Geometric Identities for Index Theory

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Homogeneous Crystals in One Slide

Definition: A homogeneous pure crystalline phase

is defined by a measure preserving ergodic dynamical system:

$$(\Omega, \mathbb{Z}^d, \tau, \mathrm{d}\mathbb{P}) \ , \quad (\Omega \text{ compact and metrizable}).$$

The dynamics of the electrons is determined by a covariant family of Hamiltonians:

$$\{H_\omega\}_{\omega\in\Omega} \ , \quad T_a H_\omega T_a^* = H_{\tau_a \omega} \ .$$

Proposition: (On the lattice)

The bounded covariant Hamiltonians on $\mathbb{C}^d \otimes \ell^2(\mathbb{Z}^d)$ take the following form:

$$H_\omega = \sum_{q\in\mathbb{Z}^d} \sum_{x\in\mathbb{Z}^d} w_q(\tau_x \omega) \otimes |x\rangle\langle x| T_q$$

When uniform magnetic fields are present, then the ordinary translations T_q are replaced by the magnetic translations.

Classification of Homogeneous Crystalline Systems

A. P. Schnyder, S. Ryu, A. Furusaki, A. W. W. Ludwig, *Classification of topological insulators and superconductors in three spatial dimensions*, Phys. Rev. **B 78**, 195125 (2008).
A. Kitaev, *Periodic table for topological insulators and superconductors*, (Advances in Theoretical Physics: Landau Memorial Conference) AIP Conference Proceedings **1134**, 22-30 (2009).
S. Ryu, A. P. Schnyder, A. Furusaki, A. W. W. Ludwig, *Topological insulators and superconductors: tenfold way and dimensional hierarchy*, New J. Phys. **12**, 065010 (2010).

j	TRS	PHS	CHS	CAZ	$0,8$	1	2	3	4	5	6	7
0	0	0	0	A	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
1	0	0	1	AIII		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$
0	$+1$	0	0	AI	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$
1	$+1$	$+1$	1	BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$		$\mathbb{Z}_2$
2	0	$+1$	0	D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$	
3	-1	$+1$	1	DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$2\,\mathbb{Z}$
4	-1	0	0	AII	$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$			
5	-1	-1	1	CII		$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$		
6	0	-1	0	C			$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	
7	$+1$	-1	1	CI				$2\,\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

- each $n \in \mathbb{Z}$ or $\mathbb{Z}_2$ defines a distinct macroscopic insulating phase: $\sigma_{xx} = 0$.
- the phases are separated by a bulk Anderson transition: $\sigma_{xx} > 0$
- $\sigma_{\|} > 0$ along any boundary cut into the crystals.

The Index Theorem for Bulk Projections (d = even)

Theorem: Let d be even and let P_ω be a covariant projection such that:

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\ \Big\langle 0 \Big| \big|[X, P_\omega]\big|^d \Big| 0 \Big\rangle < \infty$$

Let $\Gamma_1, \ldots, \Gamma_2$ be irreducible rep of $\mathcal{C}l_d$. Then, $\mathbb{P}$-almost surely

$$F_\omega = P_\omega \left(\frac{X \cdot \Gamma}{|X|} \right)_{+-} P_\omega \in \text{Fredholm class}$$

and

$$\boxed{\mathrm{Ind}\, F_\omega = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \int_\Omega \mathrm{d}\mathbb{P}_\omega \Big\langle 0 \Big| P_\omega \prod_{i=1}^{d} \imath \big[X_{\rho_i}, P_\omega\big] \Big| 0 \Big\rangle}$$

J. Bellissard, A. van Elst, H. Schulz-Baldes, *The non-commutative geometry of the quantum Hall effect*, J. Math. Phys. **35**, 5373-5451 (1994).

E. P., B. Leung, J. Bellissard, *The non-commutative n-th Chern number* ($n \geq 1$), J. Phys. A: Math. Theor. **46**, 485202 (2013).

The Index Theorem for Bulk Unitaries (d = odd)

Theorem: Let d be odd and let U_ω be a covariant unitary such that:

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\, \Big\langle 0 \Big| \big|[X, U_\omega]\big|^d \Big| 0 \Big\rangle < \infty$$

Let E_+ be the spectral projection onto the positive spectrum of $X \cdot \Gamma$. Then, $\mathbb{P}$-almost surely

$$F_\omega = E_+ U_\omega E_+ \in \text{Fredholm class}$$

and

$$\boxed{\mathrm{Ind}\, F_\omega = \Lambda_d \sum_{\rho \in S_d} (-1)^\rho \int_\Omega \mathrm{d}\mathbb{P}(\omega) \Big\langle 0 \Big| \prod_{i=1}^{d} \imath U_\omega^* \big[X_{\rho_i}, U_\omega\big] \Big| 0 \Big\rangle}$$

E. P. and H. Schulz-Baldes, *Non-commutative odd Chern numbers and topological phases of disordered chiral systems*, J. Funct. Anal. **271**, 1150-1176 (2016).

The Proof for Even Case $d = 2$

1. Condition

$$\int_\Omega \mathrm{d}\mathbb{P}(\omega)\ \left\langle 0 \middle| \left|[X, P_\omega]\right|^2 \middle| 0 \right\rangle < \infty$$

ensures that $\mathbb{P}$-almost surely:

$$(1 - F_\omega F_\omega^*)^3 \ , \quad (1 - F_\omega^* F_\omega)^3$$

are trace class.

2. Fedosov's principle applies:

$$\mathrm{Index}\ F_\omega = \mathrm{Tr}(1 - F_\omega F_\omega^*)^3 - \mathrm{Tr}(1 - F_\omega^* F_\omega)^3$$

3. Translations of ω produce only compact perturbations:

$$F_\omega - F_{\tau_x \omega} = \mathrm{Compact}$$

hence an integration over ω is allowed above.

Proof Continues

Some notations: $X = X_1 + \imath X_2$, $U = \frac{X}{|X|}$

Then, expanding the traces:

$$\mathrm{Index}\, F_\omega = -\int d\mathbb{P}(\omega)\, \mathrm{Tr}\left(P_\omega - UP_\omega U^*\right)^3 = \sum_{\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}} A(\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}) \int d\mathbb{P}(\omega)\langle \boldsymbol{0}|\Pi_\omega|\boldsymbol{x}\rangle\langle \boldsymbol{x}|\Pi_\omega|\boldsymbol{y}\rangle\langle \boldsymbol{y}|\Pi_\omega|\boldsymbol{0}\rangle$$

$$A(\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}) = \left(1 - \frac{q\overline{(q+x)}}{|q(q+x)|}\right)\left(1 - \frac{(q+x)\overline{(q+y)}}{|(q+y)(q+y)|}\right)\left(1 - \frac{(q+y)\overline{q}}{|(q+y)q|}\right)$$

And here comes the magic identity:

$$\sum_{\boldsymbol{q}} A(\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}) = 2\pi\imath(x_1y_2 - y_1x_2)$$

End result:

$$\mathrm{Index}\, F_\omega = 2\pi\imath \sum_{i,j} \epsilon_{ij} \int_\Omega d\mathbb{P}_\omega \Big\langle 0 \Big| P_\omega\left[X_i, P_\omega\right]\left[X_j, P_\omega\right] \Big| 0 \Big\rangle$$

Proof of Identity (following Verdiere)

It is easy to see that:

$$\frac{-q}{|q|}\,\overline{\frac{x-q}{|x-q|}} = e^{\imath\phi_1}, \quad \frac{x-q}{|x-q|}\,\overline{\frac{y-q}{|y-q|}} = e^{\imath\phi_2}, \quad \frac{y-q}{|y-q|}\,\overline{\frac{-q}{|q|}} = e^{\imath\phi_3}$$

and a direct calculation will show:

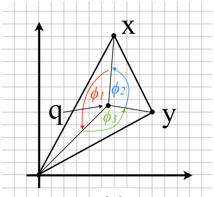
$$A(-\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}) = -2\imath(\sin\phi_1 + \sin\phi_2 + \sin\phi_3)$$

We write the summation in the following way:

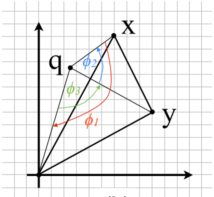
$$\frac{-1}{2\imath}\sum_{\boldsymbol{q}} A(-\boldsymbol{q},\boldsymbol{x},\boldsymbol{y}) = \sum_{\boldsymbol{q}}(\phi_1+\phi_2+\phi_3)$$
$$-\sum_{\boldsymbol{q}}(\phi_1-\sin\phi_1) + \sum_{\boldsymbol{q}}(\phi_2-\sin\phi_2) - 2\sum_{\boldsymbol{q}}(\phi_3-\sin\phi_3)$$

Note $\phi_i - \sin\phi_i$ antisymmetric w.r.t. inversion of $\boldsymbol{q}$ and:

$$\phi_1+\phi_2+\phi_3 = \begin{cases} 2\pi \text{ if } \boldsymbol{q} \text{ is inside the triangle} \\ \pi \text{ if } \boldsymbol{q} \text{ is on an edge} \\ 0 \text{ if } \boldsymbol{q} \text{ is outside the triangle} \end{cases}$$

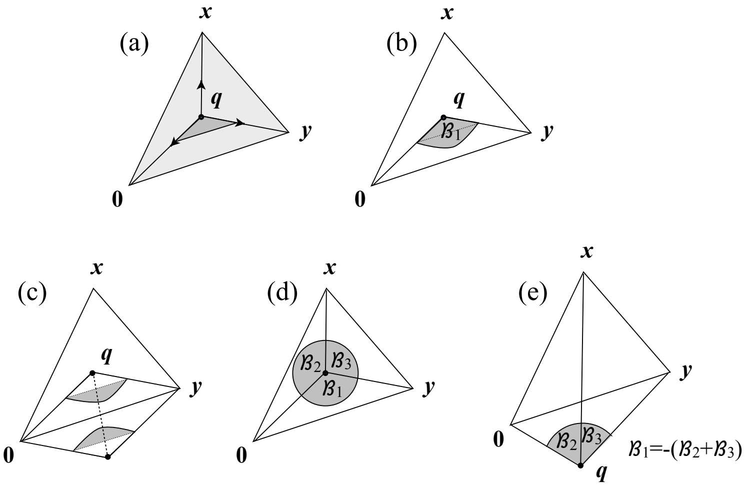


(a)



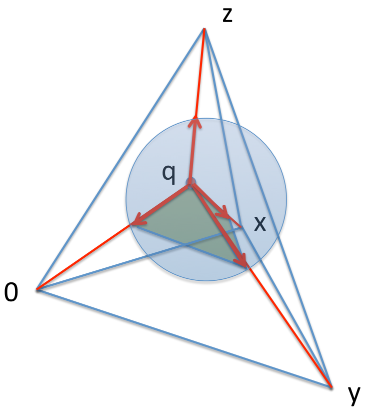
(b)

Geometric Interpretation



$$A(-\boldsymbol{q}, \boldsymbol{x}, \boldsymbol{y}) = -2\imath(\sin\phi_1 + \sin\phi_2 + \sin\phi_3).$$

Higher Dimensions



$$\sum_{\boldsymbol{q}} \sum_{\{i,j,k\}} \mathrm{Vol}(\boldsymbol{q}, \boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_k) = \mathrm{Vol}(\text{unit ball}) \times \mathrm{Vol}(\boldsymbol{0}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$$

$(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d})$ = Oriented simplex with corners $\boldsymbol{a}, \ldots, \boldsymbol{d}$

The Full Identity in Higher Dimensions

In $d = 2$ we had:

$$\Big(1 - \frac{q\overline{(q+x)}}{|q(q+x)|}\Big)\Big(1 - \frac{(q+x)\overline{(q+y)}}{|(q+y)(q+y)|}\Big)\Big(1 - \frac{(q+y)\overline{q}}{|(q+y)q|}\Big) = 2\pi\imath(x_1 y_2 - y_1 x_2)$$

For arbitrary even d:

$$\int_{\mathbb{R}^d} dx \ \mathrm{tr}\Big\{\Gamma_0 \prod_{i=1}^{d}\Big(\frac{\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_i+\boldsymbol{x})}{|\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_i+\boldsymbol{x})|} - \frac{\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_{i+1}+\boldsymbol{x})}{|\boldsymbol{\Gamma}\cdot(\boldsymbol{x}_{i+1}+\boldsymbol{x})|}\Big)\Big\} = \frac{(2\imath\pi)^{d/2}}{(d/2)!}\sum_{\rho\in\mathcal{S}_d}(-1)^{\rho}\prod_{i=1}^{d} x_{i,\rho_i}$$

It ties with the previous because of well known identity:

$$\mathrm{tr}\Big\{\Gamma_0 \prod_{i=1}^{d} \boldsymbol{\Gamma}\cdot\boldsymbol{y}_i\Big\} \ = \ (2\imath)^{d/2}(d)! \ \mathrm{Vol}[\boldsymbol{0},\boldsymbol{y}_1,\ldots,\boldsymbol{y}_d]$$

Concluding Remarks

1. The identity for $d =$ odd looks quite similar:

$$\int_{\mathbb{R}^d} \mathrm{d}x \ \mathrm{tr}\Big\{ \prod_{i=1}^{d} \Big(\frac{\mathbf{\Gamma} \cdot (\boldsymbol{x}_i + \boldsymbol{x})}{|\mathbf{\Gamma} \cdot (\boldsymbol{x}_i + \boldsymbol{x})|} - \frac{\mathbf{\Gamma} \cdot \boldsymbol{x}_{i+1} + \boldsymbol{x})}{|\mathbf{\Gamma} \cdot \boldsymbol{x}_{i+1} + \boldsymbol{x})|} \Big) \Big\} = -\frac{2^d (\imath\pi)^{(d-1)/2}}{d!!} \sum_{\rho \in \mathcal{S}_d} (-1)^\rho \prod_{i=1}^{d} x_{i,\rho_i}$$

and the proof is also very similar.

2. The topology can be encoded at the boundary, in a unitary operator if $d =$even and a projection if $d =$odd. For example:

$$\widehat{U}_\omega = \exp\left[2\pi\imath G(\widehat{H}_\omega)\right], \quad G = 0/1 \text{ below/above the bulk gap}$$

3. Then same Index Theorems can be applied at the boundary.

4. If the boundary spectrum is Anderson localized, the indices are necessarily zero (implies delocalization of boundary spectrum).

E.P. & H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators: From K-Theory to Physics*, (Springer, Berlin, 2016).