

CURVATURE AND ISOPERIMETRY IN GRAPHS

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CURVATURE AND ISOPERIMETRY IN GRAPHS

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SUMMARY

This dissertation concerns isoperimetric and functional inequalities in discrete spaces. The majority of the work concerns discrete notions of curvature. There is also discussion of volume growth in graphs and of expansion in hypergraphs.

In Chapter 2, we introduce the discrete curvature and explore the curvature-dimension (CD) inequality [7]. We improve the Cheeger inequality for graphs with non-negative curvature, a class which notably includes all Cayley graphs of abelian groups. In Chapter 3, we discuss Ollivier's definition of curvature [53] (see also Sammer [59]) via contraction of the minimum-transportation distance. We prove many results, including that a negative lower bound on graph curvature implies a bound on volume growth. In Chapter 4, we examine whether the property that a graph satisfies the CD inequality with a certain curvature parameter allows for a bound on the Ollivier curvature. For certain classes of regular graphs we find such a relationship. A consequence of this work is that we improve calculation methods for the CD inequality and compute the curvature for many graphs of general interest.

In Chapter 5, we modify methods of Agol [1] and Benson [11] to use a bound on volume growth in graphs to bound the spectral gap. In particular we use several bounds that are known from the study of discrete curvature to bound the spectral gap.

In Chapter 6, we study the expander mixing lemma in two different settings of hypergraphs. We prove inverses to the expander mixing lemmas in both cases, so that the hypergraph expander mixing lemma is tight up to a logarithmic factor. This resolves a question of Parzanchevski, Rosenthal & Tessler [56].

CHAPTER I

INTRODUCTION

1.1 Overview

This dissertation concerns isoperimetric and functional inequalities in discrete spaces. The majority of the work concerns discrete notions of curvature. There is also discussion of volume growth in graphs and of expansion in hypergraphs.

In Riemannian geometry there is a large and celebrated body of literature relating the Ricci curvature to various properties of the manifold, such as the Laplacian operator, the volume, the diameter, and various isoperimetric properties [22, 20, 14]. There has been much work in graphs and Markov chains studying the analogues of concepts that arise in Riemannian geometry, for example the Laplacian, isoperimetric constant and Cheeger inequalities [2, 3, 23]. These successes have motivated the problem of defining the discrete Ricci curvature. There have so far been several proposed definitions of discrete Ricci curvature [62, 49, 54, 7, 53, 30, 25, 9, 18]. It is generally unclear whether or not any of these notions of curvature are equivalent.

It is preferable that a notion of discrete Ricci curvature would allow for similar results to those that hold for manifolds, such as relating global isoperimetric properties to the discrete curvature. We should also hope that it is relatively easy to compute the discrete curvature. In Riemannian geometry there are many results under the hypothesis of positive (or non-negative) curvature; if we can find similar results for graphs, we would like there to be large classes of interesting graphs that *have* positive (or non-negative) curvature.

There have been many distinct definitions of the discrete Ricci curvature, each developed by taking a well-understood property of Ricci curvature in Riemannian

manifolds and adapting it to the setting of graphs and Markov chains. In this work we will mainly focus on two notions of discrete curvature. First we will discuss the curvature-dimension (CD) inequality of Bakry and Émery. The CD inequality is based on the Bochner formula, which relates the second derivatives of harmonic functions on a continuous manifold to the Ricci curvature tensor. The CD inequality was first studied specifically for graphs by Schmuckenschläger [60].

Secondly, we will discuss the Ollivier curvature, which is defined by the solutions to minimum transport problems between balls of small radius. The Ollivier curvature was first found by Ollivier (and also in the work of Sammer) [53, 59].

Here we also wish to mention the Erbar-Maas curvature, defined in [30] and further developed in [31], which relates the curvature to the convexity of entropy. The Erbar-Maas curvature has not so far allowed for particularly useful methods of calculation - recent work in this area includes [32, 29, 33].

In Chapter 2, we introduce the discrete curvature and explore the curvature-dimension inequality. We improve the Cheeger inequality for graphs with non-negative curvature, a class which notably includes all Cayley graphs of abelian groups. In Chapter 3, we discuss the Ollivier curvature. We prove many results, including that a negative lower bound on graph curvature implies a non-trivial bound on volume growth. In Chapter 4 we examine whether the property that a graph satisfies the CD inequality with a certain curvature parameter allows for a bound on the Ollivier curvature. For certain classes of regular graphs we find such a relationship. A consequence of this work is that we improve calculation methods for the CD inequality and compute the curvature for many graphs of general interest.

In Chapter 5, we modify methods of Agol [1] and Benson [11] to use a bound on volume growth in graphs to bound the spectral gap. In particular we use several bounds that are known from the study of discrete curvature to bound the spectral gap.

In Chapter 6, we study the expander mixing lemma in two different settings of hypergraphs. We prove inverses to the expander mixing lemmas in both cases, so that the hypergraph expander mixing lemma is tight up to a logarithmic factor. This resolves a question of Parzanchevski, Rosenthal & Tessler [56].

1.2 *Standard Definitions and Notation*

A graph $G = (V, E)$ has a vertex set V and an edge set E that contains 2-element subsets of V . A finite graph is one where V is a finite set.

If $\{x, y\} \in E$, we say that x and y are neighbors, denoted $x \sim y$. A common shorthand is that the edge $\{x, y\}$ may be denoted xy . The degree of a vertex x is the number of neighbors of x . A locally finite graph is one where each vertex has a finite list of neighbors. A d -regular graph is one where each graph has exactly d neighbors.

Let G be a locally finite graph. The adjacency operator A on the space $\{f : V \rightarrow \mathbb{R}\}$ is defined by the equation

$$Af(x) = \sum_{y: x \sim y} f(y),$$

and the Laplacian operator Δ on the same space is

$$\Delta f(x) = \sum_{y: y \sim x} f(x) - f(y).$$

We observe that if G is d -regular, then $\Delta = dI - A$, where I is the identity operator satisfying $If = f$ for all f .

By convention we write the eigenvalues of Δ (counting multiplicities) as $\lambda_1(\Delta), \lambda_2(\Delta), \dots$ with $\lambda_1(\Delta) \leq \lambda_2(\Delta) \leq \dots$. It is well-known that $\lambda_1(\Delta)$ is achieved by the eigenfunction $f \equiv 1$ with $\lambda_1 = 0$. The spectral gap of G is the difference between the two least eigenvalues of Δ : i.e., $\lambda_2(\Delta)$. Usually we suppress the linear operator Δ in this notation.

Sometimes the difference between the two largest eigenvalues of A is known as the spectral gap. If G is regular, these definitions are identical.

Let G be a finite graph. For a vertex subset $A \subset V$, define the edge boundary ∂A to be $\{ \{x, y\} \in E : x \in A; y \notin A \}$.

The (Cheeger) isoperimetric constant is

$$h(G) = \min_A \frac{|\partial A|}{|A|},$$

where the minimization is over all sets A with $0 < |A| \leq \frac{|V|}{2}$.

The modified Cheeger constant is

$$h^*(G) = \min_A \frac{|\partial A| \cdot |V|}{|A| \cdot |V/A|},$$

where the minimization is over all sets A where A and V/A are both non-empty. It is simple to see that $\frac{1}{2}h^*(G) \leq h(G) \leq h^*(G)$.

The Cheeger inequalities [21, 4, 63] relate λ_2 and h for a d -regular graph [3]).

$$\frac{h^2}{2d} \leq \lambda \leq 2h.$$

Alternately it is common to use the normalized adjacency operator $D^{-1/2}AD^{-1/2}$ where

$$Df(x) = \deg(x)f(x)$$

is the degree operator. Following this, the normalized Laplacian is $I - D^{-1/2}AD^{-1/2}$ and the normalized spectral gap is the difference between the two largest eigenvalues of the normalized Laplacian.

If G is d -regular, the normalized adjacency matrix is $\frac{1}{d}A$, the normalized Laplacian is $\frac{1}{d}\Delta$ and the normalized spectral gap is $\frac{1}{d}\lambda_2(\Delta)$.

Often when using these normalized operators it is useful to have the normalized isoperimetric constant $\frac{1}{d}h(G)$.

CHAPTER II

CURVATURE VIA THE CD INEQUALITY

2.1 Overview

In Riemannian geometry, the Bochner identity relates the second derivatives of a harmonic function to the Ricci curvature. Bakry and Émery generalized this relationship, introducing the curvature-dimension (CD) inequality, which allows for a definition of curvature in discrete spaces. We will say that graph G satisfies the inequality $CD(K, \infty)$ for

$$\frac{1}{2}\Delta\Gamma f - \Gamma(f, \Delta f) \geq K\Gamma(f, f) \text{ for all } f : V(G) \rightarrow \mathbb{R},$$

where Δ is the Laplacian and Γ is the field-squared operator. Full definitions are given in Section 2.2. The CD inequality was first studied specifically for graphs, as opposed to more general spaces, by Schmuckenschläger [60].

The curvature-dimension inequality has many useful properties; it is straightforward to calculate curvature and, compared to some other notions of curvature, relatively many graphs have positive or zero curvature. We discuss those topics and compute the curvature for many graphs of general interest in Chapter 4.

An issue in applying proof techniques to the CD inequality is that we do not have an analogue of the chain rule $\Gamma(\phi(f), g) = \phi'(f)\Gamma(f, g)$ (where Γ is the field-squared operator defined in Section 2.2). There have been attempts to modify the curvature-dimension inequality in order to bypass this problem, such as the CDE and CDE' inequalities of Bauer et al. [9, 40, 52]. Often these modifications do not admit as robust a class of graphs with positive curvature, and calculation of the curvature may become significantly more difficult. In this chapter we use alternate methods to prove results that hold under a bound on the CD inequality.

On manifolds, Buser's inequality shows that the Cheeger inequality lower bound on λ_2 is tight under non-negative curvature [20]. In this chapter we prove a version of Buser's inequality for graphs using the curvature-dimension inequality, arguing that $\lambda_2 \geq \frac{h^2}{2d}$ is tight up to a linear factor of d when CD curvature is non-negative. We also prove results characterizing the spectral gap and logarithmic Sobolev constant under a curvature lower bound. Similar results are found for different notions of curvature in Chapter 3 for Ollivier's curvature, in [9] for the CDE curvature, and in [31] for the Erbar-Maas curvature. In addition, we work on similar problems in Chapter 5.

The results in this chapter are also found in the author's joint work with B. Klartag, G. Kozma, and P. Tetali [43]. That work had as its kernel a long-unpublished note of Klartag and Kozma, which itself closely followed methods of M. Ledoux [45].

2.2 Definitions

Let $G = (V, E)$ be a locally finite connected graph. In our discussion of the CD curvature, the graph Laplacian will be the matrix $\Delta = \Delta(G) = A(G) - D(G)$, where $D(G)$ is the diagonal matrix of the degrees of the vertices, and $A(G)$ is the adjacency matrix of G . As an operator, its action on an $f : V \rightarrow \mathbb{R}$ can be described as:

$$\Delta f(x) = \sum_{y \sim x} (f(y) - f(x)).$$

Note that Δ is a negative semi-definite matrix.

Remark. By convention, in discussions of the CD inequality we use a rather uncommon definition of the Laplacian - more frequently $-\Delta = D - A$ will be known as the graph Laplacian. Indeed, the latter definition is given in the introduction of this work.

We will define the discrete curvature using the so-called Γ -calculus that was developed by Bakry and Émery [7].

Given functions $f, g : V \rightarrow \mathbb{R}$, we define the carré du champ (field-squared) operator Γ :

$$\Gamma(f, g)(x) = \frac{1}{2} \sum_{y \sim x} (f(x) - f(y))(g(x) - g(y)).$$

When $f = g$, the above becomes the more commonly denoted (square of the l_2 -type) discrete gradient: for each $x \in V$,

$$\Gamma(f)(x) := \Gamma(f, f)(x) = \frac{1}{2} \sum_{y \sim x} (f(x) - f(y))^2 =: \frac{1}{2} |\nabla f(x)|^2.$$

It becomes useful to define the iterated gradient

$$2\Gamma_2(f, g) = \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g).$$

By convention,

$$\Gamma_2(f) := \Gamma_2(f, f) = \frac{1}{2} \Delta\Gamma(f) - \Gamma(f, \Delta f).$$

Note that, given a measure $\pi : V \rightarrow [0, \infty)$, one can consider the expectation (with respect to π) of the above quantity, which gives us the familiar Dirichlet form associated with a graph:

$$\mathcal{E}(f, g) := \frac{1}{2} \sum_x \sum_{y \sim x} (f(x) - f(y))(g(x) - g(y))\pi(x).$$

It is useful to note an identity:

$$\sum_{x \in V} \Gamma(f, g)(x) = - \sum_{x \in V} f(x) \Delta g(x) = - \sum_{x \in V} g(x) \Delta f(x). \quad (1)$$

An additional useful local identity is:

$$\Delta(fg) = f\Delta g + 2\Gamma(f, g) + g\Delta f. \quad (2)$$

The heat kernel associated with the graph G is

$$P_t := e^{\Delta t} \quad (3)$$

for any $t \geq 0$. P_t is a positive definite matrix on \mathbb{R}^V , with $P_0 = I$. We observe that P_t commutes with Δ and with P_s , and that $\frac{\partial P_t}{\partial t} = P_t \Delta = \Delta P_t$. The matrix P_t has

only non-negative entries, so that if $f : V \rightarrow \mathbb{R}$ is a non-negative heat function, $P_t f$ is also a non-negative heat function.

The CD curvature is defined by the curvature-dimension inequality of Bakry and Émery [7]. If G is a graph and $x \in V(G)$ is a vertex, we say that G satisfies the curvature-dimension inequality $CD(K, \infty)$ at x if

$$\Gamma_2(f)(x) \geq K\Gamma(f)(x) \quad (4)$$

for all $f : V \rightarrow \mathbb{R}$. In this inequality, $K \in \mathbb{R}$ is the curvature parameter.

If G satisfies $CD(K, \infty)$ at x for every vertex x , then, as shorthand, we say G satisfies $CD(K, \infty)$ without specifying the vertex.

Definition 2.2.1. The CD curvature $\text{Ric}(G)$ of a graph G is defined as the maximum value K so that G satisfies the curvature-dimension inequality $CD(K, \infty)$.

In Chapter 4 we discuss techniques for computing the CD curvature and we give several examples of computation. Of particular interest in this chapter is Theorem 4.3.6, which states that any locally finite Cayley graph of an abelian group has non-negative curvature.

2.3 Spectral gap and curvature

Recall that λ_2 is the spectral gap of G .

Theorem 2.3.1. *Let G be a graph with curvature $\text{Ric} \geq K \geq 0$. Then $\lambda_2 \geq K$.*

A different proof of this result was previously given in [24].

Proof. We use the characterization of the heat kernel relating the first and second derivatives of $\text{Var}(P_t f)$ with respect to t , as found in [51]:

$$\lambda_2 = \min_f \frac{\mathcal{E}(-\Delta f, f)}{\mathcal{E}(f, f)},$$

so that $\alpha \leq \lambda_2$ if and only if, for any function f , we have $\alpha \cdot \mathcal{E}(f, f) \leq \mathcal{E}(-\Delta f, f)$.

By assumption, G satisfies Equation 4 with parameter K , i.e.,

$$\Delta\Gamma(f)(x) - 2\Gamma(f, \Delta f)(x) - 2K\Gamma(f)(x) \geq 0,$$

for all functions $f : V \rightarrow \mathbb{R}$ and all $x \in V$. Summing the above inequality over all vertices gives

$$0 \leq \sum_x \Delta\Gamma(f)(x) - 2 \sum_x \Gamma(\Delta f, f)(x) - 2K \sum_x \Gamma(f)(x) \quad (5)$$

$$= 2 \sum_x (\Delta f(x))^2 - K \sum_x \sum_{y \sim x} (f(y) - f(x))^2 \quad (6)$$

$$= 2 \sum_x (\Delta f(x))^2 - 2K \sum_{x \sim y} (f(y) - f(x))^2 \quad (7)$$

where in the first equality, we used Identity 1 and the fact that for any g , $\sum_x \Delta g = 0$.

Now let $|V| = n$, and recall the Dirichlet form (with respect to the measure $\pi \equiv 1$),

$$\mathcal{E}(f, f) = \sum_{x \sim y} (f(y) - f(x))^2$$

and that

$$\mathcal{E}(-\Delta f, f) = \sum_x -\Delta f(x) \sum_{y \sim x} f(y) - f(x) = \sum_x (\Delta f(x))^2.$$

Plugging this Dirichlet form into Equation 5, we find

$$2\mathcal{E}(-\Delta f, f) - 2K\mathcal{E}(f, f) \geq 0.$$

Rearranging gives

$$K\mathcal{E}(f, f) \leq \mathcal{E}(-\Delta f, f),$$

resulting in $\lambda_2 \geq K$. □

2.4 Buser-type inequalities

In this section we prove a graph version of Buser's inequality, which was originally proved for manifolds by P. Buser [20]. We will prove the following result:

Theorem 2.4.1 (Buser's inequality). *Suppose G has $\text{Ric}(G) \geq K$, for some $K \leq 0$. Then, for any subset $A \subset V$,*

$$|\partial A| \geq \frac{1}{2} \min \left\{ \sqrt{\lambda_2}, \frac{\lambda_2}{\sqrt{2|K|}} \right\} |A| \left(1 - \frac{|A|}{|V|} \right). \quad (8)$$

Recall that ∂A is the collection of all edges connecting A to $V \setminus A$.

Remark. Observe that the term $\lambda_2/\sqrt{2|K|}$ can be the minimizer only in the case $K < 0$. If $\text{Ric}(G) \geq K \geq 0$, G satisfies $CD(0, \infty)$, so we can take $K = 0$. In this case the $\sqrt{\lambda_2}$ term is the minimizer.

In order to prove the theorem, we require a bound on the heat kernel associated with G .

2.4.1 Gradient estimates

Note that in this section, we define the p -norm of a function $f : V \rightarrow \mathbb{R}$ as $\|f\|_p = (\sum_v |f(v)|^p)^{1/p}$.

Lemma 2.4.2. *Suppose G has $\text{Ric}(G) \geq K$ for some $K \in \mathbb{R}$. Then, for any $f : V \rightarrow \mathbb{R}$ and any $0 \leq t \leq 1/|2K|$,*

$$\|f - P_t f\|_1 \leq 2\sqrt{t} \|\sqrt{\Gamma(f)}\|_1.$$

Note that the restriction on t applies only when K is negative: if $K > 0$ then $\text{Ric} \geq K$ implies $\text{Ric} \geq 0$ and the lemma holds with no restriction on t .

Proof. The proof requires several identities.

1. We first prove that

$$\Gamma(P_t f) \leq e^{-2Kt} P_t(\Gamma(f)),$$

(that is, we prove that the inequality holds pointwise on V).

Remark. In fact, this identity is equivalent to the $CD(K, \infty)$ inequality. For this reason we sometimes say that the behavior of heat-flow functions on G determines the CD curvature.

Define an auxiliary function $g_s = e^{-2Ks} P_s(\Gamma(P_{t-s}f))$, a function on V . It is enough to show that $\partial g_s / \partial s$ is pointwise non-negative on $(0, t)$. We compute

$$\frac{\partial g_s}{\partial s} = e^{-2Ks} P_s [2\Gamma_2(P_{t-s}f) - 2K\Gamma(P_{t-s}f)].$$

Since P_s preserves non-negativity, it is enough to prove that

$$\Gamma_2(P_{t-s}f) - K\Gamma(P_{t-s}f) \geq 0,$$

which is true by our assumption $\text{Ric}(G) \geq K$.

2. Next we prove that

$$P_t(f^2) - (P_t f)^2 \geq \left(\int_0^t 2e^{2Ks} ds \right) \Gamma(P_t f). \quad (9)$$

We define the auxiliary function $g_s = P_s[(P_{t-s}f)^2]$; it suffices to show that $\partial g_s / \partial s \geq 2e^{2Ks} \Gamma(P_t f)$ for any $0 \leq s \leq t$. Using the local identity Equation 2, we compute that

$$\frac{\partial g_s}{\partial s} = P_s [2P_{t-s}f \cdot \Delta P_{t-s}f + 2\Gamma(P_{t-s}f)] + P_s [2P_{t-s}f \cdot (-\Delta P_{t-s}f)].$$

Hence, by Step 1, for any $0 \leq s \leq t$,

$$\frac{\partial g_s}{\partial s} = 2P_s(\Gamma(P_{t-s}f)) \geq 2e^{2Ks} \Gamma(P_t f),$$

which gives the result.

3. Denote $c_K(t) = \int_0^t 2e^{2Ks} ds$. Then $c_K(t) = (e^{2Kt} - 1)/K$ if $K \neq 0$, and $c_K(t) = 2t$ for $K = 0$. In both cases, $c_K(t) \approx 2t$ for small $t > 0$. For instance, $c_K(t) \geq t$ for $0 \leq t \leq 1/(2|K|)$. Thus, Equation 9 gives, for $0 \leq t \leq 1/(2|K|)$,

$$\max \sqrt{\Gamma(P_t f)} \leq \frac{1}{\sqrt{t}} \max \sqrt{P_t(f^2)} \leq \frac{1}{\sqrt{t}} \max |f|. \quad (10)$$

Finally we can prove the lemma. We begin by writing

$$P_t f - f = \int_0^t \frac{\partial P_s f}{\partial s} ds = \int_0^t P_s \Delta f ds.$$

It suffices to show that $\|P_s(\Delta f)\|_1 \leq s^{-1/2} \|\sqrt{\Gamma(f)}\|_1$ (since we have $\int_0^t s^{-1/2} ds = 2\sqrt{t}$). Let $\psi = \text{sgn}(P_s(\Delta f))$. Then,

$$\begin{aligned} \|P_s(\Delta f)\|_1 &= \sum_{x \in V} P_s(\Delta f)(x) \cdot \psi = \sum_{x \in V} \Delta f(x) \cdot P_s(\psi)(x) = \sum_{x \in V} -\Gamma(f, P_s(\psi))(x) \\ &\leq \sum_{x \in V} \sqrt{\Gamma(f)(x) \cdot \Gamma(P_s(\psi))(x)} \leq \|\sqrt{\Gamma(f)}\|_1 \cdot \max_{x \in V} \sqrt{\Gamma(P_s(\psi))(x)}, \end{aligned}$$

and the desired inequality follows from Equation 10, as $\max |\psi| = 1$.

□

2.4.2 Spectral gap and isoperimetry

The final step in the proof of Theorem 2.4.1 follows a proof method from [45].

Proof. Apply the previous lemma to $f = 1_A$. Note that $\Gamma(1_A)(v)$ is the number of edges in ∂A that are incident with v . We have that for any t satisfying $0 < t < 1/(2|K|)$,

$$\|1_A - P_t(1_A)\|_1 \leq 2\sqrt{t} \cdot |\partial A|.$$

Note that $0 \leq P_t(1_A) \leq 1$, hence the left-hand side may be written as follows:

$$\|1_A - P_t(1_A)\|_1 = |A| - \sum_{v \in A} P_t 1_A(v) + \sum_{v \in A^c} P_t 1_A(v) = 2 \left[|A| - \sum_{v \in V} 1_A(v) P_t 1_A(v) \right].$$

Since P_t is self-adjoint and $P_{t/2} P_{t/2} = P_t$, it follows that

$$(1/2) \|1_A - P_t 1_A\|_1 = |A| - \|P_{t/2} 1_A\|_2^2 = \|1_A\|_2^2 - \|P_{t/2} 1_A\|_2^2.$$

Let $\phi_i : 1 \leq i \leq n$ be the orthonormal eigenvectors of Δ , and let λ_i be the corresponding eigenvalues. Let $1_A = \sum a_i \phi_i$ be the spectral decomposition of A , with $\phi_1 \equiv 1/\sqrt{|V|}$ and $a_1 = |A|/\sqrt{|V|}$. Then $P_{t/2} 1_A = \sum_{i=1}^n a_i e^{-\lambda_i t/2} \phi_i$, and hence

$$(1/2) \|1_A - P_t 1_A\|_1 = \sum_i (1 - e^{-\lambda_i t}) a_i^2 \geq (1 - e^{-\lambda_2 t}) \sum_{i \geq 1} a_i^2 = (1 - e^{-\lambda_2 t}) \left(|A| - \frac{|A|^2}{|V|} \right).$$

To summarize, for any $0 < t \leq 1/(2|K|)$,

$$|\partial A| \geq \frac{1 - e^{-\lambda_2 t}}{\sqrt{t}} |A| \left(1 - \frac{|A|}{|V|}\right).$$

If $\lambda_2 \geq 2|K|$, we select $t = 1/\lambda_2 \leq 1/(2|K|)$, and deduce the theorem. On the other hand, if $\lambda_2 \leq 2|K|$, we take $t = 1/(2|K|)$. Then $1 - e^{-\lambda_2/(2|K|)} \geq \lambda_2/(4|K|)$, and the result follows. \square

Corollary 2.4.3. *Suppose a graph G has $\text{Ric}(G) \geq K$, for some $K \in \mathbb{R}$. Then*

$$\lambda_2 \leq \max\left(16h^2, 4h\sqrt{2|K|}\right).$$

In particular, if $K \geq 0$,

$$\lambda_2 \leq 16h^2.$$

Proof. Rearranging Equation 8, we see that

$$\frac{1}{2} \min\left(\sqrt{\lambda_2}, \frac{\lambda_2}{\sqrt{2|K|}}\right) \leq \frac{|\partial A||V|}{|A||V-A|} \leq \frac{2|\partial A|}{\min(|A|, |V-A|)},$$

assuming $\emptyset \subsetneq A \subsetneq V$. Minimizing over all such sets A , we see that

$$4h \geq \min\left(\sqrt{\lambda_2}, \frac{\lambda_2}{\sqrt{2|K|}}\right),$$

straightforward algebra gives the first part of the corollary. For the second part, observe that if G satisfies $CD(K, \infty)$ with $K \geq 0$, then G also satisfies $CD(0, \infty)$, taking $K = 0$ gives the result. \square

Of particular interest is the result for abelian Cayley graphs:

Corollary 2.4.4. *Let G be a Cayley graph for an abelian group. Then G has*

$$\lambda_2 \leq 16h^2.$$

The result follows directly from combining Theorem 4.3.6 with Corollary 2.4.3.

2.4.3 Logarithmic Sobolev constant and isoperimetry

We now prove a graph analogue of Theorem 5.3 from [45], relating the log-Sobolev constant ρ to an isoperimetric quantity. Consider the hypercontractive formulation of the log-Sobolev constant, as seen in e.g., [36],[28]: we define ρ to be the greatest value so that whenever $1 < r < q < \infty$ and $\sqrt{\frac{q-1}{r-1}} \leq e^{\rho t}$, then

$$n^{-1/q} \|P_t f\|_q \leq n^{-1/r} \|f\|_r .$$

Theorem 2.4.5. *Suppose G has $\text{Ric}(G) \geq K$ for some value $K \in \mathbb{R}$. Then for any subset $A \subset V$ with $|A| \leq |V|/2 = n/2$,*

$$|\partial A| \geq \frac{1}{16} \min \left(\sqrt{\rho}, \frac{\rho}{\sqrt{2|K|}} \right) |A| \log \frac{n}{|A|} .$$

Proof. As seen in the proof of Theorem 2.4.1, we observe that

$$\sqrt{t} \frac{|\partial A|}{n} \geq \frac{|A|}{n} - \frac{\|P_{t/2}(1_A)\|_2^2}{n} ,$$

if $0 < t < 1/(2|K|)$. Using the hypercontractivity property with $q = 2$ and $r = 1 + e^{-\rho t}$, we see that

$$\frac{\|P_{t/2}(1_A)\|_2^2}{n} \leq \frac{\|1_A\|_r^2}{n^{2/r}} = \left(\frac{|A|}{n} \right)^{2/r} .$$

Hence,

$$\sqrt{t} \frac{|\partial A|}{n} \geq \frac{|A|}{n} - \frac{\|P_{t/2}(1_A)\|_2^2}{n} \geq \frac{|A|}{n} - \left(\frac{|A|}{n} \right)^{2/r} .$$

As $2/r \geq 1 + \rho t/4$, whenever $0 \leq \rho t \leq 1$, and $|A|/n \leq 1$,

$$\sqrt{t} \frac{|\partial A|}{n} \geq \frac{|A|}{n} - \left(\frac{|A|}{n} \right)^{1+\rho t/4} = \frac{|A|}{n} \left(1 - \left(\frac{|A|}{n} \right)^{\rho t/4} \right) . \quad (11)$$

Let $t_0 = \min(1/2|K|, 1/\rho)$. If $|A|/n < e^{-4}$, set $t = \frac{4t_0}{\log(n/|A|)}$.

Using this value of t in Equation 11, we find

$$\begin{aligned} \frac{|\partial A|}{n} &\geq \frac{1}{\sqrt{t}} \frac{|A|}{n} (1 - e^{-\rho t_0}) \\ &\geq \frac{1}{2\sqrt{t_0}} \frac{|A|}{n} (1 - e^{-\rho t_0}) \log \left(\frac{n}{|A|} \right)^{1/2} \geq \frac{1}{4} \rho \sqrt{t_0} \frac{|A|}{n} \left(\log \frac{n}{|A|} \right)^{1/2}. \end{aligned}$$

On the other hand, if $e^{-4} \leq |A|/n \leq \frac{1}{2}$, use $t = t_0$ to obtain:

$$\frac{|\partial A|}{n} \geq \frac{1}{\sqrt{t_0}} \frac{|A|}{n} (1 - 2^{-\rho t_0/4}) \geq \frac{1}{8} \rho \sqrt{t_0} \cdot \frac{|A|}{n} \geq \frac{1}{16} \rho \sqrt{t_0} \frac{|A|}{n} \left(\log \frac{n}{|A|} \right)^{1/2},$$

where, for the second inequality, we use the fact that $1 - 2^{-x} \geq x/2$ if $0 \leq x \leq 1$. It follows that

$$\begin{aligned} \frac{|\partial A|}{n} &\geq \frac{1}{16} \rho \sqrt{\min \left(\frac{1}{2|K|}, \frac{1}{\rho} \right)} \frac{|A|}{n} \left(\log \frac{n}{|A|} \right)^{1/2} \\ &\geq \frac{1}{16} \min \left(\sqrt{\rho}, \frac{\rho}{\sqrt{2|K|}} \right) \frac{|A|}{n} \left(\log \frac{n}{|A|} \right)^{1/2}, \end{aligned}$$

proving the theorem. \square

The optimality of the above theorem (in terms of the dependence on the parameters involved) remains open at this time; in particular, we do not have tight examples. It is also natural to ask if the bound $\rho \geq K$ holds when $\text{Ric} \geq K \geq 0$, similar to the bound on λ in Theorem 2.3.1. In general this is not true, consider the complete graph on n vertices. We have seen that $\text{Ric} = 1 + \frac{n}{2}$, and it is easy to see (by considering the characteristic function of a set as a test function) that $\rho = O(\frac{n}{\log n})$ [51].

It is true that under the Erbar-Maas definition of discrete curvature, the so-called *modified* logarithmic Sobolev constant, ρ_0 can be lower bounded by the curvature, as seen in [30]. Thus it is certainly interesting to explore whether an analog of Theorem 2.3.1 is true with ρ_0 in place of λ ; recall here that ρ_0 captures the rate of decay of relative entropy of the Markov chain, relative to the equilibrium distribution, while ρ captures the hypercontractivity property of the Markov kernel. For more detailed discussion refer to [51].

CHAPTER III

CURVATURE FROM OPTIMAL TRANSPORT

3.1 *Overview*

In Chapter 2 we presented a notion of discrete curvature that is characterized by the solutions to heat flow problems. In this chapter we discuss the Ollivier curvature, which is defined by the solutions to mass transport problems.

To give a motivation for this definition, we will first briefly discuss the relation between transport and curvature in manifolds. Let M be a Riemannian manifold with points x, y and let x' and y' be the points obtained by parallel short movements from x and y respectively.

One property of the Ricci curvature between x and y is that it relates $d(x', y')$ to $d(x, y)$. If the curvature is positive, $d(x', y') < d(x, y)$. Here one can think of a sphere: there is a shorter path between x' and y' than the path parallel to the $x - y$ geodesic. On the other hand, if the curvature is negative, $d(x', y') > d(x, y)$. This is achieved on hyperbolic space, for one example.

Ollivier used this concept to help define the discrete Ricci curvature [53]. Let $B(x, r)$ be the ball of radius r around x and $B(y, r)$ similarly is a ball around y . Then the average distance between the corresponding points in $B(x, r)$ and $B(y, r)$ should determine the curvature, where two points correspond if they are at the same distance along parallel geodesics from x and y . Ollivier observed that the average distance can be replaced by the L_1 -Wasserstein distance between uniform distributions on $B(x, r)$ and $B(y, r)$, and this metric is used in definition of the so-called Ollivier curvature between x and y .

The definitions of Ollivier curvature can be applied to any metric measure space,

however, the most fruitful use has been to define curvature in graphs with the graph distance and counting measure, for example [10, 42, 19]. That will also be our focus in this chapter.

One well-known fact is that a manifold with curvature bound $\text{Ric} \geq 0$ will have the property of polynomial volume growth - that is, the volume of $B(x, r)$ is bounded by a polynomial in terms of r [14]. However, a manifold with negative curvature may not exhibit this property - again, we should think of a hyperbolic space with constant negative curvature. It is not clear whether the same is true for graphs under many notions of discrete curvature. For example, we are unaware of any work on volume growth related to the CD inequality, although there are volume growth results for the CDE' inequality [9].

In this chapter we will present results on volume growth and diameter for graphs under a bound on Ollivier curvature. In Chapter 5, we look at an application of volume growth bounds. One useful quality of the Ollivier curvature for graphs is that the curvature is relatively straightforward to calculate. In Chapter 4 we improve on techniques for calculating the Ollivier curvature on certain classes of graphs.

The results in this chapter are also found in the author's work [57], and in joint work with B. Benson and P. Tetali [12].

3.2 Definitions

Let X be a measurable metric space with metric d , and let μ, ν be two probability measures on X . The L_1 Wasserstein (also known as minimum-transport or earth-mover) distance [5] is

$$W_1(\mu, \nu) = \inf_m \int_{X \times X} d(x, y) dm(x, y),$$

where the minimum is taken over all probability measures m on $X \times X$ so that

$$\int_X m(x, y) d\nu(y) = \mu(x) \text{ and } \int_X m(x, y) d\mu(x) = \nu(y). \quad (12)$$

Qualitatively, we wish to transport the distribution μ to ν . m is a movement plan that moves probability mass $m(x, y)$ from x to y , and we choose m to minimize the average distance moved by the mass.

There is a well-known dual to the minimization problem [6]:

$$W_1(\mu, \nu) = \sup_{f \in \text{Lip}(1)} \int f d\nu - \int f d\mu. \quad (13)$$

A maximizing function for this equation is sometimes known as a *Kantorovich potential*.

Observe that if μ_x and μ_y both have finite support, both the primal and dual characterizations of $W_1(\mu_x, \mu_y)$ are linear programs on a finite set of variables. All the probability distributions we will consider in our discussion of Ollivier curvature will be of this type.

Let G be a locally finite connected graph and $x \in V(G)$ a vertex with degree d_x . For $x \in V$ and $0 \leq p \leq 1$, define a probability measure μ_x^p on V so that

$$\mu_x^p(v) = \begin{cases} p & \text{if } v = x \\ \frac{1-p}{d_x} & \text{if } v \sim x \\ 0 & \text{otherwise.} \end{cases}$$

Here, think of taking one step of a random walk starting at x and with laziness p .

Definition 3.2.1. If $x, y \in V$, the Ollivier curvature with laziness parameter p is

$$\kappa_p(x, y) = 1 - \frac{W_1(\mu_x^p, \mu_y^p)}{d(x, y)}. \quad (14)$$

Unless otherwise noted we will use laziness parameter $p = \frac{1}{2}$, and we suppress that parameter in our notation, so we have

$$\mu_x := \mu_x^{1/2} \text{ and}$$

$$\kappa(x, y) := \kappa_{1/2}(x, y).$$

The choice of parameter is to some extent not important: for any value of $p \geq \max\left(\frac{1}{d_x+1}, \frac{1}{d_y+1}\right)$, $\kappa_p(x, y)$ varies linearly with $1 - p$ [19].

For this chapter, we need some basic and well-known facts about Ollivier curvature.

Theorem 3.2.1 (Neighbors minimizing curvature (Y. Ollivier, [53])). *Suppose that $\kappa(u, v) \geq k$ whenever $u, v \in V$ are neighboring vertices. Then for any $x, y \in V$ (not necessarily neighbors), $\kappa(x, y) \geq k$ also.*

We give a quick proof due to Ollivier [53].

Proof. Observe that if $u \sim v$, $W_1(\mu_u, \mu_v) = 1 - \kappa(u, v) \leq 1 - k$.

Let $x = x_0, x_1, \dots, x_l = y$ be a geodesic path in G . Because W_1 is a metric, $W_1(\mu_x, \mu_y) \leq \sum_{i=1}^l W_1(\mu_{x_{i-1}}, \mu_{x_i}) \leq (1 - k)d(x, y)$, and $\kappa(x, y) \geq 1 - \frac{(1-k)d(x, y)}{d(x, y)} = k$. \square

In other words, it is equivalent to say that k is a global lower bound on curvature and that k is a lower bound on the curvature between each pair of neighbors.

Theorem 3.2.2 (Ollivier curvature tensorization (Y. Ollivier, [53])). *Let G be a d -regular graph, and denote $G \square G \square \dots \square G$ with r terms in the product by G^r . Suppose that for every $x, y \in V(G)$, $\kappa(x, y) \geq k$. Then for every $x', y' \in V(G^r)$, $\kappa(x', y') \geq \frac{k}{r}$.*

Again, there is a short proof from Ollivier's original work [53].

Proof. Let x and y be neighbors in G^r . By Theorem 3.2.1, it suffices to show $\kappa(x, y) > r$. Without loss of generality we may assume $x = (x_1, x_2, \dots, x_r)$ and $y = (y_1, x_2, \dots, x_r)$. Let f_1 be the Kantorovich potential satisfying

$$\int f_1 d\mu_{y_1} - \int f_1 d\mu_{x_1} = 1 - \kappa(x_1, y_1) \leq 1 - k.$$

Define $f(z_1, \dots, z_r) = f_1(z_1)$, then we see that

$$\begin{aligned} \inf f d\mu_y - \int f d\mu_x &= \frac{1}{r} \left(\int f_1 d\mu_{y_1} - \int f_1 d\mu_{x_1} \right) + \frac{r-1}{r} (f_1(y_1) - f_1(x_1)) \\ &\leq \frac{1}{r} (1 - k) + \frac{r-1}{r} = 1 - \frac{k}{r}. \end{aligned}$$

In other words, $\kappa(x, y) \geq \frac{k}{r}$. □

3.3 *Bounds on volume growth*

For manifolds, a lower bound on curvature implies an upper bound on the size of balls of radius r , where $r > 0$. The most celebrated result of this type is the Bishop-Gromov comparison theorem:

Theorem 3.3.1 (Bishop-Gromov comparison theorem [14]). *Let M be an n -dimensional Riemannian manifold with a uniform curvature bound $\text{Ric} > (n-1)k$, and let M_k be the n -dimensional space with uniform curvature $(n-1)k$. Let $x \in M$, $y \in M_k$ and $r > 0$, then*

$$\text{Vol}(B_M(x, r)) \leq \text{Vol}(B_{M_k}(y, r)), \tag{15}$$

where $B_M(x, r)$ is the ball of radius r around point x in manifold M .

A related question is whether a manifold has polynomial volume growth.

Definition 3.3.1. A manifold M has *polynomial volume growth* if there is a polynomial P_M with $\text{Vol}(B_M(x, r)) \leq P_M(r)$ for all values of $r > 0$.

Because spaces of constant zero curvature have polynomial volume growth (Euclidean space has zero curvature and the balls are Euclidean spheres), applying Bishop-Gromov tells us that any manifold with global curvature bound $\text{Ric} \geq 0$ will have polynomial volume growth.

For graphs, we will define the ball and the shell.

Definition 3.3.2. Let G be a graph and $x \in V(G)$. The ball of radius r is

$$B_x(r) = \{y \in V : d_G(x, y) \leq r\}, \quad (16)$$

and the shell of radius r is

$$d_x^{-1}(r) = \{y \in V : d_G(x, y) = r\}. \quad (17)$$

Similar to a manifold, a graph may also have the polynomial volume growth property:

Definition 3.3.3. A graph G has *polynomial volume growth* if there is a polynomial P_G so that $|B_x(r)| \leq P_G(r)$ for all $x \in V(G)$ and $r \geq 0$.

In this section we will find upper bounds on the shell volume $|d_x^{-1}(i)|$ in terms of a lower bound on Ollivier curvature. It is simple to convert such bounds into bounds on the ball volume (analogously to Bishop-Gromov) with the equation $|B_x(r)| = \sum_{i=0}^r |d_x^{-1}(i)|$.

In this area, there are some previous results due to Paeng [55].

Theorem 3.3.2 (Paeng [55]). *Let G be a graph with maximum degree D . Let r be an integer with $0 \leq r \leq \text{diam}(G)$. Assume that $\kappa(x, y) \geq k$ for all $x, y \in V$.*

$$|d_x^{-1}(r)| \leq D^r \prod_{m=0}^{r-1} \left(1 - \frac{k}{2}m\right) \quad (18)$$

These bounds are only useful in the case that $k > 0$: if we set $k = 0$ above, we see only the trivial result that $|f^{-1}(r)| \leq D^r$.

In the case $k > 0$, we see that $|d_x^{-1}(\lceil 2/k+1 \rceil)| \leq 0$; that is, G has $\lceil 2/k \rceil \leq \text{diam}(G)$. Because G is finite, G has polynomial volume growth with $|d_x^{-1}(r)| \leq |V(G)|r^0$.

We develop results that are useful in the case that G has a negative lower bound on curvature. We find that such graphs do not necessarily have polynomial volume growth. It remains an open question whether or not a bound of $k(x, y) \geq 0$ for all $x, y \in V$ implies polynomial volume growth.

Theorem 3.3.3. *Let G be a d -regular graph with $\kappa(v_1, v_2) \geq k$ for every pair of vertices v_1, v_2 . Fix $x \in V$, define $S_i = d_x^{-1}(i)$. If $i \geq 1$,*

$$|S_{i+1}| \leq \frac{d+1-2dk}{2}|S_i|.$$

Proof. First, we bound $e(S_i, S_{i+1})$, the number of edges between S_i and S_{i+1} . Let $z \in S_i$, z is adjacent to some vertex $y(z) \in S_{i-1}$. (If z is adjacent to multiple vertices in S_{i-1} , choose $y(z)$ arbitrarily from them.) Let $T(z)$ be the set of common neighbors of z and $y(z)$. Neither y nor a neighbor of y can be in S_{i+1} , so $e(z, S_{i+1}) \leq d-1-|T(z)|$. Let $T^* = \sum_{z \in S_i} |T(z)|$, so $e(S_i, S_{i+1}) \leq (d-1)|S_i| - T^*$.

Next, for each z we wish to use the Kantorovich characterization of $W_1(\mu_y, \mu_z)$. Define the following test-function f :

- $f(y) = 0$.
- $f(z) = 1$.
- $f|_{T(z)} = 0$.
- For any other neighbor v of y , $f(v) = -1$.
- Let $W(z)$ be the set of neighbors of z (besides y) that are not in $T(z)$ and are adjacent to a neighbor of y (besides z) that is not in $T(z)$. We may set $f|_{W(z)} = 0$.
- Let $U(z) = N(z) \setminus (\{y\} \cup T(z) \cup W(z))$, set $f|_{U(z)} = 1$.

- f can be made 1-Lipschitz by setting $f = 0$ on every other vertex.

We have:

$$\int f \, dm_z = 1 + \frac{|U(z)| - d}{2d} \quad \text{and} \quad \int f \, dm_y = \frac{|T(z)| + 2 - d}{2d}.$$

Combining the two, we get

$$(1 - k) \geq \int f \, d(m_z - m_y) \geq 1 + \frac{|U(z)| - 2 - |T(z)|}{2d},$$

rearranging gives

$$|T(z)| + 2 - 2dk \geq |U(z)|.$$

If a neighbor of z is not in $U(z)$, the neighbor must be either y , adjacent to y (and thus in $T(z)$), or adjacent to more than one neighbor of y (thus in $W(z)$).

Any vertex in S_{i+1} for which z is the only neighbor in S_i must be in $U(z)$. The total number U^* of vertices in S_{i+1} that are adjacent to only one vertex in S_i is at most

$$U^* \leq \sum_z |U(z)| \leq \sum_z (|T(z)| + 2 - 2dk) = T^* + |S_i|(2 - 2dk).$$

We can now see that the number of vertices in S_{i+1} that are adjacent to more than one vertex in S_i is bounded above by

$$\frac{(d-1)|S_i| - T^* - U^*}{2}.$$

This is because the total number of possible edges from S_i to these vertices is at most $e(S_i, S_{i+1}) \leq (d-1)|S_i| - T^*$ less the U^* edges that are accounted for by vertices in S_{i+1} with only one neighbor in S_i . Every other vertex must be incident to at least 2 of those $(d-1)|S_i| - T^* - U^*$ edges, so we divide by 2.

Now, we add the other U^* vertices in S_{i+1} to achieve the desired result:

$$\begin{aligned} |S_{i+1}| &\leq U^* + \frac{(d-1)|S_i| - T^* - U^*}{2} = \frac{(d-1)|S_i| - T^* + U^*}{2} \\ &\leq \frac{(d-1)|S_i| - T^* + (2-2dk)|S_i| + T^*}{2} = \frac{d+1-2dk}{2}|S_i|. \end{aligned}$$

□

Following the same proof outline, we obtain a better bound for bipartite graphs.

Theorem 3.3.4. *Let G be a d -regular bipartite graph with $\kappa(v_1, v_2) \geq k$ for every pair of vertices v_1, v_2 . Fix $x \in V$, define $S_i = d_x^{-1}(i)$. If $i \geq 1$,*

$$|S_{i+1}| \leq \frac{d - dk}{2} |S_i|.$$

Proof. First, we bound $e(S_i, S_{i+1})$, the number of edges between S_i and S_{i+1} . Let $z \in S_i$, z is adjacent to some vertex $y(z) \in S_{i-1}$. Because $y \notin S_{i+1}$, $e(z, S_{i+1}) \leq d - 1$. Clearly, $e(S_i, S_{i+1}) \leq (d - 1)|S_i|$.

Next, for each z we wish to use the Kantorovich characterization of $W_1(\mu_y, \mu_z)$. Define a test-function f :

- $f(y) = 0$.
- $f(z) = 1$.
- For any other neighbor v of y , $f(v) = -1$.
- Let $W(z)$ be the set of neighbors of z (besides y) are adjacent to a neighbor of y other than z . Set $f|_{W(z)} = 0$.
- Let $U(z) = N(z) \setminus (\{y\} \cup W(z))$, set $f|_{U(z)} = 2$.
- f can be made 1-Lipschitz by setting $f = 1$ on any other vertex in the same partite set as z and $f = 0$ on any other vertex in the same partite set as y .

We have:

$$\int f \, dm_z = 1 + \frac{2|U(z)| - d}{2d} \quad \text{and} \quad \int f \, dm_y = \frac{2 - d}{2d}.$$

Combining,

$$(1 - k) \geq \int f \, d(m_z - m_y) \geq 1 + \frac{2|U(z)| - 2}{2d},$$

resulting in

$$|U(z)| \leq 1 - dk.$$

If a neighbor of z is not in $U(z)$, it must be either y or adjacent to more than one neighbor of y (and thus in $W(z)$).

Any vertex in S_{i+1} for which z is the only neighbor in S_i must be in $U(z)$. The total number U^* of vertices in S_{i+1} that are adjacent to only one vertex in S_i is at most

$$U^* \leq \sum_z |U(z)| \leq |S_i|(1 - dk).$$

We can now bound the number of vertices in S_{i+1} that are adjacent to more than one vertex in S_i from above by

$$\frac{(d-1)|S_i| - U^*}{2}.$$

This is because the total number of possible edges from S_i to these vertices is at most $e(S_i, S_{i+1}) \leq (d-1)|S_i|$ less the U^* edges that are accounted for by vertices in S_{i+1} with only one neighbor in S_i . Each counted vertex must be incident to at least 2 of those $(d-1)|S_i| - U^*$ edges, so we divide by 2.

Now, we add the other U^* vertices to achieve the desired bound on $|S_{i+1}|$:

$$\begin{aligned} |S_{i+1}| &\leq U^* + \frac{(d-1)|S_i| - U^*}{2} = \frac{(d-1)|S_i| + U^*}{2} \\ &\leq \frac{(d-1)|S_i| + (1 - dk)|S_i|}{2} = \frac{d(1-k)}{2} |S_i|. \end{aligned}$$

□

Theorem 3.3.5. *For any d -regular graph, if $i \geq 1$,*

$$|S_i| \leq d^i \left(\frac{1 + \frac{1}{d} - 2k}{2} \right)^{i-1}.$$

For any d -regular bipartite graph, if $i \geq 1$,

$$|S_i| \leq d^i \left(\frac{1 - k}{2} \right)^{i-1}.$$

Proof. Observe $S_0 = 1$ and $S_1 = d$ for every graph. Repeated application of the previous results in this section gives the desired bounds for S_i when $i \geq 2$. \square

As far as we are aware, these are the first non-trivial bounds on volume growth under a negative bound on Ollivier curvature. A weakness in the proof method is that vertices in S_{i+1} are counted either as exactly one neighbor in $S_i(U)$, or as having several neighbors (W), but the bound on the size of U assumes the worst case - that there are a large number of vertices of type W , each having only 2 neighbors in S_i . For graphs where that assumption is correct (or close), our bound is somewhat tight. In other graphs, the average number of neighbors in S_i for any vertex in S_{i+1} can be $O(d)$. For those graphs the bound is not tight. Below we give an example of this issue.

Example 3.3.1. Let T_p be the infinite p -regular tree and T_p^q be the graph $T_p \square T_p \square \dots \square T_p$, with the product taken q times. T_p^q is pq -regular. From Theorem 4.4.3, T_p has $k(x, y) = \frac{2-p}{p}$ if $x \sim y$. By tensorization of curvature, T_p^q has $k(x, y) \geq \frac{2-p}{pq}$ whenever $x \sim y$.

Because T_p^q is bipartite, we apply the second statement of Corollary 3.3.5 to find the bound

$$|d_x^{-1}(i)| \leq (pq)^i \left(\frac{1 - \frac{2-p}{pq}}{2} \right)^{i-1} = pq \left(\frac{p(q+1)}{2} - 1 \right)^{i-1}, \quad (19)$$

so that

$$\log(|d_x^{-1}(i)|) \leq i \log \left(\frac{p(q+1)}{2} - 1 \right) + O(1). \quad (20)$$

A vertex $y \in d_x^{-1}(i)$ is characterized by the distance from x parallel to each of the q copies of T_p in the product graph, and, given those distances, by the path taken in T_p of that distance.

There are $\binom{i+q-1}{q}$ choices of what distance is travelled along each copy of T_p . At each step of any path taken along some copy of T_p , there are either p possibilities (for

the first step) or $p - 1$ possibilities (for any subsequent step). As such,

$$\binom{i + q - 1}{q} (p - 1)^i \leq |d_x^{-1}(i)| \leq \binom{i + q - 1}{q} p^q (p - 1)^{i - q}, \quad (21)$$

and

$$\log(|d_x^{-1}(i)|) = i \log(p - 1) + O(1). \quad (22)$$

Observe that q is the maximum number of neighbors that $y \in d_x^{-1}(i)$ has in $d_x^{-1}(i - 1)$. If $q = 2$ we see that the actual logarithmic volume growth bound approximately matches the bound from Theorem 3.3.5, in that both have leading term $i \log p$. On the other hand, if $q > 2$ the logarithmic volume growth bound is not tight: it is $i \log \left(\frac{p(q+1)}{2} - 1 \right)$ where the actual bound is $i \log p$.

We conjecture that T_p^q actually experiences the maximum volume growth for their curvature and regularity.

Conjecture 3.3.1. *Let G be a pq -regular graph so that if $u, v \in V(G)$, then $\kappa(u, v) \geq \frac{2-p}{pq}$. Let $x \in V(G)$ and $y \in V(T_p^q)$. Then for any $i \geq 0$,*

$$|d_x^{-1}(i)| \leq |d_y^{-1}(i)|.$$

Qualitatively, T_p^q is conjectured to fill the same role that the space of constant curvature does in the Bishop-Gromov comparison theorem. A case of this conjecture is that the d -dimensional lattice T_2^d is conjectured to have the fastest volume growth for any $2d$ -regular graph with curvature lower bound 0. If correct, this would prove that any such graph has polynomial volume growth with

$$|d_x^{-1}(i)| \leq \binom{i + d - 1}{d} 2^d \leq (2i + d - 1)^d,$$

where the right-hand side is a polynomial in terms of i .

3.4 Other results

We also present some results of interest for the Ollivier curvature.

3.4.1 Diameter bounds

It is well-known that a positive lower bound on Ollivier curvature gives an upper bound on the diameter of the graph, according to the following argument developed from [53].

Assume for every pair of neighboring vertices x, y , $\kappa(x, y) \geq k > 0$. Let x_0, \dots, x_l be a geodesic path, then

$$W_1(\mu_{x_0}, \mu_{x_l}) \leq \sum_{i=1}^l W_1(\mu_{x_{i-1}}, \mu_{x_i}) \leq l(1 - k),$$

but, considering the 1-Lipschitz function $f(y) = d(x_0, y)$,

$$W_1(\mu_{x_0}, \mu_{x_l}) \geq \int f d\mu_{x_l} - \int f d\mu_{x_0} \geq (l - \frac{1}{2}) - (\frac{1}{2}) = l - 1.$$

Thus $l(1 - k) \geq l - 1$, and $1 \geq lk$; as the diameter is achieved on a geodesic path, its length D is bounded above by $D \leq \frac{1}{k}$.

Corollary 3.4.1. *If G is a d -regular graph with positive curvature, then the diameter is bounded by $D \leq 2d$.*

For any $x, z \in V$, $\mu_x(z)$ is an integer multiple of $\frac{1}{2d}$. There is an optimal solution to the minimum-transport problem between μ_x and μ_y with the property that a multiple of $\frac{1}{2d}$ is transported along each path used - as the graph distance is integer-valued, $W_1(\mu_x, \mu_y)$ is a multiple of $\frac{1}{2d}$. If x, y are neighbors and $\kappa(x, y) > 0$, then it must be $\kappa(x, y) \geq \frac{1}{2d}$. If x, y are not neighbors, then for any $z \sim x$ along the shortest $x - y$ path, $\kappa(x, y) \geq \kappa(x, z) \geq \frac{1}{2d}$. Thus $\kappa^* \geq \frac{1}{2d}$, and $D \leq 1/\kappa^* \leq 2d$.

Remark. If d is instead an upper bound on the degree of the vertex, then for any x, y, z, w , $\mu_x(z)$ and $\mu_y(w)$ are both multiples of $\frac{1}{2d_x d_y}$, and a similar argument is possible with $\kappa(x, y) \geq \frac{1}{2d_x d_y} \geq \frac{1}{2d^2 - 2d}$ and thus $D \leq 2d^2 - 2d$.

Remark. This argument seems to require that the laziness parameter $1/2$ is an integer multiple of $\mu_x(y)$ when $x \sim y$. However, (as discussed before) for every value of

$p \geq \max\left(\frac{1}{d_x+1}, \frac{1}{d_y+1}\right)$, $\kappa_p(x, y)$ varies linearly with $1 - p$. Up to this scaling factor, we can treat a result for $\kappa_{1/2}$ as a result for κ_p as long as p satisfies the stated lower bound.

Because d -regular graphs with positive curvature have a universal bound on the diameter, there are only finitely many such graphs.

Corollary 3.4.2. *There is no infinite family of bounded-degree graphs with a lower bound on curvature of $k > 0$.*

Remark. It is known that a planar graph with bounded degree on n vertices has spectral gap $\lambda = O(1/n)$ [61]. This result tells us that among finite planar graphs with bounded degree, there is no infinite family that makes the bound $\kappa = O(1/n)$ tight, indeed, $\kappa \leq 0$ for all but finitely many bounded degree graphs.

In the case of graphs without bounded degree, straightforward calculation shows that $\kappa = 1/n$ for the star graph with n leaves, which is clearly planar.

CHAPTER IV

COMPUTATION AND COMPARISON OF DISCRETE CURVATURE

4.1 *Overview*

In this chapter we will discuss computation of the discrete curvature. We will compute the CD and Ollivier curvature (as defined in Chapters 2 and 3, respectively) for several graphs of general interest and some large classes of graphs. Qualitatively, we expect a graph will have positive curvature if, relative to the degree and size of vertex set, there exist a large number of 3-, 4-, and 5-cycles, and that a graph will have negative curvature if not. The reason for this is that a graph with small cycles will experience concentration of heat flow, and so we expect the CD curvature to be positive. Similarly, small cycles guarantee the existence of short geodesic paths between the neighbors of two chosen vertices, and so the Ollivier curvature is positive.

In general there is not a good understanding of the relationship between the CD and Ollivier curvatures, nor is there understanding of the relationships between other notions of discrete curvature. Ideally we would like to find that positive curvature in one notion implies positive curvature in another - at least for a large and interesting class of graphs. Such a relationship is known for some notions of curvature: Münch proved that $CDE'(K, n)$ implies $CD(K, n)$ [52].

As discussed in Chapter 2, when CD curvature is non-negative, $\lambda \approx h^2$: only when curvature is negative can the other Cheeger inequality tight, that is, $\lambda \approx h$. As we will discuss in Chapter 5, when the continuous curvature is non-negative, a manifold has polynomial volume growth. Otherwise exponential volume growth is possible. Under a positive bound on the CD curvature, there is a relationship between the

diameter and the spectral gap [24]. In these results and many others, 0 curvature is the critical value, and so in our analysis we are particularly interested in determining the sign of the curvature.

In this chapter, we discuss computation of the CD curvature introduced in Chapter 2. Next, we discuss computation of the Ollivier curvature introduced in Chapter 3. Finally, we examine a class of graphs on which the CD curvature and Ollivier curvature have related signs, including a characterization of some graphs where the signs of those curvatures differ.

The results in this chapter are also found in the author's joint work with B. Klartag, G. Kozma, and P. Tetali [43], and in the author's unpublished work [57].

4.2 *Calculation of CD curvature*

Recall from Chapter 2 the definitions of the Γ -calculus:

Given functions $f, g : V \rightarrow \mathbb{R}$, the carré du champ operator Γ is calculated by the expression

$$\Gamma(f, g)(x) = \frac{1}{2} \sum_{y \sim x} (f(x) - f(y))(g(x) - g(y)).$$

The iterated gradient Γ_2 is defined by the equation

$$2\Gamma_2(f, g) = \Delta\Gamma(f, g) - \Gamma(f, \Delta g) - \Gamma(\Delta f, g). \quad (23)$$

By convention,

$$\begin{aligned} \Gamma f &:= \Gamma(f, f) \text{ and} \\ \Gamma_2 f &:= \Gamma_2(f, f) = \frac{1}{2} \Delta\Gamma(f) - \Gamma(f, \Delta f). \end{aligned}$$

G has K as a lower bound on the curvature at $x \in V(G)$ if G satisfies the inequality $CD(K, \infty)$ at x :

$$\Gamma_2 f(x) \geq K\Gamma f(x) \quad (24)$$

for all functions $f : x \rightarrow \mathbb{R}$. The Ricci curvature of the graph $\text{Ric}(G)$ is the maximum value K so that G satisfies $CD(K, \infty)$ at all vertices x . Observe that if $K < \text{Ric}(G)$, G also satisfies $CD(K, \infty)$ at all vertices.

Alternately, $\text{Ric}(G)$ is the value

$$\min_{f, x: \Gamma f(x) \neq 0} \frac{\Gamma_2 f(x)}{\Gamma f(x)}.$$

For a fixed vertex x , we can characterize the functions f that minimize $\Gamma_2 f(x)/\Gamma f(x)$:

Let $x \in V$, and let $f : V \rightarrow \mathbb{R}$ be a function. Observe that Equation (24) is unchanged on adding a constant to f , so we may assume that $f(x) = 0$. We expand the right side of Equation 23. We will write d_v for the degree of v and $d(u, v)$ for the graph distance between u and v .

$$\begin{aligned} 2\Gamma_2 f(x) &= \Delta \Gamma(f)(x) - 2\Gamma(f, \Delta f)(x) \\ &= \left(\sum_{v \sim x} \Gamma(f)(v) - d_x \Gamma(f)(x) \right) - \left(\sum_{v \sim x} f(v) (\Delta f(v) - \Delta f(x)) \right) \\ &= \frac{1}{2} \left(\sum_{u \sim v \sim x} (f(u) - f(v))^2 \right) - \frac{d_x}{2} \left(\sum_{v \sim x} f^2(v) \right) + \left(\sum_{v \sim x} f(v) \right) \left(\sum_{u \sim x} f(u) \right) \quad (25) \\ &\quad - \left(\sum_{u \sim v \sim x} f(v) (f(u) - f(v)) \right) \\ &= \left(\sum_{v \sim x} f(v) \right)^2 - \frac{d_x}{2} \left(\sum_{v \sim x} f^2(v) \right) + \left(\sum_{u \sim v \sim x} \frac{f^2(u) - 4f(u)f(v) + 3f^2(v)}{2} \right) \\ &= \left(\sum_{v \sim x} f(v) \right)^2 - \left(\sum_{v \sim x} \frac{d_x + d_v}{2} f^2(v) \right) + \frac{1}{2} \left(\sum_{u \sim v \sim x} (f(u) - 2f(v))^2 \right). \quad (26) \end{aligned}$$

We break the last term of Equation 26 into the cases that $u = x$, $u \sim x$ (in which case x, u, v are a 3-clique) and $d(x, u) = 2$. In the second case, we denote by $\Delta(x, v, u)$ the property that the unordered pair (u, v) satisfies $x \sim u \sim v \sim x$. Continuing the

above expansion, we find that

$$\begin{aligned}
2\Gamma_2(f) &= \frac{1}{2} \left(\sum_{\substack{w \sim v \sim x \\ d(x,w)=2}} (f(w) - 2f(v))^2 \right) + \left(\sum_{v \sim x} f(v) \right)^2 + \left(\sum_{v \sim x} \left(2 - \frac{d_x + d_v}{2} \right) f^2(v) \right) \\
&\quad + \left(\sum_{\Delta(x,v,u)} \frac{(f(v) - 2f(u))^2 + (f(u) - 2f(v))^2}{2} \right) \\
&= \frac{1}{2} \left(\sum_{\substack{w \sim v \sim x \\ d(x,w)=2}} (f(w) - 2f(v))^2 \right) + \left(\sum_{v \sim x} f(v) \right)^2 + \left(\sum_{v \sim x} \frac{4 - d_x - d_v}{2} f^2(v) \right) \\
&\quad + \sum_{\Delta(x,v,u)} \left[2(f(v) - f(u))^2 + \frac{1}{2} (f^2(v) + f^2(u)) \right]. \tag{27}
\end{aligned}$$

Notice that $\Gamma_2 f(x)$ and $\Gamma f(x)$ depend only on the values of f on vertices v with $d(v, x) \leq 2$. Recall that we are looking for a non-zero function f that minimizes $\frac{\Gamma_2 f(x)}{\Gamma f(x)}$. Using Equation (27) we can partially characterize that function, so that for the minimizing function f we can express $\Gamma_2 f(x)$ and $\Gamma f(x)$ in terms of the values f takes only on x and the neighbors of x .

Theorem 4.2.1. *Let $f : V \rightarrow \mathbb{R}$ be a function with $\Gamma f(x) \neq 0$ that minimizes $\frac{\Gamma_2 f(x)}{\Gamma f(x)}$. As before, we may assume $f(x) = 0$. If $d(x, w) = 2$,*

$$f(w) = 2 \cdot \frac{1}{r(w)} \sum_{x \sim v \sim w} f(v), \tag{28}$$

where $r(w)$ is the number of common neighbors of w and x .

Proof. Observe that $f(w)$ only appears in the first term of Equation 27, so our objective is to choose the value of $f(w)$ that minimizes

$$\sum_{v: x \sim v \sim w} (f(w) - 2f(v))^2$$

in terms of the values $f(v)$. Elementary calculus reveals the result. \square

Observe that this result (along with the assumption $f(x) = 0$) allows us to express $\Gamma_2 f(x)$ as a quadratic form M indexed by the neighbors of x . In this setting, minimizing $K = \Gamma_2 f(x)/\Gamma f(x)$ is equivalent to the well-understood problem of minimizing

$K = \langle f, Mf \rangle / \frac{1}{2} \langle f, f \rangle$, representing f as a vector indexed by the neighbors of x . This method gives the simplest known method of deciding whether or not $CD(K, \infty)$ is satisfied for a single graph. In the rest of this chapter, we will compute the curvature for larger families of graphs.

We require two theorems that will help with computation. First, an upper bound on the Ricci curvature depending on the frequency with which triangles appear in the graph:

Theorem 4.2.2. *Let $G = (V, E)$ be a graph. If $e \in E$, let $t(e)$ denote the number of triangles containing e . Define $T := \max_e t(e)$. Then $\text{Ric}(G) \leq 2 + \frac{T}{2}$.*

Proof. Let $x \in V$ be any vertex with the minimum degree d , and consider the distance (to x) function $f(v) = \text{dist}(v, x)$. It is simple to calculate that

$$2\Gamma_2(f)(x) \stackrel{(27)}{=} d^2 + \sum_{v \sim x} \left(2 - \frac{d + d_v}{2} \right) + \sum_{\Delta(x, v, u)} 1 \leq 2d + \frac{dT}{2},$$

by observing that

$$|\Delta(x, v, u)| = \frac{1}{2} \sum_{v \sim x} t(x, v) \leq \frac{dT}{2}$$

and that $\Gamma(f)(x) = \frac{1}{2}d$. Any value of $K > 2 + \frac{T}{2}$ will not satisfy Equation 24 for the function f at vertex x , thus $\text{Ric}(G) \leq 2 + \frac{T}{2}$. \square

The other theorem is a result of Schmuckenschläger [60].

Theorem 4.2.3 (Tensorization). *Let G and H be graphs. Then*

$$\text{Ric}(G \square H) = \min\{\text{Ric}(G), \text{Ric}(H)\}.$$

We omit the proof which is straightforward.

4.3 Examples of computing CD curvature

In this section we provide bounds on the CD curvature for several graphs of general interest.

First, we prove a result which is used to compute the curvature for many interesting graphs. Second, we prove a bound on curvature for abelian Cayley graphs. Finally, we use other methods to compute the curvature for some specific graphs of interest.

4.3.1 K_3 and $K_{2,3}$ -free graphs

Definition 4.3.1. If $x \in V$ and $y, z \sim x$, we say that y and z are *linked* if there is a vertex $w \neq x$ so that $y \sim w \sim z$. We write $y \approx z$ if y and z are linked and $y \not\approx z$ if not.

Definition 4.3.2. If $x, y \in V$ and $x \sim y$, the non-linking number $N_x(y)$ is the number of other neighbors of x (i.e., besides y) that are not linked to y :

$$N_x(y) = |\{w \sim x : w \neq y, w \not\approx y\}|.$$

Recall the common convention that the complete graph on n vertices is denoted K_n and the complete bipartite graph with m vertices in one part and n in the other is $K_{m,n}$.

Theorem 4.3.1. *Let G be a d -regular graph with no subgraph isomorphic to either K_3 or $K_{2,3}$. Let $N = \max_{y \sim x} N_x(y)$.*

(i) *G satisfies $CD(\rho, \infty)$ at x if $\rho \leq \max(2 - 2N, 2 - d)$.*

(ii) *If $N > 0$, G fails $CD(\rho, \infty)$ at x if $\rho > \frac{N - N^3}{N + N^2}$.*

Proof. Before proving either part of the statement, we simplify the expression $2\Gamma_2$: because G has no subgraph isomorphic to $K_{2,3}$ or K_3 , if y, z are linked neighbors of x , there is a unique vertex w_{yz} with $d(w_{yz}, x) = 2$ and $y \sim w_{yz} \sim z$. It is also not possible that w_{yz} is adjacent to any other neighbors of x - in that case there would be a $K_{2,3}$ subgraph. Because we are interested in calculating $\text{Ric}(G)$ at x , we only need to work with functions $f : V(G) \rightarrow \mathbb{R}$ satisfying the result of Theorem 4.2.1. This means that we will take $f(w_{yz}) = f(y) + f(z)$, and if w with $w \sim y \sim x$ for

some y (and $w \neq x$) is not a linking vertex, we set $f(w) = 2f(y)$. After this step, we compute the first term of the expression for $2\Gamma_2$ found in Equation 27:

$$\begin{aligned}
& \frac{1}{2} \sum_{\substack{w \sim v \sim x \\ d(x,w)=2}} (f(w) - 2f(v))^2 \\
&= \frac{1}{2} \left(\sum_{\substack{y, z \sim x \\ y \approx z}} (f(y) - 2f(w_{yz}))^2 + (f(z) - 2f(w_{yz}))^2 \right) + \frac{1}{2} \left(\sum_{y \sim x} \sum_{\substack{w \sim y \sim x \\ w \not\approx x}} (f(w) - 2f(y))^2 \right) \\
&= \sum_{\substack{y, z \sim x \\ y \approx z}} (f(y) - f(z))^2
\end{aligned}$$

where the sum is taken over all *unordered* pairs y, z of linked neighbors of x .

(i) Substituting this expression into Equation 27, we find that

$$\begin{aligned}
2\Gamma_2 f(x) &= \sum_{\substack{y, z \sim x \\ y \approx z}} (f(y) - f(z))^2 + \left(\sum_{y \sim x} f(y) \right)^2 + (2-d) \sum_{y \sim x} f^2(y) \\
&= \sum_{y \sim x} \left((d-1 - N_x(y)) + 1 + (2-d) \right) f^2(y) + \sum_{y \not\approx z} 2f(y)f(z) \\
&\geq \sum_{y \sim x} (2 - N_x(y)) f^2(y) + - \sum_{y \not\approx z} (f^2(y) + f^2(z)) \\
&= \sum_{y \sim x} (2 - 2N_x(y)) f^2(y) \geq (2 - 2N) \sum_{y \sim x} f^2(y) = (2 - 2N) 2\Gamma f(x),
\end{aligned}$$

thus $\text{Ric}(G) \geq 2 - 2N$.

It is simple to see that $\text{Ric}(G) \geq 2 - d$:

$$2\Gamma_2 f(x) = \sum_{\substack{y, z \sim x \\ y \approx z}} (f(y) - f(z))^2 + \left(\sum_{y \sim x} f(y) \right)^2 + (2-d) \sum_{y \sim x} f^2(y) \geq (2-d) \sum_{y \sim x} f^2(y).$$

(ii) To obtain an upper bound on $\text{Ric}(G)$, we set a test-function f . We are only concerned with test-functions that minimize Γ_2 in the sense of Theorem 4.2.1. Again we write

$$2\Gamma_2 f(x) = \sum_{\substack{y, z \sim x \\ y \approx z}} (f(y) - f(z))^2 + \left(\sum_{y \sim x} f(y) \right)^2 + (2-d) \sum_{y \sim x} f^2(y).$$

Let y be a fixed neighbor of x . Define S_y to be the set of vertices $z \sim x$ with $z \neq y$ and $z \not\approx y$. Define $\overline{S_y}$ as the complement of S_y : the set of vertices $u \sim x$ with $u \approx y$.

$$\begin{aligned} 2\Gamma_2 f(x) &\leq \sum_{\substack{u,v \sim x \\ u,v \neq y}} (f(u) - f(v))^2 + \sum_{z \in \overline{S_y}} (f(y) - f(z))^2 + \left(\sum_{z \sim x} f(z) \right)^2 + (2-d) \sum_{z \sim x} f^2(z) \\ &= 2 \sum_{u \in \overline{S_y}} f^2(u) + \sum_{z \in S_y} f^2(z) + (2 - N_x(y))f^2(y) + \sum_{z \in S_y} 2f(y)f(z). \end{aligned}$$

We take the following test-function: $f = 0$ on $\overline{S_y}$, $f = -1$ on S_y and $f(y) = N_x(y)$. Set all other values as described in Theorem 4.2.1, so that we have $2\Gamma_2 f(x) \leq N_x(y) - N_x(y)^3$ and $2\Gamma f(x) = N_x(y) + N_x(y)^2$. Letting y be the vertex that maximizes $N_x(y)$ proves the theorem. \square

A corollary is the following result:

Corollary 4.3.2. *Let G be d -regular and have no subgraph isomorphic to either K_3 or $K_{2,3}$. Let $N = \max_{y \sim x} N_x(y)$.*

(i) *If $N = 0$, then G satisfies $CD(\rho, \infty)$ at x iff $\rho \leq 2$ (G is positively curved at x with $Ric(G) = 2$.)*

(ii) *If $N = 1$, then G satisfies $CD(\rho, \infty)$ at x iff $\rho \leq 0$. (G is flat at x with $Ric(G) = 0$.)*

(iii) *If $N \geq 2$, then G does not satisfy $CD(0, \infty)$ at x . (G is negatively curved at x with $Ric(G) < 0$.)*

Proof. (i) By Theorem 4.2.2, any triangle-free graph fails $CD(\rho, \infty)$ if $\rho > 2$. By Theorem 4.3.1 (i), G satisfies $CD(\rho, \infty)$ if $\rho \leq 2$.

(ii) By Theorem 4.3.1 part (i), G satisfies $CD(\rho, \infty)$ if $\rho \leq 0$. By part (ii) of the theorem, G fails $CD(\rho, \infty)$ if $\rho > 0$.

(iii) By Theorem 4.3.1 part (ii), G fails $CD(\rho, \infty)$ if $\rho > \frac{N-N^3}{N+N^2}$. Observe that $0 > \frac{N-N^3}{N+N^2}$ when $N \geq 2$. \square

Several graphs of general interest satisfy the hypotheses of Corollary 4.3.2.

- The hypercube Ω_n ($n \geq 1$), where two points are adjacent if they differ in exactly one coordinate. Ω_n has $N = 0$.
- The discrete torus C_d^n , where C_d is a cycle of length $d \geq 5$ and $n \geq 1$. C_d^n has $N = 1$.
- The discrete lattice \mathbb{Z}^n where $n \geq 1$. \mathbb{Z}^n has $N = 1$.
- The infinite d -regular tree T_d if $d \geq 2$. T_d has $N = d - 1$.
- Any d -regular graph with girth ≥ 5 if $d \geq 2$ has $N = d - 1$.
- A common example of a Catalan structure is the triangulations of a labelled n -gon. If $n \geq 6$, consider the graph of such triangulations, with edges representing the action of flipping one interior arc (changing only the two incident triangles.) Starting with triangulation x , let y and z be the triangulations obtained by flipping edges e and f of x . y and z will be unlinked if y and z are incident to the same triangle. When $n \geq 6$, x either contains a triangle with three flippable edges (each corresponding to a vertex y with $N_y(x) \geq 2$) or two adjacent triangles with two flippable edges each (so the joining edge of those triangles has $N_y(x) \geq 2$). In either case $N \geq 2$.
- The Cayley graph of the symmetric group S_n with generating set $\{(i, i + 1) : 1 \leq i \leq n\}$. If $n \geq 4$ and x is the identity, $(2, 3)$ is unlinked to $(1, 2)$ and $(3, 4)$ as a neighbor of x , so $N \geq 2$. (In fact, it can be shown that $N = 2$.)
- Given an underlying graph $H = ([n], F)$, the interchange process labels the vertices of H and at each step, the allowed moves are to exchange the labels of a pair of adjacent vertices. Let G be the graph of possible states with an edge between two states if we can move from one state to the other in a single step.

More formally, G is the Cayley graph of the subgroup of S_n with generating set $A = \{(i, j) : \{i, j\} \in F\}$.

G is always K_3 -free, and will be $K_{2,3}$ -free if and only if H is triangle-free. If $x \in V(G)$ and $a, b \in A$, $ax \approx bx$ iff a and b are not incident to the same vertex as edges of H . If H is triangle-free, N will be the maximum degree in the line graph of H .

So we can compute the CD curvature of the listed graphs:

Theorem 4.3.3. • $Ric(\Omega_n) = 2$ if $n \geq 1$.

- $Ric(C_d^n) = 0$ if $n \geq 1$ and $d \geq 5$.
- $Ric(\mathbb{Z}^n) = 0$ if $n \geq 1$.
- $Ric(T_d) < 0$ if $d \geq 3$.
- Any d -regular graph of girth ≥ 5 and $d \geq 3$ has $Ric(G) < 0$.
- The graph of triangulations of the n -gon has $Ric(G) < 0$ if $n \geq 6$.
- The Cayley graph of S_n with adjacent transpositions has $Ric < 0$.
- The interchange process on a triangle-free graph $H = ([n], F)$ has $Ric = 2$ if H is a (not necessarily perfect) matching, $Ric = 0$ if no component of H contains more than 3 vertices, and $Ric < 0$ otherwise.

The proof is to apply Corollary 4.3.2 to the graphs described above.

We can partially remove the requirement in Theorem 4.3.1 that the graph have no subgraph isomorphic to $K_{2,3}$.

First we must extend the concept of two neighbors of x being linked.

Definition 4.3.3. Let $G = (V, E)$ be a triangle-free d -regular graph and $x \in V$. Let u, w be two neighbors of x . The *linkage* of u and w is calculated by summing over all vertices $z \neq x$ for which $u \sim z \sim w$:

$$l(u, w) = \sum_z \frac{1}{|\{y : x \sim y \sim z\}|}.$$

Remark. As discussed in the proof of Theorem 4.3.1, if G contains no subgraph isomorphic to $K_{2,3}$, then there can be at most one vertex z with $z \neq x, u \sim z \sim w$ which will have $\{y : x \sim y \sim z\} = \{u, w\}$. In this case $l(u, w) = 1/2$ and (as before) we say that u and w are linked.

We prove a result similar to the $N = 0$ case of Theorem 4.3.1.

Theorem 4.3.4. *Let G be a triangle-free graph, and $x \in V(G)$. If for every pair $\{u, w\}$ of neighbors of x , $l(u, w) \geq \frac{1}{2}$, then G satisfies $CD(\rho, \infty)$ at x iff $\rho \leq 2$.*

Proof. By Theorem 4.2.2 any triangle-free graph G fails $CD(\rho, \infty)$ if $\rho > 2$.

Recall that if f is the minimizer of $\Gamma_2 f(x)/\Gamma f(x)$ with $f(x) = 0$ and $d(x, z) = 2$, then

$$f(z) = \frac{2}{\#\{y : x \sim y \sim z\}} \sum_{y: x \sim y \sim z} f(y).$$

For such a function f , (along with the assumption of a triangle-free graph), straightforward algebraic manipulation reveals a form for $\Gamma_2 f$:

$$2\Gamma_2 f(x) = \sum_{y \sim x} (3 - d)f^2(y) + \sum_{w, y \sim x} 2f(y)f(w) + \sum_{w, y \sim x} 2l(w, y)(f(w) - f(y))^2. \quad (29)$$

Because $l \geq 1/2$ we can bound this equation:

$$\begin{aligned}
2\Gamma_2 f(x) &\geq \sum_{y \sim x} (3-d)f^2(y) + \sum_{w, y \sim x} 2f(y)f(w) + \sum_{w, y \sim x} (f(w) - f(y))^2 \\
&= \sum_{y \sim x} (3-d)f^2(y) + \sum_{w, y \sim x} 2f(y)f(w) + \sum_{y \sim x} (d-1)f^2(y) - \sum_{w, y \sim x} 2f(y)f(w) \\
&= 2 \sum_{y \sim x} f^2(y) = 4\Gamma f(x),
\end{aligned}$$

and therefore $CD(2, \infty)$ is satisfied. \square

This result allows us to compute the curvature for at least two interesting graphs.

Theorem 4.3.5. • *The Cayley graph on the symmetric group S_n with generating set $\{(i, j) : 1 \leq i < j \leq n\}$ has $\text{Ric}(G) = 2$ if $n \geq 2$.*

• *The complete bipartite graph $K_{n,n}$ has $\text{Ric}(K_{n,n}) = 2$ if $n \geq 2$.*

Proof. Let G be the Cayley graph for the symmetric group S_n with an edge corresponding to any transposition of two points ($n \geq 2$). Let x be the identity. If i, j, k, m are distinct points, $l((ij), (km)) = \frac{1}{2}$ (with link $(ij)(km)$). The other case is that (ij) and (jk) share a common point. $l((ij), (jk)) = \frac{2}{3}$ with links $(ijk), (ikj)$, each of which is also adjacent to (ik) .

Now consider $K_{n,n}$ if $n \geq 2$. If y, z are two neighbors of x , y and z share $n-1$ links, each of which links n vertices, so $l(y, z) = \frac{n-1}{n}$.

Both graphs satisfy the conditions of Theorem 4.3.4, the result follows. \square

4.3.2 Finite abelian Cayley graphs

A finite abelian group is a product of cyclic groups. A misconception is that the curvature of the graph can therefore be found easily from the tensorization result of Theorem 4.2.3. In fact, a Cayley graph is determined by an underlying group *and* a generating set for that group. Here we show that a finitely generated abelian group with *any* set of generators has positive Ricci curvature - not only with the generating set inherited from a decomposition into cyclic groups.

Theorem 4.3.6. *Let X be a finitely generated abelian group, and S a finite set of generators for X . Let G be the Cayley graph corresponding to X and S . Then $\text{Ric}(G) \geq 0$.*

Recall that the Cayley graph of a group G with respect to a given set S which generates G is the graph whose vertices are the elements of G and whose edges are $\{(g, gs)\}_{g \in G, s \in S}$. Since we are interested in undirected graphs, S should be symmetric i.e. $s \in S \Rightarrow s^{-1} \in S$.

Proof. Without loss of generality, we may set x to be the identity element of X . Denote the degree of every vertex by d . As usual, let $f : G \rightarrow \mathbb{R}$ with $f(x) = 0$.

For this calculation, we prefer not to distinguish between u according to their distance from x . Starting the calculation from Equation 26 we see

$$2\Gamma_2(f)(x) = d \sum_{v \sim x} f^2(v) + \left(\sum_{v \sim x} f(v) \right)^2 + \sum_{v \sim x} \sum_{u \sim v} \left(\frac{f^2(u)}{2} - 2f(u)f(v) \right). \quad (30)$$

Because x is the identity, we observe that if $u \sim v \sim x$, there is a unique $w \sim x$ so that $u = vw$. We can express the last term of Equation 30 as

$$\begin{aligned} \sum_{v \sim x} \sum_{u \sim v} \left(\frac{f^2(u)}{2} - 2f(u)f(v) \right) &= \sum_{v \sim x} \sum_{w \sim x} \left(\frac{f^2(vw)}{2} - 2f(vw)f(v) \right) \\ &= \sum_{v \sim x} \left(\frac{f^2(v^2)}{2} - 2f(v^2)f(v) \right) + \sum_{\substack{v, w \sim x \\ v \neq w}} (f^2(vw) - 2f(vw)(f(v) + f(w))) \\ &\geq -2 \sum_{v \sim x} f^2(v) - \sum_{\substack{v, w \sim x \\ v \neq w}} (f(v) + f(w))^2 = (-d - 1) \sum_{v \sim x} f^2(v) - 2 \sum_{\substack{v, w \sim x \\ v \neq w}} f(v)f(w). \end{aligned}$$

In the last passage we used the elementary inequalities $a^2/2 - 2ab \geq -2b^2$ and $a^2 - 2ab \geq -b^2$.

Plugging this bound into Equation 30, we find that

$$2\Gamma_2(f)(x) \geq \left(\sum_{v \sim x} f(v) \right)^2 - \sum_{v \sim x} f^2(v) - 2 \sum_{\substack{v, w \sim x \\ v \neq w}} f(v)f(w) = 0.$$

This completes the proof. □

Observe that the assumption that the group is abelian is necessary. For example, in Theorem 4.3.3 we calculated that the curvature of the symmetric group with adjacent transpositions is negative when $n \geq 4$.

4.3.3 The complete graph K_n

In the following, we compute the curvature of the complete graph. Combining the curvature of K_2 with the tensorization result of [48], this provides another proof of the fact that the hypercube has curvature 2.

Theorem 4.3.7. $Ric(K_n) = 1 + \frac{n}{2}$ if $n \geq 2$.

Proof. For the complete graph on n vertices, we have, for every $x \in V$ and every $f : V \rightarrow \mathbb{R}$ such that $f(x) = 0$, from Equatio 27,

$$2\Gamma_2(f)(x) = \left(\sum_{v \sim x} f(v) \right)^2 + (3-n) \sum_{v \sim x} f^2(v) + \sum_{\substack{u, v \sim x \\ u \neq v}} \left(2(f(v) - f(u))^2 + \frac{1}{2} (f(u)^2 + f(v)^2) \right).$$

Expanding the above gives

$$\begin{aligned} & \sum_{v \sim x} f^2(v) + \sum_{\substack{u, v \sim x \\ u \neq v}} 2f(u)f(v) + (3-n) \sum_{v \sim x} f^2(v) + \frac{5}{2} \sum_{\substack{u, v \sim x \\ u \neq v}} (f^2(v) + f^2(u)) - \sum_{\substack{u, v \sim x \\ u \neq v}} 4f(u)f(v) \\ &= (4-n) \sum_{v \sim x} f^2(v) + \frac{5}{2}(n-2) \sum_{v \sim x} f^2(v) - 2 \sum_{\substack{u, v \sim x \\ u \neq v}} f(u)f(v) \\ &= \left(\frac{3n}{2} - 1 \right) \sum_{v \sim x} f^2(v) - 2 \sum_{\substack{u, v \sim x \\ u \neq v}} f(u)f(v) = \frac{3n}{2} \sum_{v \sim x} f^2(v) - \left(\sum_{v \sim x} f(v) \right)^2. \end{aligned}$$

By the Cauchy-Schwarz inequality,

$$\left(\sum_{v \sim x} f(v) \right)^2 \leq |\{v : v \sim x\}| \sum_{v \sim x} f^2(v) = (n-1) \sum_{v \sim x} f^2(v),$$

so we see that

$$\frac{3n}{2} \sum_{v \sim x} f^2(v) - \left(\sum_{v \sim x} f(v) \right)^2 \geq \left(1 + \frac{n}{2} \right) \sum_{v \sim x} f^2(v).$$

Thus $Ric \geq 1 + \frac{n}{2}$, by Theorem 4.2.2, we conclude that $Ric = 1 + \frac{n}{2}$. \square

4.3.4 Slices of the hypercube

We can consider the hypercube Ω_n as the set of 0 – 1 sequences of length n . In this setting the k -slice is the set of sequences with exactly k ones. We first consider the case of the k -slice with edges representing transposition one 0 with one 1, for example, $10010 \sim 00110$.

For some fixed value k with $1 \leq k < n$, let $G = (V, E)$ be the graph with $V = \{x \in \{0, 1\}^n : \sum_i x_i = k\}$, and $x \sim y$ whenever $|\text{supp}(x - y)| = 2$.

Theorem 4.3.8. *This graph has curvature $\text{Ric} = 1 + \frac{n}{2}$.*

Proof. Let $x \in V$. Define $s_{ij}x$ to be the vertex obtained by exchanging coordinates i and j in x . A vertex u with $d(x, u) = 2$ will be $u = s_{ij}s_{lm}x$ for some distinct coordinates i, j, l, m with $x_i = x_l = 1$, $x_j = x_m = 0$. Vertices v with $x \sim v \sim u$ are $s_{ij}x, s_{im}x, s_{lj}x, s_{lm}x$. Observe that

$$\sum_{v: x \sim v \sim u} (f(u) - 2f(v))^2 \geq 2(f(s_{ij}x) - f(s_{lm}x))^2 + 2(f(s_{im}x) - f(s_{lj}x))^2.$$

Summing over all vertices u with $d(x, u) = 2$ gives

$$\frac{1}{2} \sum_{\substack{x \sim v \sim u \\ d(x, u) = 2}} (f(u) - 2f(v))^2 \geq \sum_{\substack{v, w \sim x \\ \Delta(x, v, w)}} (f(v) - f(w))^2,$$

as for each pair $v, w \sim x$ with $v \not\sim w$, there is exactly one u with $v, w \sim u$ and $d(x, u) = 2$. (Here we use the notation $\Delta(x, v, w)$ to denote the set of unordered pairs (v, w) of distinct neighbors of x for which $v \not\sim w$.)

Also notice that any $v \sim x$ has a fixed number of triangles containing x and v : $t(\{x, v\}) = n - 2$: if $v = s_{ij}x$, the vertices that make a triangle with x and v are $s_{lj}x$ when $l \neq i$ and $x_l = x_i$, and $s_{im}x$ when $m \neq j$ and $x_m = x_j$.

Now we may compute

$$\begin{aligned}
& 2\Gamma_2(f)(x) \\
& \geq \sum_{\substack{v, w \sim x \\ \Delta(xvw)}} (f(v) - f(w))^2 + \left(\sum_{v \sim x} f(v) \right)^2 + \left(2 - d + \frac{n-2}{2} \right) \sum_{v \sim x} f(v)^2 \\
& \quad + 2 \sum_{\Delta(vwx)} (f(v) - f(w))^2 \\
& \geq \sum_{v, w \sim x} (f(v) - f(w))^2 + \left(\sum_{v \sim x} f(v) \right)^2 + \left(1 - d + \frac{n}{2} \right) \sum_{v \sim x} f(v)^2 \\
& = (d-1) \sum_{v \sim x} f(v)^2 - 2 \sum_{v, w \sim x} f(v)f(w) + \sum_{v \sim x} f(v)^2 + 2 \sum_{v, w \sim x} f(v)f(w) \\
& \quad + \left(1 - d + \frac{n}{2} \right) \sum_{v \sim x} f(v)^2 \\
& = \left(1 + \frac{n}{2} \right) \sum_{v \sim x} f(v)^2.
\end{aligned}$$

Thus $\text{Ric}(G) \geq 1 + \frac{n}{2}$. Together with Theorem 4.2.2 we get that $\text{Ric} = 1 + \frac{n}{2}$. \square

We now consider another graph on the middle slice of the hypercube, but now only allowing the transposition of adjacent entries.

We now consider G with $V = \{x \in \{-1, 1\}^{2n} : \sum_i x_i = 0\}$, where $x \sim y$ if and only if $\text{supp}(x - y)$ consists of 2 consecutive elements. Alternately, V is the set of paths in \mathbb{Z}^2 that move from $(0, 0)$ to $(2n, 0)$ with steps of $(+1, +1)$ and $(+1, -1)$, and paths x and y are neighbors if y can be achieved by transposing an adjacent $(-1, +1)$ and $(+1, -1)$ in x .

Theorem 4.3.9. $\text{Ric}(G) \geq -1$. Further, $\lim_{n \rightarrow \infty} \text{Ric}(G) = -1$.

Proof. Let $x \in V$. Let $I(x) = \{i \in \{1, \dots, 2n-1\} : x_i \neq x_{i+1}\}$, so $i \in I$ if and only if we are allowed to switch segments i and $i+1$. If $i \in I(x)$, denote by $a_i x$ the vertex obtained by making this switch. Observe $|I(x)| = \deg(x)$.

The neighbors of $a_i x$ are: $a_i(a_i x) = x$, $a_j(a_i x)$ for any $j \in I(x)$ with $|i - j| > 1$, and $a_j(a_i x)$ for any $j \notin I(x)$ with $|i - j| = 1$ and $j \neq 0, 2n$. We calculate that

$$\deg(a_i x) = \deg(x) + 2 - 2|\{j \in I(x) : |i - j| = 1\}| - 1_{i=1} - 1_{2n-1}.$$

We observe that a neighbor of the form $a_j(a_i x)$ if $j \in I(x)$ and $|i - j| > 1$ will be identical to $a_i(a_j x)$, and have $d(x, a_j a_i x) = 2$.

Now, for any function f ,

$$\begin{aligned} & \frac{1}{2} \sum_{\substack{u \sim v \sim x \\ d(x, u) = 2}} (f(u) - 2f(v))^2 \\ & \geq \frac{1}{2} \sum_{\substack{i, j \in I \\ |i - j| > 1}} (f(a_i a_j x) - 2f(a_i x))^2 + (f(a_i a_j x) - 2f(a_j x))^2 \\ & \geq \sum_{\substack{i, j \in I \\ |i - j| > 1}} (f(a_i x) - f(a_j x))^2 \\ & = \sum_{i \in I(x)} |\{j \in I(x) : |j - i| > 1\}| f^2(a_i x) - 2 \sum_{\substack{i, j \in I \\ |i - j| > 1}} f(a_i x) - f(a_j x). \end{aligned}$$

Observe that G is triangle-free. We have that

$$\begin{aligned} & 2\Gamma_2(f)(x) \\ & \geq \sum_{i \in I(x)} |\{j \in I(x) : |j - i| > 1\}| f^2(a_i x) - 2 \sum_{\substack{i, j \in I \\ |i - j| > 1}} f(a_i x) - f(a_j x) \\ & + \sum_{i \in I(x)} f^2(a_i x) + 2 \sum_{i, j \in I} f(a_i x) f(a_j x) \\ & + \sum_{i \in I(x)} \left(2 - \frac{2 \cdot \deg(x) + 2 - 2|\{j \in I(x) : |i - j| = 1\}| - 1_{i=1} - 1_{2n-1}}{2} \right) f^2(a_i x) \\ & \geq \sum_{i \in I(x)} (\#\{j \in I(x) : i \neq j\} + 2 - \deg(x)) f^2(a_i x) + 2 \sum_{\substack{i, j \in I \\ |i - j| = 1}} f(a_i x) f(a_j x) \\ & = \sum_{i \in I(x)} f^2(a_i x) + 2 \sum_{\substack{i, j \in I \\ |i - j| = 1}} f(a_i x) f(a_j x) \\ & > - \sum_{i \in I(x)} f(a_i x) + \sum_{\substack{i, j \in I \\ |i - j| = 1}} (f(a_i x) + f(a_j x))^2 \geq -2\Gamma(f)(x). \end{aligned}$$

So $\text{Ric}(G) > -1$, where we ignore a slight dependence on n in the lower order term.

Define a function with $f(+1, -1, +1, -1, \dots) = 0$ and $f(a_i x) = f(x) - x_i$, that is, if the switch lowers the path, f decreases by 1; a switch that raises the path will increase f by 1.

Using this function f and $x = (+1, -1, +1, -1, \dots)$, we find that $\text{Ric} \rightarrow -1$ as $n \rightarrow \infty$. \square

We now calculate the curvature for the subgraph G_+ that is induced on the Dyck paths, i.e., those paths that are always on or above the x -axis. Alternately, sequences in $\{\pm 1\}^{2n}$ with $\sum_{i=1}^{2n} x_i = 0$ and $\sum_{i=1}^j x_i \geq 0$ for all $j = 0, \dots, 2n$. As an aside, it is well-known that the number of Dyck paths is the Catalan number C_n .

Corollary 4.3.10. *For this subgraph G_+ , $\text{Ric}(G_+) \geq -1$. Further, $\lim_{n \rightarrow \infty} \text{Ric}(G_+) = -1$.*

Proof sketch. Let $x \in V$, and let

$$I(x) = \{i \in [2n-1] : \text{a possible move is to transpose } x_i, x_{i+1}\}.$$

If $i \in I$, let $a_i x$ be the sequence obtained by transposing x_i, x_{i+1} .

Observe that $\deg(a_i x) \leq \deg(x) + 2 - 2|\{j \in I(x) : |i - j| = 1\}| - 1_{i=1} - 1_{2n-1}$.

Using the same analysis as in the unrestricted problem, we may conclude that

$$2\Gamma_2(f)(x) \geq -2\Gamma(f)(x).$$

A similar test-function as above will prove that $\text{Ric} \leq -1 + o(1)$. We may use the same function f , and take x identical to the above example but with the first -1 and last $+1$ transposed. This will give a similar upper bound on $\text{Ric}(G_+)$. To see why, observe that the neighbors and second-neighbors of x in the unrestricted graph are all Dyck paths, so the curvature at x will be unchanged from the original. \square

4.4 Calculation of Ollivier curvature

First we recall some definitions from Chapter 3. The Ollivier curvature $\kappa(x, y)$ between two vertices x and y is defined by

$$\kappa(x, y) = 1 - \frac{W_1(\mu_x, \mu_y)}{d(x, y)},$$

where W_1 is the 1-Wasserstein metric: if μ, ν are probability distributions on X ,

$$W_1(\mu, \nu) = \inf_m \int_{X \times X} d(x, y) dm(x, y),$$

with the minimum taken over all probability measures m on $X \times X$ satisfying the equations

$$\int_X m(x, y) d\nu(y) = \mu(x) \text{ and } \int_X m(x, y) d\mu(x) = \nu(y). \quad (31)$$

Alternately, we can optimize the dual problem with a *Kantorovich potential* $f \in \text{Lip}(1)$.

$$W_1(\mu, \nu) = \sup_{f \in \text{Lip}(1)} \int_X f d\nu - \int_X f d\mu. \quad (32)$$

The probability measure μ_x on V is defined

$$\mu_x(v) = \begin{cases} \frac{1}{2} & \text{if } v = x \\ \frac{1}{2d_x} & \text{if } v \sim x \\ 0 & \text{otherwise,} \end{cases}$$

where d_x is the degree of x .

We now prove results with the same hypotheses as Theorem 4.3.1, again using $N_x(y)$ to represent the number of neighbors of x not linked to y and $N = \max_{x, y: x \sim y} N_x(y)$. We define another quantity $P_x(y)$. Let Z be the set of neighbors of x (besides y) not

linked to y and W the set of neighbors of y (besides x) not linked to x (as neighbors of y). Then

$$P_x(y) = |\{z \in Z : \exists w \in W, d(z, w) = 2\}|.$$

Observe that because $z \sim x \sim y \sim w$, $d(z, w) \leq 3$. If $d(z, w) = 1$, z would be linked to y by w , and if $d(z, w) = 0$ then $x, y, z = w$ make a triangle, which by assumption is impossible. We have defined $P_x(y)$ to distinguish between the case $d(z, w) = 3$ (as in a 6-cycle or larger) and $d(z, w) = 2$ (as in a 5-cycle).

We remark it is possible that $P_x(y) \neq P_y(x)$.

Theorem 4.4.1. *Let G be d -regular and have no subgraph isomorphic to either K_3 or $K_{2,3}$, and let x, y be vertices of G with $x \sim y$. Then*

$$\frac{1 - N_x(y)}{d} + \frac{P_x(y)}{2d} \geq \kappa(x, y) \geq \frac{1 - N_x(y)}{d}.$$

Also,

$$\frac{1 - \frac{1}{2}N_x(y)}{d} \geq \kappa(x, y) \geq \frac{1 - N_x(y)}{d}.$$

Note that in the special case $P_x(y) = 0$, both inequalities of the first statement are tight. Notably, this is the case for any bipartite graph.

Proof. The second statement follows from the first by observing $P_x(y) \leq N_x(y)$. It remains to prove the first statement.

Of the $d - 1$ neighbors of y other than x , $d - 1 - N_x(y)$ are linked to x and $N_x(y)$ are not. Because G does not have a $K_{2,3}$ subgraph, a neighbor of y may be adjacent to at most one neighbor of x (in case that vertex is linked to x). Likewise a neighbor of x may be adjacent to at most one neighbor of y .

First we give a transport plan to bound $W_1(\mu_x, \mu_y)$ from above: If $z \sim x$ ($z \neq y$) is adjacent to a vertex $w \sim y$ ($w \neq x$), move a mass $\frac{1}{2d}$ from z to w . For each of the $N_x(y)$ neighbors of x not linked to (or equal to) y , move the mass $\frac{1}{2d}$ on a path

through x and y to a neighbor of y that is not linked to x . Finally move a mass $\frac{d-1}{2d}$ from x to y . The total work for this plan is $1 + \frac{N_x(y)-1}{d}$.

Next we give a test-function to bound $W_1(\mu_x, \mu_y)$ from below:

- $f(x) = 0, f(y) = 1$.
- For the $N_x(y)$ neighbors of y not linked to x , set $f = 2$.
- For all $(d-1) - N_x(y)$ neighbors of y linked to x , set $f = 1$.
- For all $(d-1) - N_x(y)$ neighbors of x linked to y , set $f = 0$.
- For all $P_x(y)$ neighbors of x unlinked to y but at distance 2 from the set $f^{-1}(2)$, set $f = 0$.
- For all other $N_x(y) - P_x(y)$ neighbors of x , set $f = -1$.
- For other neighbors of $f^{-1}(2)$ set $f = 1$.
- Otherwise $f = 0$.

It is easy to see that f is a 1-Lipschitz function.

We can compute

$$\begin{aligned} \int f d(\mu_y - \mu_x) &= \left(1 + \frac{N_x(y) - 1}{2d}\right) - \frac{1 - (N_x(y) - P_x(y))}{2d} \\ &= 1 + \frac{2N_x(y) - P_x(y) - 2}{2d} \\ &= 1 + \frac{N_x(y) - 1}{d} - \frac{P_x(y)}{2d}. \end{aligned}$$

Combining we have

$$1 + \frac{N_x(y) - 1}{d} - \frac{P_x(y)}{2d} \leq W_1(\mu_x, \mu_y) \leq 1 + \frac{N_x(y) - 1}{d},$$

taking $\kappa(x, y) = 1 - W_1(\mu_x, \mu_y)$, we have the result. □

A result in a similar form to Corollary 4.3.2 follows immediately from the second statement of Theorem 4.4.1.

Corollary 4.4.2. *Let G be d -regular and have no subgraph isomorphic to either K_3 or $K_{2,3}$.*

- (i) *If $N_x(y) = 0$, then $\kappa(x, y) = \frac{1}{d}$.*
- (ii) *If $N_x(y) = 1$, then $\kappa(x, y) \geq 0$.*
- (iii) *If $N_x(y) \geq 2$, then $\kappa(x, y) \leq 0$.*

Remark. $\kappa > 0$ is satisfied under hypothesis (ii) for $G = C_5$, or graphs with similar structure. The issue is that if $z \sim x$ is the vertex unlinked to y and $u \sim y$ is the vertex unlinked to x , then $d(u, z)$ can be either 2 or 3. In the case $d(u, z) = 3$ we may set $f(z) = -1$ and $f = 0$ on any neighbor of z and, following the same proof method we obtain $\kappa(x, y) = 0$.

There is a related obstruction for hypothesis (iii): if the vertices $z \sim x$ that are unlinked to y and the vertices $u \sim y$ that are unlinked to x always have $d(u, z) = 3$, we may set $f(z) = -1$ and $f = 0$ on any neighbor of z and obtain $\kappa(x, y) < 0$. But in case $d(u, z) = 2$, we cannot set $f(z) = -1$ because $f(u) = 2$ and f must be 1-Lipschitz.

In the proof of Theorem 4.3.3, we computed $N_x(y)$ for many graphs. Now, combining that knowledge with the result of Theorem 4.4.1, we can calculate the Ollivier curvature for many graphs of general interest.

Theorem 4.4.3. • *If $x, y \in V(\Omega_n)$ with $x \sim y$, then $\kappa(x, y) = \frac{1}{d}$, with $N_x(y) = 0$.*

- *If $x, y \in V(C_d^n)$ with $x \sim y$, $d \geq 6$ and $n \geq 1$, then $\kappa(x, y) = 0$ with $N_x(y) = 1$.*
- *If $x, y \in V(Z^n)$ with $x \sim y$, then $\kappa(x, y) = 0$ with $N_x(y) = 1$.*
- *If $x, y \in T_d$ with $x \sim y$ and $d \geq 2$, then $\kappa(x, y) = \frac{2-d}{d}$ with $N_x(y) = d - 1$.*

- The Cayley graph of S_n ($n \geq 4$) with adjacent transpositions has $\kappa(x, y) \geq \frac{-1}{d}$ if $x \sim y$, with $N_x(y) \leq 2$.

Proof. In the proof of 4.3.3 we calculated $N_x(y)$ for each of these graphs. Because all of these graphs have $P_x(y) = 0$, the inequalities in the first statement of Theorem 4.4.1 are both tight, so we have $\kappa(x, y) = \frac{1-N_x(y)}{d}$ when $x \sim y$. \square

4.5 Comparison of discrete curvature

We now combine Corollaries 4.3.2 and 4.4.2 to show a relationship between the CD curvature and Ollivier curvature.

Theorem 4.5.1. *Let G be d -regular and have no subgraph isomorphic to either K_3 or $K_{2,3}$. Let $x \in V$.*

- (a) *If $\text{Ric}(G) > 0$ at x , then $\kappa(x, y) > 0$ for all $y \sim x$.*
- (b) *If $\text{Ric}(G) \geq 0$ at x , then $\kappa(x, y) \geq 0$ at x .*
- (c) *If $\text{Ric}(G) < 0$ at x , then there is a vertex $y \sim x$ for which $\kappa(x, y) \leq 0$.*
- (d) *If $\kappa(x, y) > 0$ for all $y \sim x$, then $\text{Ric}(G) \geq 0$ at x .*
- (e) *If $\kappa(x, y) \leq 0$ for some $y \sim x$, then $\text{Ric}(G) \leq 0$ at x .*
- (f) *If $\kappa(x, y) < 0$, then $\text{Ric}(G) < 0$ at x .*

Proof. Observe that hypotheses (i),(ii), and (iii) of Corollaries 4.3.2 and 4.4.2 are identical.

- Hypothesis (a) corresponds to hypothesis (i) of Corollary 4.3.2. In this case, hypothesis (i) of Corollary 4.4.2 is satisfied and the result follows.
- Hypothesis (b) corresponds to hypothesis (i) or (ii) of Corollary 4.3.2. In this case, hypothesis (i) or (ii) of Corollary 4.4.2 is satisfied and the result follows.
- Hypothesis (c) corresponds to hypothesis (iii) of Corollary 4.3.2. In this case, hypothesis (iii) of Corollary 4.4.2 is satisfied, this completes the proof.

- Hypothesis (d) corresponds to hypothesis (i) or (ii) of Corollary 4.4.2. In this case, hypothesis (i) or (ii) of Corollary 4.3.2 is satisfied and the result follows.
- Hypothesis (e) corresponds to hypothesis (ii) or (iii) of Corollary 4.4.2. In this case, hypothesis (ii) or (iii) of Corollary 4.3.2 is satisfied and the result follows.
- Hypothesis (f) corresponds to hypothesis (iii) of Corollary 4.4.2. In this case, hypothesis (iii) of Corollary 4.3.2 is satisfied and the result follows.

□

Remark. For the Erbar-Maas curvature there exist similar techniques for evaluating the curvature by characterizing the extent to which the graph locally resembles Ω_n ; i.e., the case $N = 0$. These methods have been developed in [32, 29, 33].

4.6 Other Results

We now present some other results that follow from the theorems in this chapter.

4.6.1 Zig-zag product

The zig-zag product is defined [58] for graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$, where

- G_1 is regular with degree $d = |V_2|$, and for each $a \in V_1$ the incident edges are indexed by V_2 - so that if $a \in V_1, x \in V_2$ we can write $a[x]$ for the neighbor of a that is reached by walking across the edge incident to a labelled x .
- G_2 is D -regular.
- The vertex set of the zig-zag product is $G_1 \times G_2$.
- If $x \sim y \sim z$ is a 2-walk in G_2 , then $(a, x) \sim (a[y], z)$.

The zig-zag product is a D^2 -regular graph that inherits its expansion properties from G_1 , which may have much larger degree. For this reason the zig-zag product is

useful in generating expander graphs of bounded degree. The question arises whether the zig-zag product inherits the curvature properties from G_1 or G_2 . In general this is not the case, we give an example of such a graph:

If G_1 and G_2 are both abelian Cayley graphs, it is not in general true that the zig-zag product will have non-negative curvature, even though both G_1 and G_2 do have non-negative curvature. As a simple example of this, take $G_1 = \Omega_d$ and $G_2 = Z_d$ with vertices $1 \dots, d$ labelled in order around the cycle, and $x \in V_1$ with $y = x[2]$, $z = x[d]$, then

- $(x, 1)$ has four neighbors - $(y, 1), (y, 3), (z, 1), (z, d - 1)$.
- $(y, 1)$ has neighbors $(x, 1), (x, 3), (y[d], 1), (y[d], d - 1)$.
- $(y, 3)$ has neighbors $(x, 1), (x, 3), (y[4], 3), (y[4], 5)$.
- $(z, 1)$ has neighbors $(x, 1), (x, d - 1), (z[2], 1), (z[2], 3)$.
- $(z, d - 1)$ has neighbors $(x, 1), (x, d - 1), (z[d - 2], d - 1), (z[d - 2], d - 3)$.

G is K_3 and $K_{2,3}$ -free, we examine which pairs of neighbors of $(x, 1)$ are not linked. We see that $(y, 3) \not\approx (z, 1), (y, 3) \not\approx (z, d - 1), (y, 1) \not\approx (z, d - 1)$. As such, G satisfies hypothesis (iii) of Corollary 4.3.2.

So G fails $CD(0, \infty)$, and simple calculation gives $\kappa((x, 1), (y, 3)) = -\frac{1}{4}$.

CHAPTER V

VOLUME GROWTH AND BUSER'S INEQUALITY

5.1 Overview

Cheeger-type inequalities for a d -regular graph relate Cheeger-type isoperimetric constants to the spectral gap of the Laplacian. These relate the algebraic and geometric expansion properties of the graph. For the outer vertex boundary (defined below), the inequalities are

$$\frac{(\sqrt{1+h_{out}}-1)^2}{2d} \leq \lambda_2 \leq h_{out}. \quad (33)$$

A problem of interest is to decide which graphs have the upper inequality tight ($\lambda_2 \approx h_{out}$) and which have the lower bound $\lambda_2 \approx h_{out}^2$ tight.

For the edge isoperimetric constant h , we prove in Chapter 2 a version of Buser's inequality, which states that under the condition of non-negative discrete Ricci curvature, the lower Cheeger inequality is tight. The proof method relies on decomposing a candidate Cheeger-optimizing vertex set as a linear sum of eigenfunctions of the Laplacian, and analyzing the behavior of those functions under the heat flow operator P_t .

In this chapter we explore an alternate proof method of Buser's inequality, which instead uses a bound on volume growth around the Cheeger-optimizing set. This is based on the original proof of P. Buser for the continuous setting [20]. Unlike the work in Chapter 2, we can extend the proof method to also bound the higher eigenvalues of the Laplacian. This follows the work of Agol and Benson [1, 11].

It is interesting to note that a bound on discrete curvature is only used in this chapter to find a volume growth function. If a bound on volume growth for graphs

exists under some other condition unrelated to curvature, that result can combine with Theorem 5.5.6 to bound the spectral gap. Similarly, if the volume growth is known for a specific graph, Theorem 5.5.6 allows us to bound the spectral gap. In particular we prove that any graph whose shells have volume bounded by a constant multiple of the volume of the isoperimetric cut-set have

$$\lambda_2 \leq ch_{out}^2,$$

so that the lower Cheeger inequality is tight up to a multiplicative factor $c = c(d)$ depending only the degree d .

The results in this chapter are from the author's joint work with B. Benson and P. Tetali [12].

5.2 *Notation*

In this chapter (contrary to Chapters 2 and 4) we use the *normalized* Laplacian Δ of a d -regular connected graph with the definition

$$\Delta f(x) = \frac{1}{d} \sum_{y \sim x} f(x) - f(y).$$

Δ is a positive semi-definite matrix whose spectral gap λ_2 is the value of the least non-zero eigenvalue.

In addition to the edge boundary of a set $A \subset V$, we define two different vertex boundaries:

The inner vertex boundary is $\partial_{in} A = \{x \in A : \exists y \sim x; y \notin A\}$.

The outer vertex boundary is $\partial_{out} A = \{y \notin A : \exists x \sim y; x \in A\}$.

Recall the Cheeger isoperimetric constant:

$$h(G) = \min \frac{\frac{1}{d} |\partial A|}{|A|}.$$

Following [15], we define analogues of this constant for the vertex boundaries:

$$h_{in}(G) = \min \frac{|\partial_{in} A|}{|A|}$$

$$h_{out}(G) = \min \frac{|\partial_{out} A|}{|A|}.$$

In all cases the minimization is over non-empty vertex sets with $|A| \leq \frac{1}{2}|V(G)|$. Observe that $h(G)$ is in this chapter the normalized Cheeger constant (in the same sense as the normalization of the Laplacian and spectral gap), and recall that because G is d -regular, each of these normalized quantities differs from their non-normalized counterparts by a factor of d .

Observe the trivial bounds $h(G) \leq h_{in}(G) \leq d \cdot h(G)$ and $h(G) \leq h_{out}(G) \leq d \cdot h(G)$.

There are a pair of Cheeger inequalities for each of these isoperimetric constants [15, 2].

$$h^2 \leq \lambda_2 \leq 2h$$

$$\frac{h_{in}^2}{4d} \leq \lambda_2 \leq h_{in}$$

$$\frac{(\sqrt{1 + h_{out}} - 1)^2}{2d} \leq \lambda_2 \leq h_{out}.$$

In this chapter we mostly work with h_{out} . Of course it is possible to use the relationship $h(G) \leq h_{out}(G) \leq d \cdot h(G)$ to re-write results containing h_{out} as results containing h .

5.3 Using volume growth to bound spectral gap in manifolds

Let M be an n -dimensional manifold and let A and B be a Cheeger-minimizing partition of M , so that their common boundary $\Sigma = \partial A = \partial B$ satisfies

$$h(M) = \frac{\text{Vol}(\Sigma)}{\min(\text{Vol}(A), \text{Vol}(B))}.$$

The minimax principle tells us that $\lambda_2(M) \leq \max \{ \lambda_1(A), \lambda_1(B) \}$ where eigenfunctions f of A (similarly B) satisfy the Dirichlet boundary conditions

$$\begin{cases} \Delta f = \mu f \text{ on } A \\ f(\Sigma) = 0. \end{cases}$$

Here, $\lambda_1(A)$ (similarly $\lambda_1(B)$) is the least non-zero value μ for which an eigenfunction exists.

We may assume that $\lambda_1(A) \geq \lambda_1(B)$.

The Rayleigh principle [41] tells us that $\lambda_1(A)$ of a manifold is achieved by minimizing the Rayleigh quotient $\int_A |\nabla f|^2 / \int_A f^2$ over functions satisfying the boundary condition $f(\Sigma) = 0$.

P. Buser sets a test-function for the Rayleigh quotient that depends linearly on the distance from Σ : $f(p) = \begin{cases} d(p, \Sigma) & \text{if } d(p, \Sigma) \leq t \\ t & \text{if } d(p, \Sigma) \geq t. \end{cases}$

Taking $A(t) = \{p \in A : d(p, \Sigma) \leq t\}$, Buser observes that

$$\begin{aligned} \int_A |\nabla f|^2 &\leq \int_{A(t)} 1 = \text{Vol}(A(t)) \text{ and} \\ \int_A f^2 &\geq \int_{A-A(t)} t^2 = t^2 (\text{Vol}(A) - \text{Vol}(A(t))). \end{aligned}$$

For any $t > 0$ satisfying $\text{Vol}(A) > \text{Vol}(A(t))$, we see that

$$\lambda_2(M) \leq \frac{\text{Vol}(A(t))}{t^2 (\text{Vol}(A) - \text{Vol}(A(t)))}$$

What remains is to bound $\text{Vol}(A(t))$. In this step Buser uses a global lower bound on Ricci curvature and the assumption that Σ is the Cheeger-optimal cut-set.

Suppose N is a compact m -submanifold of M . Assume that if N is a hypersurface or point, and assume that the planes of M containing a tangent vector of a geodesic segment which minimizes the distance to N have sectional curvatures $\geq \delta$. A

consequence of the Heintze-Karcher comparison theorem [38] is the following volume growth bound: There exists $J_\delta \in C^\infty[0, \infty)$ such that for all $\tau \geq 0$, we have

$$\text{Vol}_{n-1}(d^{-1}(\tau)) \leq \text{Vol}_{n-1}(\Sigma) J_\delta(\tau).$$

Now, using a volume growth bound $\text{Vol}(A(t)) \leq \int_0^t J(s) \text{Vol}(\Sigma) ds$ (when clear we will suppress the δ in J_δ), Buser finds the bound

$$\begin{aligned} \lambda_2 &\leq \frac{\int_0^t J(s) \text{Vol}(\Sigma) ds}{t^2 \text{Vol}(A) - t^2 \int_0^t J(s) \text{Vol}(\Sigma) ds} \\ &\leq \frac{h \int_0^t J(s) ds}{t^2 (1 - h \int_0^t J(s) ds)} \end{aligned}$$

for λ_2 in terms of the curvature (again, because J depends on curvature), the Cheeger cut-set A and boundary Σ . We do not reproduce the remainder of the proof, which is quite technical. That proof is found in [20], and the result is:

Theorem 5.3.1 (Buser's Inequality, (P. Buser 1982)). *If M is an n -dimensional manifold with $-(\delta^2)(n-1)$ as a lower bound on curvature (for some $\delta \geq 0$), then*

$$\lambda_2(M) \leq c(\delta h(M) + h(M)^2),$$

where c is a universal constant.

More recently, Benson observed that the constant in Buser's proof can be improved by optimizing over *all* possible test-functions that depend on the distance from Σ , not just those that grow linearly up to some critical distance t . Benson also observed that a similar method can give bounds on the higher eigenvalues, according to the formula

$$\lambda_{2k}(M) \leq \max(\lambda_k(A), \lambda_k(B)),$$

where $\lambda_1(M), \lambda_2(M), \dots$ are the eigenvalues of M in increasing order [11].

Here, the Rayleigh quotient is

$$\lambda_k(A) = \inf_U \sup_{f \in U} \frac{\int_A \|\nabla(f)\|^2}{\int_A f^2}$$

where U is the set of k -dimensional subspaces of $H_0^1(D)$ on which $f(\Sigma) = 0$. Limiting to only those functions f that depend on the distance from Σ , Benson obtains

$$\lambda_k(A) \leq \inf_V \sup_{f \in V} \frac{\int_0^\infty f'(s)^2 \text{Vol}(d^{-1}(s)) ds}{\int_0^\infty f(s)^2 \text{Vol}(d^{-1}(s)) ds}, \quad (34)$$

where $d^{-1}(s)$ is the set $\{p \in A : d(p, \Sigma) = s\}$ and V is the set of all k -dimensional subspaces of $H_0^1[0, \infty)$ on which $f(0) = 0$.

Given a Heintze-Karcher-type growth bound $\text{Vol}(d^{-1}(s)) \leq J(s)\text{Vol}(\Sigma)$, there is also a lower bound on $\text{Vol}(d^{-1}(s))$:

Observe that

$$\text{Vol}(A(s)) = \int_0^s \text{Vol}(d^{-1}(\tau)) d\tau \leq \text{Vol}(\Sigma) \int_0^s J(\tau) d\tau.$$

Because Σ is the Cheeger-achieving boundary and $d^{-1}(s)$ is the boundary for some other non-Cheeger-achieving partition of M ,

$$\begin{aligned} h(M) &= \frac{\text{Vol}(\Sigma)}{\min(\text{Vol}(A), \text{Vol}(B))} \text{ and} \\ h(M) &\leq \frac{\text{Vol}(d^{-1}(s))}{\min(\text{Vol}(A \setminus A(s)), \text{Vol}(B \cup A(s)))}. \end{aligned}$$

In the case that $\text{Vol}(B \cup A(s)) \geq \text{Vol}(A \setminus A(s))$,

$$\begin{aligned} h(M) &\leq \frac{\text{Vol}(d^{-1}(s))}{\text{Vol}(A \setminus A(s))}, \text{ and so} \\ \text{Vol}(d^{-1}(s)) &\geq h(M)\text{Vol}(A) - h(M)\text{Vol}(A(s)) \\ &\geq \text{Vol}(\Sigma) - h(M)\text{Vol}(\Sigma) \int_0^s J(\tau) d\tau \\ &= \text{Vol}(\Sigma) \left(1 - h(M) \int_0^s J(\tau) d\tau \right). \end{aligned}$$

In the other case, $\text{Vol}(B \cup A(s)) \leq \text{Vol}(A \setminus A(s))$. As such, $\text{Vol}(B) \leq \text{Vol}(A)$, i.e., B is the Cheeger minimizing set. We find that

$$\text{Vol}(d^{-1}(s)) \geq h(M)\text{Vol}(B \cup A(s)) \geq h(M)\text{Vol}(B) = \Sigma,$$

with the last equality being the definition of Σ .

Combining both cases, Benson achieves the lower bound

$$\text{Vol}(d^{-1}(s)) \geq \text{Vol}(\Sigma) \left(1 - h(M) \int_0^s J(\tau) d\tau \right).$$

Observe that this bound is only meaningful for values of s where

$$h(M) \int_0^s J(\tau) d\tau \leq 1.$$

Define T to be the value for which $h(M) \int_0^T J(\tau) d\tau = 1$.

With both an upper and lower bound for $\text{Vol}(d^{-1}(s))$, it is possible to plug those bounds into Equation 34, truncating the integrals at T , to obtain the bound

$$\lambda_k(A) \leq \inf_W \sup_{f \in W} \frac{\int_0^T f'(s)^2 J(s) ds}{\int_0^T u(s)^2 \left(1 - h \int_0^s J(\tau) d\tau \right) ds}, \quad (35)$$

where W is the set of k -dimensional subspaces of $H_0^1[0, T]$ in which $f(0) = 0$.

What remains is the technical problem of finding the function f that minimizes the Rayleigh quotient in Equation 35. Again we omit that discussion, which is found in [11].

5.4 *Bounds on volume growth*

In this section we discuss volume growth in graphs under a curvature lower bound. One such result is Theorem 3.3.5, which gives an upper bound on volume growth under a bound on Ollivier curvature.

We present a volume growth bound based on the CDE' curvature. The CDE' inequality is a variant of the CD inequality of Chapter 2. The CDE' inequality was introduced by Bauer et al.[9].

In further work, a volume growth bound was discovered under a lower bound on CDE' curvature:

Theorem 5.4.1. (*Horn, Lin, Liu & Yau [40]*) Let G be a locally finite graph satisfying $CDE'(n, 0)$. Then there exists a constant C depending on n such that for all $x \in V$ and any integers r, s with $r \geq s$:

$$|d_x^{-1}(r)| \leq C \left(\frac{r}{s} \right)^{\frac{\log(C)}{\log(2)}} |d_x^{-1}(s)|. \quad (36)$$

We use this bound on ball volumes to prove the following bound on shell volumes:

Corollary 5.4.2. Let G be a graph satisfying $CDE'(n, 0)$ at all vertices $x \in V(G)$. Let $\Sigma \subset V$, and let $C = C(G)$ be the constant from Theorem 5.4.1, let $r > 0$. Then

$$|d_\Sigma^{-1}(r)| \leq d|\Sigma|C(r-1)^{\frac{\log(C)}{\log(2)}} \left[C \left(\frac{r}{r-1} \right)^{\frac{\log(C)}{\log(2)}} - 1 \right].$$

Proof. Letting $s = r - 1$ in Equation 36, the estimate becomes

$$|d_x^{-1}(r)| \leq C \left(\frac{r}{r-1} \right)^{\frac{\log(C)}{\log(2)}} |d_x^{-1}(r-1)|.$$

Since we are interested in counting vertices with distance exactly r from x , we wish to consider $|d_x^{-1}(r)| - |d_x^{-1}(r-1)|$, so we subtract $|d_x^{-1}(r-1)|$ from both sides of the previous inequality to give

$$|d_x^{-1}(r)| - |d_x^{-1}(r-1)| \leq \left[C \left(\frac{r}{r-1} \right)^{\frac{\log(C)}{\log(2)}} - 1 \right] |d_x^{-1}(r-1)|.$$

In fact, we want to consider the set of vertices with distance exactly r from Σ . We can sum over all $x \in \Sigma$ on both sides of the previous equation to give

$$|d_\Sigma^{-1}(r)| \leq \sum_{x \in \Sigma} |d_x^{-1}(r)| - |d_x^{-1}(r-1)| \leq \sum_{x \in \Sigma} \left[C \left(\frac{r}{r-1} \right)^{\frac{\log(C)}{\log(2)}} - 1 \right] |d_x^{-1}(r-1)|.$$

Simplifying, we find

$$|d_\Sigma^{-1}(r)| \leq |\Sigma| \left[C \left(\frac{r}{r-1} \right)^{\frac{\log(C)}{\log(2)}} - 1 \right] \max_{x \in \Sigma} |d_x^{-1}(r-1)|.$$

Now we wish to estimate the term $\max_{x \in \Sigma} |d_x^{-1}(r-1)|$ and we will again apply Equation 36, this time with where we replace r with $r - 1$ in the formula and take $s = 1$.

As a result, our estimate becomes

$$|d_{\Sigma}^{-1}(r)| \leq d|\Sigma|C(r-1)^{\frac{\log(C)}{\log(2)}} \left[C \left(\frac{r}{r-1} \right)^{\frac{\log(C)}{\log(2)}} - 1 \right],$$

observing that $|d_x^{-1}(1)| = d$. □

5.5 Using volume growth in graphs

In this section we follow the methods from the continuous setting that were developed by B. Benson [11] and discussed in Section 5.3. First, we demonstrate an upper bound for an eigenvalue $\lambda_k(G)$ by taking the Rayleigh quotient of a function based only on distance from a cut-set Σ . Next, we optimize that quotient by treating it as a discrete Hardy-type inequality.

Remark. In this section we will use volume growth bounds of the type referenced in the previous section. These bounds are the only point in our analysis that relies on the discrete curvature. Given another volume growth result (either based on another notion of discrete curvature or unrelated to curvature), it will be possible to repeat the analysis we present here and achieve similar results.

5.5.1 Bounding eigenvalues

In this section we will establish bounds for the spectrum of the normalized Laplacian using bounds on volume growth. These methods can be viewed as the graph analogue of those used by Benson [11] in the continuous setting of Riemannian manifolds.

Our methods in this section for approximating λ_k , where $k \geq 2$, do not make any assumption about the cut-set, but the bounds we obtain will only be in terms of the generic volume growth bounds μ, ν . Later, we will give a bound for λ_2 with the assumption that Σ is the outer vertex isoperimetric optimizing cut-set.

Let $G = (V, E)$ be a graph. Let $\Sigma \subset V(G)$ be a cut-set that separates $V \setminus \Sigma$ into V^+ and V^- . Note that under this definition, it is possible that V^+ or V^- is empty. The signed distance function $d_{\Sigma} : V \rightarrow \mathbb{Z}$ is defined so that $|d_{\Sigma}(v)| = \min_{x \in \Sigma} d_G(x, v)$

where d_G is the graph distance, and the sign of d_Σ is positive on V^+ and negative on V^- .

We will assume that we have volume growth and decay bounds for the level sets of d_Σ . Specifically, let $\nu(k)$ denote a volume growth bound and $\mu(k)$ denote a uniform volume decay bound respectively. Here, for $k \in \mathbb{Z}$, the bounds $\nu(k)$ and $\mu(k)$ have the property that

$$|\Sigma|\mu(k) \leq |d_\Sigma^{-1}(k)| \leq |\Sigma|\nu(k). \quad (37)$$

Define $T^+ \in \mathbb{Z}_{>0}$ so that $\mu(k) > 0$ for all $0 \leq k \leq T^+$ and define $T^- \in \mathbb{Z}_{<0}$ so that $\mu(k) > 0$ for all $T^- \leq k \leq 0$. Let W^+ be the space of functions $g^+ : \{0, 1, 2, \dots, T^+\} \rightarrow \mathbb{R}$ such that $g^+(0) = 0$, and W^- be the space of functions $g^- : \{0, -1, -2, \dots, T^-\}$ such that $g^-(0) = 0$.

To estimate $\lambda_2(G)$, we will be interested in (the smallest positive) solutions $\rho^\pm \in \mathbb{R}^{T^\pm}$ which satisfy the respective equations

$$\sum_{i=0}^{T^+} \phi(i)^2 \cdot \nu(i) \leq 2\rho^+ \sum_{k=0}^{T^+} \left[\sum_{i=0}^k \phi(i) \right]^2 \cdot \mu(k), \quad (38)$$

$$\sum_{i=T^-}^0 \phi(i)^2 \cdot \nu(i) \leq 2\rho^- \sum_{k=T^-}^0 \left[\sum_{i=k}^0 \phi(i) \right]^2 \cdot \mu(k) \quad (39)$$

with $\phi(0) = 0$, $\phi \not\equiv 0$ and where ν and μ are the volume growth bounds defined in Equation 37. Equations of this form are called weighted discrete Hardy inequalities. For a fuller discussion of this topic refer to [50].

Theorem 5.5.1. *Let ρ^+ and ρ^- be defined by Equations 38 and 39. Then $\lambda_2(G) \leq \max\{\rho^+, \rho^-\}$.*

To estimate the higher eigenvalues, we define a symmetric matrix A^+ indexed by

$\{1, \dots, T^+\}$ so that

$$A_{i,j}^+ = \begin{cases} 2\nu(i) + \nu(i-1) + \nu(i+1) & \text{if } i = j; i < T^+ \\ 2\nu(T^+) + \nu(T^+ - 1) & \text{if } i = j = T^+ \\ -(\nu(i) + \nu(j)) & \text{if } |i - j| = 1; i, j < T^+ \\ -2\nu(T^+ - 1) - \nu(T^+) & \text{if } |i - j| = 1; i \text{ or } j = T^+ \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we define the symmetric matrix A^- indexed by $\{-1, -2, \dots, T^-\}$:

$$A_{i,j}^- = \begin{cases} 2\nu(i) + \nu(i-1) + \nu(i+1) & \text{if } i = j; i > T^- \\ 2\nu(T^-) + \nu(T^- + 1) & \text{if } i = j = T^- \\ -(\nu(i) + \nu(j)) & \text{if } |i - j| = 1; i, j > T^- \\ -2\nu(T^- + 1) - \nu(T^-) & \text{if } |i - j| = 1; i \text{ or } j = T^- \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.5.2. *For a graph G and any $k, l \in \mathbb{N}$ with $1 \leq k, l \leq \min\{T^-, T^+\}$,*

$$\lambda_{k+l}(G) \leq \frac{1}{4} \max \left\{ \frac{\rho_k^+}{\min_{0 \leq k \leq T^+} \mu(k)}, \frac{\rho_l^-}{\min_{T^- \leq k \leq 0} \mu(k)} \right\} \quad (40)$$

where ρ_k^+ and ρ_l^- are the k -th and l -th non-trivial eigenvalues of the respective equations $A^+ g^+ = \rho^+ g^+$ with $g^+ \in W^+$ and $A^- g^- = \rho^- g^-$ with $g^- \in W^-$.

Broadly, we are using estimates of volume growth and decay which act as weights and linearize the normalized Laplacian eigenvalue problem on the graph. As an example application of this theorem, we will revisit Buser's inequality. We will prove both Theorems 5.5.1 and 5.5.2 simultaneously.

Proof of Theorems 5.5.1 and 5.5.2. Using the Poincaré minimax principle for characterization of eigenvalues, we see that

$$\lambda_k(G) = \inf_U \sup_{f \in U} \frac{\langle f, \Delta f \rangle}{\langle f, f \rangle}, \quad (41)$$

where U is the set of k -dimensional subspaces of functions $f \in \mathbb{R}^V$. Expanding these inner products, we find that

$$\begin{aligned}
\langle f, \Delta f \rangle &= \sum_x f(x) \sum_{y \sim x} \frac{1}{2d} (f(x) - f(y)) \\
&= \sum_{\{x, y: x \sim y\}} \frac{1}{2d} (f(x) - f(y))^2 \\
&= \sum_x \frac{1}{4d} \sum_{y \sim x} (f(x) - f(y))^2, \text{ and} \\
\langle f, f \rangle &= \sum_x f^2(x).
\end{aligned}$$

We wish to estimate the eigenvalue $\lambda_j(G)$ of the normalized Laplacian on G by the eigenvalues of the Dirichlet Laplacian on the subgraphs $V^+ \sqcup \Sigma$ and $V^- \sqcup \Sigma$, denote them $\xi_k(V^+ \sqcup \Sigma)$ and $\xi_l(V^- \sqcup \Sigma)$, respectively. Using the Poincaré minimax principle it is possible to see that when $1 \leq k, l \leq \min\{|V^-|, |V^+|\}$, it follows that

$$\lambda_{k+l}(G) \leq \max\{\lambda_k(V^+ \sqcup \Sigma), \lambda_l(V^- \sqcup \Sigma)\}. \quad (42)$$

This result is due to Balti, following the methods developed by Benson [8, 11].

We restrict the test functions for Equation 41 in \mathbb{R}^V to functions which

- vanish on either $V^- \sqcup \Sigma$ or $V^+ \sqcup \Sigma$,
- are constant on each $d_\Sigma^{-1}(i)$,
- and are constant for values less than or equal to T^- and greater than or equal to T^+ .

We will treat these test functions as functions in W^+ or W^- , respectively. Let W_k^+ and W_l^- be, respectively, arbitrary sets of k and l -dimensional subspaces of real-valued functions of W^+ and W^- , for values $k \in \mathbb{Z}_{>0}$ and $l \in \mathbb{Z}_{<0}$. We view W^+ and W^- as inner product spaces with the following inner products:

$$\langle f, g \rangle_+ = \sum_{i=0}^{T^+} f(i)g(i) \quad \text{and} \quad \langle f, g \rangle_- = \sum_{i=T^-}^0 f(i)g(i).$$

Combining Equation 42 with the Poincaré minimax characterization for the Dirichlet eigenvalues $\xi_l(V^+ \sqcup \Sigma)$ and $\xi_l(V^- \sqcup \Sigma)$, we have that

$$\lambda_{k+l}(G) \leq \max \left\{ \inf_{W_k^+} \sup_{g \in W_k^+} \frac{\langle g, \Delta g \rangle_+}{\langle g, g \rangle_+}, \inf_{W_l^-} \sup_{g' \in W_l^-} \frac{\langle g', \Delta g' \rangle_-}{\langle g', g' \rangle_-} \right\}. \quad (43)$$

Now, for $g \in V^+ \sqcup \Sigma$, using the volume growth estimates from Equation 37, we have the estimates

$$\begin{aligned} \langle g, \Delta g \rangle_+ &= \sum_{i=0}^{T^+} \sum_{x \in d_\Sigma^{-1}(i)} \sum_{y \sim x} \frac{1}{4d} (g(i) - g(y))^2 \\ &\leq \sum_{i=0}^{T^+} \sum_{x \in d_\Sigma^{-1}(i)} \frac{1}{4} \left[(g(i) - g(i+1))^2 + (g(i) - g(i-1))^2 \right] \\ &= \frac{1}{4} \sum_{i=0}^{T^+} |d_\Sigma^{-1}(i)| \left[(g(i) - g(i+1))^2 + (g(i) - g(i-1))^2 \right] \\ &\leq \frac{1}{4} \sum_{i=0}^{T^+} \nu(i) \left[(g(i) - g(i+1))^2 + (g(i) - g(i-1))^2 \right] \end{aligned}$$

and

$$\begin{aligned} \langle g, g \rangle_+ &= \sum_{i=0}^{T^+} |d_\Sigma^{-1}(i)| g^2(i) \\ &\geq \sum_{k=0}^{T^+} \mu(i) g^2(i). \end{aligned}$$

Similar estimates hold for a function g defined on $V^- \sqcup \Sigma$.

Using these bounds in Equation 43, we find that

$$\begin{aligned} \lambda_{k+l}(G) \leq \max \left\{ \inf_{W_k^+} \sup_{g \in W_k^+} \frac{\sum_{i=0}^{T^+} \left[(g(i) - g(i+1))^2 + (g(i) - g(i-1))^2 \right] \cdot \nu(i)}{\sum_{i=0}^{T^+} g^2(i) \cdot 4\mu(i)}, \right. \\ \left. \inf_{W_l^-} \sup_{g \in W_l^-} \frac{\sum_{i=T^-}^0 \left[(g(i) - g(i+1))^2 + (g(i) - g(i-1))^2 \right] \cdot \nu(i)}{\sum_{i=T^-}^0 g^2(i) \cdot 4\mu(i)} \right\}. \quad (44) \end{aligned}$$

For estimating $\lambda_2(G)$, we take $k = l = 1$ and the Rayleigh quotient for W^+ in Equation 44 becomes

$$\inf_{g \in W^+, g \neq 0} \frac{\sum_{i=0}^{T^+} \left[(g(i) - g(i+1))^2 + (g(i) - g(i-1))^2 \right] \cdot \nu(i)}{\sum_{i=0}^{T^+} g^2(i) \cdot 4\mu(i)}. \quad (45)$$

Define $\phi(j) := g(j) - g(j-1)$ for the W^+ quotient in Equation 44. Noting that $g(j) = \sum_{i=0}^j \phi(i)$ and using the facts that $g(0) = 0$, $\phi(0) = 0$, and $g(i)$ is constant for all $i \geq T^+$ implies $\phi(i) = 0$ for all $i > T^+$. It follows from the definition of ϕ and a routine computation that the quotient in Equation 45 is bounded from above by ρ^+ in Equation 38. A similar argument verifies Equation 39. This establishes Theorem 5.5.1.

Bounding the higher eigenvalues: We now continue the argument for higher eigenvalues. Noting that

$$\sum_{i=0}^{T^+} \left[(g(i) - g(i+1))^2 + (g(i) - g(i-1))^2 \right] \nu(i) = \sum_{i=0}^{T^+-1} (g(i) - g(i+1))^2 \cdot [\nu(i) + \nu(i+1)],$$

we can now estimate Equation 44 from above using the matrices A^\pm :

$$\lambda_{k+l}(G) \leq \frac{1}{4} \max \left\{ \frac{1}{\min_{0 \leq k \leq T^+} \mu(k)} \inf_{W_k^+} \sup_{g \in W_k^+} \frac{\langle g, A^+ g \rangle_+}{\langle g, g \rangle_+}, \frac{1}{\min_{T^- \leq k \leq 0} \mu(k)} \inf_{W_l^-} \sup_{g \in W_l^-} \frac{\langle g, A^- g \rangle_-}{\langle g, g \rangle_-} \right\}. \quad (46)$$

Note that in the application of A^\pm above, we have used the fact that because $g(T^+ + 1) = g(T^+)$, it follows that

$$-2g(T^+ - 1)g(T^+ + 1)\nu(T^+ - 1) = -2g(T^+ - 1)g(T^+)\nu(T^+ - 1)$$

in the calculation.

Since A^+ and A^- are symmetric, the spectral theorem implies that there exist an orthonormal basis of T^+ real eigenfunctions of A^+ in W^+ with corresponding real eigenvalues and an orthonormal basis of T^- real eigenfunctions of A^- in W^- having real eigenvalues. It is easy to see that if $g_* \in W^+$ is an eigenfunction of A^+ with

corresponding eigenvalue ρ_* , we have

$$\frac{\langle g_*, A^+ g_* \rangle_+}{\langle g_*, g_* \rangle_+} = \rho_*. \quad (47)$$

Since the basis of eigenfunctions is orthonormal, this implies that the k -th eigenvalue of A^+ in W_k^+ , which we denote ρ_k^+ , gives the following bound:

$$\rho_k^+ \geq \frac{1}{4 \min_{0 \leq k \leq T^+} \mu(k)} \inf_{W_k^+} \sup_{g \in W_k^+} \frac{\langle g, \Delta g \rangle_+}{\langle g, g \rangle_+} \quad (48)$$

where the right term in the inequality appears in Equation 43. Since the same argument holds for the eigenfunctions and eigenvalues in W^- , we have verified Equation 40. \square

We remark that in the continuous case, one shows that analogue of the operator A can be rewritten as a Sturm-Liouville problem depending on the same parameters of the manifold as Buser's inequality. The details are found in Benson [11].

5.5.2 Applying volume growth bounds

In this section, we use $J(k)$ to denote a volume growth bound around Σ ; i.e., a function with the property that, given a fixed $\Sigma \subset V$, all choices of sets V^+, V^- , and all $k \geq 0$, $|d_\Sigma^{-1}(k)| \leq |\Sigma|J(k)$. J may depend on Σ as well as the curvature, though previously we have only presented volume growth bounds that are independent of the choice of Σ .

Remark. In this section our results are in terms of the outer vertex isoperimetric constant h_{out} . This is most natural because we use the counting measure on the vertex set. As stated before, there are simple bounds relating h_{out} to the edge isoperimetric constant h :

$$h \leq h_{out} \leq hd$$

where d is the degree of the graph. Using these inequalities it is possible to rewrite our results in terms of h .

Lemma 5.5.3. *Let $A \subset V$ be the set that achieves the outer vertex isoperimetric constant h_{out} and let $\Sigma = \partial_{out}A$. Set A to be either V^+ or V^- , and $V \setminus (A \cup \Sigma)$ to be the other; use this choice of V^\pm to define the signed distance function d_Σ . Let $k > 0$ and set $\Upsilon = d_\Sigma^{-1}(k)$. Then*

$$|\Upsilon| \geq |\Sigma| \left(1 - h_{out} \sum_{i=0}^k J(i) \right)$$

Proof. Define $C^- = \bigcup_{i < k} d_\Sigma^{-1}(i)$ and $C^+ = \bigcup_{i > k} d_\Sigma^{-1}(i)$.

(1) In the first case, suppose $|C^-| < \frac{1}{2}|V|$. Observe that $(V^- \cup \Sigma) \subset C^-$, therefore $|V^- \cup \Sigma| < \frac{1}{2}|V|$. By assumption $\frac{1}{2}|V| \leq |V \setminus A| = |(V \setminus (A \cup \Sigma)) \cup \Sigma|$, so $V^- \neq V \setminus (A \cup \Sigma)$; instead $V^- = A$.

Because $|C^-| < \frac{1}{2}|V|$, we have that $|\Upsilon| \geq h_{out}|C^-| \geq h_{out}|A| = |\Sigma|$, the result follows.

(2) In the other case, we have $|C^-| \geq \frac{1}{2}|V|$. Because C^- and C^+ are disjoint, $|C^+| \leq \frac{1}{2}|V|$. Therefore,

$$|\Upsilon| \geq h_{out}|C^+| = h_{out} \left(|V^+| - \sum_{i=1}^k |d_\Sigma^{-1}(i)| \right).$$

Observe that since $|A| \leq |V|/2$, we have that

$$\begin{aligned} \min\{|V \setminus (A \cup \Sigma)|, |A|\} &\geq \min\{|V \setminus A| - |\Sigma|, |A|\} \\ &\geq \min\{|A| - |\Sigma|, |A|\} \\ &= |A| - |\Sigma|. \end{aligned}$$

Applying the previous bound gives us

$$\begin{aligned} |\Upsilon| &\geq h_{out} \left(|V^+| - \sum_{i=1}^k |d_\Sigma^{-1}(i)| \right) \geq h_{out} \left(|A| - |\Sigma| - \sum_{i=1}^k |\Sigma| J(i) \right) \\ &= h_{out} \left(|A| - \sum_{i=0}^k |\Sigma| J(i) \right) = |\Sigma| \left(1 - h_{out} \sum_{i=0}^k J(i) \right), \end{aligned}$$

where the first equality relies on the trivial bound $J(0) \geq 1$. This proves the result. \square

Using the bounds that we have just obtained : $|\Sigma|J(k) \geq |d_{\Sigma(k)}^{-1}| \geq |\Sigma|(1 - h_{out} \sum_{i=0}^k J(i))$, and the Rayleigh quotient in Equation 44, we obtain

$$\lambda_2 \leq \inf_{W_1} \sup_{g \in W_1} \frac{\frac{1}{4} \sum_{k=0}^T J(k) \left[(g(k) - g(k+1))^2 + (g(k) - g(k-1))^2 \right]}{\sum_{k=0}^T g^2(k) (1 - h_{out} \sum_{i=0}^k J(i))}. \quad (49)$$

where T is the largest integer for which $1 > h_{out} \sum_{i=0}^T J(i)$.

Here, by assumption we have the same volume growth bounds on V^+ and V^- , so (unlike the previous section) the Rayleigh quotients are identical on both sides of the cut-set.

5.5.3 Bounds on λ_2

Of particular interest is the problem of bounding λ_2 . Indeed, the original proofs of Buser's inequality only bound λ_2 and not the higher eigenvalues $\lambda_k : k \geq 3$. [20, 44, 45].

Observe that the Rayleigh minimizing function for λ_2 must have certain properties.

Lemma 5.5.4. *The function $g(k)$ corresponding to the non-constant minimizer of $R(g)$ is monotone in k .*

Proof of Lemma 5.5.4. We will induct on k . The base case is trivial since $g(0) = 0$ by the Dirichlet boundary condition on $f^{-1}(0)$. Without loss of generality, assume that $g(1) \geq 0$, else replace $g(1)$ with $-g(1)$ and proceed to the induction step.

Assume for contradiction that g is monotone increasing up to some k in its domain, but that $g(k+1) < g(k)$. Then replacing $g(k+1)$ by $2g(k) - g(k+1)$, the numerator of $R(g)$ is unchanged as $[g(k) - (2g(k) - g(k+1))]^2 = (g(k) - g(k+1))^2$. At the same time, the denominator of $R(g)$ increases since $(2g(k) - g(k+1))^2 > g(k+1)^2$, therefore the quotient $R(g)$ decreases, contradicting the assumption that g is a non-constant minimizer of $R(g)$.

□

We are now able to bound the Rayleigh quotient within a constant factor.

To bound λ_2 , we apply Equation 49 giving the Rayleigh quotient

$$\lambda_2 \leq R := \inf_f \frac{\frac{1}{4} \sum_{k=0}^T J(k) \left[(f(k) - f(k+1))^2 + (f(k) - f(k-1))^2 \right]}{\sum_{k=1}^T f^2(k) (1 - h_{out} \sum_{i=0}^k J(i))}, \quad (50)$$

where the infimum is taken over all functions $f : \mathbb{Z} \rightarrow \mathbb{R}$ with $f(0) = 0$, $f(1) \neq 0$, $f(i) = 0$ if $i < 0$ and $f(i) = f(T)$ if $i > T$.

Theorem 5.5.5. *The bounds on $R(g)$ are $\frac{1}{16B} \leq R \leq \frac{1}{4B}$, where*

$$B = \sup_{n \geq 1} \left(\sum_{k=n}^T (1 - h_{out} \sum_{i=0}^k J(i)) \right) \left(\sum_{k=1}^n \frac{1}{J(k) + J(k-1)} \right).$$

Proof. To apply a result of L. Miclo [50], we write Equation 50 in a different form: set $g(k) = f(k) - f(k-1)$ for $k \in \mathbb{Z}$. Observe that $f(k) = \sum_{i=1}^k g(i)$. Also observe that $g(k) = 0$ if $k \leq 0$ or $k > T$. We have

$$\begin{aligned} 4R &= \inf_g \frac{\sum_{k=0}^T J(k) \left[g(k+1)^2 + g(k)^2 \right]}{\sum_{k=1}^T \left(\sum_{i=1}^k g(i) \right)^2 (1 - h_{out} \sum_{i=0}^k J(i))} \\ &= \inf_g \frac{\sum_{k=1}^T g(k)^2 \left(J(k) + J(k-1) \right)}{\sum_{k=1}^T \left(\sum_{i=1}^k g(i) \right)^2 (1 - h_{out} \sum_{i=0}^k J(i))}, \end{aligned}$$

taken over all functions $g : \mathbb{N} \rightarrow \mathbb{R}$.

To simplify we write the volume growth and decay bounds as $\mu(k) = 1 - h_{out} \sum_{i=0}^k J(i)$ and $\nu(k) = J(k) + J(k-1)$ if $1 \leq k \leq T$, and $\mu(k) = \nu(k) = 0$ if $k \geq T$. We have

$$4R = \inf_g \frac{\sum_{k=1}^T g(k)^2 \nu(k)}{\sum_{k=1}^T \left(\sum_{i=1}^k g(i) \right)^2 \mu(k)}.$$

The result follows from Proposition 1 in [50]. □

An immediate corollary is a bound on the spectral gap, obtained by combining Theorem 5.5.5 with the bound $\lambda_2 \leq R$.

Theorem 5.5.6. $\lambda_2 \leq \frac{1}{4B}$, where

$$B = \sup_{n \geq 1} \left(\sum_{k=n}^T (1 - h_{out} \sum_{i=0}^k J(i)) \right) \left(\sum_{k=1}^n \frac{1}{J(k) + J(k-1)} \right).$$

A case of particular interest is when $\Sigma = \max_{i \in \mathbb{Z}} |d_{\Sigma}^{-1}(i)|$. In this case we may set $J \equiv 1$.

Corollary 5.5.7. *If the vertex-isoperimetric cut-set Σ satisfies $\Sigma = \max_{i \in \mathbb{Z}} |d_{\Sigma}^{-1}(i)|$, then $\lambda_2 \leq \frac{27}{4} h_{out}^2 (1 + o(1))$.*

The proof is found in Example 5.6.3. Under these hypotheses the Cheeger lower bound $\lambda_2 \geq c * h_{out}^2 / d$ is tight up to a linear factor of d .

5.6 Examples of spectral gap bounds using volume growth

In this section, we use Theorem 5.5.6 to bound the second eigenvalue by the volume growth. First we obtain several general bounds depending only on the growth function $J(k)$. Second, we use these results to bound λ_2 for specific graphs where the growth function is known.

5.6.1 Examples of volume growth functions

Example 5.6.1. *If $J(i)$ is an exponential, i.e., $J(i) = c^i$ for some value $c > 1$, T satisfies*

$$\frac{c^{T+1} - 1}{c - 1} \leq \frac{1}{h_{out}} \leq \frac{c^{T+2} - 1}{c - 1}.$$

As such,

$$h_{out} \leq \frac{c - 1}{c^{T+1} - 1}.$$

Note, it is trivial that $J(i) = d \cdot (d-1)^{i-1}$ is a volume growth bound for all d -regular graphs (achieved by a tree). So we only need to consider the case $c \leq d - 1$.

If $T \geq n \geq 1$,

$$\begin{aligned} \sum_{k=n}^T (1 - h_{out} \sum_{i=0}^k J(i)) &= (T - n + 1) - \sum_{k=n}^T h_{out} \frac{c^{k+1} - 1}{c - 1} \\ &\geq (T - n + 1) - \sum_{k=n}^T \frac{c^{k+1} - 1}{c^{T+1} + 1} = (T - n + 1) \left(1 + \frac{1}{c^{T+1} - 1}\right) - \frac{c^{T+2} - c^{n+1}}{(c - 1)(c^{T+1} - 1)} \end{aligned}$$

and

$$\sum_{k=1}^n \frac{1}{J(k) + J(k-1)} = \sum_{k=1}^n \frac{1}{(c+1)c^{k-1}} = \frac{1 - c^{-n}}{c - \frac{1}{c}} = \frac{c - c^{1-n}}{c^2 - 1}$$

Combining, we have

$$B \geq \sup_{T \geq n \geq 1} \left((T - n + 1) \left(1 + \frac{1}{c^{T+1} - 1}\right) - \frac{c^{T+2} - c^{n+1}}{(c - 1)(c^{T+1} - 1)} \right) \left(\frac{c - c^{1-n}}{c^2 - 1} \right).$$

Taking $n = 1$, we find

$$\begin{aligned} B &\geq \left(T + \frac{T}{c^{T+1} - 1} - \frac{c^{T+2} - c^2}{(c - 1)(c^{T+1} - 1)} \right) \left(\frac{c - 1}{c^2 - 1} \right) \\ &\geq \left(T + \frac{T}{c^{T+1} - 1} - \frac{c}{c - 1} \right) \left(\frac{1}{c + 1} \right). \end{aligned}$$

On the other hand, for any value n satisfying $1 \leq n \leq T$,

$$\sum_{k=n}^T (1 - h_{out} \sum_{i=0}^k J(i)) \leq T$$

and

$$\sum_{k=1}^n \frac{1}{J(k) + J(k-1)} = \sum_{k=1}^n \frac{1}{(c+1)c^{k-1}} = \frac{1 - c^{-n}}{c - \frac{1}{c}} \leq \frac{1}{c - \frac{1}{c}} = \frac{c}{c^2 - 1},$$

so, combining all parts we see that

$$\left(T + \frac{T}{c^{T+1} - 1} - \frac{c}{c - 1} \right) \left(\frac{1}{c + 1} \right) \leq B \leq T \frac{c}{c^2 - 1}.$$

If $c \geq 1 + \varepsilon$, $B = \Theta(T/c)$ and $\lambda_2 = O(c/T)$.

Example 5.6.2. *Of particular interest is the case that $J(0) = 1$, $J(i) = dc^{i-1}$ if $i \geq 1$, where d is the common degree of vertices in the graph and $c > 1$. This is the form of Theorems 3.3.5 and 5.4.1. Proceeding similarly to the previous example, T satisfies*

$$1 + d \frac{c^T - 1}{c - 1} \leq \frac{1}{h_{out}} \leq 1 + d \frac{c^{T+1} - 1}{c - 1}.$$

It follows that

$$h_{out} \leq \frac{c - 1}{c - 1 + d(c^T - 1)}.$$

If $T \geq n \geq 1$,

$$\begin{aligned} \sum_{k=n}^T (1 - h_{out} \sum_{i=0}^k J(i)) &= (T - n + 1) - \sum_{k=n}^T h_{out} \frac{c - 1 + d(c^k - 1)}{c - 1} \\ &\geq (T - n + 1) - \sum_{k=n}^T \frac{c - 1 + d(c^k - 1)}{c - 1 + d(c^T - 1)} \\ &= (T - n + 1) \left(1 + \frac{d + 1 - c}{c - 1 + d(c^T - 1)}\right) - \frac{d(c^{T+1} - c^n)}{(c - 1)(c - 1 + d(c^T - 1))} \end{aligned}$$

and

$$\begin{aligned} \sum_{k=1}^n \frac{1}{J(k) + J(k-1)} &= \frac{1}{1 + d} + \sum_{k=2}^n \frac{1}{d(c + 1)c^{k-2}} \\ &= \frac{1}{1 + d} + \frac{1}{d} \cdot \frac{1 - c^{1-n}}{c - \frac{1}{c}} = \frac{1}{1 + d} + \frac{1}{d} \cdot \frac{c - c^{2-n}}{c^2 - 1}. \end{aligned}$$

Combining, we have

$$B \geq \sup_{T \geq n \geq 1} \left((T - n + 1) \left(1 + \frac{d + 1 - c}{c - 1 + d(c^T - 1)}\right) - \frac{d(c^{T+1} - c^n)}{(c - 1)(c - 1 + d(c^T - 1))} \right) \left(\frac{1}{1 + d} + \frac{1}{d} \cdot \frac{c - c^{2-n}}{c^2 - 1} \right).$$

Taking $n = 1$, we find

$$\begin{aligned} B &\geq \left(T \left(1 + \frac{d+1-c}{c-1+d(c^T-1)} \right) - \frac{d(c^{T+1}-c)}{(c-1)(c-1+d(c^T-1))} \right) \\ &\quad \left(\frac{1}{1+d} + \frac{1}{d} \cdot \frac{c-c}{c^2-1} \right) \\ &\geq \left(T + \frac{T(d+1-c)}{c-1+d(c^T-1)} - \frac{c}{c-1} \right) \left(\frac{1}{1+d} \right). \end{aligned}$$

On the other hand, if $1 \leq n \leq T$,

$$\sum_{k=n}^T (1 - h_{out} \sum_{i=0}^k J(i)) \leq T$$

and

$$\sum_{k=1}^n \frac{1}{J(k) + J(k-1)} = \frac{1}{1+d} + \frac{1}{d} \cdot \frac{c-c^{2-n}}{c^2-1} \leq \frac{1}{1+d} + \frac{1}{d} \cdot \frac{c}{c^2-1},$$

so, combining all parts we see that

$$\left(T + \frac{T(d+1-c)}{c-1+d(c^T-1)} - \frac{c}{c-1} \right) \left(\frac{1}{1+d} \right) \leq B \leq T \left(\frac{1}{1+d} + \frac{1}{d} \cdot \frac{c}{c^2-1} \right).$$

If $c \geq 1 + \varepsilon$, $B = \Theta(T/d)$ and $\lambda_2 = O(d/T)$.

Example 5.6.3.

If $J(i) = 1$ for all $i \geq 0$, then T satisfies $T + 1 \leq \frac{1}{h_{out}} \leq T + 2$.

$$\begin{aligned} B &= \sup_{n \geq 1} \left(\sum_{k=n}^T (1 - h_{out}(k+1)) \right) \left(\sum_{k=1}^n \frac{1}{2} \right) \\ &= \sup_{n \geq 1} \left((T - n + 1) - h_{out} \left(\binom{T+1}{2} - \binom{n}{2} \right) \right) \left(\frac{n}{2} \right) \\ &\geq \sup_{n \geq 1} \left(T + 1 - n - \frac{1}{T+1} \frac{T^2 + T - n^2 + n}{2} \right) \cdot \frac{n}{2} \\ &= \frac{T^2}{27} (1 \pm o(1)) = \frac{1}{27h_{out}^2} (1 \pm o(1)). \end{aligned}$$

Here the supremum is achieved when $n \approx T/3$.

We have that $\lambda_2 \leq \frac{27}{4} h_{out}^2 (1 + o(1))$. In this case the Cheeger lower bound $\lambda_2 \geq c * h_{out}^2/d$ is tight up to a linear factor of d .

5.6.2 Examples of specific graphs

Example 5.6.4. The hypercube Ω_d is commonly expressed as the graph with vertex set $\{0, 1\}^d$ and $x \sim y$ if and only if x and y disagree in exactly one coordinate. With this notation, we define the k -slice $A_k \subset V$ to be the set of vertices that are 1 in exactly k coordinates. It is clear that $|A_k| = \binom{d}{k}$.

It is known that h_{out} is achieved by the $\lfloor d/2 \rfloor$ -slice Σ , with $h_{out} = \Theta(1/\sqrt{d})$ [37]. With this choice of Σ , $d^{-1}(i) = A_{\lfloor d/2 \rfloor + i}$, and $|d^{-1}(i)| = \binom{d}{\lfloor d/2 \rfloor + i} \leq \binom{d}{\lfloor d/2 \rfloor} = |\Sigma|$. As such, we may set $J(i) = 1$, and we have $T = \lfloor \frac{1}{h_{out} + 1} \rfloor = \Theta(\sqrt{d})$.

By the results of Example 5.6.3, $\lambda_2 \leq \frac{27}{4} h_{out}^2 (1 + o(1))$, thus $\lambda_2 = O(1/d)$. It is well-known that the actual value of λ_2 is indeed $\Theta(1/d)$.

Example 5.6.5 (Discrete torus). If C_n is the n -cycle for $n \geq 3$, the discrete torus C_n^d is the $2d$ -regular graph $C_n \square C_n \dots \square C_n$. It is understood that h_{out} is achieved by the ball $B(x, \lceil \frac{n}{2} \rceil - 1)$ with $\Sigma = S(x, \lceil \frac{n}{2} \rceil)$ where x is an arbitrary fixed vertex [16]. We have that $h_{out} = \frac{2}{n} + o(\frac{1}{n})$. The level sets are $d_{\Sigma}^{-1}(i) = S(x, i + \lceil \frac{n}{2} \rceil)$ with $|d_{\Sigma}^{-1}(i)| \leq |\Sigma|$.

Similarly to the hypercube, we may use $J(i) = 1$ as in Example 5.6.3 to see that $\lambda_2 \leq \frac{27}{4} h_{out}^2 (1 + o(1))$, thus $\lambda_2 \leq \frac{27}{d^2} (1 + o(1))$. It is well-known that the actual value is $\lambda_2 = \Theta(\frac{1}{d^2})$, so our estimate is tight up to a constant.

CHAPTER VI

EXPANSION IN HYPERGRAPHS

6.1 Overview

A problem that has generated much interest is to understand the relationship between the combinatorial expansion properties of a graph and the algebraic properties of the adjacency matrix. A famous result of this type is the Cheeger inequalities [2] which we have mentioned many times in previous chapters.

Another celebrated result is the Expander Mixing Lemma, which, for a regular graph, bounds the spectral gap of the adjacency matrix with the combinatorial discrepancy.

Theorem 6.1.1 (Expander Mixing Lemma for graphs, [3]). *If G is an r -regular graph, then for any $S, T \subseteq V(G)$,*

$$|E(S, T) - \frac{r}{n} |S| |T|| \leq \lambda_2(G) \sqrt{|S| |T|}, \quad (51)$$

where $E(S, T)$ is the number of pairs $(s, t) \in S \times T$ such that $st \in E(G)$.

The discrepancy $\rho(G)$ is defined to be

$$\rho(G) = \max_{S, T} \frac{|E(S, T) - \frac{r}{n} |S| |T||}{\sqrt{|S| |T|}}, \quad (52)$$

with the maximum taken over all non-empty subsets $S, T \subset V(G)$. With this notation, the statement of the Expander Mixing Lemma simplifies to $\rho(G) \leq \lambda_2(G)$.

More recently, Bilu & Linial proved a partial converse to the Expander Mixing Lemma.

Theorem 6.1.2 (Inverse Mixing Lemma for graphs, [13]). *If G is an r -regular graph such that for every disjoint $S, T \subset V(G)$*

$$|E(S, T) - \frac{r}{n} |S| |T|| \leq \rho \sqrt{|S| |T|}, \quad (53)$$

then

$$\lambda_2(G) = O(\rho(\log(r/\rho) + 1)). \quad (54)$$

A consequence of the Inverse Mixing Lemma is that the Mixing Lemma is tight up to a logarithmic factor. Bollobás and Nikiforov constructed graphs for which the logarithmic factor is necessary, so that the Inverse Mixing Lemma is tight [17].

With the aim of better understanding expansion of hypergraphs, there have been multiple suggestions of analogues for the adjacency matrix and related concepts to hypergraphs. In this chapter we investigate the Inverse Mixing Lemma for two types of eigenvalues on a k -regular hypergraph.

The first, defined by Chung [26] using the Hodge algebra, preserves the property that the adjacency matrix is an operator that indicates whether or not two $k - 1$ -sets of vertices share an edge in common. For this eigenvalue, Parzanchevski, Rosenthal, and Tessler proved an Expander Mixing Lemma [56]. Using the so-called local-global technique of expressing a uniform hypergraph as an interacting collection of simple graphs, we find an Inverse Mixing Lemma that is tight within a logarithmic factor. We leave open the question of whether that logarithmic factor is necessary when $k > 2$.

The second, defined in [34] by Friedman and Wigderson, preserves the property that the adjacency matrix is a k -linear form. For this eigenvalue, there is an Expander Mixing Lemma due to Lenz & Mubayi [46]. We find that this eigenvalue does not allow for an interesting Inverse Mixing Lemma. However, we make a small modification to the definition of the eigenvalue. This change allows us to prove an Inverse Mixing Lemma.

The results in this chapter are also found in the author's joint work with E. Cohen, D. Mubayi, and P. Tetali [27].

6.2 *Simplicial complexes*

6.2.1 Definitions

Let V be a finite vertex set. Throughout this section we will set n to be the size of V . Any subset of V is a *cell*, and in particular a subset of V with $i + 1$ elements is known as an i -cell. The reason for this possibly counter-intuitive definition is so that an i -cell will define a simplex of dimension i . For instance, a 0-cell is a point, a 1-cell is a line segment, and so on. A set $X \subset 2^V$ is said to be a simplicial complex if whenever X contains a cell τ , X also contains every subcell of τ .

Let X be a simplicial complex on ground set V . We write X^i for the set of i -cells in X . Through this section we will let $d \geq 1$ be the largest dimension of any cell in X , that is, the largest integer for which X^d is nonempty. d is the dimension of X .

A d -dimensional simplicial complex X is said to have a *complete skeleton* if $X^i = \binom{V}{i+1}$ whenever $i < d$ - that is, X contains all cells of dimension less than d . Observe that a d -dimensional simplicial complex with a complete skeleton is completely determined by the set of d -cells X^d .

In this sense the d -dimensional simplicial complexes on ground set V with complete skeleta are equivalent to the $d + 1$ -regular hypergraphs on ground set V . A minor exception is that the hypergraph with no edges is, in this way of thinking, in correspondence with the simplicial complex with $X^i = \binom{V}{i+1}$ whenever $i < d$ and X^d empty. In order to make the equivalence complete, we will include this complex as an example of a d -dimensional simplicial complex with complete skeleton.

If S_0, \dots, S_d are any $d + 1$ subsets of V , we write $F(S_0, \dots, S_d)$ for the number of ordered $(d + 1)$ -tuples $(s_0, \dots, s_d) \in S_0 \times \dots \times S_d$ satisfying $\{s_0, \dots, s_d\} \in X^d$.

Let $\{\sigma_0, \dots, \sigma_i\}$ be an i -cell. If $i > 0$, an i -cell $\{\sigma_0, \dots, \sigma_i\}$ has two orientations given by the orderings of its vertices up to an even permutation. For each i -cell we arbitrarily choose one canonical orientation which is denoted $\sigma = (\sigma_0, \dots, \sigma_i)$ and the other orientation by $\bar{\sigma}$. Let X_{\pm}^i denote the set of all orientations of i -cells from X^i .

Note: sometimes σ is also used to refer to the *unoriented* i -cell (that is, the set of size $i + 1$) which corresponds to the oriented i -cells σ and $\bar{\sigma}$.

If σ is an oriented i -cell and $v \in V \setminus \sigma$, we write $v\sigma$ for the oriented $i + 1$ -cell $(v, \sigma_0, \dots, \sigma_i)$ and we say $v \sim \sigma$ if and only if $v\sigma \in X^{i+1}$. If $\sigma \in X^{d-1}$, then $\deg(\sigma) := |\{v \in V : v \sim \sigma\}|$ is its degree. We say that X is r -regular if $\deg(\sigma) = r$ for all $\sigma \in X^{d-1}$.

Definition 6.2.1. If $i > 0$, let Ω^i be the vector space of real-valued, skew-symmetric functions on X_{\pm}^i , i.e., functions $f : X_{\pm}^i \rightarrow \mathbb{R}$ so that $f(\bar{\sigma}) = -f(\sigma)$ for all $\sigma \in X_{\pm}^i$. Because 0-cells have only one orientation, we specially define $\Omega^0 = \mathbb{R}^V$.

For example, we can think of Ω^0 as the set of vertex weightings, and Ω^1 as the set of flow functions: that is, functions where $f(vw)$ can represent a flow from v to w . In this example, the requirement that f be skew-symmetric translates to the fact that the flow from v to w is the negative of the flow from w to v .

Definition 6.2.2. Define an inner product on Ω^i by

$$\langle f, g \rangle := \sum_{\sigma \in X^i} f(\sigma)g(\sigma), \quad (55)$$

noting that we only sum over the canonical orderings σ and not $\bar{\sigma}$. The inner product is independent of the choice of canonical orderings because $f(\sigma)g(\sigma) = f(\bar{\sigma})g(\bar{\sigma})$.

This inner product induces a norm on Ω_i : $\|f\| := \sqrt{\langle f, f \rangle}$. Given a norm on Ω_i we can also define an operator norm: if $M : \Omega^i \rightarrow \Omega^i$,

$$\|M\| := \sup_{\substack{f \in \Omega^i \\ f \neq 0}} \frac{\|Mf\|}{\|f\|}. \quad (56)$$

From now on, let X be a d -dimensional simplicial complex.

Definition 6.2.3. Define the *boundary operator* $\partial_{d-1} : \Omega^{d-1} \rightarrow \Omega^{d-2}$ by

$$(\partial_{d-1}f)(\tau) := \sum_{v \sim \tau} f(v\tau), \quad (57)$$

and let $Z_{d-1} := \ker \partial_{d-1}$.

Continuing the above examples, the boundary of a weight function is the total weight of vertices, while the boundary of a flow is the function that assigns to each vertex the net flow at that vertex. Correspondingly, Z_0 is the set of vertex-weightings with weights summing to 0, while Z_1 is the set of conservative flows.

Definition 6.2.4. For every $(d-2)$ -cell τ , define linear operators $A_\tau, J_\tau : \Omega^{d-1} \rightarrow \Omega^{d-1}$ by

$$(A_\tau f)(\sigma) := \begin{cases} \sum_{w \sim v\tau} f(w\tau) & \text{if } \sigma = v\tau \\ \sum_{w \sim v\tau} f(\overline{w\tau}) & \text{if } \sigma = \overline{v\tau} \\ 0 & \text{if } \tau \not\subset \sigma \end{cases} \quad \text{and} \quad (J_\tau f)(\sigma) := \begin{cases} \sum_{w \sim \tau} f(w\tau) & \text{if } \sigma = v\tau \\ \sum_{w \sim \tau} f(\overline{w\tau}) & \text{if } \sigma = \overline{v\tau} \\ 0 & \text{if } \tau \not\subset \sigma. \end{cases} \quad (58)$$

Let $A := \sum_{\tau \in X^{d-2}} A_\tau$ be the adjacency operator, and let $J := \sum_{\tau \in X^{d-2}} J_\tau$ be the lower Laplacian, which is sometimes also denoted by Δ^- , Δ^L , or Δ^{down} . Denote by I the identity operator on Ω^{d-1} .

Remark. The fact that we have chosen canonical orientations allows us to represent operators from $\Omega^i \rightarrow \Omega^i$ as $|X^i| \times |X^i|$ matrices, indexed by the canonical orientations of i -cells. Under this representation the inner product and norm behave as the Euclidean norm and product on $\mathbb{R}^{|X^i|}$.

We may take such a matrix for the operator $A_\tau : \Omega^{d-1} \rightarrow \Omega^{d-1}$. If we restrict this matrix to the indices for $d-1$ -cells σ where $\tau \subset \sigma$, we find an adjacency matrix of the graph induced on d -cells and $(d-1)$ -cells containing τ , with $w\tau \sim v\tau$ (i.e., a ± 1 in the coordinate corresponding to the positive orientations of $w\tau$ and $v\tau$) if and only if $wv\tau \in X^d$. The signs are determined by the orientations of $w\tau$ and $v\tau$ relative to the canonical positive orientations of those cells (for instance, if the positive orientations are $w\tau$ and $\overline{v\tau}$ then the entry is negative). This graph is commonly known as the *link* of τ .

Similarly, the matrix for J_τ would have a ± 1 for any pair of $(d-1)$ -cells (not necessarily distinct) both containing τ , with the signs determined in the same fashion.

More precisely, if $\sigma, \sigma' \in X^{d-1}$ are positively-oriented cells that differ by exactly one vertex, let $\pi_{\sigma, \sigma'}$ be the unique permutation of $\{0, \dots, d-1\}$ so that $\sigma_{\pi_{\sigma, \sigma'}(i)} = \sigma'_i$ whenever $\sigma'_i \in \sigma$. Then we can calculate the σ, σ' entry of A and J :

$$A_{\sigma, \sigma'} = \begin{cases} \text{sgn}(\pi_{\sigma, \sigma'}) & \text{if } \sigma \cup \sigma' \in X^d \\ 0 & \text{otherwise.} \end{cases} \quad \text{and} \quad J_{\sigma, \sigma'} = \begin{cases} \text{sgn}(\pi_{\sigma, \sigma'}) & \text{if } |\sigma \cup \sigma'| = d+1, \\ d & \text{if } \sigma = \sigma' \\ 0 & \text{otherwise.} \end{cases} \quad (59)$$

Note that for ease of analysis positive orientations can be chosen for any *particular* τ to make all of the signs in A_τ positive, but (depending on A) it may be impossible to maintain this property across $(d-2)$ -cells τ simultaneously and so the matrix for A might exhibit both signs.

For graphs each cell has only one orientation, so we have $X^0 = V$ and $\Omega^0 = \mathbb{R}^V$ and we can think of the usual adjacency matrix of a graph as an operator $A : \mathbb{R}^V \rightarrow \mathbb{R}^V$. Indeed, for $d = 1$ the only (-1) -cell is the empty set, and so $A = A_\emptyset$ is just the adjacency matrix of the graph, while J_\emptyset is the all-ones matrix.

Definition 6.2.5. Finally, define the *degree operator* $D : \Omega^{d-1} \rightarrow \Omega^{d-1}$ by

$$(Df)(\sigma) := \deg(\sigma)f(\sigma), \quad (60)$$

and define $\Delta^+ := D - A$.

Note that for $d = 1$, Δ^+ is the graph Laplacian.

6.2.2 Mixing lemmas for simplicial complexes

With the above definitions and notation, we can state the mixing lemma for simplicial complexes:

Theorem 6.2.1 (Mixing Lemma for simplicial complexes, [56]). *Let X be a d -dimensional complex with a complete skeleton and fix $\alpha \in \mathbb{R}$. For any disjoint sets $S_0, \dots, S_d \subseteq V$,*

$$\left| F(S_0, \dots, S_d) - \frac{\alpha}{n} |S_0| \dots |S_d| \right| \leq \rho_\alpha \sqrt{|S_0| |S_1| |S_2| \dots |S_d|}, \quad (61)$$

where $\rho_\alpha := \|(\alpha I - \Delta^+)|_{Z_{d-1}}\|$.

In the original paper [56], this result is obtained. The authors then observe that the choice of S_0 and S_1 is arbitrary. Following a symmetry argument they obtain (and state as the main result in their work) a more symmetric result:

$$\left| F(S_0, \dots, S_d) - \frac{\alpha}{n} |S_0| \dots |S_d| \right| \leq \rho_\alpha \left(|S_0| |S_1| |S_2| \dots |S_d| \right)^{\frac{d}{d+1}}.$$

Here we state the stronger version as the result of Theorem 6.2.1, as this version is required for the rest of our work.

If X is r -regular it will be most common to set $\alpha = r$, so that ρ_α is the largest non-trivial eigenvalue of $A = rI - \Delta^+$. That is, ρ_α is the largest eigenvalue in absolute value besides the $\binom{n-1}{d-1}$ eigenvalues that must be equal to r [56]. Throughout this section we will refer to the maximum absolute value of any non-trivial eigenvalue as the second-largest eigenvalue.

The main result of this section is an inverse of the mixing lemma for simplicial complexes.

Theorem 6.2.2. *Let X be an r -regular, d -dimensional complex with a complete skeleton, and suppose that for every collection of disjoint sets $S_0, \dots, S_d \subseteq V$*

$$\left| F(S_0, \dots, S_d) - \frac{r}{n} |S_0| \dots |S_d| \right| \leq \rho \sqrt{|S_0| |S_1| |S_2| \dots |S_d|}. \quad (62)$$

Then

$$\|A|_{Z_{d-1}}\| = O(\rho d(\log(r/\rho) + 1)). \quad (63)$$

Again, when the complex is regular with a complete skeleton this quantity is the second-largest eigenvalue of A .

Remark. It is possible to generalize this result by replacing the r in Equation 63 with an arbitrary value $\alpha \in \mathbb{R}$, and by replacing A with $\alpha I - \Delta^+$, as in the statement of Theorem 6.2.1.

Remark. In our proof, we do not use the full strength of the hypothesis. We will always take S_2, \dots, S_d to be singletons.

Remark. Our proof makes use of the local-global technique which is common in the theory of higher-dimensional Laplacians, and which was introduced in [35].

Proof. It is clear from the graph interpretation of A_τ that the largest eigenvalue of A_τ is r , with eigenfunction $f(v\tau) = 1$ (and $f(\overline{v}\tau) = -1$) if $v \sim \tau$ and $f(\sigma) = 0$ if $\tau \not\subset \sigma$. We will bound $\|A_\tau - \frac{r}{n}J_\tau\|$, which is an approximation of the second eigenvalue of A_τ .

First we argue that $J_\tau|_{Z_{d-1}} = 0$. To see this, consider $f \in Z_{d-1}$, $\tau \in X^{d-2}$, and $\sigma \in X^{d-1}$. If $\tau \not\subset \sigma$ then $(J_\tau f)(\sigma) = 0$. On the other hand, if $\tau \subset \sigma$ then we can write $\sigma = v\tau$ for some $v \notin \tau$. In this case

$$(J_\tau f)(\sigma) = \sum_{w \sim \tau} f(w\tau) = (\partial_{d-1} f)(\tau) = 0, \quad (64)$$

so we have $J_\tau f \equiv 0$ for every $f \in Z_{d-1}$, or in other words $J_\tau|_{Z_{d-1}} = 0$.

This allows us to say that

$$\|A|_{Z_{d-1}}\| = \left\| \left(A - \sum_{\tau \in X^{d-2}} \frac{r}{n} J_\tau \right) \Big|_{Z_{d-1}} \right\| \quad (65)$$

$$\leq \left\| A - \sum_{\tau \in X^{d-2}} \frac{r}{n} J_\tau \right\| \quad (66)$$

$$= \left\| \sum_{\tau \in X^{d-2}} (A_\tau - \frac{r}{n} J_\tau) \right\| \quad (67)$$

$$= \|A - \frac{r}{n} J\|. \quad (68)$$

What remains is to bound Equation 68. We use a lemma of Bilu & Linial:

Lemma 6.2.3 ([13]). *Let B be a symmetric, real-valued $n \times n$ matrix in which the diagonal entries are all 0. Suppose that the ℓ^1 -norm of every row of B is at most m , and also that for any vectors $x, y \in \{0, 1\}^n$ with disjoint support, we have*

$$|\langle x, By \rangle| \leq \beta \|x\| \|y\|. \quad (69)$$

Then

$$\|B\| = O(\beta(\log(m/\beta) + 1)). \quad (70)$$

We will apply this lemma to

$$B = A - \frac{r}{n} J + \frac{rd}{n} I. \quad (71)$$

Here the first two terms are the operator that is normed in Equation 68, and the final term is chosen so that the diagonal is uniformly 0.

As mentioned before, we can interpret B as a matrix indexed by positive orientations of elements in X^{d-1} . Combining the calculations of A and J in that remark, we can calculate that for each $\sigma, \sigma' \in X^{d-1}$,

$$B_{\sigma, \sigma'} = \begin{cases} \text{sgn}(\pi_{\sigma, \sigma'}) (1 - r/n) & \text{if } \sigma \cup \sigma' \in X^d \\ \text{sgn}(\pi_{\sigma, \sigma'}) (-r/n) & \text{if } \sigma \cup \sigma' \in \binom{V}{d+1} \setminus X^d \\ 0 & \text{otherwise.} \end{cases} \quad (72)$$

Then we can see that B is symmetric (because $\pi_{\sigma',\sigma} = \pi_{\sigma,\sigma'}^{-1}$), real-valued and its diagonal entries are 0. Since X is r -regular, the ℓ^1 -norm of each row σ in B is

$$\sum_{\sigma' \in X^{d-1}} |B_{\sigma,\sigma'}| = dr \left(1 - \frac{r}{n}\right) + (n - d - r)d \frac{r}{n} \leq 2dr. \quad (73)$$

Indeed, there are $(n - d)$ total sets η of size $d + 1$ containing σ , and r of those are d -cells; each such set η contains d other cells σ' such that $\sigma \cup \sigma' = \eta$.

Let $x, y : X_{\pm}^{d-1} \rightarrow \{0, \pm 1\}$ be functions in Ω^{d-1} with disjoint support, we observe that $\langle x, Iy \rangle = 0$. For each $\tau \in X^{d-2}$, define $x_{\tau}(\sigma) = x(\sigma)$ if $\tau \subset \sigma$ and $x_{\tau}(\sigma) = 0$ otherwise. Define y_{τ} similarly. Note that each $\sigma \in \text{supp } x$ is in the support of exactly d of the functions x_{τ} . Observe that

$$\langle x, By \rangle = \left\langle x, \left(\sum_{\tau \in X^{d-2}} A_{\tau} - \frac{r}{n} J_{\tau} \right) y \right\rangle + \frac{rd}{n} \langle x, Iy \rangle \quad (74)$$

$$= \sum_{\tau \in X^{d-2}} \langle x, (A_{\tau} - \frac{r}{n} J_{\tau}) y \rangle = \sum_{\tau \in X^{d-2}} \langle x_{\tau}, A_{\tau} y_{\tau} \rangle - \frac{r}{n} \langle x_{\tau}, J_{\tau} y_{\tau} \rangle. \quad (75)$$

By definition, for any fixed $\tau = (\tau_2, \dots, \tau_d)$,

$$\langle x_{\tau}, A_{\tau} y_{\tau} \rangle = \sum_{\sigma \in X^{d-1}} x_{\tau}(\sigma) (A_{\tau} y_{\tau})(\sigma) = \sum_{v \notin \tau} x(v\tau) (A_{\tau} y_{\tau})(v\tau) \quad (76)$$

$$= \sum_{v \notin \tau} x(v\tau) \sum_{w \sim v\tau} y_{\tau}(w\tau) = \sum_{\substack{v \neq w \\ v, w \notin \tau}} 1_{[vw\tau \in X^d]} x(v\tau) y(w\tau). \quad (77)$$

Using a similar decomposition for $\langle x_{\tau}, J_{\tau} y_{\tau} \rangle$ we obtain

$$\langle x_{\tau}, (A_{\tau} - \frac{r}{n} J_{\tau}) y_{\tau} \rangle = \sum_{\substack{v \neq w \\ v, w \notin \tau}} (1_{[vw\tau \in X^d]} - \frac{r}{n}) x(v\tau) y(w\tau). \quad (78)$$

We would like to interpret the first half of the sum as a number of edges and the second half as the product of the sizes of some vertex sets, since for any disjoint sets S_0, S_1 we have by assumption that

$$\left| \sum_{\substack{v \in S_0 \\ w \in S_1}} (1_{[vw\tau \in X^d]} - \frac{r}{n}) \right| = |F(S_0, S_1, \{\tau_2\}, \dots, \{\tau_d\}) - \frac{r}{n} |S_0| |S_1| |\{\tau_2\}| \dots |\{\tau_d\}| | \quad (79)$$

$$\leq \rho \sqrt{|S_0| |S_1|}. \quad (80)$$

However, this differs from what we have above by the fact that x and y may be -1 as well as 0 or 1 . First we must break the sum apart according to the signs of x and y . For each $\eta \in \{\pm 1\}$ we define

$$S_0^\eta = \{v : x(v\tau) = \eta\} \quad S_1^\eta = \{w : y(w\tau) = \eta\}. \quad (81)$$

These four sets are pairwise disjoint and $\|x_\tau\|^2 = |S_0^+| + |S_0^-|$ and $\|y_\tau\|^2 = |S_1^+| + |S_1^-|$, so now we can write

$$|\langle x_\tau, (A_\tau - \frac{r}{n} J_\tau) y_\tau \rangle| = \left| \sum_{v,w} (1_{vw\tau \in X^d} - \frac{r}{d}) x(v\tau) y(w\tau) \right| \quad (82)$$

$$= \left| \sum_{\eta_0, \eta_1 \in \{\pm 1\}} \eta_0 \eta_1 \sum_{\substack{v \in S_0^{\eta_0} \\ w \in S_1^{\eta_1}}} (1_{[vw\tau \in X^d]} - \frac{r}{n}) \right| \quad (83)$$

$$\leq \sum_{\eta_0, \eta_1 \in \{\pm 1\}} \rho \sqrt{|S_0^{\eta_0}| |S_1^{\eta_1}|} = \rho \sum_{\eta_0 \in \{\pm 1\}} \sqrt{|S_0^{\eta_0}|} \sum_{\eta_1 \in \{\pm 1\}} \sqrt{|S_1^{\eta_1}|} \quad (84)$$

$$\leq \rho \sqrt{2 \sum_{\eta_0 \in \{\pm 1\}} |S_0^{\eta_0}|} \sqrt{2 \sum_{\eta_1 \in \{\pm 1\}} |S_1^{\eta_1}|} = 2\rho \|x_\tau\| \|y_\tau\| \quad (85)$$

by Cauchy-Schwarz.

Summing over all $\tau \in X^{d-2}$ gives that

$$|\langle x, By \rangle| = \left| \sum_{\tau \in X^{d-2}} \langle x_\tau, (A_\tau - \frac{r}{n} J_\tau) y_\tau \rangle \right| \quad (86)$$

$$\leq \sum_{\tau \in X^{d-2}} |\langle x_\tau, (A_\tau - \frac{r}{n} J_\tau) y_\tau \rangle| \quad (87)$$

$$\leq \sum_{\tau \in X^{d-2}} 2\rho \sqrt{|\text{supp } x_\tau| |\text{supp } y_\tau|} \quad (88)$$

$$\leq 2\rho \sqrt{\sum_{\tau \in X^{d-2}} |\text{supp } x_\tau|} \sqrt{\sum_{\tau \in X^{d-2}} |\text{supp } y_\tau|} \quad (89)$$

$$= 2\rho \sqrt{d |\text{supp } x|} \sqrt{d |\text{supp } y|} \quad (90)$$

$$= 2\rho d \|x\| \|y\|, \quad (91)$$

where the inequality in Equation 89 follows from Cauchy-Schwarz.

Finally we can apply Lemma 6.2.3 with $m = 2rd$ and $\beta = 2\rho d$ to find

$$\left\| \left(\sum_{\tau \in X^{d-2}} A_\tau - \frac{r}{n} J_\tau \right) + \frac{rd}{n} I \right\| = O(\rho d (\log(r/\rho) + 1)). \quad (92)$$

Combining the results for each τ using the triangle inequality gives

$$\|A|_{Z_{d-1}}\| \leq \left\| \sum_{\tau \in X^{d-2}} A_\tau - \frac{r}{n} J_\tau \right\| \leq \left\| \left(\sum_{\tau \in X^{d-2}} A_\tau - \frac{r}{n} J_\tau \right) + \frac{rd}{n} I \right\| + \frac{rd}{n}. \quad (93)$$

To prove the theorem, all that remains is to argue that the second term of the above bound is bounded by $(\frac{rd}{n}) \leq 4\rho d$ so that $\frac{rd}{n} = O(\rho d (\log(r/\rho) + 1))$. (It was previously argued that this expression bounds the first term.)

As long as $\emptyset \subsetneq X^d \subsetneq \binom{V}{d+1}$, $\rho \geq \max\{1 - \frac{r}{n}, \frac{r}{n}\} \geq 1/2$ (take S_0, \dots, S_d to be singletons corresponding to a subset which is either a d -cell or not), so $\frac{rd}{n} \leq d \leq 2\rho d$ and we can replace the above bound by

$$\|A|_{Z_{d-1}}\| = O(\rho d (\log(r/\rho) + 1)). \quad (94)$$

We also need to consider the special cases of the empty and complete complexes. For the empty complex, $r = 0$ and clearly $\frac{rd}{n} = 0$. For the complete complex, $r = n - d$.

We may take S_2, \dots, S_d to be singletons and $|S_0| = |S_1| = \lceil \frac{n-d}{2} \rceil$, to get

$$\rho \geq \frac{F(S_0, \dots, S_d) - \frac{r}{n} |S_0| \dots |S_d|}{\sqrt{|S_0| |S_1| |S_2| \dots |S_d|}} = \frac{n-r}{n} \sqrt{|S_0| |S_1|} = \frac{d(n-d)}{2n} \geq \frac{1}{4} \quad (95)$$

when $1 \leq d < n$, so that $\frac{rd}{n} \leq d \leq 4\rho d$.

□

6.3 Friedman-Wigderson eigenvalues

In this section we will use a common notion of hypergraph eigenvalues that was first described by Friedman and Wigderson [34]. Ultimately we find that the traditional Friedman-Wigderson eigenvalue does not allow for an interesting inverse mixing lemma. However, we suggest a small modification to the eigenvalue that permits a higher quality of inverse mixing.

6.3.1 Notation for hypergraph eigenvalues

Throughout, let $H = (V, E(H))$ be a k -uniform hypergraph with vertex set $V = \{v_1, \dots, v_n\}$. We will only consider hypergraphs H with no loops or multiple edges, that is, $E(H) \subseteq \binom{V}{k}$. The *degree* $\deg(S) = \deg_H(S)$ of a $(k-1)$ -set S of vertices in H is the number of edges containing S . In other works this quantity is sometimes called the co-degree of S . We say that H is *r-regular* if $\deg(S) = r$ for every $(k-1)$ -set S .

For the Friedman-Wigderson eigenvalue, the higher-dimension analogue of a graph matrix is a k -linear form $M : (\mathbb{R}^n)^k \rightarrow \mathbb{R}$. If $k = 2$, we have M as a matrix, so that if $x, y \in \mathbb{R}^n$, we may evaluate $x^T M y \in \mathbb{R}$, and that the i, j entry of M satisfies $M_{i,j} = e_i^T M e_j$, where e_1, \dots, e_n are the standard basis vectors of \mathbb{R}^n . For values of k with $k > 2$, we will consider k -linear forms with an analogous property.

Definition 6.3.1 (Hypergraph adjacency form). Let $A = A_H : \prod_{i=1}^k \mathbb{R}^n \rightarrow \mathbb{R}$ be the k -linear form defined by

$$A(e_{i_1}, e_{i_2}, \dots, e_{i_k}) := \begin{cases} 1 & \text{if } \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\} \in E(H) \\ 0 & \text{otherwise} \end{cases} \quad (96)$$

for all choices of standard basis vectors $e_{i_1}, e_{i_2}, \dots, e_{i_k}$.

Definition 6.3.2. If V_1, \dots, V_k are subsets of V , then let

$$e_H(V_1, \dots, V_k) := |\{(v_1, \dots, v_k) \in V_1 \times \dots \times V_k : \{v_1, \dots, v_k\} \in E(H)\}|. \quad (97)$$

As with the adjacency form we will suppress the subscript H when the hypergraph is clear from context. If V_1, \dots, V_k are pairwise disjoint, this is the number of edges that intersect each V_i in exactly one vertex. Alternatively, if we take x^i to be the indicator vector of V_i then we could equivalently define $e(V_1, \dots, V_k) = A(x^1, \dots, x^k)$.

Let J denote the k -linear form with $J(e_{i_1}, e_{i_2}, \dots, e_{i_k}) = 1$ for all choices of standard basis vectors $e_{i_1}, e_{i_2}, \dots, e_{i_k}$. Let $K = (V, \binom{V}{k})$ denote the complete k -uniform

hypergraph on vertex set V (with corresponding adjacency form A_K which evaluates to 1 on any *distinct* standard basis vectors).

Definition 6.3.3. If $\phi : \prod_{i=1}^k \mathbb{R}^n \rightarrow \mathbb{R}$ is a k -linear form, we define the *spectral norm* of ϕ to be

$$\|\phi\| := \sup_{x_i \in \mathbb{R}^n, x_i \neq 0} \frac{|\phi(x_1, \dots, x_k)|}{\|x_1\| \dots \|x_k\|}. \quad (98)$$

In the case where ϕ is symmetric, as shown in [34] we in fact have that

$$\|\phi\| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{|\phi(x, \dots, x)|}{\|x\|^k}. \quad (99)$$

Observe that both A and J are symmetric.

Recall that the first (largest) eigenvalue of a graph can be defined as the operator norm of its adjacency matrix, $\|A_G\|$, and if the graph is r -regular then the second-largest eigenvalue is $\|A_G - \frac{r}{n}J\|$. This motivates a definition of the second eigenvalue for *hypergraphs* given by Friedman and Wigderson in [34]: if H is r -regular, we define the second eigenvalue to be

$$\lambda_2(H) := \|A - \frac{r}{n}J\|. \quad (100)$$

For any H (not necessarily r -regular), the quantity in Equation 100 is called the *second eigenvalue of H with respect to r -regularity*.

As mentioned above, λ_2 does not permit an interesting inverse to the expander mixing lemma. That requires a slightly different definition for the second eigenvalue.

Definition 6.3.4. For any $\alpha \in \mathbb{R}$, the *second eigenvalue of H with respect to α -density* is

$$\lambda_{2,\alpha}(H) := \|A - \alpha A_K\| = \sup_{x \in \mathbb{R}^n, x \neq 0} \frac{|A(x, \dots, x) - \alpha A_K(x, \dots, x)|}{\|x\|^k}. \quad (101)$$

In order to characterize the quality of mixing for a hypergraph we define the hypergraph discrepancy:

Definition 6.3.5. For a k -uniform hypergraph H , define for any value $\alpha \geq 0$

$$\rho_\alpha(H) := \max_{V_1, \dots, V_k} \frac{|e(V_1, \dots, V_k)| - \alpha |V_1| \dots |V_k|}{\sqrt{|V_1| \dots |V_k|}}, \quad (102)$$

where the maximum is taken over all k -tuples V_1, \dots, V_k of pairwise disjoint nonempty subsets of V .

Remark. Our aim is to bound the second eigenvalue in terms of $\rho = \rho_\alpha(H)$.

Unfortunately, we find that independently of ρ , $\lambda_2 = \Omega(rn^{k-2})$ with high probability for random hypergraphs with edge density r/n , making an inverse mixing lemma for λ_2 impossible. The reason for this is that the linear form $A - \frac{r}{n}J$ takes the constant value $-\frac{r}{n}$ on any list of k non-distinct elementary basis vectors. These terms can control the eigenvalue λ_2 . However, the diagonal terms do not at all help us to distinguish the quality of expansion among different r -regular and k -uniform hypergraphs - in fact, the diagonal terms are always the same constant!

This issue motivates our definition of $\lambda_{2,\alpha}$, in which we take a different k -linear form so that the value is always 0 on any list of non-distinct elementary basis vectors, and $\lambda_{2,\alpha}$ is, to a much larger extent, controlled by the particular hypergraph in question. A further discussion of this problem is found in Subsection 6.3.3.

It is natural in our definition to choose $\alpha = |E(H)| / |E(K)|$, i.e., the edge-density of H . However, to more closely parallel the Friedman-Wigderson definition one can choose $\alpha = \frac{r}{n}$. For now we will proceed without specifying a fixed value for α .

Remark. Even if one fixes α as suggested above to be the edge density of H , our definition does not quite agree with the usual definition of graph eigenvalues in the case of r -regular graphs ($k = 2$). In particular, while $\lambda(G) = \|A - \frac{r}{n}J\|$ we use $\lambda_{2,\alpha}(G) = \|A - \frac{r}{n-1}A_K\|$. However, it is easy to see that the two values differ by exactly $\frac{r}{n-1}\|I - \frac{1}{n}J\| = \frac{r}{n-1}$, which is bounded above by 1.

The following simple upper bound will come in handy in later analysis.

Proposition 6.3.1. *For any k -uniform hypergraph H with maximum degree r ,*

$$\rho_\alpha(H) \leq (r + \alpha n)n^{(k-2)/2}. \quad (103)$$

Proof. Working directly from the definition, we have

$$\rho_\alpha(H) = \max_{V_1, \dots, V_k} \frac{|e(V_1, \dots, V_k) - \alpha |V_1| \dots |V_k||}{\sqrt{|V_1| \dots |V_k|}} \quad (104)$$

$$\leq \max_{V_1, \dots, V_k} \frac{e(V_1, \dots, V_k) + \alpha |V_1| \dots |V_k|}{\sqrt{|V_1| \dots |V_k|}} \quad (105)$$

$$\leq \max_{|V_1| \geq \dots \geq |V_k|} \frac{r |V_2| \dots |V_k| + \alpha |V_1| \dots |V_k|}{\sqrt{|V_1| \dots |V_k|}} \quad (106)$$

$$= \max_{|V_1| \geq \dots \geq |V_k|} (r + \alpha |V_1|) \frac{\sqrt{|V_2| \dots |V_k|}}{\sqrt{|V_1|}} \quad (107)$$

$$\leq (r + \alpha n)n^{(k-2)/2}. \quad (108)$$

□

6.3.2 Hypergraph mixing lemmas

The following hypergraph mixing result is given in [34].

Theorem 6.3.2 (Mixing Lemma for hypergraphs, [34]). *Let H be a k -uniform hypergraph. For any choice of subsets $V_1, \dots, V_k \subset V(H)$ of vertices,*

$$\left| |e(V_1, \dots, V_k)| - \frac{k! |E(H)|}{n^k} |V_1| \dots |V_k| \right| \leq \lambda_2(H) \sqrt{|V_1| \dots |V_k|}. \quad (109)$$

Before stating and proving a converse to Theorem 6.3.2 above, we mention the mixing result using our definition of the second eigenvalue $\lambda_{2,\alpha}$, with respect to density α . The proof follows the structure of the proof of the standard Expander Mixing Lemma.

Theorem 6.3.3 (Mixing Lemma for hypergraphs). *Let H be a k -uniform hypergraph. For any choice of subsets $V_1, \dots, V_k \subset V(H)$ of vertices,*

$$|e(V_1, \dots, V_k) - \alpha e_K(V_1, \dots, V_k)| \leq \lambda_{2,\alpha}(H) \sqrt{|V_1| \dots |V_k|}. \quad (110)$$

Proof. Let $V_1, \dots, V_k \subset V(H)$. If any V_i is empty, it is clear that the inequality holds; we may assume that each V_i is nonempty. For $1 \leq i \leq k$ let $x^i \in \{0, 1\}^n$ be the indicator vector of V_i . Then

$$\frac{|e(V_1, \dots, V_k) - \alpha e_K(V_1, \dots, V_k)|}{\sqrt{|V_1| \dots |V_k|}} = \frac{|A(x^1, \dots, x^k) - \alpha A_K(x^1, \dots, x^k)|}{\prod_{i=1}^k \|x^i\|} \quad (111)$$

$$\leq \|A - \alpha A_K\| = \lambda_{2,\alpha}(H) \quad (112)$$

as desired. □

We now prove the main theorem of this section – a converse to Theorem 6.3.3:

Theorem 6.3.4 (Inverse Mixing Lemma for hypergraphs). *If H is a k -uniform hypergraph with maximum degree r and $\rho = \rho_\alpha(H)$, then*

$$\lambda_{2,\alpha}(H) = O\left(\rho(\log^{k-1}((r + \alpha n)n^{k-2}/\rho) + 1)\right). \quad (113)$$

Remark. In our discussion of hypergraphs (as opposed to simplicial complexes), the O -notation (or Θ -, Ω -, o -, \dots) suppresses a multiplicative constant that may depend on k (but not on n, ρ, r, α , or m .) One reason for this convention is that the regularity k is often treated as a fixed value which may not vary.

Remark. We have left this result in what is perhaps not its simplest form, in order to show the difference between the cases $k = 2$ and $k \geq 3$. In the case where $k = 2$ and $\alpha = \Theta(r/n)$ the dependence on n disappears and this simplifies to the classic result $\lambda_{2,\alpha} = O(\rho(\log(r/\rho) + 1))$ for graphs. For larger (but still constant) uniformity, we can still simplify the result to $\lambda_{2,\alpha} = O(\rho(\log^{k-1}((r + \alpha n)n/\rho) + 1))$.

We prove the theorem through a series of lemmas. First we show that the k -partite expansion condition suffices to give expansion for any (not necessarily disjoint) sets of vertices. Throughout, b represents a constant independent of x (but which may

depend on k, n, r, α, ρ or anything else). This proof method is similar to the proof of the (graph) inverse mixing lemma [13].

Lemma 6.3.5. *Let H be a k -uniform hypergraph on n vertices with adjacency form A , and suppose that*

$$|A(x^1, \dots, x^k) - \alpha A_K(x^1, \dots, x^k)| \leq \rho \prod_{i=1}^k \|x^i\| \quad (114)$$

for every choice of pairwise orthogonal vectors $x^1, \dots, x^k \in \{0, 1\}^n$. Then

$$|A(x^1, \dots, x^k) - \alpha A_K(x^1, \dots, x^k)| \leq \rho k^{k/2} \prod_{i=1}^k \|x^i\| \quad (115)$$

for every choice of (not necessarily orthogonal) vectors $x^1, \dots, x^k \in \{0, 1\}^n$.

Proof. Let $V_1, \dots, V_k \subseteq [n]$ be any sets of vertices. Consider an ordered partition $\mathcal{P} = P_1 \cup \dots \cup P_k$ of $[n]$ into k nonempty parts. Then

$$e(V_1, \dots, V_k) = \frac{1}{k^{n-k}} \sum_{\mathcal{P}} e(P_1 \cap V_1, \dots, P_k \cap V_k), \quad (116)$$

as every ordered edge (v_1, \dots, v_k) appears in the sum once for each partition \mathcal{P} with $v_j \in P_j$ for every j , and there are k^{n-k} such partitions (the remaining $n - k$ elements can be partitioned in any way among the k sets). Similarly, replacing H with the complete hypergraph gives

$$e_K(V_1, \dots, V_k) = \frac{1}{k^{n-k}} \sum_{\mathcal{P}} e_K(P_1 \cap V_1, \dots, P_k \cap V_k) = \frac{1}{k^{n-k}} \sum_{\mathcal{P}} \prod_i |P_i \cap V_i|. \quad (117)$$

For a fixed partition the subsets $P_i \cap V_i$ are disjoint, so by hypothesis we have

$$|e(P_1 \cap V_1, \dots, P_k \cap V_k) - \alpha e_K(P_1 \cap V_1, \dots, P_k \cap V_k)| \leq \rho \sqrt{\prod_i |P_i \cap V_i|}. \quad (118)$$

Recall that $S(n, k)$ denotes the number of ordered partitions of $[n]$ into k nonempty

sets, (i.e., the number of terms in the sum over all choices of \mathcal{P}). Then

$$|e(V_1, \dots, V_k) - \alpha e_K(V_1, \dots, V_k)| \quad (119)$$

$$\leq \frac{1}{k^{n-k}} \sum_{\mathcal{P}} |e(P_1 \cap V_1, \dots, P_k \cap V_k) - \alpha e_K(P_1 \cap V_1, \dots, P_k \cap V_k)| \quad (120)$$

$$\leq \frac{1}{k^{n-k}} \sum_{\mathcal{P}} \rho \sqrt{\prod_i |P_i \cap V_i|} \quad (121)$$

$$= \frac{\rho k! S(n, k)}{k^{n-k}} \sum_{\mathcal{P}} \frac{1}{k! S(n, k)} \sqrt{\prod_i |P_i \cap V_i|} \quad (122)$$

$$\leq \frac{\rho k! S(n, k)}{k^{n-k}} \sqrt{\frac{1}{k! S(n, k)} \sum_{\mathcal{P}} \prod_i |P_i \cap V_i|} \quad (123)$$

$$= \frac{\rho k! S(n, k)}{k^{n-k}} \sqrt{\frac{k^{n-k} e_K(V_1, \dots, V_k)}{k! S(n, k)}} \quad (124)$$

$$\leq \rho \sqrt{\frac{k! S(n, k)}{k^{n-k}}} \prod_i \sqrt{|V_i|} \leq \rho k^{k/2} \prod_i \sqrt{|V_i|}. \quad (125)$$

In the last inequality we use the fact that $k! S(n, k) \leq k^n$. The inequality in Equation 123 follows by concavity of square root.

The final result follows immediately, noting that if x^1, \dots, x^k are the indicator vectors for V_1, \dots, V_k then $e(V_1, \dots, V_k) = A(x^1, \dots, x^k)$ and $|V_i| = \|x^i\|^2$.

□

The main part of the proof is demonstrating a hypergraph version of Lemma 6.2.3. We follow the proof outline of [13] through several steps to show that if the expansion bound holds for $\{0, 1\}$ vectors then a relaxed bound holds for all real vectors.

Lemma 6.3.6. *Suppose B is a k -linear form such that*

$$|B(x^1, \dots, x^k)| \leq b \prod_{i=1}^k \|x^i\| \quad (126)$$

for every list of vectors $x^1, \dots, x^k \in \{0, 1\}^n$. Then

$$|B(x^1, \dots, x^k)| \leq 2^{k/2} b \prod_{i=1}^k \|x^i\| \quad (127)$$

for every $x^1, \dots, x^k \in \{0, \pm 1\}^n$.

Proof. Let $x^1, \dots, x^k \in \{0, \pm 1\}^n$, and decompose $x^i = x_+^i - x_-^i$ so that $x_\pm^i \in \{0, 1\}^n$ and $\text{supp } x^i = \text{supp } x_+^i \cup \text{supp } x_-^i$. Then

$$|B(x^1, \dots, x^k)| = |B(x_+^1 - x_-^1, \dots, x_+^k - x_-^k)| \quad (128)$$

$$\leq \sum_{\eta \in \{\pm\}^k} |B(x_{\eta_1}^1, \dots, x_{\eta_k}^k)| \leq \sum_{\eta \in \{\pm\}^k} b \prod_{i=1}^k \|x_{\eta_i}^i\| \quad (129)$$

$$= b \prod_{i=1}^k (\|x_+^i\| + \|x_-^i\|) \leq b \prod_{i=1}^k \sqrt{2} \|x^i\|. \quad (130)$$

□

Lemma 6.3.7. *Suppose B is a symmetric k -linear form satisfying*

$$\sum_{j=1}^n |B(e_{i_1}, \dots, e_{i_{k-1}}, e_j)| \leq m \quad (131)$$

for every $k-1$ -tuple $(i_1, \dots, i_{k-1}) \in [n]^{k-1}$ and

$$|B(x^1, \dots, x^k)| \leq b \prod_{i=1}^k \|x^i\| \quad (132)$$

for every list of vectors $x^1, \dots, x^k \in \{0, \pm 1\}^n$. Let $a \geq b/(mn^{(k-2)/2})$. Then

$$|B(x, \dots, x)| \leq b \left(\log^{k-1} \left(\frac{a^2 m^2 n^{k-2}}{b^2} \right) + \frac{k^2}{a} \right) \|x\|^k \quad (133)$$

for every $x \in \{0, \pm 2^{-\ell} : \ell \in \mathbb{N}\}^n$.

Proof. Let $x \in \{0, \pm 2^{-\ell} : \ell \in \mathbb{N}\}^n$ and write $x = \sum_{i \in \mathbb{N}} 2^{-i} x^i$ with $x^i \in \{0, \pm 1\}^n$ (the x^i have pairwise disjoint support and are hence orthogonal). Define $s_i = |\text{supp } x^i| = \|x^i\|^2$ so that

$$\|x\|_1 = \sum_{i \in \mathbb{N}} 2^{-i} s_i \quad \text{and} \quad \|x\|_2^2 = \sum_{i \in \mathbb{N}} 2^{-2i} s_i. \quad (134)$$

Note that these sums have only finitely many nonzero terms. Because

$$|B(x, \dots, x)| \leq \sum_{i \in \mathbb{N}^k} \left(\prod_{j=1}^k 2^{-i_j} \right) |B(x^{i_1}, \dots, x^{i_k})|, \quad (135)$$

we are interested in bounding the right-hand side of the inequality.

We split this sum into two parts, bounding separately the sums over the index sets

$$P = \{i \in \mathbb{N}^k : \max_j i_j - \min_j i_j < \gamma\} \quad \text{and} \quad Q = \mathbb{N}^k \setminus P \quad (136)$$

for some $\gamma \geq 0$ to be determined later. For the sum over $i \in P$ we have

$$\sum_{i \in P} \left(\prod_{j=1}^k 2^{-i_j} \right) |B(x^{i_1}, \dots, x^{i_k})| \leq \sum_{i \in P} \left(\prod_{j=1}^k 2^{-i_j} \right) b \prod_{j=1}^k \sqrt{s_{i_j}} \quad (137)$$

$$= b \sum_{i \in P} \left(\prod_j (2^{-2i_j} s_{i_j})^{k/2} \right)^{1/k} \quad (138)$$

$$\leq \frac{b}{k} \sum_{i \in P} \sum_j (2^{-2i_j} s_{i_j})^{k/2}, \quad (139)$$

where the final step uses the AM-GM inequality. Each $\ell \in \mathbb{N}$ appears at most $k(2\gamma)^{k-1}$ times in elements of P (as each time ℓ appears in some position the remaining $k-1$ terms must all be between $\ell - \gamma$ and $\ell + \gamma$), so

$$\frac{b}{k} \sum_{i \in P} \sum_j (2^{-2i_j} s_{i_j})^{k/2} \leq b(2\gamma)^{k-1} \sum_{\ell \in \mathbb{N}} (2^{-2\ell} s_\ell)^{k/2} \quad (140)$$

$$\leq b(2\gamma)^{k-1} \left(\sum_\ell 2^{-2\ell} s_\ell \right)^{k/2} \quad (141)$$

$$= b(2\gamma)^{k-1} \|x\|^k, \quad (142)$$

where we have used that $\sum_i a_i^{k/2} \leq (\sum_i a_i)^{k/2}$ for nonnegative a_i and $k \geq 2$.

Now we focus on bounding the sum over $i \in Q$. For each $i \in Q$ we reorder the indices so that i_1 is the least entry of i and i_k is the largest, without changing any other indices. Such a reordered index vector corresponds to at most k^2 non-reordered

vectors, so we have

$$\sum_{i \in Q} \left(\prod_{j=1}^k 2^{-i_j} \right) |B(x^{i_1}, \dots, x^{i_k})| \leq k^2 \sum_{i \in \mathbb{N}^{k-1}} \sum_{i_k \geq i_1 + \gamma} \left(\prod_{j=1}^k 2^{-i_j} \right) |B(x^{i_1}, \dots, x^{i_k})| \quad (143)$$

$$\leq k^2 \sum_{i \in \mathbb{N}^{k-1}} 2^{-2i_1 - \gamma} \left(\prod_{j=2}^{k-1} 2^{-i_j} \right) \sum_{i_k \in \mathbb{N}} |B(x^{i_1}, \dots, x^{i_k})|. \quad (144)$$

For fixed i_1, \dots, i_{k-1} ,

$$\sum_{i_k \in \mathbb{N}} |B(x^{i_1}, \dots, x^{i_k})| \leq \sum_{i_k \in \mathbb{N}} \sum_{\ell_1 \in \text{supp } x^{i_1}} \cdots \sum_{\ell_k \in \text{supp } x^{i_k}} |B(e_{\ell_1}, \dots, e_{\ell_k})| \quad (145)$$

$$= \sum_{\ell_1 \in \text{supp } x^{i_1}} \cdots \sum_{\ell_{k-1} \in \text{supp } x^{i_{k-1}}} \sum_{\ell_k \in [n]} |B(e_{\ell_1}, \dots, e_{\ell_k})| \quad (146)$$

$$\leq \sum_{\ell_1 \in \text{supp } x^{i_1}} \cdots \sum_{\ell_{k-1} \in \text{supp } x^{i_{k-1}}} m \quad (147)$$

$$= m \prod_{j=1}^{k-1} s_{i_j}, \quad (148)$$

where the last inequality is due to hypothesis 131. Plugging this into the bound from Equation 144 above gives

$$\sum_{i \in Q} \left(\prod_{j=1}^k 2^{-i_j} \right) |B(x^{i_1}, \dots, x^{i_k})| \leq k^2 \sum_{i \in \mathbb{N}^{k-1}} 2^{-2i_1 - \gamma} \left(\prod_{j=2}^{k-1} 2^{-i_j} \right) m \prod_{j=1}^{k-1} s_{i_j} \quad (149)$$

$$= k^2 2^{-\gamma} m \sum_{i_1 \in \mathbb{N}} 2^{-2i_1} s_{i_1} \sum_{i_2, \dots, i_{k-1}} \prod_{j=2}^{k-1} 2^{-i_j} s_{i_j} \quad (150)$$

$$= k^2 m 2^{-\gamma} \|x\|^2 \|x\|_1^{k-2} \quad (151)$$

$$\leq k^2 m n^{(k-2)/2} 2^{-\gamma} \|x\|^k, \quad (152)$$

using the fact that $\|x\|_1 \leq \sqrt{n} \|x\|$ (by Cauchy-Schwarz). Combining all the steps, we have

$$|B(x, \dots, x)| / \|x\|^k \leq b(2\gamma)^{k-1} + k^2 m n^{(k-2)/2} 2^{-\gamma}. \quad (153)$$

Finally, set $\gamma = \log(amn^{(k-2)/2}/b)$ (which is non-negative by the restriction on a) to find that

$$|B(x, \dots, x)| / \|x\|^k \leq b(\log^{k-1}(a^2 m^2 n^{k-2}/b^2) + k^2/a) \quad (154)$$

as desired. \square

Lemma 6.3.8. *Suppose B is a k -linear form such that*

$$|B(x, \dots, x)| \leq b \|x\|^k \quad (155)$$

for every $x \in \{0, \pm 2^{-\ell} : \ell \in \mathbb{N}\}^n$, and $B(e_{i_1}, \dots, e_{i_k}) = 0$ whenever i_1, \dots, i_k are not all distinct. Then $\|B\| \leq 2^k b$.

Proof. Let $x \in \mathbb{R}^n$ be a vector which maximizes $|B(x, \dots, x)| / \|x\|^k = \|B\|$. Without loss of generality, scale x so that $|x_i| \leq 1/2$ for all $i \in [n]$.

Choose a random vector $z \in \{0, \pm 2^{-\ell} : \ell \in \mathbb{N}\}^n$ by picking each coordinate z_i independently as follows:

If $x_i = 0$ then $z_i = 0$. Otherwise, write $|x_i| = 2^{\ell_i}(1 + \varepsilon_i)$ for some integer ℓ_i and some value of $\varepsilon_i \in [0, 1)$. Let $z_i = \text{sign}(x_i)2^{\ell_i}$ with probability $1 - \varepsilon_i$ and $z_i = \text{sign}(x_i)2^{\ell_i+1}$ with probability ε_i .

We can see that $\mathbb{E}[z_i] = x_i$ for all $i \in [n]$ and

$$\mathbb{E}[B(z, \dots, z)] = \sum_{i \in [n]^k} \mathbb{E}[B(z_{i_1} e_{i_1}, \dots, z_{i_k} e_{i_k})] = \sum_{i \in [n]^k} \left(\prod_{j=1}^k \mathbb{E}[z_{i_j}] \right) B(e_{i_1}, \dots, e_{i_k}) \quad (156)$$

$$= \sum_{i \in [n]^k} \left(\prod_{j=1}^k x_{i_j} \right) B(e_{i_1}, \dots, e_{i_k}) = B(x, \dots, x). \quad (157)$$

Because of this, there must be a vector z for which $|B(z, \dots, z)| \geq |B(x, \dots, x)|$. Observe that by construction $\|z\| \leq 2 \|x\|$, so

$$|B(x, \dots, x)| \leq |B(z, \dots, z)| \leq b \|z\|^k \leq 2^k b \|x\|^k. \quad (158)$$

Consequently, $\|B\| = |B(x, \dots, x)| / \|x\|^k \leq 2^k b$. \square

Finally, we put all of these lemmas together to prove the theorem.

Proof of Theorem 6.3.4. Suppose H is a k -uniform hypergraph on n vertices with maximum degree r satisfying

$$|e(V_1, \dots, V_k) - \alpha e_K(V_1, \dots, V_k)| \leq \rho \sqrt{\prod_i |V_i|} \quad (159)$$

for every choice of disjoint sets $V_1, \dots, V_k \subseteq V(H)$. By Lemma 6.3.5, the adjacency form A in fact satisfies

$$|A(x^1, \dots, x^k) - \alpha A_K(x^1, \dots, x^k)| \leq \rho k^{k/2} \prod_{i=1}^k \|x^i\| \quad (160)$$

for every $x^1, \dots, x^k \in \{0, 1\}^n$. Taking $B = A - \alpha A_K$ in Lemma 6.3.6 gives the fact that

$$|B(x^1, \dots, x^k)| \leq \rho (2k)^{k/2} \prod_{i=1}^k \|x^i\| \quad (161)$$

for all $x^1, \dots, x^k \in \{0, \pm 1\}^n$. Since for any fixed i_1, \dots, i_{k-1}

$$\sum_{j=1}^n |B(e_{i_1}, \dots, e_{i_{k-1}}, e_j)| \leq \sum_{j=1}^n |A(e_{i_1}, \dots, e_{i_{k-1}}, e_j)| + \alpha \sum_{j=1}^n |A_K(e_{i_1}, \dots, e_{i_{k-1}}, e_j)| \quad (162)$$

$$\leq r + \alpha n, \quad (163)$$

we can use $m = r + \alpha n$, $b = \rho (2k)^{k/2}$ and $a = (2k)^{k/2}$ in Lemma 6.3.7 (using Proposition 6.3.1 to guarantee the constraint on a) to find that

$$|B(x, \dots, x)| \leq \rho (2k)^{k/2} \left(\log^{k-1} \left(\frac{(d + \alpha n)^2}{\rho^2} n^{k-2} \right) + k^2 (2k)^{-k/2} \right) \|x\|^k, \quad (164)$$

for every $x \in \{0, \pm 2^{-\ell} : \ell \in \mathbb{N}\}^n$. Finally, by Lemma 6.3.8 we find that

$$\lambda_{2,\alpha}(H) = \|B\| \leq 2^{3k/2} k^{k/2} \rho \left(\log^{k-1} \left(\frac{(d + \alpha n)^2}{\rho^2} n^{k-2} \right) + k^2 (2k)^{-k/2} \right) \quad (165)$$

$$= \rho O(\log^{k-1}((r + \alpha n)n^{k-2}/\rho) + 1). \quad (166)$$

□

Open Problem: We leave as an open question whether or not the logarithmic factor $\log^{k-1}((r + \alpha n)n^{k-2}/\rho)$ in the inverse mixing lemma is necessary.

6.3.3 Comparison with the Friedman-Wigderson definition of λ_2

In this section we prove an inverse mixing lemma for the Friedman-Wigderson definition of the second eigenvalue. We will see that this result, while tight, is not as useful as Theorem 6.3.4, and we briefly discuss the reason for this.

Our proof method will involve citing our main result in the previous subsection with $\alpha = \frac{r}{n}$. In order to do this we need to describe the difference between $A - \frac{r}{n}J$ and $A - \frac{r}{n}A_K$. This k -linear form will relate to D , the diagonal form.

Definition 6.3.6. Let $D = J - A_K$ denote the k -linear form with $D(e_{i_1}, \dots, e_{i_k}) = 1$ if and only if the indices i_j are not all distinct (and 0 otherwise).

Proposition 6.3.9. $\|D\| = \Theta(n^{(k-2)/2})$.

Proof. First of all note that

$$\|D\| \geq \left| D(\vec{1}, \dots, \vec{1}) \right| / \|\vec{1}\|^k \tag{167}$$

$$= \frac{n^k - n!(n-k)!}{n^{k/2}} = \Omega(n^{(k-2)/2}). \tag{168}$$

On the other hand, for any $x \in \mathbb{R}^n$

$$|D(x, \dots, x)| \leq \sum_{i \in [n]^k} \left(\prod_{j=1}^k |x_{i_j}| \right) |D(e_{i_1}, \dots, e_{i_k})| \quad (169)$$

$$= \sum_{\substack{i \in [n]^k \\ i_j \text{ not all distinct}}} \prod_{j=1}^k |x_{i_j}| \quad (170)$$

$$\leq k^2 \sum_{i \in [n]^{k-1}} |x_{i_1}| \prod_{j=1}^{k-1} |x_{i_j}| \quad (171)$$

$$= k^2 \sum_{i_1=1}^n |x_{i_1}|^2 \prod_{j=2}^{k-1} \sum_{i_j=1}^n |x_{i_j}| \quad (172)$$

$$= k^2 \|x\|_2^2 \|x\|_1^{k-2} \quad (173)$$

$$\leq k^2 n^{(k-2)/2} \|x\|^k, \quad (174)$$

as desired. \square

Now we prove the inverse mixing lemma for $\lambda_2(H)$.

Theorem 6.3.10. *Let H be a k -uniform hypergraph with maximum degree r , and suppose that for every choice of disjoint sets $V_1, \dots, V_k \subset V(H)$,*

$$\left| e(V_1, \dots, V_k) - \frac{r}{n} \prod_i |V_i| \right| \leq \rho \sqrt{\prod_i |V_i|}. \quad (175)$$

Then

$$\lambda_2(H) = \Theta(rn^{(k-4)/2}) \pm O((\log^{k-1}(rn^{k-2}/\rho) + 1)\rho). \quad (176)$$

Proof. Set $\alpha = \frac{r}{n}$, so that $r = \Theta(\alpha n)$. Observe that if V_1, \dots, V_k are disjoint, then

$$\left| e(V_1, \dots, V_k) - \frac{r}{n} e_K(V_1, \dots, V_k) \right| = \left| e(V_1, \dots, V_k) - \frac{r}{n} \prod_i |V_i| \right| \leq \rho \sqrt{\prod_i |V_i|}. \quad (177)$$

By Theorem 6.3.4,

$$\|A_H - \frac{r}{n} A_K\| = O((\log^{k-1}(rn^{k-2}/\rho) + 1)\rho), \quad (178)$$

and hence by Proposition 6.3.9

$$\lambda_2(H) = \|A_H - \frac{r}{n}J\| \leq \frac{r}{n}\|D\| + \|A_H - \frac{r}{n}A_K\| \quad (179)$$

$$= O\left(rn^{(k-4)/2} + (\log^{k-1}(rn^{k-2}/\rho) + 1)\rho\right). \quad (180)$$

But also,

$$\|A_H - \frac{r}{n}J\| \geq \frac{r}{n}\|D\| - \|A_H - \frac{r}{n}A_K\|, \quad (181)$$

it follows that

$$\lambda_2(H) \geq \Omega(rn^{(k-4)/2}) - \rho O(\log^{k-1}(rn^{k-2}/\rho) + 1). \quad (182)$$

□

If the first term dominates in Equation 182 then the asymptotics of λ_2 are independent of ρ and so there is no interesting inverse mixing for this definition of the second eigenvalue. We now show that this is in fact typically the case by examining ρ_α for random hypergraphs.

To get some idea about the typical magnitude of ρ_α , we analyze the Erdős-Renyi random hypergraph $G(n, \alpha, k)$, in which each of the $\binom{n}{k}$ k -tuples is taken as a hyperedge independently with probability α . In hypergraphs this model was developed by Linial and Meshulam [47].

Proposition 6.3.11. *For the Erdős-Renyi random hypergraph $G = G(n, \alpha, k)$, with high probability $\rho_\alpha(G) \leq \sqrt{\frac{n \log(k+1) + n + \log(2)}{2}}$.*

Proof. For fixed disjoint sets of vertices V_1, \dots, V_k , note that $e(V_1, \dots, V_k)$ is a sum of $\prod_{i=1}^k |V_i|$ independent Bernoulli random variables each with mean α .

By Hoeffding's inequality [39], its deviation from its mean satisfies

$$\Pr \left[\left| e(V_1, \dots, V_k) - \alpha \prod_i |V_i| \right| > t \right] \leq 2e^{-2t^2 / \prod_i |V_i|}. \quad (183)$$

Plugging in $t = \rho \sqrt{\prod_i |V_i|}$ gives

$$\Pr \left[\left| e(V_1, \dots, V_k) - \alpha \prod_i |V_i| \right| > \rho \sqrt{\prod_i |V_i|} \right] \leq 2e^{-2\rho^2}. \quad (184)$$

Note that there are at most $(k+1)^n$ choices of the list of subsets V_i . Taking a union bound over all such choices, we find that as long as

$$\delta \geq 2(k+1)^n e^{-2\rho^2}, \quad \text{or equivalently} \quad \rho \geq \sqrt{\frac{n \log(k+1) + \log(2/\delta)}{2}}, \quad (185)$$

then with probability at least $1 - \delta$ the random hypergraph satisfies

$$|e(V_1, \dots, V_k) - \alpha e_K(V_1, \dots, V_k)| \leq \rho \sqrt{\prod_i |V_i|}. \quad (186)$$

for *all* choices of V_1, \dots, V_k . In particular, for $\delta = e^{-n}$ we have $\rho_\alpha(G) \leq \sqrt{\frac{n \log(k+1) + n + \log(2)}{2}}$ with probability at least $1 - e^{-n}$.

□

We can prove that this bound is tight (up to a multiplicative factor depending on k) in the case where α is constant with respect to n .

Proposition 6.3.12. *For any hypergraph H and any constant $\alpha \in [0, 1]$,*

$$\rho_\alpha(H) \geq \frac{\alpha(1-\alpha)}{\sqrt{\alpha^2 + (1-\alpha)^2}} \sqrt{n-k+1}.$$

Proof. Set V_1, \dots, V_{k-1} to be distinct singletons v_1, \dots, v_{k-1} . Define

$$S = \{v \in V \setminus \{v_1, \dots, v_{k-1}\} : \{v_1, \dots, v_{k-1}, v\} \in E(H)\}$$

$$\text{and } T = \{v \in V \setminus \{v_1, \dots, v_{k-1}\} : \{v_1, \dots, v_{k-1}, v\} \notin E(H)\}.$$

Observe that $|S| + |T| = n - k + 1$.

Also, that

$$\begin{aligned}\rho_\alpha(H) &\geq \max \left(\left| \frac{e(\{v_1\}, \dots, \{v_{k-1}\}, S) - \alpha |S|}{\sqrt{|S|}} \right|, \left| \frac{e(\{v_1\}, \dots, \{v_{k-1}\}, T) - \alpha |T|}{\sqrt{|T|}} \right| \right) \\ &= \max \left((1 - \alpha) \sqrt{|S|}, \alpha \sqrt{|T|} \right).\end{aligned}$$

If $|S| \geq (n - k + 1) \frac{\alpha^2}{(1 - \alpha)^2 + \alpha^2}$, then

$$(1 - \alpha) \sqrt{|S|} \geq \frac{\alpha(1 - \alpha)}{\sqrt{\alpha^2 + (1 - \alpha)^2}} \sqrt{n - k + 1}.$$

On the other hand, if $|S| \leq (n - k + 1) \frac{\alpha^2}{(1 - \alpha)^2 + \alpha^2}$, then

$$|T| = (n - k + 1) - |S| \geq (n - k + 1) \frac{(1 - \alpha)^2}{(1 - \alpha)^2 + \alpha^2},$$

and

$$\alpha \sqrt{|T|} \geq \frac{\alpha(1 - \alpha)}{\sqrt{\alpha^2 + (1 - \alpha)^2}} \sqrt{n - k + 1}.$$

□

Combining the previous two propositions proves the probabilistic bound on discrepancy:

Corollary 6.3.13. *For the dense Erdős-Renyi random hypergraph $G = G(n, \alpha, k)$ where $\alpha \in (0, 1)$ is constant with respect to n , with high probability $\rho_\alpha(G) = \Theta(\sqrt{n})$.*

Assume that $\alpha = \frac{r}{n}$ is a positive constant independent of n , in other words, that $r = \Theta(n)$.

We now can see that the Friedman-Wigderson eigenvalue λ_2 will, with high probability, not allow an interesting inverse mixing lemma.

Corollary 6.3.14. *For the dense Erdős-Renyi random hypergraph $G = G(n, \alpha, k)$ where $\alpha \in (0, 1)$ is constant with respect to n and $k \geq 4$, with high probability $\lambda_2 = \Theta(n^{(k-2)/2})$.*

Proof. For G that satisfies the bound in Corollary 6.3.13, the second term of Equation 182 is $\Theta(\sqrt{n} \log^{k-1}(n))$, which is dominated by the first term if $k \geq 4$. So, almost every hypergraph with $k \geq 4$ will not have an interesting inverse mixing lemma for the Friedman-Wigderson definition of the second eigenvalue. \square

It is unknown whether there is a high quality of inverse mixing for the Friedman-Wigderson eigenvalue when $k = 3$.

Remark. The reason for the low quality of inverse mixing under the Friedman-Wigderson definition is the fact that we do not allow loops, so that edges of the hypergraph must contain k distinct vertices. Coefficients of A that correspond to these loops must be 0. The dominating term of Equation 182 derives from these coefficients, which are the non-zero coefficients of the k -linear form D .

We made the choice not to allow loops in order to prove the sequence of Lemmas 6.3.5-6.3.8. In other works loops are allowed in hypergraphs; indeed, the original definition of λ_2 in [34] is for a model of hypergraph that permits loops. It is unknown whether or not λ_2 will give a higher quality of inverse mixing in that model.

A corollary of this work is a bound on the behavior of $\lambda_{2,\alpha}$ for random hypergraphs.

Corollary 6.3.15. *For the dense Erdős-Renyi random hypergraph $G = G(n, \alpha, k)$ where $\alpha \in (0, 1)$ is constant with respect to n , with high probability $\lambda_{2,\alpha}(G) = \Omega(\sqrt{n})$ and $\lambda_{2,\alpha}(G) = O(\sqrt{n} \log^{k-1} n)$.*

This follows from combining Proposition 6.3.13 with the bounds on $\lambda_{2,\alpha}$ found in Theorem 6.3.3 and Theorem 6.3.4.

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