THE EFFECT OF SCALE ON THE APPARENT ADDITIONAL MASS OF BODIES OF REVOLUTION

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THE EFFECT OF SCALE ON THE APPARENT ADDITIONAL MASS OF BODIES OF REVOLUTION

SUMMARY

A simple torsion pendulum was used to make a series of tests on various bodies of revolution, cylinders with blunt, conical, and hemispherical ends, with a length over diameter ratio of four, to determine the effect of scale on apparent additional mass. All bodies were studied for the effects of apparent additional mass for the condition of parallel flow; that is, where the flow is parallel to the longitudinal axis of revolution of the body.

Evaluation of existing three dimensional apparent additional mass formulae were studied to compare the experimental results with theory. The experimental results, checked by the theory, proved that due to increase of scale there was a linear increase of the apparent additional mass of the bodies tested, and it has been shown that the apparent additional mass coefficients remain constant for various sizes of bodies provided the shape of the body, its direction of motion, and density of the surrounding media are the same.

Curves are presented showing the relation of apparent additional mass and apparent additional mass coefficients with the mass of the displaced fluid, and the scale of the displaced fluid, respectively. As indicated, the smallest models produced slightly higher values for the apparent additional mass probably due to interference, relative accuracy

of testing procedure and other factors. Data was compared with previous tests performed by Bryan[1] and Relf and Jones[2] which cheched as closely as could be expected.

A discussion of similitude is included with emphasis on the models used in this paper. Also, an analysis of the precision of testing was presented.

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[1]Colgan Hobson Bryan, "Apparent Additional Mass Characteristics of Various Bluff Bodies of Revolution", (A masters thesis, Daniel Guggenheim School of Aeronautics, Georgia Institute of Technology, Atlanta, September, 1948), pp. 76-79.

[2]E. F. R. Relf and R. Jones, "Measurements of the Effect of Accelerations on the Longitudinal and Lateral Motion of an Airship Model," Vol. I, 1918-1919, p. 127.

INTRODUCTION

Although Green's paper[1] on pendulums could be considered the most important contribution of early works on apparent additional mass studies, Du Buat discovered, about 160 years ago, that when a sphere was immersed in water and accelerated, it possessed a mass about 0.585 times the displaced fluid in addition to its physical mass. Even though very little experimental work, and less theoretical work has been done in the field of apparent additional mass, more familiarly termed as virtual mass, entrained mass, additional mass and inertia effects, investigators including: Du Buat, Green, Bessel, Sabine, Bailey, Poisson, Plata, Stokes, Lamb, Barnes, McEarnes, Krishnayer, Cook, Taylor, Yee-Tak Yu, Munk, and various scientists at the National Advisory Committee for Aeronautics have contributed invaluable information on the subject. Towsley's[2] review of all existing data on apparent additional mass and Bryan's[3] paper on the variation of apparent additional mass with length over diameter ratio are among the most recent papers on apparent additional mass. To the present time no available data on the effect of scale on apparent additional mass has been found.

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[1]George Green, "Researches on the Vibration of Pendulums in Fluid Media", Transactions of the Royal Society of Edinburgh, Vol. 13, 1886, pp. 54-62.

[2]Melvyn F. Towsley, "Apparent Additional Mass," (Amasters thesis, Daniel Guggenheim School of Aeronautics, Georgia Institute of Technology, Atlanta, June 1947), pp. 64-70.

[3]Colgan Hobson Bryan, "Apparent Mass Characteristics of Various Bluff Bodies of Revolution," (A masters thesis, Daniel Guggenheim School of Aeronautics, Georgia Institute of Technology, Atlanta, September, 1948), pp. 1-90

When a body in a fluid media is set in motion, the fluid around the body is also set in motion. The amount of fluid that is accelerated by thebody is known as the apparent additional mass and is usually expressed as a function of the displaced fluid of the body. In a vacuum, as well as in a perfect, nonviscous, incompressible fluid, there is nothing to hinder the motion of a body, but if the body is set in motion or accelerated, a force is required to produce the acceleration of both the mass of the body plus the apparent additional mass of the fluid affected by the body. However, if the body is set in motion in a real, viscous fluid there are additional effects of viscosity and other forces that appear in a real fluid that add to the resistance of the body during acceleration. Unfortunately, it is impossible to separate these forces from the apparent additional mass effects. Previous tests have shown that the apparent additional mass varies with both acceleration and velocity so that in order to be absolutely correct in engineering calculation the apparent additional mass should be corrected for these quantities. However, to date very little experimental data has been accumulated on the variation of apparent additional mass with acceleration and velocity, and since the perfect fluid theory indicated no effect, these quantities are not usually considered. Even so, the apparent additional mass and the apparent additional mass coefficient is not negligible and should be taken into account in many engineering problems, such as: problems involving missiles, bombs, torpedoes, submarines, boats, dirigibles, airplanes, pendulum tests, and many other instances.

The apparent additional mass of a body, moving in a fluid, is a function of body shape and manner in which it is moving; since each particle of the fluid, influenced by the body, is not necessarily accelerated in the direction of the moving body. It is also an established fact that the apparent additional mass varies directly with the density of the fluid in which the body is accelerated.[4]

The primary purpose of this paper is to determine the nature of the change, if any, of the apparent additional mass when the body dimensions are increased by a constant ratio, (to define the terms used and to correlate and summarize the results of the experiments performed.) The sections of the theory that deal directly with increase in scale only, were presented.

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[4]Yee-Tak Yu, "Virtual Masses and Moments of Inertia of Disks and Cylinders in Various Liquids," Journal of Applied Physics, Vol. 13, 1942. p. 67.

SYMBOLS

Symbols to be used in this thesis unless otherwise specifiec are:

A = Frontal area of body

a = Ratio of lengths l/l_{ℓ}

B = Object at end of pendulum rod or arm

b = Ratio of times t/t_{ℓ}

c = Ratio of masses m/m_{ℓ}

D = Diameter of cylindrical bodies

e = Eccentricity of ellipsoid as in $e = \sqrt{1 - a^2/b^2}$ where a and b are the semiaxes of an ellipsoid

I = Apparent additional moment of inertia due to a single object B moving in water

I_a= Mass moment of inertia in air of the cross-arm with both objects

I_k= Mass moment of inertia of pendulum arm and attached body used to determine the torsional constant of the wire pendulum

I_o= True mass moment of inertia as measured in a vacuum

I_w= Mass moment of inertia in water of the cross-arm with both models attached

I_{xo}=Mass moment of inertia of a cylinder about its x - axis through its center of gravity

I_{yo}=Mass moment of inertia of a cylinder about its y - axis through its center of gravity

I_a'=Mass moment of inertia in air of cross-arms without bodies

I_w'=Mass moment of inertia in water of cross arms without bodies

K = A special apparent mass coefficient used in the equation $m_A = K \times C$, where C is a constant depending upon the body dimensions

K_T	=	Torsional spring constant of the wire
L	=	Distance from the center of the wire to the center of gravity of an object B
ℓ	=	Length of the cylinder
M	=	Mass of the body
m_A	=	Apparent additional mass
m_{DF}	=	Mass of displaced fluid
n	=	Internal normal used as $\frac{d\phi}{dn}$
R	=	Radius of a sphere, the value of r for $\psi = 0$
r	=	Radius of a sphere or cylinder of revolution
S	=	Surface area used as $T = -\rho\int \phi \nabla \phi \cdot ds$
T	=	Kinetic energy of the potential flow
U	=	Horizontal velocity of flow
V	=	Speed of flow or body
x,y,z	=	Cartesian coordinate axes
r,θ	=	Polar coordinate axis
$\alpha_0, \beta_0, \gamma_0$	=	Special values in Green's Integrals for $\lambda = 0$
ϕ	=	Velocity potential
$\nabla \Phi$	=	Velocity gradient
ψ	=	Stream function
ρ	=	Density of fluid
Δ	=	Precision measure of a quantity involving more than one variable
δ	=	Precision measure of a quantity with one variable

THEORY

In order to investigate the effect of scale on the apparent additional mass of bodies, the most logical approach is to express the apparent additional mass, or the apparent additional inertia, in coefficient form. These coefficients are dimensionless and are ratios of apparent mass, or inertia, to the mass, or inertia of the displaced fluid of the body in question.

The theoretical developments of the coefficients for apparent additional mass and inertia of various bodies has been presented by several authors, Bateman[1], Tuckerman[2], Munk[3], Bryan[4], and Towsley[5], but it is sufficient for the purpose of this paper to present the development of formulae for a sphere, noting that the development for other three dimensional bodies depend only on the evaluation of the velocity potential of the body and can be followed in a similar manner.

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[1]H. Bateman, "The Inertia Coefficients of an Airship in a Frictionless Fluid," U.S. National Advisory Committee for Aeronautics Technical Report, No. 164, 1923, pp. 1-16.

[2]L. B. Tuckerman, "Inertia Factors of Ellipsoids for Use in Airship Design," U. S. National Advisory Committee for Aeronautics Technical Report, No. 210, 1925, pp. 1-7.

[3]Max M. Munk, "Some Tables of the Factor of Apparent Additional Mass", U.S. National Advisory Committee for Aeronautics Technical Note, No. 197, 1924, pp. 1-7.

[4]Colgan Hobson Bryan, "Apparent Mass Characteristics of Various Bluff Bodies of Revolution," (A masters thesis, Daniel Guggenheim School of Aeronautics, Georgia Institute of Technology, Atlanta, September, 1948), pp. 12-30.

[5]Melvyn F. Towsley, "Apparent Additional Mass," (A masters thesis, Daniel Guggenheim School of Aeronautics, Georgia Institute of Technology, Atlanta, June 1947), pp. 64-70.

Bryan[6] developed an equation for the kinetic energy of the fluid involved when the motion of the fluid by the body is created from rest:

$$T = -\frac{\rho}{2}\int \phi \nabla \phi \cdot ds \tag{1}$$

where
T = kinetic energy of the fluid
ρ = density of the fluid
ϕ = velocity potential
$\nabla \phi$ = velocity gradient
dS = element of surface

By Gauss' transformation the above integral can be transformed to the surface integral

$$T = -\frac{\rho}{2}\iint \phi \frac{d\phi}{dn}\, ds \tag{2}$$

where
$\frac{d\phi}{dn}$ = internal normal of the potential function

The theoretical determination of the kinetic energy for a sphere under the same conditions as above has been presented by Munk[7]. Dr. Munk combined the flows of a doublet and a horizontal flow of constant velocity to obtain the velocity potential of a sphere,

$$\phi = U \cos \theta \, \frac{R^3}{2r^2} \tag{3}$$

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[6] *Ibid.*, p. 18

[7] W. F. Durand, Aerodynamic Theory, Vol. I (Berlin: Julius Springer, 1934), pp. 224-304.

where R = radius of the sphere, or the value of r at $\psi=0$

U = velocity of the flow in feet per second

r = radius vector

θ = space angle between x axis and radius vector

The kinetic energy of the flow may now be computed to be:[8]

$$T = \frac{\rho}{2}\int\left(\frac{U}{2} R \cos \theta\right)(U \cos \theta)(2 \pi R^2 \sin \theta)\, d\theta$$

$$T = 2/3\ \pi R^3 \frac{\rho U^2}{2} \tag{4}$$

From this equation we can see that the kinetic energy is a function of the radius of the sphere and if that radius were increased in scale, say by two, then the kinetic energy would be increased by eight times its original value provided the density and the velocity remained constant. However, since the kinetic energy is also related by the equation:

$$T = 1/2\ MU^2 \tag{5}$$

Dividing both sides of the equation by $1/2U^2$ an expression for the apparent additional mass can be found to be:

$$m_A = \frac{T}{1/2U^2} = \frac{1/3\ \pi R^3\ \rho U^2}{1/2\ U^2}$$

$$m_A = 2/3\ \pi R^3\ \rho \tag{6}$$

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[8] Ibid., p. 257

Since the apparent additional mass coefficient is the ratio of the apparent mass and the mass of the displaced fluid it is possible to divide equation (6) by the volume of a sphere times the density:

$$m_{D.F.} = \frac{4}{3} \pi R^3 \rho \tag{7}$$

Therefore:

$$k = \frac{m_A}{m_{D.F.}} = \frac{2/3 \pi R^3 \rho}{4/3 \pi R^3 \rho} = 1/2 \tag{8}$$

It can be seen that any dimension of a sphere, all of which could be expressed as a function of the radius R, could be changed and the apparent additional mass would change, but the mass of the displaced fluid would also change by an equal amount. Therefore, theoretically the geometric scale does not affect the apparent additional mass coefficient. If the mass of the body were changed, say by making the body hollow, the apparent additional mass of the body would not be affected provided the radius were kept constant. Similarly the mass of the displaced fluid would remain the same, resulting in no change of the coefficient. Since a sphere has point symmetry the direction of motion has no affect on the apparent additional mass coefficient. Therefore; theoretically, the apparent additional mass of a sphere is influenced only by the density of the fluid, but the coefficient is independent of the velocity and density and is a function of shape, and direction of the flow of the fluid.

To this date the triaxial ellipsoid, with all its special cases, is actually the only three dimensional body for which it is possible to determine the properties of apparent additional mass. By considering the ellipsoid and its direction of motion through the fluid, it is also possible to obtain two dimensional effects; such as, considering the ellipsoid of having infinite length and the flow perpendicular to its longitudinal axes. If the ellipsoid is a body of revolution, it simulates the flow about a circular cylinder; if the body has elliptical cross section, it simulates the flow of an elliptical cylinder. These two dimensional effects of apparent additional mass are discussed by various authors[9], and only the prolate and oblate ellipsoids will be discussed in this paper.

All the theoretical data on ellipsoids can be summed up in the following equations by Tuckerman[10]. Consider a triaxial ellipsoid having semi-axes of a, b, and c such that the volume is $4/3\pi abc$. Then the three apparent additional mass coefficients k_1, k_2, and k_3 for this ellipsoid, in a translational potential flow are found to be:

$$k_1 = \frac{\alpha_o}{2-\alpha_o}, \quad k_2 = \frac{B_o}{2-B_o}, \quad k_3 = \frac{\gamma_o}{2-\gamma_o} \tag{9}$$

The direction of the flow indicates which value of k represents the apparent additional mass; k_1 is for the flow parallel to the "a" dimension, k_2 for the flow parallel to the "b" dimension, and k_3 for the flow parallel to the "c" dimension.

In the above formulae α_o, B_o, and γ_o are special values for λ, (an arbitrary constant) when λ=o, in Green's integrals[11], and are expressed

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[9]Towsley, op. cit., pp. 64-66.

[10]Tuckerman, op. cit., p. 1.

[11]George Green, "Researches on the Vibration of Pendulums in Fluid Media," Transactions of the Royal Society of Edinburgh. Vol. 13, 1886, pp.54-62.

in Tuckerman's paper in elliptic integrals for the general case, however, in the special cases the elliptic integrals degenerate into algebraic, circular, hyperbolic, or other functions, or the coefficients k_1, k_2, and k_3 take on indeterminate form needing special treatment[12].

To demonstrate the effect of scale on a body in a moving potential flow, two cases of the triaxial ellipsoid as treated by Tuckerman will be used, the oblate spheroid and the prolate spheroid. Since dimensional analysis will be used to evaluate the effect of geometric scale on the apparent additional mass coefficients, it is possible to use a special case that is simpler than the general case.

Consider the oblate spheroid where a = b > c.

$$\alpha_o = \beta_o = \frac{\sqrt{1-e^2}}{e^3}\left(\sin^{-1}e - e\sqrt{1-e^2}\right) \tag{10}$$

$$\gamma_o = \frac{2\sqrt{1-e^2}}{e^3}\left(\frac{e}{\sqrt{1-e^2}} - \sin^{-1}e\right) \tag{11}$$

Where $$e = \sqrt{1-\left(\frac{c}{b}\right)^2} \quad \text{or} \quad e = \sqrt{1-\left(\frac{c}{a}\right)^2} \tag{12}$$

Therefore $$e = \sqrt{1-\left(\frac{L}{D}\right)^2} \tag{13}$$

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[12] Tuckerman, op. cit., p. 4.

The same value of "e", the eccentricity of the spheroid, is used in α_0, B_0, and γ_0, since "e" is a geometric quantity, and is independent of the direction of the flow. From the above equations it is possible to calculate the apparent additional mass coefficient of an oblate spheroid knowing only the ratio of the length and diameter, and furthermore, to determine the apparent additional mass if the actual dimensions of the body were known.

The values for α_0, B_0, and γ_0, for a prolate spheroid where $a > b = c$ are:

$$\alpha_0 = \frac{(1-e^2)}{e^3}\left[\ln\frac{1+e}{1-e} - 2e\right] \tag{14}$$

$$B_0 = \gamma_0 = \frac{(1-e^2)}{e^3}\left[\frac{e}{1-e^2} - \frac{1}{2}\ln\frac{1+e}{1-e}\right] \tag{15}$$

where $$e = \sqrt{1-\left(\frac{b}{a}\right)^2} \quad \text{or} \quad e = \sqrt{1-\left(\frac{c}{a}\right)^2} \tag{16}$$

Therefore $$e = \sqrt{1 - \frac{1}{(L/D)^2}} \tag{17}$$

When a body rotates in a translational potential flow, it becomes more important to express the apparent additional mass effects as apparent add-moment of inertia coefficients. However, the apparent additional mass coefficient can readily be obtained by dividing the inertia coefficient by the angular acceleration. The apparent additional moment of inertia can be obtained by multiplying the inertia coefficient by the inertia of the displaced fluid. These coefficients: k'_1, k'_2, and k'_3 take the form:

$$k_1' = \left(\frac{b^2 - c^2}{b^2 + c^2}\right) \frac{\gamma_o - \beta_o}{2\left(\frac{b^2 - c^2}{b^2 + c^2}\right) - (\gamma_o - \beta_o)} \tag{18}$$

with symmetrical expressions for k'_2 and k'_3.

It can be seen that the values of α_0, B_0, and γ_0, for the two cases stated, are all three functions of "e" and "e" alone, and that k_1, k_2, and k_3 are functions of α_o, β_o, and γ_o. Therefore, if it can be shown that "e" does not change with geometric scale, it follows logically that the apparent additional mass coefficients do not change with scale. It has also been noted that k'_1 does not change with a change in scale, since it is a function of "e" and a length over a length ratio. In the case of k'_1, if the geometric scale was h, or the size of the body was increased h times, then k'_1 would be:

$$k_1' = \left(\frac{h^2b^2 - h^2c^2}{h^2b^2 + h^2c^2}\right) \frac{\gamma_o - \beta_o}{2\left(\frac{h^2b^2 - h^2c^2}{h^2b^2 + h^2c^2}\right) - (\gamma_o - \beta_o)} \tag{19}$$

$$k_1' = \left(\frac{b^2 - c^2}{b^2 + c^2}\right) \frac{\gamma_o - \beta_o}{2\left(\frac{b^2 - c^2}{b^2 + c^2}\right) - (\gamma_o - \beta_o)}$$

Therefore, if the apparent additional moment of inertia coefficient changes with scale it must be through α_o, B_o, or γ_o, all of which are functions of "e".

As shown in equations (12), (13), (15), and (17) the eccentricity is a function of a ratio of lengths, or proportional to the length over diameter ratio, therefore, if the geometric scale were increased, say doubled, the length over diameter ratio would remain the same. Theoretically, it can be concluded that if the direction of motion, density of fluid, the shape of the body and other factors are kept the same and the geometric dimensions of the body are changed in scale the apparent additional mass coefficient remains constant.

LAWS OF SIMILITUDE

Two geometrically similar bodies must have a constant linear scale ratio,[1] a,

$$l/l_1 = a \qquad (1)$$

where l, l_1 are any two homologous lengths. If x, x_1, etc., are homologous coordinates for the bodies, $x/x_1 = y/y_1 = z/z_1 = a$. From equation (1) we may state that

$$(l/l_1)^2 = a^2$$

and similarly the ratios of the frontal areas A, and A_1 are also

$$\frac{A}{A_1} = a^2 \qquad (2)$$

since area is a function of a length squared.

A similar expression may be written for the ratios of these bodies that have geometric similitude, since volume is a function of the three dimensions or of the order of the ratio

$$\frac{l^3}{l_1^3} = a^3$$

and the volume ratio is $\frac{\text{Vol.}}{\text{Vol.}_1} = a^3$ (3)

It can also be seen that if the two similar bodies were immersed in the same fluid the ratios of the masses of the displaced fluid would also have the scale ratio of a^3.

$$\frac{\rho \text{Vol}}{\rho \text{Vol}_1} = a^3$$

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[1] A. F. Zahm, "Theories of Flow Similitude", U. S. National Advisory Committee for Aeronautics Technical Report, No. 287, 1928, pp. 1-7.

where ρ = density of the displaced fluid.

Therefore
$$\frac{m_{D.F.}}{m_{D.F._1}} = a^3 \tag{4}$$

Since the frontal area, volume, and mass of the displaced fluid are all dependent upon a constant scale ratio "a", it would be possible to use any of the a bove quantities in demonstrating the effect of scale. Since the apparent additional mass is usually expressed in dimensionless coefficient form by ratioing apparent additional mass, m_A, to the mass of the displaced fluid, $m_{D.F.}$. Therefore on the ratios of the masses of displaced fluid were used to express the scale of the models in this work.

If two bodies have geometric similar motions they are kinematically similar because they trace similar paths in proportional times. Let the ratio of times be:

$$t/t_1 = b \tag{5}$$

which is the same for all homologous path segments. If v, v_1 are the corresponding path speeds then the ratio of

$$\frac{v}{v_1} = \frac{\ell t_1}{\ell_1 t} = \frac{a}{b} \tag{5}$$

Likewise the ratios acceleration $\dot{v}$ and $\dot{v}_1$ are:

$$\frac{\dot{v}}{\dot{v}_1} = \frac{\ell t_1^2}{\ell_1 t^2} = \frac{\ell_1 V^2}{\ell V_1^2} = \frac{a}{b^2} \tag{6}$$

Although the constand ratios ℓ/ℓ_1, t/t_1, v/v_1, and $\frac{\dot{v}}{\dot{v}_1}$ may all be different, however it will be shown later that they are all functions of the linear scale ratio a.

In the case of the torsional pendulum, it can be shown that the ratios: ℓ / ℓ_1, t / t_1, $\frac{v}{v_1}$, $\frac{Vol.}{Vol._1}$, $\frac{m_{D.F.}}{m_{D.F._1}}$, $\frac{\dot{v}}{\dot{v}_1}$ are all functions of a.

The formula for the torsional pendulum without damping for small angles is[2]:

$$T = \frac{2\pi}{\overline{k_T/I}} \tag{7}$$

where T = period of the pendulum

K_T = spring constant of the wire

I = moment of inertia of the pendulum.

Since the dimensions of moment of inertia are $M\ell^2$ from equation (7) it follows that

$$T = C\ell \tag{8}$$

where C is an arbitrary constant. Therefore, since the period of the pendulum is proportional to a length it is possible to express the following proportion:

$$\frac{T_1}{T_2} = \frac{C\,\ell_1}{C\,\ell_2}$$

or

$$\frac{t}{t_1} = b = \frac{\ell}{\ell_1} = a$$

therefore

$$b = a \tag{9}$$

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[2]Yee-Tak Yu, "Virtual Masses of Moments of Inertia of Disks, and Cylinders in Various Liquids," Journal of Applied Physics, Vol. 13, 1942.

Thus it can be seen that a criteria for similitude for models on the torsional pendulum is that the ratios: $\frac{l}{l_1}$, $\frac{t}{t_1}$, $\frac{\dot{v}}{\dot{v}_1}$, and $\frac{v}{v_1}$ are all interrelated.

According to Zahm[3], for two bodies to be dynamically similar, they must first satisfy the laws of geometric kinematic similitude and the masses of the two figures must have a constant ratio:

$$m / m_1 = C \tag{10}$$

and all corresponding impressed forces must have their magnitudes in a constant ratio and their lines of action must be similarly located in the two systems. Since the homologous elements of the two similar configurations have resultant accelerations of the ratio of a / b^2, their impressed forces will have the ratio:

$$\frac{R}{R_1} = \frac{m\dot{v}}{m_1\dot{v}_1} = \frac{ac}{b_2} \tag{11}$$

Therefore, in order for a system to be dynamically similar, it must satisfy equation (11).

When the laws of similitude are applied, usually it is not possible to make all ratios constant. For example, in comparing two flows, it would be desirable to have the same Reynolds Number, ratio of inertia forces to viscosity forces; the same Cauchy Number, ratio of the inertia forces to the elastic forces; the same Froude Number, ratio of inertia forces to

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[3]Zahm, op. cit., p. 1.

gravity forces; the same Weber Number, ratio of inertia forces to the surface tension forces; or the same ratio of the pressure forces to the inertia forces. However, for model tests, it is not possible to have more than two of these ratios constant at the same time, therefore, the most important ratios must be chosen and the remainder neglected. In the case of the apparent additional mass models the models had the same geometric and kinematic scale, but the mass of the smaller models varied slightly from scale, since it was not deemed practical to hollow the smallest models. Further discussion of the hollow models appears in the discussion and recommendations.

DESCRIPTION OF APPARATUS

The torsional pendulum and tank used by Bryan[1] in determining apparent additional mass properties of bluff bodies of revolution was slightly modified and used to conduct these experiments. This apparatus, illustrated in Figure 1, consists of a one by one by 1/4 inch steel angle welded framework; a 0.0312 inch diameter piano wire stretched between the upper and lower members of the frame provides the torsional resistance and also supports the model assembly, which consists of a special steel clamp and two 3/16 inch diameter brass rods, eight inches long as shown in Figure 1. Twin models are threaded into the outboard ends of each rod.

Apparent additional mass properties of the bodies were measured by determining the variation of the period of oscillation of the torsional pendulum, both oscillating in air and in water. The twenty-four inch diameter by eighteen inch deep tank used for this purpose is illustrated in Figure 2. It was sanded and painted on all interior surfaces to obtain as smooth a surface as possible and to prohibit rusting. When the pendulum framework was inserted in the tank the eight inch extension rods allowed a minimum distance between the models and the tank wall of four inches. Fastening the models to the wire by placing the clamp midway between the top and bottom of the frame allowed the models to be approximately four inches below the water level when the framework was placed in the tank.

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[1]Colgan Hobson Bryan, "Apparent Additional Mass Characteristics of Various Bluff Bodies of Revolution," (A Masters thesis, Daniel Guggenheim School of Aeronautics, Georgia Institute of Technology, Atlanta, September, 1948), p. 41.

Since the amplitudes were very small in water, a projector-mirror screen system, as shown in Figure 3, was used to magnify the amplitude of the oscillating pendulum. A point source of light was obtained by inserting a two by two inch piece of aluminum (blackened on one side) with a number forty drilled aperture, in the slide holder of a hundred and fifty watt slide projector. The light was projected on a small mirror, glued on the wire of the pendulum. It was then reflected to a beaded screen approximately fifteen feet from the mirror. The mirror was placed 7.1 inches above the model clamp and 3.9 inches from the top of the frame in order to be above the top of the tank. Since the inertia effects of the morror would be corrected in the tare runs it was large enough to receive the full spot of light thereby making a clear cut image on the screen. The mirror was approximately a 1/2 inch equilateral triangle. On all runs except the runs to determine the damping constant a beaded screen was used so that more exact timing could be obtained by estimating when the light reached its maximum amplitude. When the amplitude was measured, an additional screen was used, which could be positioned more exactly with relation to the pendulum, and a three-meter stick was fastened to the screen to measure the amplitudes.

Models tested were cylinders of revolution of varying sizes, each having three different types of end configurations, blunt, conical, and hemispherical, as shown in Figures 4, 5, and 6. All models had the same length over diameter ratio of four, and had a geometric scale of 1, 2, 3, and 4, the model with a scale of one having nominal dimensions of 3/8 x 1 1/2 inches. All models were made from 17S-T aluminum alloy rod stock and constructed in the Georgia Tech model shop. The nominal dimensions were as follows; (1) 3/8 x 1 1/2 inches; (2) 3/4 x 3 inches; (3) 1 1/8 x 4 1/2 inches; (4) 1 1/2 x 6 inches. For exact dimensions see Table I, "Physical Characteristics of Models." With the exception of the smallest, all models were hollow to eliminate as much as possible deflections in the supporting rods, and had three sets of caps matched to their respective main bodies. All caps shown in Figures 4, 5, 6 and 7 had a mirror polish, while the main bodies were slightly rougher. Since the caps on one set of models seized, the facing surfaces of the caps were reduced to 1/16 of an inch. This resulted in leaking on the larger models so the caps were sealed with vaseline.

Since the models were made to fairly close tolerances it was not necessary to mass balance them, however, each model was checked for balance before it was used. To facilitate measuring the length from the center of the wire to the center of the model light scribe lines, 1/2 inch long, were placed ninety degrees from the center line of the extension rod. Two calibration cylinders as shown in Figure 1, were made of steel one inch by three inches with a threaded 3/16 diameter hole in one end. The calibration cylinders are used only to determine the spring constant of the wire. These bodies were highly polished and swung through an exceedingly small arc in order to have negligible apparent additional mass effects.

The calibration weights were weighed on a fifteen hundred gram chemical pan balance, sensitive to five milligrams and the models were weighed on a two hundred gram chemical chain balance, sensitive to one milligram. The left pan was removed from the chain balance and a support was made to hold a three inch by four inch by seven inch copper tank to enable weighing models in water. The models were supported from the balance arm by a thin silk thread, which was fastened in the threaded hole of the model for the extension arm, by a screw plug. A can with a spout about an inch from the top was first used to determine the displaced fluid, but measurement of the buoyant force proved much more accurate.

A standard stop watch, having ten seconds per revolution of the hand and calibrated in 1/10 of a second, was used. The time of testing was chosen so it did not coincide with the running of wind tunnels in the building, since it was noticed that forced frequencies were set up on the pendulum, especially when the small models were tested and the tare runs were made.

PROCEDURE

The torsional pendulum was set up as described in the Description of Apparatus, and as shown in Figure 1. A steady table was chosen and leveled, to place the pendulum and tank of water on, as shown in Figure 2. A length of 0.0312 inch diameter piano wire was stretched vertically across the pendulum frame. The diameter of the wire was arbitrarily chosen, since any size wire could be used, and the wire was tightened sufficiently to maintain the models in a horizontal plane only. The clamps, on the upper and lower cross frames of the pendulum, as shown in Figure 1, were secured as tight as possible to simulate built in ends on the wire. The steel clamp, designed by Bryan, that secured the extension arms and models to the wire, was then fastened midway between the upper and lower frames of the pendulum. Care was taken to tighten the clamp as tightly as possible to eliminate any slipping on the wire. A set of brass extension rods, shown in Figure 1, were then screwed in the clamp. The nine inch extension arms were chosen as a matter of convenience, since all models could be used on the same arm without removing the arms from the clamp. As will be shown later, the length of extension arms have no effect on the values of apparent additional mass, however, this length enters into the calculations and was measured before each run.

A light system was set up as shown in the schematic diagram in Figure 3. A small mirror was glued on the wire 7.1 inches above the center of thw wire and 3.9 inches below the top cross frame. Care was taken to allow the cement to dry thoroughly, since it was found that the twisting of the wire during the runs tended to loosen the mirror from the wire. A 100-watt Argus slide projector, placed on a nearby table and equipped with a two inch by two inch aluminum sheet slide with a number forty hole drilled in the center, was used to furnish a pin point light source. A beaded screen was hung on a wall across the room, approximately fourteen feet away, as shown in Figure 3. The pendulum was then positioned such that the light from the projector reflected from the mirror on to the screen in such a manner that the spot of light was on the screen at maximum amplitudes of the pendulum. The spot of light on the screen had a straight edge on one side, produced by focussing the light, from the projector, on the edge of the mirror. This was done to facilitate locating the exact position on the screen at which maximum amplitude of the pendulum occurred. Since the pendulum was moved in to the tank, Figure 2, and out again for each run, it was much more convenient to time the period from one maximum to the next rather than determine the equilibrium position of the swing on the screen. Although these maximum values damped rapidly in water, after a little experience it was possible to anticipate quite accurately the position of the light on the screen for the next maximum amplitude of the pendulum. The tank, shown in Figure 2, was placed on the same table and near the pendulum, making it unnecessary to change the light system when water runs were made. The tank was filled with clean water to a depth about two inches from the top, which amounted to about thirty gallons of water.

By the time the runs were made, the water in the tank was at room temperature, so that there was no variation in the spring constant of the wire, or change in density of the water due to temperature. The tank had previously been sanded and repainted inside to prevent rusting and to insure a smooth surface, thus minimumizing wall effects. The same water was used throughout the entire test program.

Calibration cylinders were accurately weighed and the dimensions were measured by a micrometer. The moment of inertia of the calibration cylinders were calculated by equations to be found in Pertinent Formulae; all calculations were done on a Friden calculator. The calibration cylinders were then screwed to the ends of the extension arms of the pendulum and allowed to swing in air. Even though the amplitudes of the pendulum were small, the calibration weights were allowed to swing as many as a thousand times. A number of these runs were made and an average of the results produced a quite accurate value of the spring constant, K_T. Although K_T was never actually calculated, there was enough data to do so by the equation:

$$K_T = 4\pi^2 \frac{I_k}{T_k^2} \tag{1}$$

As shown in Pertinent Formulas, the development of the equation for apparent additional mass when a torsional pendulum is used, can be expressed:

$$m_A = \frac{K_T}{8\pi^2 L^2} \quad T_w^2 - T_a^2 - (T_w'^2 - T_a'^2) \tag{2}$$

Upon direction substitution of K_T from equation (1), the apparent addition mass for one body becomes:

$$m_A = \frac{I_k}{2T_k^2 L^2} \quad (T_w^2 - T_a^2 - T_w'^2 + T_a'^2) \tag{3}$$

Thus equation (3) eliminates the necessity for calculating K_T, since I_K is calculated, and T_K is measured, both which remain constant, provided the wire is not changed. The same wire was used in all runs.

Standard formulae were used to calculate the moments of inertia of the clamps, rods, and calibration cylinders as shown in the Sample Calculations. However, the moment of inertia of the clamp was neglected in the calculations, since it was small compared with the total moment of inertia.

Micrometer readings on all the model dimensions were made after the caps were matched to obtain the nearest value of length over diameter of four, as shown in Table I. After the caps were matched to their respective models, they were marked with a center punch on the inside of the cap and in the hollowed out portion of the main section.

Since the large models tended to leak a small amount of water between the flange of the cap and the main section of the model, a light film of vaseline was placed on the cap each time before they were screwed into the model. This proved to be very successful. The models were then weighed in air on a 200 gram chemical chain balance; all except the largest model were weighed with the caps on, however, the largest models were weighed in sections. A special copper tank was constructed to weigh the models while completely immersed in water. The models were hung from the balance arm into the tank of water by a thin thread, fastened in the model by a small brass plug screwed into the threaded hole used by the extension arms. The models were all weighed in water with the caps on. This weight subtracted from the equivalent weight in air gave the buoyant force of the model in water, which is equal to the mass of the displaced fluid, all weighing was done to nearest milligram.

The models tested were fitted with the proper caps and mounted on the extension arms, as shown in Figures 4, 5, 6, and 7. The bodies were screwed on the extension arms as far as the threads on the arms would allow, turned into a horizontal position, and checked with a level. An engineer's scale was notched at one end to center the torsion wire and the length of the pendulum arm, from the center of the wire to the centroid of the body, indicated by a scribe line, was measured on the sixty divisions to the inch scale. With the use of reading glass it was possible to estimate a tenth of a division, making the smallest reading accurate of approximately 1/600 of an inch. The above measurements were made for both arms each time a different model was placed on the pendulum.

The pendulum, with bodies attached, was swung in air, and the period timed no less than fifteen times at a hundred swings each time for all models. Then the pendulum was submerged in the water, the light system adjusted, and swung in water and the period timed fifteen times at ten swings each, since the amplitude decreased rapidly in water.

The bodies were then taken off the pendulum arms and the tare run in water were made using the same procedure as with the models attached. The pendulum was taken out of the tank and thoroughly dried and the tare runs in air were swung. Great care was taken for the tare runs, since the period was only a little over a second. However, once the periods for the tare runs were established they were used throughout the test program, but checked occasionally.

When runs had a total of a hundred swings or more, which was the case in every instance with the exception of swinging in water, three runs of ten swings each were made. The periods of each run was recorded and averaged. This gave a good estimation of what the period should be; therefore, the pendulum was started swinging, and although the stop watch was started at the same time an electric photographic timer was set for the estimated time of ninety swings. Meanwhile the projector light was turned off to keep the projector from over heating. When the timer went off at ninety swings the projector was turned on again and on the estimated ninety-fifth swing the swings were counted until the hundredth swing was reached, and the watch stopped. The estimated period from the three runs was sufficiently accurate that the more accurate period of a hundred swings did not vary more than a tenth of the period from the estimated value. Thus it was possible to obtain the period of the pendulum accurately to a thousandth of a second. In case of the larger bodies and calibration weights this method of timing proved to be quite convenient.

PRECISION OF TESTING

Any data obtained from the measurement of physical quantities will possess a certain amount of error of one type or another. Whenever possible the magnitude or an approximation of the magnitude of the error should be determined for the results of the test to have some meaning. Errors in testing are usually classified as; instrument errors, human errors, errors of method or theoretical errors and residual errors. Emphasis is placed on residual errors, indeterminate of accidental errors, which are due to causes over which the observer has no control. These errors appear in results that differ in the last, or last two places, of a measurement with some instrument where a number of observations are recorded.

In the testing program of the apparent additional mass by the torsion pendulum all the quantities that enter into the final equation are either directly or indirectly physical measurements that are subject to the above mentioned errors. Therefore in analyzing the results from the tests in apparent additional mass from an error standpoint it is necessary to consider the following equations[1]

$$m = \frac{\sum a}{n} \qquad (1)$$

where m = arithmetical mean

$a_1, a_2, a_3, \ldots\ldots a_k, \ldots\ldots a_n$ = series of observations in a quanity.

n = number of observations.

---

[1] H. M. Goodwin, Elements of the Precision Measurements and Graphical Methods, (New York: McGraw-Hill Book Co.), 1920. pp. 1-38.

If the numerical deviations, d_1, d_2, d_3, ... d_k, ... d_n, are computed for any series of observations, their algebraic sum will be zero, since the sum of the positive deviations is equal to the sum of the negative deviations. However, if their arithmetical mean be computed, disregarding their sign, the result will be a number which expresses how much, on the average, any single observation of the series taken at random is likely to differ from the mean. This average value

$$\text{a.d.} = \frac{\sum d}{n} \tag{2}$$

and is called the average deviation of a single observation. It can also be shown that the deviation of the mean, denoted by A.D., is

$$\text{A.D.} = \frac{\text{a.d.}}{\sqrt{n}} \tag{3}$$

When the precision measure, δ , of most experimental quantities are not known, an estimation of this measure must be made and since the deviation of the mean is a good indication of the precision it is used in this case. When a relation is given, depending upon a number of physical observations, m_1, m_2, m_3, ... m_k, ... m_n, with respective precision measures, δ_1, δ_2, δ_3,..... δ_k,...... δ_n, it is possible to compute the precision measure, Δ, of the result M, by the equation

$$\Delta_k = \frac{\partial M}{\partial m_k} \cdot \delta_k \tag{4}$$

The law of least squares is then applied to the various Δ_ks to determine the error of M due to n measurements. The law of least squares[2] may be expressed as follows

$$\Delta = \sqrt{\Delta_1^2 + \Delta_2^2 + \Delta_3^2 \cdots + \Delta_k^2 \cdots\cdots + \Delta_n^2} \tag{5}$$

Consider the equation for the apparent additional mass[3]

$$m_A = \frac{I_K\, g}{2T_K^2\, L^2}(T_w^2 - T_a^2 - T_w'^2 + T_a'^2) \tag{6}$$

Since I_K was calculated for the torsion pendulum it was necessary to find the error in I_K due to a precision measure in the measurements in the following equation[4]

$$I_K = \frac{M_1}{12^3}(3r_1^2 + a_1^2) + \frac{M_2}{12^3}(3r_2^2 + a_2^2) + (M_1 + M_2)\, L^2 + \frac{M_r}{3}\, l^2 \tag{7}$$

where

M_1, M_2 = masses of the two calibration weights

M_r = mass of the two rods

r_1, r_2 = radii of the calibration cylinders in inches

a_1, a_2 = lengths of the calibration weights in inches

---

2 Ibid., p. 29.

3 Cf. post., p. 55.

4 Cf. post., p. 55.

L = distance between the center of gravity of the calibration cylinder and the center of the torsion wire

l = distance between the assumed comcentration of the mass of the rod and the center of the torsion pendulum rod

The following values were found to be:

M_1 = 291 grams	M_2 = 292 grams
r_1 = 0.494 inches	r_2 = 0.494 inches
a_1 = 3.00 inches	a_2 = 3.00 inches

Assume the following precision measurements

$\delta M_1 = \delta M_2 =$ 0.005 grams

$\delta r_1 = \delta r_2 =$ 0.0005 grams

$\delta L = \delta l =$ 0.0033 inches

$\delta a_1 = \delta a_2 =$ 0.001 inches

Noting that there are 14,589 grams per slug and that all precision measure calculations should not be carried with greater than slide rule accuracy therefore the precision measures of the moment of inertia, in slugs feet^2, of the calibration cylinders were calculated using 14,600 grams per slug.

To find the change in the moment of inertia of the calibration cylinders, Δ_m, due to the error, δ_m, in determing the masses of the weights.

$$\Delta_{m_2} = \Delta_{m_1} = \frac{\partial I_K}{\partial m_1} \cdot \delta_{m_1} = \frac{\partial I_K}{\partial m_2} \cdot \delta_{m_2}$$

$$= \frac{(3\, r_1^2 + a_1^2)\ (0.005)}{12^3\ (14,600)}$$

$$= \frac{3(0.494)^2 +)3.00)^2}{1728} \times \frac{0.005}{14,600}$$

$$\Delta_{m_2} = \Delta_{m_1} = 19.2 \times 10^{-10}$$

To find the change in the moment of inertia of the calibration cylinders, Δ_r, due to an error, δ_r, in measuring the radii of the cylinders

$$\Delta_{r_1} = \Delta_{r_2} = \frac{\partial I_K}{\partial r_1} \cdot \delta_{r_1} = \frac{\partial I_K}{\partial r_2} \cdot \delta_{r_2}$$

$$= \frac{6\ M_1\ r_1}{12^3 (14,600)} \cdot \delta_{r_1}$$

$$= \frac{6(291)(0.494)(0.0005)}{(1728)(14,600)}$$

$$\Delta_{r_1} = \Delta_{r_2} = 171 \times 10^{-10}$$

To determine the change in the moment of inertia of the calibration cylinders, Δ_1, due to an error in measuring the length, l, δ_1

$$\Delta_1 = \frac{\partial I_K}{\partial l} \cdot \delta_1 = \frac{M_r l}{3} \cdot \delta_1$$

$$= \frac{2(58.3)(7.00)(0.0033)}{3(12)^2(14,600)}$$

$$\Delta_1 = 0.427 \times 10^{-10}$$

To determine the change in the moment of inertia of the calibration cylinders, Δ_L, due to an error in measuring the length, L, δ_L

$$\Delta_L = \frac{\partial I_K}{\partial L} \cdot \delta_L$$

$$= 2(M_1 + M_2)\ L$$

$$= \frac{2(9)(582)(0.0033)}{12^2(14,600)}$$

$$\Delta_L = 16.5 \times 10^{-10}$$

By the Law of Least Squares, equation (5)

$$\Delta_I = \sqrt{2\,\Delta_{m_1}^2 + \Delta_{r_1}^2 + \Delta_1^2 + \Delta_L^2}$$

$$= 10^{-6}\sqrt{2(0.0019)^2 + 2(0.0171)^2 + (0.427)^2 + (16.5)^2}$$

$$= 10^{-6}\sqrt{271}$$

$$\Delta_I = 16.5 \times 10^{-6}$$

Since Δ_I will be used as a part of the equation for m_A it has been denoted by δ_{I_k} in that equation.

In order to determine T_K, the period of the pendulum with the calibration weights attached, the arithmetic mean, m, the average deviation, a.d., and the maximum probable deviation of the mean, A.D., must be found. Using equations (1), (2) amd (3) it is possible to obtain A.D. which will be used as the precision measure, δ_K. The following values of T_K were recorded:

$$
\begin{array}{rrcrl}
 & 7.3445 & - & 7.3495 = & 50 \\
 & 7.3475 & - & 7.3495 = & 20 \\
 & 7.3360 & - & 7.3495 = & 135 \\
 & 7.3320 & - & 7.3495 = & 175 \\
 & 7.3513 & - & 7.3495 = & 18 \\
 & 7.3521 & - & 7.3495 = & 26 \\
 & 7.3473 & - & 7.3495 = & 22 \\
 & 7.3490 & - & 7.3495 = & 5 \\
 & 7.3464 & - & 7.3495 = & 31 \\
 & 7.3489 & - & 7.3495 = & 6 \\
 & 7.3520 & - & 7.3495 = & 25 \\
 & 7.3514 & - & 7.3495 = & 19 \\
 & 7.3487 & - & 7.3495 = & 8 \\
 & 7.3491 & - & 7.3495 = & 4 \\
 & 7.3492 & - & 7.3495 = & 3 \\
 & 7.3526 & - & 7.3495 = & 31 \\
 & 7.3520 & - & 7.3495 = & 25 \\
10^{*} & \underline{7.3527} & - & 7.3495 = 32 \times 10 = & \underline{320}
\end{array}
$$

$$\sum a = 198.4370 \qquad \sum d = 635$$

$$m = \frac{\sum a}{n} = \frac{198.4370}{27} = 7.3495$$

$$a.d. = \frac{\sum d}{n} = \frac{635}{27} = 23.5$$

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{23.5}{\sqrt{27}} = 4.53$$

$$T_K = m = 7.3495 \text{ seconds} \qquad \delta_K = 0.00453$$

*This was a run with a thousand swings in lieu of a hundred swings, therefore it was conservative to use the arithmetic mean value of the thousand swing run, since the humas error in starting and stopping the watch did not enter.

The same proceedure was used for $T_{w'}$, $T_{a'}$, T_w, and T_a; however once T_K, $T_{w'}$, and $T_{a'}$ are calculated they remain the same. The values of the period of the pendulum without the bodies in water are

$$
\begin{array}{rcrcr}
1.229 & - & 1.228 & = & 1 \\
1.229 & - & 1.228 & = & 1 \\
1.230 & - & 1.228 & = & 2 \\
1.215 & - & 1.228 & = & 13 \\
1.246 & - & 1.228 & = & 18 \\
1.232 & - & 1.228 & = & 4 \\
1.228 & - & 1.228 & = & 0 \\
1.236 & - & 1.228 & = & 8 \\
1.230 & - & 1.228 & = & 2 \\
1.215 & - & 1.228 & = & 13 \\
1.231 & - & 1.228 & = & 3 \\
1.228 & - & 1.228 & = & 0 \\
1.220 & - & 1.228 & = & 8 \\
1.225 & - & 1.228 & = & 3 \\
1.225 & - & 1.228 & = & 3 \\
\hline
\end{array}
$$

$$\Sigma_a = 18.419 \qquad \Sigma_d = 67$$

$$m = T_{w'} = \frac{18.419}{15} = 1.2279$$

$$a.d. = \frac{\Sigma d}{n} = \frac{67}{15} = 4.47$$

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{4.47}{\sqrt{15}} = 1.15$$

$$T_{w'} = 1.2279 \qquad \delta_{T_{w'}} = 0.0015$$

For $T_{a'}$

$$
\begin{array}{rcrcr}
1.1321 & - & 1.1305 & = & 16 \\
1.1300 & - & 1.1305 & = & 5 \\
1.1332 & - & 1.1305 & = & 27 \\
1.1302 & - & 1.1305 & = & 3 \\
1.1297 & - & 1.1305 & = & 8 \\
1.1304 & - & 1.1305 & = & 1 \\
1.1302 & - & 1.1305 & = & 3 \\
1.1301 & - & 1.1305 & = & 4 \\
1.1293 & - & 1.1305 & = & 8 \\
\underline{1.1301} & - & 1.1305 & = & \underline{4}
\end{array}
$$

$$\Sigma_a = 11.3053 \qquad \Sigma_d = 79$$

$$m = T_{a'} = \frac{11.3053}{10} = 1.1305$$

$$a.d. = \frac{\Sigma d}{n} = \frac{79}{10} = 7.9$$

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{7.9}{\sqrt{10}} = 2.5$$

$$T_{a'} = 1.1305 \qquad \delta_{T_{a'}} = 0.0025$$

Consider the values of T_w and T_a for model number 2 - AB - B, (see Table I, Physical Characteristics of Models), or the second from the smallest group of models with blunt ends.

$$
\begin{array}{rcrcr}
3.153 & - & 3.157 & = & 4 \\
3.152 & - & 3.157 & = & 5 \\
3.165 & - & 3.157 & = & 8 \\
3.161 & - & 3.157 & = & 4 \\
3.167 & - & 3.157 & = & 10 \\
3.150 & - & 3.157 & = & 7 \\
3.168 & - & 3.157 & = & 11 \\
3.150 & - & 3.157 & = & 7 \\
3.154 & - & 3.157 & = & 3 \\
3.159 & - & 3.157 & = & 2 \\
3.158 & - & 3.157 & = & 1 \\
3.153 & - & 3.157 & = & 4 \\
3.161 & - & 3.157 & = & 4 \\
3.156 & - & 3.157 & = & 7 \\
\underline{3.153} & - & 3.157 & = & \underline{4}
\end{array}
$$

$$\Sigma_a = 47.360 \qquad \Sigma_d = 81$$

$$m = T_w = \frac{\sum a}{n} = \frac{47.360}{15} = 3.1573$$

$$a.d. = \frac{\sum d}{n} = \frac{81}{15} = 5.4$$

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{5.4}{\sqrt{15}} = 1.4$$

$T_w = 3.1573$ seconds $\qquad \delta_{T_w} = 0.00139$

For T_a

2.9623 - 2.9601 =	22
2.9605 - 2.9601 =	4
2.9611 - 2.9601 =	10
2.9613 - 2.9601 =	12
2.9609 - 2.9601 =	8
2.9586 - 2.9601 =	15
2.9579 - 2.9601 =	22
2.9590 - 2.9601 =	11
2.9601 - 2.9601 =	0
2.9590 - 2.9601 =	11

$\sum a = 29.6007 \qquad \sum d = 115$

$$m = T_a = \frac{\sum a}{n} = \frac{29.6007}{10} = 2.9601$$

$$a.d. = \frac{\sum d}{n} = \frac{115}{10} = 11.5$$

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{11.5}{\sqrt{10}} = 3.64$$

$T_a = 2.9601 \qquad \delta_{T_a} = 0.000364$

The distance from the center of the torsion wire to the center of gravity of the model, L, is

$$L = \frac{467.4}{(60)(12)} = 0.6492 \text{ feet}$$

$$L = \frac{0.2}{(60)(12)} = 0.000278 \text{ feet}$$

assuming that it is possible to deviate 0.2/60 of an inch on the scale used.

Now consider the equation for the apparent additional mass in pounds

$$m_A = \frac{I_K \, g}{2 \, T_K^2 \, L^2} \left(T_W^2 - T_a^2 - T_{W'}^2 + T_{a'}^2 \right)$$

where $g = 32.1366$ feet per seconds^2

The deviation of the apparent additional mass due to a deviation in the moment of inertia of the calibration weights, from equation (4), is

$$\Delta_{I_k} = \frac{\partial m_A}{\partial I_K} \cdot \delta_{I_K}$$

$$= \frac{g(T_W^2 - T_a^2 - T_{W'}^2 + T_{a'}^2)}{2 \, T_K^2 \, L^2} \cdot \delta_{I_K}$$

$$= \frac{(32.1)(0.974)(16.5)(10^{-6})}{2(7.35)^2(0.649)^2}$$

$$\Delta_{I_K} = 11.3 \times 10^{-6}$$

The deviation of the apparent additional mass due to a deviation in the length, L, is

$$\Delta_L = \frac{\partial m_A}{\partial L} \cdot \delta_L = \frac{I_K\, g\, \delta_L}{T_K^2\, L^3} \left(T_W^2 - T_a^2 - T_{W'}^2 + T_{a'}^2\right)$$

$$= \frac{(0.0232)(32.1)(0.000278)(0.974)}{(7.35)^2(0.649)^2}$$

$$\Delta_L = 13.7 \times 10^{-6}$$

The deviation of the apparent additional mass due to a deviation in the calibration period, T_K, is

$$\Delta_{T_K} = \frac{\partial m_A}{\partial T_K} \cdot \delta_{T_K} = \frac{I_K\, g\, \left(T_W^2 - T_a^2 - T_{W'}^2 + T_{a'}^2\right)}{T_K^3\, L^2} \cdot \delta_{T_K}$$

$$= \frac{(0.0232)(32.1)(0.974)(0.000453)}{(7.35)^3(0.649)^2}$$

$$\Delta_{T_K} = 1.97 \times 10^{-6}$$

The deviation of the apparent additional mass due to a deviation in $T_{W'}$, is

$$\Delta_{T_{W'}} = \frac{\partial m_A}{\partial T_{W'}} \cdot \delta_{T_{W'}} = \frac{I_K\, g\, T_{W'}}{T_K^2\, L^2} \cdot \delta_{T_{W'}}$$

$$= \frac{(0.0232)(32.1)(1.23)(0.00115)}{(7.35)^2(0.649)^2}$$

$$\Delta_{T_{W'}} = 46.4 \times 10^{-6}$$

The deviation of the apparent additional mass due to a deviation in $T_{a'}$, is

$$\Delta_{T_{a'}} = \frac{\partial m_A}{\partial T_{a'}} \cdot \delta_{T_{a'}} = \frac{I_K \, g \, T_{a'}}{T_K^2 \, L^2} \cdot \delta_{T_{a'}}$$

$$= (0.0328)(1.13)(0.00025)$$

$$\Delta_{T_{a'}} = 9.27 \times 10^{-6}$$

The deviation of the apparent additional mass due to a deviation in T_W, is

$$\Delta_{T_W} = \frac{\partial m_A}{\partial T_W} \cdot \delta_{T_W} = \frac{I_K \, g \, T_W}{T_K^2 \, L^2} \cdot \delta_{T_W}$$

$$= (0.328)(3.16)(0.00139)$$

$$\Delta_{T_W} = 144 \times 10^{-6}$$

The deviation of the apparent additional mass due to a deviation in T_a , is

$$\Delta_{T_a} = \frac{\partial m_A}{\partial T_a} \cdot \delta_{T_a} = \frac{I_K \, g \, T_a}{T_K^2 \, L^2} \cdot \delta_{T_a}$$

$$= (0.0328)(2.96)(0.000364)$$

$$\Delta_{T_a} = 35.4 \times 10^{-6}$$

By the Law of Least Squares it was possible to obtain the total deviation of the apparent additional mass due to the errors in the various parameters;

$$\Delta = \sqrt{\Delta I_K^2 + \Delta L^2 + \Delta T_K^2 + \Delta T_{w'}^2 + \Delta T_{a'}^2 + \Delta T_w^2 + \Delta T_a^2}$$

$$= 10^{-6}\sqrt{(11.3)^2 + (13.7)^2 + (1.97)^2 + (46.4)^2 + (8.27)^2 + (144)^2 + (35.4)^2}$$

$$\Delta = 0.000156 \text{ pounds}$$

The percentage error in the apparent additional mass is

$$\frac{\Delta}{m_A} = \frac{(0.000156)(100)}{0.01596} = 0.98\ \%$$

The above error was noticed in one of the first models, therefore an analysis was made on the last run, model 4-AB-H, where the testing proceedure had been improved.

The values of T_W and T_a for model 4-AB-H (see Table I, Physical Characteristics of Models), are used in calculating the error in the same manner as above.

For T_W

$$\begin{array}{rcrcr}
6.851 & - & 6.851 & = & 0 \\
6.848 & - & 6.851 & = & 3 \\
6.851 & - & 6.851 & = & 0 \\
6.849 & - & 6.851 & = & 2 \\
6.855 & - & 6.851 & = & 4 \\
6.852 & - & 6.851 & = & 1 \\
6.850 & - & 6.851 & = & 1 \\
6.854 & - & 6.851 & = & 3 \\
6.848 & - & 6.851 & = & 3 \\
\underline{6.848} & - & 6.851 & = & \underline{3} \\
\Sigma_a = 68.506 & & & \Sigma_d = & 20
\end{array}$$

$$m = T_W = \frac{\Sigma a}{n} = \frac{68.506}{10} = 6.8506$$

$$a.d. = \frac{\Sigma d}{n} = \frac{20}{10} = 2.0$$

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{2.0}{\sqrt{10}} = 0.633$$

$T_W = 6.8506$ seconds $\qquad \delta T_W = 0.000633$ seconds

For T_a

$$\begin{array}{rcrcr}
6.4384 & - & 6.4381 & = & 3 \\
6.4372 & - & 6.4381 & = & 9 \\
6.4376 & - & 6.4381 & = & 5 \\
6.4369 & - & 6.4381 & = & 12 \\
6.4408 & - & 6.4381 & = & 27 \\
6.4372 & - & 6.4381 & = & 9 \\
6.4398 & - & 6.4381 & = & 17 \\
\underline{6.4375} & - & 6.4381 & = & \underline{6}
\end{array}$$

$\Sigma a = 51.5054 \qquad \Sigma d = 88$

$$m = T_a = \frac{\Sigma a}{n} = \frac{51.5054}{8} = 6.4381$$

$$a.d. = \frac{\Sigma d}{n} = \frac{88}{8} = 11$$

$$A.D. = \frac{a.d.}{\sqrt{n}} = \frac{11}{\sqrt{8}} = 3.89$$

$T_a = 6.4381 \qquad \delta T_a = 0.000389$

The deviation in the apparent additional mass due to a deviation in T_a, is

$$\Delta_{T_w} = \frac{\partial m_A}{\partial T_w} \cdot \delta_{T_w} = \frac{I_K \ g \ T_w}{T_K^2 \ L^2} \cdot \delta_{T_w}$$

$$= (0.0328)(6.85)(0.000633)$$

$$\Delta_{T_w} = 142 \times 10^{-6}$$

The deviation in the apparent additional mass due to a deviation in T_a, is

$$\Delta_{T_a} = \frac{\partial m_A}{\partial T_a} \cdot \delta_{T_a} = \frac{I_K \ g \ T_a}{T_K^2 \ L^2} \cdot \delta_{T_a}$$

$$= (0.0328)(6.44)(0.000389)$$

$$\Delta_{T_a} = 79.5 \times 10^{-6}$$

The deviation of the apparent additional mass due to a deviation in L, is

$$\Delta_L = \frac{\partial m_A}{\partial L} \cdot \delta_L = \frac{I_K \ g \ \delta_L}{T_K^2 \ L^3} (T_w^2 - T_a^2 - T_{w'}^2 + T_{a'}^2)$$

$$\Delta_L = \frac{(0.0232)(32.1)(5.25)(0.000278)}{(7.35)^2(0.647)^3}$$

$$\Delta_L = 74.1 \times 10^{-6}$$

The total deviation of the apparent additional mass due to all the various parameters

$$\Delta = \sqrt{\Delta I_K^2 + \Delta L^2 + \Delta T_K^2 + \Delta T_{w'}^2 + \Delta T_{a'}^2 + \Delta T_w^2 + \Delta T_a^2}$$

$$= 10^{-6}\sqrt{(11.3)^2 + (71.4)^2 + (1.97)^2 + (46.4)^2 + (9.27)^2 + (142)^2 + (74.1)^2}$$

$$\Delta = 0.000184 \text{ pounds}$$

The percentage error in the apparent additional mass is

$$\frac{\Delta}{m_a} = \frac{(0.000184)(100)}{0.08666} = 0.21\%$$

This analysis on the precision of testing indicates that the error in calculating the apparent additional mass was in the order of 0.21% to 0.98%. In considering the error in the apparent additional mass coefficient, k, for model 2-AB-B (see Table I, Physical Characteristics of Models), the following relations were used:

$$k = \frac{m_A}{m_{D.F.}}$$

Let

$$\delta m_A = \Delta = 0.000156, \text{ from the previous calculations}$$

$$= 0.001 \text{ grams} = 2.2 \times 10^{-6} \text{ pounds}$$

$$m_A = 0.0160 \text{ pounds}$$

$$m_{D.F.} = 0.0479 \text{ pounds}$$

$$k = 0.333$$

Then

$$\Delta_{m_{D.F.}} = \frac{\partial k}{\partial m_{D.F.}} \cdot \delta_{m_{D.F.}} = \frac{m_A}{m_{D.F.}^2} \cdot \delta_{m_{D.F.}}$$

$$= \frac{(0.0160)(2.2)(10^{-6})}{(0.0479)^2}$$

$$\Delta_{m_{D.F.}} = 1.54 \times 10^{-5}$$

and

$$\Delta_{m_A} = \frac{\partial k}{\partial m_A} \cdot \delta_{m_A} = \frac{\delta_{m_A}}{m_{D.F.}}$$

$$= \frac{(0.000156)}{(0.0479)}$$

$$\Delta_{m_A} = 326 \times 10^{-5}$$

therefore

$$\Delta = \sqrt{\Delta m_{D.F.}^2 + \Delta m_A^2}$$

$$= 10^{-5}\sqrt{(1.54)^2 + (326)^2}$$

$$\Delta = 0.00326$$

The percentage error in the apparent additional mass coefficient, k, is

$$\frac{\Delta}{k} = \frac{(0.00326)(100)}{0.333} = 0.98\%$$

For the model 4-AB-H (see Table I, Physical Characteristics of Models), similar calculations were

$$\Delta_{m_{D.F.}} = \frac{\partial k}{\partial m_{D.F.}} \cdot \delta m_{D.F.} = \frac{m_A}{m_{D.F.}^2} \cdot \delta m_{D.F.}$$

$$= \frac{(0.0867)(2.2)(10^{-6})}{(0.349)^2}$$

$$\Delta_{m_{D.F.}} = 1.57 \times 10^{-6}$$

and

$$\Delta_{m_A} = \frac{\partial k}{\partial m_A} \cdot \delta m_A = \frac{\delta m_A}{m_{D.F.}}$$

$$= \frac{0.000184}{0.349}$$

$$\Delta_{m_A} = 527 \times 10^{-6}$$

Then

$$\Delta = \sqrt{\Delta m^2_{D.F.} + \Delta m^2_A}$$

The percentage error in the apparent additional mass coefficent, k, for model 4-AB-H is

$$\frac{\Delta}{k} = \frac{(0.000527)(100)}{0.248} = 0.21\,\%$$

As seen from the above calculations the accuracy in the apparent additional mass is nearly the same for the apparent additional mass coefficient due to the fact that the error in determining the mass of the displaced fluid is extremely small.

PERTINENT FORMULAE

The following formulae were used to calculate the apparent additional inertia, apparent additional mass and apparent additional mass coefficients for a torsion pendulum.

As shown by Yee-Tak Yu[1] the damping effect in air and water was small enough to neglect it entirely. Then the period of a simple torsion pendulum is[2]

$$T = 2\pi\ (I/K_T)^{1/2} \tag{1}$$

where

T = period of pendulum

I = mass moment of inertia of pendulum

K_T = spring constant of pendulum wire

therefore

$$T_K = 2\pi\ (I_K/K_T)^{1/2} \tag{2}$$

The increase in moment of inertia due to a single object B moving in water is:

$$I = 1/2\ (I_W - I_a - I_{W'} + I_{a'}) \tag{3}$$

When L is relatively long:

$$I = m_A\ L^2 \tag{4}$$

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[1]Yee-Tak Yu "Virtual Masses and Moments of Inertia of Disks and Cylinders in Various Liquids", Journal of Applied Physics, Vol.13,1942.p.67

[2]Yee-Tak Yu, loc. cit.

Although the apparent additional inertia and apparent additional mass may be calculated from the above formulae it is more convenient to express them as follows:

Solving for I in equation (1)

$$I = \frac{T^2 K_T}{4 \pi^2} \tag{5}$$

therefore

$$I_a = \frac{T_a^2 K_T}{4 \pi^2}$$

$$I_W = \frac{T_W^2 K_T}{4 \pi^2}$$

$$I_{a'} = \frac{T_{a'} K_T}{4 \pi^2}$$

$$I_{W'} = \frac{T_{W'} K_T}{4 \pi^2}$$

Solving equation (3) for K_T,

$$K_T = \frac{4 \pi^2 L_K}{T_K^2} \tag{6}$$

substituting in equation (5) and simplifying

$$I = \frac{K_T}{8 \pi^2} \left(T_W^2 - T_a^2 - T_W'^2 + T_a'^2\right) \tag{7}$$

substituting K_T from equation (6)

$$I = \frac{I_K}{2 T_K^2} \left(T_W^2 - T_a^2 - T_W'^2 + T_a'^2\right) \tag{8}$$

Substituting I from equation (8) into equation (4) and solving for m_A

$$m_A = \frac{I_K}{2 T_K^2 L^2} \left(T_W^2 - T_a^2 - T_W'^2 + T_a'^2\right) \tag{9}$$

Equation (9) eliminates calculating K_T, and the expression $\frac{I_K}{2\ T_K^2\ L^2}$ is usually constant for a series of runs. However the author choose to measure the length L for each run therefore only $\frac{I_K}{2\ T_K^2}$ was constant for a series of runs.

The formula used for the mass moment of inertia was:

$$I_K = \frac{M_1}{12}\left(3\ r_1^2 + a_1^2\right) + \frac{M_2}{12}\left(3\ r_1^2 + a_1^2\right) + (M_1 + M_2)L^2 + \qquad (10)$$

As shown by Bryan[3] the error in neglecting the clamp in the calculations for the mass moment of inertia I_K, was only 0.013c/c therefore only the bodies and the rods were used in these calculations.

The formula for the apparent additional mass coefficient is

$$k = \frac{m_A}{m_{D.F.}} \qquad (11)$$

Using the English units:

$$\text{Mass moment of inertia} = \text{slugs feet}^2$$
$$= \frac{\text{pounds seconds}^2\ \text{feet}^2}{\text{feet}}$$
$$= \text{pounds seconds}^2\ \text{feet}$$

To convert slugs to pounds mass:

1 slug = 32.136 pounds

To convert slugs to grams:

1 slug = 14,589 grams

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[3]Bryan, op.cit. p. 59

CONCLUSIONS

1. As indicated by theory, the apparent additional mass increases directly with the scale of the model. As shown in Figure 8, the apparent additional mass increased by a straight line ratio with the mass of the displaced fluid.

2. The apparent additional mass coefficients were nearly constant with the variation of scale based on the volume of the displaced fluid. This indicates that there is no scale effect on apparent additional mass coefficients. It is possible to multiply the apparent additional mass coefficient of a model by the displaced fluid of a full scale body to obtain the apparent additional mass of that body. However, the model must have similar motion, shape, and attitude as the full scale body.

3. The values of apparent additional mass and apparent additional mass coefficient of the smallest models deviated from the straight line ratios of the larger models. This deviation was probably caused by the following: (1) The interference effects of the rod and the bodies were proportionally larger, compared to apparent additional mass, for the smaller models. (2) Since the apparent additional mass was small, the experimental error was probably larger in relation to the apparent additional mass. (3) Since these bodies were not hollow compared with the larger hollow bodies, the ratio of the geometric masses differed slightly; therefore, they were not exactly dynamically similar.

4. The values of the apparent additional mass and apparent additional mass coefficient of the 3/4 by three inch bodies with conical and hemispherical ends checked very well with L/D tests performed by Bryan as shown in Figures 11 and 12, however, the blunt bodies were slightly lower than Bryan's bodies. Reruns on the blunt bodies still proved that the blunt bodies had lower apparent additional mass than previous tests indicated, which may be due to smoother surfaces and slightly more accurate testing procedure. However, when all models were tested, the average value of the apparent additional mass of the hemispherical bodies were lower than the conical bodies, while Bryan's conical bodies, with a length over diameter of four, produced the lowest apparent additional mass. This may be expected after inspection of the curves shown in Figures 11 and 12 of apparent additional mass versus length over diameter ratios of conical cylinders. At a length over diameter ratio of four the conical ends and hemispherical ends had approximately the same apparent additional mass and apparent additional mass coefficient. Since Bryan's curves cross approximately at a length over diameter ratio of four, any slight discrepancy in the precision of his testing or variations of the surface of the model would account for the difference in apparent additional mass.

5. The Reynolds number, based on length of the body, of the largest model at maximum velocity was approximately 26,000, which is quite low.

6. The wall effect of the largest bodies, for which the wall effect would be the greatest, was found to be negligible. Although the model was only four inches from the wall, the velocity was low enough and the model and walls clean enough so that swinging bodies in a much larger

body of water indicated no change in the apparent additional mass.

7. As shown in the chapter on Similitude[1] the models proved to be both geometrically and kinematically similar and very nearly dynamically similar.

8. The accuracy of the experiments was greatly increased by use of the light system, making it possible to observe very small amplitudes and was also possible to be careful that the amplitude in air and in water were approximately the same. Since the density of aluminum rod stock used to make the models may vary slightly, it was found more accurate to obtain the mass of the displaced fluid by the buoyant force method in lieu of weighing the model, dividing by the average density of the material and multiplying by the density of the water.

9. As shown in the Chapter on Precision of Testing[2], the deviation of the apparent additional mass and apparent additional mass coefficient is within the accuracy of the experiment.

10. It is concluded that the torsional pendulum is an excellent and convenient method of determing the additional masses of various bodies. Obviously, investigations to determine the effects of higher accelerations and velocities on apparent additional mass effect will probably require different apparatus. However, since the experiments

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[1] <u>Cf.</u> <u>ante.</u>, p. 20

[2] <u>Cf.</u> <u>ante.</u>, p. 51

reported here give assurance that scale has no effect on apparent additional mass characteristics, it is possible to use the torsion pendulum for at least the preliminary investigations of projectile, missile, bomb, torpedo and other phases of research where high accelerations and large bodies are experienced.

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APPENDIX

SAMPLE CALCULATIONS

In order to calculate the apparent additional mass and apparent additional mass coefficient it was first necessary to determine the mass moment of inertia of the calibration cylinders. Although the data for any model could be used, model 4-AB-H (see Table I, Physical Characteristics of Models) was choosen to illustrate the method of the calculations.

For the two steel cylinders used to calibrate the torsion pendulum:

M_1 = 291.401 grams	r_1 = 0.4938 inches
M_2 = 292.232 grams	r_2 = 0.4942 inches
M_r = 58.340 grams	a_1 = 2.9986 inches
L = 0.750 feet	a_2 = 2.9932 inches

l = 7.000 inches

From equation (10) in Pertinent Formulae[1]

$$I_K = \frac{M_1}{12}(3r_1^2 + a_1^2) + \frac{M_2}{12}(3r_2^2 + a_2^2) + (M_1 + M_2)\, g\, L^2 + \frac{M_r\, g\, l^2}{3(12)^2}$$

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[1]*Cf. ante.*, p. 55.

Since it was desired to have I_K with the dimensions pounds feet^2, the above equation must be multiplied by the acceleration due to gravity, 32.1366 feet per second^2 in Atlanta, change r_1, r_2, a_1, a_2, change 1 to feet and change grams to pounds.

Therefore

$$I_K = \frac{M_1\ g}{12^3}(3r_1^{\,2} + a_1^{\,2}) + \frac{M_2\ g}{12^3}(3r_2^{\,2} + a_2^{\,2})$$

$$+ (M_1 + M_2)\ g\ L^2 + \frac{M_r\ g\ 1^2}{3(12)^3}$$

$$I_K = \frac{(291.401)(32.1366)}{(1,728)(14,589)} \quad 3(0.4938)^2 + (2.9886)^2$$

$$+ \frac{(292.232)\ (32.1366)}{(1,728)(14,589)} \quad 3(0.4938)^2 + (2.9932)^2$$

$$+ \frac{(291.401 + 292.232)(32.1366)(0.7500)^2}{14,589}$$

$$+ \frac{(58.340)(32.1366)(7.000)^2}{(14,589)(432)}$$

$$I_K = 0.7451 \text{ pounds feet}^2$$

From equation (9) in Pertinent Formulae[2]

$$m_A = \frac{I_K}{2\ T_K^{\,2}\ L^2}\ (T_W^{\,2} - T_a^{\,2} - T_{W'}^{\,2} + T_{a'}^{\,2})$$

and from Precision of Testing the following values were obtained[3]:

T_K = 7.3495 seconds $\qquad$ T_a = 6.4381 seconds

T_W = 6.8506 seconds $\qquad$ $T_{a'}$ = 1.1305 seconds

$T_{W'}$ = 1.2279 seconds $\qquad$ L = 0.6468 feet

therefore the apparent additional mass was calculated to be

$$m_A = \frac{0.7451}{2(7.3495)^2(0.6468)^2} \quad (6.8506)^2 - (6.4381)^2 - (1.2279)^2 + (1.1305)^2$$

$$m_A = 0.08660 \text{ pounds}$$

The apparent additional mass coefficient was calculated from equation (11) in Pertinent Formulae.[4]

$$k = \frac{m_A}{m_{D.F.}} .$$

The mass of the displaced fluid of the model was (0.3492 pounds and k is:

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[2]*Cf. ante.*, p. 55.

[3]*Cf. ante.*, pp. 39-43.

[4]*Cf. ante.*, p. 55.

$$k = \frac{0.08660}{0.3492}$$

$$k = 0.2480$$

All data presented were obtained from similar calculations.

TABLE I

PHYSICAL CHARACTERISTICS OF MODELS

MODEL NUMBER	LENGTH IN.	DIAMETER IN.	L/D	WEIGHT IN AIR GMS.	WEIGHT IN WATER GMS.	BUOYANT FORCE GMS.	DISPLACED MASS LBS.
1-A-B	1.4999	0.3757	3.9902	7.8771	4.9894	2.8877	
1-B-B	1.4989	0.3756	3.9907	7.8980	5.2100	2.6880	0.006164
1-A-C	1.4919	0.3758	3.9699	6.6394	4.3857	2.2537	
1-B-C	1.4944	0.3765	3.9692	6.7074	4.4393	2.2681	0.004984
1-A-H	1.5008	0.3760	3.9941	7.2755	4.8039	2.4716	
1-B-H	1.5010	0.3769	3.9825	7.0856	4.6050	2.4806	0.005459
2-A-B	3.0000	0.7497	4.0016	55.9290	34.3852	21.5438	
2-B-B	2.9995	0.7493	4.0030	55.9026	34.3651	21.5375	0.04788
2-A-C	3.0037	0.7497	4.0113	46.1090	28.0787	18.0303	
2-B-C	3.0070	0.7493	4.0065	46.1259	28.0868	18.0391	0.03976
2-A-H	2.9959	0.7497	3.9959	50.7126	31.0398	19.6728	
2-B-H	3.0021	0.7493	4.0130	50.6421	30.9703	19.6718	0.04337

TABLE I

PHYSICAL CHARACTERISTICS OF MODELS

MODEL NUMBER	LENGTH IN.	DIAMETER IN.	L/D	WEIGHT IN AIR GMS.	WEIGHT IN WATER GMS.	BUOYANT FORCE GMS.	DISPLACED MASS LBS.
3-A-B	4.5034	1.1268	3.9966	154.1881	81.4143	72.7738	
3-B-B	4.5050	1.1268	3.9980	154.0596	80.8658	73.1938	0.1609
3-A-C	4.4975	1.1268	3.9914	119.9511	59.5348	60.4163	
3-B-C	4.5000	1.1268	3.9936	120.2311	58.9879	61.2432	0.1341
3-A-H	4.5007	1.1268	3.9942	136.3069	69.3927	66.9142	
3-B-H	4.5037	1.1268	3.9969	136.6404	69.8416	66.7988	0.1474
4-A-B	5.9897	1.5002	3.9926	320.8997	148.6506	172.2491	
4-B-B	6.0224	1.5015	4.0109	325.5701	152.8161	172.7540	0.3803
4-A-C	6.0001	1.5002	3.9995	240.8775	96.4545	144.4230	
4-B-C	6.0337	1.5015	4.0184	245.9561	100.9300	145.0261	0.3191
4-A-H	6.0101	1.5002	4.0062	281.0562	122.6750	158.3912	
4-B-H	6.0277	1.5015	4.0145	282.2646	122.9955	159.2691	0.3492

Explanation of Table I.

1. The first number is the size of the model-- "1" being the smallest and "4" the largest.
2. The second letter indicates one of a pair of models.
3. The last letter designates the type of end cap used; "B" for blunt, "C" for conical and "H" for hemispherical.

Example:

3-A-H means the third model in the series, "A" model of a pair, and fitted with hemispherical ends.

Note:

The displaced mass shown in the table was the average of both models in the pair.

TABLE II

EFFECT OF SCALE ON APPARENT ADDITIONAL MASSES AND COEFFICIENTS OF BLUNT END CYLINDRICAL BODIES MOVING LONGITUDINALLY IN WATER

MODEL NUMBER	$m_{D.F.}$ LBS.	SCALE $\frac{m_{D.F.}}{m_{D.F.} \text{ of } 1\text{-AB-B}}$	m_A LBS.	k
1-AB-B	0.006146	1.00	0.002250	0.3661
2-AB-B	0.04788	9.62	0.01595	0.3331
3-AB-B	0.1609	26.20	0.05107	0.3174
4-AB-B	0.3492	61.90	0.1275	0.3353

TABLE III

EFFECT OF SCALE ON APPARENT ADDITIONAL MASSES AND COEFFICENTS OF CONICAL END CYLINDRICAL BODIES MOVING LONGITUDINALLY IN WATER

MODEL NUMBER	$m_{D.F.}$ LBS.	SCALE $\frac{m_{D.F.}}{m_{D.F.} \text{ of 1-AB-C}}$	m_A	k
1-AB-C	0.004984	1.00	0.001936	0.3884
2-AB-C	0.03976	6.47	0.01506	0.2654
3-AB-C	0.1341	27.00	0.03680	0.2745
4-AB-C	0.3191	64.10	0.08608	0.2698

TABLE IV

EFFECT OF SCALE ON APPARENT ADDITIONAL MASSES AND COEFFICIENTS OF HEMISPHERICAL END CYLINDRICAL BODIES MOVING LONGITUDINALLY IN WATER

MODEL NUMBER	$m_{D.F.}$ LBS.	SCALE $\frac{m_{D.F.}}{m_{D.F.} \text{ of } 1\text{-AB-H}}$	m_A LBS.	k
1-AB-H	0.005459	1.00	0.001456	0.2668
2-AB-H	0.04337	7.94	0.01128	0.2602
3-AB-H	0.1474	27.00	0.03620	0.2456
4-AB-H	0.3492	63.90	0.08660	0.2480

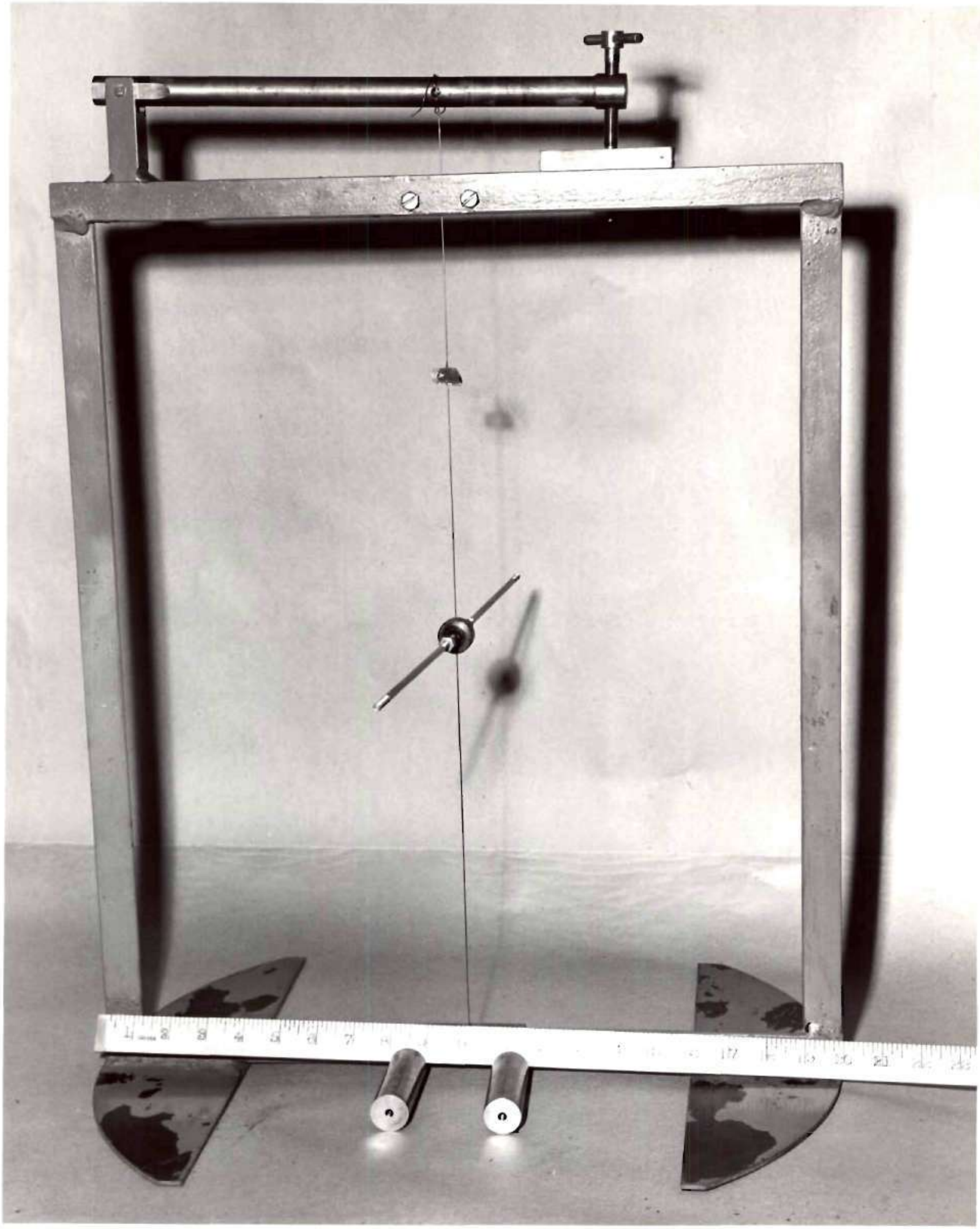


FIGURE 1

TORSION PENDULUM WITH STEEL CALIBRATION CYLINDERS

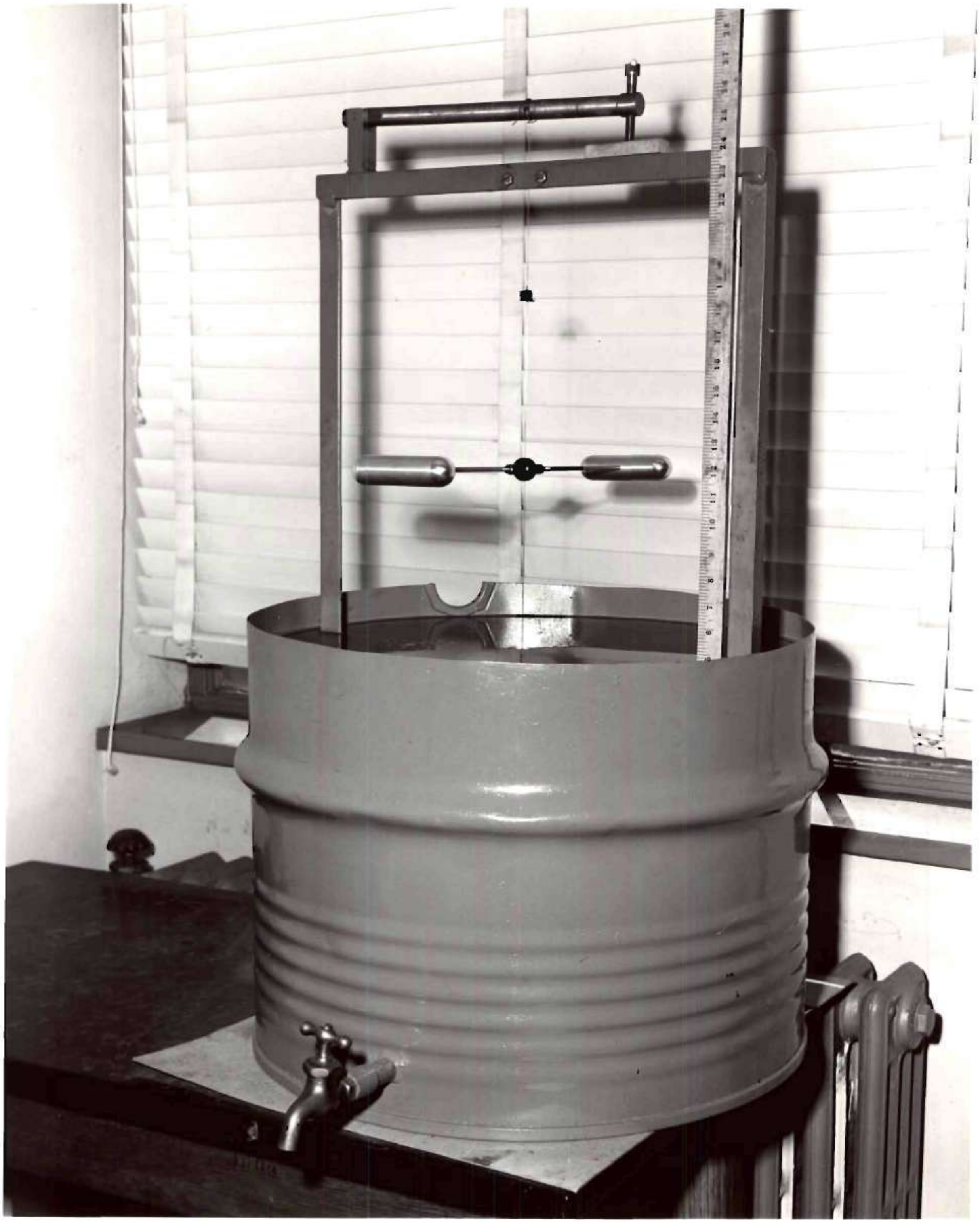


FIGURE 2

TORSION PENDULUM PARTIALLY SUBMERGED IN TANK

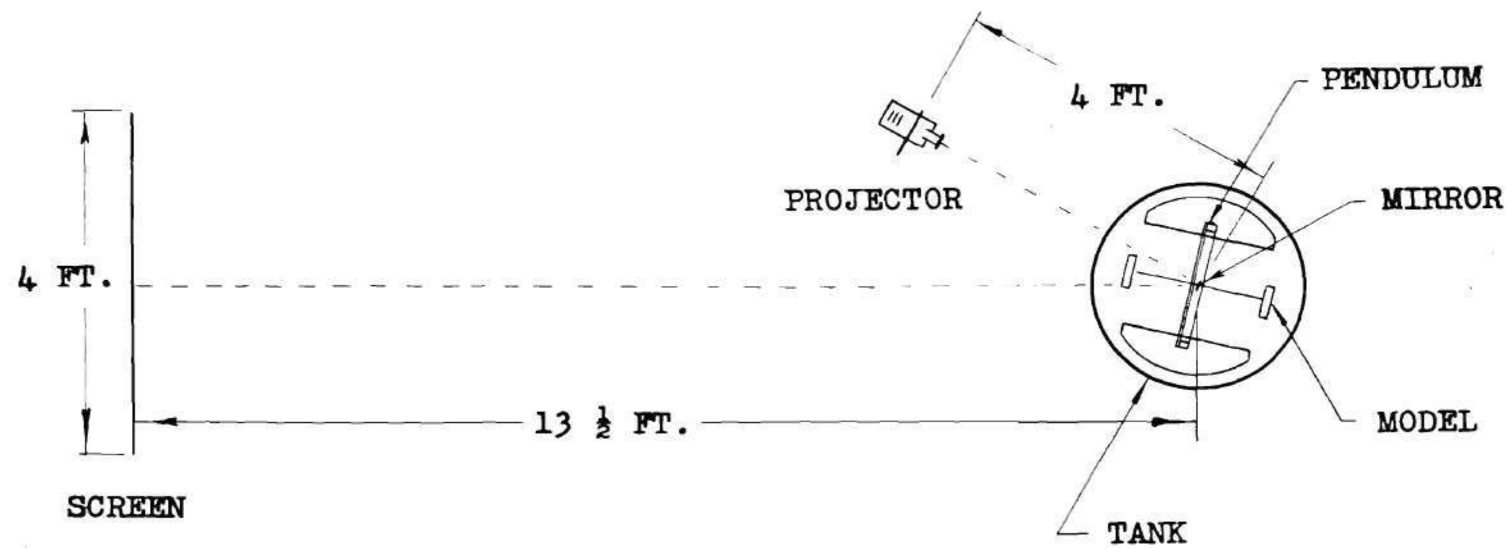


FIGURE 3

SCHEMATIC DIAGRAM OF EXPERIMENTAL EQUIPMENT

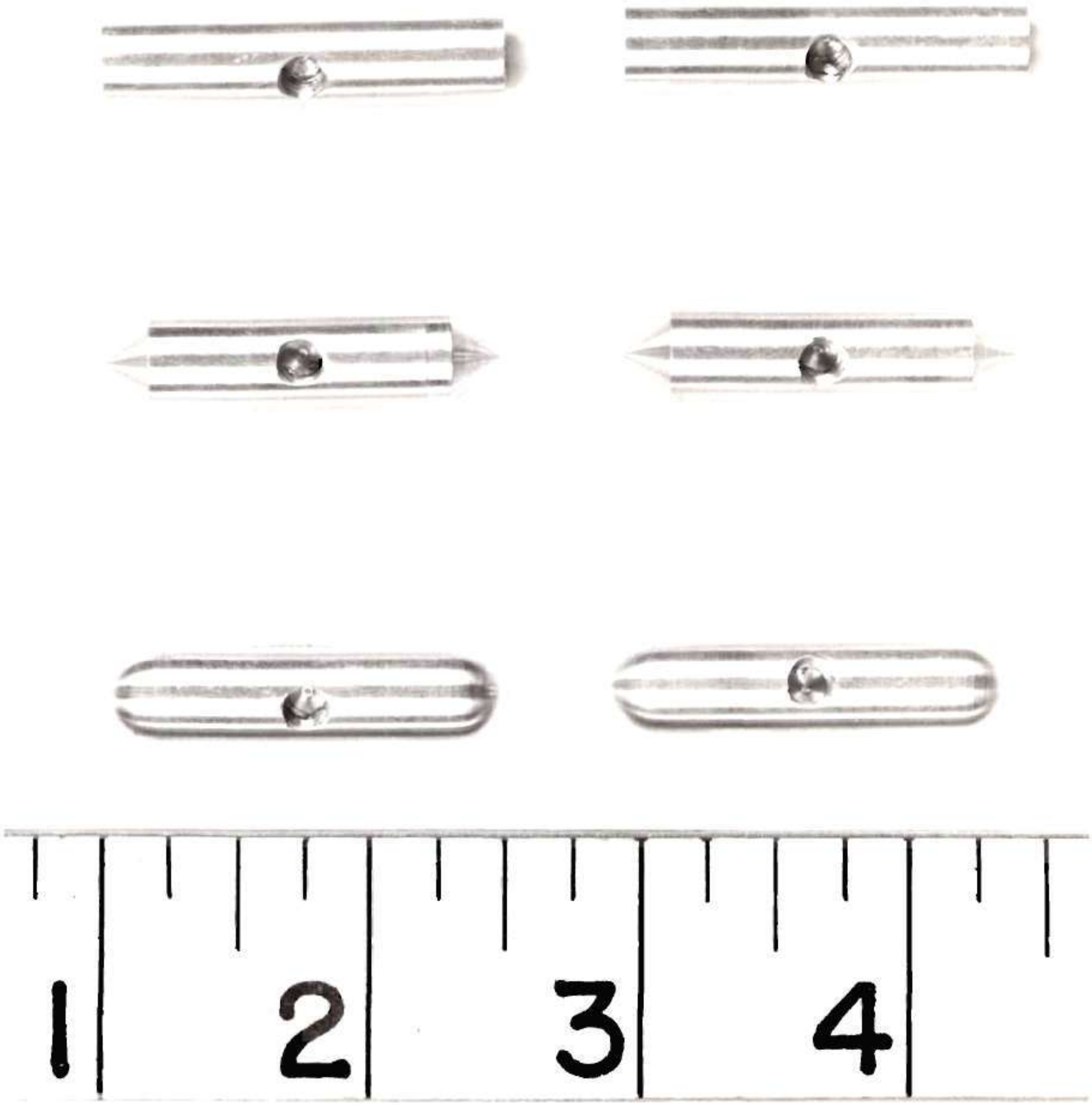


FIGURE 4

SET OF THE 3/8 x 1½ INCH MODELS

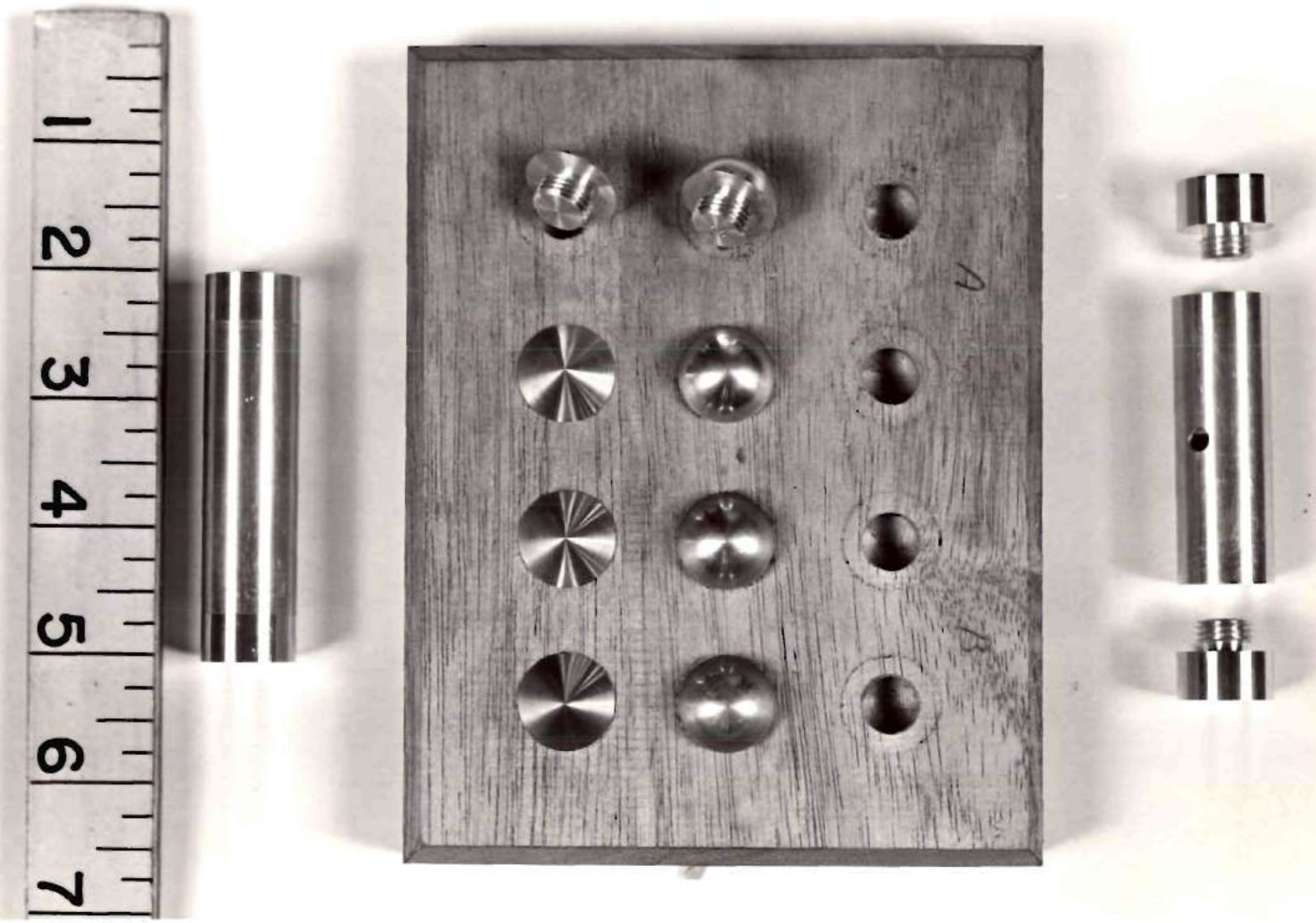


FIGURE 5

SET OF THE 3/4 x 3 INCH MODELS

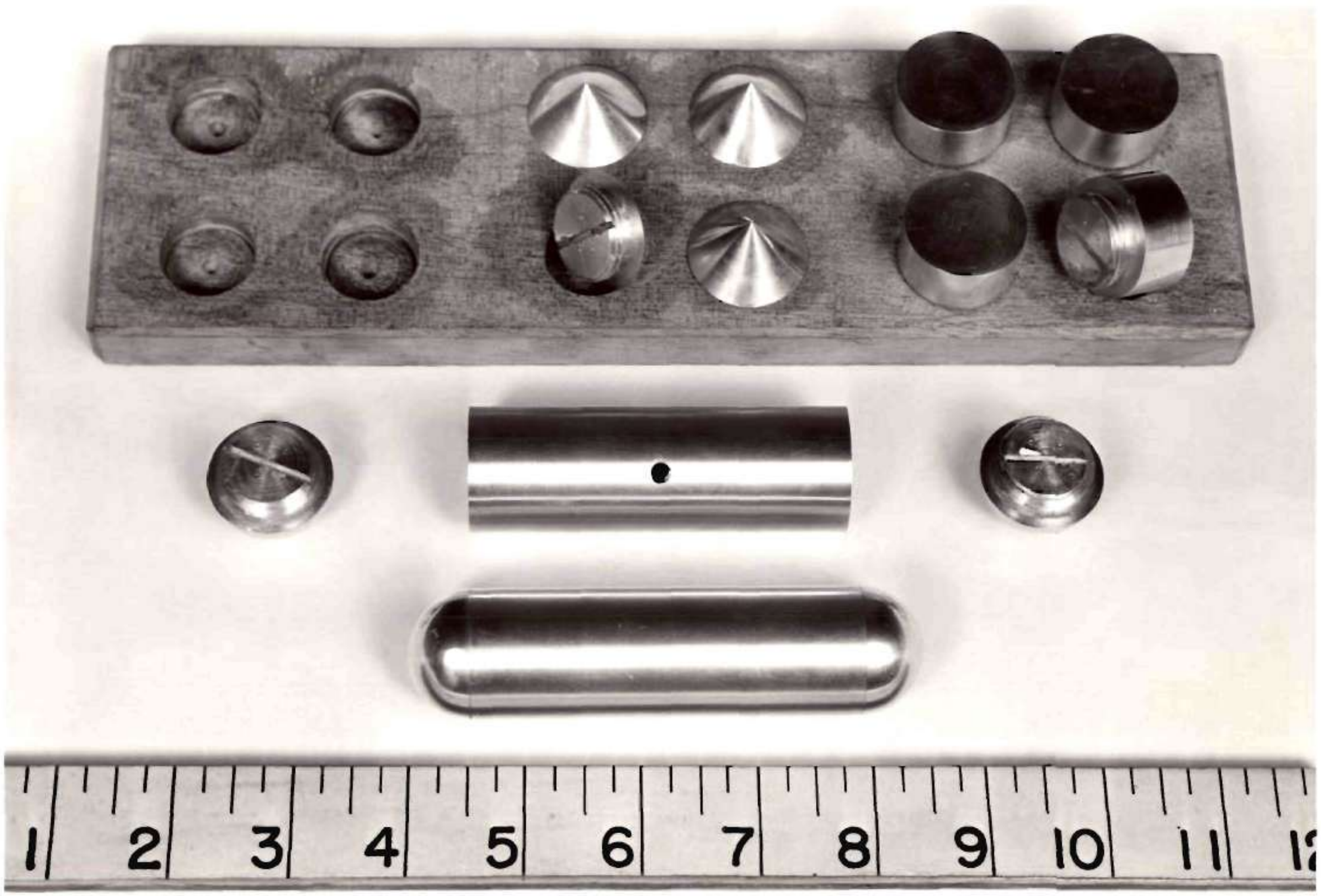


FIGURE 6

SET OF THE 1 1/8 x 4 1/2 INCH MODELS

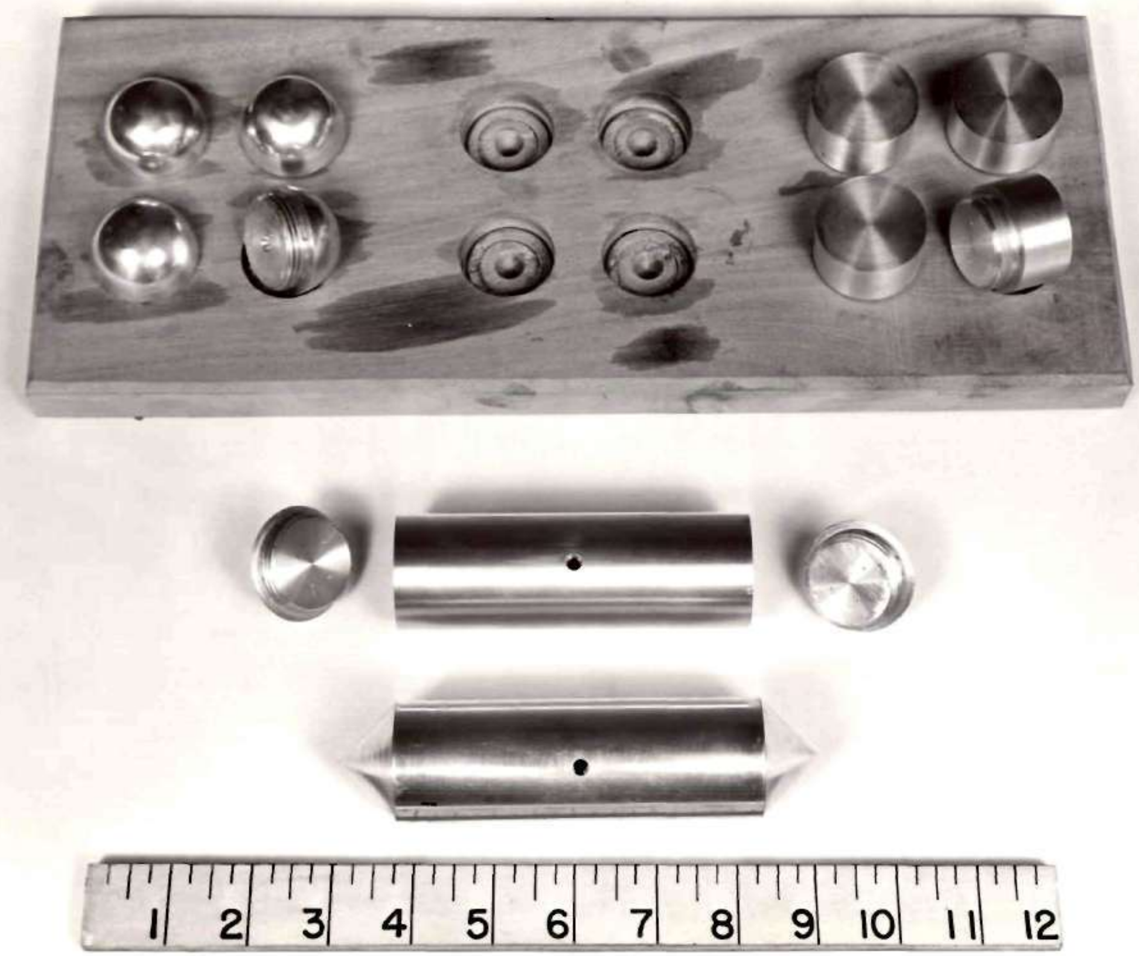


FIGURE 7

SET OF THE $1\frac{1}{2} \times 6$ INCH MODELS

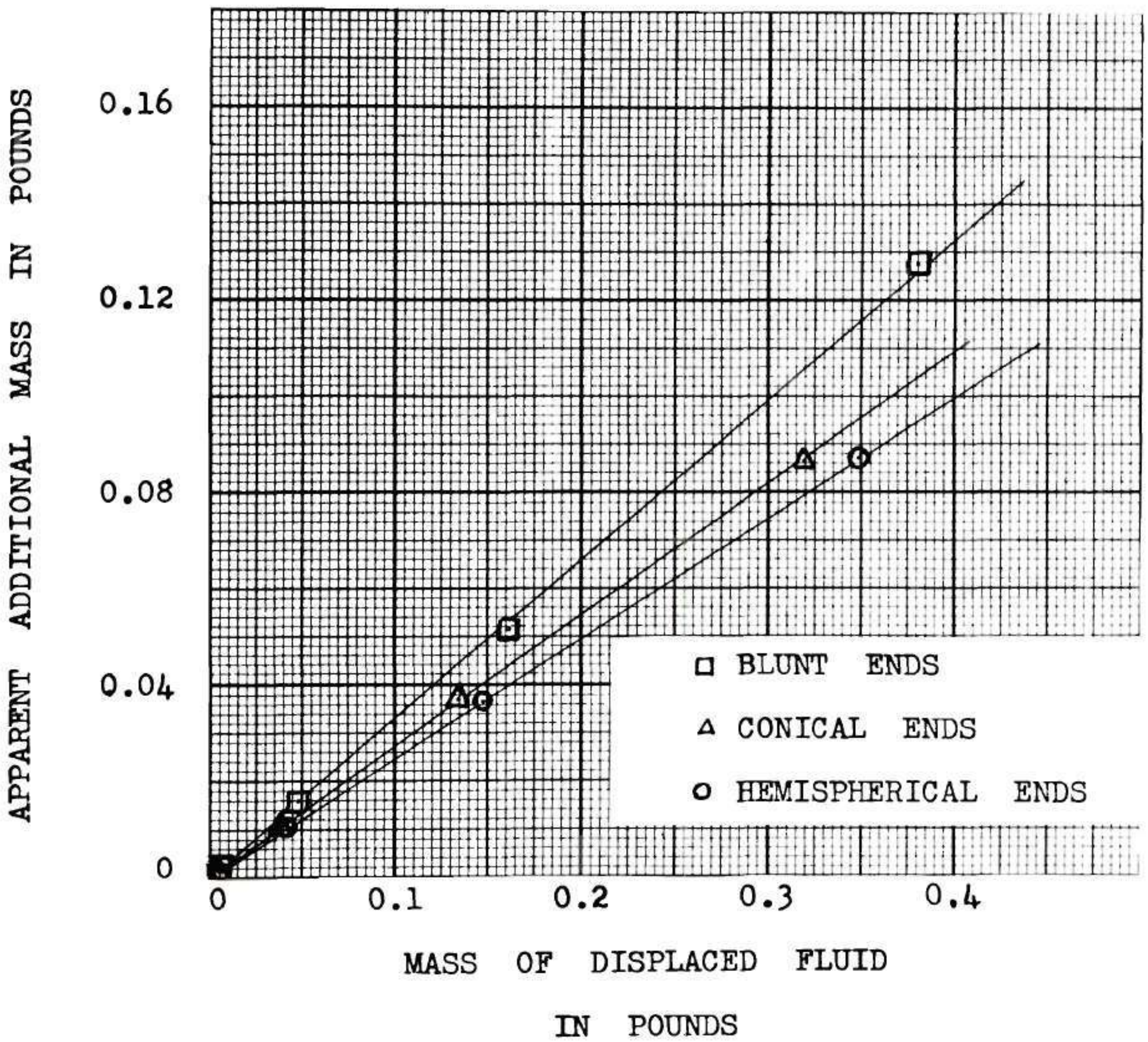


FIGURE 8

VARIATION OF APPARENT ADDITIONAL MASS WITH MASS OF DISPLACED FLUID

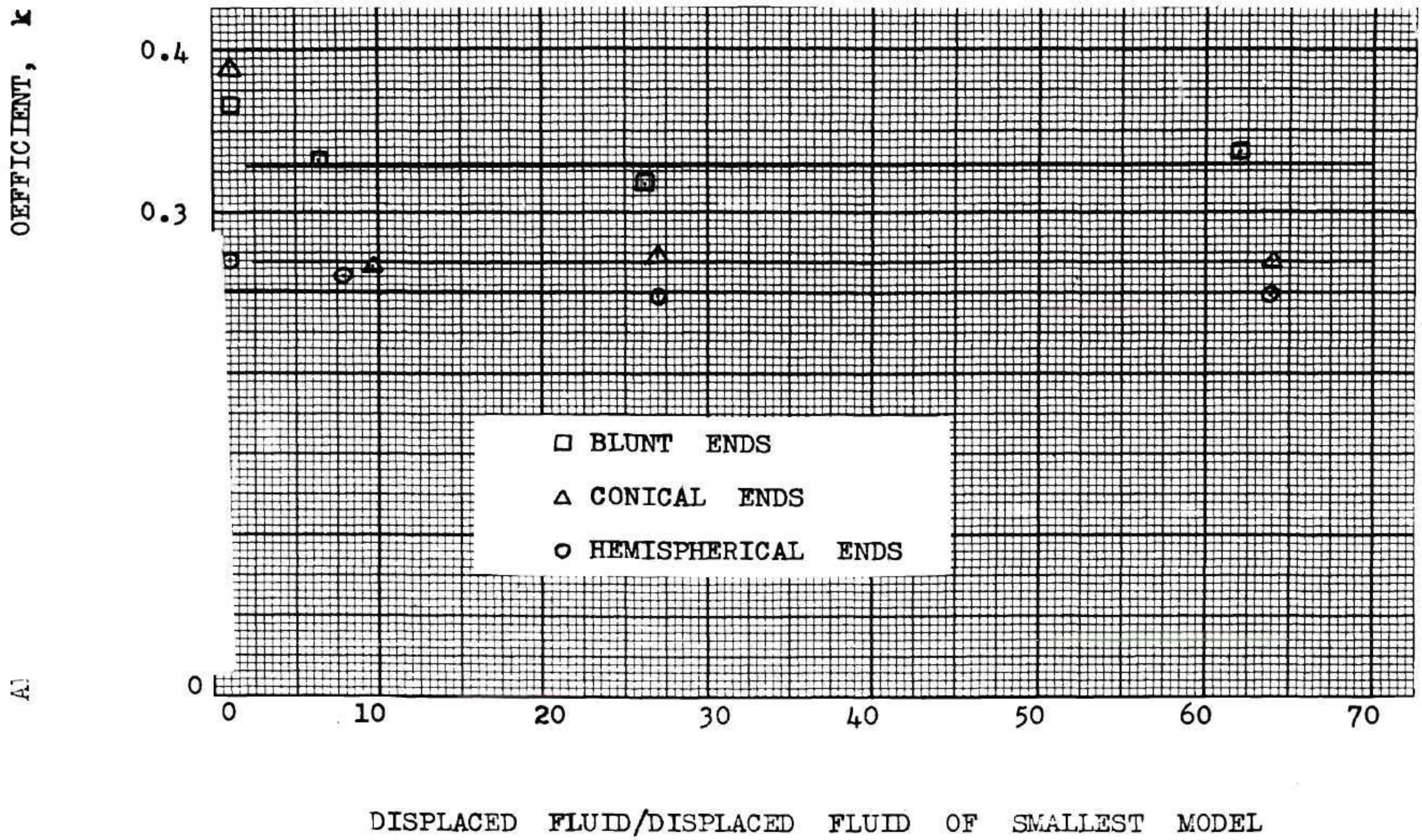
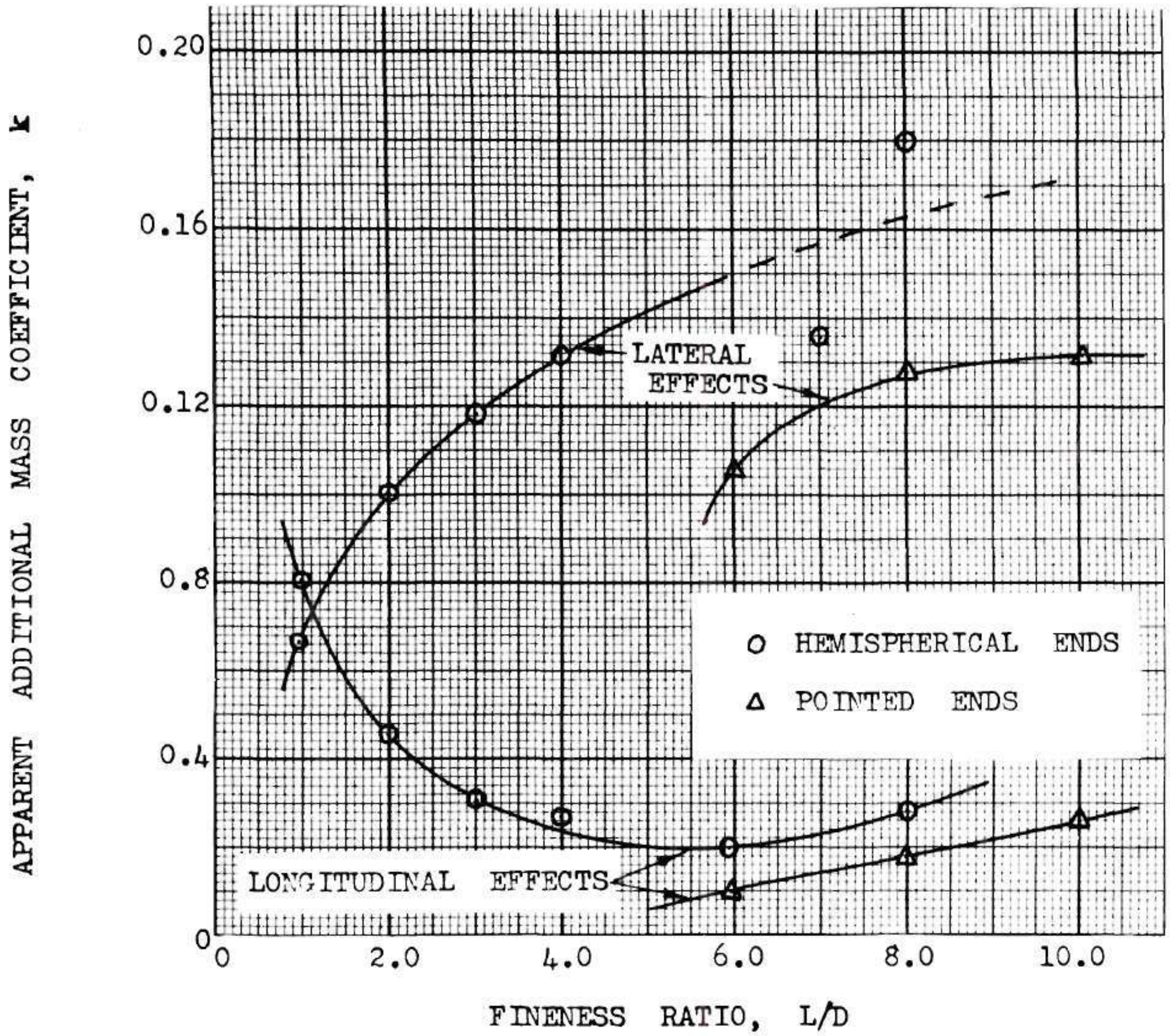


FIGURE 9

VARIATION OF APPARENT ADDITIONAL MASS COEFFICIENT WITH THE RATIO, DISPLACED FLUID/DISPLACED FLUID OF SMALLEST MODEL



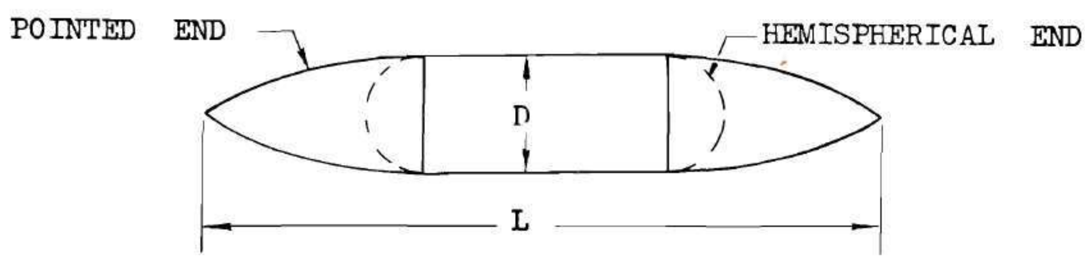


FIGURE 10

EFFECT OF FINENESS RATIO ON APPARENT ADDITIONAL MASS COEFFICIENTS FROM DATA PRESENTED BY RELF AND JONES (REFERENCE 2)

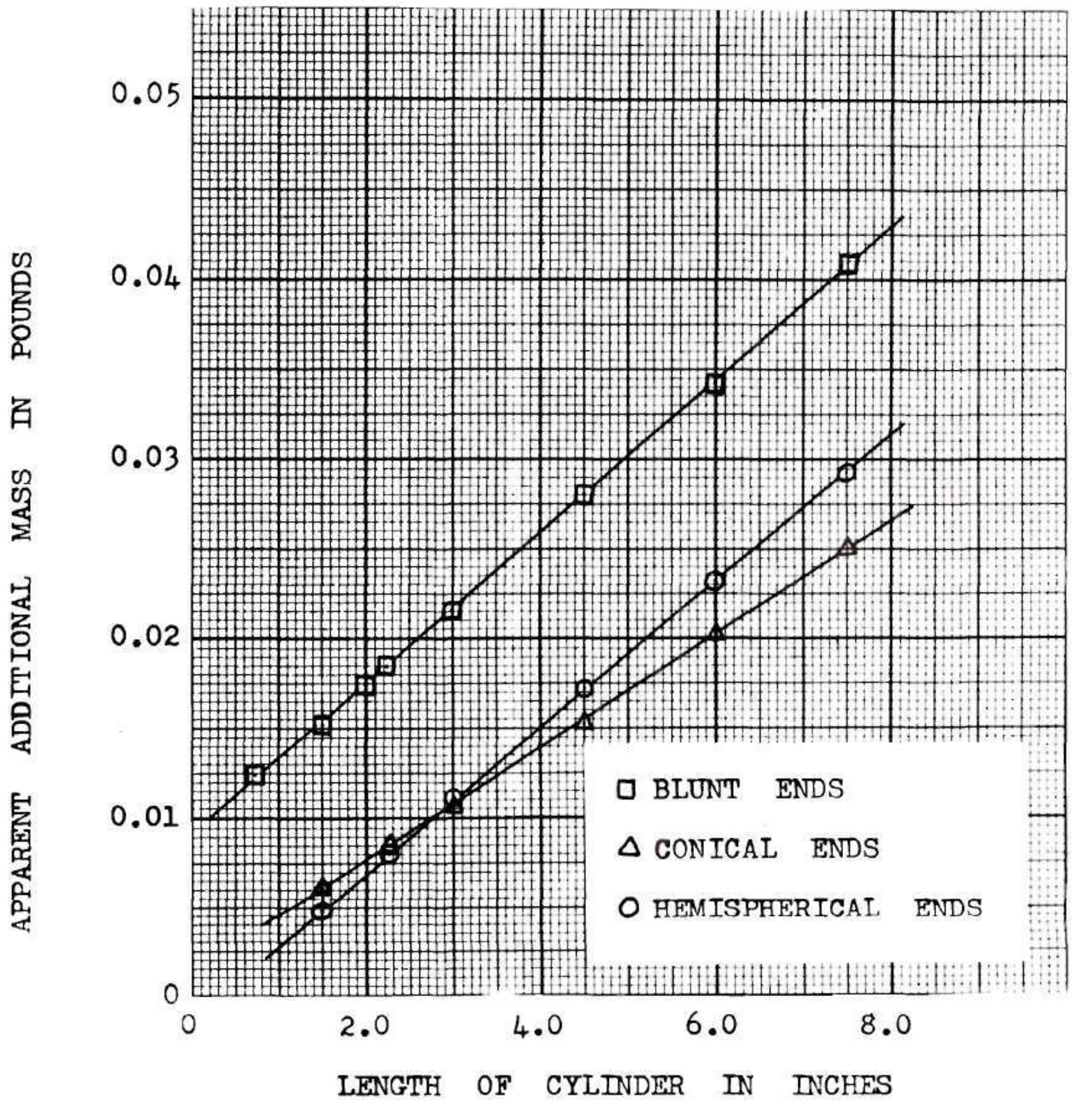


FIGURE 11

VARIATION OF APPARENT ADDITIONAL MASS WITH LENGTH FOR CYLINDERS MOVING PARALLEL TO THEIR LONGITUDINAL AXES (REFERENCE 3)

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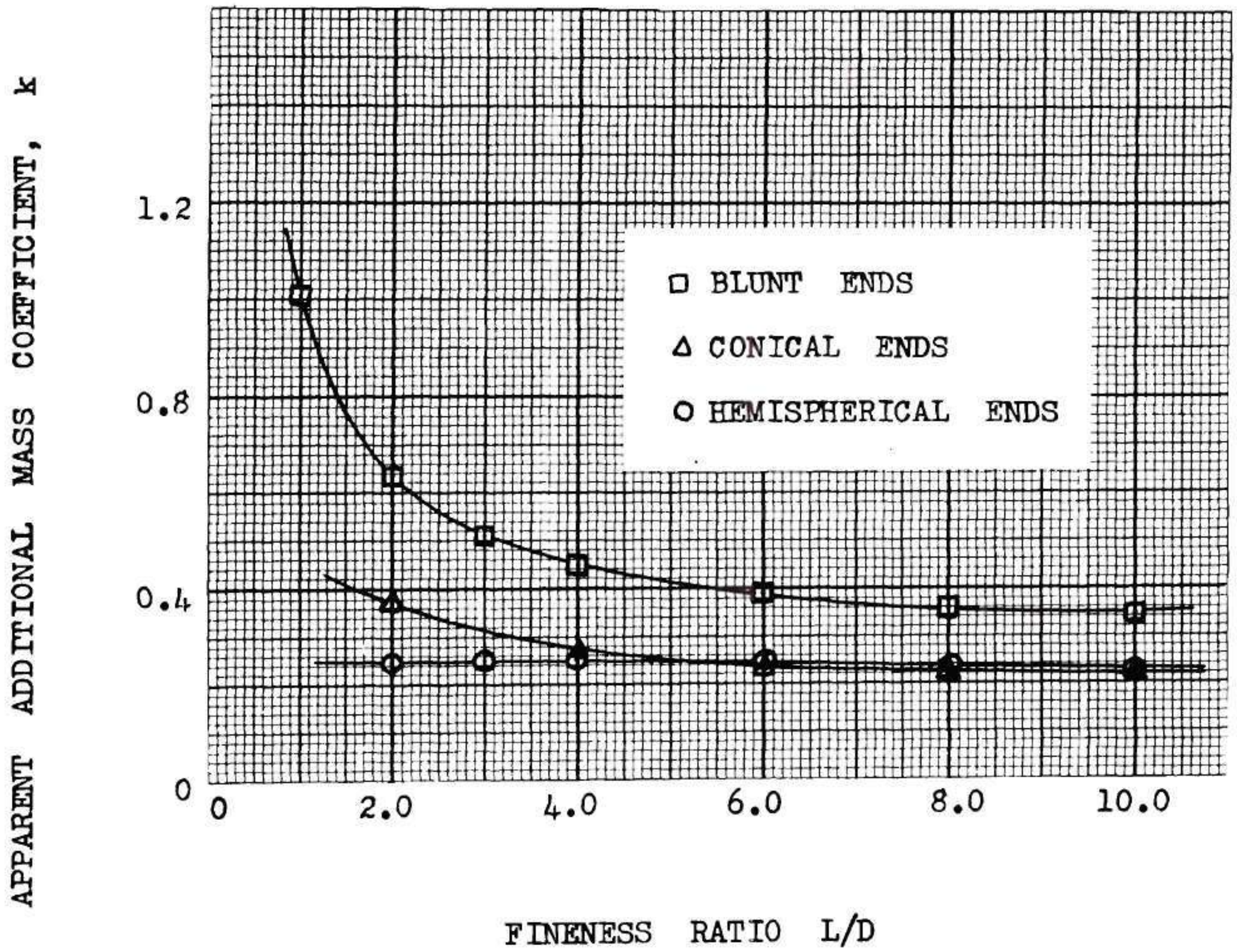

FIGURE 12

VARIATION OF APPARENT ADDITIONAL MASS COEFFICIENT WITH FINENESS RATIO FOR CYLINDERS MOVING PARALLEL TO THEIR LONGITUDINAL AXES (REFERENCE 3)