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AN ENERGY RELAXATION PRINCIPLE APPLIED TO THE ANALYSIS OF A DIELECTRICALLY COATED SLOT ANTENNA

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SUMMARY

Variational formulas have been developed and applied to the analysis of complicated electromagnetic structures where conventional techniques are difficult or impossible to utilize. Although these variational approaches make difficult problems tractable, they usually give rise to a system of nonlinear equations describing the structure. These nonlinear equations become increasingly difficult to solve as a more and more accurate analysis is attempted. To retain the advantages of a variational approach, yet avoiding the difficulties of nonlinear equations, a new variational principle which produces linear equations is presented in this dissertation. Solving the linear equations obtained from this new approach will require less work than solving the nonlinear equations from other approaches to the same problem.

The new approach is used to analyze a general class of antenna problems, and the detailed analysis for a particular antenna, namely a coated rectangular waveguide slot, is performed. The rectangular slot is in a ground plane that is transverse to the axis of the waveguide. The general equations presented in this thesis apply to plasma as well as dielectric coverings, to lossy as well as lossless coverings, and to loaded as well as unloaded waveguides. Thus, they apply to a wide variety of important antenna problems. The application of the general equations to any of these configurations would only require a change in integration scheme.

Numerical calculations are performed on four X-band slot antennas, each having an infinitely wide ground plane and a lossless dielectric covering. The calculated admittance and far field pattern of each antenna is compared with experimental measurements on similar antennas with finite ground planes. The success of the experimental verification indicates that the new procedure is practical and has wide applicability. One, three, and ten mode calculations are performed for each antenna. The test cases considered show that usually a minimum of three modes is necessary for accurate admittance predictions.

The predicted patterns are smooth curves which decrease monotonically from the maximum value, which is in a direction normal to the aperture. The measured patterns have ripples superimposed on the predicted curves. The predicted patterns tend to give the average value of these ripples. On the basis of the test cases considered, a one mode analysis seems adequate for pattern predictions.

CHAPTER I

INTRODUCTION

Solution of Electromagnetic Problems by Variational Techniques

The classical method of analyzing any electromagnetic problem is by solving Maxwell's equations subject to the boundary conditions of the system. Although such an approach is conceptually simple for any geometry, it is only for elementary configurations--those formed by coordinate system surfaces--that the mathematical analysis is also relatively easy. For such configurations it is usually possible to find a simple solution of Maxwell's equations that satisfies all of the boundary conditions. However, as the geometry becomes more complicated, the mathematical analysis also becomes more difficult. For such geometries, a simple solution usually cannot be found, and a Fourier type series approach must be used in which the fields are expanded in terms of a complete set of functions with proper weightings. Typical examples of such expansion functions are sinusoidal and hyperbolic functions in the solution of Laplace's equation in rectangular coordinates, and spherical harmonics in the solution of the wave equation in spherical coordinates.

This expansion technique also becomes very difficult to apply for the even more complicated geometries which often arise in practice. For such configurations, the variational approach is still analytically tractable. Collin (1) and Harrington (2) give good accounts of this approach, which has been extensively applied to practical electromagnetic problems during the past 30 years.

The variational approach converts a field theory problem into a calculus of variations problem by showing that the true field which exists in a system is the one that makes some particular integral stationary. A stationary formula is one that is relatively insensitive to variations in an assumed field about the correct field. The advantage of such an approach is that approximations to field quantities, as for example propagation constant and input impedance, can be obtained relatively rapidly and with much less work than is required by conventional techniques. In the variational approach a trial function containing several adjustable parameters is used to approximate the true field. By adjusting these parameters so that the integral in question is stationary, the best possible approximation to the field is obtained from among the class of functions being considered. Inclusion of more trial functions produces more accurate results but increases the effort needed to solve the resulting equations.

Variational principles have been associated with Maxwell's equations for some time. The initial principles, however, had theoretical rather than practical value. In 1900 Larmor (3) showed that the difference between the stored magnetic and electric energy densities possessed a stationary property analogous to similar expressions for mechanical systems. Henschke (4) in 1913 showed that Maxwell's equations could be derived from a particular energy function. After Henschke, most authors interested in stationary formulas for electromagnetic problems concentrated on deriving Maxwell's equations from various energy functions rather than on solving problems associated with particular geometries.

It was not until the 1940's that variational formulas found wide acceptance in the solution of practical problems. The variational method introduced by Schwinger (5) permits the handling of a large variety of problems which were very difficult, if not impossible, to solve by conventional techniques. Stationary formulas for discontinuities in waveguides and for the resonant frequency of cavities began to appear in the literature. Later, scattering problems and antenna problems were formulated in terms of variational expressions.

This dissertation presents a new variational formula having the singular characteristic of giving rise to linear algebraic equations. This characteristic is quite significant because comparable variational approaches produce nonlinear equations which rapidly become unmanageable as the number of adjustable parameters is increased. The linear equations resulting from this new approach, however, are still manageable as the number of parameters is increased, thus making possible a more precise analysis of a broad class of problems. The new approach is used to analyze a general class of antenna problems. The detailed analysis for a particular antenna, namely a waveguide slot, is demonstrated, and the experimental verification of the theoretical results is provided.

Background of the Antenna Problem

Many antennas being used today have dielectric coverings over them. Antennas under radomes and antennas under heat shields on space vehicles are two such examples. The dielectric covering is usually provided to protect the antenna from the external environment, but it also influences the electrical behavior of the antenna.

In the case of space vehicles which must travel through the earth's atmosphere, a heat shield is placed over the vehicle to protect it and its antennas from the re-entry heat. As the vehicle re-enters the atmosphere, the heat shield is ablated away, causing the thickness and the dielectric constant of the coating to change. It is important to know what influence the change in dielectric constant and the change in dielectric thickness will have upon the performance of the communication system connected to the antenna. In particular, it is necessary to know what sort of input impedance variations these changes produce in order to design matching networks for the transmitter. In addition, any change in the radiation pattern must be known in order to predict the performance of the communication system.

Because of aerodynamic considerations, a common choice of antenna for re-entry vehicles is one that can be mounted flush with the surface of the vehicle. Representative of this group is a waveguide slot antenna which uses the surface of the space vehicle as a ground plane. Such an antenna provides a wide radiation pattern giving good coverage even if the vehicle rotates somewhat. It is this antenna configuration which motivates the particular problem to be studied in this dissertation. The antenna will be analyzed using the new variational principle.

Before the advent of variational techniques, the field distribution in an antenna's aperture had to be assumed instead of being analytically calculated. For example, Silver (6) almost always assumes the form of the aperture distribution, even though it is known that these assumptions are incorrect and produce errors. However, the difficulty encountered in attempting to derive the true distribution usually prevents an exact analytic approach. Even in some recent studies (7) of slot antennas and coated slot antennas, the aperture distribution is still assumed instead of being analytically calculated using existing variational techniques. The predictions from such approaches are open to question since the assumed aperture distribution is not exact.

Variational Approaches for Slot Antennas

Several variational formulas have been developed for slot antenna problems. All of these formulas give rise to nonlinear equations when the trial field in the aperture is expanded, using more than one waveguide mode function.

Lewis (8) in 1951 presented a stationary formula for the input admittance of an open-ended rectangular waveguide with an infinite flange (ground plane) having no dielectric coating. His formula is in terms of the aperture distribution of the antenna which can have an arbitrary form. For numerical calculations, however, he assumes that only the dominant mode is present. This aperture distribution simplifies his nonlinear equation to a linear one.

Galejs (9-11) has applied Lewin's technique to plasma-covered slot antennas. His equations are nonlinear, as are Lewin's. He uses a two-term trial function, having only one adjustable parameter, for the aperture distribution. His results indicate (12) that the aperture distribution can differ significantly from that of the dominant mode alone.

In 1951 Cohen et al. (13) developed a stationary formula for a dielectrically-loaded rectangular waveguide radiating into half-space. The development was based on Schwinger's (14) approach, and the resulting variational formula was nonlinear. A dominant mode approximation was made to the aperture field.

Villeneuve (15) applied Rumsey's reaction concept (16) to slot antennas and also obtained nonlinear equations. The results he presents assume that the dominant mode alone is present.

Compton (17) in 1964 developed a variational formula similar to Lewin's for a rectangular waveguide radiating through a dielectric slab. This work was corrected by Croswell et al. (18) in 1967 to account for surface waves that Compton had neglected. Both Compton and Croswell assumed that only the dominant mode was present. Croswell made experimental measurements which show that the aperture distribution can differ markedly from dominant mode. Croswell (19) has recently extended his earlier work to include plasma, as well as dielectric, coverings. This new study uses a two mode instead of just a single mode trial field.

A single mode analysis of a dielectric coating on a circular waveguide has recently been made by Bailey and Swift (20). For a single mode trial field they show that the input admittance of the antenna can be expressed as a single integral in the circular waveguide case instead of a double integral, as in the rectangular waveguide case. Thus, the computation is simpler for the circular waveguide.

Purpose of Research

In contrast to the above approaches, this dissertation presents a variational formula which produces a system of linear instead of non-linear equations. This new variational principle is presented in Chapter II.

The resulting simplification produced by these linear equations permits a multimode instead of just a one or two mode analysis to be made of a dielectrically-coated slot antenna. This analysis is presented in Chapters III, IV, and V and is based on the variational principle of Chapter II.

Next, the far field of the slot antenna is determined using the multimode analysis of Chapters III through V. This result is presented in Chapter VI.

Finally, in Chapter VII an experimental verification is made of the analysis presented in Chapters III through VI.

CHAPTER II

THE VARIATIONAL PRINCIPLE

Notation

In this chapter a new variational formula will be presented, and a proof of its stationary character will be given. This formula applies to the general antenna configuration shown in Figures 1, 2, and 3. The antenna consists of an irregularly shaped, perfectly conducting feed structure terminating on an infinitely large, perfectly conducting sheet. The feed structure in Figure 1 may, for example, be a rectangular, circular, or elliptical cross section waveguide. The proof for a coaxial type feed structure can be handled in the same manner as used for this feed arrangement. An aperture composed of one or more irregularly shaped holes, as shown in Figure 2, is cut in the conducting sheet to let energy out of the feed region.

Covering the sheet is a series of linear, isotropic, and homogeneous slabs, each of which extends radially outward in a transverse plane to $\pm\infty$. In each region V_i $(i = 1,2,\ldots,M)$ the electrical parameters μ_i, ε_i, and σ_i are all considered to be scalar constants. The conductivity, σ_i, may or may not be zero and μ_i and ε_i can be greater than, less than, or equal to their free space values. By assumption, there is no free current or free charge anywhere. Conduction current, however, may be present.

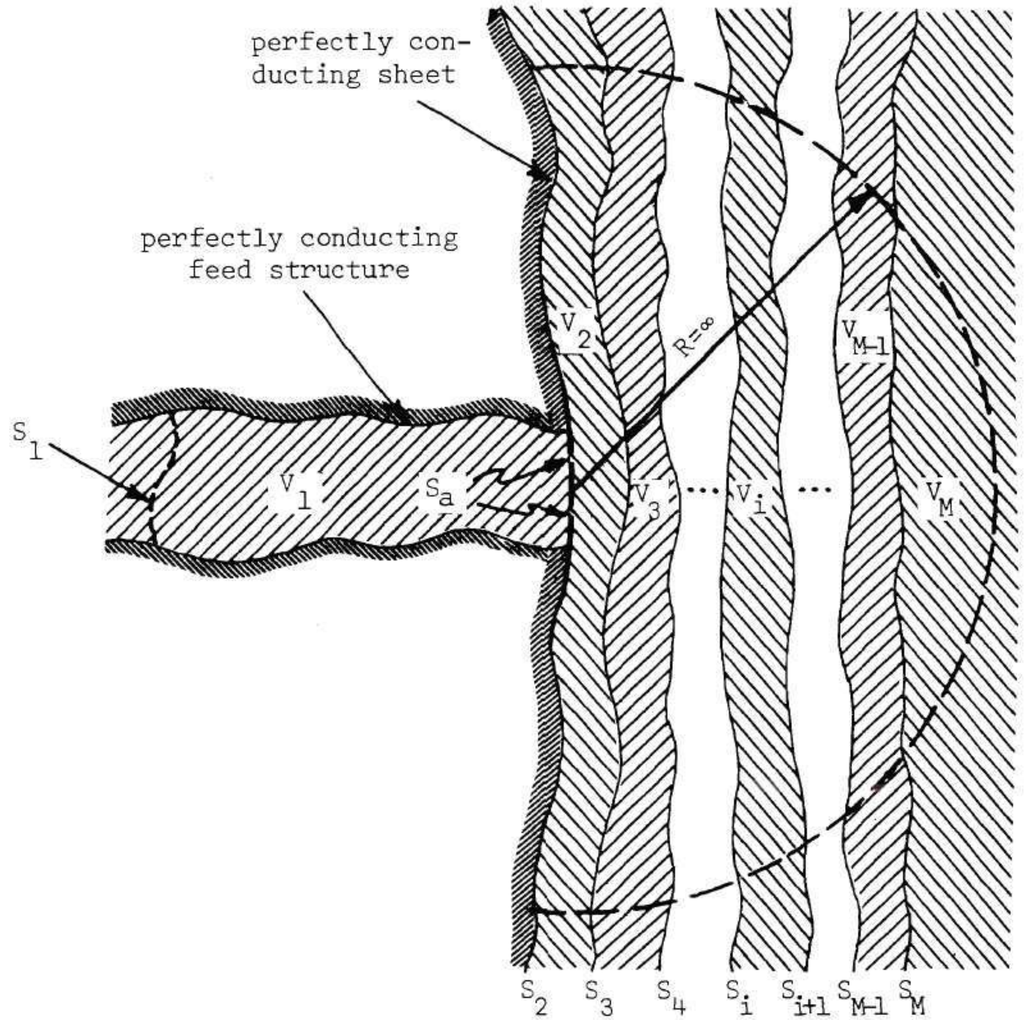

Figure 1. Side View of Antenna

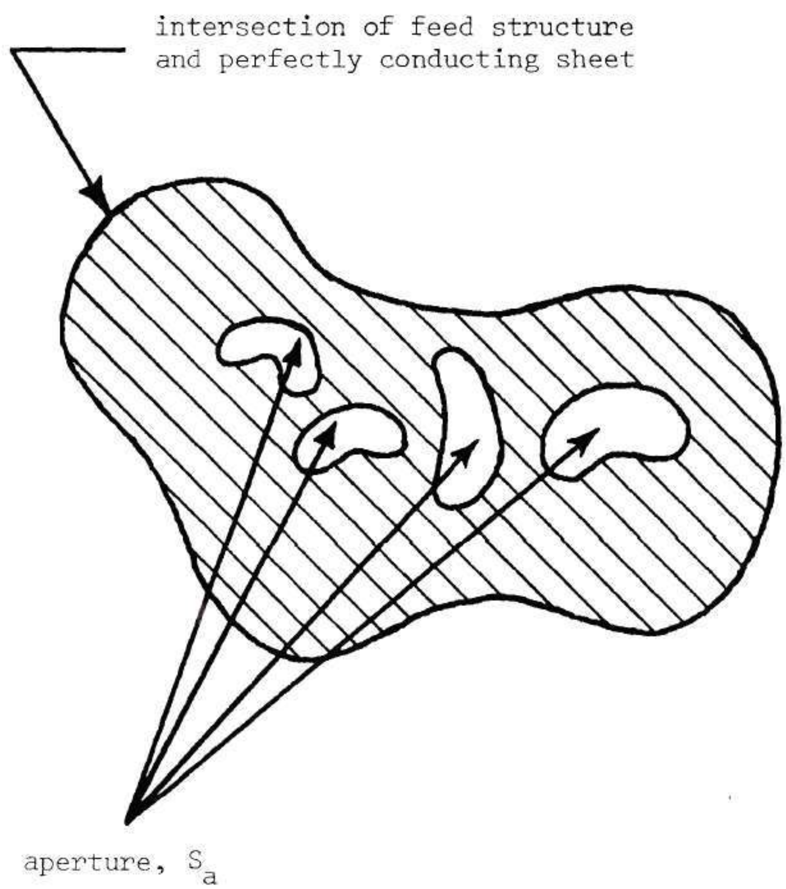

Figure 2. Front View of Antenna

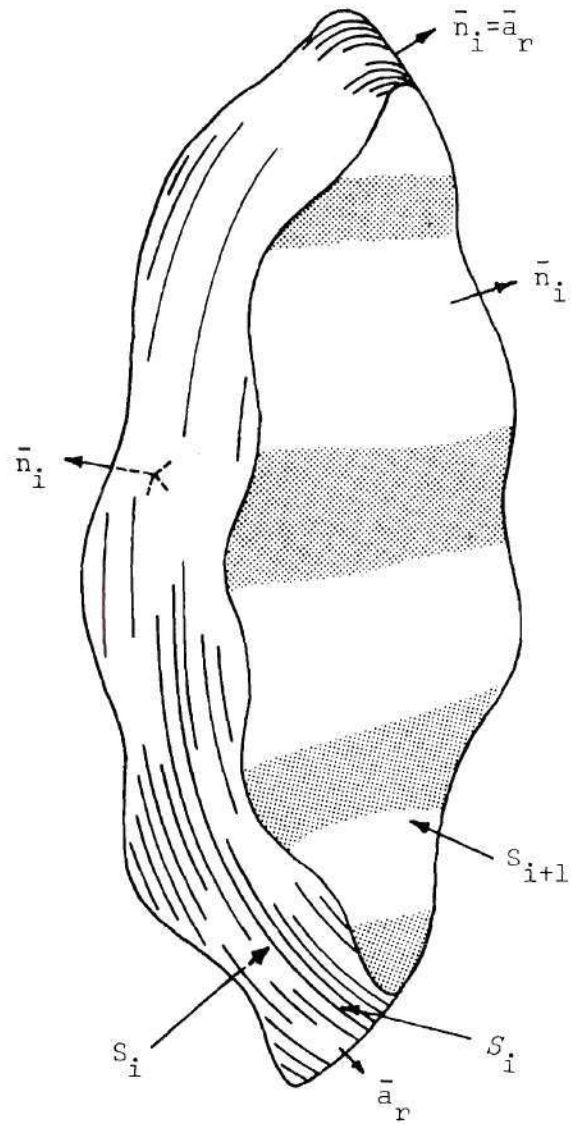

Figure 3. A Typical Volume V_i for $i = 2,3,\ldots,M-1$. The Bounding Surface of V_i is Σ_i, Which Consists of S_i, S_{i+1} and $\mathcal{S}_i$

In Figure 3, Σ_i $(i = 1,2,\ldots,M)$ is defined as a closed surface bounding the volume V_i, and $\bar{n}_i$ is a unit outward normal to Σ_i. In particular, the first closed surface Σ_1 is composed of the open surfaces S_1, S_a, the perfectly conducting feed boundary, and that portion of the perfectly conducting sheet which covers the waveguide-like feed. For theoretical purposes S_1 may be arbitrarily placed relatively to S_a. For practical applications, however, S_1 is placed many feed-structure wavelengths from S_a to facilitate constraining the trial electric field over S_1.

For $i = 2,3,\ldots,M-1$, Σ_i is composed of the open surfaces S_i, S_{i+1}, and $\mathcal{S}_i$, as shown in Figure 3. The closed surface Σ_M is composed of surfaces S_M and $\mathcal{S}_M$. For convenience, a hemispherically shaped surface of radius R is drawn about some point on S_a. The surface $\mathcal{S}_i$ $(i = 2,3,\ldots,M-1)$ is the ribbon-like surface sliced off the hemisphere by the boundaries S_i and S_{i+1} of V_i. Surface $\mathcal{S}_M$ is that portion of the hemispherical surface which lies in V_M. The unit vector $\bar{a}_r$ points in the radially outward direction and is normal to each $\mathcal{S}_i$.

The radius R in Figure 1 is initially chosen to be finite. Later, R is made very large so that each $\mathcal{S}_i$ is in the far field of the antenna, where the radiation condition (21) applies.

In each region V_i $(i = 1,2,\ldots,M)$, $\bar{E}_i$ and $\bar{H}_i$ denote, respectively, the trial electric field and the trial magnetic field which approximate the true electric field $\bar{E}$, and the true magnetic field $\bar{H}$. The true fields are the ones that satisfy Maxwell's equations and all the boundary conditions. For purposes of this analysis, Maxwell's equations in rationalized MKS units will be used, and $e^{+j\omega t}$ time variations will be assumed.

Statement and Proof of the Variational Principle

For the antenna configuration shown in Figures 1, 2, and 3, it will now be shown that the energy expression

$$W_c = -\sum_{i=1}^{M} \oiint_{\Sigma_i} \left(\frac{1}{2}\,\bar{E}_i \times \bar{H}_i\right) \cdot \bar{n}_i da + \sum_{i=2}^{M} \iint_{S_i} \left(\frac{1}{2}\,\bar{E}_i \times \bar{H}_i\right) \cdot \bar{a}_r da \quad (2\text{-}1)$$

is stationary about the true fields provided that the trial fields satisfy the following conditions:

(1) In each region V_i, $\bar{H}_i$ is obtained from $\bar{E}_i$ by

$$\bar{H}_i = \frac{\nabla \times \bar{E}_i}{-j\omega\mu_i} \quad (i = 1,2,\ldots,M)$$

(2) Each $\bar{E}_i$ is a solution of the complex vector wave equation, $\nabla \times \nabla \times \bar{E}_i = -j\omega\mu_i(\sigma_i + j\omega\epsilon_i)\bar{E}_i$, in its respective region V_i $(i = 1,2,\ldots,M)$

(3) Each $\bar{E}_i$ satisfies the radiation condition in its respective region V_i $(i = 2,3,\ldots,M)$

(4) The tangential components of the trial electric fields are continuous, i.e. $\bar{n}_i \times (\bar{E}_{i+1} - \bar{E}_i) = 0$, at each point on S_a and at each point on S_{i+1} $(i = 2,3,\ldots,M-1)$

(5) $\bar{n}_2 \times \bar{E}_2 = 0$ at each point on the perfectly-conducting, large, ground surface $S_2 - S_a$

(6) $\bar{n}_1 \times \bar{E}_1 = 0$ at each point on the perfectly-conducting, feed-structure boundary $\Sigma_1 - S_1 - S_a$

(7) $\bar{n}_1 \times \bar{E}_1 = \bar{n}_1 \times \bar{E}$ at each point on S_1.

If, in addition, it is required that

(8) The tangential components of the trial magnetic fields are continuous, i.e. $\bar{n}_i \times (\bar{H}_{i+1} - \bar{H}_i) = 0$, at each point on S_{i+1} $(i = 2,3,\ldots,M-1)$,

then the stationary energy function W_c assumes the simpler form

$$W_c = -\iint_{S_1} \left(\frac{1}{2}\,\bar{E}_1 \times \bar{H}_1\right) \cdot \bar{n}_1 da - \iint_{S_a} \left(\frac{1}{2}\,\bar{E}_1 \times \bar{H}_1\right) \cdot \bar{n}_1 da \tag{2-2}$$

$$- \iint_{S_a} \left(\frac{1}{2}\,\bar{E}_2 \times \bar{H}_2\right) \cdot \bar{n}_2 da$$

or

$$W_c = W_{c_1} + W_{c_2} + W_{c_3} \tag{2-3}$$

where

$$W_{c_1} = -\iint_{S_1} \left(\frac{1}{2}\,\bar{E}_1 \times \bar{H}_1\right) \cdot \bar{n}_1 da \tag{2-4}$$

$$W_{c_2} = -\iint_{S_a} \left(\frac{1}{2}\,\bar{E}_1 \times \bar{H}_1\right) \cdot \bar{n}_1 da \tag{2-5}$$

$$W_{c_3} = -\iint_{S_a} \left(\frac{1}{2}\,\bar{E}_2 \times \bar{H}_2\right) \cdot \bar{n}_2 da \tag{2-6}$$

Notice that the above eight conditions on the trial fields are all conditions which the true fields must satisfy. No conditions have been imposed on the trial fields which make them violate the behavior of the true fields. However, not all the conditions that the true fields satisfy have been imposed on the trial fields. If all of these conditions could be imposed, then it would be possible to solve the problem as a boundary value problem. This, however, is not possible.

To prove that W_c is stationary, let $\delta\bar{E}_i$ be the corresponding variation of $\bar{E}_i$ about $\bar{E}$; that is, let $\delta\bar{E}_i = \bar{E}_i - \bar{E}$. Let δW_c be the variation, to first order in $\delta\bar{E}_i$'s, of W_c about $\bar{E}$. Then, since the variation of a sum is the sum of the variations, it follows from Equation (2-1) that

$$\delta W_c = \sum_{i=1}^{M} \delta W_i + \sum_{i=2}^{M} \delta \mathcal{W}_i \tag{2-7}$$

where

$$\delta W_i = \delta \oiint_{\Sigma_i} \left(-\frac{1}{2}\, \bar{E}_i \times \bar{H}_i\right) \cdot \bar{n}_i da \tag{2-8}$$

and

$$\delta \mathcal{W}_i = \delta \iint_{S_i} \left(\frac{1}{2}\, \bar{E}_i \times \bar{H}_i\right) \cdot \bar{a}_r da \tag{2-9}$$

It will be expedient to first show that

$$\delta W_i = \frac{1}{j\omega\mu_i} \oint\!\!\oint_{\Sigma_i} (\bar{n}_i \times \delta\bar{E}_i) \cdot (\nabla\times\bar{E})da \tag{2-10}$$

and

$$\delta W_i = \iint_{S_i} \sqrt{\frac{\varepsilon_i}{\mu_i} - j\,\frac{\sigma_i}{\omega\mu_i}}\; (\bar{a}_r \times \delta\bar{E}_i) \cdot (\bar{a}_r \times \bar{E})da \tag{2-11}$$

To this end, it will be noted that condition one transforms Equation (2-8) to

$$\delta W_i = \delta \oint\!\!\oint_{\Sigma_i} \left[\frac{-\bar{E}_i \times (\nabla\times\bar{E}_i)}{-j2\omega\mu_i}\right] \cdot \bar{n}_i da$$

$$= \frac{1}{j2\omega\mu_i} \oint\!\!\oint_{\Sigma_i} \left[\delta\bar{E}_i \times (\nabla\times\bar{E}) + \bar{E} \times (\nabla\times\delta\bar{E}_i)\right] \cdot \bar{n}_i da$$

The subscript "i" was dropped from $\bar{E}_i$ in the last equation because the variation was taken about $\bar{E}_i = \bar{E}$. Applying the divergence theorem to the last integral in the last equation and a routine vector identity to the first, yields

$$\delta W_i = \frac{1}{j2\omega\mu_i} \oint\!\!\oint_{\Sigma_i} (\bar{n}_i \times \delta\bar{E}_i) \cdot (\nabla\times\bar{E})da \tag{2-12}$$

$$+ \frac{1}{j2\omega\mu_i} \iiint_{V_i} \nabla \cdot [\bar{E}\times(\nabla\times\delta\bar{E}_i)]dv$$

The integrand of the volume integral of Equation (2-12) may be expanded as

$$\nabla \cdot [\bar{E}\times(\nabla\times\delta\bar{E}_i)] = (\nabla\times\bar{E}) \cdot (\nabla\times\delta\bar{E}_i) - \bar{E} \cdot (\nabla\times\nabla\times\delta\bar{E}_i) \qquad (2\text{-}13)$$

Equation (2-13) may be modified by noting that

$$\nabla \cdot [\delta\bar{E}_i\times(\nabla\times\bar{E})] = (\nabla\times\delta\bar{E}_i) \cdot (\nabla\times\bar{E}) - \delta\bar{E}_i \cdot (\nabla\times\nabla\times\bar{E}) \qquad (2\text{-}14)$$

which, when substituted into Equation (2-13), yields

$$\nabla \cdot [\bar{E}\times(\nabla\times\delta\bar{E}_i)] = \nabla \cdot [\delta\bar{E}_i\times(\nabla\times\bar{E})] \qquad (2\text{-}15)$$

$$+ \delta\bar{E}_i \cdot (\nabla\times\nabla\times\bar{E}) - \bar{E} \cdot (\nabla\times\nabla\times\delta\bar{E}_i)$$

Representing the true field, the vector $\bar{E}$ satisfies the vector wave equation and, by assumption, so does $\bar{E}_i$. Hence, the difference $\delta\bar{E}_i = \bar{E}_i - \bar{E}$ also satisfies the vector wave equation since this equation is linear. Equation (2-15) thus becomes

$$\nabla \cdot [\bar{E}\times(\nabla\times\delta\bar{E}_i)] = \nabla \cdot [\delta\bar{E}_i\times(\nabla\times\bar{E})] \qquad (2\text{-}16)$$

$$+ \delta\bar{E}_i \cdot [-j\omega\mu_i(\sigma_i+j\omega\varepsilon_i)\bar{E}]$$

$$- \bar{E} \cdot [-j\omega\mu_i(\sigma_i+j\omega\varepsilon_i)\delta\bar{E}_i]$$

$$= \nabla \cdot [\delta\bar{E}_i\times(\nabla\times\bar{E})]$$

Now, substituting Equation (2-16) into the integrand of the volume integral in Equation (2-12) and applying the divergence theorem once again gives

$$\delta W_i = \frac{1}{j2\omega\mu_i} \oiint_{\Sigma_i} (\bar{n}_i \times \delta\bar{E}_i) \cdot (\nabla\times\bar{E})da + \frac{1}{j2\omega\mu_i} \oiint_{\Sigma_i} [\delta\bar{E}_i \times (\nabla\times\bar{E})] \cdot \bar{n}_i da$$

$$= \frac{1}{j\omega\mu_i} \oiint_{\Sigma_i} (\bar{n}_i \times \delta\bar{E}_i) \cdot (\nabla\times\bar{E})da$$

which proves Equation (2-10).

Equation (2-11) will be proved next. Substituting $\bar{H}_i = \nabla\times\bar{E}_i/-j\omega\mu_i$ into Equation (2-9) gives

$$\delta W_i = \delta \iint_{S_i} \left[\frac{1}{2}\bar{E}_i \times \left(\frac{\nabla\times\bar{E}_i}{-j\omega\mu_i}\right)\right] \cdot \bar{a}_r da \tag{2-17}$$

$$= \frac{1}{-j2\omega\mu_i} \iint_{S_i} \left[\delta\bar{E}_i \times (\nabla\times\bar{E}) + \bar{E}\times(\nabla\times\delta\bar{E}_i)\right] \cdot \bar{a}_r da$$

The true field $\bar{E}$ satisfies the radiation condition* over S_i, which is in the far field of the antenna. By assumption three, $\bar{E}_i$ also satisfies the radiation condition over S_i. Hence, by linearity, so does $\delta\bar{E}_i$. Thus, over S_i

$$\nabla \times \bar{E} = -j\sqrt{\omega^2\mu_i\varepsilon_i - j\omega\mu_i\sigma_i}\ \bar{a}_r \times \bar{E} \tag{2-18}$$

$$\nabla \times \delta\bar{E}_i = -j\sqrt{\omega^2\mu_i\varepsilon_i - j\omega\mu_i\sigma_i}\ \bar{a}_r \times \delta\bar{E}_i \tag{2-19}$$

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*An extension of the standard radiation condition to a lossy medium is used here with ε in the standard condition replaced by $\varepsilon - j\frac{\sigma}{\omega}$.

Substituting Equations (2-18) and (2-19) into Equation (2-17) yields

$$\delta W_i = \frac{-j\sqrt{\omega^2\mu_i\varepsilon_i - j\omega\mu_i\sigma_i}}{-j2\omega\mu_i} \iint_{S_i} [\delta\bar{E}_i\times(\bar{a}_r\times\bar{E}) + \bar{E}\times(\bar{a}_r\times\delta\bar{E}_i)] \cdot \bar{a}_r da$$

$$= \sqrt{\frac{\varepsilon_i}{\mu_i} - j\,\frac{\sigma_i}{\omega\mu_i}} \iint_{S_i} (\bar{a}_r\times\delta\bar{E}_i) \cdot (\bar{a}_r\times\bar{E}) da$$

which proves Equation (2-11).

Equations (2-10) and (2-11) may now be used to express Equation (2-7) as

$$\delta W_c = \sum_{i=1}^{M} \oiint_{\Sigma_i} (\bar{n}_i\times\delta\bar{E}_i) \cdot \left(\frac{\nabla\times\bar{E}}{j\omega\mu_i}\right) da$$

$$+ \sum_{i=2}^{M} \iint_{S_i} \sqrt{\frac{\varepsilon_i}{\mu_i} - j\,\frac{\sigma_i}{\omega\mu_i}}\, (\bar{a}_r\times\delta\bar{E}_i) \cdot (\bar{a}_r\times\bar{E}) da$$

Since $\Sigma_i = S_i + S_{i+1} + \mathcal{S}_i$ for $i = 2,3,\ldots,M$, the last expression for δW_c may be written as

$$\delta W_c = \oiint_{\Sigma_1} (\bar{n}_1\times\delta\bar{E}_1) \cdot \left(\frac{\nabla\times\bar{E}}{j\omega\mu_1}\right) da + \iint_{S_2} (\bar{n}_2\times\delta\bar{E}_2) \cdot \left(\frac{\nabla\times\bar{E}}{j\omega\mu_2}\right) da \qquad (2\text{-}20)$$

$$+ \sum_{i=2}^{M-1} \iint_{S_{i+1}} \left[(\bar{n}_i\times\delta\bar{E}_i) \cdot \left(\frac{\nabla\times\bar{E}}{j\omega\mu_i}\right) + (\bar{n}_{i+1}\times\delta\bar{E}_{i+1}) \cdot \left(\frac{\nabla\times\bar{E}}{j\omega\mu_{i+1}}\right)\right] da$$

$$+ \sum_{i=2}^{M} \iint_{S_i} \left[(\bar{n}_i \times \delta\bar{E}_i) \cdot \left(\frac{\nabla\times\bar{E}}{j\omega\mu_i}\right) + \sqrt{\frac{\varepsilon_i}{\mu_i} - j\,\frac{\sigma_i}{\omega\mu_i}}\ (\bar{a}_r \times \delta\bar{E}_i) \cdot (\bar{a}_r \times \bar{E}) \right] da$$

It will now be shown that the first sum in Equation (2-20) is zero. To this end, it will be recalled that $\bar{H} = \frac{\nabla\times\bar{E}}{-j\omega\mu_i}$ in region V_i, $\bar{H} = \frac{\nabla\times\bar{E}}{-j\omega\mu_{i+1}}$ in region V_{i+1}, and that $\bar{n}_{i+1} = -\bar{n}_i$ over S_{i+1}. Thus, the first sum in Equation (2-20) becomes

$$\sum_{i=2}^{M-1} \iint_{S_{i+1}} \left[-(\bar{n}_i \times \delta\bar{E}_i) \cdot \bar{H} + (\bar{n}_i \times \delta\bar{E}_{i+1}) \cdot \bar{H} \right] da \qquad (2\text{-}21)$$

Since $\bar{n}_i \times \delta\bar{E}_i$ and $\bar{n}_i \times \delta\bar{E}_{i+1}$ are strictly tangential to S_{i+1}, then $(\bar{n}_i \times \delta\bar{E}_i) \cdot \bar{H}$ and $(\bar{n}_i \times \delta\bar{E}_{i+1}) \cdot \bar{H}$ involve only the tangential components of $\bar{H}$. Since the tangential components of the true field are continuous over S_{i+1}, $\bar{H}$ may be factored out of Equation (2-21) and that equation rewritten as

$$\sum_{i=2}^{M-1} \iint_{S_{i+1}} [\bar{n}_i \times \delta\bar{E}_{i+1} - \bar{n}_i \times \delta\bar{E}_i] \cdot \bar{H}\, da \qquad (2\text{-}22)$$

The tangential components of the true, and by assumption four, the trial electric fields are continuous over S_{i+1}; hence, $\bar{n}_i \times (\delta\bar{E}_{i+1} - \delta\bar{E}_i) = 0$ at each point on S_{i+1}. Thus, Equation (2-22) is zero and hence, the first sum in Equation (2-20) is also zero.

Next, it will be shown that the second sum in Equation (2-20) is zero. By Equation (2-18), the second sum in Equation (2-20) may be written as

$$\sum_{i=2}^{M} \iint_{S_i} \left[(\bar{n}_i \times \delta\bar{E}_i) \cdot \left[\frac{-j\sqrt{\omega^2\mu_i\varepsilon_i - j\omega\mu_i\sigma_i}\ \bar{a}_r \times \bar{E}}{j\omega\mu_i} \right] \right.$$

$$\left. + \sqrt{\frac{\varepsilon_i}{\mu_i} - j\frac{\sigma_i}{\omega\mu_i}}\ (\bar{a}_r \times \delta\bar{E}_i) \cdot (\bar{a}_r \times \bar{E}) \right] da = 0$$

since $\bar{n}_i = \bar{a}_r$ over S_i. Thus, the second sum, as well as the first sum, in Equation (2-20) is zero. As a result, the amended form of Equation (2-20) is

$$\delta W_c = \oiint_{\Sigma_1} (\bar{n}_1 \times \delta\bar{E}_1) \cdot \left(\frac{\nabla \times \bar{E}}{j\omega\mu_1}\right) da + \iint_{S_2} (\bar{n}_2 \times \delta\bar{E}_2) \cdot \left(\frac{\nabla \times \bar{E}}{j\omega\mu_2}\right) da \qquad (2\text{-}23)$$

Now, because of condition six and the fact that $\bar{n}_1 \times \bar{E} = 0$ over $\Sigma_1 - S_1 - S_a$, $\bar{n}_1 \times \delta\bar{E}_1 = 0$ at each point on the perfectly conducting waveguide wall surface. The first integral in Equation (2-23) can be simplified, accordingly. The second integral can also be simplified since $\bar{n}_2 \times \delta\bar{E}_2 = 0$ over $S_2 - S_a$. This is true because of condition five and the fact that $\bar{n}_2 \times \bar{E} = 0$ over $S_2 - S_a$. The net result is that Equation (2-23) becomes

$$\delta W_c = \iint_{S_1} (\bar{n}_1 \times \delta\bar{E}_1) \cdot \left(\frac{\nabla \times \bar{E}}{j\omega\mu_1}\right) da + \iint_{S_a} (\bar{n}_1 \times \delta\bar{E}_1) \cdot \left(\frac{\nabla \times \bar{E}}{j\omega\mu_1}\right) da \qquad (2\text{-}24)$$

$$+ \iint_{S_a} (\bar{n}_2 \times \delta\bar{E}_2) \cdot \left(\frac{\nabla \times \bar{E}}{j\omega\mu_2}\right) da$$

Now, $\bar{n}_1 \times \delta\bar{E}_1 = \bar{n}_1 \times (\bar{E}_1 - \bar{E}) = 0$ over S_1 by condition seven; hence, the first integral in Equation (2-24) is zero. Additionally, it will be recalled that $\bar{H} = \frac{\nabla\times\bar{E}}{-j\omega\mu_1}$ in region V_1, $\bar{H} = \frac{\nabla\times\bar{E}}{-j\omega\mu_2}$ in region V_2, and that $\bar{n}_2 = -\bar{n}_1$ over S_a. These facts permit Equation (2-24) to be written as

$$\delta W_c = -\iint_{S_a} (\bar{n}_1 \times \delta\bar{E}_1) \cdot \bar{H}\, da + \iint_{S_a} (\bar{n}_1 \times \delta\bar{E}_2) \cdot \bar{H}\, da \qquad (2\text{-}25)$$

In this equation $\bar{n}_1 \times \delta\bar{E}_1$ and $\bar{n}_1 \times \delta\bar{E}_2$ are strictly tangential to S_a, and the integrands thus involve only the tangential components of $\bar{H}$, which are continuous at S_a. Hence, the two integrals in Equation (2-25) can be combined, with $\bar{H}$ as a common factor of the integrands. By condition four and the fact that the tangential components of true fields are continuous at S_a, it now follows that $\bar{n}_1 \times \delta\bar{E}_1 = \bar{n}_1 \times \delta\bar{E}_2$ at each point on S_a. Consequently,

$$\delta W_c = 0$$

which is the assertion that was to be proved.

If, in addition to conditions one through seven, condition eight is also satisfied, then W_c can be considerably simplified. First, it will be noticed that the surface integrals over S_i in Equation (2-1) cancel with their corresponding parts from the integrals over Σ_i. This permits Equation (2-1) to be written as

$$W_c = -\oiint_{\Sigma_1} \left(\frac{1}{2}\,\bar{E}_1\times\bar{H}_1\right)\cdot\bar{n}_1 da - \iint_{S_2} \left(\frac{1}{2}\,\bar{E}_2\times\bar{H}_2\right)\cdot\bar{n}_2 da \qquad (2\text{-}26)$$

$$- \sum_{i=2}^{M-1} \iint_{S_{i+1}} \left(\frac{1}{2}\,\bar{E}_i\times\bar{H}_i - \frac{1}{2}\,\bar{E}_{i+1}\times\bar{H}_{i+1}\right)\cdot\bar{n}_i da$$

where the relation $\bar{n}_{i+1} = -\bar{n}_i$ on S_{i+1} was used in the summation. But conditions four through eight require that the tangential components of the trial electric and trial magnetic fields be continuous at each point on each S_i $(i = 3,4,\ldots,M)$. Hence, the summation in (2-26) is zero. Furthermore, since the tangential components of the trial electric fields are zero over the perfect conductors (conditions five and six), it follows that Equation (2-26) may be rewritten as

$$W_c = -\iint_{S_1} \left(\frac{1}{2}\,\bar{E}_1\times\bar{H}_1\right)\cdot\bar{n}_1 da - \iint_{S_a} \left(\frac{1}{2}\,\bar{E}_1\times\bar{H}_1\right)\cdot\bar{n}_1 da$$

$$-\iint_{S_a} \left(\frac{1}{2}\,\bar{E}_2\times\bar{H}_2\right)\cdot\bar{n}_2 da$$

which concludes what was to be shown.

Comments on the Variational Principle

It should be noticed that Equation (2-2) is a stationary formula since it requires the same seven conditions that Equation (2-1) does, and since condition eight is not needed to make Equation (2-1) stationary. It should also be noticed that the expressions $\frac{1}{2}\,\bar{E}_i \times \bar{H}_i$, which appear in the generally complex functional, W_c, are not Poynting's vectors since the latter contains a conjugate of $\bar{H}$, that is $\frac{1}{2}\,\bar{E} \times \bar{H}^*$, while the terms in W_c do not. A variational principle using Poynting's vector is discussed by Paris and Hurd (22).

The variational principle states that the trial fields will adjust themselves as closely as possible to the true fields so that W_c is as close to its true value as possible. The best approximation to the fields and to W_c is obtained when there is no further change in W_c for further small changes in the trial fields.

This gives a method of obtaining approximations to the true fields by using trial fields containing several adjustable parameters. These parameters are adjusted to that set of values at which perturbations of the parameter values will produce no additional change in W_c. Then W_c is stationary. This is the same as requiring that the partial derivative of W_c with respect to a parameter be zero for each parameter. An approximation to the true fields is obtained by adjusting the parameters to the values so determined. Once the fields are known, all of the electrical characteristics of the antenna can be calculated, in principle at least.

More precisely, if trial fields of the form

$$\bar{E}_i = \sum_m a_m \bar{e}_m \tag{2-27}$$

are used, where $\{\bar{e}_m\}$ is a set of known mode functions and $\{a_m\}$ is a set of unknown mode amplitudes, then $\bar{E}_i$ will be a linear combination of the unknown a_m's. The trial magnetic field, $\bar{H}_i$, will also be a linear combination of the a_m's since it is obtained from $\bar{H}_i = \nabla\times\bar{E}_i/-j\omega\mu_i$. Thus, W_c in Equation (2-2) will be a quadratic function of the a_m's. To find the set of a_m's that makes W_c stationary, the partial derivative of W_c with respect to a_m is set equal to zero. Repeating this process for each a_m produces a system of equations in terms of the unknown a_m's. It is important to note that this system of equations will be linear since taking the partial derivative of the quadratic expression for W_c will reduce the quadratic expression to a linear one. Since the system of equations for the a_m's is linear, matrix techniques can be used to solve them, resulting in a considerable savings of time.

Comparable techniques (23-26) use stationary formulas of the form

$$Y = \frac{\iint_{S_a} f_1(\bar{E}_i,\bar{H}_i)da}{\iint_{S_a} f_2(\bar{E}_i,\bar{H}_i)da} \qquad (2\text{-}28)$$

where Y stands for the admittance of the antenna and f_1 and f_2 represent two functions. If Equation (2-27) is used for $\bar{E}_i$ in Equation (2-28), then a_m's will appear both in the numerator and in the denominator of Y. Nonlinear equations will be produced when partial derivatives of Y with respect to the a_m's are taken. Solving these nonlinear equations is much more time consuming than solving the linear ones obtained by using the approach presented in this chapter.

In addition to these comments about the new variational principle, a comment about condition seven is in order. To make $\bar{n}_1 \times \bar{E}_1 = \bar{n}_1 \times \bar{E}$ over

S_1, a surface S_1 must be found such that the tangential component of the true field is known over that surface. Then the tangential component of $\bar{E}_1$ must be forced to be equal to that tangential field over S_1. Hence, $\bar{n}_1 \times \bar{E}_1 = \bar{n}_1 \times \bar{E}$ over S_1 and condition seven is satisfied. There is one common situation which allows S_1 to be easily located, and that is when it is known that only a single mode exists in a certain region of the waveguide. Then S_1 is placed in this region, and the tangential component of $\bar{E}_1$ is forced to be equal to the tangential component of that mode over that surface.

CHAPTER III

ANALYSIS OF A DIELECTRICALLY COATED SLOT ANTENNA

Description of the Antenna

The variational principle of Chapter II will now be applied to a particular practical problem, namely the waveguide slot antenna shown in Figures 4 and 5. This antenna consists of a perfectly-conducting rectangular waveguide terminating on an infinitely large, perfectly-conducting ground plane. A rectangular slot is cut in the ground plane to couple energy from the waveguide region into the region $z > 0$. The portion of the ground plane that covers the waveguide is assumed to be infinitely thin. Two planar layers of linear, isotropic, and homogeneous materials cover the ground plane. One layer fills the region $0 < z \leqq d$ and the other layer fills the region $z > d$. The medium filling the waveguide is also assumed to be linear, isotropic, and homogeneous. All three regions of space are assumed to be charge free.

The antenna just described simulates a slot antenna under a heat shield of a re-entry vehicle. The ground plane represents the surface of the vehicle, region V_2 represents the heat shield, and region V_3 represents free space.

The antenna is operated in the following manner. A transmitter, producing a dominant mode (TE_{10}) field, is connected to the left end of the waveguide. The dominant mode wave produced by the transmitter travels undistorted through the waveguide region until it encounters

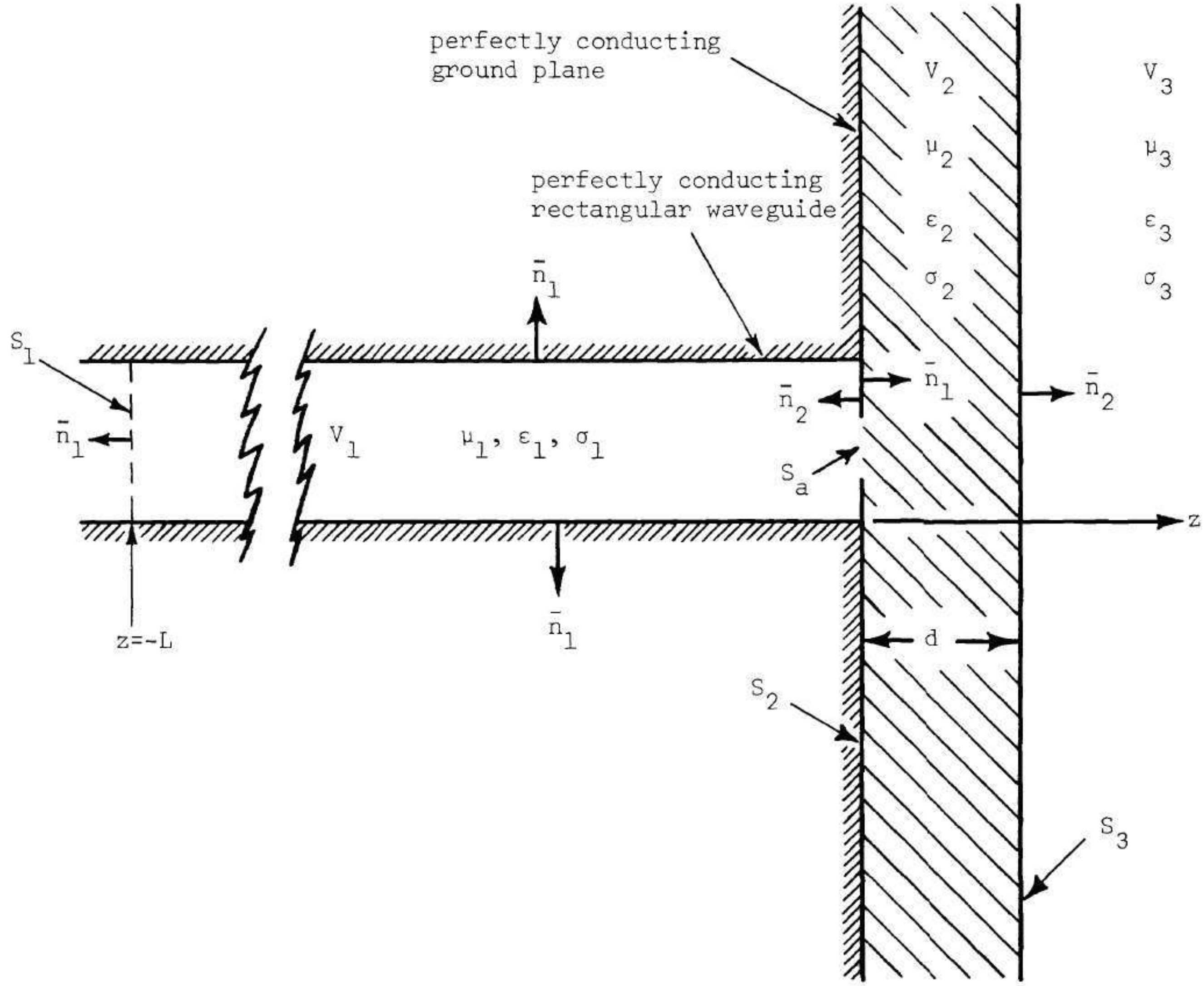


Figure 4. Side View of Coated, Rectangular Waveguide Slot Antenna

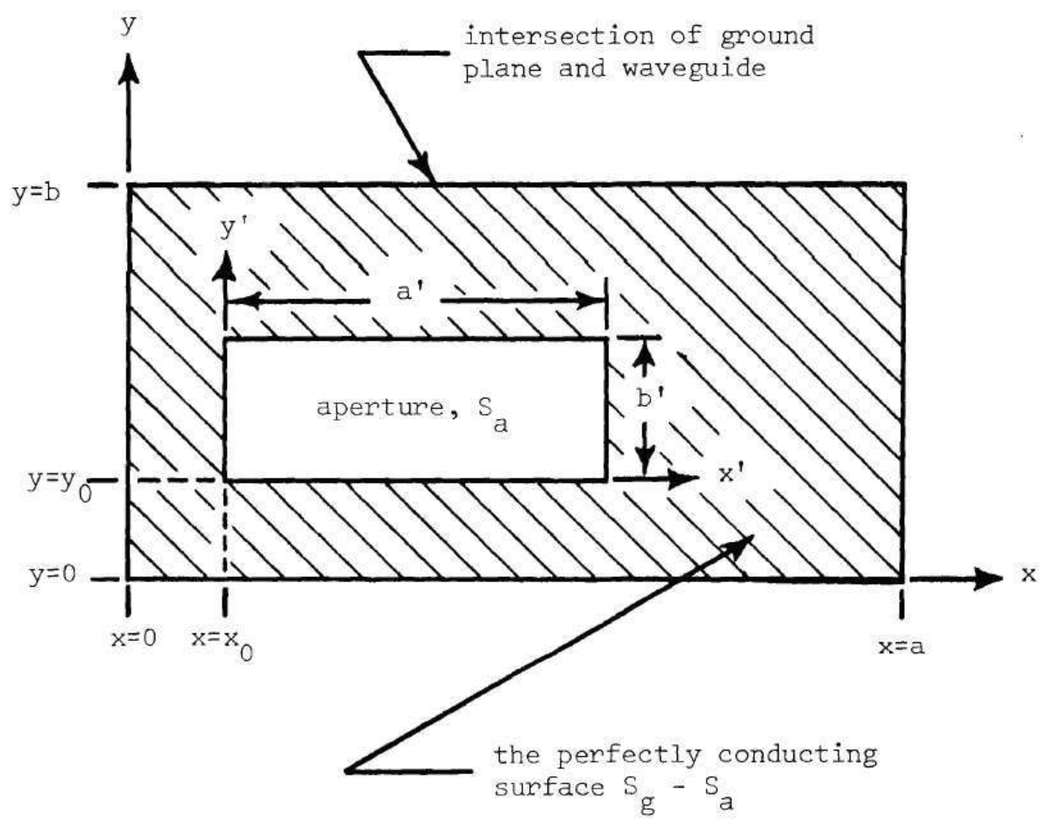


Figure 5. Front View of Slot Antenna

the discontinuity in the physical structure at $z = 0$. This discontinuity causes some of the incident energy to be reflected back down the waveguide and part of the energy to be transmitted into V_2.

The reflected field in the waveguide consists of the dominant mode plus higher order modes. The higher order modes are needed to make the tangential component of the electric field zero over the perfect conductor that covers part of the waveguide. In normal operation the higher order modes are evanescent; that is, they decay exponentially with distance away from the discontinuity. Thus, as the reflected wave moves toward the transmitter, the higher order modes decrease in amplitude until they are negligible and only the dominant mode remains. The higher order modes affect the value of the reflected dominant mode and they also affect the field radiated by the antenna. Thus, the amplitude of these modes must be determined in order to evaluate the behavior of the antenna.

Other variational approaches to this slot antenna problem have neglected the higher order modes and assumed that only the dominant mode is present. In contrast, an arbitrary number of modes can be included in the method described in this dissertation.

Before the variational principle of Chapter II can be applied, trial fields are needed in regions V_1, V_2, and V_3.

The Waveguide Region

The trial electric field in V_1 must have a zero tangential component over the perfectly conducting waveguide walls. In addition, this field should: (i) represent a dominant mode initiated from the left end of the waveguide; (ii) represent higher order modes that are evanescent from z = 0; and (iii) contain a reflected dominant mode. A set of the trial functions having all of these characteristics is the set of rectangular waveguide modes whose x and y components may be expressed as

$$E_{x_1} = \sum_{m,n} P_{m,n} \cos(A_m x) \sin(B_n y)\, e^{\alpha_{m,n} z} \tag{3-1}$$

$$E_{y_1} = (I\, e^{-j\beta z} + R\, e^{+j\beta z}) \sin(A_1 x) + \sum_{m,n} Q_{m,n} \sin(A_m x) \cos(B_n y)\, e^{\alpha_{m,n} z} \tag{3-2}$$

where

$$A_m = \frac{m\pi}{a} \tag{3-3}$$

and

$$B_n = \frac{n\pi}{b} \tag{3-4}$$

The quantities $\alpha_{m,n}$ and β in Equations (3-1) and (3-2) are the attenuation and phase constants for the various modes. These constants are complex numbers in general and must be chosen in such a way that $\bar{E}_1$ will satisfy the complex vector wave equation. They will be specified shortly. The parameters A_m and B_n have been chosen to make the tangential component of the electric field zero on the waveguide walls, in accordance with condition six of the variational principle. The usual TE and TM modes of rectangular waveguide theory have been combined in Equations (3-1) and (3-2) since they have the same transverse variations. The subscript "1" in E_{x_1} and E_{y_1} indicates that the fields are in region V_1.

The mode amplitudes I, R, $P_{m,n}$, and $Q_{m,n}$ are complex constants that must be determined. They are the complex amplitudes, that is the magnitude and phase, of the various mode functions. The analysis that follows is aimed at evaluating these constants. Once they are known, a complete picture of the antenna's field, and hence its electrical characteristics, will have been obtained. The summation indices m and n take on all positive integer values except m = n = 0, which is the trivial case, and m = 1, n = 0, which is the dominant mode and is included separately.

Next, E_{z_1} and $\bar{H}_1$ will be obtained from E_{x_1} and E_{y_1}. Condition two of the variational principle requires $\bar{E}_1$ to satisfy the vector equation, namely

$$\nabla \times \nabla \times \bar{E}_1 = k_1^2 \bar{E}_1 \qquad (3\text{-}5)$$

where

$$k_1^2 = -j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1) \qquad (3\text{-}6)$$

Taking the divergence of Equation (3-5) and remembering that the divergence of the curl is always zero yields

$$\nabla \cdot \bar{E}_1 = 0 \tag{3-7}$$

Thus, if a field satisfies the vector wave equation, it must have zero divergence. This fact will be used again in V_2 and V_3.

In rectangular coordinates, Equation (3-7) is

$$-\frac{\partial E_{z_1}}{\partial z} = \frac{\partial E_{x_1}}{\partial x} + \frac{\partial E_{y_1}}{\partial y}$$

Using Equations (3-1) and (3-2) in the last equation gives

$$-\frac{\partial E_{z_1}}{\partial z} = \sum_{m,n} [-A_m P_{m,n} - B_n Q_{m,n}] \sin(A_m x) \sin(B_n y)\, e^{\alpha_{m,n} z}$$

Integrating this last equation yields

$$E_{z_1} = \sum_{m,n} \left[\frac{A_m P_{m,n} + B_n Q_{m,n}}{\alpha_{m,n}}\right] \sin(A_m x) \sin(B_n y)\, e^{\alpha_{m,n} z} \tag{3-8}$$

All three rectangular components of the trial field $\bar{E}_1$ have now been established. To apply the variational principle, $\bar{E}_1$ must satisfy the vector wave equation.

The conditions, in addition to Equation (3-7), that the vector wave equation imposes on $\bar{E}_1$ will now be determined. By using Equation (3-7), Equation (3-5) may be simplified, in rectangular coordinates, to

$$-\nabla^2 E_{\nu_1} = k_1^2 E_{\nu_1} \tag{3-9}$$

where ν = x, y, or z. For ν = x, Equation (3-9), with the aid of Equation (3-1), becomes

$$\sum_{m,n} (A_m^2 + B_n^2 - \alpha_{m,n}^2 - k_1^2)\, P_{m,n} \cos(A_m x) \sin(B_n y)\, e^{\alpha_{m,n} z} = 0 \qquad (3\text{-}10)$$

If Equation (3-10) is to hold for any x, y, z in the waveguide and for any set of $P_{m,n}$'s, then it is necessary that

$$\alpha_{m,n} = \sqrt{A_m^2 + B_n^2 - k_1^2} \qquad (3\text{-}11)$$

In order that $\bar{E}_1$ represent higher order modes that decay or propagate with decay in the -z direction, the square root in Equation (3-11) must be chosen so that

$$\mathrm{Re}(\alpha_{m,n}) > 0 \qquad (3\text{-}12)$$

$$\mathrm{Im}(\alpha_{m,n}) \geqq 0$$

where Re and Im mean real part and imaginary part, respectively. Equations (3-11) and (3-12) specify the constant $\alpha_{m,n}$.

For ν = z, Equation (3-9) again requires that Equation (3-11) hold. For ν = y, Equation (3-9), with the aid of Equation (3-2), becomes

$$\left[\left(\frac{\pi}{a}\right)^2 + \beta^2 - k_1^2\right]\left[I e^{-j\beta z} + R e^{j\beta z}\right] \sin(A_1 x) \qquad (3\text{-}13)$$

$$+ \sum_{m,n} [A_m^2 + B_n^2 - \alpha_{m,n}^2 - k_1^2]\, Q_{m,n} \sin(A_m x) \cos(B_n y)\, e^{\alpha_{m,n} z} = 0$$

The summation in Equation (3-13) is zero because of Equation (3-11). Hence, if the remaining term in Equation (3-13) is to be zero for arbitrary x,z in the waveguide and arbitrary I and R, then

$$\beta = \sqrt{k_1^2 - \left(\frac{\pi}{a}\right)^2} \tag{3-14}$$

In order that $\bar{E}_1$ represent a dominant mode wave propagating in the +z direction, either with or without attenuation, the square root in Equation (3-14) must be selected so that

$$\mathrm{Re}(\beta) > 0 \tag{3-15}$$

$$\mathrm{Im}(\beta) \leqq 0$$

The constant β is specified by Equations (3-14) and (3-15). Equations (3-11) and (3-14) are the conditions that must be imposed for the chosen $\bar{E}_1$ to satisfy the vector wave equation.

The trial magnetic field in region V_1 is obtained by using $\bar{H}_1 = \nabla\times\bar{E}_1/-j\omega\mu_1$. The rectangular components of this equation are

$$H_{x_1} = \left[\frac{1}{-j\omega\mu_1}\right]\left[\frac{\partial E_{z_1}}{\partial y} - \frac{\partial E_{y_1}}{\partial z}\right] \tag{3-16}$$

$$H_{y_1} = \left[\frac{1}{-j\omega\mu_1}\right]\left[\frac{\partial E_{x_1}}{\partial z} - \frac{\partial E_{z_1}}{\partial x}\right] \tag{3-17}$$

Since H_{z_1} is not needed in any future calculations, it will not be evaluated. Substituting Equations (3-8) and (3-2) into Equation (3-16) yields

$$H_{x_1} = \left[\frac{j}{\omega\mu_1}\right]\left[\sum_{m,n} \left\{ \frac{A_m B_n P_{m,n} + B_n^2 Q_{m,n}}{\alpha_{m,n}} - \alpha_{m,n} Q_{m,n} \right\} \sin(A_m x)\right.$$

$$\left. \cdot \cos(B_n y) e^{\alpha_{m,n} z} + (j\beta I e^{-j\beta z} - j\beta R e^{j\beta z}) \sin(A_1 x)\right]$$

Equation (3-11) transforms the last equation to

$$H_{x_1} = \left(\frac{1}{\omega\mu_1}\right)\left[-\beta(I e^{-j\beta z} - R e^{j\beta z}) \sin(A_1 x)\right. \tag{3-18}$$

$$\left. + j \sum_{m,n} \left\{ \frac{A_m B_n P_{m,n} + (k_1^2 - A_m^2) Q_{m,n}}{\alpha_{m,n}} \right\} \sin(A_m x) \cos(B_n y) e^{\alpha_{m,n} z}\right]$$

Next, substituting Equations (3-1) and (3-8) into Equation (3-17) yields

$$H_{y_1} = \left(\frac{j}{\omega\mu_1}\right) \sum_{m,n} \left\{ \alpha_{m,n} P_{m,n} - \frac{A_m^2 P_{m,n} + A_m B_n Q_{m,n}}{\alpha_{m,n}} \right\} \cos(A_m x) \sin(B_n y) e^{\alpha_{m,n} z}$$

which in view of Equation (3-11) becomes

$$H_{y_1} = \left(\frac{-j}{\omega\mu_1}\right) \sum_{m,n} \left\{ \frac{(k_1^2-B_n^2)P_{m,n}+A_m B_n Q_{m,n}}{\alpha_{m,n}} \right\} \cos(A_m x)\sin(B_n y)e^{\alpha_{m,n} z} \qquad (3\text{-}19)$$

The Constraint at S_1

Condition seven of Chapter II requires that the trial fields E_{x_1} and E_{y_1} be exactly equal to the true fields E_x and E_y at each point on S_1. First, however, a surface must be chosen for S_1. If S_1 is selected to be a plane at $z = -L$ where $L \to \infty$, then all higher order modes will be zero over S_1. This follows because $\exp(\alpha_{m,n} z) = \exp(-\alpha_{m,n} L)$ will be negligibly small for each higher order mode. Hence, over this surface S_1 only the incident and reflected dominant mode fields will be present for both the trial and the true fields. Since the form of the electric field is now known over S_1, only its amplitude need be set in order to fix $\bar{n}_1 \times \bar{E}$ over S_1. The magnitude and phase of $\bar{n}_1 \times \bar{E}$ at $z = -L$ can be chosen to be any convenient value. This follows from the fact that the system is linear and scaling the magnitude and phase of $\bar{n}_1 \times \bar{E}$ at $z = -L$ will simply scale the magnitude and phase of all the fields by the same amount. Another way of saying this is that $\bar{n}_1 \times \bar{E}$ at $z = -L$ is the magnitude and phase reference for the system, and it may be selected to be any convenient value. For convenience, this magnitude and phase reference will be chosen as 1 volt/meter and 0 degrees, respectively. Thus, the tangential component of the true field at $z = -L$ is

$$E_x(x,y,-L) = 0 \tag{3-20}$$

$$E_y(x,y,-L) = 1 \sin(A_1 x)$$

Now, $\bar{n}_1 \times \bar{E}_1$ must be made equal to $\bar{n}_1 \times \bar{E}$ over S_1. Over this surface Equations (3-1), (3-2), (3-18), and (3-19) reduce to

$$E_{x_1} = 0 \tag{3-21}$$

$$E_{y_1} = (Ie^{j\beta L} + Re^{-j\beta L}) \sin(A_1 x)$$

$$H_{x_1} = -\frac{\beta}{\omega\mu_1}(Ie^{j\beta L} - Re^{-j\beta L}) \sin(A_1 x)$$

$$H_{y_1} = 0$$

All higher order modes are negligibly small at $z = -L$ as $L \to \infty$.

Equations (3-20) and (3-21) show that the tangential components of the electric fields over S_1 are equal provided

$$Ie^{j\beta L} + Re^{-j\beta L} = 1 \tag{3-22}$$

This relation allows the coefficient I to be set equal to $(\exp(-j\beta L) - R\exp(-j2\beta L))$ in all future equations.

Several other relationships between I and R that will be useful later can be obtained from Equation (3-22). They are

$$Ie^{j\beta L} - Re^{-j\beta L} = 1 - 2Re^{-j\beta L} \tag{3-23}$$

$$I + R = e^{-j\beta L} + R(1 - e^{-j2\beta L}) \tag{3-24}$$

and

$$I - R = e^{-j\beta L} - R(1 + e^{-j2\beta L}) \tag{3-25}$$

Evaluation of W_{c_1}

The groundwork is now ready for evaluation of the first integral in Equation (2-2), namely

$$W_{c_1} = -\iint_{S_1} \left(\frac{1}{2}\,\bar{E}_1 \times \bar{H}_1\right) \cdot \bar{n}_1 da$$

Equation (2-2) rather than Equation (2-1) will be used to calculate W_c in this chapter since the former equation is simpler. Since $\bar{n}_1 = -\bar{a}_z$ over S_1, this last equation becomes

$$W_{c_1} = \frac{1}{2}\int_0^b \int_0^a \left(E_{x_1} H_{y_1} - E_{y_1} H_{x_1}\right)\Big|_{z=-L} dxdy$$

Substituting Equation (3-21) into this last expression gives

$$W_{c_1} = \frac{1}{2}\int_0^b \int_0^a \left(\frac{\beta}{\omega\mu_1}\right)\left(Ie^{j\beta L} + Re^{-j\beta L}\right)\left(Ie^{j\beta L} - Re^{-j\beta L}\right)\sin^2(A_1 x)dxdy \tag{3-26}$$

Using Equations (3-22) and (3-23) and evaluating the integrals in Equation (3-26) yields

$$W_{c_1} = \left[\frac{1}{2\omega\mu_1}\right]\left[\frac{ab\beta}{2} - (ab\beta e^{-j\beta L})R\right] \tag{3-27}$$

Equation (3-27) represents the desired expression for W_{c_1}.

<u>The Aperture Field and the Evaluation of W_{c_2}</u>

Next, the second integral in Equation (2-2) must be evaluated, namely

$$W_{c_2} = -\iint_{S_a} (\tfrac{1}{2}\,\bar{E}_1 \times \bar{H}_1) \cdot \bar{n}_1 da$$

Since $\bar{n}_1 = \bar{a}_z$ over S_a,

$$W_{c_2} = -\frac{1}{2}\int_{y_0}^{y_0+b'}\int_{x_0}^{x_0+a'} (E_{x_1}H_{y_1} - E_{y_1}H_{x_1})\Big|_{z=0} dxdy \tag{3-28}$$

Now the surface S_g will be defined as the cross section of the waveguide at z = 0. Before Equation (3-28) can be evaluated, the appropriate parts of conditions four and six of the variational principle must be applied to $\bar{E}_1$ over S_g. In particular, E_{x_1} and E_{y_1} must be made zero over the perfect conductor $S_g - S_a$, while over S_a they must be made equal to the x and y components, respectively, of the field in the aperture. It is convenient to introduce a separate representation for the aperture field to facilitate the application of the above boundary conditions. Let the x and y components of the aperture electric field be

$$E_{x_a} = \sum_{m',n'} P'_{m',n'} \cos(A'_{m'} x') \sin(B'_{n'} y') \tag{3-29}$$

$$E_{y_a} = Q'_{1,0} \sin(A'_1 x') + \sum_{m',n'} Q'_{m',n'} \sin(A'_{m'} x') \cos(B'_{n'} y') \tag{3-30}$$

where

$$A'_{m'} = \frac{m'\pi}{a'} \tag{3-31}$$

and

$$B'_{n'} = \frac{n'\pi}{b'} \tag{3-32}$$

The subscripts m',n' cover the same range of integers that m,n do. The form of the trial aperture field was selected by analogy with the trial waveguide field. The aperture was viewed as the limiting case of a rectangular waveguide with dimensions a' by b' and a length approaching zero. The x',y' coordinate system is shown in Figure 5. The aperture mode amplitudes $Q'_{1,0}$, $Q'_{m',n'}$, and $P'_{m',n'}$ are complex constants that must be determined in the process of finding the waveguide mode amplitudes.

Since $E_{x_1} = E_{x_a}$ and $E_{y_1} = E_{y_a}$ over S_a, Equation (3-29) may be used for E_{x_1} and Equation (3-30) may be substituted for E_{y_1} in Equation (3-28). Making use of Equations (3-18) and (3-19) then yields

$$W_{c_2} = \left[\frac{1}{2\omega\mu_1}\right]\Bigg[-\beta(I-R)Q'_{1,0}\ \left\{\int_{x_0}^{x_0+a'} \sin(A_1x)\sin(A'_1x')dx\right\}\cdot\left\{\int_{y_0}^{y_0+b'} dy\right\} \quad (3\text{-}33)$$

$$-\sum_{m',n'}\beta(I-R)Q'_{m',n'}\ \left\{\int_{x_0}^{x_0+a'} \sin(A_1x)\sin(A'_{m'}x')dx\right\}\cdot\left\{\int_{y_0}^{y_0+b'} \cos(B'_{n'}y')dy\right\}$$

$$+ j\sum_{m,n}\left[\frac{k_1^2-A_m^2}{\alpha_{m,n}}\right]Q_{m,n}Q'_{1,0}\ \left\{\int_{x_0}^{x_0+a'} \sin(A_mx)\sin(A'_1x')dx\right\}\cdot\left\{\int_{y_0}^{y_0+b'} \cos(B_ny)dy\right\}$$

$$+ j\sum_{m,n}\sum_{m',n'}\left[\frac{k_1^2-A_m^2}{\alpha_{m,n}}\right]Q_{m,n}Q'_{m',n'}\left\{\int_{x_0}^{x_0+a'} \sin(A_mx)\sin(A'_{m'}x')dx\right\}$$

$$\cdot\left\{\int_{y_0}^{y_0+b'} \cos(B_ny)\cos(B'_{n'}y')dy\right\}$$

$$+ j\sum_{m,n}\left[\frac{A_mB_n}{\alpha_{m,n}}\right]P_{m,n}Q'_{1,0}\left\{\int_{x_0}^{x_0+a'} \sin(A_mx)\sin(A'_1x')dx\right\}\cdot\left\{\int_{y_0}^{y_0+b'} \cos(B_ny)dy\right\}$$

$$+ j\sum_{m,n}\sum_{m',n'}\left[\frac{A_mB_n}{\alpha_{m,n}}\right]P_{m,n}Q'_{m',n'}\left\{\int_{x_0}^{x_0+a'} \sin(A_mx)\sin(A'_{m'}x')dx\right\}$$

$$\cdot\left\{\int_{y_0}^{y_0+b'} \cos(B_ny)\cos(B'_{n'}y')dy\right\}$$

$$+ j\sum_{m,n}\sum_{m',n'}\left[\frac{k_1^2-B_n^2}{\alpha_{m,n}}\right]P_{m,n}P'_{m',n'}\left\{\int_{x_0}^{x_0+a'} \cos(A_mx)\cos(A'_{m'}x')dx\right\}$$

$$\cdot \left\{\int_{y_0}^{y_0+b'} \sin(B_n y)\sin(B'_{n'} y')dy\right\}$$

$$+ j \sum_{m,n} \sum_{m',n'} \left[\frac{A_m B_n}{\alpha_{m,n}}\right] Q_{m,n} P'_{m',n'} \left\{\int_{x_0}^{x_0+a'} \cos(A_m x)\cos(A'_{m'} x')dx\right\}$$

$$\left.\cdot \left\{\int_{y_0}^{y_0+b'} \sin(B_n y)\sin(B'_{n'} y')dy\right\}\right]$$

The integrals in Equation (3-33) are evaluated in Appendix A. Using the notation adopted there and Equation (3-25) transforms Equation (3-33) to

$$W_{c_2} = \left[\frac{1}{2\omega\mu_1}\right]\left[-\beta e^{-j\beta L} \mathrm{Int}_3(1,1)\mathrm{Int}_4(0,0)Q'_{1,0}\right. \tag{3-34}$$

$$+ \sum_{m',n'} (-\beta)e^{-j\beta L} \mathrm{Int}_3(1,m')\mathrm{Int}_4(0,n')Q'_{m',n'}$$

$$+ \beta(1+e^{-j2\beta L})\mathrm{Int}_3(1,1)\mathrm{Int}_4(0,0)RQ'_{1,0}$$

$$+ \sum_{m',n'} \beta(1+e^{-j2\beta L})\mathrm{Int}_3(1,m')\mathrm{Int}_4(0,n')RQ'_{m',n'}$$

$$+ \sum_{m,n} j\left[\frac{k_1^2-A_m^2}{\alpha_{m,n}}\right] \mathrm{Int}_3(m,1)\mathrm{Int}_4(n,0)Q_{m,n}Q'_{1,0}$$

$$+ \sum_{m,n} \sum_{m',n'} j \left[\frac{k_1^2 - A_m^2}{\alpha_{m,n}}\right] Int_3(m,m')Int_4(n,n')Q_{m,n}Q'_{m',n'}$$

$$+ \sum_{m,n} \sum_{m',n'} j \left[\frac{A_m B_n}{\alpha_{m,n}}\right] Int_1(m,m')Int_2(n,n')Q_{m,n}P'_{m',n'}$$

$$+ \sum_{m,n} j \left[\frac{A_m B_n}{\alpha_{m,n}}\right] Int_3(m,1)Int_4(n,0)P_{m,n}Q'_{1,0}$$

$$+ \sum_{m,n} \sum_{m',n'} j \left[\frac{A_m B_n}{\alpha_{m,n}}\right] Int_3(m,m')Int_4(n,n')P_{m,n}Q'_{m',n'}$$

$$\left. + \sum_{m,n} \sum_{m',n'} j \left[\frac{k_1^2 - B_n^2}{\alpha_{m,n}}\right] Int_1(m,m')Int_2(n,n')P_{m,n}P'_{m',n'} \right]$$

Equation (3-34) represents the desired expression for W_{c_2}.

It should be noted that Equation (3-34) contains both aperture mode amplitudes (primed Q's and P's) as well as waveguide mode amplitudes (unprimed Q's and P's, and R). However, when the variation is taken, W_{c_2} must be expressed in terms of waveguide mode amplitudes alone. Thus, a relationship between the two sets of mode amplitudes must be obtained so that the aperture mode amplitudes can be eliminated from W_{c_2}. This relationship can be derived by matching the tangential components of $\bar{E}_1$ and $\bar{E}_a$ over S_a and making $\bar{n}_1 \times \bar{E}_1 = 0$ over $S_g - S_a$, consistent with conditions four and six of the variational principle. Matching the x and y components of these two fields gives, respectively,

$$\sum_{m,n} P_{m,n}\cos(A_m x)\sin(B_n y) = \begin{cases} \sum_{m',n'} P'_{m',n'}\cos(A'_{m'}x')\sin(B'_{n'}y') & \text{over } S_a \\ 0 & \text{over } S_g - S_a \end{cases} \tag{3-35}$$

and

$$(I+R)\sin(A_1 x) + \sum_{m,n} Q_{m,n}\sin(A_m x)\cos(B_n y) \tag{3-36}$$

$$= \begin{cases} Q'_{1,0}\sin(A'_1 x') + \sum_{m',n'} Q'_{m',n'}\sin(A'_{m'}x')\cos(B'_{n'}y') & \text{over } S_a \\ 0 & \text{over } S_g - S_a \end{cases}$$

Equations (3-35) and (3-36) are Fourier type series for the tangential electric fields over S_g. To find relationships between the individual amplitudes, which do not depend on x and y, Equation (3-35) must be multiplied by $\cos(A_r x)$ $\sin(B_s y)$, and the resulting equation must be integrated over S_g. The subscripts r and s are integers designating a particular waveguide mode. Similarly, Equation (3-36) must be multiplied by $\sin(A_r x)$ $\cos(B_s y)$ and the resulting equation integrated over S_g. These two operations give, respectively,

$$\sum_{m,n} P_{m,n} \int_0^b\!\!\int_0^a \cos(A_m x)\cos(A_r x)\sin(B_n y)\sin(B_s y)dxdy \tag{3-37}$$

$$= \sum_{m',n'} P'_{m',n'} \int_{y_0}^{y_0+b'}\int_{x_0}^{x_0+a'} \cos(A_r x)\cos(A'_{m'}x')\sin(B_s y)\sin(B'_{n'}y')dxdy$$

and

$$(I+R)\int_0^b\int_0^a \sin(A_1x)\sin(A_rx)\cos(B_sy)dxdy \tag{3-38}$$

$$+\sum_{m,n} Q_{m,n}\int_0^b\int_0^a \sin(A_mx)\sin(A_rx)\cos(B_ny)\cos(B_sy)dxdy$$

$$= Q'_{1,0}\int_{y_0}^{y_0+b'}\int_{x_0}^{x_0+a'} \sin(A_rx)\sin(A'_1x')\cos(B_sy)dxdy$$

$$+\sum_{m',n'} Q'_{m',n'}\int_{y_0}^{y_0+b'}\int_{x_0}^{x_0+a'} \sin(A_rx)\sin(A'_{m'}x')\cos(B_sy)\cos(B'_{n'}y')dxdy$$

From Fourier series theory (27) it is known that

$$\int_0^a \cos(A_mx)\cos(A_rx)dx = \int_0^a \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{r\pi x}{a}\right)dx \tag{3-39}$$

$$= \frac{a\ \delta_{m,r}}{1+|\mathrm{sign}(r)|}$$

$$\int_0^a \sin(A_mx)\sin(A_rx)dx = \int_0^a \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{r\pi x}{a}\right)dx \tag{3-40}$$

$$= \frac{a}{2}\ \delta_{m,r}|\mathrm{sign}(r)|$$

$$\int_0^b \cos(B_ny)\cos(B_sy)dy = \int_0^b \cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{s\pi y}{b}\right)dy \tag{3-41}$$

$$= \frac{b\ \delta_{n,s}}{1+|\mathrm{sign}(s)|}$$

$$\int_0^b \sin(B_n y)\sin(B_s y)dy = \int_0^b \sin\left(\frac{n\pi y}{b}\right)\sin\left(\frac{s\pi y}{b}\right)dy \qquad (3\text{-}42)$$

$$= \frac{b}{2}\,\delta_{n,s}\,|\text{sign}(s)|$$

where

$$\delta_{m,n} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

is the Kronecker delta function and

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

If r and s are non-negative integers, as they are in the summations being used here, then the absolute value signs can be removed from all of the sign functions in Equations (3-39) through (3-42). Next, using Equations (3-39) through (3-42) to simplify the summations over m,n and the integrals of Appendix A to simplify the summations over m',n' allows Equation (3-37) to be written as

$$P_{r,s}\left(\frac{ab}{2}\right)\left[\frac{\text{sign}(s)}{1+\text{sign}(r)}\right] = \sum_{m',n'} P'_{m',n'}\text{Int}_1(r,m')\text{Int}_2(s,n') \qquad (3\text{-}43)$$

Likewise, for $r = 1 \quad s = 0$ Equation (3-38) can be written as

$$(I+R)\left(\frac{ab}{2}\right) = Q'_{1,0}\,\mathrm{Int}_3(1,1)\,\mathrm{Int}_4(0,0) + \sum_{m',n'} Q'_{m',n'}\,\mathrm{Int}_3(1,m')\,\mathrm{Int}_4(0,n') \tag{3-44}$$

Otherwise, Equation (3-38) gives

$$Q_{r,s}\left(\frac{ab}{2}\right)\left[\frac{\mathrm{sign}(r)}{1+\mathrm{sign}(s)}\right] = Q'_{1,0}\,\mathrm{Int}_3(r,1)\,\mathrm{Int}_4(s,0) + \sum_{m',n'} Q'_{m',n'}\,\mathrm{Int}_3(r,m')\,\mathrm{Int}_4(s,n') \tag{3-45}$$

Since r and s are dummy subscripts, they may be changed to m and n, respectively, without affecting the validity of Equations (3-43) through (3-45). Making this substitution, rearranging, and substituting Equation (3-24) for (I+R) into Equation (3-44) yields

$$\left(\frac{ab}{2}\right)e^{-j\beta L} + \left(\frac{ab}{2}\right)\left(1-e^{-j2\beta L}\right)R = \mathrm{Int}_3(1,1)\,\mathrm{Int}_4(0,0)\,Q'_{1,0} + \sum_{m',n'} \mathrm{Int}_3(1,m')\,\mathrm{Int}_4(0,n')\,Q'_{m',n'} \tag{3-46}$$

$$\left(\frac{ab}{2}\right)\left[\frac{\mathrm{sign}(m)}{1+\mathrm{sign}(n)}\right]Q_{m,n} = \mathrm{Int}_3(m,1)\,\mathrm{Int}_4(n,0)\,Q'_{1,0} + \sum_{m',n'} \mathrm{Int}_3(m,m')\,\mathrm{Int}_4(n,n')\,Q'_{m',n'} \tag{3-47}$$

$$\left(\frac{ab}{2}\right)\left[\frac{\text{sign}(n)}{1+\text{sign}(m)}\right]P_{m,n} = \sum_{m',n'} \text{Int}_1(m,m')\text{Int}_2(n,n')P'_{m',n'} \tag{3-48}$$

Equations (3-46) through (3-48) relate the waveguide and aperture mode amplitudes. These equations will later be solved for the aperture mode amplitudes in terms of the waveguide amplitudes so that the former amplitudes can be eliminated from all final equations.

The Fields in Regions V_2 and V_3

Now that W_{c_1} and W_{c_2} have been evaluated, only W_{c_3} in Equation (2-6) remains to be calculated. However, before this can be accomplished, trial fields must be established in regions V_2 and V_3. A logical choice of trial functions in these two regions is plane waves since the application of the boundary conditions at the planar interfaces S_2 and S_3 of Figure 4 is facilitated when plane waves are used. The expansion of a field in terms of plane waves is discussed by Borgiotti (28) and Clemmow (29).

In region V_2 of Figure 4, let the x and y components of the trial field $\bar{E}_2$ be

$$E_{x_2}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [I_x(k_x,k_y)e^{-jzk_{z_2}} + R_x(k_x,k_y)e^{jzk_{z_2}}]e^{-j[xk_x+yk_y]}dk_xdk_y \tag{3-49}$$

and

$$E_{y_2}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [I_y(k_x,k_y)e^{-jzk_{z_2}} + R_y(k_x,k_y)e^{jzk_{z_2}}]e^{-j[xk_x+yk_y]}dk_xdk_y \tag{3-50}$$

The quantities I_x, I_y, R_x, and R_y in Equations (3-49) and (3-50) are the amplitudes of the various plane wave functions. They are unknown at present and must be evaluated. It will be noticed that plane waves traveling in both the -z as well as the +z directions are included since both are present in the physical situation.

To find the z component of $\bar{E}_2$, the condition $\nabla \cdot \bar{E}_2 = 0$ expressed by Equation (3-7) may be used. In rectangular coordinates the result is

$$\frac{\partial E_{z_2}}{\partial z} = - \frac{\partial E_{x_2}}{\partial x} - \frac{\partial E_{y_2}}{\partial y}$$

Using Equations (3-49) and (3-50) in this last equation and integrating with respect to z gives

$$E_{z_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{jk_x \left[\frac{I_x e^{-jzk_{z_2}}}{-jk_{z_2}} + \frac{R_x e^{jzk_{z_2}}}{jk_{z_2}} \right] + jk_y \left[\frac{I_y e^{-jzk_{z_2}}}{-jk_{z_2}} + \frac{R_y e^{jzk_{z_2}}}{jk_{z_2}} \right] \} e^{-j[xk_x+yk_y]} dk_x dk_y$$

or

$$E_{z_2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\frac{-(k_x I_x + k_y I_y) e^{-jzk_{z_2}}}{k_{z_2}} + \frac{(k_x R_x + k_y R_y) e^{jzk_{z_2}}}{k_{z_2}} \right] \cdot e^{-j[xk_x+yk_y]} dk_x dk_y \tag{3-51}$$

All three rectangular components of $\bar{E}_2$ have now been established.

Next, $\bar{E}_2$ will be forced to satisfy the vector wave equation, as required by condition two of the variational principle. Thus,

$$\nabla \times \nabla \times \bar{E}_2 = k_2^2 \, \bar{E}_2 \tag{3-52}$$

where

$$k_2^2 = \omega^2 \mu_2 \varepsilon_2 - j\omega\mu_2\sigma_2 \tag{3-53}$$

By analogy with Equations (3-5), (3-6), (3-7), and (3-9), Equation (3-52) can be simplified to

$$-\nabla^2 E_{\nu_2} = k_2^2 E_{\nu_2} \tag{3-54}$$

where ν = x, y, or z. By using Equations (3-49), (3-50), and (3-51) it can be seen that for all three components of $\bar{E}_2$, Equation (3-54) reduces to

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_x^2 + k_y^2 + k_{z_2}^2 - k_2^2) \mathrm{Intg}_\nu \, dk_x dk_y = 0$$

where Intg_ν is the integrand in Equations (3-49), (3-50), or (3-51). The only way this last equation can hold for arbitrary x, y, and z is if

$$k_x^2 + k_y^2 + k_{z_2}^2 = k_2^2 = \omega^2 \mu_2 \varepsilon_2 - j\omega\mu_2\sigma_2$$

From this it follows at once that

$$k_{z_2} = \sqrt{k_2^2 - k_x^2 - k_y^2} \tag{3-55}$$

where the square root is selected so that

$$\mathrm{Re}(k_{z_2}) \geqq 0 \tag{3-56}$$

$$\mathrm{Im}(k_{z_2}) \leqq 0$$

This choice of square root ensures that the radiation condition and the passive properties of medium V_2 are satisfied since it causes the outward-going plane wave to be attenuated as it travels.

The trial magnetic field in region V_2 is obtained by using $\bar{H}_2 = \nabla\times\bar{E}_2/-j\omega\mu_2$. The rectangular components of this equation are

$$H_{x_2} = \left[\frac{1}{-j\omega\mu_2}\right]\left[\frac{\partial E_{z_2}}{\partial y} - \frac{\partial E_{y_2}}{\partial z}\right] \tag{3-57}$$

$$H_{y_2} = \left[\frac{1}{-j\omega\mu_2}\right]\left[\frac{\partial E_{x_2}}{\partial z} - \frac{\partial E_{z_2}}{\partial x}\right] \tag{3-58}$$

Since H_{z_2} is not needed in future calculations, it will not be determined. Using Equations (3-50) and (3-51) in Equation (3-57) yields

$$H_{x_2} = \left[\frac{1}{-j\omega\mu_2}\right] \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[j\left[\frac{k_x k_y I_x + k_y^2 I_y}{k_{z_2}}\right] e^{-jzk_{z_2}} - j\left[\frac{k_x k_y R_x + k_y^2 R_y}{k_{z_2}}\right] e^{jzk_{z_2}} \right.$$

$$\left. + jk_{z_2} I_y e^{-jzk_{z_2}} - jk_{z_2} R_y e^{jzk_{z_2}} \right] e^{-j[xk_x + yk_y]} dk_x dk_y$$

Inserting the result expressed by Equation (3-55) into this equation gives

$$H_{x_2} = \left[\frac{-1}{\omega\mu_2}\right] \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left[\frac{k_x k_y I_x}{k_{z_2}} + \frac{(k_2^2 - k_x^2) I_y}{k_{z_2}}\right] e^{-jzk_{z_2}} \right. \tag{3-59}$$

$$\left. - \left[\frac{k_x k_y R_x}{k_{z_2}} + \frac{(k_2^2 - k_x^2) R_y}{k_{z_2}}\right] e^{jzk_{z_2}} \right] e^{-j[xk_x + yk_y]} dk_x dk_y$$

Next, using Equations (3-49) and (3-51) in Equation (3-58) yields

$$H_{y_2} = \left[\frac{1}{-j\omega\mu_2}\right] \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[-jk_{z_2} I_x e^{-jzk_{z_2}} + jk_{z_2} R_x e^{jzk_{z_2}} \right.$$

$$\left. - j\left[\frac{k_x^2 I_x + k_x k_y I_y}{k_{z_2}}\right] e^{-jzk_{z_2}} + j\left[\frac{k_x^2 R_x + k_x k_y R_y}{k_{z_2}}\right] e^{jzk_{z_2}} \right] e^{-j[xk_x + yk_y]} dk_x dk_y$$

or, in view of Equation (3-55),

$$H_{y_2} = \left[\frac{1}{\omega\mu_2}\right] \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left[\frac{(k_2^2-k_y^2)I_x}{k_{z_2}} + \frac{k_x k_y I_y}{k_{z_2}}\right] e^{-jzk_{z_2}} \right. \tag{3-60}$$

$$\left. - \left[\frac{(k_2^2-k_y^2)R_x}{k_{z_2}} + \frac{k_x k_y R_y}{k_{z_2}}\right] e^{jzk_{z_2}}\right] e^{-j[xk_x+yk_y]} dk_x dk_y$$

The trial electric and magnetic fields have now been established in region V_2.

In addition to trial fields in V_2, trial fields for V_3 are also needed. Again, plane wave expansions will be used to represent these fields. Let the x and y components of $\bar{E}_3$ be

$$E_{x_3}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_x(k_x,k_y) e^{-j[xk_x+yk_y+zk_{z_3}]} dk_x dk_y \tag{3-61}$$

and

$$E_{y_3}(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} T_y(k_x,k_y) e^{-j[xk_x+yk_y+zk_{z_3}]} dk_x dk_y \tag{3-62}$$

Equations (3-61) and (3-62) contain plane waves traveling only in the +z direction in accordance with the radiation condition.

Now, the requirement $\nabla \cdot \bar{E}_3 = 0$ gives

$$\frac{\partial E_{z_3}}{\partial z} = - \frac{\partial E_{x_3}}{\partial x} - \frac{\partial E_{y_3}}{\partial y}$$

which, in view of Equations (3-61) and (3-62), becomes

$$E_{z_3} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[- \frac{k_x T_x}{k_{z_3}} - \frac{k_y T_y}{k_{z_3}} \right] e^{-j[xk_x + yk_y + zk_{z_3}]} dk_x dk_y \tag{3-63}$$

Next, $\bar{E}_3$ must be forced to satisfy the complex vector wave equation. By analogy with Equations (3-52) through (3-56) it can be seen that if $\bar{E}_3$ is to satisfy the wave equation, then

$$k_{z_3} = \sqrt{k_3^2 - k_x^2 - k_y^2} \tag{3-64}$$

where

$$k_3^2 = \omega^2 \mu_3 \epsilon_3 - j\omega\mu_3\sigma_3 \tag{3-65}$$

and

$$\mathrm{Re}(k_{z_3}) \geqq 0 \tag{3-66}$$

$$\mathrm{Im}(k_{z_3}) \leqq 0$$

The trial magnetic field in region V_3 is obtained by using

$\bar{H}_3 = \nabla\times\bar{E}_3/-j\omega\mu_3$. The rectangular components of $\bar{H}_3$ are thus

$$H_{x_3} = \left[\frac{1}{-j\omega\mu_3}\right]\left[\frac{\partial E_{z_3}}{\partial y} - \frac{\partial E_{y_3}}{\partial z}\right] \tag{3-67}$$

$$H_{y_3} = \left[\frac{1}{-j\omega\mu_3}\right]\left[\frac{\partial E_{x_3}}{\partial z} - \frac{\partial E_{z_3}}{\partial x}\right] \tag{3-68}$$

Since H_{z_3} will not be needed in future calculations, it will not be evaluated. Substituting Equations (3-62) and (3-63) into Equation (3-67) yields

$$H_{x_3} = \left[\frac{1}{-j\omega\mu_3}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{jk_xk_yT_x}{k_{z_3}} + \frac{jk_y^2T_y}{k_{z_3}} + jk_{z_3}T_y\right]e^{-j[xk_x+yk_y+zk_{z_3}]}dk_xdk_y$$

Using Equation (3-64) in this last equation gives

$$H_{x_3} = \left[\frac{-1}{\omega\mu_3}\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{k_xk_yT_x}{k_{z_3}} + \frac{(k_3^2-k_x^2)T_y}{k_{z_3}}\right]e^{-j[xk_x+yk_y+zk_{z_3}]}dk_xdk_y \tag{3-69}$$

Likewise, substituting Equations (3-61) and (3-63) into Equation (3-68) yields

$$H_{y_3} = \left(\frac{1}{\omega\mu_3}\right)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\frac{(k_3^2-k_y^2)T_x}{k_{z_3}} + \frac{k_xk_yT_y}{k_{z_3}}\right]e^{-j[xk_x+yk_y+zk_{z_3}]}dk_x\,dk_y \tag{3-70}$$

This completes the specification of the trial electric and magnetic fields in regions V_2 and V_3.

Next, the tangential components of the trial fields on either side of the z = d plane (surface S_3 of Chapter II) must be made equal. This is necessary because of conditions four and eight of the variational principle. It can be seen from Equations (3-49) and (3-61) that if $E_{x_2} = E_{x_3}$ at z = d for all x and y, then

$$I_x e^{-jdk_{z_2}} + R_x e^{jdk_{z_2}} = T_x e^{-jdk_{z_3}} \tag{3-71}$$

Similarly, if $E_{y_2} = E_{y_3}$ at z = d for all x and y, then Equations (3-50) and (3-62) require that

$$I_y e^{-jdk_{z_2}} + R_y e^{jdk_{z_2}} = T_y e^{-jdk_{z_3}} \tag{3-72}$$

Next, it must be required that $H_{x_2} = H_{x_3}$ and $H_{y_2} = H_{y_3}$ at z = d. Using Equations (3-59), (3-60), (3-69), and (3-70), leads to

$$\left[\frac{\mu_3}{\mu_2}\right]\left[\frac{k_x k_y I_x}{k_{z_2}} + \frac{(k_2^2-k_x^2)I_y}{k_{z_2}}\right]e^{-jdk_{z_2}} \tag{3-73}$$

$$-\left[\frac{\mu_3}{\mu_2}\right]\left[\frac{k_x k_y R_x}{k_{z_2}} + \frac{(k_2^2-k_x^2)R_y}{k_{z_2}}\right]e^{jdk_{z_2}} = \left[\frac{k_x k_y T_x}{k_{z_3}} + \frac{(k_3^2-k_x^2)T_y}{k_{z_3}}\right]e^{-jdk_{z_3}}$$

and

$$\left[\frac{\mu_3}{\mu_2}\right]\left[\frac{(k_2^2-k_y^2)I_x}{k_{z_2}} + \frac{k_x k_y I_y}{k_{z_2}}\right]e^{-jdk_{z_2}} \tag{3-74}$$

$$- \left[\frac{\mu_3}{\mu_2}\right]\left[\frac{(k_2^2-k_y^2)R_x}{k_{z_2}} + \frac{k_x k_y R_y}{k_{z_2}}\right]e^{jdk_{z_2}} = \left[\frac{(k_3^2-k_y^2)T_x}{k_{z_3}} + \frac{k_x k_y T_y}{k_{z_3}}\right]e^{-jdk_{z_3}}$$

Finally, Equations (3-49) and (3-50) may be evaluated at $z = 0$ to give, respectively,

$$E_{x_2}(x,y,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (I_x+R_x)e^{-jxk_x}\, e^{-jyk_y}\, dk_x dk_y$$

and

$$E_{y_2}(x,y,0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (I_y+R_y)e^{-jxk_x}\, e^{-jyk_y}\, dk_x dk_y$$

Each of the last two equations represents a double Fourier transform. Inverse transforming them gives

$$A_x(k_x,k_y) = I_x + R_x = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{x_2}(x,y,0)e^{jxk_x}\, e^{jyk_y}\, dxdy \tag{3-75}$$

$$A_y(k_x,k_y) = I_y + R_y = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_{y_2}(x,y,0)e^{jxk_x}\, e^{jyk_y}\, dxdy \tag{3-76}$$

The newly defined quantities A_x and A_y represent the Fourier transform of the tangential components of the electric field at z = 0.

It should now be noticed that Equations (3-71) through (3-76) constitute six equations in terms of the six unknowns I_x, I_y, R_x, R_y, T_x, and T_y. These equations are solved in Appendix B in terms of the soon-to-be-evaluated quantities A_x and A_y. From Appendix B the solution of these six equations is

$$I_x = \frac{N_x}{D} \tag{3-77}$$

$$I_y = \frac{N_y}{D} \tag{3-78}$$

$$R_x = A_x - I_x \tag{3-79}$$

$$R_y = A_y - I_y \tag{3-80}$$

$$T_x = e^{jdk_{z_3}}[A_x e^{jdk_{z_2}} - 2jI_x \sin(dk_{z_2})] \tag{3-81}$$

$$T_y = e^{jdk_{z_3}}[A_y e^{jdk_{z_2}} - 2jI_y \sin(dk_{z_2})] \tag{3-82}$$

where N_x, N_y, and D are defined as

$$N_x = A_x\{\cos^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2 k_{z_2}k_{z_3} + \left[\frac{\mu_3}{\mu_2}\right]k_y^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right] \tag{3-83}$$

$$-\sin^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + k_{z_2}k_{z_3} + \left[\frac{\mu_3}{\mu_2}\right]k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2 + \left[\frac{k_2}{k_3}\right]^2 k_{z_3}^2\right] + k_{z_2}k_{z_3}\right.$$

$$\left.\cdot\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right]\right]\} + A_y k_x k_y\left[\frac{\mu_3}{\mu_2}\right]\left[\left[\frac{k_2}{k_3}\right]^2 - 1\right]$$

$$N_y = A_x k_x k_y\left[\frac{\mu_3}{\mu_2}\right]\left[\left[\frac{k_2}{k_3}\right]^2 - 1\right] + A_y\ \{\cos^2(dk_{z_2}) \tag{3-84}$$

$$\cdot\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2 k_{z_2}k_{z_3} + \left[\frac{\mu_3}{\mu_2}\right]k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$-\sin^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + k_{z_2}k_{z_3} + \left[\frac{\mu_3}{\mu_2}\right]k_y^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\ \sin(dk_{z_2})\cos(dk_{z_2}) \left[\left[\frac{\mu_3}{\mu_2}\right] \left[k_{z_2}^2 + \left[\frac{k_2}{k_3}\right]^2 k_{z_3}^2 \right] + k_{z_2} k_{z_3} \left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2 \right] \right] \}$$

$$D = 2k_{z_2} k_{z_3} \left[\left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2 \cos^2(dk_{z_2}) - \sin^2(dk_{z_2}) \right] \tag{3-85}$$

$$+ j\ 2 \left[\frac{\mu_3}{\mu_2}\right] \left[\left[\frac{k_2}{k_3}\right]^2 k_{z_3}^2 + k_{z_2}^2 \right] \sin(dk_{z_2})\cos(dk_{z_2})$$

Now that the plane wave amplitude coefficients have been determined, W_{c_3} can be evaluated. From Equation (2-6), W_{c_3} is

$$W_{c_3} = -\iint_{S_a} \left(\frac{1}{2}\,\bar{E}_2 \times \bar{H}_2\right) \cdot \bar{n}_2 da$$

The integration over S_a can be extended to cover the entire surface S_2, that is, the entire x-y plane, because $\bar{n}_2 \times \bar{E}_2 = 0$ over the perfectly conducting surface $S_2 - S_a$. Noting that $\bar{n}_2 = -\bar{a}_z$ over S_2, the last expression for W_{c_3} then becomes

$$W_{c_3} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{2}\,\bar{E}_2 \times \bar{H}_2\right) \cdot \bar{a}_z \Big|_{z=0} dxdy$$

or

$$W_{c_3} = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (E_{x_2}H_{y_2} - E_{y_2}H_{x_2})\Big|_{z=0} \, dxdy \tag{3-86}$$

Before substituting the trial fields into Equation (3-86), it will be helpful to keep the integrations on k_x and k_y which appear in Equations (3-59) and (3-60) distinct from the k_x, k_y integrations appearing in Equations (3-49) and (3-50). This will be done by using primes on k_x and k_y in Equations (3-49) and (3-50). Then substituting Equations (3-49), (3-50), (3-59), and (3-60) into Equation (3-86) gives, after changing the order of integration,

$$\begin{aligned} W_{c_3} = \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} & \{[\hat{I}_x + \hat{R}_x]\left[\frac{1}{\omega\mu_2}\right] \\ & \cdot \left[\frac{(k_2^2-k_y^2)(I_x-R_x)}{k_{z_2}} + \frac{k_xk_y(I_y-R_y)}{k_{z_2}}\right] - [\hat{I}_y+\hat{R}_y]\left[\frac{-1}{\omega\mu_2}\right] \\ & \cdot \left[\frac{k_xk_y(I_x-R_x)}{k_{z_2}} + \frac{(k_2^2-k_x^2)(I_y-R_y)}{k_{z_2}}\right]\} \, e^{-jx(k_x+k'_x)} \\ & \cdot e^{-jy(k_y+k'_y)} dx\, dy\, dk'_x\, dk'_y\, dk_x\, dk_y \end{aligned} \tag{3-87}$$

where the caret (^) has the meaning

$$\hat{I}_x = I_x(k'_x, k'_y) \tag{3-88}$$

$$\hat{I}_y = I_y(k_x',k_y')$$

$$\hat{R}_x = R_x(k_x',k_y')$$

$$\hat{R}_y = R_y(k_x',k_y')$$

The caretted terms are needed because of the change from k_x,k_y to k_x',k_y' in E_{x_2} and E_{y_2}.

The integrands over x and y in Equation (3-87) can be evaluated immediately by noting that (30)

$$\int_{-\infty}^{\infty} e^{-jr(a+b)}dr = 2\pi\delta(-a-b)$$

where $\delta(x)$ is the Dirac delta function. Hence,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-jx(k_x+k_x')}e^{-jy(k_y+k_y')}dxdy \tag{3-89}$$

$$= (2\pi)^2\delta(-k_x-k_x')\delta(-k_y-k_y')$$

Equation (3-89) is the evaluation of the x and y integrations in Equation (3-87) because x and y only enter Equation (3-87) exponentially as given in Equation (3-89). After Equation (3-89) is used in Equation (3-87), the k_x',k_y' integrations can be performed immediately using the well-known properties of the delta function. Then W_{c_3} becomes

$$W_{c_3} = \frac{(2\pi)^2}{2\omega\mu_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{z_2}^{-1} \{[\tilde{I}_x + \tilde{R}_x][(k_2^2 - k_y^2)(I_x - R_x) \tag{3-90}$$

$$+ k_x k_y (I_y - R_y)] + [\tilde{I}_y + \tilde{R}_y][k_x k_y (I_x - R_x) + (k_2^2 - k_x^2)(I_y - R_y)]\} \, dk_x dk_y$$

where tilde (˜) means

$$\tilde{I}_x = I_x(-k_x, -k_y) \tag{3-91}$$

$$\tilde{I}_y = I_y(-k_x, -k_y)$$

$$\tilde{R}_x = R_x(-k_x, -k_y)$$

$$\tilde{R}_y = R_y(-k_x, -k_y)$$

$$\tilde{A}_x = A_x(-k_x, -k_y)$$

$$\tilde{A}_y = A_y(-k_x, -k_y)$$

Using Equations (3-77) through (3-80) in Equation (3-90) yields

$$W_{c_3} = \frac{(2\pi)^2}{2\omega\mu_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k_{z_2}^{-1} \left[\tilde{A}_x (k_2^2 - k_y^2) \left[\frac{2N_x}{D} - A_x \right] + \tilde{A}_x k_x k_y \right.$$

$$\cdot \left[\frac{2N_y}{D} - A_y \right] + \tilde{A}_y k_x k_y \left[\frac{2N_x}{D} - A_x \right] + \tilde{A}_y (k_2^2 - k_x^2)$$

$$\left. \cdot \left[\frac{2N_y}{D} - A_y \right] \right] dk_x dk_y$$

Rearranging terms in this last equation gives

$$W_{c_3} = \frac{(2\pi)^2}{2\omega\mu_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\frac{1}{Dk_{z_2}}\right]\Big[\{(k_2^2-k_x^2)(2N_y-DA_y) + k_xk_y \tag{3-92}$$

$$\cdot\ (2N_x-DA_x)\}\ \tilde{A}_y + \{(k_2^2-k_y^2)(2N_x-DA_x) + k_xk_y(2N_y-DA_y)\}\ \tilde{A}_x\Big]dk_x dk_y$$

In order to simplify Equation (3-92), it should be noticed that

$$2N_x - DA_x = 2A_x \ \ \{\cos^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_y^2 \tag{3-93}$$

$$\cdot \left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right] - \sin^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2})k_{z_2}k_{z_3}\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right]\ \}$$

$$+ 2A_y k_x k_y \left[\frac{\mu_3}{\mu_2}\right]\left[\left[\frac{k_2}{k_3}\right]^2 - 1\right]$$

and that

$$2N_y - DA_y = 2A_x k_x k_y \left[\frac{\mu_3}{\mu_2}\right]\left[\left[\frac{k_2}{k_3}\right]^2 - 1\right] + 2A_y\ \{\cos^2(dk_{z_2}) \tag{3-94}$$

$$\cdot \left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right] - \sin^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2\right.$$

$$\left. + \left[\frac{\mu_3}{\mu_2}\right]k_y^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right] + j\ \sin(dk_{z_2})\cos(dk_{z_2})k_{z_2}k_{z_3}$$

$$\cdot \left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] \}$$

By means of Equations (3-93) and (3-94) the first expression in the numerator of Equation (3-92) can be simplified as follows:

$$\tilde{A}_y \left[(k_2^2-k_x^2)(2N_y-DA_y) + k_xk_y(2N_x-DA_x)\right]$$

$$= 2\tilde{A}_yA_x \ \{k_xk_y\left[\frac{\mu_3}{\mu_2}\right](k_2^2-k_x^2)\left[\left[\frac{k_2}{k_3}\right]^2 - 1\right][\cos^2(dk_{z_2}) + \sin^2(dk_{z_2})]$$

$$+ \cos^2(dk_{z_2})k_xk_y\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_y^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$- \sin^2(dk_{z_2})k_xk_y\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\ \sin(dk_{z_2})\cos(dk_{z_2})k_xk_yk_{z_2}k_{z_3}\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] \}$$

$$+ 2\tilde{A}_y A_y \{k_x^2 k_y^2 \left[\frac{\mu_3}{\mu_2}\right]\left[\left[\frac{k_2}{k_3}\right]^2 - 1\right] [\cos^2(dk_{z_2}) + \sin^2(dk_{z_2})]$$

$$+ \cos^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right][k_2^2-k_x^2]$$

$$- \sin^2(dk_{z_2})\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_y^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right] [k_2^2-k_x^2]$$

$$+ j \sin(dk_{z_2})\cos(dk_{z_2})k_{z_2}k_{z_3}(k_2^2-k_x^2)\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] \}$$

Rearranging terms in this last equation yields

$$\tilde{A}_y[(k_2^2-k_x^2)(2N_y-DA_y) + k_x k_y(2N_x-DA_x)] \tag{3-95}$$

$$= 2\tilde{A}_y A_x \{\cos^2(dk_{z_2})k_x k_y\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2\left[\frac{k_2}{k_3}\right]^2$$

$$- \sin^2(dk_{z_2})k_x k_y\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2 + k_2^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j \sin(dk_{z_2})\cos(dk_{z_2})k_x k_y k_{z_2} k_{z_3}\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] \}$$

$$+ 2A_y\tilde{A}_y \{\cos^2(dk_{z_2})\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2(k_2^2-k_x^2) + k_x^2 k_{z_2}^2\left[1-\left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$- \sin^2(dk_{z_2}) \left[\frac{\mu_3}{\mu_2}\right] \left[k_{z_2}^2 (k_2^2 - k_x^2) + k_y^2 k_2^2 \left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j \sin(dk_{z_2})\cos(dk_{z_2}) k_{z_2} k_{z_3} (k_2^2 - k_x^2) \left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] \}$$

Next, Equations (3-93) and (3-94) transform the second term in the integrand of Equation (3-92) to

$$\tilde{A}_x[(k_2^2 - k_y^2)(2N_x - DA_x) + k_x k_y (2N_y - DA_y)]$$

$$= 2A_x \tilde{A}_x \ \{k_x^2 k_y^2 \left[\frac{\mu_3}{\mu_2}\right] \left[\left[\frac{k_2}{k_3}\right]^2 - 1\right] [\cos^2(dk_{z_2}) + \sin^2(dk_{z_2})]$$

$$+ \cos^2(dk_{z_2}) \left[\left[\frac{\mu_3}{\mu_2}\right] k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right] k_y^2 \left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right] [k_2^2 - k_y^2]$$

$$- \sin^2(dk_{z_2}) \left[\left[\frac{\mu_3}{\mu_2}\right] k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right] k_x^2 \left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right] [k_2^2 - k_y^2]$$

$$+ j \sin(dk_{z_2})\cos(dk_{z_2}) k_{z_2} k_{z_3} \left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] [k_2^2 - k_y^2] \ \}$$

$$+ 2A_y\tilde{A}_x \{k_x k_y \left[\frac{\mu_3}{\mu_2}\right]\left[\left[\frac{k_2}{k_3}\right]^2 - 1\right][k_2^2-k_y^2][\cos^2(dk_{z_2}) + \sin^2(dk_{z_2})]$$

$$+ \cos^2(dk_{z_2})k_x k_y\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$- \sin^2(dk_{z_2})k_x k_y\left[\left[\frac{\mu_3}{\mu_2}\right]k_{z_2}^2 + \left[\frac{\mu_3}{\mu_2}\right]k_y^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2})k_x k_y k_{z_2} k_{z_3}\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right]\ \}$$

Rearranging terms in this last equation yields

$$\tilde{A}_x[(k_2^2-k_y^2)(2N_x-DA_x) + k_x k_y(2N_y-DA_y)] \tag{3-96}$$

$$= 2A_x\tilde{A}_x \{\cos^2(dk_{z_2})\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2(k_2^2-k_y^2) + k_y^2 k_{z_2}^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$- \sin^2(dk_{z_2})\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2(k_2^2-k_y^2) + k_x^2 k_2^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j \sin(dk_{z_2})\cos(dk_{z_2})k_{z_2}k_{z_3}(k_2^2-k_y^2)\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] \}$$

$$+ 2A_y\tilde{A}_x \; \{\cos^2(dk_{z_2})k_x k_y \left[\frac{\mu_3}{\mu_2}\right] k_{z_2}^2 \left[\frac{k_2}{k_3}\right]^2$$

$$- \sin^2(dk_{z_2})k_x k_y \left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2 + k_2^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j \sin(dk_{z_2})\cos(dk_{z_2})k_x k_y k_{z_2} k_{z_3}\left[1 + \left[\frac{\mu_3 k_2}{\mu_2 k_3}\right]^2\right] \}$$

If the numerator and denominator of Equation (3-92) are both divided by $2k_{z_2}^2$, then W_{c_3} --with the help of Equations (3-95), (3-96), and (3-85)--can be rewritten as

$$W_{c_3} = \frac{(2\pi)^2}{2\omega\mu_2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left[\left[\frac{N_1}{Den}\right]A_x\tilde{A}_x + \left[\frac{N_2}{Den}\right]A_y\tilde{A}_y \right. \tag{3-97}$$

$$\left. + \left[\frac{N_3}{Den}\right]\left(A_x\tilde{A}_y + A_y\tilde{A}_x\right)\right]dk_x dk_y$$

where N_1, N_2, N_3, and Den are defined as

$$N_1 = \cos^2(dk_{z_2})\left[\frac{\mu_3}{\mu_2}\right]\left[k_2^2 - k_y^2\left[\frac{k_2}{k_3}\right]^2\right] \tag{3-98}$$

$$- \left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right]^2\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2(k_2^2-k_y^2) + k_2^2k_x^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right]\cos(dk_{z_2})k_{z_3}(k_2^2-k_y^2)\left[1 + \left[\frac{\mu_3}{\mu_2}\right]^2\left[\frac{k_2}{k_3}\right]^2\right]$$

$$N_2 = \cos^2(dk_{z_2})\left[\frac{\mu_3}{\mu_2}\right]\left[k_2^2 - k_x^2\left[\frac{k_2}{k_3}\right]^2\right] \tag{3-99}$$

$$- \left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right]^2\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2(k_2^2-k_x^2) + k_2^2k_y^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right]\cos(dk_{z_2})k_{z_3}(k_2^2-k_x^2)\left[1 + \left[\frac{\mu_3}{\mu_2}\right]^2\left[\frac{k_2}{k_3}\right]^2\right]$$

$$N_3 = k_xk_y\{\cos^2(dk_{z_2})\left[\frac{\mu_3}{\mu_2}\right]\left[\frac{k_2}{k_3}\right]^2 \tag{3-100}$$

$$- \left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right]^2\left[\frac{\mu_3}{\mu_2}\right]\left[k_{z_2}^2 + k_2^2\left[1 - \left[\frac{k_2}{k_3}\right]^2\right]\right]$$

$$+ j\left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right]\cos(dk_{z_2})k_{z_3}\left[1 + \left[\frac{\mu_3}{\mu_2}\right]^2\left[\frac{k_2}{k_3}\right]^2\right]\ \}$$

and

$$\text{Den} = k_{z_3}\left[\left[\frac{\mu_3}{\mu_2}\right]^2\left[\frac{k_2}{k_3}\right]^2\cos^2(dk_{z_2}) - \sin^2(dk_{z_2})\right] \tag{3-101}$$

$$+ j\left[\frac{\mu_3}{\mu_2}\right]\left[\left[\frac{k_2}{k_3}\right]^2 k_{z_3}^2 + k_{z_2}^2\right]\left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right]\cos(dk_{z_2})$$

It should be noticed that Equation (3-97) expresses W_{c_3} in terms of the known quantities N_1, N_2, N_3, Den, and in terms of the unevaluated quantities A_x and A_y. Before W_{c_3} can be used, the Fourier transforms A_x and A_y must be evaluated. To this end, it will be noticed that condition five of the variational principle requires $E_{x_2} = E_{y_2} = 0$ over the perfect conductor $S_2 - S_a$, while condition four requires $E_{x_2} = E_{x_a}$ and $E_{y_2} = E_{y_a}$ over S_a. Accordingly, Equations (3-75) and (3-76) become, respectively,

$$A_x(k_x,k_y) = \left[\frac{1}{2\pi}\right]^2 \int_{y_0}^{y_0+b'} \int_{x_0}^{x_0+a'} E_{x_a}(x,y)e^{jxk_x}\, e^{jyk_y}\, dxdy \tag{3-102}$$

$$A_y(k_x,k_y) = \left[\frac{1}{2\pi}\right]^2 \int_{y_0}^{y_0+b'} \int_{x_0}^{x_0+a'} E_{y_a}(x,y)e^{jxk_x} e^{jyk_y} \, dxdy \quad (3\text{-}103)$$

Next, substituting Equations (3-29) and (3-30) into Equations (3-102) and (3-103) yields

$$A_x(k_x,k_y) = \left[\frac{1}{2\pi}\right]^2 \sum_{m',n'} P'_{m',n'} \left\{\int_{x_0}^{x_0+a'} \cos(A'_{m'}x')e^{jxk_x} \, dx\right\} \quad (3\text{-}104)$$

$$\cdot \left\{\int_{y_0}^{y_0+b'} \sin(B'_{n'}y')e^{jyk_y} \, dy\right\}$$

(3-105)

$$A_y(k_x,k_y) = \left[\frac{1}{2\pi}\right]^2 Q'_{1,0} \left\{\int_{x_0}^{x_0+a'} \sin(A'_1x')e^{jxk_x} \, dx\right\}\left\{\int_{y_0}^{y_0+b'} e^{jyk_y} \, dy\right\}$$

$$+ \left[\frac{1}{2\pi}\right]^2 \sum_{m',n'} Q'_{m',n'} \left\{\int_{x_0}^{x_0+a'} \sin(A'_{m'}x')e^{jxk_x} \, dx\right\}$$

$$\cdot \left\{\int_{y_0}^{y_0+b'} \cos(B'_{n'}y')e^{jyk_y} \, dy\right\}$$

The integrals involved in Equations (3-104) and (3-105) are evaluated in Appendix C. Using the notation adopted there allows Equations (3-104) and (3-105) to be written as

$$A_x(k_x,k_y) = \left[\frac{1}{2\pi}\right]^2 e^{jx_0k_x} e^{jy_0k_y} \sum_{m',n'} P'_{m',n'} \tag{3-106}$$

$$\cdot\ Ic_x(m',k_x)Is_y(n',k_y)$$

$$A_y(k_x,k_y) = \left[\frac{1}{2\pi}\right]^2 e^{jx_0k_x} e^{jy_0k_y} [Q'_{1,0}Is_x(1,k_x) \tag{3-107}$$

$$\cdot\ Ic_y(0,k_y) + \sum_{m',n'} Q'_{m',n'}Is_x(m',k_x)Ic_y(n',k_y)]$$

Before Equations (3-106) and (3-107) are used in Equation (3-97), $\tilde{A}_x$ and $\tilde{A}_y$ must be evaluated. From Equation (3-91), it is clear that

$$\tilde{A}_x = A_x(-k_x,-k_y) \tag{3-108}$$

$$\tilde{A}_y = A_y(-k_x,-k_y) \tag{3-109}$$

while from Appendix C

$$Is_x(m',-k_x) = Is_x^*(m',k_x) \tag{3-110}$$

$$Ic_x(m',-k_x) = Ic_x^*(m',k_x) \tag{3-111}$$

$$Is_y(n',-k_y) = Is_y^*(n',k_y) \tag{3-112}$$

$$Ic_y(n',-k_y) = Ic_y^*(n',k_y) \tag{3-113}$$

Using Equations (3-106), (3-107), and (3-110) through (3-113) in Equations (3-108) and (3-109) yields

$$\tilde{A}_x = \left[\frac{1}{2\pi}\right]^2 e^{-jx_0k_x} e^{-jy_0k_y} \sum_{mm',nn'} P'_{mm',nn'} Ic_x^*(mm',k_x) \cdot Is_y^*(nn',k_y) \tag{3-114}$$

$$\tilde{A}_y = \left[\frac{1}{2\pi}\right]^2 e^{-jx_0k_x} e^{-jy_0k_y} \Bigg[Q'_{1,0} Is_x^*(1,k_x) Ic_y^*(0,k_y) + \sum_{mm',nn'} Q'_{mm',nn'} Is_x^*(mm',k_x) Ic_y^*(nn',k_y)\Bigg] \tag{3-115}$$

It should be noticed that the summations in Equations (3-114) and (3-115) have been changed from m',n' to mm',nn' in order to keep the subscripts distinct in later manipulations. The primes on mm' and nn' indicate that the summations are over the aperture modes.

Equations (3-106), (3-107), (3-114), and (3-115) are now ready to be substituted into Equation (3-97). It should be noticed that the exponential terms in A_x and A_y will cancel the corresponding exponential terms in $\tilde{A}_x$ and $\tilde{A}_y$ when these four terms are placed in Equation (3-97). Thus, Equation (3-97) becomes

$$W_{c_3} = \left[\frac{1}{2\omega\mu_1}\right]\left[\frac{\mu_1}{\mu_2}\right](2\pi)^2 \left[\frac{1}{2\pi}\right]^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Bigg[\tag{3-116}$$

$$\sum_{m',n'} \sum_{mm',nn'} P'_{m',n'} P'_{mm',nn'} \left[\frac{N_1}{Den}\right] Ic_x(m',k_x) Is_y(n',k_y) Ic_x^*(mm',k_x) Is_y^*(nn',k_y)$$

$$+ \sum_{m',n'} \sum_{mm',nn'} Q'_{m',n'} Q'_{mm',nn'} \left[\frac{N_2}{Den}\right] Is_x(m',k_x) Ic_y(n',k_y) Is_x^*(mm',k_x) Ic_y^*(nn',k_y)$$

$$+ Q'_{1,0} Q'_{1,0} \left[\frac{N_2}{Den}\right] Is_x(1,k_x) Ic_y(0,k_y) Is_x^*(1,k_x) Ic_y^*(0,k_y)$$

$$+ \sum_{mm',nn'} Q'_{1,0} Q'_{mm',nn'} \left[\frac{N_2}{Den}\right] Is_x(1,k_x) Ic_y(0,k_y) Is_x^*(mm',k_x) Ic_y^*(nn',k_y)$$

$$+ \sum_{m',n'} Q'_{m',n'} Q'_{1,0} \left[\frac{N_2}{Den}\right] Is_x(m',k_x) Ic_y(n',k_y) Is_x^*(1,k_x) Ic_y^*(0,k_y)$$

$$+ \sum_{m',n'} P'_{m',n'} Q'_{1,0} \left[\frac{N_3}{Den}\right] Ic_x(m',k_x) Is_y(n',k_y) Is_x^*(1,k_x) Ic_y^*(0,k_y)$$

$$+ \sum_{m',n'} \sum_{mm',nn'} P'_{m',n'} Q'_{mm',nn'} \left[\frac{N_3}{Den}\right] Ic_x(m',k_x) Is_y(n',k_y) Is_x^*(mm',k_x) Ic_y^*(nn',k_y)$$

$$+ \sum_{mm',nn'} Q'_{1,0} P'_{mm',nn'} \left[\frac{N_3}{Den}\right] Is_x(1,k_x) Ic_y(0,k_y) Ic_x^*(mm',k_x) Is_y^*(nn',k_y)$$

$$+ \sum_{m',n'} \sum_{mm',nn'} Q'_{m',n'} P'_{mm',nn'} \left[\frac{N_3}{Den}\right] Is_x(m',k_x) Ic_y(n',k_y)$$

$$\left. Ic_x^*(mm',k_x) Is_y^*(nn',k_y) \right] dk_x dk_y$$

To reduce the size of the region over which the integrand in Equation (3-116) must be integrated, the present integration over all four quadrants of the k_x - k_y plane will be transformed into an integration over the first quadrant alone. To do this, Equation (3-116) will first be represented as

$$W_{c_3} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x,k_y)dk_x dk_y \tag{3-117}$$

where $G(k_x,k_y)$ represents the entire integrand of Equation (3-116). Next, the integral in Equation (3-117) is split into four parts as follows:

$$W_{c_3} = \int_0^{\infty} \int_0^{\infty} G(k_x,k_y)dk_x dk_y + \int_0^{\infty} \int_{-\infty}^{0} G(k_x,k_y)dk_x dk_y \tag{3-118}$$

$$+ \int_{-\infty}^{0} \int_{-\infty}^{0} G(k_x,k_y)dk_x dk_y + \int_{-\infty}^{0} \int_0^{\infty} G(k_x,k_y)dk_x dk_y$$

Now, making a change of variables in the last three integrals on the right and then collecting terms transforms Equation (3-118) to

$$W_{c_3} = \int_0^{\infty} \int_0^{\infty} [G(k_x,k_y) + G(-k_x,k_y) + G(-k_x,-k_y) \tag{3-119}$$

$$+ G(k_x,-k_y)]dk_x dk_y$$

This equation can be considerably simplified by making the following observations. From Equations (3-98), (3-55), and (3-64), it can be seen that

$$N_1(k_x,k_y) = N_1(-k_x,k_y) = N_1(-k_x,-k_y) = N_1(k_x,-k_y) \qquad (3\text{-}120)$$

while from Equations (3-99), (3-55), and (3-64), it can be seen that

$$N_2(k_x,k_y) = N_2(-k_x,k_y) = N_2(-k_x,-k_y) = N_2(k_x,-k_y) \qquad (3\text{-}121)$$

From Equations (3-101), (3-55), and (3-64) it follows that

$$\mathrm{Den}(k_x,k_y) = \mathrm{Den}(-k_x,k_y) = \mathrm{Den}(-k_x,-k_y) = \mathrm{Den}(k_x,-k_y) \qquad (3\text{-}122)$$

while from Equations (3-100), (3-55), and (3-64) it follows that

$$N_3(k_x,k_y) = N_3(-k_x,-k_y) = -N_3(-k_x,k_y) = -N_3(k_x,-k_y) \qquad (3\text{-}123)$$

Using Equations (3-119) through (3-123) in Equation (3-116) yields

$$W_{c_3} = \left[\frac{1}{2\omega\mu_1}\right]\left[\frac{\mu_1}{\mu_2}\right]\left[\frac{1}{2\pi}\right]^2 \int_0^\infty \int_0^\infty \Bigg[\qquad (3\text{-}124)$$

$$Q'_{1,0}Q'_{1,0}\left[\frac{N_2}{\mathrm{Den}}\right] \{Is_x(1,k_x)Ic_y(0,k_y)Is_x^*(1,k_x)Ic_y^*(0,k_y)$$

$$+ Is_x(1,-k_x)Ic_y(0,-k_y)Is_x^*(1,-k_x)Ic_y^*(0,-k_y)$$

$$+ \, Is_x(1,-k_x)Ic_y(0,k_y)Is_x^*(1,-k_x)Ic_y^*(0,k_y)$$

$$+ \, Is_x(1,k_x)Ic_y(0,-k_y)Is_x^*(1,k_x)Ic_y^*(0,-k_y) \, \}$$

$$+ \sum_{mm',nn'} Q'_{1,0}Q'_{mm',nn'}\left[\frac{N_2}{Den}\right] \{Is_x(1,k_x)Ic_y(0,k_y)Is_x^*(mm',k_x)Ic_y^*(nn',k_y)$$

$$+ \, Is_x(1,-k_x)Ic_y(0,-k_y)Is_x^*(mm',-k_x)Ic_y^*(nn',-k_y)$$

$$+ \, Is_x(1,-k_x)Ic_y(0,k_y)Is_x^*(mm',-k_x)Ic_y^*(nn',k_y)$$

$$+ \, Is_x(1,k_x)Ic_y(0,-k_y)Is_x^*(mm',k_x)Ic_y^*(nn',-k_y) \, \}$$

$$+ \sum_{mm',nn'} Q'_{1,0}P'_{mm',nn'}\left[\frac{N_3}{Den}\right] \{Is_x(1,k_x)Ic_y(0,k_y)Ic_x^*(mm',k_x)Is_y^*(nn',k_y)$$

$$+ \, Is_x(1,-k_x)Ic_y(0,-k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',-k_y)$$

$$- \, Is_x(1,-k_x)Ic_y(0,k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',k_y)$$

$$- \, Is_x(1,k_x)Ic_y(0,-k_y)Ic_x^*(mm',k_x)Is_y^*(nn',-k_y) \, \}$$

$$+ \sum_{m',n'} Q'_{m',n'}Q'_{1,0}\left[\frac{N_2}{Den}\right] \{Is_x(m',k_x)Ic_y(n',k_y)Is_x^*(1,k_x)Ic_y^*(0,k_y)$$

$$+ \, Is_x(m',-k_x)Ic_y(n',-k_y)Is_x^*(1,-k_x)Ic_y^*(0,-k_y)$$

$$+ Is_x(m',-k_x)Ic_y(n',k_y)Is_x^*(1,-k_x)Ic_y^*(0,k_y)$$

$$+ Is_x(m',k_x)Ic_y(n',-k_y)Is_x^*(1,k_x)Ic_y^*(0,-k_y) \}$$

$$+ \sum_{m',n'} \sum_{mm',nn'} Q'_{m',n'} Q'_{mm',nn'} \left[\frac{N_2}{Den}\right] \{Is_x(m',k_x)Ic_y(n',k_y)$$

$$\cdot Is_x^*(mm',k_x)Ic_y^*(nn',k_y)$$

$$+ Is_x(m',-k_x)Ic_y(n',-k_y)Is_x^*(mm',-k_x)Ic_y^*(nn',-k_y)$$

$$+ Is_x(m',-k_x)Ic_y(n',k_y)Is_x^*(mm',-k_x)Ic_y^*(nn',k_y)$$

$$+ Is_x(m',k_x)Ic_y(n',-k_y)Is_x^*(mm',k_x)Ic_y^*(nn',-k_y) \}$$

$$+ \sum_{m',n'} \sum_{mm',nn'} Q'_{m',n'} P'_{mm',nn'} \left[\frac{N_3}{Den}\right] \{Is_x(m',k_x)Ic_y(n',k_y)$$

$$\cdot Ic_x^*(mm',k_x)Is_y^*(nn',k_y)$$

$$+ Is_x(m',-k_x)Ic_y(n',-k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',-k_y)$$

$$- Is_x(m',-k_x)Ic_y(n',k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',k_y)$$

$$- Is_x(m',k_x)Ic_y(n',-k_y)Ic_x^*(mm',k_x)Is_y^*(nn',-k_y) \}$$

$$+ \sum_{m',n'} P'_{m',n'} Q'_{1,0} \left[\frac{N_3}{Den}\right] \{ Ic_x(m',k_x) Is_y(n',k_y) Is^*_x(1,k_x) Ic^*_y(0,k_y)$$

$$+ Ic_x(m',-k_x) Is_y(n',-k_y) Is^*_x(1,-k_x) Ic^*_y(0,-k_y)$$

$$- Ic_x(m',-k_x) Is_y(n',k_y) Is^*_x(1,-k_x) Ic^*_y(0,k_y)$$

$$- Ic_x(m',k_x) Is_y(n',-k_y) Is^*_x(1,k_x) Ic^*_y(0,-k_y) \}$$

$$+ \sum_{m',n'} \sum_{mm',nn'} P'_{m',n'} Q'_{mm',nn'} \left[\frac{N_3}{Den}\right] \{ Ic_x(m',k_x) Is_y(n',k_y)$$

$$\cdot Is^*_x(mm',k_x) Ic^*_y(nn',k_y)$$

$$+ Ic_x(m',-k_x) Is_y(n',-k_y) Is^*_x(mm',-k_x) Ic^*_y(nn',-k_y)$$

$$- Ic_x(m',-k_x) Is_y(n',k_y) Is^*_x(mm',-k_x) Ic^*_y(nn',k_y)$$

$$- Ic_x(m',k_x) Is_y(n',-k_y) Is^*_x(mm',k_x) Ic^*_y(nn',-k_y) \}$$

$$+ \sum_{m',n'} \sum_{mm',nn'} P'_{m',n'} P'_{mm',nn'} \left[\frac{N_1}{Den}\right] \{ Ic_x(m',k_x) Is_y(n',k_y)$$

$$\cdot Ic^*_x(mm',k_x) Is^*_y(nn',k_y)$$

$$+ Ic_x(m',-k_x)Is_y(n',-k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',-k_y)$$

$$+ Ic_x(m',-k_x)Is_y(n',k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',k_y)$$

$$\left. + Ic_x(m',k_x)Is_y(n',-k_y)Ic_x^*(mm',k_x)Is_y^*(nn',-k_y)\}\right] dk_x dk_y$$

The second bracketed term in Equation (3-124) can be simplified with the aid of Equations (3-110) through (3-113) as follows:

$$Is_x(1,k_x)Ic_y(0,k_y)Is_x^*(mm',k_x)Ic_y^*(nn',k_y) \tag{3-125}$$

$$+ Is_x(1,-k_x)Ic_y(0,-k_y)Is_x^*(mm',-k_x)Ic_y^*(nn',-k_y)$$

$$+ Is_x(1,-k_x)Ic_y(0,k_y)Is_x^*(mm',-k_x)Ic_y^*(nn',k_y)$$

$$+ Is_x(1,k_x)Ic_y(0,-k_y)Is_x^*(mm',k_x)Ic_y^*(nn',-k_y)$$

$$= 2\ Re[Is_x(1,k_x)Ic_y(0,k_y)Is_x^*(mm',k_x)Ic_y^*(nn',k_y)$$

$$+ Is_x(1,k_x)Ic_y^*(0,k_y)Is_x^*(mm',k_x)Ic_y(nn',k_y)]$$

$$= 2\ Re[Is_x(1,k_x)Is_x^*(mm',k_x)\ \{Ic_y(0,k_y)Ic_y^*(nn',k_y)$$

$$+ Ic_y^*(0,k_y)Ic_y(nn',k_y)\}]$$

$$= 4\ \mathrm{Re}[Is_x(1,k_x)Is_x^*(mm',k_x)]\ \mathrm{Re}[Ic_y(0,k_y)Ic_y^*(nn',k_y)]$$

In a similar manner the third bracketed term in Equation (3-124) can be simplified with the aid of Equations (3-110) through (3-113) as follows:

$$Is_x(1,k_x)Ic_y(0,k_y)Ic_x^*(mm',k_x)Is_y^*(nn',k_y) \tag{3-126}$$

$$+ Is_x(1,-k_x)Ic_y(0,-k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',-k_y)$$

$$- Is_x(1,-k_x)Ic_y(0,k_y)Ic_x^*(mm',-k_x)Is_y^*(nn',k_y)$$

$$- Is_x(1,k_x)Ic_y(0,-k_y)Ic_x^*(mm',k_x)Is_y^*(nn',-k_y)$$

$$= 2\ \mathrm{Re}[Is_x(1,k_x)Ic_y(0,k_y)Ic_x^*(mm',k_x)Is_y^*(nn',k_y)$$

$$- Is_x(1,k_x)Ic_y^*(0,k_y)Ic_x^*(mm',k_x)Is_y(nn',k_y)]$$

$$= 2\ \mathrm{Re}[Is_x(1,k_x)Ic_x^*(mm',k_x)\ \{Ic_y(0,k_y)Is_y^*(nn',k_y)$$

$$- Ic_y^*(0,k_y)Is_y(nn',k_y)\}]$$

$$= 2\ \mathrm{Re}[Is_x(1,k_x)Ic_x^*(mm',k_x)(-2j)\mathrm{Im}\{Ic_y^*(0,k_y)Is_y(nn',k_y)\}]$$

$$= 4\ \mathrm{Im}[Is_x(1,k_x)Ic_x^*(mm',k_x)]\mathrm{Im}[Ic_y^*(0,k_y)Is_y(nn',k_y)]$$

The remaining bracketed terms in Equation (3-124) can be simplified by analogy with Equations (3-125) and (3-126). Using these simplifications in Equation (3-124) yields

$$W_{c_3} = \left[\frac{1}{2\omega\mu_1}\right]\left[\frac{\mu_1}{\mu_2}\right]\left[\frac{1}{\pi^2}\right]\int_0^\infty\int_0^\infty \Big[\tag{3-127}$$

$$Q'_{1,0}Q'_{1,0}\left[\frac{N_2}{Den}\right]\mathrm{Re}\{Is_x(1,k_x)Is_x^*(1,k_x)\}\ \mathrm{Re}\{Ic_y(0,k_y)Ic_y^*(0,k_y)\}$$

$$+\sum_{mm',nn'} Q'_{1,0}Q'_{mm',nn'}\left[\frac{N_2}{Den}\right]\ \mathrm{Re}\{Is_x(1,k_x)Is_x^*(mm',k_x)\}$$

$$\cdot\ \mathrm{Re}\{Ic_y(0,k_y)Ic_y^*(nn',k_y)\}$$

$$+\sum_{mm',nn'} Q'_{1,0}P'_{mm',nn'}\left[\frac{N_3}{Den}\right]\mathrm{Im}\{Is_x(1,k_x)Ic_x^*(mm',k_x)\}$$

$$\cdot\ \mathrm{Im}\{Ic_y^*(0,k_y)Is_y(nn',k_y)\}$$

$$+\sum_{m',n'} Q'_{m',n'}Q'_{1,0}\left[\frac{N_2}{Den}\right]\mathrm{Re}\{Is_x(m',k_x)Is_x^*(1,k_x)\}\mathrm{Re}\{Ic_y(n',k_y)Ic_y^*(0,k_y)\}$$

$$+\sum_{m',n'}\ \sum_{mm',nn'} Q'_{m',n'}Q'_{mm',nn'}\left[\frac{N_2}{Den}\right]\mathrm{Re}\{Is_x(m',k_x)Is_x^*(mm',k_x)\}$$

$$\cdot\ \mathrm{Re}\{Ic_y(n',k_y)Ic_y^*(nn',k_y)\}$$

$$+\sum_{m',n'}\ \sum_{mm',nn'} Q'_{m',n'}P'_{mm',nn'}\left[\frac{N_3}{Den}\right]\mathrm{Im}\{Is_x(m',k_x)Ic_x^*(mm',k_x)\}$$

$$\cdot \ \mathrm{Im}\{Ic_y^*(n',k_y)Is_y(nn',k_y)\}$$

$$+ \sum_{m',n'} P'_{m',n'}Q'_{1,0}\left[\frac{N_3}{Den}\right] \mathrm{Im}\{Ic_x(m',k_x)Is_x^*(1,k_x)\}$$

$$\cdot \ \mathrm{Im}\{Is_y^*(n',k_y)Ic_y(0,k_y)\}$$

$$+ \sum_{m',n'} \sum_{mm',nn'} P'_{m',n'}Q'_{mm',nn'}\left[\frac{N_3}{Den}\right] \mathrm{Im}\{Ic_x(m',k_x)Is_x^*(mm',k_x)\}$$

$$\cdot \ \mathrm{Im}\{Is_y^*(n',k_y)Ic_y(nn',k_y)\}$$

$$+ \sum_{m',n'} \sum_{mm',nn'} P'_{m',n'}P'_{mm',nn'}\left[\frac{N_1}{Den}\right] \mathrm{Re}\{Ic_x(m',k_x)Ic_x^*(mm',k_x)\}$$

$$\left. \cdot \ \mathrm{Re}\{Is_y(n',k_y)Is_y^*(nn',k_y)\}\right] dk_x dk_y$$

Equation (3-127) is the general form of W_{c_3} for an arbitrary number of modes and an arbitrary set of medium parameters. It should be noticed that the plane wave amplitude coefficients have been eliminated from W_{c_3} and that the only unknown quantities in Equation (3-127) are the aperture mode amplitudes. The quantities N_1, N_2, N_3, and Den are given by Equations (3-98) through (3-101), while Is_x, Ic_x, Is_y, and Ic_y are given in Appendix C. It should also be noticed that the original, doubly infinite, six-fold integral expression for W_{c_3}, Equation (3-87), has been reduced to a singly infinite, double integral expression, Equation (3-127). This reduction considerably simplifies the evaluation of W_{c_3}.

Although the form of W_{c_3} has been considerably simplified, the integrals in Equation (3-127) still cannot be evaluated in closed form; they have to be numerically integrated. In contrast, the integrals in W_{c_1} and W_{c_2} are evaluated in closed form. Equation (3-127) must be further changed to facilitate the numerical integration. This point is covered in Chapter V when regions V_2 and V_3 are both lossless or only slightly lossy.

Before taking up the numerical integration, a matrix solution will now be obtained for the waveguide and aperture mode amplitudes in terms of the general expressions for W_{c_1}, W_{c_2}, and W_{c_3}, which have been derived in this chapter.

CHAPTER IV

MATRIX SOLUTION FOR THE MODE AMPLITUDES

In Chapter III the energy expressions W_{c_1}, W_{c_2}, and W_{c_3} were evaluated for the slot antenna shown in Figures 4 and 5, assuming that the waveguide is excited by a dominant mode wave and that all higher order modes are evanescent. These three energy expressions are needed to determine the mode amplitudes, using the variational principle of Chapter II. According to this principle, the set of trial waveguide mode amplitudes that makes W_c stationary is obtained by setting the partial derivative of W_c with respect to one of the waveguide mode amplitudes equal to zero. Repeating this process for each waveguide mode amplitude produces a system of equations in terms of these unknown amplitudes. The solution of this system of equations determines the set of trial amplitudes which best approximates the true waveguide mode amplitudes. In this chapter it will be shown that these equations are linear and consequently can be solved using matrix techniques. The fact that the equations resulting from this new approach can be solved by matrix methods is significant. It implies that more accurate results can be obtained more easily by using the new variational principle than by using comparable approaches which produce nonlinear equations.

To calculate the partial derivatives of W_c, the partial derivatives of W_{c_1}, W_{c_2}, and W_{c_3} must first be obtained, since $W_c = W_{c_1} + W_{c_2} + W_{c_3}$. As a prelude to calculating these derivatives, it should be noticed that the trial electric and magnetic fields in regions V_2 and V_3 and in the aperture can all be expressed as a linear combination of the waveguide mode amplitudes. This statement follows from the principle of superposition and the fact that the waveguide field has been represented--in Equations (3-1) and (3-2)--as a linear combination of the trial waveguide mode amplitudes. If this set of amplitudes--that is, R, $Q_{m,n}$ and $P_{m,n}$--is designated by the set $\{x_i\}$, then, by superposition, each trial $\bar{E}_i$ and $\bar{H}_i$ in W_{c_1}, W_{c_2}, and W_{c_3} can be expressed as a linear combination of the x_i's. Hence, W_{c_1}, W_{c_2}, and W_{c_3} are all quadratic functions of the x_i's, because each of these energy expressions is formed by the product $\bar{E}_i \times \bar{H}_i$.

Since each energy expression is a quadratic function of the x_i's, then W_{c_1}, for example, can be written as

$$W_{c_1} = K_1 + \sum_{\ell=1}^{N} c_{1_\ell} x_\ell + \sum_{\ell=1}^{N} x_\ell \left(\sum_{j=1}^{N} m_{1_{\ell j}} x_j \right) \tag{4-1}$$

where K_1 and each c_{1_ℓ} and $m_{1_{\ell j}}$ are constants independent of each x_ℓ. It is assumed in Equation (4-1) that N of the x_i's are being used to approximate the true waveguide field. Equation (4-1) is the most general quadratic form that W_{c_1} can have. For convenience in manipulation Equation (4-1) will be rewritten in matrix form as

$$W_{c_1} = K_1 + C_1]^T \cdot X] + X]^T \cdot [M_1] \cdot X] \tag{4-2}$$

In this chapter the following matrix notation will be used. A bracket to the right of a quantity--for example, $C_1]$ --represents a column matrix, the i^{th} *element of* which is c_{1_i}. Brackets on both sides of a quantity--for example, $[M_1]$--represent a square matrix, the ij^{th} element of which is $m_{1_{ij}}$. The transpose of a column matrix or a square matrix will be denoted by a superscript T. Since N of the x_i's are used in Equation (4-1), $C_1]$ has the dimensions $1 \times N$, while $[M_1]$ has the dimensions $N \times N$ in Equation (4-2). By analogy with Equation (4-2), W_{c_2} and W_{c_3} can be written as

$$W_{c_2} = K_2 + C_2]^T \cdot X] + X]^T \cdot [M_2] \cdot X] \tag{4-3}$$

$$W_{c_3} = K_3 + C_3]^T \cdot X] + X]^T \cdot [M_3] \cdot X] \tag{4-4}$$

The variational theorem developed in Chapter II states that the set of x_i's that makes W_c stationary is the one that is the solution of the system of equations

$$G] = 0] \tag{4-5}$$

where $G]$ is the gradient of W_c; that is, $G]$ is the column matrix whose i^{th} element is

$$g_i = \frac{\partial W_c}{\partial x_i} = \frac{\partial W_{c_1}}{\partial x_i} + \frac{\partial W_{c_2}}{\partial x_i} + \frac{\partial W_{c_3}}{\partial x_i} \tag{4-6}$$

Since W_{c_1}, W_{c_2}, and W_{c_3} all have the same quadratic form, only the partial derivative of one of them needs to be examined, as for example $\partial W_{c_1}/\partial x_i$, in order to determine $\partial W_c/\partial x_i$ in Equation (4-6). From Equation (4-1) it can be seen that

$$\frac{\partial W_{c_1}}{\partial x_i} = \frac{\partial}{\partial x_i}\left[K_1 + \sum_{\ell=1}^{N} c_{1_\ell} x_\ell + \sum_{\ell=1}^{N} x_\ell \left(\sum_{j=1}^{N} m_{1_{\ell j}} x_j\right)\right]$$

$$= c_{1_i} + \sum_{j=1}^{N} m_{1_{ij}} x_j + \sum_{\ell=1}^{N} x_\ell m_{1_{\ell i}}$$

$$= c_{1_i} + \sum_{j=1}^{N} m_{1_{ij}} x_j + \sum_{j=1}^{N} m_{1_{ji}} x_j$$

Converting this last equation to matrix form and then using it along with the analogous results for $\partial W_{c_2}/\partial x_i$ and $\partial W_{c_3}/\partial x_i$ in Equation (4-6) yields

$$G] = (C_1] + C_2] + C_3]) + ([M_1] + [M_2] \tag{4-7}$$

$$+ [M_3] + [M_1]^T + [M_2]^T + [M_3]^T) \cdot X]$$

Defining $C_0]$ and $[M_0]$ as

$$C_0] = C_1] + C_2] + C_3] \tag{4-8}$$

and

$$[M_0] = [M_1] + [M_2] + [M_3] + [M_1]^T + [M_2]^T + [M_3]^T \qquad (4\text{-}9)$$

allows Equation (4-7) to be written as

$$G] = C_0] + [M_0] \cdot X] \qquad (4\text{-}10)$$

From Equations (4-5) and (4-10) it can be seen that the set of waveguide mode amplitudes, X], that causes W_c to be stationary, is given by

$$X] = -[M_0]^{-1} \cdot C_0] \qquad (4\text{-}11)$$

Thus, to find the solution matrix X], the matrices $C_1]$, $C_2]$, $C_3]$, $[M_1]$, $[M_2]$, and $[M_3]$ must first be found and then used in Equations (4-8), (4-9), and (4-11).

It will be noticed that W_{c_1} has already been expressed in terms of X] in Equation (3-27). From this equation it can be seen that

$$W_{c_1} = K_1 + C_1]^T \cdot X] \qquad (4\text{-}12)$$

Since $[M_1]$ is zero it can be eliminated from all equations. It should also be noticed from Equation (3-27) that $c_{1_i} = 0$ for all i except for the i corresponding to R.

From Equation (3-34) it can be seen that W_{c_2} has been expressed as

$$W_{c_2} = C_2']^T \cdot X'] + X]^T \cdot [M_2'] \cdot X'] \tag{4-13}$$

when X'] stands for the set of aperture mode amplitudes, that is, $Q'_{m',n'}$ and $P'_{m',n'}$. The number of aperture modes, x_i''s, must be the same as the number of waveguide modes, x_i's, in order to perform the matrix operations in this chapter. Primes on matrices do not represent differentiation but are used to designate matrices which are different from the unprimed ones.

From Equations (3-46) through (3-48) it can be seen that the relationship between X] and X'] is of the form

$$C_5] + [M_5] \cdot X] = [M_6] \cdot X'] \tag{4-14}$$

Using Equation (4-14), it can be seen that

$$X'] = [M_6]^{-1} \cdot C_5] + [M_6]^{-1} \cdot [M_5] \cdot X] \tag{4-15}$$

Next, substituting Equation (4-15) into Equation (4-13) allows W_{c_2} to be written as

$$W_{c_2} = \{ C_2']^T + X]^T \cdot [M_2']\} \cdot \{[M_6]^{-1} \cdot C_5] + [M_6]^{-1} \cdot [M_5] \cdot X]\}$$

or

$$W_{c_2} = \{C_2']^T \cdot [M_6]^{-1} \cdot C_5]\} + \{C_2']^T \cdot [M_6]^{-1} \cdot [M_5] \cdot X]\} \qquad (4\text{-}16)$$

$$+ \{X]^T \cdot [M_2'] \cdot [M_6]^{-1} \cdot C_5]\} + \{X]^T \cdot [M_2'] \cdot [M_6]^{-1}$$

$$\cdot [M_5] \cdot X]\}$$

Since each term in Equation (4-16) is a scalar, any term in that equation may be replaced by its transpose without affecting the equation. Remembering that for any matrices [A] and [B]

$$([A] \cdot [B])^T = [B]^T \cdot [A]^T$$

it can be seen that the third term in Equation (4-16) can be rewritten as

$$X]^T \cdot [M_2'] \cdot [M_6]^{-1} \cdot C_5] = \{X]^T \cdot [M_2'] \cdot [M_6]^{-1} \cdot C_5]\}^T \qquad (4\text{-}17)$$

$$= \{[M_2'] \cdot [M_6]^{-1} \cdot C_5]\}^T \cdot X]$$

$$= \{C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_2']^T\} \cdot X]$$

Using Equation (4-17) in Equation (4-16) yields

$$W_{c_2} = \{C_2']^T \cdot [M_6]^{-1} \cdot C_5]\} \qquad (4\text{-}18)$$

$$+ \{C_2']^T \cdot [M_6]^{-1} \cdot [M_5] + C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_2']^T\} \cdot X]$$

$$+ X]^T \cdot \{[M_2'] \cdot [M_6]^{-1} \cdot [M_5]\} \cdot X]$$

Comparing Equation (4-3) and Equation (4-18), it can be seen that

$$K_2 = C_2']^T \cdot [M_6]^{-1} \cdot C_5] \tag{4-19}$$

and

$$C_2]^T = C_2']^T \cdot [M_6]^{-1} \cdot [M_5] + C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_2']^T$$

or

$$C_2] = [M_5]^T \cdot ([M_6]^{-1})^T \cdot C_2'] + [M_2'] \cdot [M_6]^{-1} \cdot C_5] \tag{4-20}$$

and

$$[M_2] = [M_2'] \cdot [M_6]^{-1} \cdot [M_5] \tag{4-21}$$

The functionals W_{c_1} and W_{c_2} have now been expressed in terms of X] alone, instead of both X] and X']. Next, W_{c_3} must be expressed in terms of X] alone. From Equation (3-127) it can be seen that W_{c_3} has been expressed as

$$W_{c_3} = X']^T \cdot [M_3'] \cdot X'] \qquad (4\text{-}22)$$

Using Equation (4-15) in Equation (4-22) yields

$$W_{c_3} = \{C_5]^T \cdot ([M_6]^{-1})^T + X]^T \cdot [M_5]^T \cdot ([M_6]^{-1})^T\}$$

$$\cdot [M_3'] \cdot \{[M_6]^{-1} \cdot C_5] + [M_6]^{-1} \cdot [M_5] \cdot X]\}$$

or

$$W_{c_3} = C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5] + X]^T \cdot [M_5]^T \qquad (4\text{-}23)$$

$$\cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5] + C_5]^T \cdot ([M_6]^{-1})^T$$

$$\cdot [M_3'] \cdot [M_6]^{-1} \cdot [M_5] \cdot X] + X]^T \cdot \{[M_5]^T \cdot ([M_6]^{-1})^T$$

$$\cdot [M_3'] \cdot [M_6]^{-1} \cdot [M_5]\} \cdot X]$$

Taking the transpose of the second term in Equation (4-23)--which is a scalar--and collecting terms permits W_{c_3} to be written as

$$W_{c_3} = \{C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5]\} \qquad (4\text{-}24)$$

$$+ (\{[M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5]\}^T$$

$$+ C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot [M_5]) \cdot X]$$

$$+ X]^T \cdot \{[M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot [M_5]\} \cdot X]$$

Comparing Equations (4-4) and (4-24), it can be seen that

$$K_3 = C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5] \qquad (4\text{-}25)$$

and

$$C_3]^T = \{[M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5]\}^T$$

$$+ C_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot [M_5]$$

$$C_3] = [M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5]$$

$$+ [M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3']^T \cdot [M_6]^{-1} \cdot C_5]$$

$$= [M_5]^T \cdot ([M_6]^{-1})^T \cdot \{[M_3'] + [M_3']^T\} \cdot [M_6]^{-1} \cdot C_5]$$

or, since $[M_3']$ is symmetric, then the last equation may be written as

$$C_3] = 2[M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5] \qquad (4\text{-}26)$$

In addition,

$$[M_3] = [M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot [M_5] \tag{4-27}$$

From the results of this section it can be seen that the following steps must be taken to determine the waveguide mode amplitudes $X]$:

1) Using Equation (3-27), determine $C_1]$ so that

$$W_{c_1} = K_1 + C_1]^T \cdot X] \tag{4-28}$$

2) Next, using Equation (3-34), determine $C_2']$ and $[M_2']$ so that

$$W_{c_2} = C_2']^T \cdot X'] + X]^T \cdot [M_2'] \cdot X'] \tag{4-29}$$

3) Then using Equation (3-127), determine $[M_3']$ so that

$$W_{c_3} = X']^T \cdot [M_3'] \cdot X'] \tag{4-30}$$

4) Next, using Equations (3-46) through (3-48), determine $C_5]$, $[M_5]$, and $[M_6]$ so that

$$C_5] + [M_5] \cdot X] = [M_6] \cdot X'] \tag{4-31}$$

5) Then determine $C_2]$ and $[M_2]$, using Equations (4-20) and (4-21), respectively, as

$$C_2] = [M_5]^T \cdot ([M_6]^{-1})^T \cdot C_2'] + [M_2'] \cdot [M_6]^{-1} \cdot C_5] \qquad (4\text{-}32)$$

and

$$[M_2] = [M_2'] \cdot [M_6]^{-1} \cdot [M_5] \qquad (4\text{-}33)$$

6) Next, determine $C_3]$ and $[M_3]$, using Equations (4-26) and (4-27) as

$$C_3] = 2[M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot C_5] \qquad (4\text{-}34)$$

and

$$[M_3] = [M_5]^T \cdot ([M_6]^{-1})^T \cdot [M_3'] \cdot [M_6]^{-1} \cdot [M_5] \qquad (4\text{-}35)$$

7) Then using Equations (4-8) and (4-9) and noting from Equation (4-28) that $[M_1] = 0$, form

$$C_0] = C_1] + C_2] + C_3] \qquad (4\text{-}36)$$

and

$$[M_0] = [M_2] + [M_3] + [M_2]^T + [M_3]^T \qquad (4\text{-}37)$$

(8) Finally, using Equation (4-11), the waveguide mode amplitudes are obtained from

$$X] = -[M_0]^{-1} \cdot C_0] \tag{4-38}$$

It should be noticed that the value of X] in Equation (4-38) will depend on the choice of L. Changing L changes the phase and magnitude reference plane S_1 of Chapter II, which in turn changes the phase and magnitude of each mode. This variation of X] with L can be removed by normalizing X] with respect to one of the mode amplitudes. For example, X], as given by Equation (4-38), can be divided by the amplitude of the incident dominant mode. This normalization causes all the mode amplitudes to be referenced to a unit magnitude, zero phase incident dominant mode at $z = 0$, as can be seen from Equation (3-2). Such a normalized X] is independent of L. This last statement is borne out by numerical results.

CHAPTER V

NUMERICAL INTEGRATION OF W_{c_3}

Up to this point all equations are valid for arbitrary medium parameters in regions V_1, V_2, and V_3. Hence, the expressions for W_{c_1}, W_{c_2}, and W_{c_3}, as given by Equations (3-27), (3-34), and (3-127), can be applied to a wide variety of physical problems. They can be applied, for example, to plasma, as well as dielectric coverings and to lossy, as well as lossless coverings. Since the coefficients in W_{c_1} and W_{c_2} are expressed in closed form, these two equations can be used directly, no matter what values the medium parameters have. However, the coefficients in W_{c_3} are given as integrals which must be integrated numerically. It is difficult, if not impossible, to devise a numerical integration scheme which will efficiently handle any combination of medium parameters. Hence, the range of these parameters must be limited in any single study. Since the object of this dissertation is to analyze a dielectric coating on a slot antenna, V_2 is restricted to be a dielectric. Equation (3-127) will be modified in this chapter so that it can be numerically integrated more easily for this case.

A Change of Variables

The first modification is the reduction in size of the region of integration. This will be done by making a change of variables from rectangular coordinates--k_x,k_y--to polar coordinates--ρ,ψ--using

$$k_x = k_0 \rho \cos \psi \tag{5-1}$$

$$k_y = k_0 \rho \sin \psi \tag{5-2}$$

The wave number, or propagation constant of free space, k_0, is defined as

$$k_0 = \omega\sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} \tag{5-3}$$

where μ_0 and ε_0 are the free space permeability and permittivity, respectively, and c is the speed of light in vacuum. Letting $F(k_x,k_y)$ represent the entire integrand of Equation (3-127) permits W_{c_3} to be written as

$$W_{c_3} = \int_0^\infty \int_0^\infty F(k_x,k_y)dk_x dk_y \tag{5-4}$$

Using Equations (5-1) and (5-2) in the last equation produces

$$W_{c_3} = \int_0^\infty \int_0^{\pi/2} F(k_0\rho\cos\psi, k_0\rho\sin\psi)k_0^2\rho d\psi d\rho \tag{5-5}$$

The change of variables just employed has, in effect, converted one of the infinite integrals in Equation (5-4) into the finite integral in Equation (5-5). This finite integral is easier to evaluate numerically than is the infinite integral.

Before continuing, the quantities k_2^2, k_3^2, $k_{z_2}^2$, and $k_{z_3}^2$ that appear in Chapter III will be rewritten in a form which is more useful in this chapter. From Equations (3-53) and (5-3) it can be seen that

$$k_2^2 = k_0^2\left(\frac{\omega^2\mu_2\varepsilon_2 - j\omega\mu_2\sigma_2}{\omega^2\mu_0\varepsilon_0}\right)$$

$$= k_0^2\left[\mu_{r_2}\varepsilon_{r_2} - j\mu_{r_2}\left(\frac{\sigma_2}{\omega\varepsilon_0}\right)\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_2}}\right)\right] = k_0^2\ \mu_{r_2}\varepsilon_{r_2}\left(1 - j\ \frac{\sigma_2}{\omega\varepsilon_2}\right)$$

or

$$k_2^2 = k_0^2\ \mu_{r_2}\varepsilon_{r_2}(1 - j\tan\delta_2) \tag{5-6}$$

The loss tangent of medium V_2 is defined as

$$\tan\delta_2 = \frac{\sigma_2}{\omega\varepsilon_2} \tag{5-7}$$

and μ_{r_2} and ε_{r_2} are the relative permeability and permittivity, respectively, of region V_2. From Equations (3-53) and (3-65) it can be seen that k_2^2 and k_3^2 have the same form. Thus, by analogy with Equation (5-6).

$$k_3^2 = k_0^2\mu_{r_3}\varepsilon_{r_3}(1 - j\tan\delta_3) \tag{5-8}$$

The loss tangent of medium V_3 is defined as

$$\tan\delta_3 = \frac{\sigma_3}{\omega\varepsilon_3}$$

and μ_{r_3} and ε_{r_3} are the relative permeability and permittivity, respectively, of region V_3. Next, applying Equations (5-1), (5-2), and (5-6) to Equation (3-55) gives

$$k_{z_2}^2 = k_2^2 - k_x^2 - k_y^2$$

$$= k_0^2[\mu_{r_2}\varepsilon_{r_2}(1-j\tan\delta_2) - \rho^2(\cos^2\psi+\sin^2\psi)]$$

or

$$k_{z_2}^2 = k_0^2[\mu_{r_2}\varepsilon_{r_2}(1-j\tan\delta_2) - \rho^2] \tag{5-9}$$

From Equations (3-55) and (3-64) it can be seen that k_{z_2} and k_{z_3} have the same form. Hence, by analogy with Equation (5-9),

$$k_{z_3}^2 = k_0^2[\mu_{r_3}\varepsilon_{r_3}(1-j\tan\delta_3) - \rho^2] \tag{5-10}$$

It is important to notice from Equations (5-1) and (5-2) that k_x and k_y are functions of both ρ and ψ. Also, it can be seen from Equations (5-9) and (5-10) that k_{z_2} and k_{z_3} are functions of ρ alone. Thus, the function Den defined by Equation (3-101) depends on ρ alone and not of both ρ and ψ. In contrast, the numerator terms N_1, N_2, and N_3, given by Equations (3-98) through (3-100), are all functions of both ρ and ψ since all of them contain k_x and k_y, as well as k_{z_2} and k_{z_3}. As a result, Equation (5-5) may be written as

$$W_{c_3} = \int_0^{\infty} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi d\rho \tag{5-11}$$

where Num(ψ,ρ) represents the total numerator of Equation (3-127) after using the change of variable given in Equations (5-1) and (5-2).

Surface Wave Poles

An examination of Equation (3-101) reveals that the function Den can be written as the product of two terms in the following manner:

$$\mathrm{Den} = \left(\frac{\mu_3}{\mu_2}\right)^2 \left(\frac{k_2}{k_3}\right)^2 k_{z_3} \cos^2(d\,k_{z_2}) - k_{z_3} \sin^2(d\,k_{z_2})$$

$$+ \left[\frac{\sin(d\,k_{z_2})}{k_{z_2}}\right] \cos(d\,k_{z_2}) \left[j\left(\frac{\mu_3}{\mu_2}\right)\left(\frac{k_2}{k_3}\right)^2 k_{z_3}^2 + j\left(\frac{\mu_3}{\mu_2}\right) k_{z_2}^2\right]$$

$$= \left[j\left(\frac{\mu_3}{\mu_2}\right)\left(\frac{k_2}{k_3}\right)^2 k_{z_3} \cos(d\,k_{z_2}) - k_{z_2} \sin(d\,k_{z_2})\right]$$

$$\cdot \left[k_{z_3} \left[\frac{\sin(d\,k_{z_2})}{k_{z_2}}\right] - j\left(\frac{\mu_3}{\mu_2}\right) \cos(d\,k_{z_2})\right]$$

or

$$\mathrm{Den} = D_{TM} D_{TE} \tag{5-12}$$

where D_{TM} and D_{TE} are defined as

$$D_{TM} = j\left(\frac{\mu_3}{\mu_2}\right)\left(\frac{k_2}{k_3}\right)^2 k_{z_3} \cos(d\ k_{z_2}) - k_{z_2} \sin(d\ k_{z_2}) \tag{5-13}$$

$$D_{TE} = k_{z_3}\left[\frac{\sin(d\ k_{z_2})}{k_{z_2}}\right] - j\left(\frac{\mu_3}{\mu_2}\right)\cos(d\ k_{z_2}) \tag{5-14}$$

If either of the two terms in Equation (5-12) is zero in the region of integration $0 \leq \rho < \infty$, then the integrand of Equation (5-11) contains a pole. These poles are called surface wave poles because (31) they represent waves traveling parallel to the interface between regions V_2 and V_3 and because the energy of these waves is confined to a region near the interface. By analogy with similar equations studied by Collin (32), the zeros of D_{TM} give rise to what are called TM (transverse magnetic) surface waves, and zeros of D_{TE} give rise to what are called TE (transverse electric) surface waves. Whether the pole is TM or TE is of no consequence in evaluating the residue at the pole. Hence, the nature of the pole will not be pursued further. It is necessary only to remember that when the function Den is zero in the interval $0 \leq \rho < \infty$, residue contributions must be included in the evaluation of Equation (5-11).

Restriction of Problem to Low-Loss Dielectric Coverings

Since the object of this dissertation is to analyze a dielectric coating on a slot antenna, region V_2 is assumed to be a dielectric.

Actually, it will be assumed, for the remainder of this chapter, that $\mu_{r_2}\varepsilon_{r_2} > \mu_{r_3}\varepsilon_{r_3}$ and that the loss tangents of regions V_2 and V_3 are both much less than one; region V_1 may still have arbitrary medium parameters. These restrictions, which include the low-loss dielectric covering as a special case, are needed to simplify the numerical technique used to evaluate W_{c_3}. The motivation for the low-loss restriction will now be discussed.

In Appendix D it is shown that if $\tan\delta_2 = \tan\delta_3 = 0$, $\mu_{r_2}\varepsilon_{r_2} > \mu_{r_3}\varepsilon_{r_3}$, and the layer thickness $d \neq 0$, then surface wave poles are present and they only occur in the interval

$$\sqrt{\mu_{r_3}\varepsilon_{r_3}} \leq \rho \leq \sqrt{\mu_{r_2}\varepsilon_{r_2}} \tag{5-15}$$

In addition, the poles are simple, and the number, n_{pole}, of poles for this case is shown to be

$$n_{pole} = \text{entier}(4df\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}\,/c) + 1 \tag{5-16}$$

where entier(x) is the greatest integer function. This equation shows that at least one pole is always present, even as $d \to 0$.

The ρ integration in Equation (5-11) may be viewed as a contour integral along the real ρ axis. Since poles lie on this path, the ρ integration contour must be deformed. Deforming the contour is permissible as long as the residue contributions from the semicircular deflections about the poles are included. Whether the contour should be deflected above or below the poles can be determined from convergence considerations; that is, when V_3 is lossless, the integrals (along the deflected contour) in Equations (3-61) and (3-62) should converge as $z \to \infty$. These integrals will converge when the imaginary part of k_{z_3} is non-positive, as required by Equation (3-66).

If ρ is temporarily allowed to be complex and ρ_{re} and ρ_{im} are its real and imaginary parts, respectively, then Equation (5-10), with $\tan\delta_3 = 0$, becomes

$$k_{z_3} = k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}-\rho_{re}^2+\rho_{im}^2-j2\rho_{re}\rho_{im}}$$

According to the definition of k_{z_3} given in Chapter III, k_{z_3} will lie in the first quadrant whenever the quantity under the radical in the last equation is in either the first or second quadrant. In addition, k_{z_3} will lie in the fourth quadrant (or on the negative imaginary axis) when the quantity under the radical lies in either the third or fourth quadrant (or on negative real axis). Hence, the expression under the radical in the last equation must have a non-positive imaginary part if k_{z_3} is to have a non-positive imaginary part. Since ρ_{re} is positive, this requirement is fulfilled when ρ_{im} is non-negative. Thus, the path of integration, C, must be deflected above the poles, as shown in Figure 6.

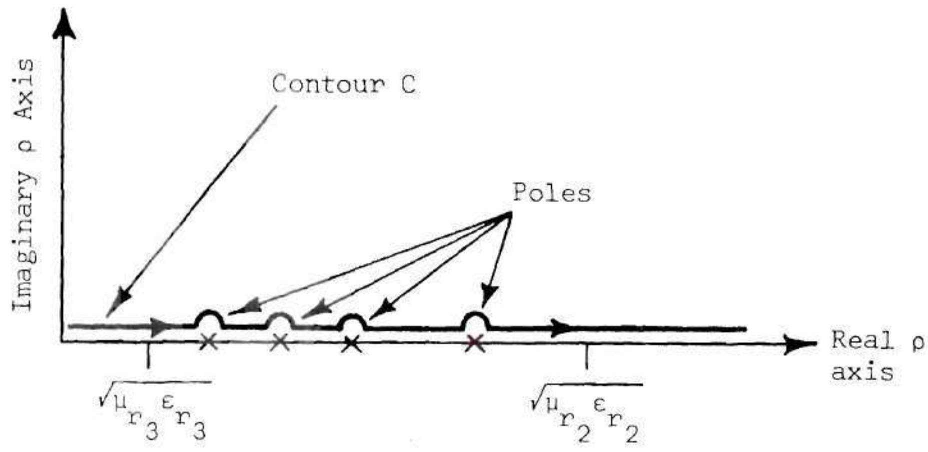


Figure 6. Contour of Integration in the Complex ρ Plane when $\tan\delta_2 = \tan\delta_3 = 0$

If $\mathrm{Den}(\rho)$ is zero on the positive real ρ axis at the points ρ_i for $i = 1,2,\ldots,n_{pole}$, then according to residue theory the contribution to Equation (5-11) from the semicircular deflections is

$$-\frac{1}{2}(2\pi j)\sum_{i=1}^{n_{pole}} (\text{Residue at } \rho_i) \qquad (5\text{-}17)$$

The minus sign is needed because the pole is encircled clockwise rather than counter-clockwise, and the $\frac{1}{2}$ is needed because only half of the pole is encircled. Integrating half way around each pole gives exactly half of the residue since the poles are simple.

If V_2 is lossy, the poles move off the real ρ axis. The path of integration then does not have to be deflected near the poles, and residue theory can no longer be applied to evaluate the contribution to the integral near the poles. If the poles are close to the real ρ axis, the integrand will have a sharp peak near each pole. The numerical evaluation of the area under the peaks is difficult and is not attempted in this dissertation. Instead, it is assumed that V_2 and V_3 are either lossless or low-loss so that the surface wave poles lie exactly or essentially on the real ρ axis. For the remainder of this chapter it will be assumed that the poles lie exactly on the real ρ axis for both the lossless and the low-loss cases.

The evaluation of W_{c_3}, when a single pole located at $\rho = \rho_1$ is present, will now be considered. By utilizing Equation (5-17), the ρ integration in Equation (5-11) may be written in the following manner:

$$W_{c_3} = \int_0^{\rho_1-\Delta} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi d\rho + \int_{\rho_1-\Delta}^{\rho_1} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi d\rho \tag{5-18}$$

$$- \pi j(\text{Residue at } \rho_1) + \int_{\rho_1}^{\rho_1+\Delta} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi d\rho$$

$$+ \int_{\rho_1+\Delta}^{\infty} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi d\rho$$

where Δ is a small positive number. If Δ is small enough, then

$$\int_{\rho_1-\Delta}^{\rho_1} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi d\rho + \int_{\rho_1}^{\rho_1+\Delta} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi d\rho = 0 \tag{5-19}$$

since in the immediate vicinity of a pole, Num will be a constant and Den will be an odd function. Utilizing this last equation in Equation (5-18) gives

$$W_{c_3} = \int_0^{\rho_1-\Delta} \int_0^{\pi/2} \frac{\text{Num}(\psi,\rho)}{\text{Den}(\rho)} \, d\psi d\rho + \int_{\rho_1+\Delta}^{\infty} \int_0^{\pi/2} \frac{\text{Num}(\psi,\rho)}{\text{Den}(\rho)} \, d\psi d\rho \qquad (5\text{-}20)$$

$$- \pi j(\text{Residue at } \rho_1)$$

Thus, W_{c_3} may be evaluated by integrating to within Δ of each side of the pole and then adding on $(-\pi j)$ times the residue at the pole. The elimination of the left side of Equation (5-19) from W_{c_3} eliminates the need to evaluate an infinite integrand at $\rho = \rho_1$.

The residue of the simple pole at $\rho = \rho_1$ can be obtained as follows (33):

$$\begin{matrix}\text{the residue} \\ \text{at } \rho_1\end{matrix} = \lim_{\rho\to\rho_1} \left[(\rho-\rho_1) \int_0^{\pi/2} \frac{\text{Num}(\psi,\rho)}{\text{Den}(\rho)} \, d\psi \right]$$

Application of L'Hospital's rule to the last equation gives

$$\begin{matrix}\text{the residue} \\ \text{at } \rho_1\end{matrix} = \lim_{\rho\to\rho_1} \int_0^{\pi/2} \left[\frac{(\rho-\rho_1) \dfrac{\partial(\text{Num})}{\partial\rho} + \text{Num}}{\dfrac{d(\text{Den})}{d\rho}} \right] d\psi$$

or

$$\underset{\text{at } \rho_1}{\text{the residue}} = \int_0^{\pi/2} \frac{\text{Num}(\psi,\rho_1)}{\text{Den}'(\rho_1)}\, d\psi \tag{5-21}$$

where Den' is defined as the derivative of Den with respect to ρ. It follows from Equation (5-12) that

$$\text{Den}' = D_{TM}'(\rho)\, D_{TE}(\rho) + D_{TM}(\rho)\, D_{TE}'(\rho) \tag{5-22}$$

where the primes on D_{TM}' and D_{TE}' indicate differentiation with respect to ρ. To evaluate D_{TM}', Equation (5-13) may be used to obtain

$$D_{TM}'(\rho) = j\left(\frac{\mu_3}{\mu_2}\right)\left(\frac{k_2}{k_3}\right)^2 \left[\left(\frac{dk_{z_3}}{d\rho}\right)\cos(dk_{z_2}) - k_{z_3}\, d\, \sin(dk_{z_2})\left(\frac{dk_{z_2}}{d\rho}\right)\right] - \left(\frac{dk_{z_2}}{d\rho}\right)\sin(dk_{z_2}) - k_{z_2}\, d\, \cos(d\, k_{z_2})\left(\frac{dk_{z_2}}{d\rho}\right) \tag{5-23}$$

The derivatives in the last equation may be evaluated by noting from Equation (5-9) that

$$2\, k_{z_2} \frac{d\, k_{z_2}}{d\rho} = -2\, k_0^2 \rho$$

or

$$\frac{d\,k_{z_2}}{d\rho} = -\frac{k_0^2\rho}{k_{z_2}} \tag{5-24}$$

Since k_{z_2} and k_{z_3} have the same form, as can be seen from Equations (5-9) and (5-10), it follows from the last equation that

$$\frac{d\,k_{z_3}}{d\rho} = -\frac{k_0^2\rho}{k_{z_3}} \tag{5-25}$$

Using the last two equations in Equation (5-23) shows that

$$D'_{TM}(\rho) = j\left(\frac{\mu_3}{\mu_2}\right)\left(\frac{k_2}{k_3}\right)^2 k_0^2\rho\left[\left(\frac{k_{z_3}}{k_{z_2}}\right)d\sin(d\,k_{z_2}) - \frac{\cos(d\,k_{z_2})}{k_{z_3}}\right] + k_0^2\rho\left[d\cos(d\,k_{z_2}) + \frac{\sin(d\,k_{z_2})}{k_{z_2}}\right] \tag{5-26}$$

Next, D'_{TE} will be calculated. From Equation (5-14) it can be seen that

$$D'_{TE}(\rho) = \frac{d\,k_{z_3}}{d\rho}\left[\frac{\sin(d\,k_{z_2})}{k_{z_2}}\right] + k_{z_3}\left[k_{z_2}\,d\cos(d\,k_{z_2})\left(\frac{d\,k_{z_2}}{d\rho}\right) - \sin(d\,k_{z_2})\left(\frac{d\,k_{z_2}}{d\rho}\right)\right]\left[\frac{1}{k_{z_2}^2}\right] + j\left(\frac{\mu_3}{\mu_2}\right)d\sin(d\,k_{z_2})\left(\frac{d\,k_{z_2}}{d\rho}\right)$$

Applying Equations (5-24) and (5-25) to the last equation gives

$$D'_{TE}(\rho) = -k_0^2\rho\left[\frac{\sin(d\,k_{z_2})}{k_{z_2}}\right]\left[j\left(\frac{\mu_3}{\mu_2}\right)d + \frac{1}{k_{z_3}}\right] \tag{5-27}$$

$$+ k_0^2\rho\left(\frac{k_{z_3}}{k_{z_2}^2}\right)\left[\frac{\sin(d\,k_{z_2})}{k_{z_2}} - d\cos(d\,k_{z_2})\right]$$

Equations (5-22), (5-26), and (5-27) permit Den' to be evaluated when it is needed.

Combining Equations (5-20) and (5-21) reveals that, when only one pole is present, W_{c_3} is given by

$$W_{c_3} = \int_0^{\rho_1-\Delta}\int_0^{\pi/2} \frac{\text{Num}(\psi,\rho)}{\text{Den}(\rho)}\,d\psi d\rho + \int_{\rho_1+\Delta}^{\infty}\int_0^{\pi/2} \frac{\text{Num}(\psi,\rho)}{\text{Den}(\rho)}\,d\psi d\rho \tag{5-28}$$

$$-\pi j\int_0^{\pi/2} \frac{\text{Num}(\psi,\rho_1)}{\text{Den}'(\rho_1)}\,d\psi$$

The evaluation of W_{c_3}, when more than one pole is present, can be obtained by analogy with Equation (5-28). If n_{pole} surface wave poles are present and if they are located at $\rho = \rho_1, \rho_2, \ldots, \rho_{n_{pole}}$ where $\rho_1 < \rho_2 < \cdots < \rho_{n_{pole}}$, then W_{c_3} becomes

$$W_{c_3} = \int_0^{\rho_1-\Delta} Ig\ d\rho + \int_{\rho_1+\Delta}^{\rho_2-\Delta} Ig\ d\rho + \int_{\rho_2+\Delta}^{\rho_3-\Delta} Ig\ d\rho \tag{5-29}$$

$$+ \cdots + \int_{\rho_{n_{pole}-1}+\Delta}^{\rho_{n_{pole}}-\Delta} Ig\ d\rho + \int_{\rho_{n_{pole}}+\Delta}^{\infty} Ig\ d\rho$$

$$- \pi j \sum_{i=1}^{n_{pole}} \left[\int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho_i)}{\mathrm{Den}'(\rho_i)}\, d\psi \right]$$

where Ig is defined as

$$Ig = \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)}{\mathrm{Den}(\rho)}\, d\psi \tag{5-30}$$

$$= \int_0^{\pi/2} F(k_0\rho\cos\psi,\ k_0\rho\sin\psi) k_0^2 \rho\, d\psi$$

Equation (5-5) was used in obtaining the last expression. From Equation (5-15) it is noticed that

$$\sqrt{\mu_{r_3}\varepsilon_{r_3}} \leqq \rho_1 < \rho_2 < \cdots < \rho_{n_{pole}} \leqq \sqrt{\mu_{r_2}\varepsilon_{r_2}}$$

It is now convenient to split the first and the next to the last integrals in Equation (5-29) and rewrite that equation as

$$W_{c_3} = \int_0^{\sqrt{\mu_{r_3}\varepsilon_{r_3}}} Ig\, d\rho + \int_{\sqrt{\mu_{r_3}\varepsilon_{r_3}}}^{\rho_1-\Delta} Ig\, d\rho \tag{5-31}$$

$$+ \sum_{i=1}^{n_{pole}-1} \int_{\rho_i+\Delta}^{\rho_{i+1}-\Delta} Ig\, d\rho + \int_{\rho_{n_{pole}}+\Delta}^{\sqrt{\mu_{r_2}\varepsilon_{r_2}}} Ig\, d\rho$$

$$+ \int_{\sqrt{\mu_{r_2}\varepsilon_{r_2}}}^{\infty} Ig\, d\rho - \pi j \sum_{i=1}^{n_{pole}} \left[\int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho_i)}{\mathrm{Den}'(\rho_i)}\, d\psi\right]$$

To remove the integrable singularity discussed in Appendix D (when $\mu_{r_2}\varepsilon_{r_2} = \mu_{r_3}\varepsilon_{r_3}$ or when $d = 0$), a change of variables will be applied to the first integral and the next to the last integral in Equation (5-31). Letting

$$\rho = \sqrt{\mu_{r_3}\varepsilon_{r_3}}\, \sin\alpha \tag{5-32}$$

in the first integral and

$$\rho = \sqrt{\mu_{r_2}\varepsilon_{r_2}}\, \cosh\alpha \tag{5-33}$$

in the next to the last integral converts the first integral in Equation (5-31) to

$$\int_0^{\sqrt{\mu_{r_3}\varepsilon_{r_3}}} \int_0^{\pi/2} F(k_0\rho\cos\psi,\ k_0\rho\sin\psi)k_0^2\rho d\psi d\rho \tag{5-34}$$

$$= \int_0^{\pi/2}\int_0^{\pi/2} F(k_{0_3}\ \sin\alpha\ \cos\psi,\ k_{0_3}\ \sin\alpha\ \sin\psi)k_{0_3}^2\ \sin\alpha\ \cos\alpha\ d\psi\ d\alpha$$

and the next to the last integral to

$$\int_{\sqrt{\mu_{r_2}\varepsilon_{r_2}}}^{\infty} \int_0^{\pi/2} F(k_0\rho\cos\psi,\ k_0\rho\sin\psi)k_0^2\rho d\psi d\rho \tag{5-35}$$

$$= \int_0^{\infty}\int_0^{\pi/2} F(k_{0_2}\cosh\alpha\cos\psi,\ k_{0_2}\cosh\alpha\sin\psi)k_{0_2}^2\ \sinh\alpha\ \cosh\alpha\ d\psi d\alpha$$

where k_{0_2} and k_{0_3} are defined as

$$k_{0_2} = k_0\sqrt{\mu_{r_2}\varepsilon_{r_2}} \tag{5-36}$$

$$k_{0_3} = k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}} \tag{5-37}$$

The remaining integrals on ρ will now be modified. These integrals will be altered so that a more accurate numerical answer can be obtained for the contributions to W_{c_3} from the vicinity of the poles. Since the integrands become very large near the poles, many sample points in the numerical integration scheme must be placed near the poles. This can be accomplished for the second integral in Equation (5-31) by letting

$$\rho = \rho_1 - \alpha^2$$

in that integral. This gives

$$\int_{\sqrt{\mu_{r_3}\varepsilon_{r_3}}}^{\rho_1-\Delta} \int_0^{\pi/2} F(k_0\rho\cos\psi,\ k_0\rho\sin\psi)k_0^2\rho\, d\psi\, d\rho \tag{5-38}$$

$$= \int_{\sqrt{\Delta}}^{\sqrt{\rho_1-\sqrt{\mu_{r_3}\varepsilon_{r_3}}}} \int_0^{\pi/2} F(k_0[\rho_1-\alpha^2]\cos\psi, k_0[\rho_1-\alpha^2]\sin\psi)k_0^2[\rho_1-\alpha^2]2\alpha\, d\psi\, d\alpha$$

A uniform partitioning of the α axis causes the sample points on the ρ axis to be bunched near the pole at $\rho = \rho_1$. Next, letting

$$\rho = \alpha^2 + \rho_i$$

in the i^{th} sum in the third integral in Equation (5-31) produces

$$\sum_{i=1}^{n_{pole}-1} \int_{\rho_i+\Delta}^{\rho_{i+1}-\Delta} \int_0^{\pi/2} F(k_0\rho\cos\psi,\ k_0\rho\sin\psi)k_0^2\rho\, d\psi\, d\rho \tag{5-39}$$

$$= \sum_{i=1}^{n_{pole}-1} \int_{\sqrt{\Delta}}^{\sqrt{\rho_{i+1}-\Delta-\rho_i}} \int_0^{\pi/2} F(k_0^2[\alpha^2+\rho_i]\cos\psi,$$

$$k_0[\alpha^2+\rho_i]\sin\psi)k_0^2[\alpha^2+\rho_i]2\alpha d\psi d\alpha$$

Finally, letting

$$\rho = \alpha^2 + \rho_{n_{pole}}$$

in the fourth integral in Equation (5-31) yields

$$\int_{\rho_{n_{pole}}+\Delta}^{\sqrt{\mu_{r_2}\varepsilon_{r_2}}} \int_0^{\pi/2} F(k_0\rho\cos\psi,\ k_0\rho\sin\psi)k_0^2\rho d\psi d\rho \tag{5-40}$$

$$= \int_{\sqrt{\Delta}}^{\sqrt{\sqrt{\mu_{r_2}\varepsilon_{r_2}}-\rho_{n_{pole}}}} \int_0^{\pi/2} F(k_0[\alpha^2+\rho_{n_{pole}}]\cos\psi,$$

$$k_0[\alpha^2+\rho_{n_{pole}}]\sin\psi)k_0^2[\alpha^2+\rho_{n_{pole}}]2\alpha d\psi d\alpha$$

From Equations (5-4), (5-34), (5-35), (5-38), (5-39), and (5-40), it can be seen that

$$W_{c_3} = \int_0^\infty \int_0^\infty F(k_x,k_y)dk_x dk_y \tag{5-41}$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} F(k_{0_3}\sin\alpha\cos\psi,\ k_{0_3}\sin\alpha\sin\psi)k_{0_3}^2 \sin\alpha\cos\alpha\ d\psi d\alpha$$

$$+ \int_{\sqrt{\Delta}}^{\sqrt{\rho_1-\sqrt{\mu_{r_3}\varepsilon_{r_3}}}} \int_0^{\pi/2} F(k_0[\rho_1-\alpha^2]\cos\psi,\ k_0[\rho_1-\alpha^2]\sin\psi)k_0^2[\rho_1-\alpha^2]2\alpha d\psi d\alpha$$

$$+ \sum_{i=1}^{n_{pole}-1} \int_{\sqrt{\Delta}}^{\sqrt{\rho_{i+1}-\Delta-\rho_i}} \int_0^{\pi/2} F(k_0[\alpha^2+\rho_i]\cos\psi,\ k_0[\alpha^2+\rho_i]\sin\psi)$$

$$\cdot\ k_0^2[\alpha^2+\rho_i]2\alpha d\psi d\alpha + \int_{\sqrt{\Delta}}^{\sqrt{\sqrt{\mu_{r_2}\varepsilon_{r_2}} - \rho_{n_{pole}}}} \int_0^{\pi/2} F(k_0[\alpha^2+\rho_{n_{pole}}]\cos\psi,$$

$$k_0[\alpha^2+\rho_{n_{pole}}]\sin\psi)k_0^2[\alpha^2+\rho_{n_{pole}}]2\alpha d\psi d\alpha + \int_0^\infty \int_0^{\pi/2} F(k_{0_2}\cosh\alpha\cos\psi,$$

$$k_{0_2}\cosh\alpha\sin\psi)k_{0_2}^2 \sinh\alpha\cosh\alpha\ d\psi d\alpha - \pi j \sum_{i=1}^{n_{pole}} \left[\int_0^{\pi/2} \frac{Num(\psi,\rho_i)}{Den'(\rho_i)} d\psi\right]$$

Equation (5-41) presents the manner in which the integrals in Equation (3-127) can be evaluated when V_2 and V_3 are lossless, or low-loss, and when $\mu_{r_2}\varepsilon_{r_2} > \mu_{r_3}\varepsilon_{r_3}$. An arbitrary number of surface wave poles can be included. The changes of variables used in obtaining Equation (5-41) were selected to speed the numerical integration.

CHAPTER VI

THE FAR FIELD OF THE ANTENNA

In this chapter the far field of the antenna will be evaluated, using an asymptotic technique known as the method of stationary phase. This technique permits the integrals representing field to be explicitly evaluated at large distances from the aperture.

Each of the rectangular components of the trial electric field in region V_3 has been expressed as a double integral of the form

$$I = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(k_x,k_y) e^{-j[xk_x+yk_y+zk_{z_3}]} dk_x \, dk_y \qquad (6\text{-}1)$$

where I represents either E_{x_3}, E_{y_3}, or E_{z_3}, and the integrand in Equation (6-1) represents the integrand of either Equation (3-61), (3-62), or (3-63). To calculate the rectangular components of $\bar{E}_3$ in the far field of the antenna, Equation (6-1) must be evaluated at large distances from the aperture. This evaluation will be performed using the method of stationary phase. To expedite the application of this technique, the observation point (x,y,z) will be expressed in spherical coordinates (r,θ,ϕ) by means of the familiar transformation

$$x = r \sin\theta \cos\phi \qquad (6\text{-}2)$$

$$y = r \sin\theta \sin\phi$$

$$z = r \cos\theta$$

It is also convenient to let

$$k_x = k_0 \rho \cos\psi \tag{6-3}$$

$$k_y = k_0 \rho \sin\psi$$

The polar angle θ is measured from the z axis in Figure 4, and the azimuthal angle ϕ is measured from the x axis in Figure 5. The change of variables in Equation (6-3) is the same one used in Chapter V.

If V_3 is lossy, then the far field of the antenna is zero because all of the "radiated" energy is dissipated as heat loss in V_3. To have a non-zero far field, it will be assumed for this chapter that region V_3 is lossless. Setting $\tan\delta_3 = 0$ in Equation (5-10) and employing Equation (3-66) shows that

$$k_{z_3} = \begin{cases} k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}-\rho^2} & \text{if} \quad \rho < \sqrt{\mu_{r_3}\varepsilon_{r_3}} \\ -jk_0\sqrt{\rho^2-\mu_{r_3}\varepsilon_{r_3}} & \text{if} \quad \rho \geq \sqrt{\mu_{r_3}\varepsilon_{r_3}} \end{cases} \tag{6-4}$$

By using Equations (6-2) and (6-3), the exponent in Equation (6-1) can be rewritten as

$$-jk_0 r[\sin\theta \cos\phi \; \rho \cos\psi + \sin\theta \sin\phi \; \rho \sin\psi + \cos\theta(k_{z_3}/k_0)]$$

$$= -jk_0 r[\rho \sin\theta \cos(\psi-\phi) + \cos\theta(k_{z_3}/k_0)]$$

Applying this last equation, along with Equations (6-3) and (6-4), to Equation (6-1) gives

$$I = I_1 + I_2 \tag{6-5}$$

where I_1 and I_2 are defined as

$$I_1 = \int_0^{\sqrt{\mu_{r_3}\varepsilon_{r_3}}} \int_0^{2\pi} k_0^2 \rho f(k_0\rho\cos\psi,\ k_0\rho\sin\psi) \tag{6-6}$$

$$\cdot\ e^{-jk_0 r[\rho\sin\theta\cos(\psi-\phi)+\cos\theta\sqrt{\mu_{r_3}\varepsilon_{r_3}-\rho^2}]}\ d\psi\ d\rho$$

$$I_2 = \int_{\sqrt{\mu_{r_3}\varepsilon_{r_3}}}^{\infty} \int_0^{2\pi} k_0^2 \rho f(k_0\rho\cos\psi,\ k_0\rho\sin\psi) \tag{6-7}$$

$$\cdot\ e^{-jk_0 r[\rho\sin\theta\cos(\psi-\phi)\ -\ j\cos\theta\sqrt{\rho^2-\mu_{r_3}\varepsilon_{r_3}}\ ]}\ d\psi\ d\rho$$

Next, letting $\rho = \sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \sin\tau$ in Equation (6-6) and letting $\rho = \sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \cosh\tau$ in Equation (6-7) permits I_1 and I_2 to be written as

$$I_1 = \int_0^{\pi/2} e^{-jk_{0_3} r\cos\theta\cos\tau}\ \{ \int_0^{2\pi} k_{0_3}^2 \sin\tau\ \cos\tau\ h(\tau,\psi) \tag{6-8}$$

$$\cdot\, e^{-jk_{0_3} r \sin\tau \sin\theta \cos(\psi-\phi)} d\psi\} \, d\tau$$

$$I_2 = \int_0^\infty e^{-k_{0_3} r\cos\theta\sinh\tau} \{\int_0^{2\pi} k_{0_3}^2 \sinh\tau \cosh\tau \, g(\tau,\psi) \tag{6-9}$$

$$\cdot\, e^{-jk_{0_3} \cosh\tau\sin\theta\cos(\psi-\phi)} d\psi\} \, d\tau$$

where h and g are defined as

$$h(\tau,\psi) = f(k_{0_3} \sin\tau\cos\psi,\ k_{0_3} \sin\tau\sin\psi) \tag{6-10}$$

$$g(\tau,\psi) = f(k_{0_3} \cosh\tau\cos\psi,\ k_{0_3} \cosh\tau\sin\psi) \tag{6-11}$$

and k_{0_3} is given by Equation (5-37). Equations (6-8) and (6-9) will now be evaluated in the far field of the antenna by using the principle of stationary phase (34), (35). According to this principle,

$$\lim_{\nu\to\infty} \int_a^b f(x) e^{j\nu g(x)} dx = \begin{cases} \text{(i) } O\left(\frac{1}{\nu}\right) \text{ if } g'(x) \neq 0 \text{ for } a \leqq x \leqq b & \text{(6-12)} \\[2ex] \text{(ii) } \sqrt{\dfrac{2\pi}{\nu |g''(x_0)|}}\, f(x_0) e^{j\nu g(x_0) + j\frac{\pi}{4}\operatorname{sign}(g''(x_0))} & \\ \quad \text{if } g'(x_0) = 0 \text{ but } g''(x_0) \neq 0 \text{ and } a < x_0 < b & \\[2ex] \text{(iii) } \frac{1}{2} \text{ of (ii) if } x_0 = a \text{ or } x_0 = b & \end{cases}$$

where $O\left(\frac{1}{\nu}\right)$ means "is of the order of $\frac{1}{\nu}$." The sign function was defined immediately before Equation (3-43). Equation (6-12) applies if

1) a, b, ν, and x are all real.
2) f and g are independent of ν and are analytic functions when their arguments are complex.
3) $g(x)$ is real valued for x real and $a \leq x \leq b$.

The point x_0, which causes $g'(x_0)$ to be zero in the interval $[a,b]$, is called a stationary phase point of $g(x)$. If more than one stationary point exists in the interval $[a,b]$, then the integral in Equation (6-12) is equal to the sum of all the stationary phase point contributions.

The Evaluation of I_1

The two integrals in Equation (6-8) will now be evaluated by making two applications of the method of stationary phase. Utilizing the notation of Equation (6-12), the following identifications can be made for the ψ integration in Equation (6-8):

$$\nu = k_{0_3} r \tag{6-13}$$

$$g = g(\psi) = -\sin\tau \sin\theta \cos(\psi-\phi) \tag{6-14}$$

Hence,

$$g' = \sin\tau \sin\theta \sin(\psi-\phi) \tag{6-15}$$

$$g'' = \sin\tau \sin\theta \cos(\psi-\phi) \tag{6-16}$$

The *far field* of the antenna will be defined to occur when $k_{0_3} r \to \infty$, which is in agreement with the requirement on ν in Equation (6-12).

To determine the stationary phase points of $g(\psi)$, it will be assumed that $\tau \neq 0$ and $\theta \neq 0$. The case of $\tau = 0$ need not be considered since $\sin\tau$ is zero when $\tau = 0$. Hence, the integrand of Equation (6-8) is zero and cannot contribute to I_1 when $\tau = 0$. The case of $\theta = 0$ will be treated as a limiting case later. Thus, assuming that $\tau \neq 0$ and that $\theta \neq 0$, it can be seen from Equation (6-15) that $g'(\psi_0) = 0$ when

$$\psi_0 = \phi + n\pi \text{ where } n = 0, \pm 1, \pm 2, \cdots \tag{6-17}$$

The numbers ψ_0 are the set of possible stationary phase points of Equation (6-14). Since ψ and ϕ are restricted to the intervals $0 \leqq \psi \leqq 2\pi$ and $0 \leqq \phi \leqq 2\pi$, it follows that the permissible solutions of Equation (6-17) are

$$\psi_0 = \begin{cases} 0,\ \pi,\ 2\pi & \text{if } \phi = 0 \\ \phi,\ \phi+\pi & \text{if } 0 < \phi < \pi \\ 0,\ \pi,\ 2\pi & \text{if } \phi = \pi \\ \phi,\ \phi-\pi & \text{if } \pi < \phi < 2\pi \\ 0,\ \pi,\ 2\pi & \text{if } \phi = 2\pi \end{cases} \tag{6-18}$$

Equation (6-8) will now be evaluated for each of the cases given in Equation (6-18).

Case 1. For this case it will be assumed that $0 < \phi < \pi$, $\tau \neq 0$, and $\theta \neq 0$. From Equation (6-18) it is seen that $\psi_0 = \phi$ and $\psi_0 = \phi + \pi$ are the stationary phase points for this case. Because there are two stationary phase points, the application of Equation (6-12) along with Equations (6-13) through (6-16) to the ψ integration in Equation (6-8) gives

$$I_1 = I_{11} + I_{12} \tag{6-19}$$

where I_{11} and I_{12} are defined as

$$I_{11} = \int_0^{\pi/2} \left[\sqrt{\frac{2\pi}{k_{0_3} r|\sin\tau \sin\theta|}} \; k_{0_3}^2 \sin\tau \cos\tau \; h(\tau,\phi) \cdot e^{-jk_{0_3} r\sin\tau\sin\theta} \; e^{j\frac{\pi}{4}\,\mathrm{sign}(\sin\tau\sin\theta)} \right] e^{-jk_{0_3} r\cos\theta\cos\tau} \, d\tau$$

$$I_{12} = \int_0^{\pi/2} \left[\sqrt{\frac{2\pi}{k_{0_3} r|\sin\tau \sin\theta|}} \; k_{0_3}^2 \sin\tau \cos\tau \; h(\tau,\phi+\pi) \cdot e^{jk_{0_3} r\sin\tau\sin\theta} \; e^{j\frac{\pi}{4}\,\mathrm{sign}(-\sin\tau\sin\theta)} \right] e^{-jk_{0_3} r\cos\theta\cos\tau} \, d\tau$$

Both $\sin\tau$ and $\sin\theta$ are positive in the last two equations since τ and θ are restricted to the intervals

$$0 < \tau \leqq \frac{\pi}{2} \tag{6-20}$$

$$0 < \theta \leqq \frac{\pi}{2}$$

It should be remembered that $\theta = 0$ and $\tau = 0$ have been excluded from consideration. The simplification of I_1 and I_2 yields

$$I_{11} = \int_0^{\pi/2} \sqrt{\frac{2\pi\sin\tau}{k_{0_3} r\sin\theta}}\, k_{0_3}^2 \cos\tau\; h(\tau,\phi) e^{j\frac{\pi}{4}} e^{-jk_{0_3} r\cos(\tau-\theta)}\, d\tau \tag{6-21}$$

$$I_{12} = \int_0^{\pi/2} \sqrt{\frac{2\pi\sin\tau}{k_{0_3} r\sin\theta}}\, k_{0_3}^2 \cos\tau\; h(\tau,\phi+\pi) e^{-j\frac{\pi}{4}} e^{-jk_{0_3} r\cos(\tau+\theta)}\, d\tau \tag{6-22}$$

Equation (6-19) is valid in the far field of the antenna, that is, as $k_{0_3} r \to \infty$.

The stationary phase technique will next be applied to I_{11}. Following the notation of Equation (6-12), it will be observed that for the τ integration in Equation (6-21)

$$\nu = k_{0_3} r \tag{6-23}$$

$$g = g(\tau) = -\cos(\tau-\theta) \tag{6-24}$$

Thus, $g'(\tau_0) = 0$ implies that

$$\tau_0 = \theta + n\pi \text{ where } n = 0, \pm 1, \pm 2, \ldots \tag{6-25}$$

The numbers τ_0 represent the possible stationary phase points of Equation (6-24). Since τ and θ are restricted to the intervals given by Equation (6-20), it follows from Equation (6-25) that $\tau = \theta$ is the only permissible stationary phase point for Equation (6-24). Thus, as $k_{0_3} r \to \infty$, Equations (6-12), (6-21), (6-23), and (6-24) combine to give

$$I_{11} = \sqrt{\frac{2\pi}{k_{0_3} r}} \{ \sqrt{\frac{2\pi \sin\theta}{k_{0_3} r \sin\theta}}\, k_{0_3}^2 \cos\theta\, h(\theta,\phi) e^{j\frac{\pi}{4}} \} e^{-jk_{0_3} r} e^{+j\frac{\pi}{4}}$$

or

$$I_{11} = j\, 2\pi\, k_{0_3} \cos\theta\, h(\theta,\phi) \frac{e^{-jk_{0_3} r}}{r} \tag{6-26}$$

when $0 < \phi < \pi$. Equation (6-26) is the far field evaluation of I_{11}. If $\theta = \frac{\pi}{2}$, then the stationary phase point is $\tau_0 = \frac{\pi}{2}$, which is at one end of the τ integration interval. In this situation Equation (6-12) says that I_{11} is one half of the value given in Equation (6-26). However, $I_{11} = 0$ at $\theta = \frac{\pi}{2}$. Hence, Equation (6-26) is still valid when $\theta = \frac{\pi}{2}$.

Next, I_{12} will be evaluated, using the principle of stationary phase. Applying the notation of Equation (6-12) to Equation (6-22) shows that

$$\nu = k_{0_3} r$$

$$g = g(\tau) = -\cos(\tau+\theta) \tag{6-27}$$

Thus, $g'(\tau_0) = 0$ implies that

$$\tau_0 = -\theta + n\pi \text{ where } n = 0, \pm 1, \pm 2, \ldots$$

The numbers τ_0 represent the possible stationary phase points of Equation (6-27). Since τ and θ are restricted to the intervals given in Equation (6-20), it follows that except when $\theta = \frac{\pi}{2}$ the last equation has no solutions. Thus, $I_{12} = O\left([k_{0_3} r]^{-3/2}\right)$, which is negligible compared to $I_{11} = O\left([k_{0_3} r]^{-1}\right)$ as $k_{0_3} r \to \infty$. When $\theta = \frac{\pi}{2}$, $\tau_0 = \frac{\pi}{2}$ and the integrand of Equation (6-22) is zero. Thus I_{12} does not contribute to I_1 for $\theta = \frac{\pi}{2}$ either.

It then follows from Equation (6-19) that $I_1 = I_{11}$. This, combined with Equation (6-26), gives

$$I_1 = j\, 2\pi\, k_{0_3} \cos\theta\, h(\theta,\phi) \frac{e^{-jk_{0_3} r}}{r} \tag{6-28}$$

This last equation is the far field evaluation of I_1 when $0 < \phi < \pi$

and $0 < \theta \leq \frac{\pi}{2}$. The object of the preceding manipulations was to replace the double integral representation of I_1, namely Equation (6-8), by an explicit function representation, namely Equation (6-28), as $k_{0_3} r \to \infty$. The method of stationary phase made this replacement possible. The explicit function representation of I_1 is, of course, easier to interpret and manipulate.

Equation (6-28) has only been shown to be valid when $0 < \phi < \pi$ and $0 < \theta \leq \frac{\pi}{2}$. It will now be shown that this equation is also valid for the remaining values of ϕ and θ.

Case 2. For this case it will be assumed that $\pi < \phi < 2\pi$, $\tau \neq 0$, and $\theta \neq 0$. Then, from Equation (6-18), the stationary phase points for the ψ integration in Equation (6-8) are $\psi_0 = \phi$ and $\psi_0 = \phi - \pi$. By analogy with Case 1, the far field evaluation of I_1 for this case is given by Equation (6-19), where I_{11} is given by Equation (6-21) and I_{12} is given by Equation (6-22), with $h(\tau,\phi+\pi)$ replaced by $h(\tau,\phi-\pi)$. The stationary phase point $\psi_0 = \phi$ produces I_{11}, while $\psi_0 = \phi - \pi$ produces I_{12}. The fact that g, g', and g", as given by Equations (6-14), (6-15), and (6-16), have the same values at $\psi = \phi + \pi$ as they do at $\psi = \phi - \pi$ was used in obtaining I_{12} for this case.

The far field evaluation of I_{11} and I_{12} is the same for this case as it is in Case 1, since the stationary phase points are determined by the exponential term in the integrands--and not by the arguments of the function h. Thus, I_{12} is again zero, as compared with I_{11}, and Equation (6-28) is again the far field evaluation of I_1.

Case 3. For this case it will be assumed that $\phi = 0$, $\tau \neq 0$, and $\theta \neq 0$. Equation (6-18) then gives $\psi_0 = 0$, $\psi_0 = \pi$, and $\psi_0 = 2\pi$ as the stationary phase points of the ψ integration in I_1. Applying Equations (6-12) through (6-16) to Equation (6-8) gives, for this case,

$$I_1 = I_{13} + I_{14} \tag{6-29}$$

where I_{13} and I_{14} are defined as

$$I_{13} = \int_0^{\pi/2} \left[\frac{1}{2}\sqrt{\frac{2\pi}{k_{0_3} r|\sin\tau\ \sin\theta|}}\ k_{0_3}^2 \sin\tau\ \cos\tau\ h(\tau,0)\right. \tag{6-30}$$

$$\left.\cdot\ e^{-jk_{0_3} r\sin\tau\sin\theta}\ e^{j\frac{\pi}{4}\,\mathrm{sign}(\sin\tau\sin\theta)}\right] e^{-jk_{0_3} r\cos\theta\cos\tau}\ d\tau$$

$$+ \int_0^{\pi/2} \left[\frac{1}{2}\sqrt{\frac{2\pi}{k_{0_3} r|\sin\tau\ \sin\theta|}}\ k_{0_3}^2 \sin\tau\ \cos\tau\ h(\tau,2\pi)\right.$$

$$\left.\cdot\ e^{-jk_{0_3} r\sin\tau\sin\theta}\ e^{j\frac{\pi}{4}\,\mathrm{sign}(\sin\tau\sin\theta)}\right] e^{-jk_{0_3} r\cos\theta\cos\tau}\ d\tau$$

$$I_{14} = \int_0^{\pi/2} \left[\sqrt{\frac{2\pi}{k_{0_3} r|\sin\tau\ \sin\theta|}}\ k_{0_3}^2\ \sin\tau\ \cos\tau\ h(\tau,\pi)\right.$$

$$\cdot\ e^{jk_{0_3} r\sin\tau\sin\theta}\ e^{j\frac{\pi}{4}\,\mathrm{sign}(-\sin\tau\sin\theta)}\Bigg]\, e^{-jk_{0_3} r\cos\theta\cos\tau}\, d\tau$$

Since $\sin\tau$ and $\sin\theta$ are both positive in the intervals given in Equation (6-20), the last equation may be simplified to

$$I_{14} = \int_0^{\pi/2} \sqrt{\frac{2\pi \sin\tau}{k_{0_3} r \sin\theta}}\, k_{0_3}^2 \cos\tau\ h(\tau,\pi) e^{-j\frac{\pi}{4}}\ e^{-jk_{0_3} r\cos(\tau+\theta)}\, d\tau \qquad (6\text{-}31)$$

Equation (6-10) shows that $h(\tau,2\pi) = h(\tau,0)$. Hence, Equation (6-30) can be rewritten as

$$I_{13} = \int_0^{\pi/2} \sqrt{\frac{2\pi \sin\tau}{k_{0_3} r \sin\theta}}\, k_{0_3}^2 \cos\tau\ h(\tau,0) e^{j\frac{\pi}{4}}\ e^{-jk_{0_3} r\cos(\tau-\theta)}\, d\tau \qquad (6\text{-}32)$$

It should now be noticed that I_{13}, in Equation (6-32), is identical to I_{11}, in (6-21) with $\phi = 0$. Similarly, I_{14} is identical to I_{12} with $\phi = 0$, as can be seen from Equations (6-22) and (6-31). Hence, the far field evaluation of I_{13} is given by Equation (6-26) with $\phi = 0$, and the far field evaluation of I_{14} is zero. Thus, I_1 for Case 3 is given by Equation (6-28) with $\phi = 0$.

Case 4. For this case it will be assumed that $\phi = 2\pi$, $\tau \neq 0$, and $\theta \neq 0$. From Equation (6-18) it can be seen that the stationary phase points for this case are the same as those when $\phi = 0$. Since Equations (6-14), (6-15), and (6-16) have the same values at $\phi = 0$ as they do when $\phi = 2\pi$, it follows that the far field evaluation of I_1 for Case 4 is identical to the corresponding evaluation in Case 3, with the exception that $h(\theta,0)$ is replaced by $h(\theta,2\pi)$. Thus, Equation (6-28) with $\phi = 2\pi$ is the evaluation of I_1 for this case as $k_{0_3} r \to \infty$.

Case 5. For this case it will be assumed that $\phi = \pi$, $\tau \neq 0$, and $\theta \neq 0$. According to Equation (6-18), the stationary phase points for the ψ integration in Equation (6-8) for this case are $\psi_0 = 0$, $\psi_0 = \pi$, and $\psi_0 = 2\pi$. The application of Equations (6-12) through (6-16) to Equation (6-8) gives, for this case,

$$I_1 = I_{15} + I_{16}$$

where I_{15} and I_{16} are defined as

$$I_{15} = \int_0^{\pi/2} \left[\frac{1}{2} \sqrt{\frac{2\pi}{k_{0_3} r |\sin\tau \, \sin\theta|}} \, k_{0_3}^2 \sin\tau \, \cos\tau \, h(\tau,0) \right. \tag{6-33}$$

$$\left. \cdot \, e^{jk_{0_3} r\sin\tau\sin\theta} \, e^{j \frac{\pi}{4} \operatorname{sign}(-\sin\tau\sin\theta)} \right] e^{-jk_{0_3} r\cos\theta\cos\tau} \, d\tau$$

$$+ \int_0^{\pi/2} \left[\frac{1}{2} \sqrt{\frac{2\pi}{k_{0_3} r |\sin\tau \, \sin\theta|}} \, k_{0_3}^2 \sin\tau \, \cos\tau \, h(\tau,2\pi) \right.$$

$$\cdot\, e^{jk_{0_3} r\sin\tau\sin\theta}\, e^{j\frac{\pi}{4}\,\mathrm{sign}(-\sin\tau\sin\theta)}\Bigg]\, e^{-jk_{0_3} r\cos\theta\cos\tau}\, d\tau$$

$$I_{16} = \int_0^{\pi/2} \left[\sqrt{\frac{2\pi}{k_{0_3} r|\sin\tau\ \sin\theta|}}\ k_{0_3}^2\ \sin\tau\ \cos\tau\ h(\tau,\pi) \right.$$

$$\cdot\, e^{-jk_{0_3} r\sin\tau\sin\theta}\, e^{j\frac{\pi}{4}\,\mathrm{sign}(\sin\tau\sin\theta)}\Bigg]\, e^{-jk_{0_3} r\cos\theta\cos\tau}\, d\tau$$

Since $\sin\tau$ and $\sin\theta$ are both positive in the intervals given in Equation (6-20), the last equation can be simplified to

$$I_{16} = \int_0^{\pi/2} \sqrt{\frac{2\pi\ \sin\tau}{k_{0_3} r\ \sin\theta}}\ k_{0_3}^2 \cos\tau\ h(\tau,\pi)e^{j\frac{\pi}{4}}\ e^{-jk_{0_3} r\cos(\tau-\theta)}\, d\tau \qquad (6\text{-}34)$$

Equation (6-10) shows that $h(\tau,2\pi) = h(\tau,0)$. Hence, I_{15} can be rewritten as

$$I_{15} = \int_0^{\pi/2} \sqrt{\frac{2\pi\ \sin\tau}{k_{0_3} r\ \sin\theta}}\ k_{0_3}^2 \cos\tau\ h(\tau,2\pi)e^{-j\frac{\pi}{4}}\ e^{-jk_{0_3} r\cos(\tau+\theta)}\, d\tau \qquad (6\text{-}35)$$

From this last equation it is seen that I_{15} is identical to I_{12}, as given by Equation (6-22) with $\phi = \pi$. In addition, I_{16} is identical to I_{11} with $\phi = \pi$, as can be seen from Equations (6-21) and (6-34). Hence, the far field evaluation of I_1 for this case is identical to that of Case 1 with $\phi = \pi$. Consequently, Equation (6-28), with $\phi = \pi$, applies to this case.

Cases 1 through 4 have shown that Equation (6-28) is the far field evaluation of I_1, for $0 \leqq \phi \leqq 2\pi$ and for $0 < \theta \leqq \frac{\pi}{2}$. Since the electric field should be a continuous function, Equation (6-28) is defined to also apply when $\theta = 0$. Thus, Equation (6-28) applies to all far field observation points that lie to the right of the ground plane shown in Figure 4.

It is now necessary to evaluate I_2 so that I, as given by Equation (6-5), will be known in the far field of the antenna.

The Evaluation of I_2

From Equation (6-9) it can be seen that I_2 approaches zero exponentially as $k_{0_3} r \to \infty$ as long as $\cos\theta \sinh\tau \neq 0$. Hence, only when $\cos\theta \sinh\tau = 0$ can I_2 be non-zero in the far field. The point $\tau = 0$ cannot contribute to I_2 since the integrand of I_2 is zero at this point. Hence, only when $\theta = \frac{\pi}{2}$, that is along the ground plane, will I_2 be non-zero and contribute to I. When V_2 is a lossless dielectric, Equations (5-15) and (6-7) show that all of the contributions from the surface wave poles enter I_2, and not I_1. Since $I_2 = 0$, if $\theta \neq \frac{\pi}{2}$, these pole contributions can only affect the far field pattern at $\theta = \frac{\pi}{2}$. But this is exactly the region where the theoretical pattern cannot accurately predict the measured pattern. The disagreement arises because the physical ground plane and dielectric sheet must be finite rather than infinite in their transverse (to the z axis) dimensions. Since the model cannot accurately predict the physical pattern at $\theta = \frac{\pi}{2}$, the effect of I_2 on I at $\theta = \frac{\pi}{2}$ will be neglected.

Combining this statement with Equations (6-1), (6-5), (6-10), and (6-28), shows that when V_3 is lossless,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(k_x,k_y)e^{-j[xk_x+yk_y+zk_{z_3}]} dk_x\, dk_y \tag{6-36}$$

$$= j\, 2\pi\, k_{0_3} \cos\theta\, f(k_{0_3}\sin\theta\cos\phi,\ k_{0_3}\sin\theta\sin\phi)\, \frac{e^{-jk_{0_3}r}}{r}$$

in the far field of the antenna, that is as $k_{0_3} r \to \infty$. The right side of Equation (6-36) is the dominant term, that is the r^{-1} term, in the asymptotic expansion of the left side of that same equation. Those terms containing r^{-2}, r^{-3}, r^{-4}, etc., have not been included on the right side of Equation (6-36) since they are essentially zero, as compared to the r^{-1} term that was retained. Equation (6-36) has also been derived by Borgiotti (36) but in a manner different from the one presented here. In addition, his derivation does not consider the effect of surface wave poles, while the derivation of this chapter does.

Equation (6-36) will now be applied to E_{x_3}, E_{y_3}, and E_{z_3}. By comparing Equations (3-61), (3-62), and (3-63) with Equation (6-36) it can be seen that in the far field of the antenna

$$E_{x_3} = j\,2\pi\,k_{0_3}\,\cos\theta\,T_x(k_{0_3}\sin\theta\cos\phi,\,k_{0_3}\sin\theta\sin\phi)\,\frac{e^{-jk_{0_3}r}}{r} \tag{6-37}$$

$$E_{y_3} = j\,2\pi\,k_{0_3}\,\cos\theta\,T_y(k_{0_3}\sin\theta\cos\phi,\,k_{0_3}\sin\theta\sin\phi)\,\frac{e^{-jk_{0_3}r}}{r} \tag{6-38}$$

$$E_{z_3} = -j\,2\pi\,k_{0_3}\sin\theta[\cos\phi T_x(k_{0_3}\sin\theta\cos\phi,\,k_{0_3}\sin\theta\sin\phi) \tag{6-39}$$

$$+ \sin\phi\,T_y(k_{0_3}\sin\theta\cos\phi,\,k_{0_3}\sin\theta\sin\phi)]\,\frac{e^{-jk_{0_3}r}}{r}$$

These rectangular components of $\bar{E}_3$ will next be converted to their equivalent spherical components. This will be done since in most far field measurements it is the spherical components, rather than the rectangular components, which are measured. The transformation from rectangular to spherical components (37) is

$$E_{r_3} = E_{x_3}\,\sin\theta\,\cos\phi + E_{y_3}\sin\theta\,\sin\phi + E_{z_3}\,\cos\theta \tag{6-40}$$

$$E_{\theta_3} = E_{x_3}\,\cos\theta\,\cos\phi + E_{y_3}\cos\theta\,\sin\phi - E_{z_3}\,\sin\theta \tag{6-41}$$

$$E_{\phi_3} = -E_{x_3}\,\sin\phi + E_{y_3}\,\cos\phi \tag{6-42}$$

Applying Equations (3-37), (3-38), and (3-39) to Equation (6-40) produces

$$E_{r_3} = j\,2\pi\,k_{0_3}\,\frac{e^{-jk_{0_3}r}}{r}\,[\sin\theta\,\cos\theta\,\cos\phi\,T_x$$

$$+ \sin\theta \cos\theta \sin\phi \; T_y - \sin\theta \cos\theta \cos\phi \; T_x - \sin\theta \cos\theta \sin\phi \; T_y]$$

or

$$E_{r_3} = 0 \tag{6-43}$$

which is the expected result in the far field.

A similar substitution into Equation (6-41) gives

$$E_{\theta_3} = j \, 2\pi \, k_{0_3} \frac{e^{-jk_{0_3} r}}{r} [\cos^2\theta\cos\phi T_x + \cos^2\theta\sin\phi T_y$$

$$+ \sin^2\theta\cos\phi T_x + \sin^2\theta\sin\phi T_y]$$

or

$$E_{\theta_3} = j \, 2\pi \, k_{0_3} [\cos\phi T_x(k_{0_3} \sin\theta\cos\phi, \, k_{0_3} \sin\theta\sin\phi) \tag{6-44}$$

$$+ \sin\phi T_y(k_{0_3} \sin\theta\cos\phi, \, k_{0_3} \sin\theta\sin\phi)] \frac{e^{-jk_{0_3} r}}{r}$$

Next, using Equations (6-37) and (6-38) in Equation (6-42) yields

$$E_{\phi_3} = j \, 2\pi \, k_{0_3} \cos\theta[-\sin\phi T_x(k_{0_3} \sin\theta\cos\phi, \, k_{0_3} \sin\theta\sin\phi) \tag{6-45}$$

$$+ \cos\phi T_y(k_{0_3} \sin\theta\cos\phi, \, k_{0_3} \sin\theta\sin\phi)] \frac{e^{-jk_{0_3} r}}{r}$$

The absolute values of E_{r_3}, E_{θ_3}, and E_{ϕ_3} are

$$|E_{r_3}| = 0 \tag{6-46}$$

$$|E_{\theta_3}| = \left(\frac{2\pi k_{0_3}}{r}\right) |\cos\phi T_x + \sin\phi T_y| \tag{6-47}$$

$$|E_{\phi_3}| = \left(\frac{2\pi k_{0_3} \cos\theta}{r}\right) |-\sin\phi T_x + \cos\phi T_y| \tag{6-48}$$

Equations (6-46), (6-47), and (6-48) represent the desired far field evaluation of the trial field $\bar{E}_3$. The last three equations are given because physical measuring equipment responds to the absolute value of the field, rather than to the field itself. These three equations can be evaluated in terms of the mode amplitudes by using Equations (3-81), (3-82), (3-77), (3-78), (3-106), and (3-107).

The purpose of this chapter was to explicitly evaluate the double integral plane wave representation of $\bar{E}_3$ in the far field of the antenna. Equations (6-43) through (6-48) present the results of this evaluation. It should be remembered that the only restriction made on the medium parameters in obtaining these equations was that region V_3 was lossless. Hence, the far field evaluation presented in this chapter applies to arbitrary medium parameters in regions V_1 and V_2 and to arbitrary μ_3 and ε_3 as long as $\sigma_3 = 0$.

CHAPTER VII

COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

In order to demonstrate the validity of the slot antenna analysis of the preceding chapters, four antennas were examined using this analysis, and the corresponding antennas were constructed and tested. The test data obtained from these antennas was compared with the predicted behavior, and the results of this comparison are presented in this chapter.

Selection of Test Antennas

The antennas selected for examination were chosen on the basis of availability of microwave equipment for making measurements, the desire to examine a variety of configurations, and the need to avoid excessive computation time during numerical calculations. Measurements and calculations were performed at X-band (8 - 12.4 GHz) not only because the necessary microwave equipment was available for this frequency range but also because this band is often used in practice.

Two different slot sizes were examined both with and without dielectric coverings. Besides demonstrating the validity of the analysis, this variety of configurations also gives an indication of the relative importance of higher modes for the different geometries.

To reduce the numerical computation time, the slot was placed in the center of the waveguide for the examples presented in this chapter. Under these circumstances the geometry, as well as the excitation, is symmetrical about the center of the waveguide. Because of this symmetry, only symmetric higher order modes can be excited. From Equations (3-1), (3-2), (3-29), and (3-30), it can be seen that to preserve symmetry the amplitudes of the m,n^{th} mode must be zero except when m is odd and n is even simultaneously. This fact allows a large number of higher order modes to be removed from consideration and produces a correspondingly simpler numerical problem.

Numerical Calculations

The equations from the preceding chapters for the low loss covering were programmed in GTL for the Burroughs B5500 computer and in Algol for the Univac U1108 computer. The programming was initially done for the U1108, but it was necessary to change to the B5500 when it was discovered that round-off error on the U1108 during matrix inversion was excessive as the numbers of modes was increased to ten. The eleven significant figures carried by the B5500 continued to yield accurate matrix inverses as the number of modes was increased to ten, while the eight significant figures of the U1108 Algol did not.

When a dielectric covering is present, the location of the surface wave poles, that is the real zeros of Den, must be determined. The zeros were found using Muller's method, as modified by Frank (38).

To obtain an indication of the speed of "convergence," calculations were performed using one, three, and ten modes. The mode numbers used in each case are given in Table 1.

Table 1. The m,n Values of the Modes Used in Numerical Calculations

1 Mode Case	3 Mode Case	10 Mode Case
1,0	1,0	1,0
	3,0	3,0
	1,2	1,2
		5,0
		3,2
		5,2
		7,0
		7,2
		9,0
		1,4

The modes used in the calculations were selected on the basis of lowest cutoff frequency, which is a standard practice. Because of symmetry, only symmetric modes were included. Each mode number includes both TE and TM modes simultaneously.

Numerical calculations were performed for the following configurations:

1) An open ended ($a' = a$, $b' = b$) X-band waveguide radiating into free space ($d = 0$ and V_3 = free space).

2) An open ended X-band waveguide covered by the polyethylene slab $\varepsilon_{r_2} = 2.25$, $\tan\delta_2 = 0$, $d = 0.3201$ cm. Region V_3 was free space.

3) An X-band slot antenna with dimensions $a' = 0.7a$, $b' = 0.8b$ centered in the guide ($x_0 = (a-a')/2$, $y_0 = (b-b')/2$) and radiating into free space.

4) The slot antenna of 3) covered by the polyethylene slab $\varepsilon_{r_2} = 2.25$, $\tan\delta_2 = 0$, $d = 0.3201$ cm. Region V_3 was free space.

The calculated input admittance for each of these configurations referred to the plane $z = 0$ and normalized with respect to the waveguide admittance, $Y_0 = \beta/\omega\mu_1$, is presented in Figures 7 through 10, along with the measured values. The calculated admittance, Y/Y_0, is equal to $[1 - (R/I)]/[1 + (R/I)]$ where R is obtained from the matrix solution for the mode amplitudes and I is given in terms of R by Equation (3-24). Figures 11 through 18 present some typical calculated radiation patterns along with the corresponding measured patterns. Patterns were only measured between 8 and 10 GHz since this was the range of the antenna testing equipment.

Experimental Antennas

The purpose of the experimental portion of this thesis was to demonstrate the validity of the analytical work, and not to construct precision antennas. Consequently, the construction and measuring techniques employed were less than ideal, but the results indicate that they were sufficient to accomplish the purpose.

The ground plane of the open ended X-band waveguide antenna was a 1/32-inch thick, brass sheet 30.1 cm. square. The waveguide was placed in the center of the ground plane. The polyethylene slab which covered this antenna was approximately 1/8-inch thick (measured to be 0.3201 cm.) and was 30.5 cm. square. It was taped to the ground plane. The ground plane for the X-band slot antenna was a 1/4-inch thick copper plate 30.5 cm. square. The waveguide was placed in the center of the ground plane. The iris portion of the ground plane that covered this waveguide was 0.01 inches thick. The same polyethylene slab was used

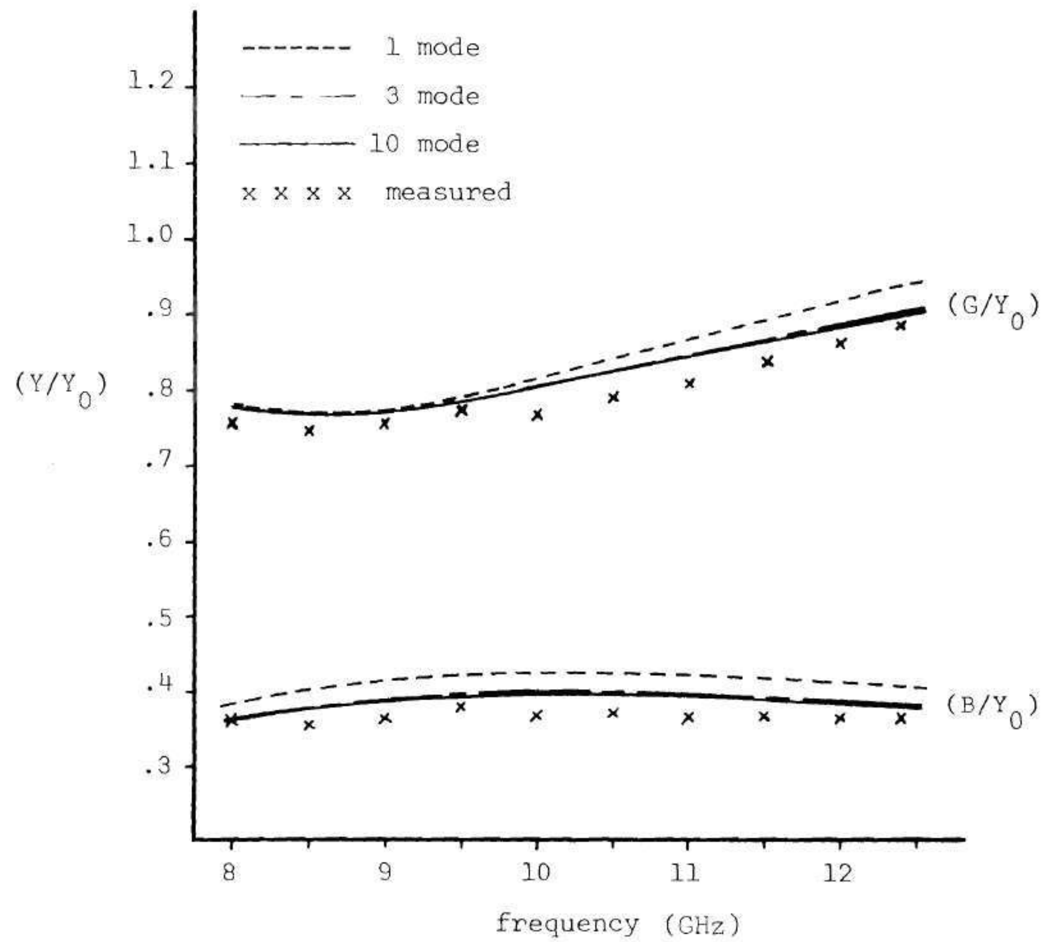
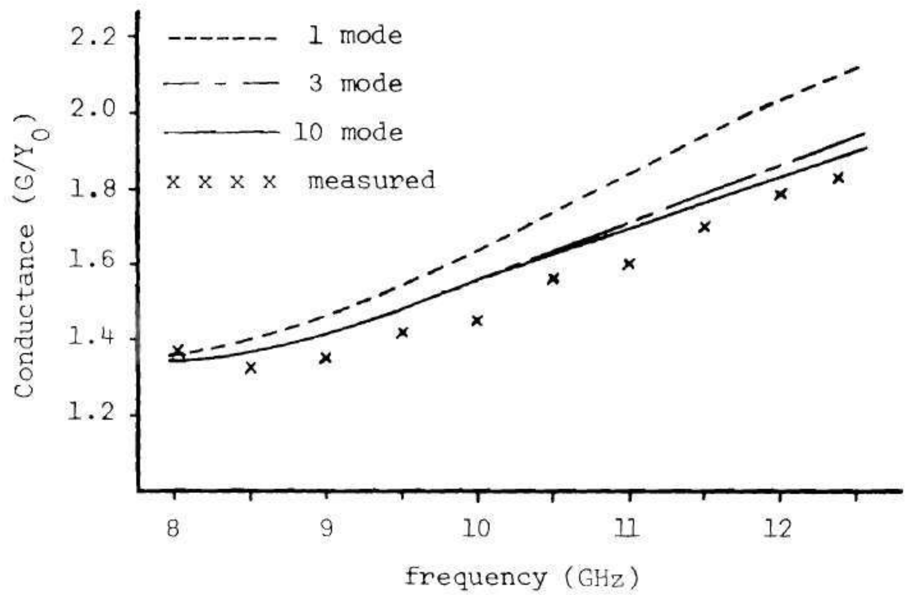


Figure 7. Input Admittance of Rectangular Waveguide Radiating into Free Space



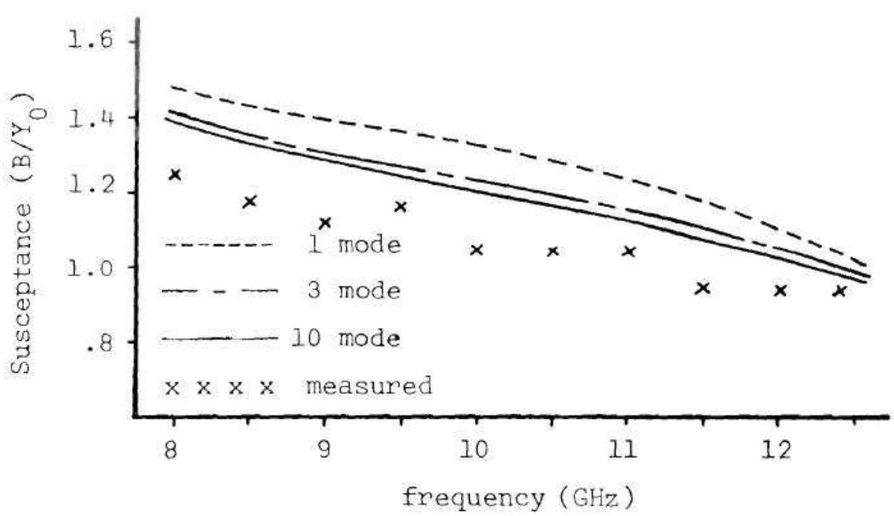


Figure 8. Input Admittance of Rectangular Waveguide Under the Dielectric Slab $\varepsilon_{r_2} = 2.25$, $d = 0.3201$ cm

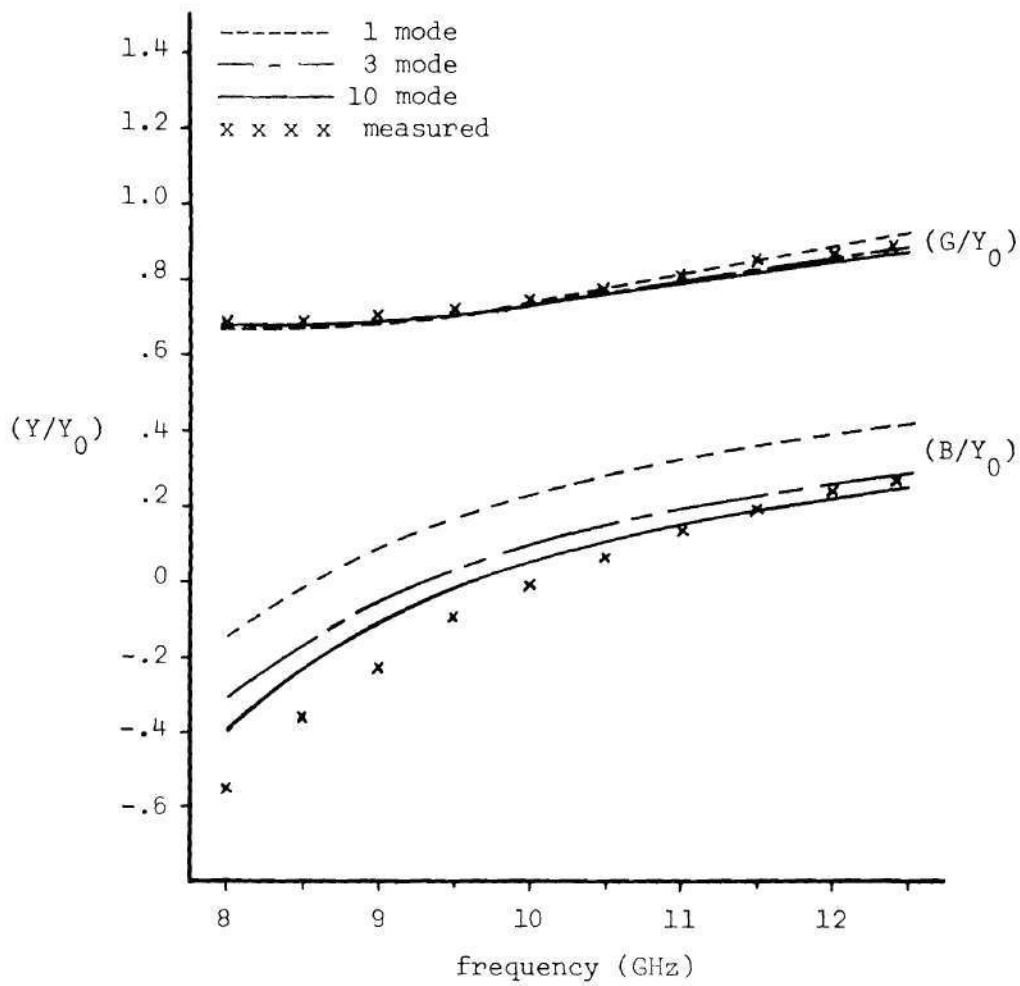


Figure 9. Input Admittance of Slot Antenna Radiating into Free Space

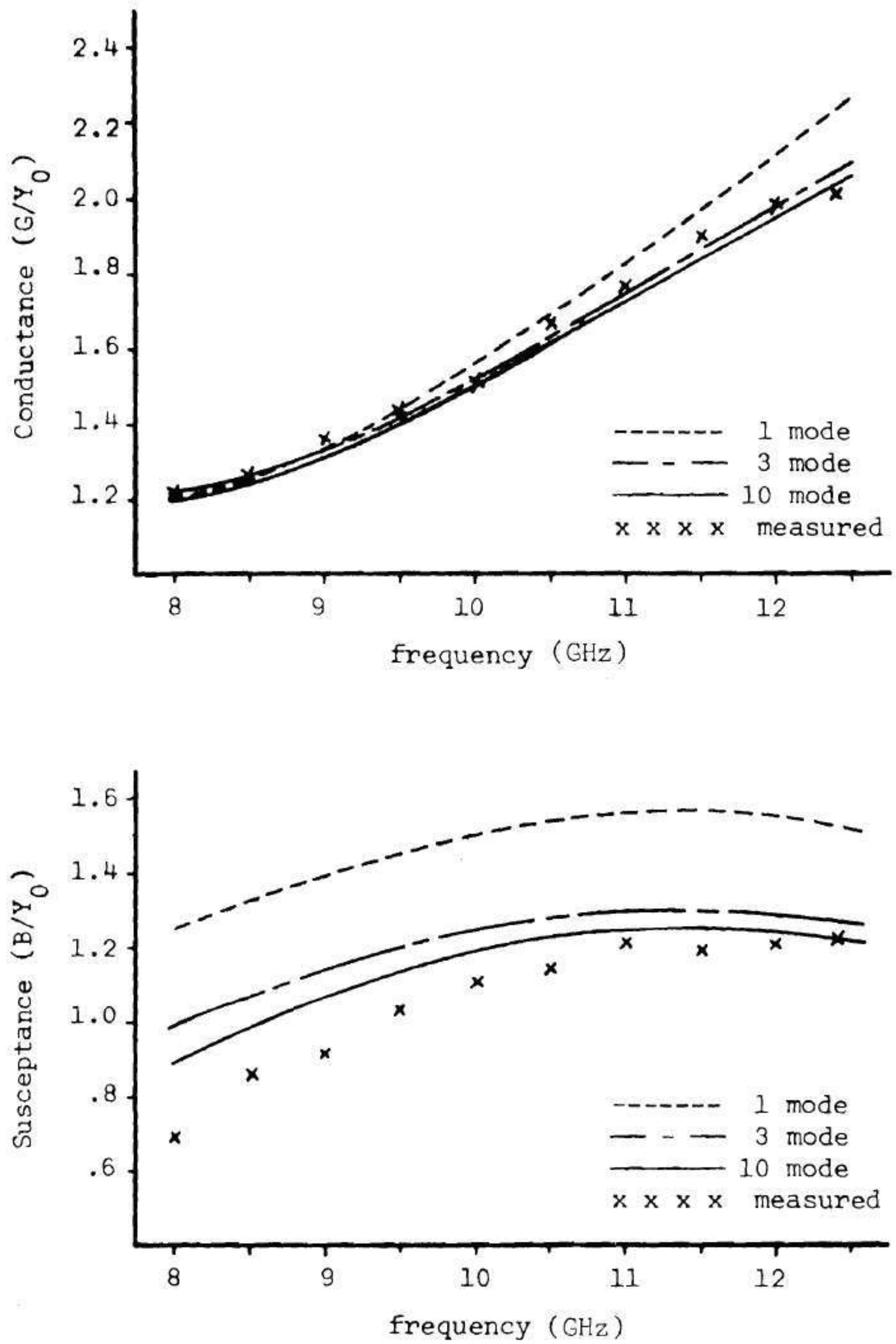


Figure 10. Input Admittance of Slot Antenna Under the Dielectric Slab $\varepsilon_{r_2} = 2.25$, $d = 0.3201$ cm

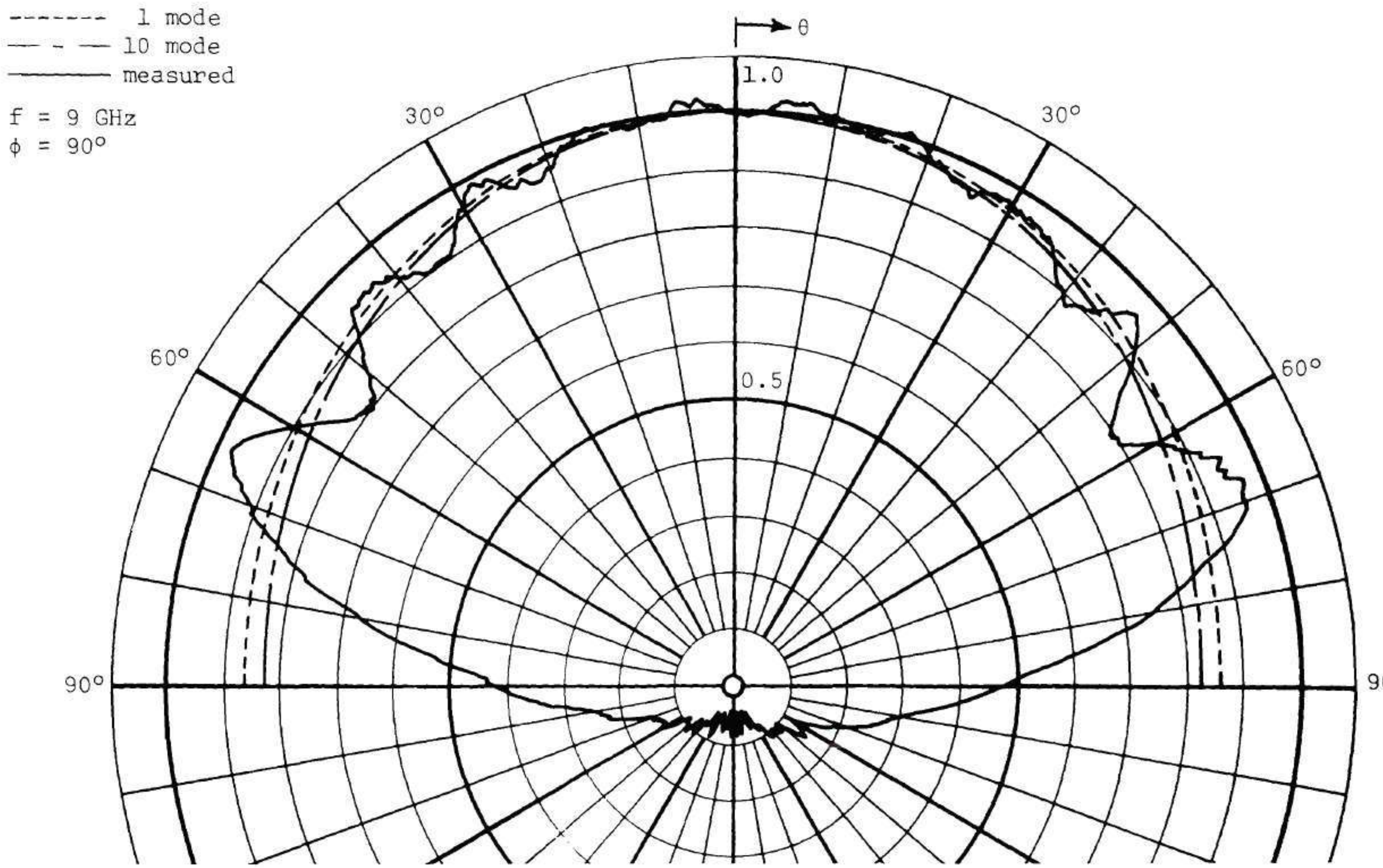


Figure 11. Far Field E Plane Pattern of Rectangular
Waveguide Radiating into Free Space

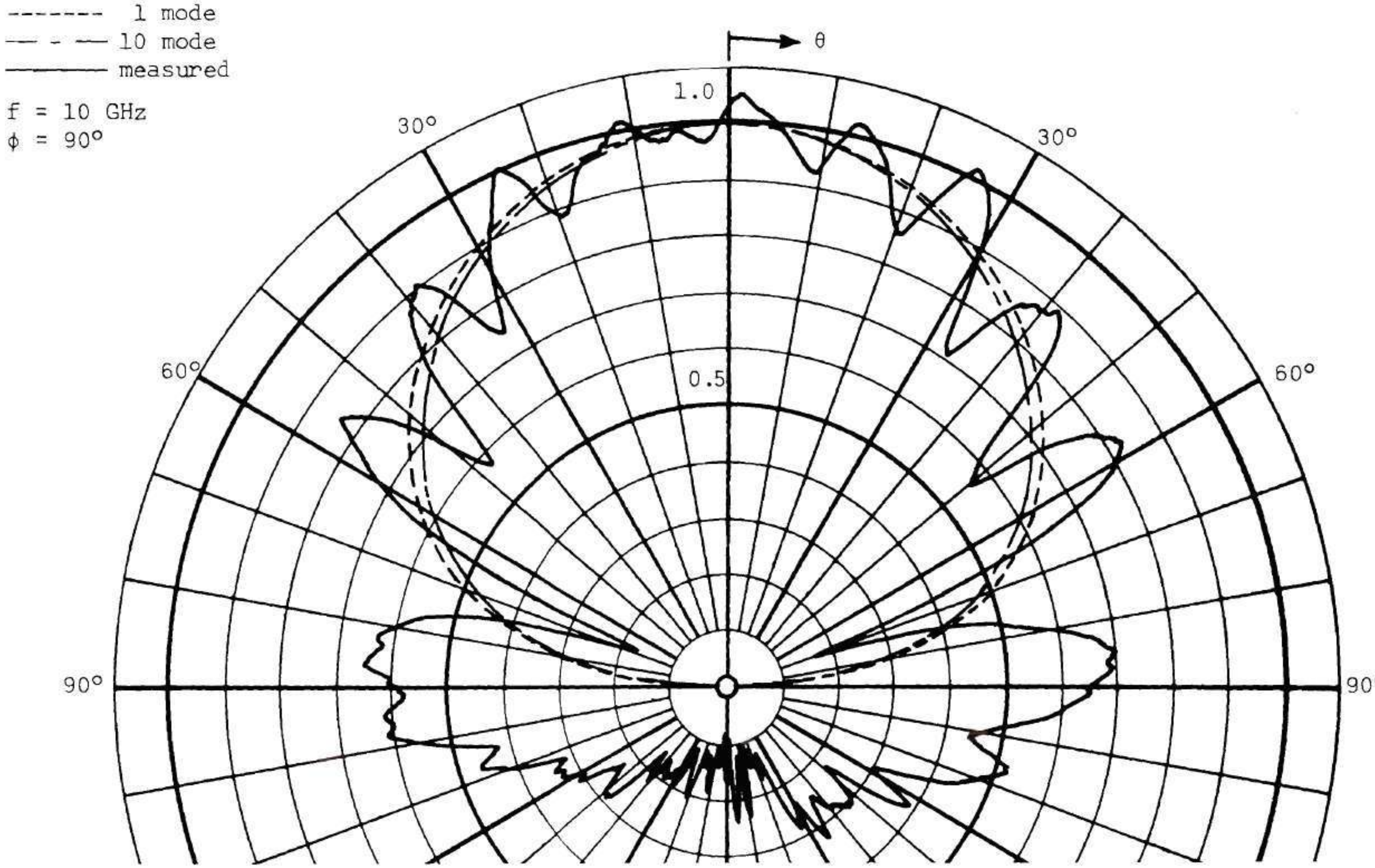


Figure 12. Far Field E Plane Pattern of Rectangular Waveguide Under the Dielectric Slab $\varepsilon_{r_2} = 2.25$, d = 0.3201 cm

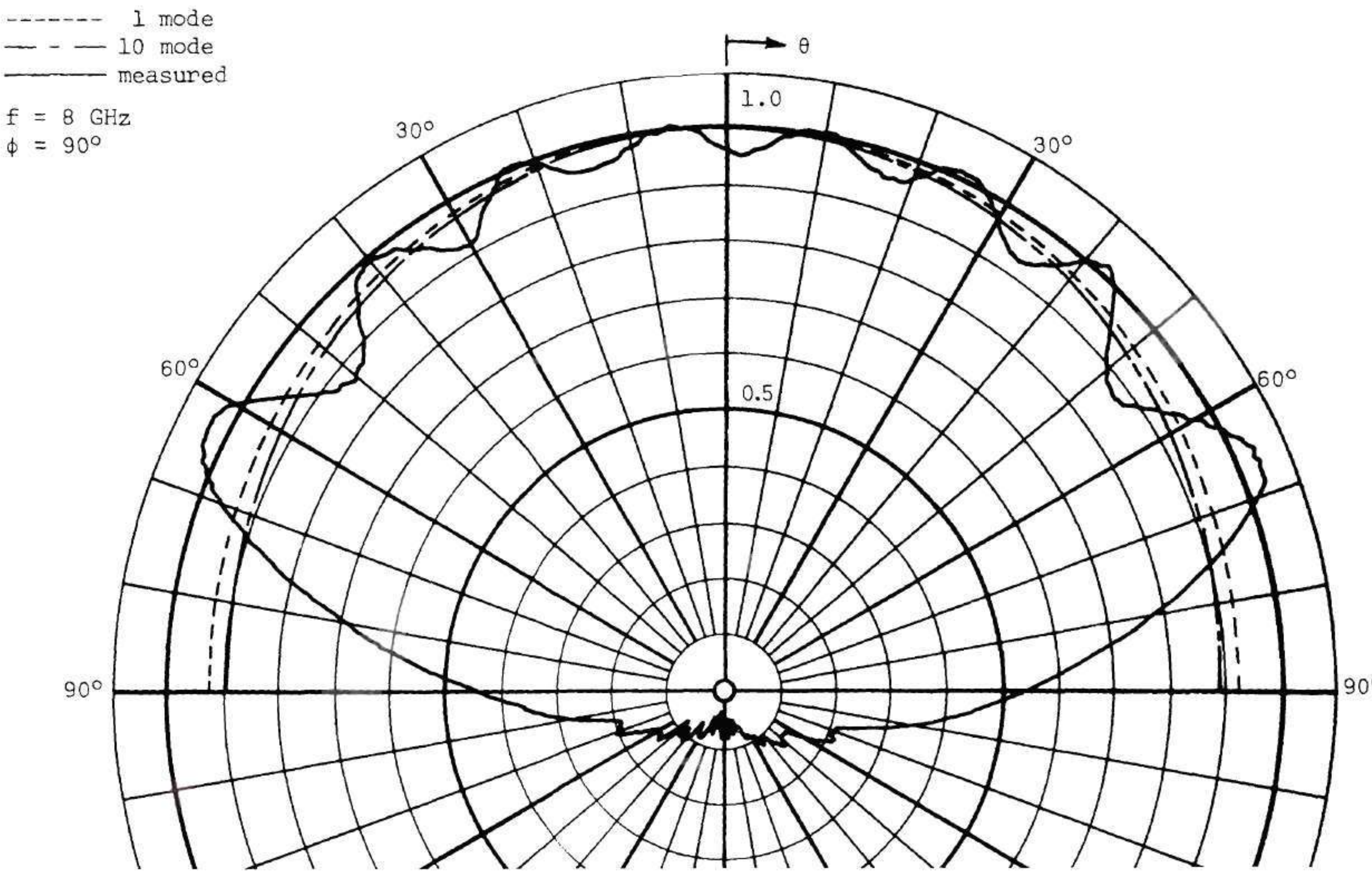


Figure 13. Far Field E Plane Pattern of Slot Antenna Radiating into Free Space

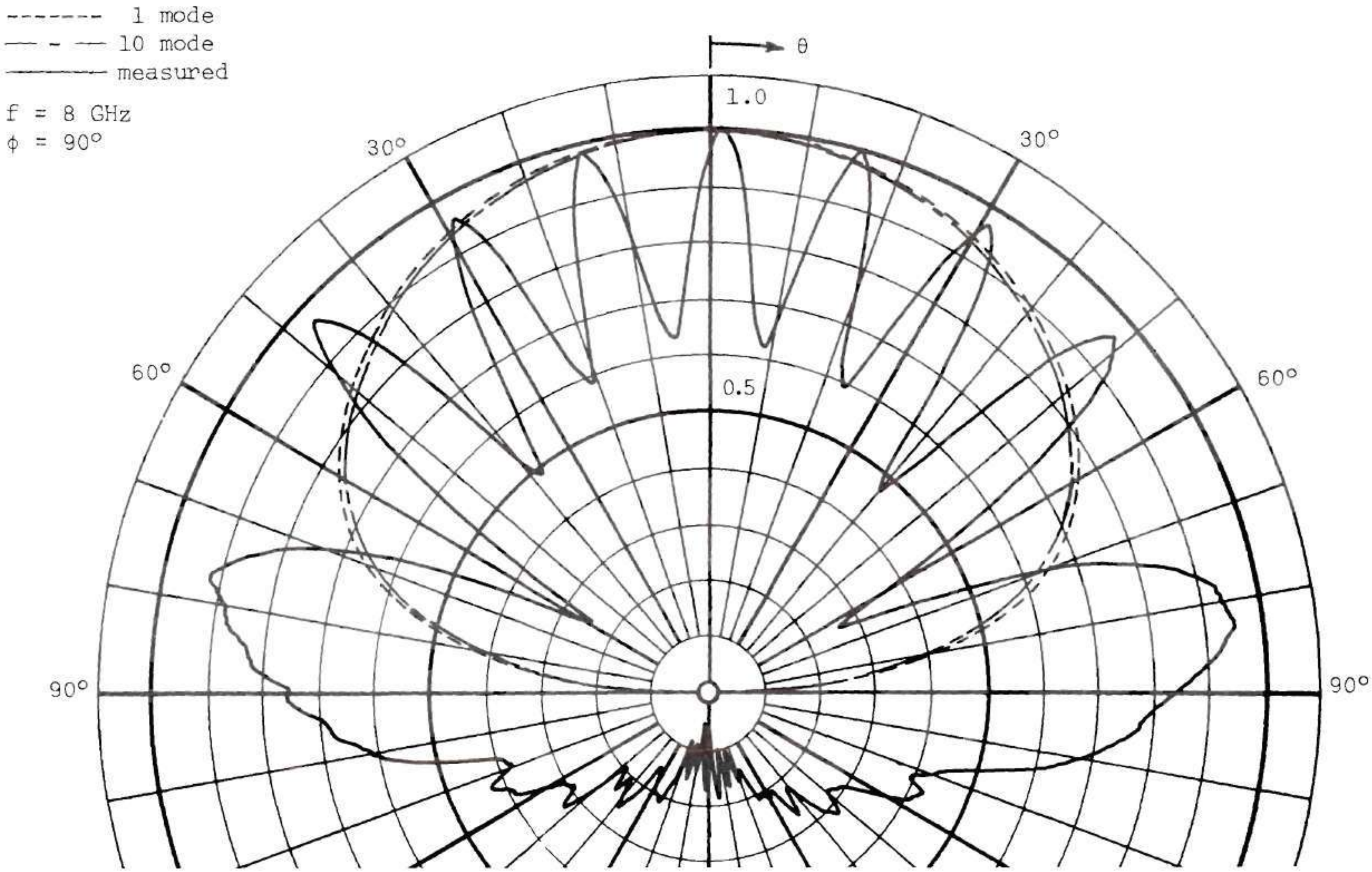


Figure 14. Far Field E Plane Pattern of Slot Antenna Under the Dielectric Slab $\varepsilon_{r_2} = 2.25$, $d = 0.3201$ cm

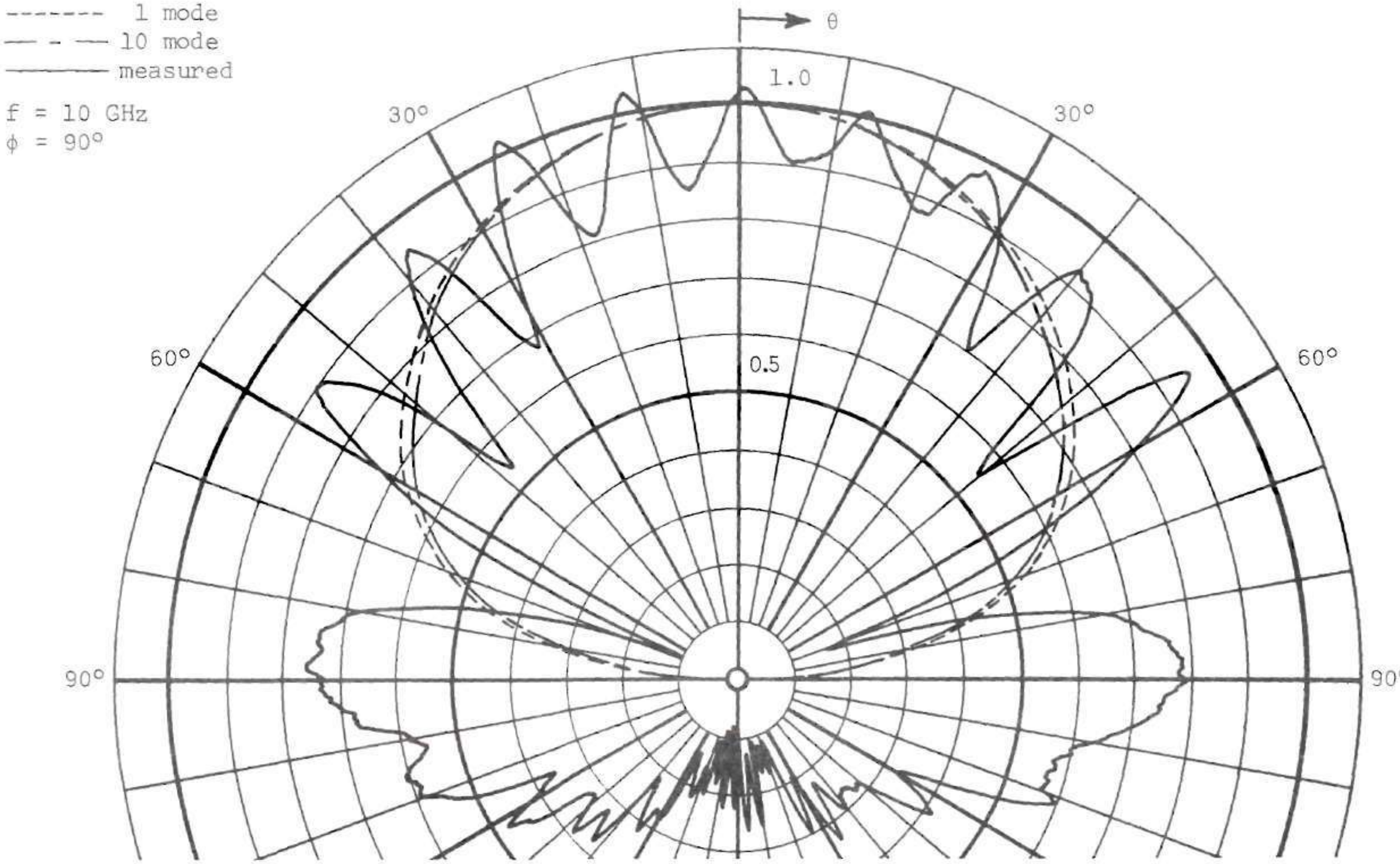


Figure 15. Far Field E Plane Pattern of Slot Antenna Under the Dielectric Slab $\varepsilon_{r_2} = 2.25$, $d = 0.3201$ cm

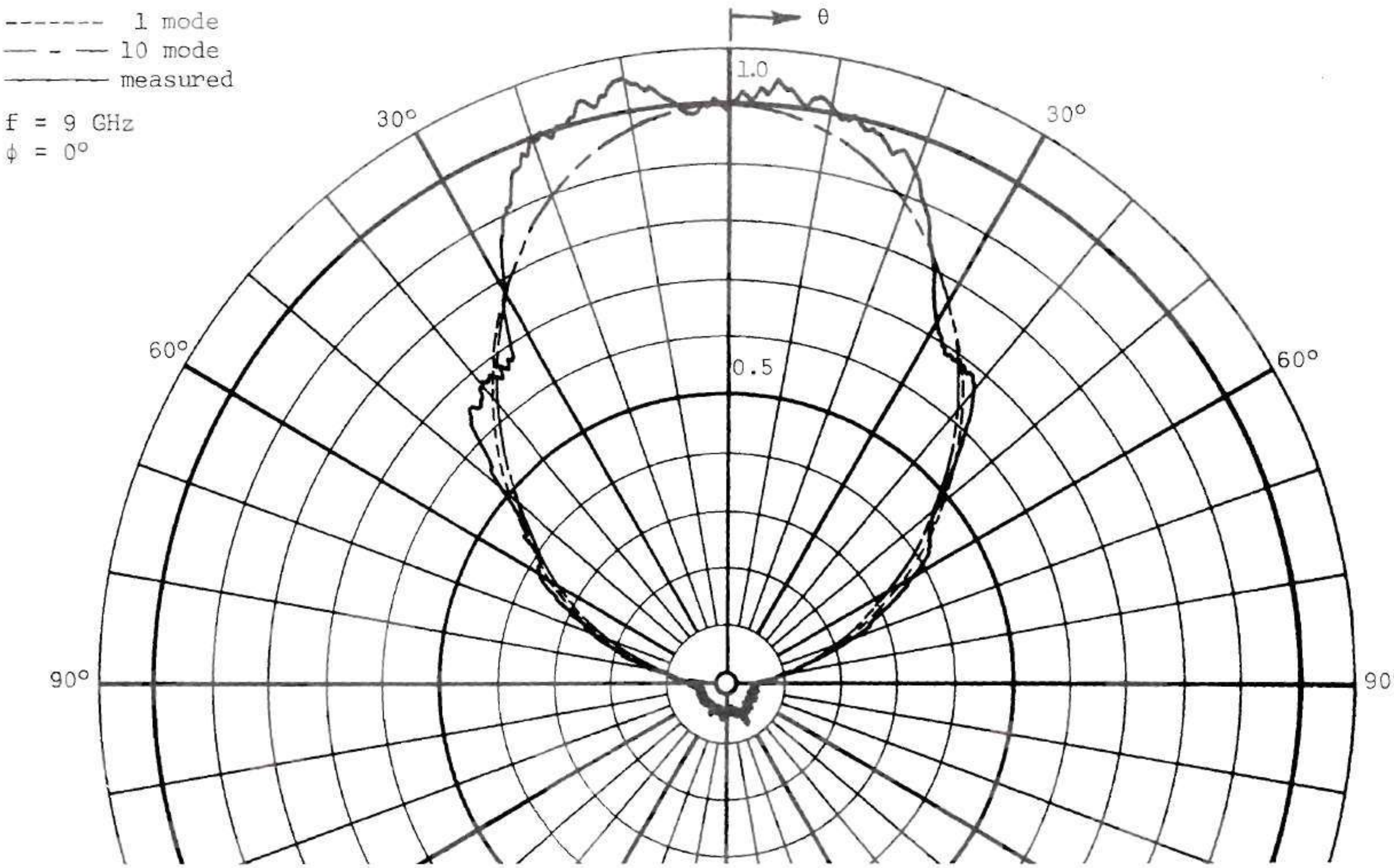


Figure 16. Far Field H Plane Pattern of Rectangular Waveguide Radiating into Free Space

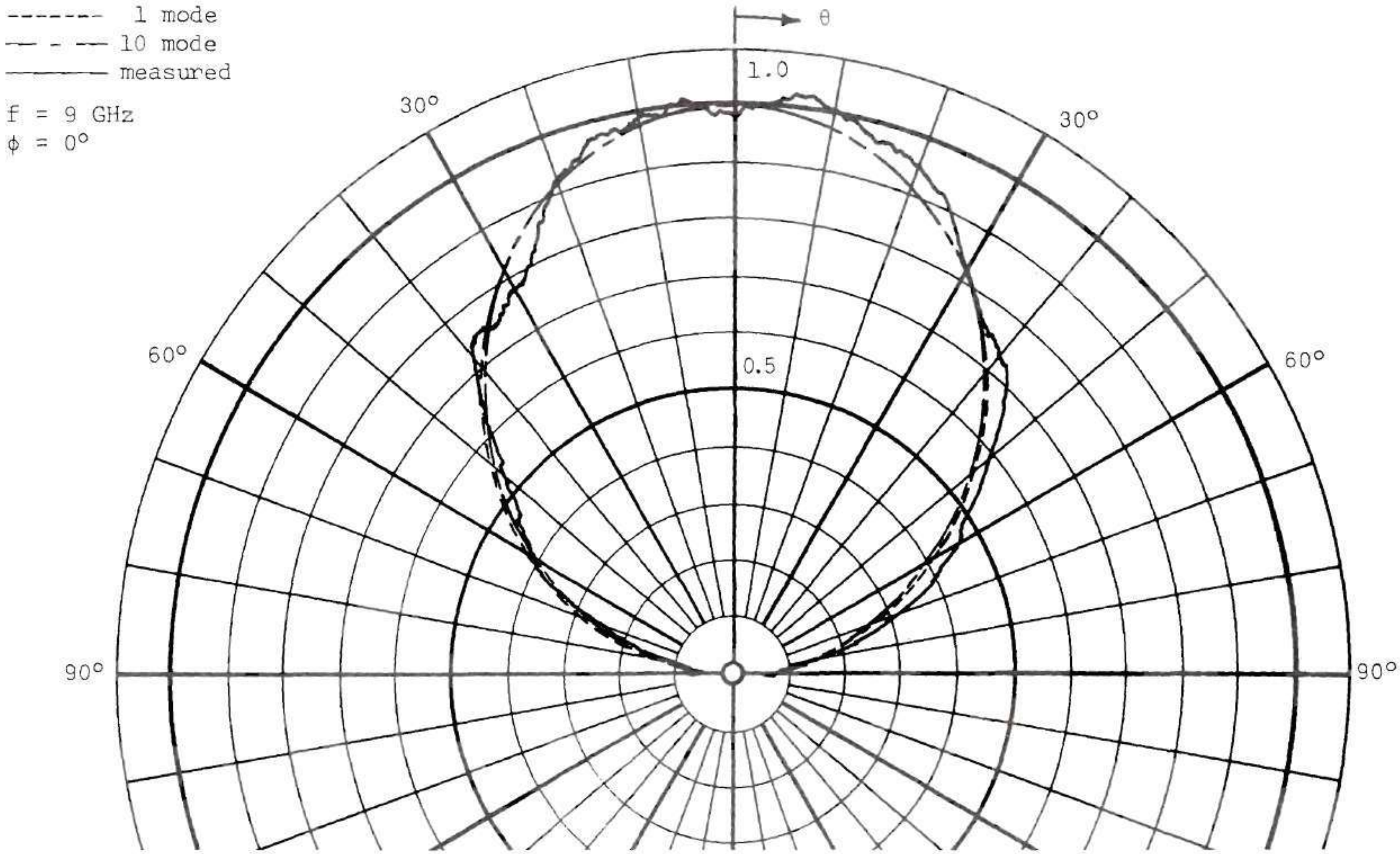


Figure 17. Far Field H Plane Pattern of Slot Antenna Radiating into Free Space

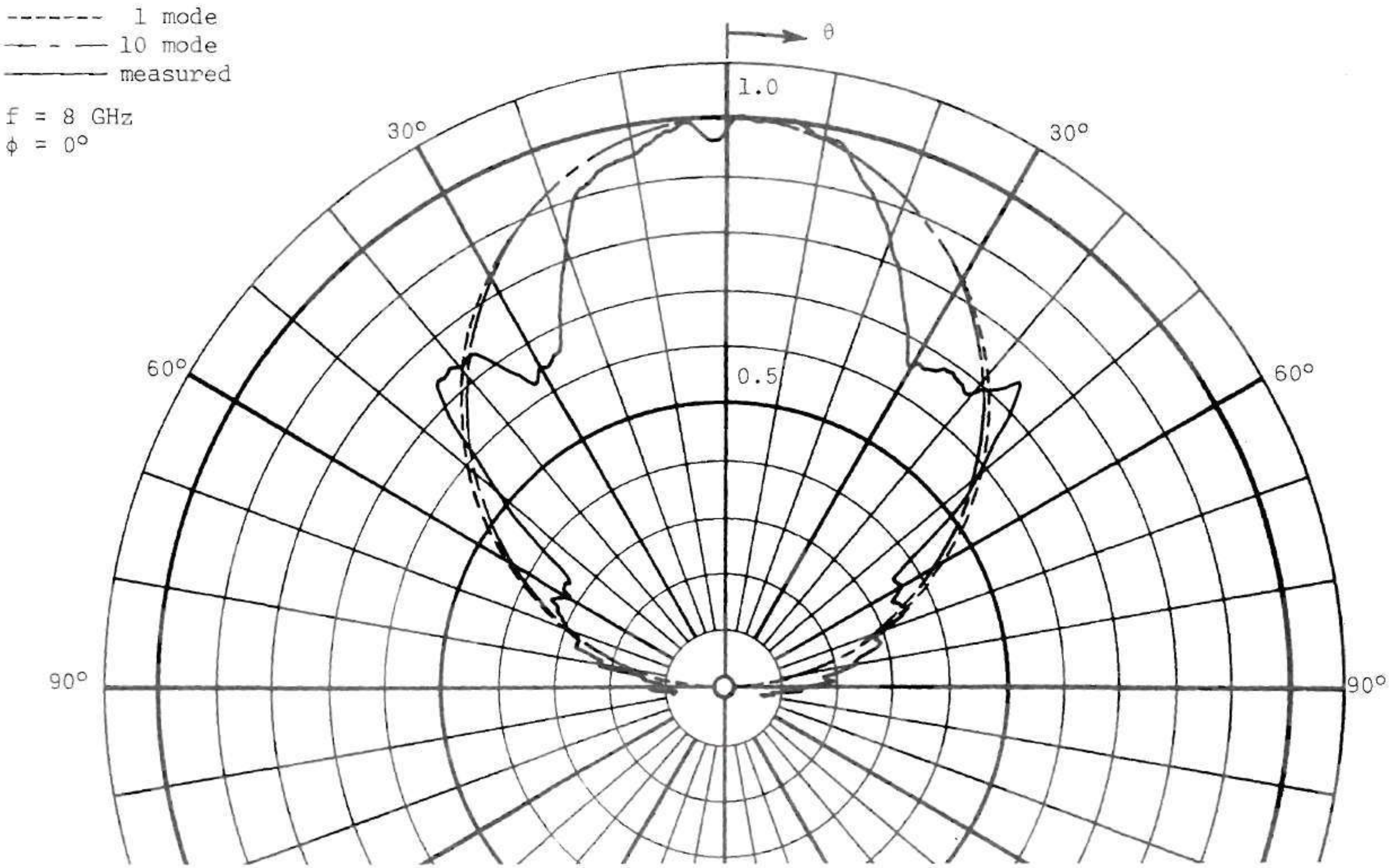


Figure 18. Far Field H Plane Pattern of Slot Antenna Under the Dielectric Slab $\varepsilon_{r_2} = 2.25$, d = 0.3201 cm

for this antenna as was employed for the open-ended waveguide.

The input admittance of these antennas was measured using conventional slotted waveguide techniques (39). Only the E and H plane patterns of the antennas were measured, as is common practice.

Discussion of Data

The admittance of the X-band waveguide radiating into free space is shown in Figure 7. In this figure it is seen that the three-mode and ten-mode curves are essentially the same and that the one-mode curve differs only slightly from the three-mode. The predicted admittance agrees well with the measured values. The admittance of this same antenna with a polyethylene coating is shown in Figure 8. Again, there is little difference between the three-mode and ten-mode curves. The measured values of conductance and susceptance fall below the predicted values for this antenna.

The input admittance of the slot antenna both with and without a polyethylene covering is presented in Figures 9 and 10. For both antennas the predicted conductance agrees very well with the measured values. The predicted susceptance, however, of both of these antennas shows a systematic divergence from the measured values as the frequency is decreased. The differences in the admittance curves are due to several causes and will be treated following the discussion of the patterns.

All of the predicted patterns are smooth curves, which decrease monotonically from the maximum value, which is in a direction normal to the aperture. For comparison purposes, all calculated radiation patterns have been normalized to have a value of one in the direction $\theta = 0$, that is, in the direction normal to the aperture. The measured patterns contain ripples which fluctuate above and below the predicted curves. Since these ripples can all be attributed to the same cause, the differences between the predicted and the measured patterns will be discussed first.

Effect of Finite Size of Ground Plane on Radiation Pattern

The radiation patterns of the experimental antenna and the theoretical model will not be identical because the ground plane of the physical antenna is finite in size, while the ground plane of the theoretical model is infinitely large. However, it should be expected that the patterns of the two antennas should become more and more alike as the finite ground plane is made larger and larger. The effect of ground plane size on the radiation pattern has been investigated by Wait (40) using a thin elliptic cylinder and by Frood and Wait (41) for an infinitely long strip. Their work shows that ripples are created in the pattern because of the edges and that the amplitude of these ripples is smaller for larger ground planes.

Using the method of Dorne and Lazarus (42), the effect of the edges on the radiation pattern can be approximated by means of elementary point sources at the edges of the finite ground plane. First, the aperture with the infinite ground plane is approximated as an isotropic radiator of strength E_0. Next, the aperture with the finite ground plane is approximated by a point source E_0 at the aperture and by two point sources, each of strength ΓE_0, as shown in Figure 19. The width of the ground plane is W. The constant Γ determines the strength of the edge sources. By symmetry, the sources at the edges are equal in

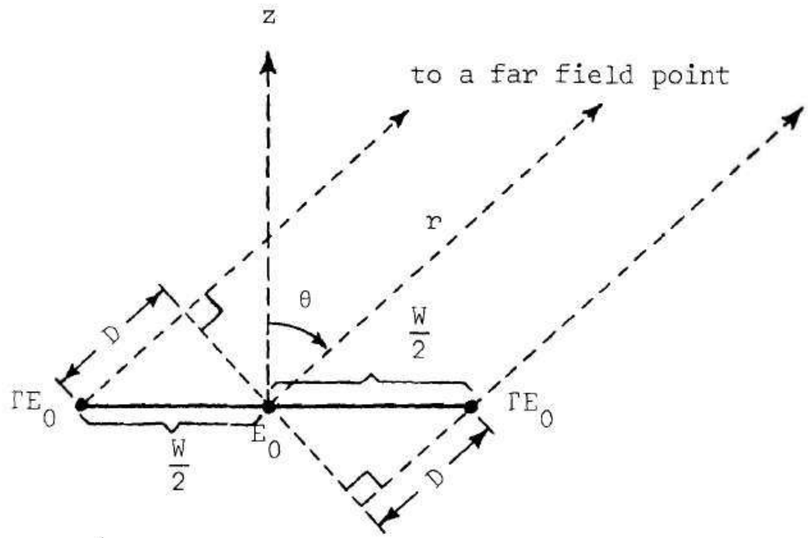


Figure 19. Location of Point Sources for Study of Edge Effects on Radiation Pattern

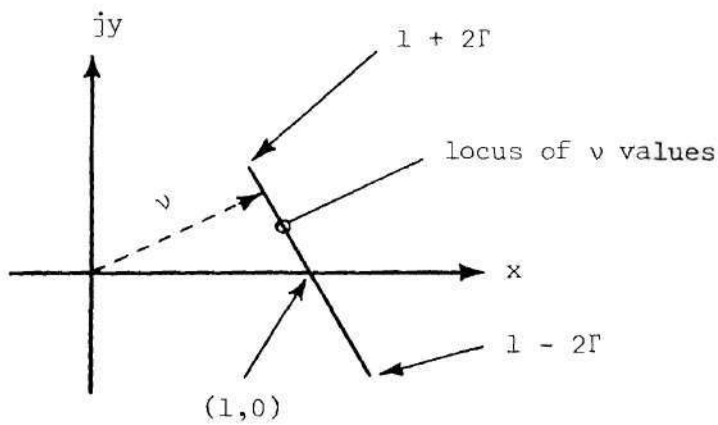


Figure 20. Representation of ν in the Complex Plane

magnitude and are in phase. From Figure 19, the far field from the three point sources is seen to be

$$E = \Gamma E_0 \frac{e^{-jk_0(r+D)}}{r} + E_0 \frac{e^{-jk_0 r}}{r} + \Gamma E_0 \frac{e^{-jk_0(r-D)}}{r}$$

$$= E_0 \frac{e^{-jk_0 r}}{r} [1 + \Gamma(e^{jk_0 D} + e^{-jk_0 D})]$$

$$= E_0 \frac{e^{-jk_0 r}}{r} [1 + 2\Gamma \cos(k_0 D)]$$

where k_0 is the phase constant of free space. But, $D = (W/2) \sin\theta$. Hence,

$$k_0 D = \left(\frac{2\pi}{\lambda_0}\right)\left(\frac{W}{2}\right)\sin\theta = \pi\left(\frac{W}{\lambda_0}\right)\sin\theta$$

Thus,

$$E = E_0 \frac{e^{-jk_0 r}}{r} [1 + 2\Gamma \cos(\pi\left(\frac{W}{\lambda_0}\right)\sin\theta)] \qquad (7\text{-}1)$$

The locations of the minimum and maximum values of this electric field will now be determined. From Equation (7-1) it can be seen that E is directly proportional to ν where ν is defined as

$$\nu = 1 + 2\Gamma \cos(\pi[\frac{W}{\lambda_0}]\sin\theta) \qquad (7\text{-}2)$$

Thus $|E|$ and $|\nu|$ have their minimum and maximum values at the same values of θ. In the complex plane, ν can be represented as shown in Figure 20. From Equation (7-2) and Figure 20 it can be seen that $|\nu|$, and hence $|E|$, has its longest and shortest lengths when

$$\cos(\pi[\frac{W}{\lambda_0}]\sin\theta) = \pm 1$$

or when

$$\sin\theta_m = n(\frac{\lambda_0}{W}) \quad \text{for } n = 0, \pm 1, \pm 2, \ldots \tag{7-3}$$

The angles θ_m locate the directions of the local minimum and maximum of the pattern. The ground planes of the experimental antennas were approximately 30.5 cm. wide; the locations of the minima and maxima of the patterns for this ground plane size are given in Table 2.

Table 2. Predicted Location of the Minima and Maxima of the Pattern in Degrees when W = 30.5 cm.

n	θ_m for 8 GHz	θ_m for 9 GHz	θ_m for 10 GHz
0	0	0	0
1	7	6	6
2	14	13	11
3	22	19	17
4	29	26	23
5	38	33	29
6	47	41	36
7	59	50	43
8	79	61	52
9	-	79	62
10	-	-	79

Since the patterns are symmetric about $\theta = 0$, only the minima and maxima on one side of $\theta = 0$ need to be examined. The angles given in Table 2 are in excellent agreement with the measured E plane values given in Figures 11 through 15. This agreement confirms that the ripples in the measured pattern are the result of scattering of energy by the edges of the ground plane. These figures also show that the calculated pattern, which is based on an infinite ground plane, approximately predicts the average value of the ripples. The predicted patterns should represent large ground planes better than small ones since the field scattered by the edges should be smaller the farther the edges are from the aperture.

The predicted H plane patterns show very little energy near $\theta = 90°$. Hence, there should be only a small amount of energy (compared to the E plane scattered energy) scattered by the edges which are parallel to the narrow dimension of the slot. This scattered field must be added to the predicted H plane pattern to obtain (approximately) the measured H plane pattern. Since the scattered field is small, it will be influential only when the predicted field is also small. Thus, the scattered field should be most influential near the ground plane. This is the effect that is observed.

Placing the dielectric on the ground plane tends to cause more energy to be stored near the ground plane. Hence, more energy is available at the edge to be scattered, and so a larger ripple in the pattern is expected with the dielectric than without. An examination of Figures 13 and 14 bears out this point.

It should also be observed from Figures 11 through 18 that the predicted patterns for the one mode case differ only slightly from those for the ten mode case. Hence, as far as pattern predictions go, a one mode analysis appears quite adequate for the configurations considered.

Mechanical Tolerances and Input Admittance

Figures 7 through 10 show a good agreement between calculated and measured admittances indicating that the equations and computer program are correct. These figures also show that a substantial improvement in predicted admittance, especially susceptances, can be obtained by using more than one mode.

The agreement between measured and calculated admittance is worse in Figure 8 than it is in Figure 7. This is due to the fact that the thin ground plane for the open ended waveguide was warped near the aperture, and consequently the dielectric slab was not in good contact with it. In addition, the measurement of the admittance could be in error by 10 to 20 per cent because the shift in the null of the standing wave pattern going from short circuit to load was small (about 0.10 to 0.20 cm.). Since the error in measuring this null shift was about 0.02 cm., a 10 to 20 per cent maximum error in measured admittance could be expected. For the other antennas the shift in null was much larger, and so the maximum percentage error was smaller.

The measured conductance of the slot and covered slot antennas agrees quite well with the predicted values as can be seen in Figures 9 and 10. However, these figures also show a systematic divergence of the measured and calculated susceptances as the frequency is decreased.

This deviation can be attributed to the sensitivity of the susceptance to the dimensions of the antenna. As shown in Tables 3 and 4 (which were obtained by using the new variational approach), small changes in the width of the slot produce large changes in (B/Y_0) at 8 GHz, but only small change in (B/Y_0) at 12.5 GHz. At the same time, (G/Y_0) remains relatively constant as width of the slot is changed.

This is the type of effect that occurs in Figures 9 and 10. The sensitivity in (B/Y_0) to slot width can be attributed to the fact that the iris is behaving approximately like an inductor (43), whose inductance increases rapidly as the frequency is decreased. Hence, small changes in the iris size produce large changes in its inductance and hence large changes in the susceptance of the antenna.

Comparing Tables 3 and 4 with Figures 9 and 10 reveals that calculations based on an a' of 0.60 inches would have produced the measured values of admittance. Since the measured width of the slot was within a few thousandths of an inch of 0.63 inches instead of 0.60 inches, the deviations in susceptance cannot entirely be attributed to an incorrectly machined slot width. Tables 3 and 4, however, do indicate that small errors in mechanical dimensions such as in the inside width of the waveguide, the finite thickness of the iris, and the width of the slot could produce the deviation in susceptance that was observed.

Table 3. Calculated Variations of Admittance with Slot Width for the X-band Slot Antenna with the Polyethylene Covering*

f in GHz	a' in Inches	(Y/Y_0)
8.0	0.63	1.20 + j 0.887
8.0	0.62	1.20 + j 0.827
8.0	0.60	1.18 + j 0.697
12.5	0.63	2.06 + j 1.23
12.5	0.62	2.06 + j 1.22
12.5	0.60	2.06 + j 1.19

*Using 10 modes and b' = 0.7b = 0.32 inches.

Table 4. Calculated Variations of Admittance with Slot Width for the X-band Slot Antenna Radiating into Free Space*

f in GHz	a' in Inches	(Y/Y_0)
8.0	0.63	0.679 - j 0.396
8.0	0.62	0.675 - j 0.460
8.0	0.60	0.667 - j 0.598
12.5	0.63	0.878 + j 0.253
12.5	0.62	0.876 + j 0.233
12.5	0.60	0.871 + j 0.186

*Using 10 modes and b' = 0.7b = 0.32 inches.

Discussion of Results

The purpose of this chapter was to demonstrate that the slot antenna analysis of the preceding chapters is correct. This verification was accomplished by showing the agreement of predicted and measured results for four slot antenna configurations. The results of this chapter indicate that the infinite ground plane analysis can be successfully applied to determine the admittance of a slot antenna with a finite size ground plane. The pattern predictions are not as accurate as the admittance predictions because of the diffraction by the edges, especially when a dielectric covering is present. Better agreement should be obtained as the ground plane is made wider and as the dielectric is made more lossy. If the dielectric is lossy, the energy at the edge of the dielectric slab tends to be smaller, and so the scattered field is smaller.

CHAPTER VIII

CONCLUSIONS

The new variational principle presented in this thesis produces a system of linear equations rather than nonlinear equations, as comparable variational approaches do. The simplification produced by these linear equations makes feasible a multimode analysis of a large class of electromagnetic problems, while only a one or two mode analysis is usually practical using nonlinear equations. The new variational principle thus permits more accurate studies to be made more quickly than is possible using comparable variational approaches. A considerable amount of mathematical manipulation was used in applying the variational principle to the coated waveguide slot antenna. This amount of manipulation is typical of all similar variational approaches.

Even though the numerical examples presented in Chapter VII involved only lossless dielectric coverings, the general expressions given for the energy functions apply to much broader situations. They are applicable, for example, to lossy as well as lossless coatings, to plasma as well as dielectric coatings, and to loaded as well as unloaded waveguide configurations. Hence, by simply modifying the integration scheme used by the author, a large variety of important antenna problems can be studied with potentially greater accuracy than was previously possible.

The success of the experimental verification of the new procedure indicates that it is practical and has wide applicability. The test cases considered show that admittance predictions based on an infinite ground plane model can predict within experimental error the admittance of a slot antenna having a ground plane that is only eight or ten wavelengths wide. Usually a minimum of three modes is necessary for accurate admittance predictions.

The presence of an iris over the mouth of the waveguide can substantially increase the number of modes required for precise admittance predictions. A general rule concerning the number of modes required in this case is difficult because of the variety of shapes that the iris can assume.

Pattern predictions based on the infinitely wide ground plane model are less accurate than are the admittance predictions. The pattern appears to be much more sensitive to the edge diffraction than is the admittance. Since the theoretical calculations do not take into account diffraction from the edges of the finite ground plane, the measured pattern will not be the predicted pattern but will be the predicted pattern with ripples superimposed upon it. These ripples are closer together and smaller in amplitude the larger the ground plane. When a dielectric is placed on the ground plane, the ripples in the pattern remain in essentially the same place, but their amplitude increases. However, the larger the loss tangent of the dielectric, the smaller is the amplitude of the ripples. The numerical examples show that the calculated pattern based on a one mode analysis differs negligibly from that of a ten mode analysis. Hence, a one mode analysis seems adequate for pattern predictions.

The new variational principle was shown to produce a stationary formula for a particular complex energy function; no analysis was performed concerning the stationarity of the input admittance of the antenna. However, calculations on slot antennas that were not presented in the thesis produced the same values of admittance that Croswell's (44) stationary admittance formula did. This result indicates that the method used in this dissertation might also be stationary or nearly stationary for the input admittance. Future work should be done along this line. In addition, the new technique could be used to advantage in studying mutual coupling between two or more slots.

APPENDICES

APPENDIX A

EVALUATION OF THE INTEGRALS IN W_{c_2}

The following integrals are needed in the evaluation of W_{c_2}:

$$Int_1(m,m') = \int_{x_0}^{x_0+a'} \cos(A_m x)\ \cos(A'_{m'} x')dx \qquad (A-1)$$

$$Int_2(n,n') = \int_{y_0}^{y_0+b'} \sin(B_n y)\ \sin(B'_{n'} y')dy \qquad (A-2)$$

$$Int_3(m,m') = \int_{x_0}^{x_0+a'} \sin(A_m x)\ \sin(A'_{m'} x')dx \qquad (A-3)$$

$$Int_4(n,n') = \int_{y_0}^{y_0+b'} \cos(B_n y)\ \cos(B'_{n'} y')dy \qquad (A-4)$$

where

$$A_m = \frac{m\pi}{a}\ , \quad A'_{m'} = \frac{m'\pi}{a'} \qquad (A-5)$$

and

$$B_n = \frac{n\pi}{b} , \qquad B'_{n'} = \frac{n'\pi}{b'} \tag{A-6}$$

and

$$x' = x - x_0 , \quad y' = y - y_0 \tag{A-7}$$

These integrals will be evaluated in this appendix.

Using the trigonometric identity $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$ along with Equations (A-5) and (A-7) transforms Equation (A-1) to

$$Int_1(m,m') = \frac{1}{2} \int_{x_0}^{x_0+a'} \{\cos\left(\frac{m\pi x}{a} + \frac{m'\pi[x-x_0]}{a'}\right) + \cos\left(\frac{m\pi x}{a} - \frac{m'\pi[x-x_0]}{a'}\right)\}dx \tag{A-8}$$

If $\frac{m}{a} \neq \frac{m'}{a'}$, then

$$Int_1(m,m') = \frac{1}{2} \left\{\frac{\sin\left(\frac{m\pi x}{a} + \frac{m'\pi[x-x_0]}{a'}\right)}{\left(\frac{m\pi}{a} + \frac{m'\pi}{a'}\right)} + \frac{\sin\left(\frac{m\pi x}{a} - \frac{m'\pi[x-x_0]}{a'}\right)}{\left(\frac{m\pi}{a} - \frac{m'\pi}{a'}\right)}\right\}\Bigg|_{x=x_0}^{x_0+a'}$$

or

$$\text{Int}_1(m,m') = [\frac{1}{2\pi}]\left[\frac{\sin\left(\frac{m}{a}\,\pi[x_0+a'] + m'\pi\right) - \sin\left(\frac{m}{a}\,\pi x_0\right)}{\left(\frac{m}{a} + \frac{m'}{a'}\right)}\right. \tag{A-9}$$

$$\left. + \frac{\sin\left(\frac{m}{a}\,\pi[x_0+a']-m'\pi\right) - \sin\left(\frac{m}{a}\,\pi x_0\right)}{\left(\frac{m}{a} - \frac{m'}{a'}\right)}\right]$$

If $\frac{m}{a} = \frac{m'}{a'} \neq 0$, the integral of the second term in Equation (A-8) is

$$\frac{1}{2}\int_{x_0}^{x_0+a'} \cos\left(\frac{m\pi x}{a} - \frac{m'\pi[x-x_0]}{a'}\right)dx = \frac{1}{2}\int_{x_0}^{x_0+a'} \cos\left(\frac{m\pi x_0}{a}\right)dx = \frac{a'}{2}\cos\left(\frac{m}{a}\,\pi x_0\right)$$

while from Equation (A-9), the integral of the first term in Equation (A-8) for this case is

$$[\frac{1}{2\pi}][\frac{a}{2m}][\sin(\frac{m}{a}\,\pi x_0 + \frac{m'}{a'}\,\pi a' + m'\pi) - \sin(\frac{m}{a}\,\pi x_0)]$$

$$= [\frac{a}{4\pi m}][\sin(\frac{m}{a}\,\pi x_0)\cos(m'2\pi) + \cos(\frac{m}{a}\,\pi x_0)\sin(m'2\pi) - \sin(\frac{m}{a}\,\pi x_0)]=0$$

since m' is an integer. Thus, if $\frac{m}{a} = \frac{m'}{a'} \neq 0$,

$$\text{Int}_1(m,m') = \frac{a'}{2}\cos(\frac{m}{a}\,\pi x_0) \tag{A-10}$$

Using Equations (A-1) and (A-5) it can be seen that if $\frac{m}{a} = \frac{m'}{a'} = 0$, then

$$\mathrm{Int}_1(m,m') = a' \tag{A-11}$$

Equations (A-9), (A-10), and (A-11) are the evaluation of Equation (A-1).

To evaluate Equation (A-2), the trigonometric identity $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$ may be used along with Equations (A-6) and (A-7) to obtain

$$\mathrm{Int}_2(n,n') = \frac{1}{2}\int_{y_0}^{y_0+b'} \Big[\cos\Big(\frac{n\pi y}{b} - \frac{n'\pi[y-y_0]}{b'}\Big) - \cos\Big(\frac{n\pi y}{b} + \frac{n'\pi[y-y_0]}{b'}\Big)\Big]dy \tag{A-12}$$

Comparing the terms in Equation (A-12) with those in Equation (A-8) shows that the integrals in Equation (A-12) can be evaluated by analogy with the integrals in Equation (A-8). Hence, if $\frac{n}{b} \neq \frac{n'}{b'}$,

$$\mathrm{Int}_2(n,n') = \Big[\frac{1}{2\pi}\Big]\left[\frac{\sin\big(\frac{n}{b}\,\pi[y_0+b'] - n'\pi\big) - \sin\big(\frac{n}{b}\,\pi y_0\big)}{\big(\frac{n}{b} - \frac{n'}{b'}\big)} - \frac{\sin\big(\frac{n}{b}\,\pi[y_0+b'] + n'\pi\big) - \sin\big(\frac{n}{b}\,\pi y_0\big)}{\big(\frac{n}{b} + \frac{n'}{b'}\big)}\right] \tag{A-13}$$

while, if $\frac{n}{b} = \frac{n'}{b'} \neq 0$,

$$\mathrm{Int}_2(n,n') = \frac{b'}{2}\cos\left(\frac{n}{b}\,\pi y_0\right) \tag{A-14}$$

From Equations (A-2) and (A-6) it can be seen that if $\frac{n}{b} = \frac{n'}{b'} = 0$, then

$$\mathrm{Int}_2(n,n') = 0 \tag{A-15}$$

The evaluation of Int_3 can be performed by analogy with Int_2. From Equation (A-13) it follows that, if $\frac{m}{a} \neq \frac{m'}{a'}$,

$$\mathrm{Int}_3(m,m') = [\frac{1}{2\pi}]\left[\frac{\sin\left(\frac{m}{a}\,\pi[x_0+a'] - m'\pi\right) - \sin\left(\frac{m}{a}\,\pi x_0\right)}{\left(\frac{m}{a} - \frac{m'}{a'}\right)} - \frac{\sin\left(\frac{m}{a}\,\pi[x_0+a'] + m'\pi\right) - \sin\left(\frac{m}{a}\,\pi x_0\right)}{\left(\frac{m}{a} + \frac{m'}{a'}\right)}\right] \tag{A-16}$$

but if $\frac{m}{a} = \frac{m'}{a'} \neq 0$, then Equation (A-14) gives

$$\mathrm{Int}_3(m,m') = \frac{a'}{2}\cos\left(\frac{m}{a}\,\pi x_0\right) \tag{A-17}$$

From Equations (A-3) and (A-5) it can be seen that if $\frac{m}{a} = \frac{m'}{a'} = 0$, then

$$\mathrm{Int}_3(m,m') = 0 \tag{A-18}$$

The evaluation of Int_4 can be performed by analogy with Int_1.

From Equation (A-9) it follows that if $\frac{n}{b} \neq \frac{n'}{b'}$,

$$\mathrm{Int}_4(n,n') = [\tfrac{1}{2\pi}]\left[\frac{\sin\left(\frac{n}{b}\,\pi[y_0+b'] + n'\pi\right) - \sin\left(\frac{n}{b}\,\pi y_0\right)}{\left(\frac{n}{b} + \frac{n'}{b'}\right)}\right. \tag{A-19}$$

$$\left. + \frac{\sin\left(\frac{n}{b}\,\pi[y_0+b'] - n'\pi\right) - \sin\left(\frac{n}{b}\,\pi y_0\right)}{\left(\frac{n}{b} - \frac{n'}{b'}\right)}\right]$$

while if $\frac{n}{b} = \frac{n'}{b'} \neq 0$, then Equation (A-10) gives

$$\mathrm{Int}_4(n,n') = \frac{b'}{2}\cos\left(\frac{n}{b}\,\pi y_0\right) \tag{A-20}$$

From Equations (A-4) and (A-6) it can be seen that if $\frac{n}{b} = \frac{n'}{b'} = 0$, then

$$\mathrm{Int}_4(n,n') = b' \tag{A-21}$$

This completes the evaluation of the four integrals Int_1, Int_2, Int_3, and Int_4.

APPENDIX B

DETERMINATION OF THE PLANE WAVE AMPLITUDE COEFFICIENTS

In this appendix the simultaneous equations for I_x, I_y, R_x, R_y, T_x, and T_y that appear in Chapter III will be solved. For convenience, this system of equations, that is, Equations (3-71) through (3-76), will be reproduced here. They are

$$I_x e^{-jdk_{z_2}} + R_x e^{jdk_{z_2}} = T_x e^{-jdk_{z_3}} \tag{B-1}$$

$$I_y e^{-jdk_{z_2}} + R_y e^{jdk_{z_2}} = T_y e^{-jdk_{z_3}} \tag{B-2}$$

$$\left(\frac{\mu_3}{\mu_2}\right)\left[\frac{k_x k_y I_x}{k_{z_2}} + \frac{(k_2^2-k_x^2)I_y}{k_{z_2}}\right]e^{-jdk_{z_2}} \tag{B-3}$$

$$-\left(\frac{\mu_3}{\mu_2}\right)\left[\frac{k_x k_y R_x}{k_{z_2}} + \frac{(k_2^2-k_x^2)R_y}{k_{z_2}}\right]e^{jdk_{z_2}} = \left[\frac{k_x k_y T_x}{k_{z_3}} + \frac{(k_3^2-k_x^2)T_y}{k_{z_3}}\right]e^{-jdk_{z_3}}$$

$$\left(\frac{\mu_3}{\mu_2}\right)\left[\frac{(k_2^2-k_y^2)I_x}{k_{z_2}} + \frac{k_x k_y I_y}{k_{z_2}}\right]e^{-jdk_{z_2}} \tag{B-4}$$

$$-\left(\frac{\mu_3}{\mu_2}\right)\left[\frac{(k_2^2-k_y^2)R_x}{k_{z_2}}+\frac{k_x k_y R_y}{k_{z_2}}\right]e^{jdk_{z_2}}=\left[\frac{(k_3^2-k_y^2)T_x}{k_{z_3}}+\frac{k_x k_y T_y}{k_{z_3}}\right]e^{-jdk_{z_3}}$$

$$I_x + R_x = A_x \tag{B-5}$$

$$I_y + R_y = A_y \tag{B-6}$$

Substituting Equation (B-5) into Equation (B-1) yields

$$I_x e^{-jdk_{z_2}} + (A_x - I_x)e^{jdk_{z_2}} = T_x e^{-jdk_{z_3}}$$

or

$$T_x e^{-jdk_{z_3}} = -2jI_x \sin(dk_{z_2}) + A_x e^{jdk_{z_2}} \tag{B-7}$$

Next, substituting Equation (B-6) into Equation (B-2) yields

$$I_y e^{-jdk_{z_2}} + (A_y - I_y)e^{jdk_{z_2}} = T_y e^{-jdk_{z_3}}$$

or

$$T_y e^{-jdk_{z_3}} = -2jI_y \sin(dk_{z_2}) + A_y e^{jdk_{z_2}} \tag{B-8}$$

Then, substituting Equations (B-5) through (B-8) into Equation (B-3) gives

$$\left(\frac{\mu_3}{\mu_2}\right)k_{z_3}[k_xk_yI_x + (k_2^2-k_x^2)I_y]e^{-jdk_{z_2}}$$

$$- \left(\frac{\mu_3}{\mu_2}\right)k_{z_3}[k_xk_y(A_x-I_x) + (k_2^2-k_x^2)(A_y-I_y)]e^{jdk_{z_2}}$$

$$= k_xk_yk_{z_2}[-2jI_x\sin(dk_{z_2}) + A_xe^{jdk_{z_2}}]$$

$$+ (k_3^2-k_x^2)k_{z_2}[-2jI_y\sin(dk_{z_2}) + A_ye^{jdk_{z_2}}]$$

or

$$I_x[k_xk_yk_{z_3}\left(\frac{\mu_3}{\mu_2}\right)2\cos(dk_{z_2}) + 2jk_xk_yk_{z_2}\sin(dk_{z_2})] \tag{B-9}$$

$$+ I_y[(k_2^2-k_x^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)2\cos(dk_{z_2}) + 2j(k_3^2-k_x^2)k_{z_2}\sin(dk_{z_2})]$$

$$= A_x[k_xk_yk_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + k_xk_yk_{z_2}]e^{jdk_{z_2}}$$

$$+ A_y[(k_2^2-k_x^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + (k_3^2-k_x^2)k_{z_2}]e^{jdk_{z_2}}$$

A similar substitution of Equations (B-5) through (B-8) into Equation (B-4) yields

$$I_x[(k_2^2-k_y^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)2\cos(dk_{z_2}) + 2j(k_3^2-k_y^2)k_{z_2}\sin(dk_{z_2})] \tag{B-10}$$

$$+ I_y[k_x k_y k_{z_3}(\frac{\mu_3}{\mu_2}) 2\cos(dk_{z_2}) + 2jk_x k_y k_{z_2} \sin(dk_{z_2})]$$

$$= A_x[(k_2^2-k_y^2)k_{z_3}(\frac{\mu_3}{\mu_2}) + (k_3^2-k_y^2)k_{z_2}]e^{jdk_{z_2}}$$

$$+ A_y[k_x k_y k_{z_3}(\frac{\mu_3}{\mu_2}) + k_x k_y k_{z_2}]e^{jdk_{z_2}}$$

Equations (B-9) and (B-10) form a pair of equations in the two unknowns I_x and I_y. In solving these equations for I_x and I_y it will be convenient to use matrix notation and write them in the form

$$a_{11}I_x + a_{12}I_y = c_1 \tag{B-11}$$

$$a_{21}I_x + a_{11}I_y = c_2$$

where

$$a_{11} = 2k_x k_y[k_{z_3}(\frac{\mu_3}{\mu_2})\cos(dk_{z_2}) + jk_{z_2}\sin(dk_{z_2})] \tag{B-12}$$

$$a_{12} = 2[(k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2})\cos(dk_{z_2}) + j(k_3^2-k_x^2)k_{z_2}\sin(dk_{z_2})] \tag{B-13}$$

$$a_{21} = 2[(k_2^2-k_y^2)k_{z_3}(\frac{\mu_3}{\mu_2})\cos(dk_{z_2}) + j(k_3^2-k_y^2)k_{z_2}\sin(dk_{z_2})] \tag{B-14}$$

$$c_1 = A_x k_x k_y[k_{z_3}(\frac{\mu_3}{\mu_2}) + k_{z_2}][\cos(dk_{z_2}) + j\sin(dk_{z_2})] \tag{B-15}$$

$$+ A_y[(k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2}) + (k_3^2-k_x^2)k_{z_2}][\cos(dk_{z_2}) + j\sin(dk_{z_2})]$$

and

$$c_2 = A_x[(k_2^2-k_y^2)k_{z_3}(\frac{\mu_3}{\mu_2}) + (k_3^2-k_y^2)k_{z_2}][\cos(dk_{z_2}) + j\sin(dk_{z_2})] \quad \text{(B-16)}$$

$$+ A_y k_x k_y[k_{z_3}(\frac{\mu_3}{\mu_2}) + k_{z_2}][\cos(dk_{z_2}) + j\sin(dk_{z_2})]$$

From matrix theory the solutions of Equation (B-11) are

$$I_x = \frac{\begin{vmatrix} c_1 & a_{12} \\ c_2 & a_{11} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{vmatrix}} = \frac{n_x}{\Delta} \quad \text{(B-17)}$$

and

$$I_y = \frac{\begin{vmatrix} a_{11} & c_1 \\ a_{21} & c_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{vmatrix}} = \frac{n_y}{\Delta} \quad \text{(B-18)}$$

where

$$n_x = c_1 a_{11} - c_2 a_{12} \tag{B-19}$$

$$n_y = c_2 a_{11} - c_1 a_{21} \tag{B-20}$$

and

$$\Delta = a_{11}^2 - a_{12} a_{21} \tag{B-21}$$

The denominator Δ will be explicitly determined next. Using Equations (B-12), (B-13), and (B-14) in Equation (B-21) gives

$$\Delta = 4k_x^2 k_y^2 [k_{z_3}^2 \left(\frac{\mu_3}{\mu_2}\right)^2 \cos^2(dk_{z_2}) + 2jk_{z_2} k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) \tag{B-22}$$

$$\cdot \sin(dk_{z_2}) \cos(dk_{z_2}) - k_{z_2}^2 \sin^2(dk_{z_2})]$$

$$- 4(k_2^2 - k_x^2)(k_2^2 - k_y^2) k_{z_3}^2 \left(\frac{\mu_3}{\mu_2}\right)^2 \cos^2(dk_{z_2}) + 4(k_3^2 - k_x^2)(k_3^2 - k_y^2)$$

$$\cdot k_{z_2}^2 \sin^2(dk_{z_2}) - 4jk_{z_2} k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) [(k_2^2 - k_x^2)(k_3^2 - k_y^2)$$

$$+ (k_3^2 - k_x^2)(k_2^2 - k_y^2)] \sin(dk_{z_2}) \cos(dk_{z_2})$$

But, by Equation (3-55),

$$(k_2^2 - k_x^2)(k_2^2 - k_y^2) = k_2^2 (k_2^2 - k_x^2 - k_y^2) + k_x^2 k_y^2 \tag{B-23}$$

$$= k_2^2 k_{z_2}^2 + k_x^2 k_y^2$$

Similarly,

$$(k_3^2 - k_x^2)(k_3^2 - k_y^2) = k_3^2 k_{z_3}^2 + k_x^2 k_y^2 \tag{B-24}$$

while

$$(k_2^2 - k_x^2)(k_3^2 - k_y^2) = k_2^2(k_3^2 - k_y^2) - k_3^2 k_x^2 + k_x^2 k_y^2 \tag{B-25}$$

$$(k_3^2 - k_x^2)(k_2^2 - k_y^2) = k_3^2(k_2^2 - k_y^2) - k_2^2 k_x^2 + k_x^2 k_y^2 \tag{B-26}$$

Using Equations (B-23) through (B-26) in Equation (B-22) yields

$$\Delta = 4\{-\cos^2(dk_{z_2})[k_{z_3}^2 \left(\frac{\mu_3}{\mu_2}\right)^2 k_2^2 k_{z_2}^2] + \sin^2(dk_{z_2})[k_{z_2}^2 k_3^2 k_{z_3}^2]$$

$$- j\sin(dk_{z_2})\cos(dk_{z_2}) k_{z_2} k_{z_3} \left(\frac{\mu_3}{\mu_2}\right)[k_2^2 k_{z_3}^2 + k_3^2 k_{z_2}^2]\}$$

or

$$\Delta = -4k_3^2 k_{z_2}^2 k_{z_3}^2 \left[\left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2 \cos^2(dk_{z_2}) - \sin^2(dk_{z_2})\right] \tag{B-27}$$

$$- j4k_3^2 k_{z_2} k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) \left[\left(\frac{k_2}{k_3}\right)^2 k_{z_3}^2 + k_{z_2}^2\right] \sin(dk_{z_2})\cos(dk_{z_2})$$

Next, n_x will be determined. Substituting Equations (B-12), (B-13), (B-15), and (B-16) into Equation (B-19) yields

$$n_x = 2A_x k_x^2 k_y^2 [k_{z_3}(\frac{\mu_3}{\mu_2}) + k_{z_2}][k_{z_3}(\frac{\mu_3}{\mu_2})\cos^2(dk_{z_2}) - k_{z_2}\sin^2(dk_{z_2})$$

$$+ j\big(k_{z_2} + k_{z_3}(\frac{\mu_3}{\mu_2})\big)\sin(dk_{z_2})\cos(dk_{z_2})]$$

$$+ 2k_x k_y A_y [(k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2}) + (k_3^2-k_x^2)k_{z_2}][k_{z_3}(\frac{\mu_3}{\mu_2})\cos^2(dk_{z_2})$$

$$- k_{z_2}\sin^2(dk_{z_2}) + j\big(k_{z_2} + k_{z_3}(\frac{\mu_3}{\mu_2})\big)\sin(dk_{z_2})\cos(dk_{z_2})]$$

$$- 2A_x[(k_2^2-k_y^2)k_{z_3}(\frac{\mu_3}{\mu_2}) + (k_3^2-k_y^2)k_{z_2}][(k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2})\cos^2(dk_{z_2})$$

$$- (k_3^2-k_x^2)k_{z_2}\sin^2(dk_{z_2}) + j\big((k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2}) + (k_3^2-k_x^2)$$

$$\cdot k_{z_2}\big)\sin(dk_{z_2})\cos(dk_{z_2})]$$

$$- 2A_y k_x k_y [k_{z_3}(\frac{\mu_3}{\mu_2}) + k_{z_2}][(k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2})\cos^2(dk_{z_2})$$

$$- (k_3^2-k_x^2)k_{z_2}\sin^2(dk_{z_2}) + j\big((k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2})$$

$$+ (k_3^2-k_x^2)k_{z_2}\big)\sin(dk_{z_2})\cos(dk_{z_2})]$$

or

$$n_x = 2A_x\Big[\cos^2(dk_{z_2})\ \{k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 k_x^2k_y^2 + k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)k_x^2k_y^2 \tag{B-28}$$

$$- k_{z_3}^2\ \left(\frac{\mu_3}{\mu_2}\right)^2 (k_2^2-k_x^2)(k_2^2-k_y^2) - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_2^2-k_x^2)(k_3^2-k_y^2)\}$$

$$- \sin^2(dk_{z_2})\{k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)k_x^2k_y^2 + k_{z_2}^2\ k_x^2k_y^2 - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)$$

$$\cdot\ (k_3^2-k_x^2)(k_2^2-k_y^2) - k_{z_2}^2\ (k_3^2-k_x^2)(k_3^2-k_y^2)\}$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2})\ \{k_{z_2}^2\ k_x^2k_y^2 + 2k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)k_x^2k_y^2 + k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 k_x^2k_y^2$$

$$- k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 (k_2^2-k_x^2)(k_2^2-k_y^2) - k_{z_2}^2\ (k_3^2-k_x^2)(k_3^2-k_y^2)$$

$$- k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)\left((k_2^2-k_x^2)(k_3^2-k_y^2) + (k_3^2-k_x^2)(k_2^2-k_y^2)\right)\}\Big]$$

$$+ 2k_xk_yA_y\Big[\cos^2(dk_{z_2})\ \{k_{z_3}^2\ \left(\frac{\mu_3}{\mu_2}\right)^2(k_2^2-k_x^2) + k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_3^2-k_x^2)$$

$$- k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 (k_2^2-k_x^2) - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_2^2-k_x^2)\} - \sin^2(dk_{z_2})$$

$$\cdot\ \{k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_2^2-k_x^2) + k_{z_2}^2\ (k_3^2-k_x^2) - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_3^2-k_x^2)$$

$$- k_{z_2}^2 (k_3^2-k_x^2)\} + j\sin(dk_{z_2})\cos(dk_{z_2})\{ \Big((k_2^2-k_x^2)k_{z_3}\Big(\frac{\mu_3}{\mu_2}\Big)$$

$$+ (k_3^2-k_x^2)k_{z_2}\Big)\Big(k_{z_2} + k_{z_3}(\frac{\mu_3}{\mu_2})\Big) - \Big((k_2^2-k_x^2)k_{z_3}(\frac{\mu_3}{\mu_2})$$

$$+ (k_3^2-k_x^2)k_{z_2}\Big)\Big(k_{z_2} + k_{z_3}(\frac{\mu_3}{\mu_2})\Big)\}\Big]$$

Using Equations (B-23) through (B-26) in Equation (B-28) gives

$$n_x = -2A_x\Big[\cos^2(dk_{z_2}) \{k_2^2 k_{z_2}^2 k_{z_3}^2 \Big(\frac{\mu_3}{\mu_2}\Big)^2 + k_{z_2}k_{z_3}\Big(\frac{\mu_3}{\mu_2}\Big)(k_2^2[k_3^2-k_y^2] - k_3^2k_x^2)\}$$

$$- \sin^2(dk_{z_2}) \{k_{z_2}k_{z_3}\Big(\frac{\mu_3}{\mu_2}\Big)\Big(k_3^2[k_2^2-k_y^2] - k_2^2k_x^2\Big) + k_3^2k_{z_2}^2 k_{z_3}^2 \}$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2}) \{k_3^2k_{z_2}^2 k_{z_3}^2 + k_2^2k_{z_2}^2 k_{z_3}^2 \Big(\frac{\mu_3}{\mu_2}\Big)^2$$

$$+ k_{z_2}k_{z_3}\Big(\frac{\mu_3}{\mu_2}\Big)(k_2^2k_{z_3}^2 + k_3^2k_{z_2}^2)\}\Big]$$

$$+ 2A_y k_x k_y k_{z_2} k_{z_3}\Big(\frac{\mu_3}{\mu_2}\Big)\Big(k_3^2-k_2^2\Big)[\cos^2(dk_{z_2}) + \sin^2(dk_{z_2})]$$

or

$$n_x = -2k_3^2 k_{z_2} k_{z_3} A_x \{\cos^2(dk_{z_2}) \Big[\left(\frac{\mu_3}{\mu_2}\right) k_{z_2}^2 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2 \tag{B-29}$$

$$\cdot k_{z_2} k_{z_3} + \left(\frac{\mu_3}{\mu_2}\right) k_y^2 \left(1 - \left(\frac{k_2}{k_3}\right)^2\right)\Big]$$

$$- \sin^2(dk_{z_2}) \Big[\left(\frac{\mu_3}{\mu_2}\right) k_{z_2}^2 + k_{z_2} k_{z_3} + \left(\frac{\mu_3}{\mu_2}\right) k_x^2 \left(1 - \left(\frac{k_2}{k_3}\right)^2\right)\Big]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2}) \Big[\left(\frac{\mu_3}{\mu_2}\right) \left(k_{z_2}^2 + \left(\frac{k_2}{k_3}\right)^2 k_{z_3}^2\right) + k_{z_2} k_{z_3}$$

$$\cdot \left(1 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2\right)\Big]\} - 2k_3^2 k_{z_2} k_{z_3} A_y k_x k_y \left(\frac{\mu_3}{\mu_2}\right) \Big[\left(\frac{k_2}{k_3}\right)^2 - 1\Big]$$

Now n_y will be determined. Using Equations (B-12), (B-14), (B-15), and (B-16) in Equation (B-20) gives

$$n_y = 2k_x k_y A_x [(k_2^2 - k_y^2) k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) + (k_3^2 - k_y^2) k_{z_2}][k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) \cos^2(dk_{z_2})$$

$$- k_{z_2} \sin^2(dk_{z_2}) + j\left(k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) + k_{z_2}\right) \sin(dk_{z_2}) \cos(dk_{z_2})]$$

$$+ 2A_y k_x^2 k_y^2 [k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) + k_{z_2}][k_{z_3} \left(\frac{\mu_3}{\mu_2}\right) \cos^2(dk_{z_2}) - k_{z_2} \sin^2(dk_{z_2})$$

$$+ j\left(k_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + k_{z_2}\right)\sin(dk_{z_2})\cos(dk_{z_2})]$$

$$- 2A_x k_x k_y [k_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + k_{z_2}][(k_2^2-k_y^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)\cos^2(dk_{z_2})$$

$$- (k_3^2-k_y^2)k_{z_2}\sin^2(dk_{z_2}) + j\left((k_2^2-k_y^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)\right.$$

$$\left.+ (k_3^2-k_y^2)k_{z_2}\right)\sin(dk_{z_2})\cos(dk_{z_2})]$$

$$- 2A_y[(k_2^2-k_x^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + (k_3^2-k_x^2)k_{z_2}][(k_2^2-k_y^2)k_{z_3}$$

$$\cdot \left(\frac{\mu_3}{\mu_2}\right)\cos^2(dk_{z_2}) - (k_3^2-k_y^2)k_{z_2}\sin^2(dk_{z_2})$$

$$+ j\left((k_2^2-k_y^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + (k_3^2-k_y^2)k_{z_2}\right)\sin(dk_{z_2})\cos(dk_{z_2})]$$

or

$$n_y = 2A_x k_x k_y \{\cos^2(dk_{z_2})[k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2(k_2^2-k_y^2) + k_{z_2}k_{z_3} \tag{B-30}$$

$$\cdot \left(\frac{\mu_3}{\mu_2}\right)(k_3^2-k_y^2) - k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2(k_2^2-k_y^2) - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_2^2-k_y^2)]$$

$$- \sin^2(dk_{z_2})[k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_2^2-k_y^2) + k_{z_2}^2(k_3^2-k_y^2)$$

$$- k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_3^2-k_y^2) - k_{z_2}^2(k_3^2-k_y^2)]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2})[(k_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + k_{z_2})\left((k_2^2-k_y^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)\right.$$

$$\left.+ (k_3^2-k_y^2)k_{z_2}\right) - \left(k_{z_3}\left(\frac{\mu_3}{\mu_2}\right) + k_{z_2}\right)\left((k_2^2-k_y^2)k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)\right.$$

$$\left.+ (k_3^2-k_y^2)k_{z_2}\right)]\}$$

$$+ 2A_y\ \{\cos^2(dk_{z_2})[k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 k_x^2k_y^2 + k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)k_x^2k_y^2$$

$$- k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2(k_2^2-k_x^2)(k_2^2-k_y^2) - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_3^2-k_x^2)(k_2^2-k_y^2)]$$

$$- \sin^2(dk_{z_2})[k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)k_x^2k_y^2 + k_{z_2}^2k_x^2k_y^2 - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)$$

$$\cdot (k_2^2-k_x^2)(k_3^2-k_y^2) - k_{z_2}^2(k_3^2-k_x^2)(k_3^2-k_y^2)]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2})[k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 k_x^2k_y^2 + 2k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)k_x^2k_y^2 + k_{z_2}^2$$

$$\cdot\ k_x^2k_y^2 - k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2(k_2^2-k_x^2)(k_2^2-k_y^2) - k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)$$

$$\cdot\ \left((k_3^2-k_x^2)(k_2^2-k_y^2) + (k_2^2-k_x^2)(k_3^2-k_y^2)\right)$$

$$- k_{z_2}^2(k_3^2-k_x^2)(k_3^2-k_y^2)]\}$$

Using Equations (B-23) through (B-26) transforms Equation (B-30) to

$$n_y = 2A_x k_x k_y k_{z_2} k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)(k_3^2-k_2^2)[\cos^2(dk_{z_2}) + \sin^2(dk_{z_2})]$$

$$- 2A_y\ \{\cos^2(dk_{z_2})[k_2^2k_{z_2}^2k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 + k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)\left(k_3^2(k_2^2-k_y^2)-k_2^2k_x^2\right)]$$

$$- \sin^2(dk_{z_2})[k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)\left(k_2^2(k_3^2-k_y^2) - k_3^2k_x^2\right) + k_3^2k_{z_2}^2k_{z_3}^2]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2})[k_2^2k_{z_2}^2k_{z_3}^2\left(\frac{\mu_3}{\mu_2}\right)^2 + k_{z_2}k_{z_3}\left(\frac{\mu_3}{\mu_2}\right)$$

$$\cdot\ (k_2^2k_{z_3}^2 + k_3^2k_{z_2}^2) + k_3^2k_{z_2}^2k_{z_3}^2]\}$$

or

$$n_y = -2k_3^2 k_{z_2} k_{z_3} A_x k_x k_y \left(\frac{\mu_3}{\mu_2}\right) \left[\left(\frac{k_2}{k_3}\right)^2 - 1 \right] \tag{B-31}$$

$$- 2k_3^2 k_{z_2} k_{z_3} A_y \; \{\cos^2(dk_{z_2}) \left[\left(\frac{\mu_3}{\mu_2}\right) k_{z_2}^2 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2 k_{z_2} k_{z_3} \right.$$

$$\left. + \left(\frac{\mu_3}{\mu_2}\right) k_x^2 \left(1 - \left(\frac{k_2}{k_3}\right)^2\right) \right]$$

$$- \sin^2(dk_{z_2}) \left[\left(\frac{\mu_3}{\mu_2}\right) k_{z_2}^2 + k_{z_2} k_{z_3} + \left(\frac{\mu_3}{\mu_2}\right) k_y^2 \left(1 - \left(\frac{k_2}{k_3}\right)^2\right) \right]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2}) \left[\left(\frac{\mu_3}{\mu_2}\right) \left(k_{z_2}^2 + \left(\frac{k_2}{k_3}\right)^2 k_{z_3}^2\right) + k_{z_2} k_{z_3} \right.$$

$$\left. \cdot \left(1 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2\right) \right] \}$$

Now define D, N_x, and N_y as

$$D = \frac{\Delta}{-2k_3^2 k_{z_2} k_{z_3}} \tag{B-32}$$

$$N_x = \frac{n_x}{-2k_3^2 k_{z_2} k_{z_3}} \tag{B-33}$$

$$N_y = \frac{n_y}{-2k_3^2 k_{z_2} k_{z_3}} \tag{B-34}$$

Using these last three equations, Equations (B-29) and (B-31) become, respectively,

$$I_x = \frac{N_x}{D} \tag{B-35}$$

$$I_y = \frac{N_y}{D} \tag{B-36}$$

The amplitude coefficients of the remaining plane wave terms can be obtained from I_x and I_y by using Equations (B-5) through (B-8). These equations give, respectively,

$$R_x = A_x - I_x \tag{B-37}$$

$$R_y = A_y - I_y \tag{B-38}$$

$$T_x = e^{jdk_{z_3}}[A_x e^{jdk_{z_2}} - 2jI_x \sin(dk_{z_2})] \tag{B-39}$$

$$T_y = e^{jdk_{z_3}}[A_y e^{jdk_{z_2}} - 2jI_y \sin(dk_{z_2})] \tag{B-40}$$

The explicit form of D can be obtained by substituting Equation (B-27) into Equation (B-32). This operation yields

$$D = 2k_{z_2}k_{z_3}\left[\left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2 \cos^2(dk_{z_2}) - \sin^2(dk_{z_2})\right] \tag{B-41}$$

$$+ j2\left(\frac{\mu_3}{\mu_2}\right)\left[\left(\frac{k_2}{k_3}\right)^2 k_{z_3}^2 + k_{z_2}^2\right]\sin(dk_{z_2})\cos(dk_{z_2})$$

The numerator term N_x will now be evaluated by substituting Equation (B-29) into Equation (B-33). This operation yields

$$N_x = A_x \{\cos^2(dk_{z_2})\left[\left(\frac{\mu_3}{\mu_2}\right)k_{z_2}^2 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2 k_{z_2}k_{z_3} + \left(\frac{\mu_3}{\mu_2}\right)\right. \tag{B-42}$$

$$\left.\cdot\ k_y^2\left(1 - \left(\frac{k_2}{k_3}\right)^2\right)\right] - \sin^2(dk_{z_2})\left[\left(\frac{\mu_3}{\mu_2}\right)k_{z_2}^2 + k_{z_2}k_{z_3}\right.$$

$$\left. + \left(\frac{\mu_3}{\mu_2}\right)k_x^2\left(1 - \left(\frac{k_2}{k_3}\right)^2\right)\right] + j\sin(dk_{z_2})\cos(dk_{z_2})\left[\left(\frac{\mu_3}{\mu_2}\right)\right.$$

$$\left.\cdot\ \left(k_{z_2}^2 + \left(\frac{k_2}{k_3}\right)^2 k_{z_3}^2\right) + k_{z_2}k_{z_3}\left(1 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2\right)\right]\}$$

$$+ A_y k_x k_y\left(\frac{\mu_3}{\mu_2}\right)\left[\left(\frac{k_2}{k_3}\right)^2 - 1\right]$$

Finally, substituting Equation (B-31) into Equation (B-34) yields

$$N_y = A_x k_x k_y \left(\frac{\mu_3}{\mu_2}\right) \left[\left(\frac{k_2}{k_3}\right)^2 - 1 \right] \tag{B-43}$$

$$+ A_y \,\{\cos^2(dk_{z_2}) \left[\left(\frac{\mu_3}{\mu_2}\right) k_{z_2}^2 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2 k_{z_2} k_{z_3} + \left(\frac{\mu_3}{\mu_2}\right) k_x^2 \left(1 - \left(\frac{k_2}{k_3}\right)^2\right) \right]$$

$$- \sin^2(dk_{z_2}) \left[\left(\frac{\mu_3}{\mu_2}\right) k_{z_2}^2 + k_{z_2} k_{z_3} + \left(\frac{\mu_3}{\mu_2}\right) k_y^2 \left(1 - \left(\frac{k_2}{k_3}\right)^2\right) \right]$$

$$+ j\sin(dk_{z_2})\cos(dk_{z_2}) \left[\left(\frac{\mu_3}{\mu_2}\right) \left(k_{z_2}^2 + \left(\frac{k_2}{k_3}\right)^2 k_{z_3}^2\right) + k_{z_2} k_{z_3} \right.$$

$$\left. \cdot \left(1 + \left(\frac{\mu_3 k_2}{\mu_2 k_3}\right)^2\right) \right] \}$$

Equations (B-35) through (B-43) constitute the solution of the plane wave amplitude coefficients in terms of A_x and A_y. The quantities A_x and A_y are the Fourier transforms of the x and y components of the aperture electric-field distribution. These transforms will be determined elsewhere.

APPENDIX C

EVALUATION OF THE INTEGRALS IN A_x AND A_y

The following integrals are needed in the evaluation of A_x and A_y in terms of the aperture field:

$$I_1 = Is_x(m',k_x)e^{jx_0k_x} = \int_{x_0}^{x_0+a'} \sin(A'_{m'}x')e^{jxk_x}\,dx \qquad \text{(C-1)}$$

$$= \int_{x_0}^{x_0+a'} \sin\left[\frac{m'\pi(x-x_0)}{a'}\right]e^{jxk_x}\,dx$$

$$I_2 = Ic_x(m',k_x)e^{jx_0k_x} = \int_{x_0}^{x_0+a'} \cos(A'_{m'}x')e^{jxk_x}\,dx \qquad \text{(C-2)}$$

$$= \int_{x_0}^{x_0+a'} \cos\left[\frac{m'\pi(x-x_0)}{a'}\right]e^{jxk_x}\,dx$$

$$I_3 = Is_y(n',k_y)e^{jy_0k_y} = \int_{y_0}^{y_0+b'} \sin(B'_{n'}y')e^{jyk_y}\,dy \qquad \text{(C-3)}$$

$$= \int_{y_0}^{y_0+b'} \sin\left[\frac{n'\pi(y-y_0)}{b'}\right]e^{jyk_y}\,dy$$

$$I_4 = Ic_y(n',k_y)e^{jy_0k_y} = \int_{y_0}^{y_0+b'} \cos(B_n',y')e^{jyk_y}\,dy \qquad \text{(C-4)}$$

$$= \int_{y_0}^{y_0+b'} \cos\left[\frac{n'\pi(y-y_0)}{b'}\right]e^{jyk_y}\,dy$$

These integrals will be evaluated in this appendix.

Let k be a purely real number and let n be a positive integer. Then, according to Churchill (45)

$$\int_0^\pi e^{jxk}\sin(nx)dx = \begin{cases} \dfrac{n[1-(-1)^n e^{j\pi k}]}{[n^2-k^2]} & \text{if } n \neq |k| \\ j\left(\frac{\pi}{2}\right)\text{sign}(k) & \text{if } n = |k| \text{ and } n \neq 0 \\ 0 & \text{if } n = 0 \end{cases} \qquad \text{(C-5)}$$

and

$$\int_0^\pi e^{jxk}\cos(nx)dx = \begin{cases} \dfrac{-jk[1-(-1)^n e^{j\pi k}]}{[n^2-k^2]} & \text{if } n \neq |k| \\ \dfrac{\pi}{2} & \text{if } n = |k| \text{ and } n \neq 0 \\ \pi & \text{if } n = k = 0 \end{cases} \qquad \text{(C-6)}$$

Integral I_1 can be converted into the form of Equation (C-5) by using the following change of variables:

$$r = \frac{\pi(x-x_0)}{a'} \qquad \text{or} \qquad x = \left(\frac{a'}{\pi}\right)r + x_0 \tag{C-7}$$

Then Equation (C-1) yields

$$I_1 = e^{jx_0k_x}\left(\frac{a'}{\pi}\right)\int_0^{\pi} e^{jr\left(\frac{a'k_x}{\pi}\right)} \sin(m'r)dr \tag{C-8}$$

Using Equation (C-5) and the definition of Is_x in Equation (C-1) transforms Equation (C-8) to

$$Is_x(m',k_x) = \begin{cases} 0 & \text{if } m' = 0 \\ j\left(\frac{a'}{2}\right)\text{sign}\left(\frac{a'k_x}{\pi}\right) & \text{if } m' = \left|\frac{a'k_x}{\pi}\right| \text{ and } m' \neq 0 \\ \dfrac{a'm'[1 - (-1)^{m'} e^{ja'k_x}]}{\pi[(m')^2 - (\frac{a'k_x}{\pi})^2]} & \text{if } m' \neq \left|\frac{a'k_x}{\pi}\right| \end{cases} \tag{C-9}$$

It should be noticed that I_3 can be evaluated by direct analogy with I_1. Using this analogy, along with Equations (C-3) and (C-9), easily gives

$$Is_y(n',k_y) = \begin{cases} 0 & \text{if } n' = 0 \\ j\left(\frac{b'}{2}\right)\text{sign}\left(\frac{b'k_y}{\pi}\right) & \text{if } n' = \left|\frac{b'k_y}{\pi}\right| \text{ and } n' \neq 0 \\ \dfrac{b'n'[1 - (-1)^{n'} e^{jb'k_y}]}{\pi[(n')^2 - \left(\frac{b'k_y}{\pi}\right)^2]} & \text{if } n' \neq \left|\frac{b'k_y}{\pi}\right| \end{cases} \quad \text{(C-10)}$$

To evaluate the integral I_2 in Equation (C-2), the change of variables given in Equation (C-7) is again used to arrive at the result

$$I_2 = e^{jx_0k_x} \left(\frac{a'}{\pi}\right) \int_0^{\pi} e^{jr\left(\frac{a'k_x}{\pi}\right)} \cos(m'r)dr \quad \text{(C-11)}$$

Next, Equation (C-6) is used to evaluate Equation (C-11). Then, in view of the definition of Ic_x as given in Equation (C-2), it is clear that

$$Ic_x(m',k_x) = \begin{cases} a' & \text{if } m' = \left(\frac{a'k_x}{\pi}\right) = 0 \\ \frac{a'}{2} & \text{if } m' = \left|\frac{a'k_x}{\pi}\right| \text{ and } m' \neq 0 \\ \dfrac{-j(a')^2k_x[1 - (-1)^{m'} e^{ja'k_x}]}{\pi^2[(m')^2 - \left(\frac{a'k_x}{\pi}\right)^2]} & \text{if } m' \neq \left|\frac{a'k_x}{\pi}\right| \end{cases} \quad \text{(C-12)}$$

It should be noticed that I_4 can be evaluated by direct analogy with I_2. Using this analogy, along with Equations (C-4) and (C-12), easily gives

$$
Ic_y(n',k_y) = \begin{cases} b' & \text{if } n' = \left(\frac{b'k_y}{\pi}\right) = 0 \\[2ex] \frac{b'}{2} & \text{if } n' = \left|\frac{b'k_y}{\pi}\right| \text{ and } n' \neq 0 \\[2ex] \frac{-j(b')^2 k_y [1 - (-1)^{n'} e^{jb'k_y}]}{\pi^2[(n')^2 - \left(\frac{b'k_y}{\pi}\right)^2]} & \text{if } n' \neq \left|\frac{b'k_y}{\pi}\right| \end{cases} \qquad \text{(C-13)}
$$

This completes the evaluation of I_1, I_2, I_3, and I_4. In closing, it should be noticed that since a', b', n', m', k_x, and k_y are all purely real, Equations (C-9), (C-10), (C-12), and (C-13) imply that

$$Is_x(m',-k_x) = Is_x^*(m',k_x) \qquad \text{(C-14)}$$

$$Ic_x(m',-k_x) = Ic_x^*(m',k_x) \qquad \text{(C-15)}$$

$$Is_y(n',-k_y) = Is_y^*(n',k_y) \qquad \text{(C-16)}$$

$$Ic_y(n',-k_y) = Ic_y^*(n',k_y) \qquad \text{(C-17)}$$

where the asterisk represents complex conjugation.

APPENDIX D

SURFACE WAVE POLES

WHEN REGIONS V_2 AND V_3 ARE LOSSLESS

The purpose of this appendix is to determine when and how many surface wave poles are present if regions V_2 and V_3 in Figure 4 of Chapter III are both lossless. This information is needed during the numerical integration of W_{c_3}. More specifically, the following three statements will be proved in this appendix provided $\tan\delta_2 = \tan\delta_3 = 0$.

1) If $\mu_{r_2}\varepsilon_{r_2} < \mu_{r_3}\varepsilon_{r_3}$ and $d \neq 0$, then no surface wave poles or other singularities exist for any real ρ.

2) If $\mu_{r_2}\varepsilon_{r_2} = \mu_{r_3}\varepsilon_{r_3}$ or $d = 0$, then no surface wave poles exist for any real ρ; but there is an integrable singularity at $\rho = \sqrt{\mu_{r_3}\varepsilon_{r_3}}$.

3) If $\mu_{r_2}\varepsilon_{r_2} > \mu_{r_3}\varepsilon_{r_3}$ and $d \neq 0$, then surface wave poles are present, and they occur only in the interval $\sqrt{\mu_{r_3}\varepsilon_{r_3}} \leq \rho \leq \sqrt{\mu_{r_2}\varepsilon_{r_2}}$.

In addition, it will be proved that if statement three is true, then the total number of surface wave poles present is given by

$$n_{pole} = \text{entier}(4df\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}\,/c) + 1 \qquad \text{(D-1)}$$

where entier(x) is the greatest integer function, $f = \omega/2\pi$, and c is the speed of light in vacuum. The quantity ρ, which is used above, is one of the integration variables in Equation (5-11).

As pointed out in Chapter V, the surface wave poles occur when $\mathrm{Den}(\rho) = 0$ in the interval $0 \leqq \rho < \infty$. Before examining Den in this interval, the quantities k_2, k_3, k_{z_2}, and k_{z_3}, which appear in Den, must be evaluated for the case when regions V_2 and V_3 are lossless. Setting $\tan\delta_2 = \tan\delta_3 = 0$ in Equations (5-6) and (5-8) gives

$$k_2^2 = k_0^2 \mu_{r_2} \varepsilon_{r_2} \tag{D-2}$$

$$k_3^2 = k_0^2 \mu_{r_3} \varepsilon_{r_3} \tag{D-3}$$

Next, letting $\tan\delta_2 = 0$ in Equations (5-9) and using Equation (3-56) yields

$$k_{z_2} = \begin{cases} k_0\sqrt{\mu_{r_2}\varepsilon_{r_2} - \rho^2} & \text{if } \rho^2 < \mu_{r_2}\varepsilon_{r_2} \\ -jk_0\sqrt{\rho^2 - \mu_{r_2}\varepsilon_{r_2}} & \text{if } \rho^2 \geqq \mu_{r_2}\varepsilon_{r_2} \end{cases} \tag{D-4}$$

It can be seen from Equations (5-9) and (5-10) that k_{z_2} and k_{z_3} have the same form. Thus, by analogy with Equation (D-4),

$$k_{z_3} = \begin{cases} k_0\sqrt{\mu_{r_3}\varepsilon_{r_3} - \rho^2} & \text{if } \rho^2 < \mu_{r_3}\varepsilon_{r_3} \\ -jk_0\sqrt{\rho^2 - \mu_{r_3}\varepsilon_{r_3}} & \text{if } \rho^2 \geqq \mu_{r_3}\varepsilon_{r_3} \end{cases} \tag{D-5}$$

Equation (5-12) is

$$\mathrm{Den} = D_{TM} D_{TE} \tag{D-6}$$

while the substitution of Equations (D-2) and (D-3) into Equations (5-13) and (5-14) yields

$$D_{TM} = j\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right) k_{z_3} \cos(dk_{z_2}) - k_{z_2} \sin(dk_{z_2}) \tag{D-7}$$

$$D_{TE} = k_{z_3} \left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right] - j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right) \cos(dk_{z_2}) \tag{D-8}$$

The zeros of Den can be found by examining the zeros of D_{TM} and D_{TE}, since the zeros of these two terms are also the zeros of Den. This will be done shortly. From Equations (D-4) through (D-8) it can also be seen that Den is an even function of ρ. Thus, if ρ_i is a zero of Den, then so is $-\rho_i$. Only the zeros on the positive ρ axis are of interest, however, since only they contribute residue terms to Equation (5-11).

The following lemma will now be proved as a first step in verifying Statements 1, 2, and 3.

Lemma:

When regions V_2 and V_3 are lossless, the real zeros of Equations (D-7) and (D-8) always occur in the intervals

$$\min\{\mu_{r_2}\varepsilon_{r_2}, \mu_{r_3}\varepsilon_{r_3}\} \leq \rho^2 \leq \max\{\mu_{r_2}\varepsilon_{r_2}, \mu_{r_3}\varepsilon_{r_3}\} \qquad \text{(D-9)}$$

where min{ } and max{ } are the minimum and maximum values, respectively, of the list of quantities inside the brackets. This lemma will be proved by eliminating all other possibilities.

Proof:

Case i. For this case it will be assumed that

$$\rho^2 < \min\{\mu_{r_2}\varepsilon_{r_2}, \mu_{r_3}\varepsilon_{r_3}\} \qquad \text{(D-10)}$$

Then Equations (D-4) and (D-5) become

$$k_{z_2} = k_0\sqrt{\mu_{r_2}\varepsilon_{r_2}-\rho^2} \qquad \text{(D-11)}$$

$$k_{z_3} = k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}-\rho^2} \qquad \text{(D-12)}$$

It should be noticed from Equation (D-11) that k_{z_2} is purely real, implying that $\cos(dk_{z_2})$ and $\sin(dk_{z_2})$ are also purely real. In addition, k_{z_3} is purely real. Hence, the first term in Equation (D-7) is purely imaginary and the second term in that equation purely real. For D_{TM} to be zero, both its real and imaginary parts must be zero simultaneously for the same value of ρ. But k_{z_2} and k_{z_3} are both non-zero in the region given by Equation (D-10), while $\cos(dk_{z_2})$ and $\sin(dk_{z_2})$ are not zero simultaneously. Hence D_{TM} cannot be zero in the region under consideration.

Similar arguments apply to Equation (D-8) except that there the first term is purely real and the second term is purely imaginary. Thus, it has been shown that neither D_{TM} nor D_{TE} has any zeros whatsoever in the interval $\rho^2 < \min\{\mu_{r_2}\varepsilon_{r_2}, \mu_{r_3}\varepsilon_{r_3}\}$ when V_2 and V_3 are lossless.

Case ii. For this case it will be assumed that

$$\rho^2 > \max\{\mu_{r_2}\varepsilon_{r_2}, \mu_{r_3}\varepsilon_{r_3}\} \tag{D-13}$$

Then Equations (D-4) and (D-5) become

$$k_{z_2} = -jk_0\sqrt{\rho^2-\mu_{r_2}\varepsilon_{r_2}} = -jK_{z_2} \tag{D-14}$$

$$k_{z_3} = -jk_0\sqrt{\rho^2-\mu_{r_3}\varepsilon_{r_3}} = -jK_{z_3} \tag{D-15}$$

where the quantities K_{z_2} and K_{z_3} are defined as the following purely real, strictly positive quantities:

$$K_{z_2} = k_0\sqrt{\rho^2-\mu_{r_2}\varepsilon_{r_2}} \tag{D-16}$$

$$K_{z_3} = k_0\sqrt{\rho^2-\mu_{r_3}\varepsilon_{r_3}} \tag{D-17}$$

From Equations (D-14) and (D-16) along with the identities

$$\sin(x+jy) = \sin(x)\cosh(y) + j\cos(x)\sinh(y)$$

$$\cos(x+jy) = \cos(x)\cosh(y) - j\sin(x)\sinh(y)$$

it can be seen that

$$\sin(dk_{z_2}) = \sin(-jdK_{z_2}) = -j\sinh(dK_{z_2}) \qquad \text{(D-18)}$$

$$\cos(dk_{z_2}) = \sin(-jdK_{z_2}) = \cosh(dK_{z_2}) \qquad \text{(D-19)}$$

Next, substituting Equations (D-18), (D-19), (D-14), and (D-15) into Equation (D-7) gives

$$D_{TM} = j\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right)\left(-jK_{z_3}\right)\cosh(dK_{z_2}) - (-jK_{z_2})[-j\sinh(dK_{z_2})]$$

or

$$D_{TM} = \left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right)K_{z_3}\cosh(dK_{z_2}) + K_{z_2}\sinh(dK_{z_2}) \qquad \text{(D-20)}$$

A similar substitution into Equation (D-8) yields

$$D_{TE} = -j\left[\frac{K_{z_3}}{K_{z_2}}\sinh(dK_{z_2}) + \left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cosh(dK_{z_2})\right] \qquad \text{(D-21)}$$

Since K_{z_2} is purely real and strictly positive, $\sinh(dK_{z_2})$ and $\cosh(dK_{z_2})$ are also real, positive numbers. Thus, both terms in Equation (D-20) are purely real and strictly positive, implying that D_{TM} is never zero in the region defined by Equation (D-13). Similar remarks apply to the bracketed terms in Equation (D-21). Thus, neither D_{TM} nor D_{TE} has any zeros at all in the region $\rho^2 > \max\{\mu_{r_2}\varepsilon_{r_2}, \mu_{r_3}\varepsilon_{r_3}\}$ when V_2 and V_3 are lossless.

Combining the results of Cases i and ii shows that the zeros of D_{TM} and D_{TE} must occur in the region given by Equation (D-9).

Q.E.D.

It has not yet been proved that either D_{TM} or D_{TE} has any zeros. It has simply been shown that if they do, then these zeros can only occur in the region given by Equation (D-9).

Now D_{TM} and D_{TE} will be examined more closely in the region of Equation (D-9). It will be shown that no surface wave poles exist in this region when $\mu_{r_2}\varepsilon_{r_2} \leq \mu_{r_3}\varepsilon_{r_3}$, but that they do exist when $\mu_{r_2}\varepsilon_{r_2} > \mu_{r_3}\varepsilon_{r_3}$. The following situation will be considered first.

Case 1. For this case it will be assumed that

$$\mu_{r_2}\varepsilon_{r_2} < \mu_{r_3}\varepsilon_{r_3}, \tag{D-22}$$

$$\mu_{r_2}\varepsilon_{r_2} \leq \rho^2 \leq \mu_{r_3}\varepsilon_{r_3},$$

$$\text{and} \quad d \neq 0$$

Under these conditions Equations (D-4) and (D-5) become

$$k_{z_2} = -jk_0\sqrt{\rho^2-\mu_{r_2}\varepsilon_{r_2}} = -jK_{z_2} \tag{D-23}$$

$$k_{z_3} = k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}-\rho^2} \tag{D-24}$$

where K_{z_2} is defined by Equation (D-16). Substituting Equations (D-23), (D-24), (D-18), and (D-19) into Equation (D-7) gives

$$D_{TM} = j\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right)k_{z_3}\cosh(dK_{z_2}) - [-jK_{z_2}][-j\sinh(dK_{z_2})]$$

or

$$D_{TM} = j\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right)k_{z_3}\cosh(dK_{z_2}) + K_{z_2}\sinh(dK_{z_2}) \tag{D-25}$$

A similar substitution into Equation (D-8) yields

$$D_{TE} = \left(\frac{k_{z_3}}{K_{z_2}}\right)\sinh(dK_{z_2}) - j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cosh(dK_{z_2}) \tag{D-26}$$

Since K_{z_2} is purely real, both $\cosh(dK_{z_2})$ and $\sinh(dK_{z_2})$ are purely real. This information and the fact that k_{z_3} is also purely real indicate that the first term in Equation (D-25) is purely imaginary, while the second is purely real. By equating real and imaginary parts to zero, it can be seen that the right side of Equation (D-25) is zero only if $k_{z_3}\cosh(dK_{z_2})$ and $K_{z_2}\sinh(dK_{z_2})$ are zero simultaneously. Since $\cosh(dK_{z_2})$ is never zero in the interval under consideration and since

$d \neq 0$, k_{z_3} and K_{z_2} must be zero simultaneously if D_{TM} is to be zero. But these two terms cannot be zero for the same ρ because $\mu_{r_2}\varepsilon_{r_2} \neq \mu_{r_3}\varepsilon_{r_3}$ by assumption. Thus D_{TM} is never zero if the conditions given in Equation (D-22) hold.

Next, it should be noticed that the right side of Equation (D-26) is not zero since $\cosh(dK_{z_2})$ is not zero. Thus, neither D_{TM} nor D_{TE} is zero when Equation (D-22) holds. Combining this result with the Lemma proves Statement 1.

Case 2. For this case it will be assumed that

$$\mu_{r_2}\varepsilon_{r_2} > \mu_{r_3}\varepsilon_{r_3} \tag{D-27}$$

$$\mu_{r_3}\varepsilon_{r_3} \leqq \rho^2 \leqq \mu_{r_2}\varepsilon_{r_2}$$

$$\text{and } d \neq 0$$

Under these conditions Equations (D-4) and (D-5) become

$$k_{z_2} = k_0\sqrt{\mu_{r_2}\varepsilon_{r_2} - \rho^2} \tag{D-28}$$

$$k_{z_3} = -jk_0\sqrt{\rho^2 - \mu_{r_3}\varepsilon_{r_3}} = -jK_{z_3} \tag{D-29}$$

where K_{z_3} is defined by Equation (D-17). Substituting the last two equations into Equation (D-7) gives

$$D_{TM} = \left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right) K_{z_3} \cos(dk_{z_2}) - k_{z_2} \sin(dk_{z_2}) \tag{D-30}$$

A similar substitution into Equation (D-8) yields

$$D_{TE} = -j\left[K_{z_3}\left[\frac{\sin(dk_{z_2})}{k_{z_2}}\right] + \left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2})\right] \tag{D-31}$$

Thus D_{TM} is zero when

$$\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right) K_{z_3} \cos(dk_{z_2}) = k_{z_2} \sin(dk_{z_2})$$

or since $d \neq 0$, when

$$\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right) dK_{z_3} = dk_{z_2} \tan(dk_{z_2}) \tag{D-32}$$

Similarly, D_{TE} is zero when

$$\left(\frac{\mu_{r_2}}{\mu_{r_3}}\right) dK_{z_3} = -dk_{z_2} \cot(dk_{z_2}) \tag{D-33}$$

It is important to notice that all terms appearing in Equations (D-32) and (D-33) are purely real.

To aid in the examination of Equation (D-32), a real variable x will be defined, for this appendix alone, as

$$x = dk_{z_2} = dk_0\sqrt{\mu_{r_2}\varepsilon_{r_2}-\rho^2} \tag{D-34}$$

This x should not be confused with the x of rectangular coordinates. Next, y_1 will be defined as

$$y_1(x) = x\tan(x) = dk_{z_2}\tan(dk_{z_2}) \tag{D-35}$$

It should be noticed that y_1 is simply the right side of Equation (D-32). Now the left side of Equation (D-32) will be expressed in terms of x.

It should be observed from Equation (D-34) that

$$\rho^2 = \mu_{r_2}\varepsilon_{r_2} - \left(\frac{x}{dk_0}\right)^2 \tag{D-36}$$

Defining the left side of Equation (D-32) as y_2 and making use of Equation (D-17) shows that

$$y_2(x) = \left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right)dK_{z_3} = \left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right)dk_0\sqrt{\rho^2-\mu_{r_3}\varepsilon_{r_3}}$$

Substituting Equation (D-36) into this last equation yields

$$y_2(x) = \left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right) dk_0 \sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3} - \left(\frac{x}{dk_0}\right)^2} \qquad \text{(D-37)}$$

As a function of x, y_2 is an ellipse.

Since y_1 and y_2 are the right and left sides, respectively, of Equation (D-32), the solutions of Equation (D-32), and hence the zeros of D_{TM}, correspond to the intersections of the curves $y_1(x)$ and $y_2(x)$. A typical plot of y_1 and y_2 versus x is shown in Figure 21. For the situation shown in that figure, D_{TM} has two real zeros since y_1 and y_2 intersect twice. There is always at least one solution, that is, intersection of y_1 and y_2, and it occurs somewhere in the interval $0 \leqq x < \frac{\pi}{2}$. This proves Statement 3.

It will now be convenient to let the "radius" of y_2 increase. From Figure 21 it can then be seen that new intersections of y_1 and y_2 occur whenever $x = n\pi$, where $n = 1,2,3,\ldots$, and $y_2 = 0$ simultaneously. The total number of intersections is $n + 1$. This information, combined with Equation (D-37), indicates that new solutions occur when

$$y_2(n\pi) = 0 = \left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right) dk_0 \sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3} - \left(\frac{n\pi}{dk_0}\right)^2}$$

or when

$$dk_0 = \frac{n\pi}{\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}}$$

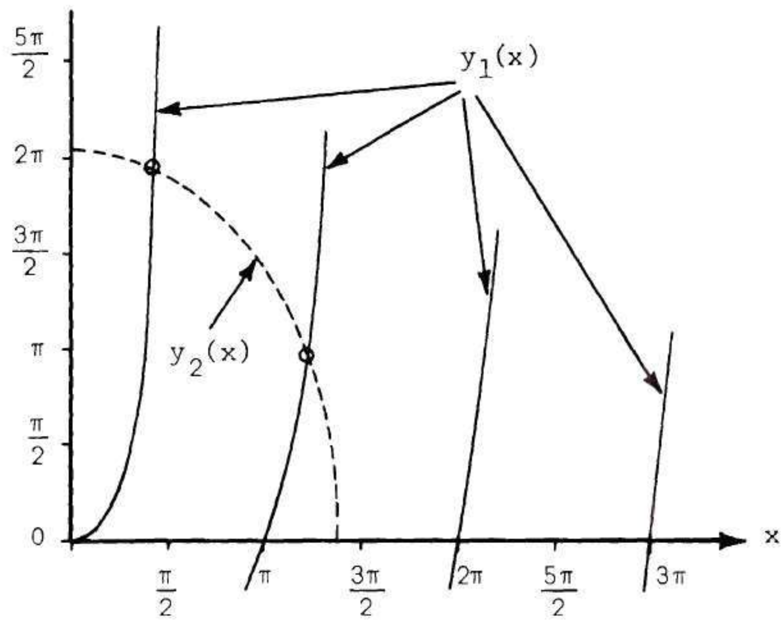


Figure 21. Graphical Location of the Real Zeros of D_{TM}

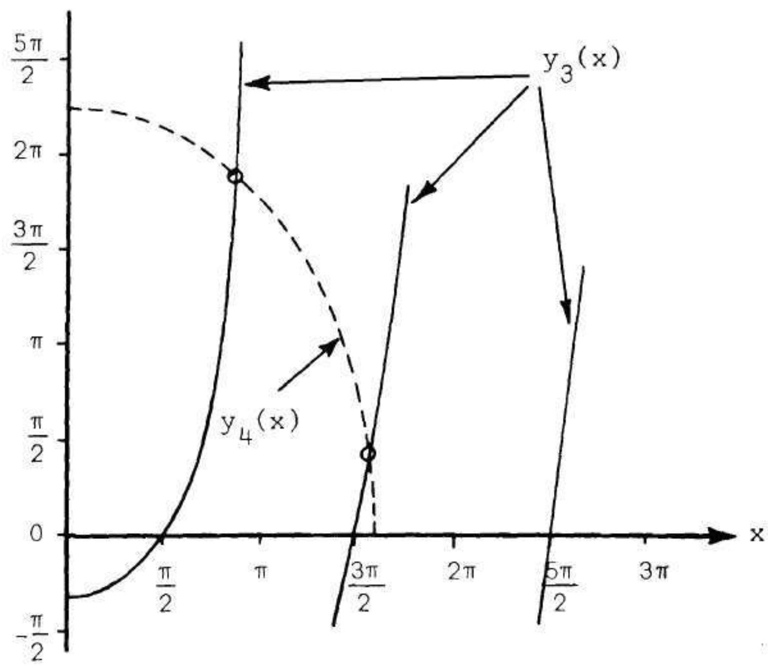


Figure 22. Graphical Location of the Real Zeros of D_{TE}

Substituting Equation (5-3) into this last equation and noting that $d \neq 0$ gives

$$f_{TM_{n+1}} = \frac{nc}{2d\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}} \text{ for } n = 1,2,3,... \tag{D-38}$$

where $f_{TM_{n+1}}$ is defined as the frequency at which the n + 1'st TM surface wave begins to propagate. It should be noticed from this last equation that if f is the operating frequency and if $f_{TM_{n+1}} \leqq f < f_{TM_{n+2}}$ where $n = 1,2,3,...$, then n + 1 of the TM surface poles exist. If $f < f_{TM_1}$, then only one TM surface wave pole exists. Equation (D-38) and the last comment imply that n + 1 of the TM poles are present if

$$\frac{nc}{2d\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}} \leqq f < \frac{(n+1)c}{2d\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}}$$

or if

$$n \leqq \frac{2df\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}}{c} < n + 1$$

From this last equation it can be seen that the number, n_{TM}, of TM surface wave poles that are present is given by

$$n_{TM} = \text{entier } (2df\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}/c) + 1 \tag{D-39}$$

Although Equation (D-39) does not locate the TM poles, it does show how many exist. This information is very important since in most numerical schemes for finding the zeros of a function the number of zeros to be found must be specified before the zero-finding is initiated. Equation (D-39) provides this information for D_{TM}.

Now the number of zeros in D_{TE} will be determined. Using the definition of x given in Equation (D-34), y_3 and y_4 will be defined as

$$y_3(x) = -x\cot(x) = -dk_{z_2}\cot(dk_{z_2}) \qquad \text{(D-40)}$$

and

$$y_4(x) = \left(\frac{\mu_{r_2}}{\mu_{r_3}}\right)dK_{z_3} = \left(\frac{\mu_{r_2}}{\mu_{r_3}}\right)dk_0\sqrt{\rho^2-\mu_{r_3}\varepsilon_{r_3}}$$

Substituting Equation (D-36) into the last equation gives

$$y_4(x) = \left(\frac{\mu_{r_2}}{\mu_{r_3}}\right)dk_0\sqrt{\mu_{r_2}\varepsilon_{r_2}-\mu_{r_3}\varepsilon_{r_3}-\left(\frac{x}{dk_0}\right)^2} \qquad \text{(D-41)}$$

From this last equation it can be seen that y_4 as a function of x is an ellipse. Since y_3 and y_4 are the right and left sides, respectively, of Equation (D-33), the zeros of D_{TE} correspond to the intersections of y_3 and y_4. A typical plot of y_3 and y_4 versus x is shown in Figure 22. For the situation shown, D_{TE} has two zeros since there are two intersections of y_3 and y_4. However, if y_4 intersects the x axis at an

$x < \frac{\pi}{2}$, then y_3 and y_4 will not intersect, and there will be no TE surface wave poles.

It will now be convenient to let the "radius" of y_4 increase. It can then be seen from Figure 22 that new intersections of y_3 and y_4 occur when $x = \frac{\pi}{2}$ (2n-1), for n = 1,2,3,..., and $y_4 = 0$ simultaneously. The total number of intersections is n. This information, combined with Equation (D-41), implies that new solutions occur when

$$y_4\left(\frac{\pi}{2}\,[2n-1]\right) = 0 = \left(\frac{\mu_{r_2}}{\mu_{r_3}}\right) dk_0 \sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3} - \left[\frac{\frac{\pi}{2}[2n-1]}{dk_0}\right]^2}$$

or when

$$dk_0 = \frac{\frac{\pi}{2}[2n-1]}{\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}}$$

Substituting Equation (5-3) into this last equation and remembering that $d \neq 0$ gives

$$f_{TE_n} = \frac{c(2n-1)}{4d\sqrt{\mu_{r_2}\varepsilon_{r_2} - \mu_{r_3}\varepsilon_{r_3}}} \text{ for } n = 1,2,3,\ldots \qquad \text{(D-42)}$$

where f_{TE_n} is defined as the frequency at which the n^{th} TE surface wave begins to propagate. It should be observed from the last equation that if $f_{TE_n} \leqq f < f_{TE_{n+1}}$, where n = 1,2,3,..., then n of the TE surface wave poles are present. However, if $f < f_{TE_1}$, then no TE poles are present. These comments and Equation (D-42) indicate that n of the TE poles are present if

$$\frac{c(2n-1)}{4d\sqrt{\mu_{r_2}\varepsilon_{r_2}-\mu_{r_3}\varepsilon_{r_3}}} \leqq f < \frac{c(2n+1)}{4d\sqrt{\mu_{r_2}\varepsilon_{r_2}-\mu_{r_3}\varepsilon_{r_3}}}$$

or if

$$2n - 1 \leqq \frac{4df}{c}\sqrt{\mu_{r_2}\varepsilon_{r_2}-\mu_{r_3}\varepsilon_{r_3}} < 2n + 1$$

or if

$$n \leqq \frac{2df\sqrt{\mu_{r_2}\varepsilon_{r_2}-\mu_{r_3}\varepsilon_{r_3}}}{c} + \frac{1}{2} < n + 1$$

From this last equation it can be seen that the number, n_{TE}, of TE surface wave poles present is given by

$$n_{TE} = \text{entier}\left(\frac{2df\sqrt{\mu_{r_2}\varepsilon_{r_2}-\mu_{r_3}\varepsilon_{r_3}}}{c} + \frac{1}{2}\right) \qquad \text{(D-43)}$$

The total number of surface wave poles, n_{pole}, is

$$n_{pole} = n_{TM} + n_{TE} \qquad \text{(D-44)}$$

To proceed further, the following identity is needed:

$$\text{entier}(x) + \text{entier}(x + \tfrac{1}{2}) = \text{entier}(2x) \tag{D-45}$$

This identity can be easily proved graphically. Applying Equations (D-39), (D-43), and (D-45) to Equation (D-44) shows that Equation (D-1) is indeed valid. Thus far, Statements 1 and 3, as well as Equation (D-1), have been verified. Only Statement 2 remains to be proved.

Case 3. If $d = 0$, no surface wave poles should exist since then there is no interface between regions V_2 and V_3 to support such waves. This statement can be shown mathematically, as follows. Setting $d = 0$ in Equations (D-6), (D-7), and (D-8) gives

$$\text{Den}\Big|_{d=0} = D_{TM}\Big|_{d=0} \; D_{TE}\Big|_{d=0} = \left[j\left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right) k_{z_3} \right] \left[-j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right) \right]$$

or

$$\text{Den}\Big|_{d=0} = \left(\frac{\varepsilon_{r_2}}{\varepsilon_{r_3}}\right)\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right) k_{z_3} \tag{D-46}$$

Equation (D-46) represents an integrable singularity at $k_{z_3} = 0$, that is at $\rho = \sqrt{\mu_{r_3}\varepsilon_{r_3}}$, instead of a pole. This can be shown by analogy with the next case. Since this is the only singularity in Den, no surface wave poles are present when $d = 0$. It should be noted that if $d = 0$, then the value of $\mu_{r_2}\varepsilon_{r_2}$ has no significance at all.

Case 4. It will next be assumed that

$$\mu_{r_2}\varepsilon_{r_2} = \mu_{r_3}\varepsilon_{r_3} \tag{D-47}$$

$$\text{and} \quad d \neq 0$$

For this case it will be shown that Den has an integrable singularity at $\rho = \sqrt{\mu_{r_2}\mu_{r_2}} = \sqrt{\mu_{r_3}\mu_{r_3}}$ and that no poles are present. To do this, it should first be observed from Equations (D-4) and (D-5) that

$$k_{z_2} = k_{z_3} = \begin{cases} k_0\sqrt{\mu_{r_3}\varepsilon_{r_3} - \rho^2} & \text{if } \rho^2 \leqq \mu_{r_3}\varepsilon_{r_3} \\ -jk_0\sqrt{\rho^2 - \mu_{r_3}\varepsilon_{r_3}} & \text{if } \rho^2 > \mu_{r_3}\varepsilon_{r_3} \end{cases} \tag{D-48}$$

Next, Equations (D-2) and (D-3) for this case become

$$k_2 = k_3 = k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}} \tag{D-49}$$

Substituting Equations (D-48) and (D-49) into Equations (D-7) and (D-8) gives, respectively.

$$D_{TM} = j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)k_{z_2}\cos(dk_{z_2}) - k_{z_2}\sin(dk_{z_2}) \tag{D-50}$$

$$D_{TE} = \sin(dk_{z_2}) - j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) \tag{D-51}$$

The last two equations permit Equation (D-6) to be written as

$$\text{Den} = -k_{z_2}\left[j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) - \sin(dk_{z_2})\right]^2 \qquad \text{(D-52)}$$

The bracketed term in Equation (D-52) can be shown to be non-zero in the following manner. If $\rho \leqq \sqrt{\mu_{r_3}\varepsilon_{r_3}}$, then k_{z_2} is purely real, as can be seen from Equation (D-48); and, consequently, both $\cos(dk_{z_2})$ and $\sin(dk_{z_2})$ are also purely real. Then, since the sine and cosine functions are never zero simultaneously, it follows that the bracketed term in Equation (D-52) is never zero when $\rho \leqq \sqrt{\mu_{r_3}\varepsilon_{r_3}}$.

Next consider the interval $\rho > \sqrt{\mu_{r_3}\varepsilon_{r_3}}$, in which k_{z_2} is a strictly negative, purely imaginary number, as indicated in Equation (D-48). Equations (D-18) and (D-19) then show that the bracketed term in Equation (D-52) is a strictly positive, purely imaginary number. Thus, the bracketed term under consideration is never zero for $\rho > \sqrt{\mu_{r_3}\varepsilon_{r_3}}$. This paragraph and the preceding one prove that the bracketed term in Equation (D-52) is non-zero for all real ρ. Therefore, the only singularity Den can have is when $k_{z_2} = 0$.

It will now be shown that k_{z_2} in Equation (D-52) introduces into W_{c_3} an integrable singularity instead of a pole. From Equation (5-11) it can be seen that

$$W_{c_3} = \int_0^\infty \int_0^{\pi/2} \frac{\text{Num}(\psi,\rho)}{\text{Den}(\rho)}\, d\psi\, d\rho$$

By using Equation (D-52) and splitting the ρ integration, the last equation becomes

$$W_{c_3} = \int_0^{\sqrt{\mu_{r_3}\varepsilon_{r_3}}} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)\; d\psi\; d\rho}{-k_{z_2}\left[j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) - \sin(dk_{z_2})\right]^2} \tag{D-53}$$

$$+ \int_{\sqrt{\mu_{r_3}\varepsilon_{r_3}}}^{\infty} \int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\rho)\; d\psi\; d\rho}{-k_{z_2}\left[j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) - \sin(dk_{z_2})\right]^2}$$

Now let

$$\rho = \sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \sin\alpha$$

in the first integral in Equation (D-53) and, in the second, let

$$\rho = \sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \cosh\alpha$$

Applying these last two equations to Equation (D-48) reveals that

$$k_{z_2} = k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \cos\alpha$$

in the first integral of Equation (D-53), while in the second integral

$$k_{z_2} = -jk_0\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \sinh\alpha$$

Substituting these last four equations into Equation (D-53) yields

$$W_{c_3} = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \sin\alpha)\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \cos\alpha\ d\psi\ d\alpha}{-k_0\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \cos\alpha\left[j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) - \sin(dk_{z_2})\right]^2}$$

$$+ \int_0^{\infty}\int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \cosh\alpha)\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \sinh\alpha\ d\psi\ d\alpha}{jk_0\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \sinh\alpha\left[j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) - \sin(dk_{z_2})\right]^2}$$

or

$$W_{c_3} = \int_0^{\pi/2}\int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \sin\alpha)\ d\psi\ d\alpha}{-k_0\left[j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) - \sin(dk_{z_2})\right]^2} \qquad \text{(D-54)}$$

$$+ \int_0^{\infty}\int_0^{\pi/2} \frac{\mathrm{Num}(\psi,\sqrt{\mu_{r_3}\varepsilon_{r_3}}\ \cosh\alpha)\ d\psi\ d\alpha}{jk_0\left[j\left(\frac{\mu_{r_3}}{\mu_{r_2}}\right)\cos(dk_{z_2}) - \sin(dk_{z_2})\right]^2}$$

It will now be observed that the denominators in Equation (D-54) have no singularities since the bracketed term in the denominators has already been shown to be non-zero. Thus, the singularity at $\rho = \sqrt{\mu_{r_3} \varepsilon_{r_3}}$ has been removed. It has also been shown that W_{c_3} has no other singularities for Case 4. This last comment, combined with Case 3, proves Statement 2.

Thus, Statements 1, 2, and 3, as well as Equation (D-1), have all been verified, which was the intent of this appendix. It should also be noticed that the real zeros of Den are simple. This can be seen by observing that the curves y_1 and y_2 in Figure 21 are not tangent when they intersect. Hence, $(y_1-y_2)^{-1}$ can have only simple poles. Similar remarks apply to Figure 22.

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