigham Young University, we have been studying approximately invariant manifolds in general infinite dimensional dynamical systems. Approximately invariant manifolds may come from numerical simulations, from true invariant manifolds of approximate systems, or from some asymptotic analysis. We proved that if an approximately invariant manifold is also approximately normally hyperbolic, then there exists a nearby true normally hyperbolic invariant manifold. center-stable and center-unstable manifolds and stable and unstable foliations are also constructed which characterize the structure in the neighborhoods of the approximately invariant manifolds.

0.6. **Reaction-diffusion equation.** With P. Bates and K. Lu, we also studied PDE singular perturbation problems using invariant manifolds for semiflows in Banach spaces. We considered a singular parabolic equations. Its formal asymptotic analysis hints that there are solutions which are very much localized in space with the peak profile given by a ground state with steady or dynamic peak locations. previous, there had been many work addressing the existence of this type of stationary spike solutions. Their construction was very much based on local analysis in the phase (function) space near a spike profile with some special location of the peak. We construct dynamic spike solutions instead. This spike solutions evolve slowly in time and we rigorously found the leading order motions of the peaks.

With Prof. Junping Shi, at College of William and Mary, we are seeking potential collaborations on reaction-diffusion equations. We found that in certain models from the material sciences, traveling waves with exponential speed, instead of constant speed, appears naturally. They also carry transition layers and appear to be stable.